Notes on Some (0,2) Supersymmetric Theories in Two Dimensions

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(ABSTRACT)

This thesis is devoted to a discussion of two-dimensional theories with (0,2) supersymmetry. Examples of two-dimensional (0,2) gauged linear sigma models (GLSMs) are constructed for various spaces including Grassmannians, complete intersections in Grassmannians, and non-complete intersections such as Pfaffians. Generalizations of (2,2) Toda dual theories to (0,2) Toda-like theories are also discussed and some examples are given, including products of projective spaces and del Pezzo surfaces. Correlation functions are computed to show the examples are the correct mirror models.

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(General Audience Abstract)

This thesis is devoted to a discussion in a specific branch of string theory in particle physics. String theory is a model of elementary particles, which are currently described by a theory called the Standard Model. String theory is a more fundamental theory than the Standard Model in two ways: it incorporates general relativity, *i.e.*, the theory of gravity; and it is a ultraviolet theory of the Standard Model, or equivalently, the Standard model is seen as a low energy approximation of string theory. This thesis is concerned with the quantum mechanics of string theory, described by quantum field theory along a two-dimensional worldsheet swept out by a one-dimensional string as it propagates in time. Specifically, this thesis explores examples of two-dimensional worldsheet theories with a technical property known as (0,2) supersymmetry.

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Chapter 1

Introduction

We currently describe elementary particle physics with the Standard Model. The Standard Model describes the electromagnetic, weak, and strong interactions, as well as classifying all known elementary particles. It is renormalizable and mathematically self-consistent, however despite making successful experimental predictions, it does leave some phenomena unexplained. In particular, general relativity is not incorporated. Instead, general relativity is understood as a low-energy effective field theory, which breaks down at energies and curvatures that are comparable to the Planck scale. One of the main tasks in physics is to find a way to combine the theory of gravity and the Standard Model in a mathematically consistent way valid at higher scales. In particular, the theory must be renormalizable, like the Standard Model itself. However, general relativity is not renormalizable. Physicists have tried many ways to construct a consistent theory of quantum gravity, but it was not accomplished until string theory emerged.

String theory appeared as a theory of the strong nuclear force in the late 1960s. Among other things, this theory predicts a massless spin two particle, which was unsuitable for a theory of nuclear physics. However, this property made string theory a promising candidate for a quantum theory of gravity. As its name would suggest, string theory replaces point-like particles with one-dimensional objects called strings. The vibrational modes are interpreted as different elementary particles with its mass, charge, and other properties, since on large distance scales, a two-dimensional string behaves like an ordinary particle. A Feynman diagram in string theory is a two-dimensional smooth surface, so it avoids the singularities in loop diagrams of particles. The zero-distance behavior is therefore irrelevant in string theory, and this makes string theory very attractive as a theory of quantum gravity.

Among various string theories, heterotic string theory is a hybrid combining elements of superstring and bosonic string theories. Its worldsheet theory treats left-movers as bosonic strings propagating in 26 dimensional space and right-movers as superstrings propagating in ten-dimensional space. Advantages of heterotic string theory include the fact that it describes both bosons and fermions as well as a nonabelian ($E_8 \times E_8$ or $Spin(32)/Z_2$) gauge symmetry.

Superstring theories are ten-dimensional theories whereas the spacetime of the observed universe is four-dimensional, so one immediately has a question of how observed four-dimensional physics is related to an underlying ten-dimensional physics. One proposal is to assume that the 10-manifold $\mathbf{M}^{10} = M^4 \times K$, where M^4 is an external non-compact spacetime, and K is the internal manifold with dimension six.

In addition, we want the low-energy theory to have other properties. For example, supersymmetry is one possible solution to the hierarchy problem, so we want to preserve it at the compactification scale. Furthermore, we are primarily interested in only unbroken $\mathcal{N} = 1$ supersymmetry in four dimensions because in four-dimensional supersymmetric theories with $\mathcal{N} \geq 2$ supersymmetry, the massless fermions transform in a real representation of the gauge group, which is not observed in nature [4].

With some appropriate assumptions and algebra, one can show that in order to preserve $\mathcal{N} = 1$ supersymmetry in four dimensions, the following conditions must be obeyed on ten-dimensional spacetime:

$$\delta\psi_M = D_M\eta = 0$$
$$\delta\chi^a = \Gamma^{ij}F^a_{ij}\eta$$

with supersymmetry parameter η. ψ_M is a Majorana gravitino, χ^a is a gaugino, and F^a_{ij} is the Yang-Mills field strength, $M = 1, 2, \ldots, 10$.

The first condition

$$\delta\psi_M = D_M\eta = 0$$

implies the integrability condition

$$[D_M, D_N]\eta = 0,$$

in other words,

$$R_{MNPQ}\Gamma^{PQ}\eta = 0,$$

with R_{MNPQ} the Riemann tensor. It further implies that the external spacetime M^4 must be flat Minkowski space as one assumes that M^4 is maximally symmetric.

What do the conditions say about the internal space? On the internal space K, it admits a spinor field η such that $D_i\eta = 0$, $i = 5, 6, \ldots, 10$. This implies that K admits a metric of $SU(3)$ holonomy. It can be shown that a Kähler manifold of vanishing first Chern class always admits a Kähler metric of $SU(N)$ holonomy [4]. A Kähler manifold of vanishing first Chern class is called a Calabi-Yau manifold. So the conditions for unbroken supersymmetry impose additional conditions on both M^4 and K: M^4 is Minkowski, as one hopes; and K is a Calabi-Yau manifold.

One often writes (2,2) supersymmetry for $\mathcal{N} = 2$ supersymmetry in $D = 2$. The numbers (a,b) means that the number of left-moving supercharges is a and the number of right-hand supercharges is b. One can read off that for (0,2) supersymmetry, there is no left-moving supersymmetry and there are two right-moving supercharges, which is a generalization of (2,2) supersymmetry. It can be shown [1] that to get a four-dimensional theory with $\mathcal{N} = 1$ supersymmetry, one needs a worldsheet theory with (0,2) supersymmetry. However, historically far more work has been done on theories with (2,2) supersymmetry, leaving many unanswered questions in (0,2) supersymmetry.

This thesis focuses on generalizations of (2,2) worldsheet supersymmetry to (0,2) supersymmetry, for those are important in compactification of heterotic strings. The next chapter following the introduction is about the construction of (0,2) version of GLSMs on various spaces: Grassmannians, complete intersections inside Grassmannians, and Pfaffians. This is used in [90] for tests of two-dimensional gauge dualities.

Chapters 3 and 4 concern (0,2) generalizations of mirror symmetry. This is a symmetry in which sigma models on two different spaces are secretly equivalent to one another. In this thesis, we focus on mirror symmetry between A-twisted nonlinear sigma models on positively-curved (Fano) spaces and B-twisted Landau-Ginzburg theories, and their (0,2) generalizations. We wish to build more examples of (0,2) supersymmetric version of mirror symmetry as (0,2) mirror symmetry is not well understood currently. Chapter 3 proposes a (0,2) generalization of mirror symmetry for products of projective spaces, and chapter 4 for del Pezzo surfaces.

Chapter 2

Construction of anomaly-free models on various spaces

In this chapter we will construct examples of gauged linear sigma models (GLSMs) [2] with (0,2) supersymmetry, probing geometries which have not previously been explored with GLSMs, following other recent developments in the field [5–14, 16–18, 20–34]. The results in section 2.2, 2.3, 2.5 were partially published in [90], and describe our own work.

One special property of two-dimensional field theories is that gauge fields can be integrated out. This is because in two dimensions, gauge fields do not have propagating degrees of freedom. After integrating them out, we are left with a nonlinear sigma model (NLSM). Therefore the study of GLSMs can be turned into a study of the geometries of target spaces of NLSMs. This can be useful for two-dimensional dualities. If two different GLSMs renormalization group (RG) flow to the same weakly-coupled nonlinear sigma model, then in principle one has shown that they have the same IR limit. This is actually a two-dimensional analogue of a Seiberg duality, which implies that Higgs moduli spaces, chiral rings, and global symmetries match.

Topological field theory often plays a important role in mirror symmetry, which exchanges the A model and the B model. Recall in a supersymmetric nonlinear sigma model with worldsheet Σ and target space X, with action

$$\frac{1}{\alpha'}\int_{\Sigma} d^2z\left((g_{\mu\nu}+iB_{\mu\nu})\,\partial\phi^{\mu}\overline{\partial}\phi^{\nu} \;+\; \frac{i}{2}g_{\mu\nu}\psi_+^{\mu}D_{\overline{z}}\psi_+^{\nu} \;+\; \frac{i}{2}g_{\mu\nu}\psi_-^{\mu}D_{z}\psi_-^{\nu} \;+\; R_{i\overline{\jmath}k\overline{l}}\psi_+^{i}\psi_+^{\overline{\jmath}}\psi_-^{k}\psi_-^{\overline{l}}\right),$$

fermions live on

$$\begin{aligned}
\psi_+^{i} &\in \Gamma_{C^\infty}\left(K_\Sigma^{1/2}\otimes\phi^*T^{1,0}X\right), &\qquad \psi_-^{i} &\in \Gamma_{C^\infty}\left(\overline{K}_\Sigma^{1/2}\otimes(\phi^*T^{0,1}X)^*\right),\\
\psi_+^{\overline{\imath}} &\in \Gamma_{C^\infty}\left(K_\Sigma^{1/2}\otimes(\phi^*T^{1,0}X)^*\right), &\qquad \psi_-^{\overline{\imath}} &\in \Gamma_{C^\infty}\left(\overline{K}_\Sigma^{1/2}\otimes\phi^*T^{0,1}X\right),
\end{aligned}$$

where ψ_+ indicates right-moving fermions, ψ_- indicates left-moving fermions. K_Σ and $\overline{K}_\Sigma$ are canonical and anti-canonical line bundles of Σ, and ϕ^* is the pullback map from X to Σ.

The topological A model is a twist of the ordinary supersymmetric nonlinear sigma model, with action [54]

$$\frac{1}{\alpha'}\int_{\Sigma} d^2z \left((g_{\mu\nu} + iB_{\mu\nu})\, \partial\phi^{\mu}\overline{\partial}\phi^{\nu} \;+\; \frac{i}{2} g_{\mu\nu}\psi_+^{\mu} D_{\overline{z}}\psi_+^{\nu} \;+\; \frac{i}{2} g_{\mu\nu}\psi_-^{\mu} D_z\psi_-^{\nu} \;+\; R_{i\overline{\jmath}k\overline{l}}\psi_+^i\psi_+^{\overline{\jmath}}\psi_-^k\psi_-^{\overline{l}} \right).$$

The twist turns worldsheet fermions into worldsheet scalars and vectors as follows [19]:

$$\begin{aligned} \psi_+^i &\in \Gamma_{C^\infty}\left(\phi^* T^{1,0}X\right), & \psi_-^i &\in \Gamma_{C^\infty}\left(\overline{K}_\Sigma \otimes \left(\phi^* T^{0,1}X\right)^*\right), \\ \psi_+^{\overline{\imath}} &\in \Gamma_{C^\infty}\left(K_\Sigma \otimes \left(\phi^* T^{1,0}X\right)^*\right), & \psi_-^{\overline{\imath}} &\in \Gamma_{C^\infty}\left(\phi^* T^{0,1}X\right). \end{aligned}$$

The heterotic analogue of the A model, known as the A/2 model, is a twist of the $(0,2)$ nonlinear sigma model [54]

$$\frac{1}{\alpha'}\int_{\Sigma} d^2z \left((g_{\mu\nu} + iB_{\mu\nu})\, \partial\phi^{\mu}\overline{\partial}\phi^{\nu} \;+\; \frac{i}{2} g_{\mu\nu}\psi_+^{\mu} D_{\overline{z}}\psi_+^{\nu} \;+\; \frac{i}{2} h_{\alpha\beta}\lambda_-^{\alpha} D_z\lambda_-^{\beta} \;+\; F_{i\overline{\jmath}a\overline{b}}\psi_+^i\psi_+^{\overline{\jmath}}\lambda_-^a\lambda_-^{\overline{b}} \right),$$

in which the fermions couple to bundles as follows:

$$\begin{aligned} \psi_+^i &\in \Gamma_{C^\infty}\left(\phi^* T^{1,0}X\right), & \lambda_-^a &\in \Gamma_{C^\infty}\left(\overline{K}_\Sigma \otimes \phi^*\overline{\mathcal{E}}^*\right), \\ \psi_+^{\overline{\imath}} &\in \Gamma_{C^\infty}\left(K_\Sigma \otimes \left(\phi^* T^{1,0}X\right)^*\right), & \lambda_-^{\overline{a}} &\in \Gamma_{C^\infty}\left(\phi^*\overline{\mathcal{E}}\right), \end{aligned}$$

where $\mathcal{E}$ is a holomorphic vector bundle on X.

We begin in section 2.2 by discussing some toy (0,2) GLSMs on Grassmannians, as basic examples and warm-ups for later constructions. We relate gauge anomaly cancellation to cohomological conditions on Chern class in Grassmannians, and discuss the details of several examples. In section 2.3 we outline some constructions of nonabelian (0,2) theories corresponding to complete intersections in Grassmannians and affine Grassmannians. In section 2.5, we construct (0,2) models on Pfaffians.

2.1 Overview

In this section, we will review ordinary (2,2) gauged linear sigma models (GLSMs) in two dimensions as well as (0,2) GLSMs in order to provide background for the analysis of the following sections.

2.1.1 Review of superspace

Since $\mathcal{N} = 2$ supersymmetry in two dimensions can be obtained by dimensional reduction from $\mathcal{N} = 1$ supersymmetry in four dimensions, we begin with a brief introduction to $\mathcal{N} = 1$

supersymmetry in four dimensions to make this thesis more nearly self-contained. Working in superspace $\mathbb{R}^{1,3|4}$ with coordinates $x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}$, we define the supersymmetry operators [40]

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^m},$$
$$\overline{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\sigma^m_{\alpha\dot{\alpha}}\theta^\alpha\frac{\partial}{\partial x^m},$$

where α and $\dot{\alpha}$ are the two chiralities of spinors. Also one writes $\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta,\ \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$, where ϵ is the antisymmetric tensor with $\epsilon^{12} = -\epsilon_{12} = 1$, and similarly for dotted indices. Here, σ^m are Pauli matrices.

Supersymmetric covariant derivatives commute with the supersymmetric operators [40]

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^m},$$
$$\overline{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\sigma^m_{\alpha\dot{\alpha}}\theta^\alpha\frac{\partial}{\partial x^m}.$$

Using the supersymmetric covariant derivatives, one can define chiral superfields by $\overline{D}_{\dot{\alpha}}\Phi = 0$ and antichiral superfields by $D_\alpha\bar{\Phi} = 0$. They can be expanded in terms of $y^m = x^m + i\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}$ [40]

$$\Phi(x,\theta) = \phi(y) + \sqrt{2}\theta^\alpha\psi_\alpha(y) + \theta^\alpha\theta_\alpha F(y), \tag{2.1}$$
$$\overline{\Phi}(x,\theta) = \bar{\phi}(\bar{y}) + \sqrt{2}\bar{\theta}_{\dot{\alpha}}\overline{\psi}^{\dot{\alpha}}(\bar{y}) + \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}F(\bar{y}). \tag{2.2}$$

Next we consider gauge theories. One then needs to promote the differential operators D_α, $\overline{D}_{\dot{\alpha}}$, and $\partial_m = \partial/\partial x^m$ to gauge covariant derivatives $\mathcal{D}_\alpha$, $\bar{\mathcal{D}}_{\dot{\alpha}}$, and $\mathcal{D}_m$. We begin by defining a vector superfield to be a real superfield

$$V = V^\dagger. \tag{2.3}$$

For the abelian gauge group $U(1)$, V is invariant under gauge transformation

$$V \to V + i(\Lambda - \overline{\Lambda}), \tag{2.4}$$

where Λ is a chiral superfield. A chiral superfield Φ of charge Q transforms as

$$\Phi \to \exp(-iQ\Lambda)\cdot\Phi.$$

One can partially fix this gauge invariance by using the transformations above to put V in the form [40]

$$V = -\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}A_m + i\theta^\alpha\theta_\alpha\bar{\theta}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} - i\bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\theta^\alpha\lambda_\alpha + \frac{1}{2}\theta^\alpha\theta_\alpha\bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}D. \tag{2.5}$$

This is called Wess-Zumino gauge.

One can construct a chiral superfield which contains the gauge invariant field strength [40]

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V.$$

Then the most general gauge invariant Lagrangian for a vector superfield is

$$\begin{aligned}\mathcal{L}_V &= \int d^2\theta \mathrm{Tr}(W_\alpha W^\alpha + h.c.),\\ &= -\frac{1}{4}F^a_{mn}F^{mna} - i\lambda^a\sigma^m\mathcal{D}_m\overline{\lambda}^a + \frac{1}{2}(D^a)^2.\end{aligned}$$

This model is usually called $\mathcal{N}=1$ supersymmetric Yang-Mills theory.

After coupling to matter, the most general possible classical action takes the form

$$\begin{aligned}\mathcal{L} &= -g_{i\bar{j}}\mathcal{D}_m\phi^i\mathcal{D}^m\overline{\phi}^{\bar{j}} - ig_{i\bar{j}}\overline{\psi}^{\bar{j}}\overline{\sigma}^m\mathcal{D}_m\psi^i - i\overline{\lambda}^a\overline{\sigma}^m\mathcal{D}_m\lambda^a - \frac{1}{4}F^a_{mn}F^{mna} - \frac{1}{2}(D^a)^2\\ &+ \sqrt{2}g_{i\bar{j}}(X^{ia}\overline{\psi}^{\bar{j}}\overline{\lambda}^a + \overline{X}^{\bar{j}a}\psi^i\lambda^a) - \frac{1}{2}\mathcal{D}_iW_j\psi^i\psi^j - \frac{1}{2}\overline{\mathcal{D}_{\bar{i}}}\overline{W}_{\bar{j}}\overline{\psi}^{\bar{i}}\overline{\psi}^{\bar{j}}\\ &- g^{i\bar{j}}W_i\overline{W}_{\bar{j}} + \frac{1}{4}\mathcal{R}_{i\bar{j}k\bar{l}}\psi^i\psi^k\overline{\psi}^{\bar{j}}\overline{\psi}^{\bar{l}},\end{aligned}$$

where we set the gauge coupling constant to 1 and integrated out the auxiliary field F^i. Geometrically, ϕ's are maps into local coordinates on some space, known as the target space and required by supersymmetry to be Kähler, and X's are components of a holomorphic Killing vector field on the target space generating the gauge isometry. All derivatives are covariant:

$$\mathcal{D}_m\phi^i = \partial_m\phi^i - A^a_m X^{ia}, \tag{2.6}$$

$$\mathcal{D}_m\psi^i = \partial_m\psi^i + \Gamma^i_{jk}\mathcal{D}_m\phi^j\psi^k - A^a_m\partial_j X^{ia}\psi^j, \tag{2.7}$$

$$\mathcal{D}_m\lambda^a = \partial_m\lambda^a - t^{abc}A^b_m\lambda^c, \tag{2.8}$$

where t^{abc} are the structure constants of the gauge Lie algebra. When the target space is simply $\mathbb{C}^n$ with its natural Kähler potential $\sum|\phi^i|^2$, we get an important special case known as a gauged linear sigma model. This is the four-dimensional analogue of the two-dimensional theory we will deal with in the following discussion.

2.1.2 (2,2) supersymmetry in two dimensions

We can use dimensional reduction to get $\mathcal{N}=2$ supersymmetry in two dimensions from $\mathcal{N}=1$ supersymmetry in four dimensions [77]. We will work in the superspace $\mathbb{R}^{1,1|4}$ with coordinates $(x_0, x_1, \theta^{\pm}, \bar{\theta}^{\pm})$. The transformation laws with parameters $\epsilon_{\pm}$,$\bar{\epsilon}_{\pm}$ for the component fields are [2]

$$\begin{aligned}
\delta\phi &= \sqrt{2}\,(\epsilon_+\psi_- - \epsilon_-\psi_+)\,, \\
\delta\psi_+ &= i\sqrt{2}(D_0 + D_1)\phi\bar{\epsilon}_- + \sqrt{2}\epsilon_+ F - 2Q\phi\bar{\sigma}\,\bar{\epsilon}_+, \\
\delta\psi_- &= -i\sqrt{2}(D_0 - D_1)\phi\bar{\epsilon}_+ + \sqrt{2}\epsilon_- F + 2Q\phi\sigma\,\bar{\epsilon}_-, \\
\delta F &= -i\sqrt{2}\bar{\epsilon}_+(D_0 - D_1)\psi_+ - i\sqrt{2}\bar{\epsilon}_-(D_0 + D_1)\psi_-, \\
&\quad + 2Q(\bar{\epsilon}_+\bar{\sigma}\psi_- + \bar{\epsilon}_-\sigma\psi_+) + 2iQ\phi(\bar{\epsilon}_-\bar{\lambda}_+ - \bar{\epsilon}_+\bar{\lambda}_-),
\end{aligned}$$

and the transformation laws of the antichiral multiplet $\overline{\Phi}$ are the complex conjugate of these.

The transformation laws of the vector multiplet are [2]

$$\begin{aligned}
\delta A_m &= i\bar{\epsilon}\sigma_m\lambda + i\epsilon\sigma_m\bar{\lambda}, \\
\delta\sigma &= -i\sqrt{2}\bar{\epsilon}_+\lambda_- - i\sqrt{2}\epsilon_-\bar{\lambda}_+, \\
\delta\bar{\sigma} &= -i\sqrt{2}\epsilon_+\bar{\lambda}_- - i\sqrt{2}\bar{\epsilon}_-\lambda_+, \\
\delta D &= -\bar{\epsilon}_+(\partial_0 - \partial_1)\lambda_+ - \bar{\epsilon}_-(\partial_0 + \partial_1)\lambda_- + \epsilon_+(\partial_0 - \partial_1)\bar{\lambda}_+ + \epsilon_-(\partial_0 + \partial_1)\bar{\lambda}_-, \\
\delta\lambda_+ &= i\epsilon_+ D + \sqrt{2}(\partial_0 + \partial_1)\bar{\sigma}\epsilon_- - F_{01}\epsilon_+, \\
\delta\lambda_- &= i\epsilon_- D + \sqrt{2}(\partial_0 - \partial_1)\sigma\epsilon_+ + F_{01}\epsilon_-, \\
\delta\bar{\lambda}_+ &= -i\bar{\epsilon}_+ D + \sqrt{2}(\partial_0 + \partial_1)\sigma\bar{\epsilon}_- - F_{01}\bar{\epsilon}_+, \\
\delta\bar{\lambda}_- &= -i\bar{\epsilon}_- D + \sqrt{2}(\partial_0 - \partial_1)\bar{\sigma}\bar{\epsilon}_+ + F_{01}\bar{\epsilon}_-,
\end{aligned}$$

with $F_{01} = \partial_0 A_1 - \partial_1 A_0$.

Chiral superfields in two dimensions are defined the same way as in four dimensions:

$$\bar{D}_{\pm}\Phi = 0.$$

As a result, it also has the same expansion

$$\Phi = \phi + \sqrt{2}\theta\psi + \theta\theta F.$$

Note $\psi(\psi_-, \psi_+)$ is a two-dimensional Dirac spinor with two chiral components ψ_- and ψ_+.

One of the properties of two-dimensional theories is the existence of twisted chiral superfields:

$$\overline{D}_+\Sigma = D_-\Sigma = 0.$$

One should note that in two dimensions the basic gauge invariant field strength of the superspace gauge field is a twisted chiral superfield:

$$\Sigma = \frac{1}{2\sqrt{2}}\{\overline{\mathcal{D}}_+, \mathcal{D}_-\}.$$

It is easy to check that the above the superfield is indeed a twisted chiral superfield.

Lagrangian

In general, Lagrangians of GLSMs in two dimensions are of the form

$$L = L_K + L_W + L_{gauge} + L_{D,\theta},$$

where the terms represent the kinetic energy of chiral superfields, the superpotential interactions, the gauge kinetic energy, and the Fayet-Iliopoulos term and theta angle, respectively. We will give the details next.

Assume for simplicity that the gauge group is $U(1)$, and the charges of the k chiral superfields Φ_i are Q_i. As mentioned before, Φ_i can be treated as coordinates on the target space $\mathbb{C}^k$. As a matter of fact, the kinetic energy is totally determined by the geometry of the target space, *i.e.*, Kähler metric. The Lagrangian for the kinetic energy is [2]

$$\begin{aligned} L_K =& \int d^2y\, d^4\theta \sum_i \overline{\Phi}_i \Phi_i, \\ =& \sum_i \int d^2y\big(- D_\rho\overline{\phi}_i D^\rho \phi_i + i\overline{\psi}_{-,i}(D_0 + D_1)\psi_{-,i} + i\overline{\psi}_{+,i}(D_0 - D_1)\psi_{+,i} + |F_i|^2 \\ & - 2\overline{\sigma}\sigma Q_i^2 \overline{\phi}_i\phi_i - \sqrt{2}Q_i \left(\overline{\sigma}\overline{\psi}_{+i}\psi_{-i} + \sigma\overline{\psi}_{-i}\psi_{+i}\right) + DQ_i\overline{\phi}_i\phi_i \\ & - i\sqrt{2}Q_i\overline{\phi}_i(\psi_{-,i}\lambda_+ - \psi_{+,i}\lambda_-) - i\sqrt{2}Q_i\phi_i(\overline{\lambda}_-\overline{\psi}_{+,i} - \overline{\lambda}_+\overline{\psi}_{-,i})\big), \end{aligned}$$

where the worldsheet metric is $ds^2 = -(dy^0)^2 + (dy^1)^2$.

The second part of the Lagrangian involves the superpotential $W(\Phi_i)$, which is a holomorphic function on the target manifold: [2]

$$\begin{aligned} L_W =& - \int d^2y d\theta^+ d\theta^- W(\Phi_i)|_{\bar{\theta}^+=\bar{\theta}^-=0} - h.c., \\ =& - \int d^2y \left(F_i \frac{\partial W}{\partial \phi_i} + \frac{\partial^2 W}{\partial\phi_i \partial\phi_j}\psi_{-,i}\psi_{+,j} \right) - h.c. \end{aligned}$$

The gauge kinetic energy terms are constructed from the twisted chiral superfields Σ_a defined as above. Up to a coupling constant, the gauge kinetic energy is [2]

$$\begin{aligned} L_{gauge} =& - \int d^2y d^4\theta \overline{\Sigma}_a \Sigma_a, \\ =& \int d^2y\big(\frac{1}{2}{F_{01,a}}^2 + \frac{1}{2}{D_a}^2 + i\overline{\lambda}_{+,a}(\partial_0 - \partial_1)\lambda_{+,a} \\ & + i\overline{\lambda}_{-,a}(\partial_0 + \partial_1)\lambda_{-,a} - |\partial_\rho\sigma_a|^2\big). \end{aligned}$$

One can also add supersymmetric total derivatives to the action. For example,

$$-r \int d^2y\ D$$

which is known as the Fayet-Iliopoulos term. Another coupling involves an angular variable θ,

$$\frac{\theta}{2\pi}\int dv = \frac{\theta}{2\pi}\int d^2yF_{01}.$$

θ is called the theta angle, since the quantity $\int dv/2\pi$ measures the first Chern class of the abelian gauge field and is always integral. So the last term in Lagrangian is [2]

$$L_{D,\theta} = \int d^2y\left(-rD + \frac{\theta}{2\pi}F_{01}\right).$$

2.1.3 (0,2) supersymmetry in two dimensions

One can generalize (2,2) supersymmetry to (0,2) supersymmetry, by omitting the left-moving supersymmetry. In such theories, the supercharges are now Q_+, $\overline{Q}_+$, and the corresponding superspace is $\mathbb{R}^{1,1|2}$ with coordinates $(x_0, x_1, \theta^+, \bar{\theta}^+)$.

The supersymmetry operators are the same:

$$Q_+ = \frac{\partial}{\partial\theta^+} + i\bar{\theta}^+\left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}\right),$$
$$\overline{Q}_+ = -\frac{\partial}{\partial\bar{\theta}^+} - i\theta^+\left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}\right).$$

The supersymmetry derivatives are

$$D_+ = \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+\left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}\right),$$
$$\overline{D}_+ = -\frac{\partial}{\partial\bar{\theta}^+} + i\theta^+\left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}\right),$$

which can be used to define chiral superfields as

$$\bar{\mathcal{D}}_+\Phi = 0.$$

Chiral superfields can be expanded in terms of superspace coordinates:

$$\Phi = \phi + \sqrt{2}\theta^+\psi_+ - i\theta^+\bar{\theta}^+(\partial_0 + \partial_1)\phi.$$

One can see from the expansion above that ϕ has a superpartner ψ_+ as a right-moving fermion, but no left-moving fermion ψ_- as in (2,2) case. This is because (0,2) supersymmetry only has right-moving supersymmetry transformations.

One can have analogous left-moving fermions by defining a so called Fermi superfield Λ_- obeying

$$\overline{\mathcal{D}}_+\Lambda_- = \sqrt{2}E,$$

where E is a chiral superfield. The expansion of the Fermi multiplet is [2]

$$\Lambda_- = \lambda_- - \sqrt{2}\theta^+ G - i\theta^+\bar{\theta}^+(\partial_0 + \partial_1)\lambda_- - \sqrt{2}\bar{\theta}^+ E.$$

The lowest component λ_- plays the role of the left-moving fermion when E is taken appropriate value. Notice that $E = E(\Phi_i)$ is a holomorphic function of some chiral superfields Φ_i, one has the expansion [2]

$$E(\Phi_i) = E(\phi_i) + \sqrt{2}\theta^+ \frac{\partial E}{\partial \phi_i}\psi_{+,i} - i\theta^+\bar{\theta}^+(\partial_0 + \partial_1)E(\phi_i).$$

A (0,2) version of vector superfields can be obtained by dimensional reduction from $\mathcal{N} = 1$ four-dimensional vector superfields. In Wess-Zumino gauge these can be expanded as [2]

$$V = A_0 - A_1 - 2i\theta^+\overline{\lambda}_- - 2i\bar{\theta}^+\lambda_- + 2\theta^+\bar{\theta}^+ D.$$

where A_0, A_1 are components of the two-dimensional gauge field, λ_- is the left-moving gaugino, and D is the real scalar auxiliary field.

The basic gauge invariant field strength is [2]

$$\Upsilon = -\lambda_- i\theta^+(D - F_{01}) - i\bar{\theta}^+(\mathcal{D}_0 + \mathcal{D}_1)\lambda_-,$$

where $F_{01} = \partial_0 A_1 - \partial_1 A_0 - i[A_0, A_1]$ is the gauge field strength, and $\mathcal{D}_m = \partial_m - iA_m$ is the gauge covariant derivatives.

Lagrangian

The Lagrangian for (0,2) GLSM has five parts

$$L = L_{gauge} + L_{ch} + L_F + L_{D,\theta} + L_J,$$

which we will describe next.

The first term is the kinetic energy of the gauge field strength [2]:

$$\begin{aligned} L_{gauge} &= \int d^2y d\theta^+ d\bar{\theta}^+ \mathrm{Tr}\overline{\Upsilon}\Upsilon, \\ &= \int d^2y \left(\frac{1}{2}{F_{01}}^2 + i\overline{\lambda}_-(\partial_0 + \partial_1)\lambda_- + \frac{1}{2}D^2\right). \end{aligned}$$

The second and the third terms come from the chiral multiplet and Fermi multiplet respectively [2]:

$$L_{ch} = -\frac{i}{2}\int d^2y\, d^2\theta \overline{\Phi}(\mathcal{D}_0 - \mathcal{D}_1)\Phi = \int d^2y \left(-|D_\alpha\phi|^2 + \overline{\psi}_+ i(D_0 - D_1)\psi_+ \right.$$
$$\left. -iQ\sqrt{2}\overline{\phi}\lambda_-\psi_+ + iQ\sqrt{2}\overline{\psi}_+\overline{\lambda}_-\phi + QD\overline{\phi}\phi\right),$$
$$L_F = -\frac{1}{2}\int d^2y d^2\theta\, \overline{\Lambda}_-\Lambda_-,$$
$$= \int d^2y \left(i\overline{\lambda}_-(D_0 + D_1)\lambda_- + |G|^2 - |E(\phi_i)|^2 - \left(\overline{\lambda}_-\frac{\partial E}{\partial \phi_i}\psi_{+,i} + \frac{\partial \bar{E}}{\partial \bar{\phi}_i}\overline{\psi}_{+,i}\lambda_-\right)\right).$$

If one can find chiral superfields J^a satisfying

$$E_a J^a = 0,$$

then from the definition of the Fermi superfield

$$\overline{\mathcal{D}}_+\Lambda_{-,a} = \sqrt{2}E_a,$$

one has

$$\overline{\mathcal{D}}_+(\Lambda_{-,a}J^a) = 0.$$

So we can introduce another term in the action; it is the $(0,2)$ analog of the superpotential [2]:

$$L_J = -\frac{1}{\sqrt{2}}\int d^2y d\theta^+ \Lambda_{-,a}J^a|_{\bar{\theta}^+=0} - h.c.,$$
$$= -\int d^2y \left(G_a J^a(\phi_i) + \lambda_{-,a}\psi_{+,i}\frac{\partial J^a}{\partial \phi_i}\right) - h.c.$$

We will see this superpotential correctly reproduces the (2,2) superpotential. Let

$$\Lambda_- = \frac{1}{\sqrt{2}}\mathcal{D}_-\Phi|_{\theta^-=\bar{\theta}^-=0},$$

then [2]

$$E = iQ\sqrt{2}\Sigma'\Phi',$$

where Q is the charge of Φ, and $\Phi' = \Phi|_{\theta^-=\bar{\theta}^-=0}$. It can be checked that $J^i = \partial W/\partial \Phi'_i$ satisfies $J^i E_i = 0$ if $W(\Phi')$ is a homogeneous polynomial of degree zero. So the (0,2) superpotential is now of form

$$W_{(0,2)} = \frac{\partial W}{\partial \Phi'_i}\Lambda_{-,i}.$$

This corresponds to a (2,2) model with gauge invariant superpotential W.

Finally one can write the Fayet-Iliopoulos term in the same way as in (2,2) cases [2]

$$\begin{aligned} L_{D,\theta} =& \frac{t}{4}\int d^2y d\theta^+ \Upsilon|_{\bar\theta^+=0} + h.c., \\ =& \frac{it}{2}\int d^2y (D - iF_{01}) + h.c.. \end{aligned}$$

2.1.4 Massless fermions

Let us analyze the vacua of the sigma model. As it is known that the massless fermions in the low energy theory are tangent to the vacua, one can use a sequence to describe the tangent bundle on the manifold of vacua in (2,2) model, and to describe the vector bundle on the vacua in (0,2) model. We focus on the regime $r \gg 0$, in which the vacua can be analyzed semiclassically.

We begin by writing down the bosonic potential

$$U(\phi_i, \sigma) = \frac{1}{2e^2}D^2 + \sum_i |F_i|^2 + 2\bar\sigma\sigma \sum_i {Q_i}^2 |\phi_i|^2.$$

The fields D and F_i are auxiliary fields and so can be integrated out:

$$\begin{aligned} D &= -e^2\left(\sum_i Q_i|\phi_i|^2 - r\right), \\ F_i &= \frac{\partial W}{\partial \phi_i}. \end{aligned}$$

Projective space

We first consider a simple example. Consider $N+1$ chiral superfields charged under a single $U(1)$, all with charges $Q_i = 1$, and with vanishing superpotential ($W = 0$). Then the bosonic potential is

$$U(\phi_i, \sigma) = \frac{1}{2e^2}D^2 + 2\bar\sigma\sigma \sum_i {Q_i}^2 |\phi_i|^2.$$

So the vacua are given by the vanishing of the potential:

$$\sum_i |\phi_i|^2 = r, \quad \sigma = 0.$$

This solutions form a sphere S^{2N+1}. By modding out the $U(1)$ gauge symmetry, we see the space of the vacua is $\mathbb{CP}^N = S^{2N+1}/U(1)$.

This analysis is only classical. There is a one-loop log-divergent diagram which contributes to the renormalization of r. The β-function is proportional to $\sum Q_i$, and the sign is such that $r(\mu)$ decreases in the infrared [3]. On the other hand, the r parameter does not RG flow in the GLSM for Calabi-Yau, because Calabi-Yau condition is that $\sum Q_i = 0$, which implies that the one-loop correction to r vanishes.

Masses for fermions come from Yukawa couplings $\overline{\lambda}^i \psi^i \sigma$ and $\lambda^i \overline{\psi}^i \overline{\sigma}$. One can easily see that if ψ_i is of form $f\phi_i$, for f any function on $\mathbb{CP}^N$, then ψ_i is massive; otherwise, ψ_i is massless, which means they live on the tangent bundle of $\mathbb{CP}^N$. We can describe this physical phenomenon with the Euler sequence [3]:

$$0 \to \mathcal{O} \xrightarrow{\otimes \phi^i} \mathcal{O}(1)^{\oplus N+1} \to T_{\mathbb{CP}^N} \to 0.$$

The left-moving fermions λ^i in a (2,2) theory are interpreted in the same fashion as the ψ^i.

Calabi-Yau hypersurface in weighted projective space

We now add a superpotential and another field. Let

$$\begin{aligned} J^i =& P \frac{\partial G}{\partial S_i}, \\ J^0 =& G, \end{aligned}$$

where S_i and P are chiral superfields with charge w_i and $-d = -\sum_i w_i$ respectively, and G is a homogeneous polynomial with degree d. We also require that the hypersurface $G = 0$ be smooth. This is equivalent to say that $W = PG(S_i)$ in (2,2) language. Then the potential energy is [2]

$$U(s_i, p) = \frac{e^2}{2} \left(\sum_i |s_i|^2 - n|p|^2 - r \right)^2 + |G|^2 + |p|^2 \sum_i \left| \frac{\partial G}{\partial s_i} \right|^2 + 2|\sigma|^2 \left(\sum_i |s_i|^2 + n^2 |p|^2 \right).$$

Let us analyze the semiclassical vacua. The D terms require

$$\sum_i |s_i|^2 - n|p|^2 - r = 0$$

which for $r \gg 0$ implies that the s_i cannot all vanish. From the smoothness condition, $\partial G / \partial s_i$ cannot all vanish. So

$$\begin{aligned} G &= 0, \\ p_i &= 0, \\ \sigma &= 0, \\ \sum_i |s_i|^2 &= r. \end{aligned}$$

The space of solutions is similar to that in 2.1.4 except that the chiral superfields now have different charges and one more condition $G = 0$. What we have now is a hypersurface G in a weighted projective space $W\mathbb{P}^4 = \mathbb{C}^5/\sim$, where

$$(z_1, \ldots, z_n) \sim (\lambda^{w_1} z_1, \ldots, \lambda^{w_n} z_n).$$

Notice G is Calabi-Yau since the charge condition $d = \sum_i w_i$ is equivalent to $c_1(T) = 0$.

The mass of right-moving fermions ψ^inow comes from two Yukawa coupling terms $\overline{\lambda}^i \psi^i \sigma$ and $\gamma \psi^i (\partial G/\partial s^i)$. Hence the massless right-moving fermions from the cohomology of the sequence [3]

$$0 \to \mathcal{O} \xrightarrow{f} \bigoplus_i \mathcal{O}(w_i) \xrightarrow{g} \mathcal{O}(d) \to 0,$$

where $f(l) = (w_1 s_1 l, \ldots, w_n s_n l)$ and $g(u_1, \ldots, u_n) = \sum u_i (\partial G/\partial s^i)$. This cohomology is actually the tangent bundle over the hypersurface: $T = \ker(g)/\mathrm{im}(f)$.

The same argument applies to left-moving fermions λ^i.

Deformation of the tangent bundle

We generalize the superpotential to

$$\begin{aligned} J^i =& P F_i, \\ J^0 =& G, \end{aligned}$$

where

$$\sum w_i s^i F_i(s) = dG(s).$$

We also generalize the charge of Λ^a to be n_a and that of P to be $-m = -\sum_a n_a$. By examining the Yukawa couplings, one can see the right-moving fermions ψ^i still live on the tangent bundle to the hypersurface G; but the left-moving fermions λ^i now transform as sections of a vector bundle V over G, which can be realized by the cohomology of a short sequence [3]:

$$0 \to \mathcal{O} \xrightarrow{f} \bigoplus_i \mathcal{O}(w_i) \xrightarrow{g'} \mathcal{O}(d) \to 0,$$

where $g'(u_1, \ldots, u_n) = \sum u_i F^i(s)$. The vector bundle is described by $V = \ker(g)/\mathrm{im}(f)$. The charge condition $m = \sum_a n_a$ implies $c_1(V) = 0$.

Since we are dealing with a theory with chiral fermions coupled to a gauge field, we need to ensure that there is no gauge anomaly. The anomaly cancellation condition is

$$\mathrm{ch}_2(T) = \mathrm{ch}_2(V).$$

In the current case, $c_1(V) = c_1(T) = 0$. It further implies that

$$c_2(V) = c_2(T).$$

In physics language, it constrains the charges:

$$\frac{1}{2}\left(\sum_a n_a^2 - m^2\right) = \frac{1}{2}\left(\sum_i w_i^2 - d^2\right).$$

2.2 Examples on ordinary Grassmannians

Two-dimensional (2,2) GLSMs for Grassmannians have been discussed in [11, 35], and for flag manifolds in [13]. Briefly, the Grassmannian $G(k, n)$ is constructed via a $U(k)$ gauge theory with n chiral superfields in the fundamental representation.

Two-dimensional (0,2) theories describing bundles on $G(k, n)$ can be built from $U(k)$ gauge theories with n (0,2) chiral superfields in the fundamental and suitable matter to describe the gauge bundle. These form the prototype for other constructions: understanding (0,2) Grassmannian constructions is essential to understand (0,2) Pfaffian constructions, for example, and will also be important in our analysis of dualities.

In this section, we will outline some general aspects of (0,2) GLSMs and their relation to cohomology and bundles on Grassmannians, as simple toy models to illustrate various phenomena. This section was previously published in [90].

2.2.1 Anomaly cancellation and Chern classes

We shall begin by considering anomaly cancellation in nonabelian (0,2) models, and its relation to cohomology of the underlying space. In two-dimensional gauge theories, anomaly cancellation requires, schematically,

$$\sum_{R_{\rm left}} {\rm tr}(T^a T^b) = \sum_{R_{\rm right}} {\rm tr}(T^a T^b). \tag{2.9}$$

More concretely, in terms of the Casimirs discussed in appendix A, we have the following conditions:

$$\sum_{R_{\rm left}} \dim(R_{\rm left}){\rm Cas}_2(R_{\rm left}) = \sum_{R_{\rm right}} \dim(R_{\rm right}){\rm Cas}_2(R_{\rm right}), \tag{2.10}$$

$$\sum_{R_{\rm left}} \dim(R_{\rm left})({\rm Cas}_1(R_{\rm left}))^2 = \sum_{R_{\rm right}} \dim(R_{\rm right})({\rm Cas}_1(R_{\rm right}))^2. \tag{2.11}$$

The first condition is the $u(k)^2$ gauge anomaly condition, the second the $u(1)^2$ condition; there is no $u(1)-su(k)$ condition, as elements of the Lie algebra of $su(k)$ are traceless. Note for $SU(n)$ gauge theories, the second condition is automatically satisfied, because of the fact that $\mathrm{Cas}_1(R)=0$ for any representation R of $SU(n)$.

For example, consider a (0,2) GLSM with right-moving chiral superfields Φ, P, left-moving Fermi superfields Λ, Γ, and (left-moving) gauginos. The gauge anomaly cancellation conditions are given by

$$\begin{aligned}
&\sum_{R_\Lambda}\dim(R_\Lambda)\mathrm{Cas}_2(R_\Lambda)+\dim(\mathrm{adj})\mathrm{Cas}_2(\mathrm{adj}) \\
&\qquad\qquad = \sum_{R_\Phi}\dim(R_\Phi)\mathrm{Cas}_2(R_\Phi)+\sum_{R_P}\dim(R_P)\mathrm{Cas}_2(R_P), \\
&\sum_{R_\Lambda}\dim(R_\Lambda)(\mathrm{Cas}_1(R_\Lambda))^2+\dim(\mathrm{adj})(\mathrm{Cas}_1(\mathrm{adj}))^2 \\
&\qquad\qquad = \sum_{R_\Phi}\dim(R_\Phi)(\mathrm{Cas}_1(R_\Phi))^2+\sum_{R_P}\dim(R_P)(\mathrm{Cas}_1(R_P))^2.
\end{aligned} \tag{2.12}$$

In principle, anomaly cancellation in the UV GLSM implies

$$\mathrm{ch}_2(E)=\mathrm{ch}_2(TX) \tag{2.13}$$

in the IR NLSM on the space X, and in general is slightly stronger than the IR condition (see for example [36][section 6.5] for examples of anomalous GLSMs associated mathematically to anomaly-free IR geometries).

Furthermore, as discussed in appendix A, the Chern classes are determined by the Casimirs: for any given representation λ,

$$c_1(\mathcal{O}(\lambda)) \;=\; \frac{d_\lambda \mathrm{Cas}_1(\lambda)}{k}\sigma_{\square}, \tag{2.14}$$

$$\begin{aligned}
\mathrm{ch}_2(\mathcal{O}(\lambda)) \;&=\; (1/2)c_1(\mathcal{O}(\lambda))^2 \;-\; c_2(\mathcal{O}(\lambda)), \\
&=\; d_\lambda \mathrm{Cas}_2(\lambda)\left[-\frac{1}{k^2-1}\sigma_{\substack{\square\\ \square}} \;+\; \frac{1}{2k(k+1)}\sigma_{\square}^2\right] \\
&\qquad + d_\lambda \mathrm{Cas}_1(\lambda)^2\left[\frac{1}{k(k^2-1)}\sigma_{\substack{\square\\ \square}} \;+\; \frac{1}{2k(k+1)}\sigma_{\square}^2\right],
\end{aligned} \tag{2.15}$$

where d_λ is the dimension of representation λ.

Let us apply this to heterotic geometries, and check that the Casimir conditions above imply the mathematical matching of Chern classes and characters. Specifically, consider a bundle $\mathcal{E}$ defined by the kernel

$$0 \longrightarrow \mathcal{E} \longrightarrow \oplus_i\mathcal{O}(\lambda_i) \longrightarrow \oplus_a\mathcal{O}(\lambda_a) \longrightarrow 0. \tag{2.16}$$

This is defined by a set of Fermi superfields Λ in the representations λ_i, chiral superfields P in representations dual to λ_a, and a (0,2) superpotential encoding the second nontrivial map. Mathematically, using the additivity properties of Chern characters, we have

$$\begin{aligned} \mathrm{ch}_2(\mathcal{E}) &= \mathrm{ch}_2\left(\oplus_i \mathcal{O}(\lambda_i)\right) \; - \; \mathrm{ch}_2\left(\oplus_a \mathcal{O}(\lambda_a)\right), \\ &= \sum_i \mathrm{ch}_2\left(\mathcal{O}(\lambda_i)\right) \; - \; \sum_a \mathrm{ch}_2\left(\mathcal{O}(\lambda_a)\right). \end{aligned}$$

The tangent bundle of the Grassmannian $G(k,n)$ is defined as the cokernel

$$0 \longrightarrow S^* \otimes S \longrightarrow S^* \otimes \mathcal{O}^n \longrightarrow S^* \otimes Q \; = \; TG(k,n) \longrightarrow 0,$$

and so

$$\begin{aligned} \mathrm{ch}_2(TG(k,n)) &= \mathrm{ch}_2\left(S^* \otimes \mathcal{O}^n\right) \; - \; \mathrm{ch}_2\left(S^* \otimes S\right), \\ &= n\, \mathrm{ch}_2\left(S^*\right) \; - \; \mathrm{ch}_2\left(S^* \otimes S\right). \end{aligned}$$

The anomaly-cancellation condition is given by

$$\mathrm{ch}_2(TG(k,n)) \; = \; \mathrm{ch}_2(\mathcal{E}),$$

which is equivalent to

$$\sum_i \mathrm{ch}_2\left(\mathcal{O}(\lambda_i)\right) \; + \; \mathrm{ch}_2\left(S^* \otimes S\right) \; = \; \sum_a \mathrm{ch}_2\left(\mathcal{O}(\lambda_a)\right) \; + \; n\, \mathrm{ch}_2\left(S^*\right).$$

Writing ch_2 in terms of Casimirs as in equation (2.15) above, we see that the mathematical anomaly-cancellation condition above is satisfied if and only if the physical gauge anomaly constraints (2.12) are satisfied, as expected.

Now, let us turn to the A/2 pseudo-topological field theory. As discussed in [36, 37], for a gauge bundle $\mathcal{E}$ over a space X, in addition to the anomaly-cancellation condition one must also impose the constraint

$$\wedge^{\mathrm{top}} \mathcal{E}^* \; \cong \; K_X,$$

which implies $c_1(\mathcal{E}) = c_1(TX)$. For the gauge bundle defined by (2.16) over $X = G(k,n)$, this constraint becomes

$$c_1\left(\oplus_i \mathcal{O}(\lambda_i)\right) \; - \; c_1\left(\oplus_a \mathcal{O}(\lambda) a)\right) \; = \; c_1\left(S^* \otimes \mathcal{O}^n\right) \; - \; c_1\left(S^* \otimes S\right),$$

which can easily be checked to be equivalent to the statement

$$\sum_{R_\Lambda} \dim(R_\Lambda) \mathrm{Cas}_1(R_\Lambda) + \dim(\mathrm{adj}) \mathrm{Cas}_1(\mathrm{adj}) = \sum_{R_\Phi} \dim(R_\Phi) \mathrm{Cas}_1(R_\Phi) + \sum_{R_P} \dim(R_P) \mathrm{Cas}_1(R_P),$$

or more simply,

$$\sum_{R_{\mathrm{left}}} \dim(R_{\mathrm{left}}) \mathrm{Cas}_1(R_{\mathrm{left}}) \; = \; \sum_{R_{\mathrm{right}}} \dim(R_{\mathrm{right}}) \mathrm{Cas}_1(R_{\mathrm{right}}).$$

2.2.2 Examples on $G(2,4)$

In table 2.1, we list examples of bundles $\mathcal{E}$ of the form

$$0 \longrightarrow \mathcal{E} \longrightarrow \oplus^m \mathcal{O}(\lambda_{A1}, \lambda_{B1}) \longrightarrow \oplus^n \mathcal{O}(\lambda_{A2}, \lambda_{B2}) \longrightarrow 0 \tag{2.17}$$

on $G(2,4)$, satisfying anomaly cancellation. For simplicity we have chosen to focus on bundles defined by kernels; however, nonabelian (0,2) GLSMs can also be used to describe cokernels and cohomologies of monads. As those constructions are simple generalizations, we omit their discussion.

In table 2.1, we have used the notation $\mathcal{O}(\lambda_1, \lambda_2)$ to indicate a vector bundle on $G(2,4)$ defined by the (λ_1, λ_2) representation of $U(2)$ $(\lambda_1 \geq \lambda_2)$. See appendix A for our conventions.

In the first entry in table 2.1, we have a map $\mathcal{O}(-2,-2)^5 \to \mathcal{O}(3,3)^2$. The elements of this map are provided by sections of

$$\mathcal{O}(5,5) \;=\; (\det S^*)^5 \,.$$

A section of $\det S^*$ is a baryon constructed from the chiral superfields defining the Grassmannian $G(2,4)$, *i.e.* an operator of the form

$$B_{ij} \;\equiv\; \epsilon_{ab} \phi_i^a \phi_j^b,$$

where in this case $i, j \in \{1, \cdots, 4\}$. Therefore, the maps in the bundle in the first entry in table 2.1 are provided by degree five polynomials in the B^{ij}, and the (0,2) superpotential is then of the form

$$W \;=\; \Lambda_\alpha p_\gamma f_5^{\alpha\gamma}(B_{ij}),$$

where Λ's are Fermi superfields in representation $(-2,-2)$, p's are chiral superfields in the representation dual to $(3,3)$, and f_5 is a degree five polynomial.

However, there is a potential issue with the tables: it may be that not all of the entries can be realized by nontrivial maps. In some cases, it may not be possible to find explicit maps between the second and third entries in the short exact sequence, or the data given does not define a bundle, merely a sheaf.

Table 2.1: Anomaly-free examples on $G(2,4)$

m	$(\lambda_{A1}, \lambda_{B1})$	n	$(\lambda_{A2}, \lambda_{B2})$	rank
5	(-2, -2)	2	(3 , 3)	3
5	(-7, -7)	3	(9 , 9)	2
4	(-5, -5)	2	(7 , 7)	2
18	(-5, -5)	7	(8 , 8)	11
18	(-3, -3)	10	(4 , 4)	8
3	(-3, -3)	1	(5 , 5)	2
4	(-2, -3)	1	(-3 , -5)	5
4	(-2, -3)	1	(5 , 3)	5
13	(-2, -2)	2	(5 , 5)	11
3	(-1, -1)	1	(-1 , -1)	2
3	(-1, -1)	1	(1 , 1)	2
6	(-1, -1)	1	(2 , 2)	5
11	(-1, -1)	1	(3 , 3)	10
18	(-1, -1)	1	(4 , 4)	17
4	(0, -1)	1	(1 , -1)	5
2	(1, -2)	1	(2 , -2)	3
4	(1, 0)	1	(1 , -1)	5
3	(1, 1)	1	(1 , 1)	2
6	(1, 1)	1	(2 , 2)	5
11	(1, 1)	1	(3 , 3)	10
18	(1, 1)	1	(4 , 4)	17
2	(2, -1)	1	(2 , -2)	3
5	(2, 2)	2	(3 , 3)	3
13	(2, 2)	2	(5 , 5)	11
4	(3, 2)	1	(-3 , -5)	5
4	(3, 2)	1	(5 , 3)	5
3	(3, 3)	1	(5 , 5)	2
4	(5, 5)	2	(7 , 7)	2
18	(5, 5)	7	(8 , 8)	11
5	(7, 7)	3	(9 , 9)	2

2.2.3 Examples on $G(2,7)$

For $G(2,7)$, it is difficult to construct a bundle $\mathcal{E}$ to fit in the short exact sequence like 2.17. In this section we list examples of bundles $\mathcal{E}$ defined by short exact sequences of the form:

$$0 \to \mathcal{E} \to \oplus^{m_1}\mathcal{O}(\lambda_{A1}, \lambda_{B1}) \oplus^{m_2}\mathcal{O}(\lambda_{A2}, \lambda_{B2}) \to \oplus^{n_1}\mathcal{O}(\lambda_{A3}, \lambda_{B3}) \oplus^{n_2}\mathcal{O}(\lambda_{A4}, \lambda_{B4}) \to 0. \quad (2.18)$$

Note that $\mathcal{O}(a,b)$ indicates a vector bundle defined by representation (a,b) of $U(2)$, which need not be a line bundle.

In the following tables we list examples of bundles $\mathcal{E}$ of rank 3, 4, and 5 on $G(2,7)$, satisfying anomaly cancellation. In some cases, it may not be possible to find explicit maps between the second and third entries in the short exact sequence, or the data given does not define a bundle, merely a sheaf.

Table 2.2: Vector bundles of rank 3

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$
2	(-5, -5)	5	(-1, -2)	5	(-2 , -2)	2	(-3, -4)
2	(-5, -5)	2	(0, -2)	1	(-1 , -2)	1	(-3, -5)
2	(-5, -5)	2	(0, -2)	1	(-1 , -3)	1	(-4, -5)
2	(-5, -5)	5	(0, -1)	5	(0 , 0)	2	(-3, -4)
2	(-5, -5)	3	(0, -1)	2	(0 , 0)	3	(-4, -4)
4	(-5, -5)	3	(0, -1)	2	(-3 , -3)	5	(-4, -4)
2	(-5, -5)	3	(1, -1)	2	(0 , -2)	1	(-4, -5)
4	(-5, -5)	3	(1, 0)	2	(-3 , -3)	5	(-4, -4)
2	(-5, -5)	2	(2, -5)	1	(5 , -4)	5	(-4, -4)
2	(-5, -5)	2	(2, 0)	1	(-1 , -2)	1	(-3, -5)
2	(-5, -5)	2	(2, 0)	1	(-1 , -3)	1	(-4, -5)
2	(-5, -5)	5	(2, 1)	5	(-2 , -2)	2	(-3, -4)
1	(-4, -5)	3	(-1, -3)	1	(3 , 0)	4	(-4, -4)
1	(-4, -5)	3	(-1, -3)	1	(0 , -3)	4	(-4, -4)
1	(-4, -5)	3	(1, -1)	1	(1 , -2)	4	(-3, -3)
3	(-4, -5)	5	(2, -2)	4	(1 , -3)	2	(-2, -5)
3	(-4, -5)	2	(2, -2)	1	(0 , -4)	2	(-2, -5)
3	(-4, -4)	3	(-3, -4)	2	(3 , 3)	4	(-5, -5)
3	(-4, -4)	3	(-3, -4)	2	(-3 , -3)	4	(-5, -5)
1	(-4, -4)	3	(-2, -3)	2	(0 , 0)	2	(-5, -5)
1	(-4, -4)	3	(-2, -3)	2	(-3 , -3)	2	(-4, -4)
2	(-4, -4)	2	(-1, -3)	1	(2 , 1)	1	(-3, -5)
2	(-4, -4)	2	(-1, -3)	1	(-1 , -2)	1	(-3, -5)
2	(-4, -4)	2	(-1, -3)	1	(-1 , -3)	1	(-4, -5)
3	(-4, -4)	1	(-1, -3)	1	(-4 , -4)	1	(-4, -5)
1	(-4, -4)	3	(-1, -2)	2	(-2 , -2)	2	(-3, -3)
2	(-4, -4)	4	(0, -2)	3	(1 , -1)	1	(-4, -5)
2	(-4, -4)	4	(0, -2)	1	(-1 , -2)	3	(-1, -3)
5	(-4, -4)	1	(0, -2)	1	(-1 , -2)	3	(-5, -5)
3	(-4, -4)	3	(0, -1)	1	(-1 , -1)	5	(-3, -3)
2	(-4, -4)	3	(0, -1)	3	(-2 , -2)	2	(-3, -3)
2	(-4, -4)	3	(1, -1)	1	(-1 , -2)	2	(-1, -3)
3	(-4, -4)	1	(1, -1)	1	(-2 , -2)	1	(-4, -5)
3	(-4, -4)	3	(1, 0)	1	(-1 , -1)	5	(-3, -3)
2	(-4, -4)	3	(1, 0)	3	(-2 , -2)	2	(-3, -3)
2	(-4, -4)	4	(2, 0)	1	(-1 , -2)	3	(-1, -3)
5	(-4, -4)	1	(2, 0)	1	(-1 , -2)	3	(-5, -5)
1	(-4, -4)	3	(2, 1)	2	(-2 , -2)	2	(-3, -3)
3	(-4, -4)	1	(3, 1)	1	(-4 , -4)	1	(-4, -5)
4	(-4, -4)	1	(3, 1)	2	(-4 , -4)	1	(-4, -5)

Table 2.3: Vector bundles of rank 3 (continued)

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$
5	(-4, -4)	1	(3, 1)	3	(-4 , -4)	1	(-4, -5)
1	(-4, -4)	3	(3, 2)	2	(-3 , -3)	2	(-4, -4)
1	(-3, -5)	3	(-2, -2)	1	(-4 , -4)	1	(-4, -5)
1	(-3, -5)	4	(-1, -1)	2	(-2 , -2)	1	(-4, -5)
1	(-3, -5)	3	(0, 0)	1	(-2 , -2)	1	(-4, -5)
3	(-3, -5)	4	(1, -1)	2	(0 , -3)	5	(-3, -4)
1	(-3, -5)	4	(1, 1)	2	(-2 , -2)	1	(-4, -5)
1	(-3, -5)	3	(2, 2)	1	(-4 , -4)	1	(-4, -5)
3	(-3, -4)	1	(-3, -3)	1	(2 , 2)	3	(-5, -5)
3	(-3, -4)	1	(-3, -3)	1	(-2 , -2)	3	(-5, -5)
3	(-3, -4)	3	(-2, -2)	2	(-3 , -3)	4	(-4, -4)
5	(-3, -4)	1	(-1, -4)	2	(0 , 0)	3	(-3, -5)
1	(-3, -4)	3	(-1, -4)	1	(2 , -2)	2	(-3, -5)
2	(-3, -4)	1	(-1, -2)	1	(0 , 0)	2	(-5, -5)
3	(-3, -4)	5	(-1, -1)	5	(0 , 0)	3	(-5, -5)
1	(-3, -4)	2	(0, -1)	1	(-2 , -2)	2	(-3, -3)
3	(-3, -4)	4	(0, 0)	5	(-2 , -2)	2	(-5, -5)
3	(-3, -4)	1	(1, -3)	1	(2 , -2)	3	(-5, -5)
1	(-3, -4)	3	(1, -2)	1	(2 , -2)	2	(-1, -3)
1	(-3, -4)	2	(1, 0)	1	(-2 , -2)	2	(-3, -3)
1	(-3, -4)	3	(2, -5)	3	(-1 , -4)	1	(5, -5)
3	(-3, -4)	5	(2, -2)	4	(1 , -3)	2	(-1, -4)
3	(-3, -4)	2	(2, -2)	1	(0 , -4)	2	(-1, -4)
3	(-3, -4)	3	(2, 2)	2	(-3 , -3)	4	(-4, -4)
1	(-3, -4)	3	(3, -4)	3	(0 , -3)	1	(5, -5)
4	(-3, -3)	3	(-2, -3)	5	(2 , 2)	2	(-5, -5)
4	(-3, -3)	3	(-2, -3)	5	(-2 , -2)	2	(-5, -5)
1	(-3, -3)	3	(-2, -3)	3	(-3 , -3)	1	(-4, -4)
4	(-3, -3)	1	(-1, -3)	1	(-2 , -3)	2	(-4, -4)
1	(-3, -3)	3	(-1, -2)	3	(1 , 1)	1	(-4, -4)
1	(-3, -3)	3	(-1, -2)	3	(-1 , -1)	1	(-4, -4)
2	(-3, -3)	4	(-1, -2)	5	(-2 , -2)	1	(-2, -3)
2	(-3, -3)	3	(0, -1)	1	(0 , 0)	4	(-2, -2)
2	(-3, -3)	3	(0, -1)	4	(0 , 0)	1	(-4, -4)
3	(-3, -3)	3	(0, -1)	5	(0 , 0)	1	(-5, -5)
1	(-3, -3)	3	(0, -1)	3	(-1 , -1)	1	(-2, -2)
4	(-3, -3)	1	(1, -1)	1	(0 , -1)	2	(-4, -4)
4	(-3, -3)	1	(1, -1)	2	(-2 , -2)	1	(-3, -4)
1	(-3, -3)	3	(1, 0)	3	(-1 , -1)	1	(-2, -2)
2	(-3, -3)	4	(2, -5)	1	(5 , -4)	3	(1, -5)

Table 2.4: Vector bundles of rank 3 (continued)

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$
2	(-3, -3)	4	(2, -5)	1	(4 , -5)	3	(1, -5)
2	(-3, -3)	4	(2, 1)	5	(-2 , -2)	1	(-2, -3)
4	(-3, -3)	1	(3, 1)	1	(-2 , -3)	2	(-4, -4)
1	(-3, -3)	3	(3, 2)	3	(-3 , -3)	1	(-4, -4)
1	(-2, -5)	5	(-1, -2)	3	(-1 , -3)	2	(-4, -4)
2	(-2, -5)	1	(2, -2)	1	(1 , -4)	2	(-4, -5)
1	(-2, -5)	5	(2, 1)	3	(-1 , -3)	2	(-4, -4)
1	(-2, -4)	5	(-2, -2)	3	(1 , 1)	1	(-4, -5)
1	(-2, -4)	5	(-2, -2)	3	(-1 , -1)	1	(-4, -5)
4	(-2, -4)	3	(-1, -3)	2	(3 , 0)	5	(-3, -4)
4	(-2, -4)	3	(-1, -3)	2	(0 , -3)	5	(-3, -4)
2	(-2, -4)	2	(-1, -1)	1	(-1 , -2)	1	(-3, -5)
2	(-2, -4)	2	(-1, -1)	1	(-1 , -3)	1	(-4, -5)
2	(-2, -4)	2	(1, 1)	1	(-1 , -2)	1	(-3, -5)
2	(-2, -4)	2	(1, 1)	1	(-1 , -3)	1	(-4, -5)
1	(-2, -3)	2	(-2, -3)	2	(-3 , -3)	1	(-4, -4)
3	(-2, -3)	3	(-2, -2)	1	(1 , 1)	5	(-3, -3)
3	(-2, -3)	4	(-2, -2)	5	(0 , 0)	2	(-5, -5)
3	(-2, -3)	3	(-2, -2)	1	(-1 , -1)	5	(-3, -3)
1	(-2, -3)	3	(-1, -3)	1	(3 , 0)	4	(-3, -3)
1	(-2, -3)	3	(-1, -3)	1	(0 , -3)	4	(-3, -3)
1	(-2, -3)	2	(-1, -2)	2	(1 , 1)	1	(-4, -4)
1	(-2, -3)	2	(-1, -2)	1	(0 , 0)	2	(-3, -3)
1	(-2, -3)	2	(-1, -2)	2	(-1 , -1)	1	(-4, -4)
3	(-2, -3)	2	(-1, -1)	1	(0 , 0)	4	(-3, -3)
3	(-2, -3)	4	(-1, -1)	5	(-2 , -2)	2	(-3, -3)
5	(-2, -3)	1	(0, -3)	3	(-1 , -3)	2	(-4, -4)
2	(-2, -3)	1	(0, -1)	1	(-2 , -2)	2	(-3, -3)
3	(-2, -3)	1	(0, 0)	2	(-1 , -1)	2	(-4, -4)
2	(-2, -3)	1	(1, 0)	1	(-2 , -2)	2	(-3, -3)
3	(-2, -3)	4	(1, 1)	5	(-2 , -2)	2	(-3, -3)
3	(-2, -3)	2	(2, -4)	2	(3 , -3)	3	(-4, -4)
5	(-2, -3)	1	(2, -2)	2	(-1 , -2)	2	(-1, -4)
3	(-2, -3)	1	(2, -2)	1	(0 , -3)	1	(-1, -4)
3	(-2, -3)	1	(3, -3)	3	(-3 , -3)	1	(2, -4)
5	(-2, -3)	1	(3, 0)	3	(-1 , -3)	2	(-4, -4)
1	(-2, -3)	2	(3, 2)	2	(-3 , -3)	1	(-4, -4)
3	(-2, -2)	1	(-1, -3)	1	(2 , 1)	1	(-4, -4)
4	(-2, -2)	1	(-1, -3)	2	(0 , 0)	1	(-3, -4)
2	(-2, -2)	3	(-1, -3)	2	(1 , -1)	1	(-4, -5)

Table 2.5: Vector bundles of rank 3 (continued)

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$
3	(-2, -2)	1	(-1, -3)	1	(-1 , -2)	1	(-4, -4)
3	(-2, -2)	3	(-1, -2)	4	(1 , 1)	2	(-3, -3)
2	(-2, -2)	3	(-1, -2)	3	(0 , 0)	2	(-3, -3)
3	(-2, -2)	3	(-1, -2)	4	(-1 , -1)	2	(-3, -3)
1	(-2, -2)	3	(0, -1)	2	(0 , 0)	2	(-1, -1)
3	(-2, -2)	1	(1, -1)	1	(-1 , -2)	1	(-2, -2)
5	(-2, -2)	1	(1, -1)	1	(-1 , -2)	3	(-2, -2)
2	(-1, -5)	3	(0, -1)	2	(0 , -4)	3	(-4, -4)
2	(-1, -5)	3	(1, 0)	2	(0 , -4)	3	(-4, -4)
1	(-1, -4)	5	(-1, -2)	2	(-2 , -2)	3	(-1, -3)
2	(-1, -4)	1	(2, -2)	2	(-1 , -2)	1	(0, -5)
1	(-1, -4)	5	(2, 1)	2	(-2 , -2)	3	(-1, -3)
2	(-1, -3)	5	(-1, -3)	2	(3 , 0)	5	(-2, -3)
2	(-1, -3)	5	(-1, -3)	2	(0 , -3)	5	(-2, -3)
3	(-1, -3)	1	(-1, -2)	1	(2 , -1)	4	(-3, -3)
3	(-1, -3)	1	(-1, -2)	1	(1 , -2)	4	(-3, -3)
1	(-1, -3)	4	(-1, -1)	2	(0 , 0)	1	(-2, -3)
1	(-1, -3)	4	(-1, -1)	1	(-1 , -2)	2	(-2, -2)
1	(-1, -3)	4	(0, 0)	1	(0 , -1)	2	(-2, -2)
2	(-1, -3)	5	(1, -1)	5	(0 , -1)	2	(0, -3)
1	(-1, -3)	4	(1, 1)	1	(-1 , -2)	2	(-2, -2)
5	(-1, -2)	2	(-1, -1)	2	(1 , 0)	5	(-2, -2)
3	(-1, -2)	2	(-1, -1)	2	(0 , 0)	3	(-2, -2)
5	(-1, -2)	2	(-1, -1)	2	(0 , -1)	5	(-2, -2)
3	(-1, -2)	2	(0, -4)	1	(-2 , 2)	2	(-1, -4)
1	(-1, -2)	2	(0, -1)	1	(0 , 0)	2	(-1, -1)
2	(-1, -2)	1	(0, -1)	2	(-1 , -1)	1	(-2, -2)
2	(-1, -2)	1	(1, 0)	2	(-1 , -1)	1	(-2, -2)
5	(-1, -2)	1	(2, -2)	2	(0 , -1)	2	(0, -3)
3	(-1, -2)	1	(2, -2)	1	(1 , -2)	1	(0, -3)
4	(-1, -2)	1	(2, -2)	2	(1 , -2)	1	(-2, -3)
3	(-1, -2)	1	(3, -3)	3	(-1 , -1)	1	(2, -4)
5	(-1, -1)	1	(0, -2)	3	(0 , 0)	1	(-1, -2)
2	(-1, -1)	2	(0, -2)	1	(1 , -1)	1	(-1, -2)
2	(-1, -1)	4	(0, -1)	5	(0 , 0)	1	(0, -1)
3	(-1, -1)	3	(0, -1)	5	(0 , 0)	1	(-1, -1)
2	(-1, -1)	2	(3, -4)	5	(0 , 0)	1	(4, -5)
2	(0, -5)	1	(0, -4)	1	(5 , -2)	3	(-3, -4)
2	(0, -5)	1	(0, -4)	3	(-3 , -4)	1	(2, -5)
1	(0, -4)	4	(0, -1)	1	(0 , -1)	2	(0, -3)

Table 2.6: Vector bundles of rank 3 (continued)

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$
1	(0, -4)	5	(0, -1)	2	(0 , -1)	2	(0, -3)
5	(0, -4)	3	(0, -1)	4	(2 , -2)	2	(-2, -5)
1	(0, -4)	3	(0, -1)	1	(0 , -3)	1	(0, -3)
1	(0, -4)	2	(2, -5)	2	(-1 , -4)	1	(4, -5)
1	(0, -4)	2	(2, -3)	3	(0 , -1)	1	(2, -5)
4	(0, -4)	2	(2, -2)	3	(-1 , -3)	3	(1, -4)
1	(0, -4)	2	(3, -4)	2	(0 , -3)	1	(4, -5)
3	(0, -4)	2	(5, -5)	2	(-3 , -4)	3	(4, -5)
1	(0, -3)	5	(0, -1)	3	(1 , -1)	2	(-2, -2)
3	(0, -3)	1	(0, -1)	1	(2 , -2)	2	(-1, -3)
2	(0, -3)	1	(2, -2)	2	(0 , -1)	1	(1, -4)
3	(0, -1)	1	(1, -3)	3	(-1 , -1)	1	(2, -2)
3	(0, -1)	4	(1, -3)	3	(2 , -2)	2	(0, -3)
5	(0, -1)	1	(1, -2)	2	(0 , 0)	3	(1, -1)
1	(0, -1)	3	(1, -2)	2	(1 , -1)	1	(2, -2)
3	(0, -1)	2	(2, -4)	3	(-2 , -2)	2	(3, -3)
1	(0, -1)	3	(3, -4)	3	(1 , -2)	1	(5, -5)
5	(1, -4)	2	(2, -5)	2	(4 , -4)	5	(0, -4)
3	(2, -3)	2	(3, -4)	5	(1 , -2)	1	(5, -5)
1	(3, 2)	2	(3, 2)	2	(-3 , -3)	1	(-4, -4)

Table 2.7: Vector bundles of rank 4

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$
1	(-5, -5)	5	(-2, -3)	2	(2 , 1)	3	(-5, -5)
1	(-5, -5)	5	(-2, -3)	3	(1 , 1)	2	(-4, -5)
1	(-5, -5)	5	(-2, -3)	3	(-1 , -1)	2	(-4, -5)
1	(-5, -5)	3	(-2, -3)	1	(-3 , -3)	2	(-5, -5)
2	(-5, -5)	4	(-1, -2)	1	(1 , 0)	4	(-4, -4)
2	(-5, -5)	4	(-1, -2)	1	(0 , -1)	4	(-4, -4)
1	(-5, -5)	4	(-1, -2)	1	(-2 , -3)	3	(-3, -3)
3	(-5, -5)	5	(-1, -2)	5	(-3 , -3)	2	(-3, -4)
2	(-5, -5)	5	(0, -1)	4	(0 , 0)	2	(-3, -4)
2	(-5, -5)	4	(0, -1)	1	(-2 , -3)	4	(-3, -3)
3	(-5, -5)	3	(0, -1)	1	(-3 , -3)	4	(-4, -4)
2	(-5, -5)	4	(1, -1)	1	(0 , -3)	3	(-2, -3)
2	(-5, -5)	4	(1, 0)	1	(-2 , -3)	4	(-3, -3)
3	(-5, -5)	3	(1, 0)	1	(-3 , -3)	4	(-4, -4)
1	(-5, -5)	4	(2, 1)	1	(-2 , -3)	3	(-3, -3)
3	(-5, -5)	5	(2, 1)	5	(-3 , -3)	2	(-3, -4)
1	(-5, -5)	3	(3, 2)	1	(-3 , -3)	2	(-5, -5)
1	(-4, -5)	3	(-2, -3)	1	(-3 , -4)	2	(-5, -5)
3	(-4, -5)	4	(-2, -2)	1	(-3 , -3)	5	(-5, -5)
1	(-4, -5)	3	(-1, -3)	1	(-1 , -4)	3	(-4, -4)
2	(-4, -5)	5	(-1, -2)	2	(1 , 1)	4	(-3, -4)
1	(-4, -5)	3	(-1, -2)	1	(1 , 0)	2	(-5, -5)
1	(-4, -5)	3	(-1, -2)	1	(0 , -1)	2	(-5, -5)
2	(-4, -5)	5	(-1, -2)	2	(-1 , -1)	4	(-3, -4)
2	(-4, -5)	5	(-1, -2)	4	(-2 , -3)	2	(-5, -5)
3	(-4, -5)	5	(-1, -2)	2	(-3 , -3)	5	(-3, -4)
3	(-4, -5)	4	(0, -4)	2	(-3 , -3)	4	(-1, -5)
1	(-4, -5)	5	(0, -1)	2	(-1 , -1)	3	(-2, -3)
3	(-4, -5)	4	(0, 0)	2	(-3 , -3)	4	(-5, -5)
1	(-4, -5)	4	(1, -3)	2	(3 , -2)	2	(-2, -4)
1	(-4, -5)	4	(1, -3)	2	(2 , -3)	2	(-2, -4)
1	(-4, -5)	3	(1, -1)	3	(-2 , -2)	1	(-1, -4)
1	(-4, -5)	5	(1, 0)	2	(-1 , -1)	3	(-2, -3)
3	(-4, -5)	5	(2, -2)	2	(-3 , -3)	5	(0, -4)
2	(-4, -5)	5	(2, 1)	4	(-2 , -3)	2	(-5, -5)
3	(-4, -5)	5	(2, 1)	2	(-3 , -3)	5	(-3, -4)
3	(-4, -5)	4	(2, 2)	1	(-3 , -3)	5	(-5, -5)
1	(-4, -5)	3	(3, -4)	1	(-1 , -4)	2	(3, -5)
1	(-4, -5)	3	(3, 1)	1	(-1 , -4)	3	(-4, -4)
1	(-4, -5)	3	(3, 2)	1	(-3 , -4)	2	(-5, -5)

Table 2.8: Vector bundles of rank 4 (continued)

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$
1	(-4, -5)	3	(4, -3)	1	(-1 , -4)	2	(3, -5)
3	(-4, -5)	4	(4, 0)	2	(-3 , -3)	4	(-1, -5)
4	(-4, -4)	3	(-3, -4)	1	(3 , 3)	5	(-5, -5)
4	(-4, -4)	3	(-3, -4)	1	(-3 , -3)	5	(-5, -5)
1	(-4, -4)	3	(-2, -3)	1	(0 , 0)	2	(-5, -5)
1	(-3, -5)	4	(0, 0)	1	(-2 , -2)	1	(-4, -5)
4	(-3, -4)	2	(-3, -3)	1	(3 , 2)	4	(-5, -5)
4	(-3, -4)	2	(-3, -3)	1	(-2 , -3)	4	(-5, -5)
2	(-3, -4)	3	(-2, -3)	2	(1 , 1)	2	(-4, -5)
2	(-3, -4)	3	(-2, -3)	2	(-1 , -1)	2	(-4, -5)
1	(-3, -4)	3	(-2, -3)	2	(-3 , -3)	1	(-4, -5)
3	(-3, -4)	4	(-2, -2)	4	(-3 , -3)	2	(-5, -5)
1	(-3, -4)	3	(-1, -3)	1	(3 , 0)	3	(-4, -4)
1	(-3, -4)	3	(-1, -3)	1	(0 , -3)	3	(-4, -4)
5	(-3, -4)	1	(-1, -2)	2	(1 , 1)	3	(-4, -5)
5	(-3, -4)	1	(-1, -2)	2	(-1 , -1)	3	(-4, -5)
2	(-3, -4)	2	(-1, -2)	1	(-1 , -2)	2	(-5, -5)
2	(-3, -4)	5	(-1, -2)	4	(-2 , -3)	2	(-3, -3)
4	(-3, -4)	4	(-1, -2)	5	(-2 , -3)	2	(-5, -5)
4	(-3, -4)	1	(-1, -2)	2	(-3 , -3)	2	(-4, -5)
3	(-3, -4)	1	(-1, -2)	1	(-3 , -4)	2	(-5, -5)
2	(-3, -4)	1	(-1, -2)	1	(-5 , -5)	1	(-5, -5)
3	(-3, -4)	5	(-1, -1)	4	(0 , 0)	3	(-5, -5)
3	(-3, -4)	3	(-1, -1)	1	(-3 , -3)	4	(-4, -4)
1	(-3, -4)	3	(0, -1)	1	(-1 , -2)	2	(-3, -3)
3	(-3, -4)	5	(0, -1)	5	(-1 , -2)	2	(-5, -5)
3	(-3, -4)	5	(0, 0)	5	(-2 , -2)	2	(-5, -5)
1	(-3, -4)	3	(1, -1)	3	(-2 , -2)	1	(0, -3)
1	(-3, -4)	3	(1, 0)	1	(-1 , -2)	2	(-3, -3)
3	(-3, -4)	5	(1, 0)	5	(-1 , -2)	2	(-5, -5)
3	(-3, -4)	3	(1, 1)	1	(-3 , -3)	4	(-4, -4)
3	(-3, -4)	4	(2, -2)	4	(1 , -3)	2	(-5, -5)
3	(-3, -4)	1	(2, -2)	1	(0 , -4)	2	(-5, -5)
2	(-3, -4)	5	(2, 1)	4	(-2 , -3)	2	(-3, -3)
4	(-3, -4)	4	(2, 1)	5	(-2 , -3)	2	(-5, -5)
4	(-3, -4)	1	(2, 1)	2	(-3 , -3)	2	(-4, -5)
3	(-3, -4)	1	(2, 1)	1	(-3 , -4)	2	(-5, -5)
2	(-3, -4)	1	(2, 1)	1	(-5 , -5)	1	(-5, -5)

Table 2.9: Vector bundles of rank 4 (continued)

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$
3	(-3, -4)	4	(2, 2)	4	(-3 , -3)	2	(-5, -5)
1	(-3, -4)	3	(3, -4)	1	(0 , -3)	2	(3, -5)
1	(-3, -4)	3	(3, 2)	2	(-3 , -3)	1	(-4, -5)
1	(-3, -4)	3	(4, -3)	1	(0 , -3)	2	(3, -5)
2	(-3, -3)	4	(-2, -4)	1	(2 , -1)	3	(-4, -5)
2	(-3, -3)	4	(-2, -4)	1	(1 , -2)	3	(-4, -5)
2	(-3, -3)	4	(-2, -4)	3	(-3 , -4)	1	(-2, -5)
2	(-3, -3)	3	(-2, -3)	1	(2 , 2)	3	(-4, -4)
3	(-3, -3)	5	(-2, -3)	5	(1 , 1)	2	(-4, -5)
2	(-3, -3)	3	(-2, -3)	2	(1 , 1)	2	(-5, -5)
2	(-3, -3)	4	(-2, -3)	1	(1 ,0)	4	(-4, -4)
2	(-3, -3)	4	(-2, -3)	1	(0 , -1)	4	(-4, -4)
3	(-3, -3)	5	(-2, -3)	5	(-1 , -1)	2	(-4, -5)
2	(-3, -3)	3	(-2, -3)	2	(-1 , -1)	2	(-5, -5)
2	(-3, -3)	3	(-2, -3)	1	(-2 , -2)	3	(-4, -4)
1	(-3, -3)	3	(-2, -3)	2	(-3 , -3)	1	(-5, -5)
2	(-3, -3)	4	(-1, -3)	3	(2 , 1)	1	(-2, -5)
2	(-3, -3)	4	(-1, -3)	3	(-1 , -2)	1	(-2, -5)
4	(-3, -3)	1	(-1, -3)	1	(-2 , -2)	1	(-4, -5)
2	(-3, -3)	4	(-1, -3)	3	(-2 , -3)	1	(-1, -4)
1	(-3, -3)	5	(-1, -2)	3	(1 , 1)	2	(-2, -3)
2	(-3, -3)	3	(-1, -2)	3	(1 ,1)	1	(-5, -5)
1	(-3, -3)	5	(-1, -2)	2	(1 , 0)	3	(-3, -3)
1	(-3, -3)	5	(-1, -2)	2	(0 , -1)	3	(-3, -3)
1	(-3, -3)	5	(-1, -2)	3	(-1 , -1)	2	(-2, -3)
1	(-3, -3)	3	(-1, -2)	1	(-1 , -1)	2	(-3, -3)
4	(-3, -3)	3	(-1, -2)	1	(-1 , -1)	5	(-3, -3)
2	(-3, -3)	3	(-1, -2)	3	(-1 , -1)	1	(-5, -5)
2	(-3, -3)	3	(-1, -2)	3	(-2 , -2)	1	(-4, -4)
2	(-3, -3)	4	(0, -2)	3	(1 , 0)	1	(-1, -4)
2	(-3, -3)	4	(0, -2)	3	(0 , -1)	1	(-1, -4)
5	(-3, -3)	1	(0, -2)	1	(-2 , -3)	2	(-4, -4)
2	(-3, -3)	3	(0, -1)	3	(0 , 0)	1	(-4, -4)
3	(-3, -3)	3	(0, -1)	4	(0 , 0)	1	(-5, -5)
2	(-3, -3)	4	(0, -1)	1	(0 , -1)	4	(-2, -2)
1	(-3, -3)	4	(0, -1)	3	(-1 , -1)	1	(-1, -2)
2	(-3, -3)	4	(0, -1)	4	(-1 , -1)	1	(-2, -3)
2	(-3, -3)	3	(0, -1)	1	(-2 , -2)	3	(-2, -2)
3	(-3, -3)	3	(0, -1)	4	(-2 , -2)	1	(-3, -3)

Table 2.10: Vector bundles of rank 4 (continued)

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$
2	(-3, -3)	4	(1, -1)	1	(1 , -2)	3	(-1, -2)
1	(-3, -3)	4	(1, 0)	3	(-1 , -1)	1	(-1, -2)
2	(-3, -3)	4	(1, 0)	4	(-1 , -1)	1	(-2, -3)
2	(-3, -3)	3	(1, 0)	1	(-2 , -2)	3	(-2, -2)
3	(-3, -3)	3	(1, 0)	4	(-2 , -2)	1	(-3, -3)
5	(-3, -3)	1	(2, 0)	1	(-2 , -3)	2	(-4, -4)
2	(-3, -3)	3	(2, 1)	3	(-2 , -2)	1	(-4, -4)
2	(-3, -3)	2	(3, -4)	4	(-2 , -2)	1	(4, -5)
2	(-3, -3)	4	(3, 1)	3	(-2 , -3)	1	(-1, -4)
1	(-3, -3)	3	(3, 2)	2	(-3 , -3)	1	(-5, -5)
2	(-3, -3)	4	(4, 2)	3	(-3 , -4)	1	(-2, -5)
1	(-2, -5)	3	(-1, -2)	1	(0 , -3)	2	(-5, -5)
1	(-2, -5)	3	(2, -3)	4	(-3 , -3)	2	(2, -4)
1	(-2, -5)	3	(3, -2)	4	(-3 , -3)	2	(2, -4)
4	(-2, -4)	2	(3, 3)	3	(-3 , -4)	1	(-2, -5)
1	(-2, -3)	4	(-2, -3)	2	(2 , 1)	2	(-5, -5)
1	(-2, -3)	4	(-2, -3)	2	(-1 , -2)	2	(-5, -5)
1	(-2, -3)	2	(-2, -3)	1	(-3 , -3)	1	(-5, -5)
3	(-2, -3)	1	(-1, -4)	1	(3 , 0)	2	(-5, -5)
3	(-2, -3)	1	(-1, -4)	1	(0 , -3)	2	(-5, -5)
1	(-2, -3)	4	(-1, -2)	2	(1 , 1)	2	(-2, -3)
4	(-2, -3)	1	(-1, -2)	2	(1 , 1)	2	(-3, -4)
2	(-2, -3)	1	(-1, -2)	1	(1 , 1)	1	(-5, -5)
3	(-2, -3)	4	(-1, -2)	4	(1 , 0)	2	(-5, -5)
3	(-2, -3)	4	(-1, -2)	4	(0 , -1)	2	(-5, -5)
1	(-2, -3)	4	(-1, -2)	2	(-1 , -1)	2	(-2, -3)
4	(-2, -3)	1	(-1, -2)	2	(-1 , -1)	2	(-3, -4)
2	(-2, -3)	1	(-1, -2)	1	(-1 , -1)	1	(-5, -5)
1	(-2, -3)	3	(-1, -2)	1	(-1 , -2)	2	(-3, -3)
5	(-2, -3)	1	(-1, -2)	3	(-1 , -2)	2	(-5, -5)
2	(-2, -3)	2	(-1, -2)	1	(-2 , -3)	2	(-3, -3)
1	(-2, -3)	2	(-1, -2)	1	(-3 , -3)	1	(-3, -3)
3	(-2, -3)	3	(-1, -1)	1	(-1 , -1)	4	(-3, -3)
4	(-2, -3)	3	(-1, -1)	1	(-1 , -2)	5	(-3, -3)
4	(-2, -3)	2	(-1, -1)	1	(-2 , -3)	4	(-3, -3)
3	(-2, -3)	2	(-1, -1)	1	(-3 , -3)	3	(-3, -3)
3	(-2, -3)	2	(0, -3)	2	(1 , 1)	2	(-1, -4)
3	(-2, -3)	2	(0, -3)	2	(2 , -1)	2	(-5, -5)
3	(-2, -3)	2	(0, -3)	2	(-1 , -1)	2	(-1, -4)
3	(-2, -3)	2	(0, -3)	2	(1 , -2)	2	(-5, -5)
3	(-2, -3)	1	(0, -3)	2	(-3 , -3)	1	(-1, -4)

Table 2.11: Vector bundles of rank 4 (continued)

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$
1	(-2, -3)	4	(0, -1)	2	(-1 , -1)	2	(-1, -2)
1	(-2, -3)	2	(0, -1)	1	(-1 , -1)	1	(-3, -3)
2	(-2, -3)	2	(0, -1)	1	(-1 , -2)	2	(-3, -3)
3	(-2, -3)	2	(0, 0)	2	(-1 , -1)	2	(-4, -4)
3	(-2, -3)	4	(1, -2)	2	(-1 , -1)	4	(0, -3)
3	(-2, -3)	2	(1, -2)	2	(0 , -3)	2	(-3, -3)
1	(-2, -3)	4	(1, 0)	2	(-1 , -1)	2	(-1, -2)
1	(-2, -3)	2	(1, 0)	1	(-1 , -1)	1	(-3, -3)
2	(-2, -3)	2	(1, 0)	1	(-1 , -2)	2	(-3, -3)
4	(-2, -3)	3	(1, 1)	1	(-1 , -2)	5	(-3, -3)
4	(-2, -3)	2	(1, 1)	1	(-2 , -3)	4	(-3, -3)
3	(-2, -3)	2	(1, 1)	1	(-3 , -3)	3	(-3, -3)
3	(-2, -3)	1	(2, -5)	1	(4 , -3)	2	(-5, -5)
3	(-2, -3)	1	(2, -5)	1	(3 , -4)	2	(-5, -5)
1	(-2, -3)	4	(2, -2)	2	(0 , -2)	2	(2, -3)
3	(-2, -3)	4	(2, -1)	2	(-1 , -1)	4	(0, -3)
3	(-2, -3)	2	(2, -1)	2	(0 , -3)	2	(-3, -3)
2	(-2, -3)	2	(2, 1)	1	(-2 , -3)	2	(-3, -3)
1	(-2, -3)	2	(2, 1)	1	(-3 , -3)	1	(-3, -3)
3	(-2, -3)	2	(3, -4)	2	(-1 , -1)	2	(2, -5)
3	(-2, -3)	1	(3, -4)	2	(-3 , -3)	1	(2, -5)
3	(-2, -3)	1	(3, 0)	2	(-3 , -3)	1	(-1, -4)
1	(-2, -3)	2	(3, 2)	1	(-3 , -3)	1	(-5, -5)
3	(-2, -3)	2	(4, -3)	2	(-1 , -1)	2	(2, -5)
3	(-2, -3)	1	(4, -3)	2	(-3 , -3)	1	(2, -5)
4	(-2, -2)	1	(-1, -3)	1	(0 , 0)	1	(-3, -4)
4	(-2, -2)	3	(0, -1)	5	(-1 , -1)	1	(-3, -3)
4	(-2, -2)	2	(1, -3)	2	(-1 , -2)	1	(1, -4)
4	(-2, -2)	3	(1, 0)	5	(-1 , -1)	1	(-3, -3)
4	(-2, -2)	2	(3, -1)	2	(-1 , -2)	1	(1, -4)
1	(-1, -5)	3	(-1, -2)	1	(3 , -1)	2	(-5, -5)
1	(-1, -5)	3	(-1, -2)	1	(1 , -3)	2	(-5, -5)
2	(-1, -5)	4	(0, 0)	2	(-3 , -4)	1	(0, -5)
2	(-1, -4)	1	(0, -2)	1	(-1 , -2)	1	(-1, -5)
2	(-1, -4)	1	(2, 0)	1	(-1 , -2)	1	(-1, -5)
3	(-1, -3)	1	(-1, -2)	3	(-2 , -2)	1	(-1, -4)
4	(-1, -3)	2	(-1, -1)	1	(0 , -3)	3	(-2, -3)
4	(-1, -3)	2	(1, 1)	1	(0 , -3)	3	(-2, -3)
3	(-1, -3)	1	(2, 1)	3	(-2 , -2)	1	(-1, -4)
1	(-1, -2)	3	(-1, -2)	2	(1 , 1)	1	(-2, -3)
1	(-1, -2)	2	(-1, -2)	1	(1 , 1)	1	(-3, -3)

Table 2.12: Vector bundles of rank 4 (continued)

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$
1	(-1, -2)	4	(-1, -2)	2	(1 , 0)	2	(-3, -3)
1	(-1, -2)	4	(-1, -2)	2	(0 , -1)	2	(-3, -3)
1	(-1, -2)	3	(-1, -2)	2	(-1 , -1)	1	(-2, -3)
1	(-1, -2)	2	(-1, -2)	1	(-1 , -1)	1	(-3, -3)
4	(-1, -2)	2	(-1, -1)	1	(1 , 0)	4	(-2, -2)
4	(-1, -2)	2	(-1, -1)	1	(0 , -1)	4	(-2, -2)
4	(-1, -2)	1	(-1, -1)	3	(-1 , -1)	1	(-2, -3)
3	(-1, -2)	1	(-1, -1)	2	(-1 , -1)	1	(-3, -3)
1	(-1, -2)	4	(0, -4)	2	(2 , 0)	2	(0, -5)
3	(-1, -2)	2	(0, -4)	2	(2 , -2)	2	(-5, -5)
1	(-1, -2)	4	(0, -4)	2	(0 , -2)	2	(0, -5)
1	(-1, -2)	4	(0, -4)	2	(1 , -4)	2	(-2, -4)
3	(-1, -2)	1	(0, -3)	1	(2 , -1)	2	(-3, -3)
3	(-1, -2)	5	(0, -3)	5	(2 , -1)	2	(-5, -5)
3	(-1, -2)	1	(0, -3)	1	(1 , -2)	2	(-3, -3)
3	(-1, -2)	5	(0, -3)	5	(1 , -2)	2	(-5, -5)
1	(-1, -2)	3	(0, -1)	1	(0 , -1)	2	(-1, -1)
5	(-1, -2)	1	(0, -1)	3	(0 , -1)	2	(-3, -3)
1	(-1, -2)	2	(0, -1)	1	(-1 , -1)	1	(-1, -1)
2	(-1, -2)	2	(0, -1)	2	(-1 , -1)	1	(-1, -2)
3	(-1, -2)	1	(1, -2)	2	(-1 , -1)	1	(0, -3)
1	(-1, -2)	2	(1, 0)	1	(-1 , -1)	1	(-1, -1)
2	(-1, -2)	2	(1, 0)	2	(-1 , -1)	1	(-1, -2)
3	(-1, -2)	1	(2, -1)	2	(-1 , -1)	1	(0, -3)
2	(-1, -2)	1	(3, 2)	1	(-3 , -3)	1	(-3, -3)
1	(-1, -2)	2	(4, 3)	1	(-5 , -5)	1	(-5, -5)
2	(-1, -1)	4	(0, -2)	3	(1 , 0)	1	(0, -3)
2	(-1, -1)	4	(0, -2)	3	(0 , -1)	1	(0, -3)
4	(-1, -1)	2	(1, -3)	1	(3 , -2)	2	(-1, -2)
4	(-1, -1)	2	(1, -3)	2	(-1 , -2)	1	(2, -3)
2	(-1, -1)	3	(3, 2)	1	(-3 , -3)	3	(-3, -3)
1	(0, -4)	3	(0, -1)	1	(2 , -2)	2	(-3, -3)
2	(0, -3)	1	(0, -2)	1	(1 , -3)	1	(-2, -3)
1	(0, -3)	2	(2, -4)	1	(3 , -2)	1	(2, -5)
1	(0, -3)	2	(2, -4)	1	(2 , -3)	1	(2, -5)
1	(0, -2)	3	(4, -5)	1	(2 , -4)	2	(5, -5)
3	(0, -1)	2	(1, -3)	2	(2 , -2)	2	(-2, -2)
3	(0, -1)	4	(1, -3)	4	(2 , -2)	2	(-3, -3)
2	(1, 1)	3	(3, 2)	1	(-3 , -3)	3	(-3, -3)
2	(2, 1)	1	(3, 2)	1	(-3 , -3)	1	(-3, -3)
1	(2, 1)	2	(4, 3)	1	(-5 , -5)	1	(-5, -5)

Table 2.13: Vector bundles of rank 4 (continued)

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$
1	(3, 2)	2	(3, 2)	1	(-3 , -3)	1	(-5, -5)
2	(3, 3)	4	(4, 2)	3	(-3 , -4)	1	(-2, -5)

2.2.4 Examples on $G(2, N)$

More generally, we can use sequence (2.17) to describe $\mathcal{E}$ on $G(2, N)$, satisfying anomaly cancellation conditions. In the table 2.22, we list examples of $\mathcal{E}$ on $G(2, N)$, for N from 5 to 20. In some cases, it may not be possible to find explicit maps between the second and third entries in the short exact sequence, or the data given does not define a bundle, merely a sheaf.

Table 2.14: Vector bundles of rank 5

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$
4	(-5, -5)	4	(-1, -3)	1	(1 , 0)	3	(-3, -5)
4	(-5, -5)	4	(-1, -3)	1	(0 , -1)	3	(-3, -5)
2	(-5, -5)	4	(0, -1)	1	(0 , -1)	3	(-4, -4)
2	(-5, -5)	3	(0, -1)	1	(-4 , -4)	2	(-4, -4)
3	(-5, -5)	3	(0, -1)	3	(-4 , -4)	1	(-5, -5)
4	(-5, -5)	3	(1, -1)	1	(0 , -1)	2	(-3, -5)
2	(-5, -5)	3	(1, 0)	1	(-4 , -4)	2	(-4, -4)
3	(-5, -5)	3	(1, 0)	3	(-4 , -4)	1	(-5, -5)
2	(-5, -5)	2	(3, -4)	3	(-4 , -4)	1	(4, -5)
3	(-4, -5)	5	(-1, -1)	3	(-4 , -4)	3	(-5, -5)
5	(-4, -5)	5	(0, -3)	4	(-1 , -4)	3	(-3, -5)
3	(-4, -5)	5	(0, 0)	2	(-3 , -3)	4	(-5, -5)
3	(-4, -5)	2	(1, -2)	1	(0 , -4)	4	(-5, -5)
3	(-4, -5)	5	(1, 1)	3	(-4 , -4)	3	(-5, -5)
3	(-4, -5)	2	(2, -1)	1	(0 , -4)	4	(-5, -5)
5	(-4, -5)	5	(3, 0)	4	(-1 , -4)	3	(-3, -5)
5	(-4, -4)	3	(-3, -4)	3	(-5 , -5)	3	(-5, -5)
4	(-4, -4)	4	(-2, -4)	1	(-3 , -4)	3	(-3, -5)
1	(-4, -4)	4	(-2, -3)	1	(-2 , -3)	2	(-5, -5)
2	(-4, -4)	3	(-2, -3)	1	(-4 , -4)	2	(-5, -5)
1	(-4, -4)	3	(-2, -3)	1	(-5 , -5)	1	(-5, -5)
4	(-4, -4)	3	(-1, -3)	1	(1 , 0)	2	(-3, -5)
4	(-4, -4)	3	(-1, -3)	1	(0 , -1)	2	(-3, -5)
4	(-4, -4)	5	(-1, -3)	1	(-2 , -3)	4	(-2, -4)
1	(-4, -4)	3	(-1, -2)	1	(1 , 1)	1	(-5, -5)
1	(-4, -4)	3	(-1, -2)	1	(-1 , -1)	1	(-5, -5)
3	(-4, -4)	3	(-1, -2)	2	(-2 , -2)	2	(-5, -5)
1	(-4, -4)	4	(-1, -2)	1	(-2 , -3)	2	(-3, -3)
1	(-4, -4)	4	(0, -1)	2	(-1 , -1)	1	(-2, -3)
5	(-4, -4)	3	(0, -1)	3	(-1 , -1)	3	(-5, -5)
4	(-4, -4)	3	(0, -1)	3	(-2 , -2)	2	(-5, -5)
5	(-4, -4)	2	(1, -3)	2	(-3 , -4)	1	(0, -5)
4	(-4, -4)	5	(1, -1)	4	(-1 , -3)	1	(-2, -3)
4	(-4, -4)	4	(1, -1)	3	(-1 , -3)	1	(-3, -4)
4	(-4, -4)	2	(1, -1)	1	(-2 , -3)	1	(-3, -5)
1	(-4, -4)	4	(1, 0)	2	(-1 , -1)	1	(-2, -3)
5	(-4, -4)	3	(1, 0)	3	(-1 , -1)	3	(-5, -5)
3	(-4, -4)	3	(2, 1)	2	(-2 , -2)	2	(-5, -5)
1	(-4, -4)	4	(2, 1)	1	(-2 , -3)	2	(-3, -3)
5	(-4, -4)	2	(3, -1)	2	(-3 , -4)	1	(0, -5)

Table 2.15: Vector bundles of rank 5 (continued)

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$
4	(-4, -4)	5	(3, 1)	1	(-2 , -3)	4	(-2, -4)
2	(-4, -4)	3	(3, 2)	1	(-4 , -4)	2	(-5, -5)
1	(-4, -4)	3	(3, 2)	1	(-5 , -5)	1	(-5, -5)
4	(-4, -4)	4	(4, 2)	1	(-3 , -4)	3	(-3, -5)
5	(-4, -4)	3	(4, 3)	3	(-5 , -5)	3	(-5, -5)
1	(-3, -5)	5	(-1, -1)	1	(-3 , -3)	1	(-4, -5)
1	(-3, -5)	5	(-1, -1)	1	(-3 , -4)	1	(-5, -5)
1	(-3, -5)	5	(0, 0)	1	(-2 , -2)	1	(-4, -5)
1	(-3, -5)	5	(1, -2)	4	(0 , -2)	1	(0, -5)
1	(-3, -5)	5	(1, -2)	1	(2 , -3)	4	(-1, -3)
2	(-3, -5)	5	(1, -2)	5	(-1 , -3)	1	(0, -5)
1	(-3, -5)	5	(1, 1)	1	(-3 , -3)	1	(-4, -5)
1	(-3, -5)	5	(1, 1)	1	(-3 , -4)	1	(-5, -5)
1	(-3, -5)	5	(2, -1)	4	(0 , -2)	1	(0, -5)
2	(-3, -5)	5	(2, -1)	5	(-1 , -3)	1	(0, -5)
3	(-3, -4)	3	(-2, -2)	2	(-4 , -4)	2	(-5, -5)
3	(-3, -4)	2	(-1, -4)	1	(4 , 0)	4	(-5, -5)
3	(-3, -4)	2	(-1, -4)	1	(0 , -4)	4	(-5, -5)
5	(-3, -4)	2	(-1, -4)	1	(-1 , -4)	3	(-3, -5)
5	(-3, -4)	1	(-1, -4)	1	(-3 , -5)	2	(-3, -5)
1	(-3, -4)	3	(-1, -3)	2	(-2 , -2)	1	(-2, -5)
1	(-3, -4)	3	(-1, -3)	1	(-1 , -4)	2	(-4, -4)
3	(-3, -4)	3	(-1, -3)	3	(-2 , -4)	1	(-5, -5)
3	(-3, -4)	5	(-1, -1)	3	(-3 , -3)	3	(-4, -4)
3	(-3, -4)	3	(-1, -1)	3	(-4 , -4)	1	(-5, -5)
2	(-3, -4)	2	(0, -4)	1	(0 , -5)	1	(-3, -5)
5	(-3, -4)	3	(0, -3)	2	(2 , -1)	3	(-3, -5)
5	(-3, -4)	3	(0, -3)	2	(1 , -2)	3	(-3, -5)
3	(-3, -4)	5	(0, -2)	5	(-1 , -3)	1	(-5, -5)
3	(-3, -4)	1	(0, -2)	1	(-3 , -5)	1	(-5, -5)
3	(-3, -4)	5	(0, 0)	5	(-3 , -3)	1	(-5, -5)
3	(-3, -4)	3	(1, -3)	3	(0 , -4)	1	(-5, -5)
3	(-3, -4)	2	(1, -1)	2	(-2 , -4)	1	(-4, -4)
3	(-3, -4)	5	(1, 1)	3	(-3 , -3)	3	(-4, -4)
3	(-3, -4)	3	(1, 1)	3	(-4 , -4)	1	(-5, -5)
3	(-3, -4)	1	(2, -2)	1	(-1 , -5)	1	(-5, -5)
3	(-3, -4)	5	(2, 0)	5	(-1 , -3)	1	(-5, -5)
3	(-3, -4)	1	(2, 0)	1	(-3 , -5)	1	(-5, -5)
3	(-3, -4)	3	(2, 2)	2	(-4 , -4)	2	(-5, -5)

Table 2.16: Vector bundles of rank 5 (continued)

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$
3	(-3, -4)	3	(3, -1)	3	(0 , -4)	1	(-5, -5)
1	(-3, -4)	3	(3, 1)	1	(-1 , -4)	2	(-4, -4)
3	(-3, -4)	3	(3, 1)	3	(-2 , -4)	1	(-5, -5)
3	(-3, -4)	5	(4, -4)	5	(3 , -5)	1	(-5, -5)
2	(-3, -4)	2	(4, 0)	1	(0 , -5)	1	(-3, -5)
5	(-3, -4)	1	(4, 1)	1	(-3 , -5)	2	(-3, -5)
3	(-3, -4)	5	(4, 4)	3	(-5 , -5)	3	(-5, -5)
2	(-3, -3)	5	(-1, -2)	3	(-2 , -2)	2	(-2, -3)
3	(-3, -3)	3	(-1, -2)	3	(-2 , -2)	1	(-5, -5)
5	(-3, -3)	1	(0, -2)	1	(-2 , -2)	1	(-4, -5)
2	(-3, -3)	4	(0, -1)	1	(-1 , -2)	3	(-2, -2)
5	(-3, -3)	1	(1, -1)	1	(-1 , -1)	1	(-4, -5)
2	(-3, -3)	4	(1, 0)	1	(-1 , -2)	3	(-2, -2)
2	(-3, -3)	2	(2, -5)	1	(5 , -4)	3	(-4, -4)
2	(-3, -3)	2	(2, -5)	3	(-4 , -4)	1	(4, -5)
5	(-3, -3)	1	(2, 0)	1	(-2 , -2)	1	(-4, -5)
2	(-3, -3)	5	(2, 1)	3	(-2 , -2)	2	(-2, -3)
3	(-3, -3)	3	(2, 1)	3	(-2 , -2)	1	(-5, -5)
1	(-2, -5)	5	(-2, -3)	1	(3 , 1)	2	(-3, -5)
1	(-2, -5)	5	(-2, -3)	1	(-1 , -3)	2	(-3, -5)
1	(-2, -5)	4	(-1, -4)	1	(3 , -2)	3	(-3, -5)
1	(-2, -5)	4	(-1, -4)	1	(2 , -3)	3	(-3, -5)
1	(-2, -5)	4	(-1, -3)	1	(0 , -4)	3	(-3, -4)
1	(-2, -5)	4	(0, -2)	3	(-2 , -3)	1	(0, -4)
1	(-2, -5)	5	(0, -1)	2	(1 , -1)	1	(-3, -5)
3	(-2, -5)	5	(0, -1)	2	(1 , -2)	3	(-3, -5)
5	(-2, -5)	5	(0, -1)	4	(-1 , -4)	3	(-3, -5)
3	(-2, -5)	2	(1, -2)	1	(2 , -3)	3	(-3, -5)
1	(-2, -5)	4	(1, -2)	3	(-1 , -3)	1	(1, -4)
5	(-2, -5)	2	(1, -1)	1	(2 , -3)	5	(-3, -5)
5	(-2, -5)	5	(1, 0)	4	(-1 , -4)	3	(-3, -5)
1	(-2, -5)	4	(2, -1)	3	(-1 , -3)	1	(1, -4)
1	(-2, -5)	4	(2, 0)	3	(-2 , -3)	1	(0, -4)
1	(-2, -4)	3	(-1, -2)	1	(-1 , -3)	1	(-5, -5)
1	(-2, -4)	5	(-1, -1)	1	(-2 , -2)	1	(-3, -4)
1	(-2, -4)	5	(-1, -1)	1	(-2 , -3)	1	(-4, -4)
1	(-2, -4)	3	(0, -1)	1	(1 , -1)	1	(-5, -5)
1	(-2, -4)	5	(1, 1)	1	(-2 , -2)	1	(-3, -4)
1	(-2, -4)	5	(1, 1)	1	(-2 , -3)	1	(-4, -4)
1	(-2, -4)	3	(2, 1)	1	(-1 , -3)	1	(-5, -5)

Table 2.17: Vector bundles of rank 5 (continued)

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$
3	(-2, -3)	4	(-2, -2)	2	(1 , 1)	3	(-4, -4)
3	(-2, -3)	5	(-2, -2)	4	(1 , 1)	2	(-5, -5)
4	(-2, -3)	1	(-2, -2)	1	(1 , 0)	2	(-5, -5)
4	(-2, -3)	1	(-2, -2)	1	(0 , -1)	2	(-5, -5)
3	(-2, -3)	4	(-2, -2)	2	(-1 , -1)	3	(-4, -4)
3	(-2, -3)	5	(-2, -2)	4	(-1 , -1)	2	(-5, -5)
3	(-2, -3)	5	(-2, -2)	3	(-3 , -3)	3	(-3, -3)
5	(-2, -3)	3	(-1, -4)	3	(3 , 1)	2	(-2, -5)
3	(-2, -3)	2	(-1, -4)	1	(4 , 0)	4	(-4, -4)
5	(-2, -3)	3	(-1, -4)	3	(-1 , -3)	2	(-2, -5)
3	(-2, -3)	2	(-1, -4)	1	(0 , -4)	4	(-4, -4)
3	(-2, -3)	2	(-1, -3)	1	(-2 , -2)	2	(-2, -4)
4	(-2, -3)	2	(-1, -1)	1	(1 , 0)	3	(-4, -4)
4	(-2, -3)	2	(-1, -1)	1	(0 , -1)	3	(-4, -4)
3	(-2, -3)	5	(-1, -1)	3	(-2 , -2)	3	(-3, -3)
5	(-2, -3)	2	(-1, -1)	3	(-2 , -2)	2	(-3, -4)
3	(-2, -3)	2	(-1, -1)	1	(-2 , -2)	2	(-4, -4)
3	(-2, -3)	3	(-1, -1)	3	(-2 , -2)	1	(-5, -5)
2	(-2, -3)	2	(0, -4)	1	(4 , -1)	1	(-3, -5)
3	(-2, -3)	1	(0, -4)	1	(-3 , -3)	1	(-1, -5)
2	(-2, -3)	2	(0, -4)	1	(1 , -4)	1	(-3, -5)
5	(-2, -3)	3	(0, -3)	3	(-1 , -3)	2	(-1, -4)
3	(-2, -3)	2	(0, -2)	2	(-1 , -3)	1	(-4, -4)
3	(-2, -3)	1	(0, -2)	1	(-1 , -3)	1	(-5, -5)
3	(-2, -3)	3	(0, 0)	2	(-1 , -1)	2	(-4, -4)
5	(-2, -3)	1	(0, 0)	2	(-1 , -2)	2	(-5, -5)
3	(-2, -3)	5	(0, 0)	4	(-2 , -2)	2	(-3, -3)
3	(-2, -3)	2	(0, 0)	2	(-3 , -3)	1	(-4, -4)
3	(-2, -3)	1	(0, 0)	1	(-3 , -3)	1	(-5, -5)
3	(-2, -3)	2	(1, -3)	1	(-2 , -2)	2	(0, -4)
3	(-2, -3)	2	(1, -2)	1	(2 , -2)	4	(-3, -3)
3	(-2, -3)	2	(1, -2)	4	(-2 , -2)	1	(0, -4)
5	(-2, -3)	4	(1, -2)	3	(0 , -3)	3	(-1, -3)
5	(-2, -3)	2	(1, -2)	3	(-1 , -3)	1	(-1, -4)
1	(-2, -3)	3	(1, -1)	1	(1 , -2)	2	(-2, -2)
3	(-2, -3)	3	(1, -1)	3	(0 , -2)	1	(-5, -5)
3	(-2, -3)	5	(1, 1)	3	(-2 , -2)	3	(-3, -3)
5	(-2, -3)	2	(1, 1)	3	(-2 , -2)	2	(-3, -4)
3	(-2, -3)	2	(1, 1)	1	(-2 , -2)	2	(-4, -4)

Table 2.18: Vector bundles of rank 5 (continued)

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$
3	(-2, -3)	3	(1, 1)	3	(-2 , -2)	1	(-5, -5)
3	(-2, -3)	5	(2, -2)	5	(1 , -3)	1	(-3, -3)
3	(-2, -3)	2	(2, -1)	4	(-2 , -2)	1	(0, -4)
5	(-2, -3)	4	(2, -1)	3	(0 , -3)	3	(-1, -3)
5	(-2, -3)	2	(2, -1)	3	(-1 , -3)	1	(-1, -4)
3	(-2, -3)	2	(2, 0)	2	(-1 , -3)	1	(-4, -4)
3	(-2, -3)	1	(2, 0)	1	(-1 , -3)	1	(-5, -5)
3	(-2, -3)	5	(2, 2)	3	(-3 , -3)	3	(-3, -3)
3	(-2, -3)	2	(3, -1)	1	(-2 , -2)	2	(0, -4)
5	(-2, -3)	3	(3, 0)	3	(-1 , -3)	2	(-1, -4)
3	(-2, -3)	2	(4, -4)	1	(-4 , -4)	2	(3, -5)
3	(-2, -3)	1	(4, -4)	1	(3 , -5)	1	(-5, -5)
3	(-2, -3)	1	(4, 0)	1	(-3 , -3)	1	(-1, -5)
3	(-2, -3)	1	(4, 4)	1	(-5 , -5)	1	(-5, -5)
5	(-2, -2)	1	(-1, -3)	1	(-2 , -2)	1	(-3, -4)
4	(-2, -2)	2	(-1, -3)	1	(-1 , -3)	1	(-3, -4)
5	(-2, -2)	1	(-1, -3)	1	(-2 , -3)	1	(-4, -4)
5	(-2, -2)	3	(-1, -2)	3	(1 , 1)	3	(-3, -3)
2	(-2, -2)	3	(-1, -2)	2	(1 , 1)	1	(-4, -4)
5	(-2, -2)	3	(-1, -2)	5	(1 , 1)	1	(-5, -5)
1	(-2, -2)	4	(-1, -2)	1	(1 , 0)	2	(-3, -3)
1	(-2, -2)	4	(-1, -2)	1	(0 , -1)	2	(-3, -3)
5	(-2, -2)	3	(-1, -2)	3	(-1 , -1)	3	(-3, -3)
2	(-2, -2)	3	(-1, -2)	2	(-1 , -1)	1	(-4, -4)
5	(-2, -2)	3	(-1, -2)	5	(-1 , -1)	1	(-5, -5)
4	(-2, -2)	4	(0, -2)	3	(1 , -1)	1	(-3, -4)
1	(-2, -2)	4	(0, -1)	1	(0 , -1)	2	(-1, -1)
1	(-2, -2)	3	(0, -1)	1	(-1 , -1)	1	(-1, -1)
2	(-2, -2)	3	(0, -1)	2	(-1 , -1)	1	(-2, -2)
5	(-2, -2)	2	(1, -3)	1	(2 , -3)	2	(-2, -3)
4	(-2, -2)	5	(1, -1)	1	(0 , -1)	4	(0, -2)
4	(-2, -2)	2	(1, -1)	1	(0 , -1)	1	(-1, -3)
5	(-2, -2)	1	(1, -1)	1	(0 , -1)	1	(-4, -4)
5	(-2, -2)	1	(1, -1)	1	(-2 , -2)	1	(-2, -3)
1	(-2, -2)	3	(1, 0)	1	(-1 , -1)	1	(-1, -1)
2	(-2, -2)	3	(1, 0)	2	(-1 , -1)	1	(-2, -2)
1	(-2, -2)	2	(3, -4)	2	(-1 , -1)	1	(4, -5)
5	(-2, -2)	1	(3, 1)	1	(-2 , -3)	1	(-4, -4)
5	(-2, -2)	3	(3, 2)	3	(-3 , -3)	3	(-3, -3)
3	(-2, -2)	3	(4, 3)	2	(-4 , -4)	2	(-5, -5)
2	(-1, -5)	5	(0, 0)	2	(-3 , -4)	1	(0, -5)
2	(-1, -4)	3	(0, -3)	3	(-1 , -3)	1	(0, -5)

Table 2.19: Vector bundles of rank 5 (continued)

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$
1	(-1, -4)	4	(0, -2)	3	(-1 , -2)	1	(0, -4)
1	(-1, -4)	5	(0, -1)	1	(1 , -1)	2	(-1, -3)
2	(-1, -4)	5	(0, -1)	3	(1 , -1)	1	(-2, -5)
1	(-1, -4)	3	(1, -4)	2	(4 , -2)	3	(-4, -4)
5	(-1, -4)	2	(1, -4)	3	(3 , -2)	3	(-3, -5)
5	(-1, -4)	2	(1, -4)	3	(2 , -3)	3	(-3, -5)
5	(-1, -4)	2	(1, -4)	3	(-1 , -3)	3	(0, -5)
1	(-1, -4)	3	(1, -4)	2	(2 , -4)	3	(-4, -4)
2	(-1, -4)	3	(1, -2)	3	(-1 , -3)	1	(1, -4)
3	(-1, -4)	2	(1, -2)	3	(-1 , -3)	1	(0, -5)
1	(-1, -4)	4	(1, -1)	3	(0 , -1)	1	(0, -4)
5	(-1, -4)	1	(1, -1)	1	(1 , -4)	4	(-2, -4)
1	(-1, -4)	3	(2, -3)	3	(-2 , -2)	2	(2, -4)
2	(-1, -4)	3	(2, -1)	3	(-1 , -3)	1	(1, -4)
3	(-1, -4)	2	(2, -1)	3	(-1 , -3)	1	(0, -5)
1	(-1, -4)	4	(2, 0)	3	(-1 , -2)	1	(0, -4)
1	(-1, -4)	3	(3, -2)	3	(-2 , -2)	2	(2, -4)
2	(-1, -4)	3	(3, 0)	3	(-1 , -3)	1	(0, -5)
1	(-1, -4)	5	(4, -4)	2	(2 , -4)	3	(4, -5)
5	(-1, -4)	2	(4, -1)	3	(-1 , -3)	3	(0, -5)
1	(-1, -4)	5	(4, 3)	1	(-3 , -5)	2	(-3, -5)
2	(-1, -3)	4	(-1, -1)	1	(1 , -1)	1	(-3, -4)
1	(-1, -3)	5	(-1, -1)	1	(-1 , -1)	1	(-2, -3)
1	(-1, -3)	5	(-1, -1)	1	(-1 , -2)	1	(-3, -3)
2	(-1, -3)	4	(-1, -1)	1	(-1 , -3)	1	(-2, -3)
3	(-1, -3)	1	(0, -1)	1	(1 , -2)	2	(-4, -4)
2	(-1, -3)	3	(0, -1)	2	(0 , -2)	1	(-4, -4)
3	(-1, -3)	3	(0, -1)	3	(0 , -2)	1	(-5, -5)
3	(-1, -3)	1	(0, -1)	2	(-2 , -2)	1	(-1, -4)
1	(-1, -3)	5	(0, 0)	1	(-1 , -2)	1	(-2, -2)
1	(-1, -3)	5	(1, -2)	4	(1 , -1)	1	(1, -4)
1	(-1, -3)	5	(1, -2)	4	(0 , -2)	1	(2, -3)
2	(-1, -3)	3	(1, 0)	2	(0 , -2)	1	(-4, -4)
3	(-1, -3)	3	(1, 0)	3	(0 , -2)	1	(-5, -5)
3	(-1, -3)	1	(1, 0)	2	(-2 , -2)	1	(-1, -4)
1	(-1, -3)	5	(1, 1)	1	(-1 , -2)	1	(-3, -3)
2	(-1, -3)	4	(1, 1)	1	(-1 , -3)	1	(-2, -3)
1	(-1, -3)	5	(2, 2)	1	(-2 , -3)	1	(-4, -4)
3	(-1, -3)	4	(3, -4)	2	(4 , -4)	3	(1, -4)
3	(-1, -2)	3	(-1, -1)	1	(-1 , -1)	3	(-2, -2)

Table 2.20: Vector bundles of rank 5 (continued)

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$
4	(-1, -2)	2	(-1, -1)	1	(-1 , -2)	3	(-2, -2)
3	(-1, -2)	2	(-1, -1)	1	(-2 , -2)	2	(-2, -2)
3	(-1, -2)	1	(0, -4)	1	(3 , -1)	1	(-5, -5)
3	(-1, -2)	1	(0, -4)	1	(1 , -3)	1	(-5, -5)
5	(-1, -2)	2	(0, -3)	1	(2 , -1)	3	(-1, -3)
5	(-1, -2)	2	(0, -3)	1	(1 , -2)	3	(-1, -3)
3	(-1, -2)	2	(0, -2)	2	(1 , -1)	1	(-4, -4)
3	(-1, -2)	5	(0, -2)	5	(1 , -1)	1	(-5, -5)
3	(-1, -2)	1	(0, -2)	1	(-1 , -1)	1	(-1, -3)
5	(-1, -2)	1	(0, 0)	2	(0 , -1)	2	(-3, -3)
3	(-1, -2)	3	(0, 0)	2	(-1 , -1)	2	(-2, -2)
4	(-1, -2)	1	(0, 0)	2	(-1 , -1)	1	(-2, -3)
3	(-1, -2)	1	(0, 0)	1	(-1 , -1)	1	(-3, -3)
3	(-1, -2)	3	(1, -3)	3	(2 , -2)	1	(-5, -5)
3	(-1, -2)	3	(1, -1)	1	(-1 , -1)	3	(0, -2)
3	(-1, -2)	2	(1, -1)	2	(0 , -2)	1	(-2, -2)
4	(-1, -2)	2	(1, 1)	1	(-1 , -2)	3	(-2, -2)
3	(-1, -2)	2	(1, 1)	1	(-2 , -2)	2	(-2, -2)
3	(-1, -2)	1	(4, -4)	1	(-1 , -1)	1	(3, -5)
5	(-1, -1)	1	(0, -2)	1	(1 , 0)	1	(-2, -2)
4	(-1, -1)	4	(0, -2)	3	(1 , -1)	1	(-2, -3)
5	(-1, -1)	1	(0, -2)	1	(0 , -1)	1	(-2, -2)
5	(-1, -1)	1	(1, -1)	1	(0 , -1)	1	(-1, -1)
2	(-1, -1)	3	(2, 1)	1	(-2 , -2)	2	(-2, -2)
3	(-1, -1)	3	(4, 3)	3	(-4 , -4)	1	(-5, -5)
1	(0, -5)	5	(1, -2)	2	(2 , -3)	3	(-1, -3)
4	(0, -5)	5	(1, -2)	5	(2 , -3)	3	(-3, -5)
2	(0, -5)	5	(1, -2)	3	(-1 , -3)	3	(1, -4)
2	(0, -5)	5	(2, -1)	3	(-1 , -3)	3	(1, -4)
2	(0, -4)	2	(0, -1)	1	(1 , -1)	1	(0, -5)
1	(0, -4)	3	(1, -4)	1	(2 , -1)	2	(1, -5)
1	(0, -4)	3	(1, -4)	1	(1 , -2)	2	(1, -5)
1	(0, -4)	3	(3, 2)	1	(-3 , -3)	1	(-1, -5)
2	(0, -3)	3	(0, -1)	1	(2 , -2)	4	(-2, -2)
1	(0, -2)	3	(0, -1)	1	(1 , -1)	1	(-1, -1)
2	(0, -2)	3	(0, -1)	2	(1 , -1)	1	(-2, -2)
1	(0, -2)	3	(4, 3)	1	(-3 , -5)	1	(-5, -5)
5	(0, -1)	1	(1, -2)	1	(1 , -1)	2	(1, -1)
5	(0, -1)	2	(1, -2)	3	(1 , -1)	1	(1, -2)
3	(0, -1)	2	(3, -5)	2	(4 , -4)	1	(-4, -4)
3	(0, -1)	3	(3, -5)	3	(4 , -4)	1	(-5, -5)

Table 2.21: Vector bundles of rank 5 (continued)

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$
2	(0, 0)	3	(3, 2)	2	(-3 , -3)	1	(-4, -4)
1	(0, 0)	3	(3, 2)	1	(-3 , -3)	1	(-5, -5)
3	(1, -4)	5	(1, -2)	4	(2 , -3)	3	(-1, -3)
5	(1, -4)	1	(4, -4)	2	(1 , -3)	3	(2, -5)
2	(1, -3)	3	(3, -3)	4	(1 , -2)	1	(4, -5)
3	(1, -3)	1	(4, -4)	3	(0 , -2)	1	(4, -5)
1	(1, -2)	4	(1, -2)	3	(1 , -1)	1	(2, -3)
2	(1, 1)	3	(2, 1)	1	(-2 , -2)	2	(-2, -2)
3	(1, 1)	3	(4, 3)	3	(-4 , -4)	1	(-5, -5)
1	(2, -3)	4	(3, -4)	2	(0 , -2)	3	(4, -4)
1	(2, 0)	3	(4, 3)	1	(-3 , -5)	1	(-5, -5)
5	(2, 2)	1	(3, 1)	1	(-2 , -3)	1	(-4, -4)
5	(2, 2)	3	(3, 2)	3	(-3 , -3)	3	(-3, -3)
3	(2, 2)	3	(4, 3)	2	(-4 , -4)	2	(-5, -5)
3	(3, 2)	1	(4, 0)	1	(-3 , -3)	1	(-1, -5)
3	(3, 2)	1	(4, 4)	1	(-5 , -5)	1	(-5, -5)
1	(4, 1)	5	(4, 3)	1	(-3 , -5)	2	(-3, -5)
3	(4, 3)	5	(4, 4)	3	(-5 , -5)	3	(-5, -5)

Table 2.22: Anomaly-free examples on $G(2, N)$

N	m	$(\lambda_{A1}, \lambda_{B1})$	n	$(\lambda_{A2}, \lambda_{B2})$	rank
5	5	(0, -1)	1	(1 , -1)	7
5	5	(1, 0)	1	(1 , -1)	7
6	6	(0, -1)	1	(1 , -1)	9
6	6	(1, 0)	1	(1 , -1)	9
7	7	(0, -1)	1	(1 , -1)	11
7	7	(1, 0)	1	(1 , -1)	11
8	8	(-3, -4)	1	(-7 , -9)	13
8	8	(-3, -4)	1	(9 , 7)	13
8	1	(-1, -3)	2	(-2 , -2)	1
8	1	(-1, -3)	2	(2 , 2)	1
8	8	(0, -1)	1	(1 , -1)	13
8	8	(1, 0)	1	(1 , -1)	13
8	1	(3, 1)	2	(-2 , -2)	1
8	1	(3, 1)	2	(2 , 2)	1
8	8	(4, 3)	1	(-7 , -9)	13
8	8	(4, 3)	1	(9 , 7)	13
9	9	(-2, -3)	1	(-5 , -7)	15
9	9	(-2, -3)	1	(7 , 5)	15
9	5	(-1, -2)	2	(-3 , -3)	8
9	13	(-1, -2)	2	(-2 , -4)	20
9	5	(-1, -2)	2	(3 , 3)	8
9	13	(-1, -2)	2	(4 , 2)	20
9	9	(0, -1)	1	(1 , -1)	15
9	9	(1, 0)	1	(1 , -1)	15
9	5	(2, 1)	2	(-3 , -3)	8
9	13	(2, 1)	2	(-2 , -4)	20
9	5	(2, 1)	2	(3 , 3)	8
9	13	(2, 1)	2	(4 , 2)	20
9	9	(3, 2)	1	(-5 , -7)	15
9	9	(3, 2)	1	(7 , 5)	15
10	2	(-6, -8)	2	(-8 , -9)	2
10	2	(-6, -8)	2	(9 , 8)	2
10	2	(-2, -4)	2	(-3 , -4)	2
10	2	(-2, -4)	2	(4 , 3)	2
10	2	(0, -2)	2	(0 , -1)	2
10	2	(0, -2)	2	(1 , 0)	2
10	10	(0, -1)	1	(1 , -1)	17
10	10	(1, 0)	1	(1 , -1)	17
10	2	(2, 0)	2	(0 , -1)	2

Table 2.23: Anomaly-free examples on $G(2, N)$ (continued)

N	m	$(\lambda_{A1}, \lambda_{B1})$	n	$(\lambda_{A2}, \lambda_{B2})$	rank
10	2	(4, 2)	2	(-3 , -4)	2
10	2	(4, 2)	2	(4 , 3)	2
10	2	(8, 6)	2	(-8 , -9)	2
10	2	(8, 6)	2	(9 , 8)	2
11	2	(-4, -6)	1	(-8 , -9)	4
11	2	(-4, -6)	1	(9 , 8)	4
11	2	(0, -2)	1	(0 , -1)	4
11	2	(0, -2)	1	(1 , 0)	4
11	11	(0, -1)	1	(1 , -1)	19
11	11	(1, 0)	1	(1 , -1)	19
11	2	(2, 0)	1	(0 , -1)	4
11	2	(2, 0)	1	(1 , 0)	4
11	2	(6, 4)	1	(-8 , -9)	4
11	2	(6, 4)	1	(9 , 8)	4
12	5	(-6, -8)	3	(-8 , -10)	6
12	5	(-6, -8)	3	(10 , 8)	6
12	4	(-4, -6)	2	(-6 , -8)	6
12	2	(-4, -6)	4	(-6 , -6)	2
12	2	(-4, -6)	4	(6 , 6)	2
12	4	(-4, -6)	2	(8 , 6)	6
12	3	(-2, -4)	1	(-4 , -6)	6
12	2	(-2, -4)	3	(-4 , -4)	3
12	2	(-2, -4)	3	(4 , 4)	3
12	3	(-2, -4)	1	(6 , 4)	6
12	2	(-1, -3)	2	(-3 , -3)	4
12	2	(-1, -3)	2	(3 , 3)	4
12	12	(-1, -2)	1	(-3 , -5)	21
12	12	(-1, -2)	1	(5 , 3)	21
12	2	(0, -2)	5	(0 , 0)	1
12	3	(0, -2)	1	(2 , 0)	6
12	12	(0, -1)	1	(1 , -1)	21
12	12	(1, 0)	1	(1 , -1)	21
12	3	(2, 0)	1	(0 , -2)	6

Table 2.24: Anomaly-free examples on $G(2, N)$ (continued)

N	m	$(\lambda_{A1}, \lambda_{B1})$	n	$(\lambda_{A2}, \lambda_{B2})$	rank
12	2	(2, 0)	5	(0 , 0)	1
12	3	(2, 0)	1	(2 , 0)	6
12	12	(2, 1)	1	(-3 , -5)	21
12	12	(2, 1)	1	(5 , 3)	21
12	2	(3, 1)	2	(-3 , -3)	4
12	2	(3, 1)	2	(3 , 3)	4
12	3	(4, 2)	1	(-4 , -6)	6
12	2	(4, 2)	3	(-4 , -4)	3
12	2	(4, 2)	3	(4 , 4)	3
12	3	(4, 2)	1	(6 , 4)	6
12	4	(6, 4)	2	(-6 , -8)	6
12	2	(6, 4)	4	(-6 , -6)	2
12	2	(6, 4)	4	(6 , 6)	2
12	4	(6, 4)	2	(8 , 6)	6
12	5	(8, 6)	3	(-8 , -10)	6
12	5	(8, 6)	3	(10 , 8)	6
13	13	(0, -1)	1	(1 , -1)	23
13	13	(1, 0)	1	(1 , -1)	23
14	1	(-5, -8)	2	(-9 , -9)	2
14	1	(-5, -8)	2	(9 , 9)	2
14	1	(-1, -4)	2	(-3 , -3)	2
14	1	(-1, -4)	2	(3 , 3)	2
14	1	(0, -3)	2	(-1 , -1)	2
14	3	(0, -3)	1	(0 , -4)	7
14	1	(0, -3)	2	(1 , 1)	2
14	3	(0, -3)	1	(4 , 0)	7
14	14	(0, -1)	1	(1 , -1)	25
14	7	(1, -2)	3	(2 , -2)	13
14	14	(1, 0)	1	(1 , -1)	25
14	7	(2, -1)	3	(2 , -2)	13

Table 2.25: Anomaly-free examples on $G(2, N)$ (continued)

N	m	$(\lambda_{A1}, \lambda_{B1})$	n	$(\lambda_{A2}, \lambda_{B2})$	rank
14	3	(3, 0)	1	(0 , -4)	7
14	1	(3, 0)	2	(1 , 1)	2
14	3	(3, 0)	1	(4 , 0)	7
14	1	(4, 1)	2	(-3 , -3)	2
14	1	(4, 1)	2	(3 , 3)	2
14	1	(8, 5)	2	(-9 , -9)	2
14	1	(8, 5)	2	(9 , 9)	2
15	15	(0, -1)	1	(1 , -1)	27
15	15	(1, 0)	1	(1 , -1)	27
16	16	(-2, -3)	1	(-7 , -9)	29
16	16	(-2, -3)	1	(9 , 7)	29
16	3	(-1, -3)	7	(-2 , -2)	2
16	3	(-1, -3)	7	(2 , 2)	2
16	3	(0, -2)	1	(-1 , -1)	8
16	3	(0, -2)	1	(1 , 1)	8
16	16	(0, -1)	1	(1 , -1)	29
16	16	(1, 0)	1	(1 , -1)	29
16	3	(2, 0)	1	(-1 , -1)	8
16	3	(2, 0)	1	(1 , 1)	8
30	14	(0, -2)	3	(3 , 0)	30
30	14	(2, 0)	3	(3 , 0)	30
30	3	(3, 0)	1	(-1 , -3)	9
30	3	(3, 0)	1	(3 , 1)	9
30	14	(4, 2)	3	(-4 , -7)	30
30	14	(4, 2)	3	(7 , 4)	30
24	2	(3, 0)	6	(-1 , -1)	2
24	8	(3, 0)	3	(0 , -4)	17
20	4	(4, 2)	2	(-7 , -7)	10
20	4	(4, 2)	2	(7 , 7)	10
10	2	(2, 0)	2	(1 , 0)	2
14	1	(3, 0)	2	(-1 , -1)	2
20	9	(3, 1)	2	(5 , 2)	19

Table 2.26: Anomaly-free examples on $G(2, N)$ (continued)

N	m	$(\lambda_{A1}, \lambda_{B1})$	n	$(\lambda_{A2}, \lambda_{B2})$	rank
16	3	(3, 1)	7	(-2 , -2)	2
16	3	(3, 1)	7	(2 , 2)	2
16	16	(3, 2)	1	(-7 , -9)	29
16	16	(3, 2)	1	(9 , 7)	29
17	13	(-1, -2)	2	(-5 , -5)	24
17	13	(-1, -2)	2	(5 , 5)	24
17	17	(0, -1)	1	(1 , -1)	31
17	17	(1, 0)	1	(1 , -1)	31
17	13	(2, 1)	2	(-5 , -5)	24
17	13	(2, 1)	2	(5 , 5)	24
18	14	(-1, -2)	6	(-3 , -3)	22
18	14	(-1, -2)	6	(3 , 3)	22
18	6	(0, -2)	1	(0 , -3)	14
18	6	(0, -2)	1	(3 , 0)	14
18	18	(0, -1)	1	(1 , -1)	33
18	18	(1, 0)	1	(1 , -1)	33
18	6	(2, 0)	1	(0 , -3)	14
18	6	(2, 0)	1	(3 , 0)	14
18	14	(2, 1)	6	(-3 , -3)	22
18	14	(2, 1)	6	(3 , 3)	22
19	19	(0, -1)	1	(1 , -1)	35
19	19	(1, 0)	1	(1 , -1)	35
20	4	(-2, -4)	2	(-7 , -7)	10
20	4	(-2, -4)	2	(7 , 7)	10
20	9	(-1, -3)	2	(-2 , -5)	19
20	5	(-1, -3)	4	(-2 , -3)	7
20	5	(-1, -3)	4	(3 , 2)	7
20	9	(-1, -3)	2	(5 , 2)	19
20	4	(0, -2)	2	(-1 , -1)	10
20	4	(0, -2)	2	(1 , 1)	10
20	20	(0, -1)	1	(1 , -1)	37
20	20	(1, 0)	1	(1 , -1)	37
20	4	(2, 0)	2	(-1 , -1)	10
20	4	(2, 0)	2	(1 , 1)	10
20	9	(3, 1)	2	(-2 , -5)	19
20	5	(3, 1)	4	(-2 , -3)	7
20	5	(3, 1)	4	(3 , 2)	7

Table 2.27: Anomaly-free examples on $G(2, N)$ (continued)

N	m	$(\lambda_{A1}, \lambda_{B1})$	n	$(\lambda_{A2}, \lambda_{B2})$	rank
22	1	(-4, -8)	2	(-6 , -7)	1
22	1	(-4, -8)	2	(7 , 6)	1
22	5	(-1, -3)	2	(-3 , -4)	11
22	5	(-1, -3)	2	(4 , 3)	11
22	1	(0, -4)	2	(-1 , -2)	1
22	1	(0, -4)	2	(2 , 1)	1
22	5	(3, 1)	2	(-3 , -4)	11
22	5	(3, 1)	2	(4 , 3)	11
22	1	(4, 0)	2	(-1 , -2)	1
22	1	(4, 0)	2	(2 , 1)	1
22	1	(8, 4)	2	(-6 , -7)	1
22	1	(8, 4)	2	(7 , 6)	1
24	16	(-4, -6)	11	(-5 , -7)	15
24	16	(-4, -6)	11	(7 , 5)	15
24	2	(-3, -6)	6	(-5 , -5)	2
24	2	(-3, -6)	6	(5 , 5)	2
24	7	(-3, -5)	8	(-4 , -5)	5
24	7	(-3, -5)	8	(5 , 4)	5
24	5	(-1, -3)	3	(-4 , -4)	12
24	5	(-1, -3)	12	(-2 , -2)	3
24	5	(-1, -3)	12	(2 , 2)	3
24	5	(-1, -3)	3	(4 , 4)	12
24	1	(0, -4)	2	(-2 , -2)	3
24	1	(0, -4)	2	(2 , 2)	3
24	2	(0, -3)	6	(-1 , -1)	2
24	8	(0, -3)	3	(0 , -4)	17
24	2	(0, -3)	6	(1 , 1)	2
24	8	(0, -3)	3	(4 , 0)	17
24	5	(0, -2)	3	(-1 , -1)	12
24	10	(0, -2)	2	(0 , -3)	22
24	5	(0, -2)	3	(1 , 1)	12
24	10	(0, -2)	2	(3 , 0)	22
24	12	(1, -2)	5	(2 , -2)	23
24	12	(2, -1)	5	(2 , -2)	23
24	5	(2, 0)	3	(-1 , -1)	12
24	10	(2, 0)	2	(0 , -3)	22
24	5	(2, 0)	3	(1 , 1)	12
24	10	(2, 0)	2	(3 , 0)	22

Table 2.28: Anomaly-free examples on $G(2, N)$ (continued)

N	m	$(\lambda_{A1}, \lambda_{B1})$	n	$(\lambda_{A2}, \lambda_{B2})$	rank
24	2	(3, 0)	6	(1 , 1)	2
24	8	(3, 0)	3	(4 , 0)	17
24	5	(3, 1)	3	(-4 , -4)	12
24	5	(3, 1)	12	(-2 , -2)	3
24	5	(3, 1)	12	(2 , 2)	3
24	5	(3, 1)	3	(4 , 4)	12
24	1	(4, 0)	2	(-2 , -2)	3
24	1	(4, 0)	2	(2 , 2)	3
24	7	(5, 3)	8	(-4 , -5)	5
24	7	(5, 3)	8	(5 , 4)	5
24	2	(6, 3)	6	(-5 , -5)	2
24	2	(6, 3)	6	(5 , 5)	2
24	16	(6, 4)	11	(-5 , -7)	15
24	16	(6, 4)	11	(7 , 5)	15
27	6	(0, -2)	1	(-1 , -2)	16
27	6	(0, -2)	1	(2 , 1)	16
27	6	(2, 0)	1	(-1 , -2)	16
27	6	(2, 0)	1	(2 , 1)	16
28	6	(0, -2)	1	(-2 , -2)	17
28	6	(0, -2)	4	(-1 , -1)	14
28	6	(0, -2)	4	(1 , 1)	14
28	6	(0, -2)	1	(2 , 2)	17
28	6	(2, 0)	1	(-2 , -2)	17
28	6	(2, 0)	4	(-1 , -1)	14
28	6	(2, 0)	4	(1 , 1)	14
28	6	(2, 0)	1	(2 , 2)	17
30	14	(-2, -4)	3	(-4 , -7)	30
30	14	(-2, -4)	3	(7 , 4)	30
30	3	(0, -3)	1	(-1 , -3)	9
30	3	(0, -3)	1	(3 , 1)	9
30	14	(0, -2)	3	(0 , -3)	30

2.3 Examples on $G(2,4)[4]$

Part of this section was previously published in [90]. To build a (0,2) GLSM for a complete intersection, we follow a pattern similar to that in abelian (0,2) GLSMs: for each hypersurface $\{G_a = 0\}$ (degree d_a) in the complete intersection, we add a Fermi superfield Γ^a, charged under $\det U(k)$ with charge $-kd_a$ (*i.e.* couples to bundle $(\det S^*)^{-d_a} = \mathcal{O}(-d_a, -d_a)$), and a (0,2) superpotential term

$$W \;=\; \Gamma^a G_a(\phi).$$

Integrating out the auxiliary field in Γ^a forces the vacua to lie along $\{G_a = 0\}$. The reason for the charge assignments lies in how the polynomials G_a are defined. Specifically, these are functions of baryons in the $U(k)$ theory (*i.e.* homogeneous coordinates in the Plücker embedding),

$$B_{i_1 \cdots i_k} \;=\; \epsilon_{a_1 \cdots a_k} \phi^{a_1}_{i_1} \cdots \phi^{a_k}_{i_k},$$

which each have $\det U(k)$ charge k.

In this language, the Calabi-Yau condition for a complete intersection of hypersurfaces in $G(k,n)$ is that the sum of the degrees of the hypersurfaces equals n:

$$\sum_a d_a \;=\; n.$$

In table 2.29 we list anomaly-free examples of bundles $\mathcal{E}$ of the form

$$0 \;\to\; \mathcal{E} \;\to\; \oplus^{m_1}\mathcal{O}(\lambda_{A1}, \lambda_{B1}) \oplus^{m_2} \mathcal{O}(\lambda_{A2}, \lambda_{B2}) \;\to\; \oplus^{n_1}\mathcal{O}(\lambda_{A3}, \lambda_{B3}) \oplus^{n_2} \mathcal{O}(\lambda_{A4}, \lambda_{B4}) \;\to\; 0$$

on $G(2,4)[4]$ with $c_1(\mathcal{E}) = 0$. In some cases, it may not be possible to find explicit maps between the second and third entries in the short exact sequence, or the data given does not define a bundle, merely a sheaf. For bundles of the form above,

$$\begin{aligned} c_1(\mathcal{E}) &= m_1 c_1(\mathcal{O}(\lambda_{A1}, \lambda_{B1})) \;+\; m_2 c_1(\mathcal{O}(\lambda_{A2}, \lambda_{B2})) \\ &\qquad -\; n_1 c_1(\mathcal{O}(\lambda_{A3}, \lambda_{B3})) \;-\; n_2 c_1(\mathcal{O}(\lambda_{A4}, \lambda_{B4})), \\ &\propto d_{(\lambda_{A1}, \lambda_{B1})} \mathrm{Cas}_1(\lambda_{A1}, \lambda_{B1}) \;+\; d_{(\lambda_{A2}, \lambda_{B2})} \mathrm{Cas}_1(\lambda_{A2}, \lambda_{B2}) \\ &\qquad -\; d_{(\lambda_{A3}, \lambda_{B3})} \mathrm{Cas}_1(\lambda_{A3}, \lambda_{B3}) \;-\; d_{(\lambda_{A4}, \lambda_{B4})} \mathrm{Cas}_1(\lambda_{A4}, \lambda_{B4}). \end{aligned}$$

Let us examine carefully the first entry in table 2.29. The field content of the (0,2) GLSM pertinent to anomalies is as follows:

- 1 Fermi superfield in representation (1,0) (for the middle term defining $\mathcal{E}$),
- 5 Fermi superfields in representation (2,1) (for the middle term defining $\mathcal{E}$),
- 1 Fermi superfield Γ in representation (-4,-4) (for the hypersurface),

Table 2.29: Anomaly-free examples on $G(2,4)[4]$

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$	rank
1	(1, 0)	5	(2, 1)	1	(3, 1)	2	(3, 2)	5
1	(-3, -3)	4	(3, 2)	1	(5, 3)	1	(5, 5)	5
2	(-1, -1)	4	(3, 2)	2	(3, 3)	1	(5, 3)	5
1	(0, -1)	3	(3, 2)	1	(4, 2)	1	(5, 5)	4
1	(0, 0)	5	(2, 2)	2	(3, 3)	1	(4, 4)	3
2	(0, 0)	5	(2, 2)	2	(3, 3)	1	(4, 4)	4
3	(0, 0)	5	(2, 2)	2	(3, 3)	1	(4, 4)	5
4	(0, 0)	5	(2, 2)	2	(3, 3)	1	(4, 4)	6
5	(0, 0)	5	(2, 2)	2	(3, 3)	1	(4, 4)	7
1	(0, 0)	3	(3, 3)	1	(4, 4)	1	(5, 5)	2
2	(0, 0)	3	(3, 3)	1	(4, 4)	1	(5, 5)	3
3	(0, 0)	3	(3, 3)	1	(4, 4)	1	(5, 5)	4
4	(0, 0)	3	(3, 3)	1	(4, 4)	1	(5, 5)	5
5	(0, 0)	3	(3, 3)	1	(4, 4)	1	(5, 5)	6
1	(0, 0)	4	(4, 3)	4	(4, 4)	1	(5, 3)	2
2	(0, 0)	4	(4, 3)	4	(4, 4)	1	(5, 3)	3
3	(0, 0)	4	(4, 3)	4	(4, 4)	1	(5, 3)	4
4	(0, 0)	4	(4, 3)	4	(4, 4)	1	(5, 3)	5
5	(0, 0)	4	(4, 3)	4	(4, 4)	1	(5, 3)	6
1	(1, -1)	4	(3, -1)	4	(2, 2)	1	(5, -2)	11
1	(1, -1)	4	(4, 3)	1	(4, 4)	2	(5, 3)	4
4	(1, 0)	5	(1, 1)	1	(3, 1)	1	(3, 3)	9
1	(1, 0)	3	(2, 1)	1	(3, 1)	1	(4, 4)	4
2	(1, 0)	2	(4, 3)	1	(4, 4)	1	(5, 3)	4
3	(1, 1)	5	(1, 1)	1	(2, 2)	2	(3, 3)	5
4	(1, 1)	4	(1, 1)	1	(2, 2)	2	(3, 3)	5
1	(1, 1)	5	(1, 1)	1	(2, 2)	1	(4, 4)	4
2	(1, 1)	4	(1, 1)	1	(2, 2)	1	(4, 4)	4
3	(1, 1)	3	(1, 1)	1	(2, 2)	1	(4, 4)	4
5	(1, 1)	5	(2, 0)	4	(2, 2)	2	(3, 0)	8
2	(1, 1)	5	(2, 2)	1	(3, 3)	3	(3, 3)	3
2	(1, 1)	5	(2, 2)	2	(3, 3)	2	(3, 3)	3
5	(1, 1)	2	(2, 2)	1	(3, 3)	2	(3, 3)	4
3	(1, 1)	2	(2, 2)	1	(3, 3)	1	(4, 4)	3

Table 2.30: Anomaly-free examples on $G(2,4)[4]$ (continued)

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$	rank
2	(1, 1)	4	(3, 0)	3	(2, 2)	2	(4, 0)	5
5	(1, 1)	3	(4, -2)	1	(2, 2)	2	(5, -2)	9
2	(2, 0)	2	(2, 1)	1	(2, 2)	1	(4, 1)	5
2	(2, 1)	5	(2, 2)	2	(3, 2)	2	(3, 3)	3
1	(2, 2)	4	(2, 2)	2	(3, 3)	1	(4, 4)	2
2	(2, 2)	3	(2, 2)	2	(3, 3)	1	(4, 4)	2
5	(2, 2)	1	(3, 0)	2	(3, 3)	1	(4, 1)	3
2	(2, 2)	4	(3, 2)	5	(3, 3)	1	(4, 2)	2
4	(2, 2)	1	(3, 2)	2	(3, 3)	1	(4, 3)	2
2	(2, 2)	2	(4, 1)	3	(3, 3)	1	(5, 1)	2
1	(2, 2)	2	(4, 4)	1	(5, 5)	1	(5, 5)	1
1	(3, 3)	2	(3, 3)	1	(4, 4)	1	(5, 5)	1
2	(3, 3)	1	(4, 3)	1	(4, 4)	1	(5, 4)	1
1	(4, 3)	3	(4, 3)	4	(4, 4)	1	(5, 3)	1
2	(4, 3)	2	(4, 3)	4	(4, 4)	1	(5, 3)	1

- 1 left-moving gaugino in the adjoint,
- 4 chiral superfields in the fundamental (1,0) (defining the Grassmannian),
- 1 chiral superfield in the dual of (3,1) (corresponding to the last term defining $\mathcal{E}$),
- 2 chiral superfields in the dual of (3,2) (corresponding to the last term defining $\mathcal{E}$).

It is straightforward to check that this field content is anomaly-free, and defines a theory with $c_1(\mathcal{E}) = 0$.

As another consistency check, let us compute the left and right central charges of the IR limits of the GLSM, applying c-extremization[1] as discussed in [39] (see also [32] for other recent applications). Briefly, the basic idea is that the central charge can be determined using the fact that the symmetry that becomes the R-symmetry in the IR SCFT will extremize trial central charges determined by anomalies. Consider for example the first entry in table 2.29. From the matter content listed above and the anomaly

$$c_R \;=\; 3\mathrm{Tr}\,\gamma^3 RR,$$

one has the trial right-moving central charge

$$c_R \;=\; 3(8(R_\phi - 1)^2 - 12R_\Lambda^2 + 7(R_P - 1)^2 - R_\Gamma^2 - 4),$$

[1] This is closely analogous to a-maximization in four-dimensional theories [38].

where the R's denote charges under the left $U(1)$. We need to find R charges that extremize c_R. Furthermore, from the superpotential terms, there are constraints. Specifically, terms of the form

$$\int d\theta^+ \Gamma G$$

yield

$$-1 + R_\Gamma + 8R_\Phi \; = \; 0,$$

and terms of the form

$$\int d\theta^+ \Lambda P F$$

yield constraints

$$\begin{aligned} -1 + R_\Lambda + R_P + 3R_\Phi &= 0, \\ -1 + R_\Lambda + R_P + 4R_\Phi &= 0, \\ -1 + R_\Lambda + R_P + R_\Phi &= 0, \\ -1 + R_\Lambda + R_P + 2R_\Phi &= 0. \end{aligned}$$

Extremizing the central charge gives

$$R_\Phi = 0, \quad R_\Gamma = 1, \quad R_\Lambda = 0, \quad R_P = 1,$$

which results in $c_R = 9$. The other central charge, c_L, can be computed from $c_R - c_L = \mathrm{Tr}\gamma^3 = 8 - 12 + 7 - 1 - 4 = -2$, yielding altogether $(c_R, c_L) = (9, 11)$ for the first entry, exactly right to describe a (0,2) theory on a 3-fold with a bundle of rank 5. Proceeding in a similar fashion, the central charges of the other entries in table 2.29 are computed to be

$$(c_R, c_L) \; = \; (9, 11), \quad (9, 14), \quad (9, 9), \quad (9, 10), \quad (9, 9),$$

respectively, exactly correct for the given ranks and dimensions. We take this as evidence for the existence of nontrivial IR fixed points in these theories.

2.4 Examples on $G(2,7)[1^7]$

The second example we consider is a complete intersection of 7 degree 1 hypersurfaces in $G(2,7)$. It is also a Calabi-Yau subspace and the bundle $\mathcal{E}$ is the kernel in the following short exact sequence:

$$0 \;\to\; \mathcal{E} \;\to\; \oplus^{m_1}\mathcal{O}(\lambda_{A1}, \lambda_{B1}) \oplus^{m_2} \mathcal{O}(\lambda_{A2}, \lambda_{B2}) \;\to\; \oplus^{n_1}\mathcal{O}(\lambda_{A3}, \lambda_{B3}) \oplus^{n_2} \mathcal{O}(\lambda_{A4}, \lambda_{B4}) \;\to\; 0$$

on $G(2,7)[1^7]$ with $c_1(\mathcal{E}) = 0$.

Table 2.31: Anomaly-free examples on $G(2,7)[1^7]$

m_1	$(\lambda_{A1}, \lambda_{B1})$	m_2	$(\lambda_{A2}, \lambda_{B2})$	n_1	$(\lambda_{A3}, \lambda_{B3})$	n_2	$(\lambda_{A4}, \lambda_{B4})$	rank
3	(-2, -2)	4	(3, 2)	1	(5, 4)	1	(5, 5)	8
1	(-1, -2)	4	(3, 2)	1	(3, 3)	2	(4, 3)	5
3	(-1, -1)	5	(2, 1)	1	(2, 2)	2	(3, 2)	8
3	(-1, -1)	3	(2, 1)	1	(2, 2)	1	(4, 4)	7
2	(-1, -1)	3	(3, 2)	3	(3, 3)	1	(4, 4)	4
5	(-1, -1)	4	(3, 2)	2	(3, 3)	1	(5, 4)	9
3	(-1, -1)	4	(4, 3)	4	(4, 4)	1	(5, 4)	5
1	(0, -1)	3	(2, 0)	1	(2, 2)	1	(3, 0)	6
1	(0, -1)	2	(2, 1)	1	(2, 2)	1	(3, 3)	4
3	(0, -1)	5	(2, 2)	1	(3, 3)	1	(4, 4)	9
3	(0, -1)	4	(3, 3)	1	(4, 4)	1	(5, 5)	8
1	(0, 0)	4	(1, 0)	1	(1, 1)	1	(2, 1)	6
2	(0, 0)	4	(1, 0)	1	(1, 1)	1	(2, 1)	7
3	(0, 0)	4	(1, 0)	1	(1, 1)	1	(2, 1)	8
4	(0, 0)	4	(1, 0)	1	(1, 1)	1	(2, 1)	9
5	(0, 0)	4	(1, 0)	1	(1, 1)	1	(2, 1)	10
1	(0, 0)	3	(1, 0)	1	(1, 1)	1	(2, 2)	5
2	(0, 0)	3	(1, 0)	1	(1, 1)	1	(2, 2)	6
3	(0, 0)	3	(1, 0)	1	(1, 1)	1	(2, 2)	7
4	(0, 0)	3	(1, 0)	1	(1, 1)	1	(2, 2)	8
5	(0, 0)	3	(1, 0)	1	(1, 1)	1	(2, 2)	9
1	(1, -1)	5	(1, 1)	1	(2, 1)	1	(2, 2)	5
2	(1, 0)	5	(1, 0)	4	(1, 1)	1	(2, 0)	7
3	(1, 0)	4	(1, 0)	4	(1, 1)	1	(2, 0)	7
1	(1, 0)	3	(1, 0)	1	(1, 1)	1	(2, 1)	5
2	(1, 0)	2	(1, 0)	1	(1, 1)	1	(2, 1)	5
1	(1, 0)	2	(1, 0)	1	(1, 1)	1	(2, 2)	4
3	(1, 0)	2	(2, -1)	2	(1, 1)	1	(3, -1)	7
2	(1, 1)	1	(2, 0)	1	(2, 1)	1	(2, 2)	2

Table 2.31 gives the anomaly free examples of such bundle. In some cases, it may not be possible to find explicit maps between the second and third entries in the short exact sequence, or the data given does not define a bundle, merely a sheaf.

We examine carefully the first entry in table 2.31. The field content of the (0,2) GLSM pertinent to anomalies is as follows:

- 3 Fermi superfields in representation (-2,-2) (for the middle term defining $\mathcal{E}$),
- 4 Fermi superfields in representation (3,2) (for the middle term defining $\mathcal{E}$),
- 7 Fermi superfields Γ in representation (-1,-1) (for the complete intersection of hypersurfaces),
- 1 left-moving gaugino in the adjoint,
- 7 chiral superfields in the fundamental (1,0) (defining the Grassmannian),
- 1 chiral superfield in the dual of (5,4) (corresponding to the last term defining $\mathcal{E}$),
- 1 chiral superfield in the dual of (5,5) (corresponding to the last term defining $\mathcal{E}$).

It is straightforward to check that this field content is anomaly-free, and defines a theory with $c_1(\mathcal{E}) = 0$.

2.5 Pfaffian constructions

For a long time, it was thought that only hypersurfaces and complete intersections could be realized in GLSMs. This is because for general non-complete intersections, the number of generators d is greater than the codimensison r of the intersection, and if we construct the GLSM in the usual manner, we would end up with a target space that is a rank $d-r$ bundle over the intersection. Generalizing initial work of [11], H. Jockers, V. Kumar, J. Lapan, D. Morrison, M. Romo constructed GLSMs for non-complete-intersections known as Pfaffians in [10]. We will give a short review of the Pfaffian model in the following context. Part of this section was previously published in [90].

2.5.1 Review of (2,2) constructions

The paper [10] built two models of (2,2) GLSMs associated to a given Pfaffian variety, called the PAX and PAXY models. A Pfaffian variety is defined as follows. First, consider homogeneous coordinates ϕ_a on an ambient toric variety V and define an m by n matrix $A(\phi)$ with entries given by quasi-homogeneous polynomials in ϕ. A Pfaffian variety is then defined by the locus in V where the rank of $A(\phi)$ is k or less. Such a Pfaffian variety is determined by $\binom{m}{k+1}\binom{n}{k+1}$ polynomials, and it can be shown that the codimension in V is $(m-k)(n-k)$. The pertinent question is how to impose the rank conditions in the language of GLSMs. The paper [10] gave two ways to construct the GLSM and corresponding models are called PAX and PAXY model.

The first model we consider is the PAX model. The gauge group of this model is $U(n-k)$. Besides the chiral superfield Φ defining the toric variety, we add two chiral superfields P, X, where X transforms as n copies of the fundamental[2] of $U(n-k)$ and P as n copies of the antifundamental of $U(n-k)$, together with a (2,2) superpotential

$$W \;=\; \mathrm{tr}\, PAX \tag{2.19}$$

where $A(\Phi)$ is a $n \times n$ matrix with entries are polynomial in Φ. P and X also have charges under the abelian gauge symmetry defining the toric variety, so in effect, the model describes a superpotential over a bundle with fibers that are the total spaces of

$$S^{\oplus n} \;\longrightarrow\; G(n-k, n) \tag{2.20}$$

fibered over the given toric variety. All charges are required to be such that the superpotential (2.19) is neutral.

The D-terms give a constraint of the form

$$XX^{\dagger} \;-\; P^{\dagger}P \;=\; rI,$$

where r is a Fayet-Iliopoulos parameter associated to the overall $U(1)$. The change of r, in a similar way in [77], gives two phases in the (2,2) GLSM. They are closely related. Without loss of generality, we shall take $r \gg 0$. The F-terms give constraints of the form

$$AX \;=\; 0, \quad PA \;=\; 0, \quad P(dA)X \;=\; 0.$$

The first constraint defines the variety

$$Z \;\equiv\; \{(\phi, x) \,|\, A(\phi)x \;=\; 0\},$$

which is our desired (resolution of a) Pfaffian. Under a smoothness assumption, the second two F-term constraints imply $P = 0$, as discussed in B. Thus, the low-energy physics is merely a nonlinear sigma model with target space Z. The same analysis applies when $r \ll 0$, when the roles of X and P are reversed.

Let us see what Calabi-Yau condition looks like physically in a PAX model. First, note that the fibers (2.20) are already Calabi-Yau, so we merely need a constraint on charges of the abelian gauge symmetries defining the underlying toric variety.

[2] To make our (0,2) conventions cleaner, we have made a trivial convention flip with respect to [10], in that P and X are defined in opposite representations.

As we saw in section 2.1.4, the space will be Calabi-Yau if the sum of the $U(1)$ charges vanishes, for each $U(1)$ defining the underlying toric variety. For example, suppose the underlying toric variety is a projective space, $\mathbb{P}^m$ for some m. Let p_i denote the $U(1)$ of the i-th fundamental in P, and x_i the $U(1)$ charge of the i-th antifundamental in X. Then the Calabi-Yau condition can be succinctly stated as the condition

$$\sum_i (n-k)p_i \;+\; \sum_i (n-k)x_i \;+\; m+1 \;=\; 0, \tag{2.21}$$

where we have used the fact that the fundamentals and antifundamentals both have dimension $n-k$.

Another way to achieve rank condition is a PAXY model. In this theory, one needs a gauge group $U(k)$. We also add two other chiral superfields called $\tilde{X}$, $\tilde{Y}$ and $\tilde{P}$ besides Φ who defines the toric variety. $\tilde{X}$ transforms in fundamental representation of $U(k)$ while $\tilde{Y}$ transforms in antifundamental way and $\tilde{P}$ is an $n \times n$ matrix of neutral chiral superfields. The (2,2) superpotential is now

$$W \;=\; \mathrm{tr}\, \tilde{P}\left(A \;-\; \tilde{Y}\tilde{X}\right). \tag{2.22}$$

Here also, $\tilde{P}$, $\tilde{X}$, $\tilde{Y}$ are charged under the abelian gauge symmetry defining the underlying toric variety, with charges such that the superpotential (2.22) is gauge invariant.

The Calabi-Yau condition for the PAXY model is similar to that of the PAX model. Assume the toric variety is $\mathbb{P}^m$, as before, then the Calabi-Yau condition is

$$k\sum_i x_i \;+\; k\sum_i y_i \;+\; k\sum_i p_i \;+\; m \;+\; 1 \;=\; 0, \tag{2.23}$$

where we have used the fact that the fundamentals and antifundamentals have dimension k.

2.5.2 More general (0,2) examples

We begin by rewriting the (2,2) PAX and PAXY models in (0,2) language to better understand (0,2) models on Pfaffians.

Let us begin with the (0,2) PAX model. The (2,2) chiral superfield X in the original theory splits into (0,2) chiral superfield X and Fermi superfields Λ_X, all describing n copies of the fundamental; and the (2,2) chiral superfield P splits into (0,2) chiral superfield P and Fermi superfield Λ_P, describing n copies of the antifundamental. Let Φ, Λ_Φ denote the (0,2) chiral, Fermi superfields associated to the (2,2) Φ defining the underlying toric variety. This decomposition of the (2,2) theory also gives rise to an adjoint-valued (0,2) chiral Σ, originating in the (2,2) gauge multiplet. Then we can write down the (0,2) theory in a $U(n-k)$ gauge using fields P, Λ_P, X, Λ_X, Φ, Λ_Φ, obeying

$$\overline{D}_+\Lambda_P \;\propto\; \Sigma P$$

(and similarly for other Fermi superfields).

The (0,2) superpotential is

$$W \;=\; \mathrm{tr}\left(\Lambda_P A(\Phi) X \;+\; P A(\Phi) \Lambda_X \;+\; P \frac{\partial A(\Phi)}{\partial \Phi^\alpha} \Lambda_\Phi^\alpha X \right).$$

In the phase $r \gg 0$, one can interpret Λ_P as acting as a Lagrange multiplier, forcing $AX = 0$, and Λ_X, Λ_Φ^α as describing the fermions in which the gauge bundle lives.

Mathematically, one can use a short sequence to describe the tangent bundle of the Pfaffian as the cohomology:

$$\begin{aligned} 0 \;\longrightarrow\; \mathcal{O}^r \oplus (S^* \otimes S) \;\stackrel{*_1}{\longrightarrow}\; & \oplus_{a,\alpha} \mathcal{O}((0,0), q_{a,\alpha}) \oplus_i \mathcal{O}((1,0), x_{a,i}) \\ & \stackrel{*_2}{\longrightarrow}\; \oplus_i \mathcal{O}((1,0), -p_{a,i}) \;\longrightarrow\; 0, \end{aligned} \tag{2.24}$$

where

$$*_1 \;=\; \left[\begin{array}{cc} q_{a,\alpha}\Phi^\alpha & 0 \\ x_a X & X \end{array} \right], \quad *_2 \;=\; \left[\frac{\partial A}{\partial \Phi^\alpha} X, A \right].$$

As a consistency check, note that the composition of the two maps above has the form

$$*_2 *_1 \;=\; \left[q_{a,\alpha}\Phi^\alpha \frac{\partial A}{\partial \Phi^\alpha} X \;+\; x_a A X, AX \right] \;=\; \left[p_a AX, AX \right],$$

which vanishes on the Pfaffian, as expected. The sequence also has physically interpretation on each terms: The $\mathcal{O}^r \oplus S^* \otimes S$ is determined by the gauginos; the other terms are determined by remaining fermions.

One can read off the Calabi-Yau condition from the short sequence:

$$-\,(n-k) \sum_i p_{a,i} \;=\; (n-k) \sum_i x_{a,i} \;+\; \sum_\alpha q_{a,\alpha} \tag{2.25}$$

for each a, which reduces to the Calabi-Yau condition in eq.2.21 discussed previously in (2,2) models.

One can slightly generalize the case we just discussed to a (0,2) deformation of the tangent bundle of the Pfaffian. It would be described by a theory with the same matter content, but different (0,2) superpotential

$$W \;=\; \mathrm{tr}\left(\Lambda_P A(\Phi) X \;+\; P A(\Phi) \Lambda_X \;+\; P \left(\frac{\partial A(\Phi)}{\partial \Phi^\alpha} \;+\; G_\alpha(\Phi) \right) \Lambda_\Phi^\alpha X \right),$$

where

$$q_{a,\alpha} \Phi^\alpha G_\alpha \;=\; 0$$

for each a. This is described by a monad of the same form as in equation (2.24), but the second map is different

$$*_1 \;=\; \left[\begin{array}{cc} q_{a,\alpha}\Phi^\alpha & 0 \\ x_a X & X \end{array} \right], \quad *_2 \;=\; \left[\left(\frac{\partial A}{\partial \Phi^\alpha} X \;+\; G_\alpha \right), A \right].$$

It is easily seen that it reduces to the tangent bundle if $G_a = 0$.

One can further generalize (0,2) model as a bundle using a kernel of a short exact sequence based on the PAX model. We do it as follows: First, to build the Pfaffian itself, we will need a $U(n-k)$ gauge theory, n chiral superfields in the fundamental, forming an $n \times (n-k)$ matrix denoted X, and n Fermi superfields in the antifundamental, forming an $n \times (n-k)$ matrix of Fermi superfields denoted Λ_0. Then, to describe a bundle $\mathcal{E}$ as a kernel, say,

$$0 \longrightarrow \mathcal{E} \longrightarrow \oplus_\beta \mathcal{O}((\lambda_{\beta 1}, \lambda_{\beta 2}), q_{a,\beta}) \stackrel{F_\beta^\gamma}{\longrightarrow} \oplus_\gamma \mathcal{O}((\lambda_{\gamma 1}, \lambda_{\gamma 2}), q_{a,\gamma}) \longrightarrow 0,$$

we add a set of Fermi superfields Λ^β in the $(\lambda_{\beta 1}, \lambda_{\beta 2})$ representation of $U(n-k)$ and with charges $q_{a,\beta}$ under the abelian gauge symmetry $U(1)^r$ defining the toric variety, along with a set of chiral superfields P_γ in the $U(n-k)$ representation dual to $(\lambda_{\gamma 1}, \lambda_{\gamma 2})$ and with charges $-q_{a,\gamma}$ under the abelian gauge symmetry defining the toric variety. In addition, we have a (0,2) superpotential

$$W \;=\; \mathrm{tr}\, \left(\Lambda_0 A(\Phi) X \;+\; \Lambda^\beta F_\beta^\gamma(\Phi) P_\gamma \right).$$

All representations we choose need to satisfy gauge anomaly cancellation for this $U(n-k) \times U(1)^r$ gauge theory. (Given the kernel construction above, GLSMs for bundles built as cokernels and as cohomologies of monads are very straightforward, and so for brevity are omitted.)

We will see the low energy physics in this theory. The D-terms constraint from $U(2)$ is of the form

$$X X^\dagger \;+\; \sum_\gamma P_\gamma^\dagger P_\gamma \;=\; rI,$$

and the X's are not all zero from the previous reasoning in section 2.2.2 for suitable bundle representations. The F terms constraint gives us

$$AX \;=\; 0,$$

which describes the underlying Pfaffian variety. So long as the nontrivial map determined by F_β^γ is surjective, the P_γ chiral superfields will all become massive, leaving us with a gauge bundle contained within the associated Fermi superfields, as expected.

To impose the Calabi-Yau condition, *i.e.*, a bundle with $c_1(\mathcal{E}) = 0$, we impose the conditions

$$\sum_\beta d_{\lambda\beta} \mathrm{Cas}_1(\lambda_{\beta 1}, \lambda_{\beta 2}) \;=\; \sum_\gamma d_{\lambda\gamma} \mathrm{Cas}_1(\lambda_{\gamma 1}, \lambda_{\gamma 2}),$$

$$\sum_{\beta} q_{a,\beta} \;=\; \sum_{\gamma} q_{a,\gamma}.$$

Now let us turn to the PAXY model. We shall write the (2,2) model in (0,2) language following the same convention as previously for the PAX model. First of all we have a guage group $U(k)$ and (0,2) chiral superfields $\tilde{P}$, $\tilde{X}$, $\tilde{Y}$, Φ^{α}, (0,2) Fermi superfields $\Lambda_{\tilde{P}}$, $\Lambda_{\tilde{X}}$, $\Lambda_{\tilde{Y}}$, Λ_{Φ}^{α}, and a (0,2) superpotential of the form

$$W \;=\; \mathrm{tr}\left(\Lambda_{\tilde{P}}\left(A \,-\, \tilde{Y}\tilde{X}\right) \,+\, \tilde{P}\left(\frac{\partial A}{\partial \Phi^{\alpha}}\Lambda_{\Phi}^{\alpha} \,-\, \Lambda_{\tilde{Y}}\tilde{X} \,-\, \tilde{Y}\Lambda_{\tilde{X}}\right)\right). \tag{2.26}$$

A (0,2) theory describing a deformation of the tangent bundle is defined by the superpotential

$$W \;=\; \mathrm{tr}\left(\Lambda_{\tilde{P}}\left(A \,-\, \tilde{Y}\tilde{X}\right) \,+\, \tilde{P}\left(\left(\frac{\partial A}{\partial \Phi^{\alpha}} \,+\, G_{\alpha}\right)\Lambda_{\Phi}^{\alpha} \,-\, \Lambda_{\tilde{Y}}\tilde{X} \,-\, \tilde{Y}\Lambda_{\tilde{X}}\right)\right). \tag{2.27}$$

The gauge bundle can also be described by a short exact sequence as a kernel $\mathcal{E}$

$$0 \;\longrightarrow\; \mathcal{E} \;\longrightarrow\; \oplus_{\beta}\mathcal{O}((\lambda_{\beta 1},\lambda_{\beta 2}),q_{a,\beta}) \;\stackrel{F_{\beta}^{\gamma}}{\longrightarrow}\; \oplus_{\gamma}\mathcal{O}((\lambda_{\gamma 1},\lambda_{\gamma 2}),q_{a,\gamma}) \;\longrightarrow\; 0 \tag{2.28}$$

over the Pfaffian. We can describe this following the PAXY pattern as follows. Given the abelian gauge theory for the toric variety, we add a $U(k)$ gauge theory with

- n chiral superfields in the fundamental, forming a matrix $\tilde{X}$,
- n chiral superfields in the antifundamental, forming a matrix $\tilde{Y}$,
- an $n \times n$ matrix of neutral Fermi superfields Λ_0,
- a set of Fermi superfields Λ^{β} in the $(\lambda_{\beta 1},\lambda_{\beta 2})$ representation of $U(k)$, with charges $q_{a,\beta}$ under the abelian gauge symmetry defining the toric variety,
- a set of chiral superfields P_{γ} in the $U(k)$ representation dual to $(\lambda_{\gamma 1},\lambda_{\gamma 2})$ and with charges $-q_{a,\gamma}$ under the abelian gauge symmetry defining the toric variety,
- and finally a (0,2) superpotential

$$W \;=\; \mathrm{tr}\left(\Lambda_0\left(A(\Phi) \,-\, \tilde{Y}\tilde{X}\right) \,+\, \Lambda^{\beta}F_{\beta}^{\gamma}(\Phi)P_{\gamma}\right).$$

Note that although the data defining the bundle is formally very similar to that in the PAX construction, the representations given in the short exact sequence (2.28) are representations of $U(k)$, whereas the representations given in the analogue for the PAX construction are representations of $U(n-k)$.

2.5.3 Examples

Listed in table 2.32 are some examples of (0,2) models on Pfaffians. In some cases, it may not be possible to find explicit maps between the second and third entries in the short exact sequence, or the data given does not define a bundle, merely a sheaf. The Pfaffians themselves are all constructed via the (0,2) PAX model for gauge bundle kernels, as Pfaffians of a 4×4 matrix A, defined as the locus where the rank of A is less than or equal to 2. Hence, we have a $U(4-2) = U(2)$ gauge theory. The Pfaffians are subvarieties of $\mathbb{P}^7$, so for the PAX construction we have fibered

$$S^{\oplus 4} \longrightarrow G(2,4)$$

over $\mathbb{P}^7$, with the fibering defined by the statement that the n antifundamentals[3] X have $U(1)$ charge 0 and the fundamentals Λ_0 have $U(1)$ charge -1. The chiral superfields defining $\mathbb{P}^7$ have charge 1, and the entries of the matrix A are of degree 1. It is straightforward to check that the resulting Pfaffian is Calabi-Yau, from the criteria given earlier, and applying the methods of *e.g.* [13] we see that these are 3-folds.

Table 2.32 lists data for bundles over the total space of the $(S^4 \to G(2,4))$-bundle over $\mathbb{P}^7$. We have restricted to bundles built as kernels. (More general cases are straightforward, and so are left as exercises.) Bundles are kernels of the form

$$\begin{aligned} 0 \longrightarrow \mathcal{E} \longrightarrow & \oplus^{m_1}\mathcal{O}((\lambda_{A1},\lambda_{B1}),Q_1) \oplus^{m_2} \mathcal{O}((\lambda_{A2},\lambda_{B2}),Q_2) \\ & \longrightarrow \oplus^{n_1}\mathcal{O}((\lambda_{A3},\lambda_{B3}),Q_3) \oplus^{n_2} \mathcal{O}((\lambda_{A4},\lambda_{B4}),Q_4) \longrightarrow 0. \end{aligned}$$

For each special homogeneous bundle appearing, we give both a representation of $U(2)$ and also a charge under the $U(1)$ defining the $\mathbb{P}^7$. Conventions are such that $U(2)$ representation $(\lambda_{Ai}, \lambda_{Bi})$ has $\mathbb{P}^7$ $U(1)$ charge Q_i, a fact we have indicated above in subscripts. All of the examples in table 2.32 have $c_1(\mathcal{E}) = 0$.

For completeness, let us describe the first example in table 2.32 in detail. It describes a theory containing charged left-moving fermions as:

- 5 Fermi superfields in the ((0,0),-1), for part of the gauge bundle,
- 2 Fermi superfields in the ((2,2),0), for part of the gauge bundle,
- Λ_0: 4 Fermi superfields in the ((1,0),-1),
- 1 $U(2) \times U(1)$ gaugino,

and charged right-moving fermions as:

[3] Our conventions in the table are flipped relative to the earlier discussion: X is here a set of antifundamentals rather than fundamentals, and Λ_0 is a set of fundamentals rather than antifundamentals. The choice is arbitrary.

- X: 4 chiral superfields in the ((0,-1),0),
- 2 chiral superfields in the dual of ((2,2),-1), for part of the gauge bundle,
- 1 chiral superfield in the dual of ((1,-1),-1), for part of the gauge bundle,
- 8 chiral superfields in the ((0,0),+1), describing homogeneous coordinates on $\mathbb{P}^7$.

There are several gauge anomaly cancellation conditions that must be obeyed: the Cas_2 condition and $(\text{Cas}_1)^2$ conditions for $U(2)$ gauge anomaly cancellation, plus a q^2 condition for solely the extra $U(1)$ for $\mathbb{P}^7$, plus a mixed $U(1) - U(1)$ condition involving products of the general form $q\text{Cas}_1$.

As a consistency check, let us work out central charges of the theories in table 2.32, using c-extremization [39] as discussed earlier in section 2.3. Let us work through the first entry in detail, and summarize results for the rest of the entries. Given the field content, it is straightforward to show that the right-moving central charge ansatz provided by the identity

$$c_R \;=\; 3\text{Tr}\,\gamma^3 RR$$

has the form

$$c_R \;=\; 3\left(8(R_\phi - 1)^2 - 7R_\Lambda^2 + 8(R_X - 1)^2 + 5(R_P - 1)^2 - 8R_{\Lambda_0}^2 - 5\right),$$

where $R_\Phi, R_\Lambda, R_X, R_P, R_{\Lambda_0}$ denote the R-charge of $\Phi, \Lambda, X, P, \Lambda_0$, respectively. Furthermore, from the superpotential terms

$$\int d\theta^+ \Lambda A(\Phi) X,$$

we have the constraint

$$-1 + R_{\Lambda_0} + R_\Phi + R_X \;=\; 0,$$

and from the superpotential terms

$$\int d\theta^+ \Lambda P F,$$

we have the constraints

$$\begin{aligned}
-1 + R_\Lambda + R_P &= 0,\\
-1 + R_\Lambda + R_P + 4R_\Phi &= 0,\\
-1 + R_\Lambda + R_P - 4R_\Phi &= 0.
\end{aligned}$$

Extremizing the central charge yields

$$R_\Phi = 0, \quad R_{\Lambda_0} = 0, \quad R_\Lambda = 0, \quad R_P = 1, \quad R_X = 1,$$

Table 2.32: Anomaly-free (0,2) models on Pfaffians inside $\mathbb{P}^7$

Q_1	m_1	$(\lambda_{A1}, \lambda_{B1})$	Q_2	m_2	$(\lambda_{A2}, \lambda_{B2})$	Q_3	n_1	$(\lambda_{A3}, \lambda_{B3})$	Q_4	n_2	$(\lambda_{A4}, \lambda_{B4})$	rank
-1	5	(0, 0)	0	2	(2, 2)	-1	1	(1, -1)	-1	2	(2, 2)	2
-1	2	(-1, -2)	3	2	(0, -1)	-4	1	(-2, -2)	2	2	(0, -2)	1
0	2	(-2, -2)	1	5	(0, 0)	1	2	(-2, -2)	1	1	(1, -1)	2
0	1	(-2, -2)	2	4	(0, 0)	2	1	(-2, -2)	2	1	(1, -1)	1
-3	1	(-2, -2)	1	4	(1, -1)	-1	1	(-2, -2)	2	1	(2, -2)	7
-3	2	(0, -1)	-1	2	(2, 1)	-4	1	(-2, -2)	-2	2	(2, 0)	1
-4	2	(-2, -2)	0	5	(0, -1)	-4	1	(-1, -2)	0	2	(0, -2)	4
-1	4	(-1, -1)	2	2	(0, 0)	-3	1	(-1, -1)	1	1	(0, -2)	2
0	4	(-1, -1)	4	1	(0, 0)	-2	1	(-1, -1)	2	1	(0, -2)	1
0	4	(-1, -1)	1	4	(0, 0)	1	1	(-1, -1)	1	1	(0, -2)	4
0	2	(-1, -1)	2	5	(0, 0)	2	2	(-1, -1)	2	1	(1, -1)	2
1	2	(-1, -1)	4	4	(0, 0)	3	2	(-1, -1)	4	1	(1, -1)	1
0	1	(-1, -1)	4	4	(0, 0)	4	1	(-1, -1)	4	1	(1, -1)	1
-5	2	(-1, -1)	2	4	(1, -1)	-3	2	(-1, -1)	4	1	(2, -2)	7
-2	4	(0, 0)	0	1	(2, 2)	-2	1	(-1, -1)	-2	1	(2, 0)	1
-1	4	(-1, -1)	2	2	(0, 0)	-1	1	(0, -2)	3	1	(-1, -1)	2
0	5	(-1, -1)	2	2	(0, 0)	0	1	(0, -2)	2	2	(-1, -1)	2
0	4	(-1, -1)	4	1	(0, 0)	0	1	(0, -2)	4	1	(-1, -1)	1
0	4	(-1, -1)	1	4	(0, 0)	1	1	(0, -2)	1	1	(-1, -1)	4
1	5	(-1, -1)	4	1	(0, 0)	1	1	(0, -2)	3	2	(-1, -1)	1
-5	1	(-1, -2)	1	5	(0, -1)	-2	2	(0, -2)	3	2	(0, -1)	2
0	1	(-2, -2)	2	4	(0, -1)	2	2	(0, -2)	2	2	(0, 0)	1
0	1	(-2, -2)	2	4	(0, 0)	2	1	(0, -2)	2	1	(1, 1)	1
-4	2	(1, -2)	0	1	(1, 1)	-4	1	(0, -2)	-4	1	(2, -2)	1
1	2	(-1, -2)	3	2	(1, 0)	2	2	(0, -2)	4	1	(2, 2)	1
-2	1	(-1, -2)	0	4	(0, -1)	-2	1	(0, -1)	0	2	(0, -2)	2
1	2	(0, -1)	4	5	(0, 0)	3	2	(0, -1)	4	1	(1, -1)	2
0	1	(0, -1)	4	5	(0, 0)	4	1	(0, -1)	4	1	(1, -1)	2
-2	1	(-2, -2)	0	4	(0, -1)	-2	1	(0, 0)	0	2	(0, -2)	2
0	1	(-2, -2)	2	4	(0, -1)	2	2	(0, 0)	2	2	(0, -2)	1
-3	2	(0, -1)	0	1	(1, -1)	-4	2	(0, 0)	-1	1	(1, -2)	1
0	4	(0, -1)	1	3	(1, -1)	1	1	(0, 0)	1	2	(1, -2)	8

Table 2.33: Anomaly-free (0,2) models on Pfaffians inside $\mathbb{P}^7$ (continued)

Q_1	m_1	$(\lambda_{A1}, \lambda_{B1})$	Q_2	m_2	$(\lambda_{A2}, \lambda_{B2})$	Q_3	n_1	$(\lambda_{A3}, \lambda_{B3})$	Q_4	n_2	$(\lambda_{A4}, \lambda_{B4})$	rank
-4	1	(1, -1)	-1	2	(1, 0)	-4	1	(0, 0)	-3	1	(2, -1)	2
-1	3	(1, -1)	0	4	(1, 0)	-1	1	(0, 0)	-1	2	(2, -1)	8
-2	4	(1, 0)	0	1	(2, 2)	-2	2	(0, 0)	-2	2	(2, 0)	1
-1	2	(-1, -2)	0	1	(1, -1)	-1	1	(1, -2)	0	2	(-2, -2)	1
-3	5	(0, -1)	3	1	(1, 0)	-5	1	(1, -2)	-2	2	(-1, -1)	6
0	4	(0, -1)	1	3	(1, -1)	1	2	(1, -2)	1	1	(0, 0)	8
1	2	(0, -1)	4	1	(1, -1)	3	1	(1, -2)	4	1	(0, 0)	2
1	1	(1, -2)	4	5	(0, 0)	3	1	(1, -2)	4	1	(1, -1)	2
-3	1	(-2, -2)	-1	4	(0, 0)	-2	1	(1, -1)	-1	1	(-2, -2)	1
0	2	(-2, -2)	1	5	(0, 0)	1	1	(1, -1)	1	2	(-2, -2)	2
0	1	(-2, -2)	2	4	(0, 0)	2	1	(1, -1)	2	1	(-2, -2)	1
0	4	(-1, -1)	1	1	(1, 0)	0	1	(1, -1)	1	1	(-1, -2)	1
-5	2	(-1, -1)	-2	4	(0, 0)	-4	1	(1, -1)	-3	2	(-1, -1)	1
0	4	(-1, -1)	1	4	(0, 0)	0	1	(1, -1)	1	4	(-1, -1)	1
0	2	(-1, -1)	2	5	(0, 0)	2	1	(1, -1)	2	2	(-1, -1)	2
-1	1	(-1, -1)	2	5	(0, 0)	2	1	(1, -1)	3	1	(-1, -1)	2
0	1	(-1, -1)	4	4	(0, 0)	4	1	(1, -1)	4	1	(-1, -1)	1
0	2	(0, -1)	2	4	(0, 0)	0	1	(1, -1)	2	2	(0, -1)	1
-1	1	(0, -1)	2	4	(0, 0)	0	1	(1, -1)	3	1	(0, -1)	1
0	1	(0, -1)	4	5	(0, 0)	4	1	(1, -1)	4	1	(0, -1)	2
0	1	(1, -2)	2	4	(0, 0)	0	1	(1, -1)	2	1	(1, -2)	1
-4	5	(0, 0)	0	1	(1, 0)	-4	1	(1, -1)	-4	1	(1, 0)	2
-4	5	(0, 0)	-1	2	(1, 0)	-4	1	(1, -1)	-3	2	(1, 0)	2
-4	4	(0, 0)	0	1	(1, 1)	-4	1	(1, -1)	-4	1	(1, 1)	1
-4	4	(0, 0)	-1	2	(1, 1)	-4	1	(1, -1)	-3	2	(1, 1)	1
-2	5	(0, 0)	0	2	(1, 1)	-2	1	(1, -1)	-2	2	(1, 1)	2
-4	5	(0, 0)	-1	1	(2, -1)	-4	1	(1, -1)	-3	1	(2, -1)	2
-2	4	(0, 0)	0	1	(2, 2)	-2	1	(1, -1)	-2	1	(2, 2)	1
-4	5	(0, 0)	0	1	(1, 0)	-4	1	(1, 0)	-4	1	(1, -1)	2
-2	4	(0, 0)	1	1	(1, 0)	-3	1	(1, 0)	0	1	(1, -1)	1
-2	4	(0, 0)	0	2	(1, 0)	-2	2	(1, 0)	0	1	(1, -1)	1

Table 2.34: Anomaly-free (0,2) models on Pfaffians inside $\mathbb{P}^7$ (continued)

Q_1	m_1	$(\lambda_{A1}, \lambda_{B1})$	Q_2	m_2	$(\lambda_{A2}, \lambda_{B2})$	Q_3	n_1	$(\lambda_{A3}, \lambda_{B3})$	Q_4	n_2	$(\lambda_{A4}, \lambda_{B4})$	rank
-1	5	(1, 0)	5	1	(2, 1)	-3	2	(1, 0)	2	2	(2, 0)	2
0	1	(-2, -2)	2	4	(0, 0)	2	1	(1, 1)	2	1	(0, -2)	1
-4	4	(0, 0)	0	1	(1, 1)	-4	1	(1, 1)	-4	1	(1, -1)	1
-2	5	(0, 0)	1	1	(1, 1)	-3	1	(1, 1)	-2	1	(1, -1)	2
-2	5	(0, 0)	0	2	(1, 1)	-2	2	(1, 1)	-2	1	(1, -1)	2
-1	4	(0, 0)	0	4	(1, 1)	-1	4	(1, 1)	0	1	(1, -1)	1
2	4	(0, 0)	5	2	(1, 1)	3	2	(1, 1)	4	1	(1, -1)	1
-3	1	(0, -1)	3	5	(1, 0)	2	2	(1, 1)	5	1	(2, -1)	6
-4	1	(0, 0)	0	4	(1, 1)	-4	1	(1, 1)	0	1	(2, 0)	1
-4	1	(0, 0)	-1	5	(1, 1)	-3	2	(1, 1)	-1	1	(2, 0)	1
-2	2	(0, 0)	1	4	(1, 1)	-3	1	(1, 1)	1	1	(2, 0)	2
-2	2	(0, 0)	0	5	(1, 1)	-2	2	(1, 1)	0	1	(2, 0)	2
-1	4	(0, 0)	0	4	(1, 1)	-1	1	(1, 1)	-1	1	(2, 0)	4
-4	2	(1, -2)	0	1	(1, 1)	-4	1	(2, -2)	-4	1	(0, -2)	1
-2	4	(1, -1)	5	2	(1, 1)	-4	1	(2, -2)	3	2	(1, 1)	7
0	1	(-1, -1)	4	2	(2, -1)	4	1	(2, -2)	4	1	(2, 0)	1
-1	4	(1, -1)	3	1	(2, 2)	-2	1	(2, -2)	1	1	(2, 2)	7
-1	3	(1, -1)	0	4	(1, 0)	-1	2	(2, -1)	-1	1	(0, 0)	8
0	1	(1, -1)	3	2	(1, 0)	1	1	(2, -1)	4	2	(0, 0)	1
-2	4	(0, 0)	0	1	(2, -1)	-2	1	(2, -1)	0	1	(1, -1)	1
-2	4	(0, 0)	0	1	(2, 2)	-2	1	(2, 0)	-2	1	(-1, -1)	1
-2	4	(1, 0)	0	1	(2, 2)	-2	2	(2, 0)	-2	2	(0, 0)	1
0	4	(1, 0)	2	1	(2, 2)	0	2	(2, 0)	2	1	(0, 0)	2
0	4	(1, 0)	2	1	(2, 1)	0	2	(2, 0)	2	1	(1, 0)	2
-4	1	(0, 0)	0	4	(1, 1)	-2	1	(2, 0)	2	1	(1, 1)	1
-1	4	(0, 0)	0	4	(1, 1)	-1	1	(2, 0)	-1	1	(1, 1)	4
-2	2	(0, 0)	1	4	(1, 1)	-1	1	(2, 0)	3	1	(1, 1)	2
0	1	(-1, -1)	4	2	(2, -1)	4	1	(2, 0)	4	1	(2, -2)	1
0	5	(1, 0)	4	2	(2, 2)	0	2	(2, 0)	4	1	(2, 1)	4
-3	2	(1, 0)	1	2	(2, 1)	-2	2	(2, 0)	4	1	(2, 2)	1
-1	1	(0, -1)	0	4	(1, 1)	-1	1	(2, 1)	0	1	(1, -1)	1
-2	4	(0, 0)	0	1	(2, 2)	-2	1	(2, 2)	-2	1	(1, -1)	1
-1	5	(0, 0)	0	2	(2, 2)	-1	2	(2, 2)	-1	1	(1, -1)	2
1	4	(0, 0)	3	1	(2, 2)	1	1	(2, 2)	2	1	(1, -1)	1
0	1	(1, -1)	1	2	(2, 1)	0	2	(2, 2)	1	1	(2, -1)	1

and the result, $c_R = 9$, is consistent with an IR description as a nonlinear sigma model on a Calabi-Yau 3-fold, as expected. Using

$$c_R - c_L \; = \; \mathrm{Tr}\gamma^3 \; = \; 8 - 7 + 8 + 5 - 8 - 5 \; = \; 1,$$

we compute $c_L = 8$, consistent with a rank 2 bundle on a Calabi-Yau 3-fold. Proceeding in the same fashion, one finds that all other central charges in the models listed in table 2.32 are consistent with a bundle on a Calabi-Yau 3-fold of the indicated rank. This supports the conclusion that the PAX models listed do indeed RG flow to the indicated (0,2) nonlinear sigma models.

Chapter 3

Toda-like mirrors on product of projective spaces

This chapter concerns a generalization of mirror symmetry, known as '(0,2) mirror symmetry', as it relates UV descriptions of theories with (0,2) supersymmetry, just as ordinary mirror symmetry relates UV descriptions of theories with (2,2) supersymmetry. This chapter was published in [84], and sections 3.2, 3.3 of this chapter represent our contribution to that paper.

Mirror symmetry is a relation between, in simple cases, pairs of Calabi-Yau manifolds in which string propagation on one is equivalent to string propagation on the other. If string propagation on a Calabi-Yau manifold M is equivalent to string propagation on a Calabi-Yau manifold W, then we say M and W are mirror manifolds. The idea of mirror symmetry can be traced back to T-duality, in which string propagation on a circle with radius R is equivalent to string propagation on a circle with radius $1/R$. Mirror manifolds appear in mirror string theories as target spaces of two-dimensional sigma models. As they define physically the same theory, the dimension of mirror manifolds should be the same; in particular, if strings compactified on two internal manifolds are equivalent, then the dimension of the internal manifold should be the same:

$$\dim(M) = \dim(W).$$

Furthermore, the massless particles in the four-dimensional external space are counted by the cohomology of the internal space. So another property that mirror manifolds have is

$$\sum \dim H^*_{dR}(M) = \sum \dim H^*_{dR}(W),$$

where H^*_{dR} is the cohomology. More specifically, mirror symmetry exchanges cohomology of degree (p,q) with cohomology of degree $(n-p,q)$, n the complex dimension, so one have

$$\dim H^{p,q}(M) = \dim H^{n-p,q}(W).$$

This can be viewed as rotating the Hodge diamond. There is also a notion of mirror symmetry for non-Calabi-Yau spaces. Here, in typical examples, the mirror to an A-twisted theory on a non-Calabi-Yau space X will be a B-twisted Landau-Ginzburg model, known as a Toda dual, as explained in [15] and reviewed later in this chapter.

One conjecture of the generalization of the ordinary mirror symmetry is the (0,2) mirror symmetry. Recall a (0,2) supersymmetric nonlinear sigma model is typically defined by a complex Kähler manifold X and holomorphic vector bundle $\mathcal{E} \to X$ obeying

$$\mathrm{ch}_2(\mathcal{E}) \;=\; \mathrm{ch}_2(TX),$$

known as the Green-Schwarz or anomaly cancellation condition. In addition, to define the A/2-twist, we must also require that

$$\det \mathcal{E}^* \;\cong\; K_X.$$

For example, if $\mathcal{E} = TX$, both of these conditions are trivially satisfied. There is also a B/2-twist, which requires instead

$$\det \mathcal{E} \;\cong\; K_X.$$

If $\mathcal{E} = TX$ and $K_X^{\otimes 2}$ is trivial, these conditions are satisfied, which match the conditions for consistency of the closed-string B model [63]. The A/2 and B/2 twists are closely related: the A/2 twist of a nonlinear sigma model defined by $(X, \mathcal{E})$ is equivalent to the B/2 twist of a nonlinear sigma model defined by $(X, \mathcal{E}^*)$ [63].

For X a Calabi-Yau, the simplest version of (0,2) mirror symmetry asserts that the pair $(X, \mathcal{E})$ define the same (0,2) SCFT as another pair $(X', \mathcal{E}')$, satisfying the same two conditions above, where X' is Calabi-Yau. This duality also exchanges the A/2 and B/2 twists, in the sense that the A/2 twist of the nonlinear sigma model defined by $(X, \mathcal{E})$ is equivalent to the B/2 twist of the nonlinear sigma model defined by $(X', \mathcal{E}')$.

(0,2) mirror symmetry reduces to (2,2) mirror symmetry if $\mathcal{E}$ is taken to be TX. Instead of exchanging (p, q) forms, (0,2) mirror symmetry exchanges bundle-valued differential forms:

$$H^j(X_1, \wedge^i \mathcal{E}_1) \longleftrightarrow H^j(X_2, (\wedge^i \mathcal{E}_2)^\vee).$$

If $\mathcal{E}$ reduces to TX, then the above relation reduces to

$$H^{i,j}(X_1) \longleftrightarrow H^{n-i,j}(X_2).$$

In fact, the bundle-valued differential forms enable us to define a quantum sheaf cohomology ring [53], *i.e.*, there is a deformation of classical cup product structure on $H^\bullet(X, \wedge^\bullet \mathcal{E}^\vee)$. This structure arises from the correlation functions in A/2 twisted model:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \sum_d \int_{\mathcal{M}_d} H^{p_1}(\mathcal{M}_d, \wedge^{q_1} \mathcal{E}^\vee) \wedge \cdots \wedge H^{p_m}(\mathcal{M}_d, \wedge^{q_m} \mathcal{E}^\vee).$$

Although (0,2) mirror symmetry has not been developed to nearly the same extent as ordinary mirror symmetry, a number of crucial results do exist. One of the first accomplishments was a numerical scan through anomaly-free examples demonstrating the existence of pairs of (0,2) theories with matching spectrum computations [47], giving strong evidence for the existence of (0,2) mirrors. Other work includes a version [48] of the old Greene-Plesser orbifold construction [49], work on GLSM-based dualities [50], and most recently, a proposal for a generalization of Batyrev's construction involving reflexively plain polytopes [51]. In addition, there has been considerable work on quantum sheaf cohomology [52–72], the (0,2) analogue of ordinary quantum cohomology.

All that said, many basic gaps remain. For example, there is not yet a systematic description of (0,2) Landau-Ginzburg mirrors to (0,2) nonlinear sigma models on Fano spaces, aside from a special case discussed in [50]. This chapter is a first pass at filling that gap.

In this chapter, we will be concerned with duals in cases where X is not Calabi-Yau. Specifically, we will consider duals to A/2 twists of nonlinear sigma models on Fano manifolds X, which will correspond to B/2 twists of certain (0,2) Landau-Ginzburg models.

For (2,2) theories, such dualities are well-known as Toda duals to Fano spaces. For (0,2) theories, one special case was worked out in [50], corresponding to particular deformations of the tangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$. The point of this paper is to construct (0,2) Landau-Ginzburg mirrors to more general tangent bundle deformations of arbitrary products of projective spaces, as deformations of (2,2) Landau-Ginzburg mirrors, and in so doing, pave the way for an understanding of such duals to arbitrary Fano manifolds.

We check our ansatz for (0,2) duals by comparing correlation functions of B/2 twists of the proposed (0,2) Landau-Ginzburg mirrors to correlation functions in A/2-twisted nonlinear sigma models, which can be computed as in [53, 54, 67, 70]. In particular, those nonlinear sigma models compute quantum sheaf cohomology, a generalization of ordinary quantum cohomology. Recall that in a (2,2) supersymmetric nonlinear sigma model, the ordinary quantum cohomology is generated additively by

$$H^\bullet(X, \wedge^\bullet T^*X),$$

with T^*X the cotangent bundle of X. In the (0,2) case, the analogue (known as the quantum sheaf cohomology ring) is generated additively by

$$H^\bullet(X, \wedge^\bullet \mathcal{E}^*)$$

instead. Quantum sheaf cohomology was first introduced in [52], and the subject has been further developed in a number of works including [53–72].

3.1 Review of Toda duals in ordinary mirror symmetry

Let us quickly review ordinary Toda duals to A-twisted (2,2) supersymmetric nonlinear sigma models on projective spaces. First, recall that in the A-twisted[1] nonlinear sigma model on $\mathbb{P}^n$, all BRST-cohomology classes of local operators are generated by a single operator ψ, corresponding to a degree-two cohomology class on $\mathbb{P}^n$, with correlation functions of the form

$$\begin{aligned} \langle \psi^n \rangle &= 1, \\ \langle \psi^{2n+1} \rangle &= q, \\ \langle \psi^{n+d(n+1)} \rangle &= q^d, \end{aligned}$$

and OPE (quantum cohomology relation) $\psi^{n+1} = q$.

The mirror Toda theory is a B-twisted Landau-Ginzburg theory with superpotential of the form

$$W = \exp(Y_1) + \exp(Y_2) + \cdots + \exp(Y_n) + q\exp(-Y_1 - Y_2 - \cdots - Y_n).$$

(In effect, because of the exponentials, the superpotential is defined over $(\mathbb{C}^\times)^n$.) We define

$$X_i = e^{Y_i},$$

so that the superpotential can be written in the simpler form

$$W \;=\; X_1 + X_2 + \cdots + X_n + \frac{q}{X_1 \cdots X_n},$$

bearing in mind that the fundamental fields are Y_i.

As the superpotential is over a vector space, the correlation functions in this[2] theory are

$$\langle F_1 \cdots F_n \rangle = \sum_{dW=0} \frac{F_1 \cdots F_n}{H},$$

where $H = \det(\partial_i \partial_j W)$ (with derivatives computed with respect to Y's).

Solving the constraint $dW = 0$ (for derivatives with respect to the fundamental fields Y), one finds that the classical vacua are given by

$$X_1 = X_2 = \cdots = X_n \equiv X, \quad X = qX^{-n}.$$

[1] The reader should note that we do not couple this theory to worldsheet gravity – throughout this paper, we consider only topological field theories, not topological string theories.

[2] Here we are considering Landau-Ginzburg models over vector spaces, for which this correlation function can be found in [73]. See [74] for a discussion of correlation functions in more general B-twisted Landau-Ginzburg models. The computation in this section, demonstrating how quantum cohomology appears in Toda duals, is also described in [75], as a prelude to the discussion of Toda duals to (2,2) theories on smooth Fano Deligne-Mumford stacks.

In particular, the vacua are given by X such that $X^{n+1} = q$, which is the defining relation of the quantum cohomology ring of $\mathbb{P}^n$. (This is no accident, and in fact, is an important property we will apply later in working out duals to (0,2) theories.) Furthermore, after restriction to the classical vacua, the Hessian H is easily computed to be

$$H \;=\; (n+1)X^n.$$

Thus, the correlation functions of this model are

$$\langle X^m \rangle = \sum \frac{X^m}{(n+1)X^n},$$

where the sum runs over X's solving $X^{n+1} = q$, *i.e.* $(n+1)$th roots of q. This expression can only be nonvanishing when $m - n$ is divisible by $n+1$. We find that the nonzero correlation functions are[3]

$$\begin{aligned}\langle X^n \rangle &= 1,\\ \langle X^{2n+1} \rangle &= q,\\ \langle X^{n+d(n+1)} \rangle &= q^d,\end{aligned}$$

matching the A model correlation functions if we identify X with ψ.

In the rest of this paper, we shall describe an ansatz for Toda-like duals to (0,2) nonlinear sigma models on certain Fano spaces with deformations of the tangent bundle, generalizing the discussion above off the (2,2) locus, which we will check by comparing correlation functions (and quantum sheaf cohomology relations).

3.2 Toda-like duals to $\mathbb{P}^1 \times \mathbb{P}^1$

This section was previously published in [84].

3.2.1 The (0,2) NLSM

In the case of $X = \mathbb{P}^1 \times \mathbb{P}^1$, one can describe a general deformation $\mathcal{E}$ of the tangent bundle as the cokernel of the following sequence:

$$0 \;\longrightarrow\; \mathcal{O} \otimes \mathcal{O} \;\xrightarrow{E}\; \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \;\longrightarrow\; \mathcal{E} \;\longrightarrow\; 0,$$

[3] In principle we should write the correlation functions in terms of $X_1, \cdots, X_n$; however, since the X_i coincide on the set of vacua, and the correlation functions are computed by summing over vacua, it is an immediate result that

$$\langle f(X_1, \cdots, X_n) \rangle \;=\; \langle f(X, \cdots, X) \rangle,$$

so without loss of generality we merely write the correlation functions in terms of powers of X.

where

$$E = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix},$$

with A, B, C, D 2×2 matrices and

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

are homogeneous coordinates on the two $\mathbb{P}^1$ factors. The tangent bundle corresponds to $A = D = I$, and $B = C = 0$. For more general A, B, C, D, the vector bundle is (generically) a deformation of the tangent bundle. In this model, it has been argued in [53, 54, 67, 70] that the OPE ring relations in the A/2 twist (defining the quantum sheaf cohomology ring) are given by

$$\det(A\psi + B\tilde{\psi}) = q_1, \tag{3.1}$$

$$\det(C\psi + D\tilde{\psi}) = q_2. \tag{3.2}$$

Correlation functions in A/2 twisted theories on $\mathbb{P}^1 \times \mathbb{P}^1$ with a deformation of the tangent bundle can be computed in several ways. One method is to use direct Cech techniques to compute sheaf cohomology products on $\mathbb{P}^1 \times \mathbb{P}^1$, as has been discussed in *e.g.* [52, 56, 71]. Another method is to use GLSM-based Coulomb branch results, as described in [70]. A third, more recent, method is to use residue formulas obtained via localization, as in [67]. In this last approach, correlation functions in the A/2 twisted theory on $\mathbb{P}^1 \times \mathbb{P}^1$ are of the form

$$\begin{aligned} &\langle f(\psi, \tilde{\psi}) \rangle \\ &= \sum_{k_1, k_2} q_1^{k_1} q_2^{k_2} \mathrm{JKG} - \mathrm{Res} \left(\frac{1}{\det(A\psi + B\tilde{\psi})^{k_1+1}} \frac{1}{\det(C\psi + D\tilde{\psi})^{k_2+1}} f(\psi, \tilde{\psi}) \right). \end{aligned}$$

However one computes the correlation functions, the results have the following form, in terms of the matrices A, B, C, D above. Let

$$a = \det(A), b = \det(B), c = \det(C), d = \det(D),$$

$$e = \det(A + B), \quad f = \det(C + D).$$

Define

$$\mu = e - a - b, \quad \nu = f - c - d,$$

$$\begin{aligned} \phi_1 &= \nu b - \mu d \; = \; ad + bf - de - bc, \\ \phi_2 &= ad - bc, \\ \phi_3 &= \mu c - \nu a \; = \; ad + ce - af - bc, \end{aligned}$$

$$\Delta = \phi_2^2 - \phi_1\phi_3,$$
$$= (c-d)(bc-ad)e + cde^2 + (a-b)(ad-bc)f - (bc+ad)ef + abf^2.$$

Then the two-point correlation functions, for example, can be expressed as:

$$\langle\psi\psi\rangle = \frac{\phi_1}{\Delta}, \quad \langle\psi\tilde{\psi}\rangle = \frac{\phi_2}{\Delta}, \quad \langle\tilde{\psi}\tilde{\psi}\rangle = \frac{\phi_3}{\Delta}. \tag{3.3}$$

Higher-point correlation functions have a similar form. We list four-point functions in this A/2-twisted theory in section 3.2.2. More general correlation functions at genus zero are straightforward to compute with residue techniques, but the resulting expressions are rather unwieldy, so we do not include them in this paper.

3.2.2 A/2 correlation functions on $\mathbb{P}^1 \times \mathbb{P}^1$

In this appendix we list the two- and four-point correlation functions for A/2 twisted non-linear sigma models on $\mathbb{P}^1 \times \mathbb{P}^1$ with a deformation $\mathcal{E}$ of the tangent bundle, defined by

$$0 \longrightarrow \mathcal{O} \otimes \mathcal{O} \xrightarrow{E} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0,$$

defined as in section 3.2.1 by four matrices A, B, C, D.

In writing the correlation functions, we use the following notation:

$$a = \det(A), b = \det(B), c = \det(C), d = \det(D),$$

$$e = \det(A+B), \quad f = \det(C+D),$$
$$\mu = e - a - b, \quad \nu = f - c - d,$$

$$\phi_1 = \nu b - \mu d \ = \ ad + bf - de - bc,$$
$$\phi_2 = ad - bc,$$
$$\phi_3 = \mu c - \nu a \ = \ ad + ce - af - bc,$$

$$\Delta = \phi_2^2 - \phi_1\phi_3,$$
$$= (c-d)(bc-ad)e + cde^2 + (a-b)(ad-bc)f - (bc+ad)ef + abf^2.$$

The two-point correlation functions are given by

$$\langle\psi\psi\rangle = \frac{\phi_1}{\Delta}, \quad \langle\psi\tilde{\psi}\rangle = \frac{\phi_2}{\Delta}, \quad \langle\tilde{\psi}\tilde{\psi}\rangle = \frac{\phi_3}{\Delta}. \tag{3.4}$$

The four-point correlation functions are given by

$$
\begin{aligned}
\langle\psi\psi\psi\psi\rangle_{10} &= \frac{\phi_1}{\Delta^2}\left(\nu\phi_1+2\phi_2 d\right) = \frac{1}{\Delta^2}(\phi_1((f-c)\phi_1+ad^2+d^2e-bcd-bdf)),\\
\langle\psi\psi\psi\tilde{\psi}\rangle_{10} &= \frac{1}{\Delta^2}\left(-\phi_1^2 c+\phi_2^2 d\right),\\
\langle\psi\psi\tilde{\psi}\tilde{\psi}\rangle_{10} &= \frac{\phi_2}{\Delta^2}\left(\phi_3 d-\phi_1 c\right),\\
\langle\psi\tilde{\psi}\tilde{\psi}\tilde{\psi}\rangle_{10} &= \frac{1}{\Delta^2}\left(\phi_3^2 d-\phi_2^2 c\right),\\
\langle\tilde{\psi}\tilde{\psi}\tilde{\psi}\tilde{\psi}\rangle_{10} &= \frac{\phi_3}{\Delta^2}\left(\nu\phi_3+2\phi_2 c\right),\\
&= \frac{1}{\Delta^2}(\phi_3(ce(c+d-f)+bc(c-d+f)+a((d-f)^2-c(d+f)))),\\
\langle\psi\psi\psi\psi\rangle_{01} &= -\frac{\phi_1}{\Delta^2}\left(\mu\phi_1+2\phi_2 b\right),\\
&= \frac{1}{\Delta^2}(-\phi_1(2b(ad-bc)-d(a+b-e)^2+b(a+b-e)(c+d-f))),\\
\langle\psi\psi\psi\tilde{\psi}\rangle_{01} &= \frac{1}{\Delta^2}\left(\phi_1^2 a-\phi_2^2 b\right),\\
\langle\psi\psi\tilde{\psi}\tilde{\psi}\rangle_{01} &= \frac{\phi_2}{\Delta^2}\left(-\phi_3 b+\phi_1 a\right),\\
\langle\psi\tilde{\psi}\tilde{\psi}\tilde{\psi}\rangle_{01} &= \frac{1}{\Delta^2}\left(-\phi_3^2 b+\phi_2^2 a\right),\\
\langle\tilde{\psi}\tilde{\psi}\tilde{\psi}\tilde{\psi}\rangle_{01} &= -\frac{\phi_3}{\Delta^2}\left(\mu\phi_3-2\phi_2 a\right) = \frac{1}{\Delta^2}(\phi_3((e-b)\phi_3+a^2d+a^2f-abc-acd)),
\end{aligned}
$$

where the subscripts 10 and 01 denote contributions from the degree one sector on either $\mathbb{P}^1$ factor:

$$\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle = q_1\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle_{10}+q_2\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle_{01}.$$

3.2.3 The Toda-like mirror theory

We claim the mirror theory to the A/2 twisted theory just described, is a (0,2) Landau-Ginzburg model, defined by a (0,2) superpotential of the form

$$W = FJ+\tilde{F}\tilde{J}, \tag{3.5}$$

where F and $\tilde{F}$ are Fermi superfields, and

$$
\begin{aligned}
J &= X^{-1}(\det(AX+B\tilde{X})-q_1) = aX+b\frac{\tilde{X}^2}{X}+\mu\tilde{X}-\frac{q_1}{X},\\
\tilde{J} &= \tilde{X}^{-1}(\det(CX+D\tilde{X})-q_2) = d\tilde{X}+c\frac{X^2}{\tilde{X}}+\nu X-\frac{q_2}{\tilde{X}},
\end{aligned}
$$

for $X = \exp(Y)$, $\tilde{X} = \exp(\tilde{Y})$, where Y, $\tilde{Y}$ are the fundamental fields, and

$$a = \det A, \quad b = \det B, \quad c = \det C, \quad d = \det D,$$

$$\mu = \det(A + B) - \det A - \det B,$$
$$\nu = \det(C + D) - \det C - \det D,$$

for A, B, C, D the matrices defining the tangent bundle deformation of the A/2 theory.

We will check the ansatz above by comparing correlation functions between the original A/2 theory and the B/2 twist of the Landau-Ginzburg theory above, but first, let us make a few quick observations.

As one consistency check, note that for

$$A = D = I, \ B = C = 0,$$

then the vector bundle $\mathcal{E}$ is the tangent bundle, and the theory has (2,2) supersymmetry. This also can be seen from the (0,2) superpotential

$$W = F\left(X - \frac{q_1}{X}\right) + \tilde{F}\left(\tilde{X} - \frac{q_2}{\tilde{X}}\right),$$

which matches the (2,2) superpotential in this case.

As another check, note that the space of classical vacua of this theory ($J = \tilde{J} = 0$) matches the space of solutions to the quantum sheaf coholomogy ring relations:

$$\det(AX + B\tilde{X}) = q_1, \tag{3.6}$$
$$\det(CX + D\tilde{X}) = q_2. \tag{3.7}$$

Now, let us compute and compare genus zero correlation functions. Given a B/2-twisted Landau-Ginzburg model with superpotential W over a vector space or a product of $\mathbb{C}^{\times}$'s, correlation functions at genus zero are given by[4] [69]

$$\langle \phi^{i_1}(x_1) \cdots \phi^{i_k}(x_k) \rangle = \sum_{J_i(\phi)=0} \phi^{i_1}(x_1) \cdots \phi^{i_k}(x_k) [\det_{i,j} J_{i,j}]^{-1} \tag{3.8}$$

[4] Correlation functions for more general B/2-twisted Landau-Ginzburg models are discussed in [66]. In passing, we should comment on the absence of worldsheet instanton corrections to the formulas above. On the (2,2) locus, the Toda duals to A model topological field theories are B-twisted, and correlation functions in the B model do not have worldsheet instanton corrections. In the present case, our Toda-like mirrors to A/2 model pseudo-topological field theories are B/2 twisted. Unlike the (2,2) case, however, in general B/2 twisted models can and will receive worldsheet instanton corrections.

However, our Toda-like theories are defined by superpotentials over algebraic tori, *i.e.* $(\mathbb{C}^{\times})^n$, and there are no non-constant holomorphic maps from $\mathbb{P}^1$ (or any projective variety) to an algebraic torus. All holomorphic maps are constant maps, hence there are no worldsheet instanton corrections in these theories [76]. Thus, we need only compute classically in the B/2 model, just as in ordinary Toda mirrors.

where the sum is over classical vacua.

Using the formula above for B/2-twisted Landau-Ginzburg correlation functions, one finds that the two-point correlation functions in this model are given by

$$\begin{aligned} \langle XX \rangle &= \Delta^{-1}(b\nu - d\mu), \\ \langle X\tilde{X} \rangle &= \Delta^{-1}(ad - bc), \\ \langle \tilde{X}\tilde{X} \rangle &= \Delta^{-1}(c\mu - a\nu), \end{aligned}$$

where

$$\Delta = b^2c^2 - 2abcd + a^2d^2 + cd\mu^2 - (bc + ad)\mu\nu + ab\nu^2.$$

These match the A/2 correlation functions in equation (3.3), if we identify X with ψ and $\tilde{X}$ with $\tilde{\psi}$.

We also checked that all four-point functions for general A, B, C, D (as listed in appendix 3.2.4) match the results from the A/2 model. For the special case in which $\det B = \det C = 0$, we have checked that all correlation funcions up to ten-point correlation functions and one twelve-point correlation function $\langle X^6 \tilde{X}^6 \rangle$ match the results from the A/2 model.

Beyond special cases, there is also a general argument that all correlation functions must match. We will utilize a formula for the A/2 model correlation functions given in [70][section 3.4], which is similar in form to the formula above for B/2 Landau-Ginzburg model correlation functions, and argue that after some algebra, the formula for A/2 correlation functions in [70] matches the formula for B/2 correlation functions above. As a result, all correlation functions in our B/2-twisted Landau-Ginzburg model will necessarily match those of the A/2 nonlinear sigma model.

Let us describe this argument for general matching correlation functions. From [70][section 3.4], all correlation functions in an A/2-twisted (0,2) nonlinear sigma model on $\mathbb{P}^1 \times \mathbb{P}^1$, at genus zero, take the form

$$\begin{aligned} &\langle f(\psi, \tilde{\psi}) \rangle \\ &= \sum_{\psi, \tilde{\psi} | \mathcal{J}_a = 0} f(\psi, \tilde{\psi}) \left[\left(\det_{a,b} \mathcal{J}_{a,b} \right) \det \left(A\psi + B\tilde{\psi} \right) \det \left(C\psi + D\tilde{\psi} \right) \right]^{-1}, \\ &= \sum_{\psi, \tilde{\psi} | \mathcal{J}_a = 0} f(\psi, \tilde{\psi}) \det \left[\begin{array}{cc} \partial_\psi \det(A\psi + B\tilde{\psi}) & \partial_{\tilde{\psi}} \det(A\psi + B\tilde{\psi}) \\ \partial_\psi \det(C\psi + D\tilde{\psi}) & \partial_{\tilde{\psi}} \det(C\psi + D\tilde{\psi}) \end{array} \right]^{-1} \end{aligned}$$

where the $\mathcal{J}_a$ (not to be confused with the J, $\tilde{J}$ we used in our dual theory earlier) are defined by

$$\begin{aligned} \mathcal{J}_1 &= \ln \left(q_1^{-1} \det \left(A\psi + B\tilde{\psi} \right) \right), \\ \mathcal{J}_2 &= \ln \left(q_2^{-1} \det \left(C\psi + D\tilde{\psi} \right) \right). \end{aligned}$$

To compare the correlation functions above with the B/2 correlation functions in our dual theory, which take a similar form, first note that the constraint $\tilde{J}_a = 0$ implies

$$\det(A\psi + B\tilde{\psi}) \;=\; q_1, \quad \det(C\psi + D\tilde{\psi}) \;=\; q_2,$$

the quantum sheaf cohomology relations and also the relations defining the vacua of the B/2 Landau-Ginzburg model. Then, matching follows as a consequence of

$$\det_{i,j} J_{i,j} \;=\; \left(\det_{a,b} \mathcal{J}_{a,b}\right) \det\left(A\psi + B\tilde{\psi}\right) \det\left(C\psi + D\tilde{\psi}\right),$$

or more explicitly

$$\det \begin{bmatrix} \partial_Y \left(aX + b\tilde{X}^2/X + \mu\tilde{X} - q_1/X\right) & \partial_{\tilde{Y}} \left(aX + b\tilde{X}^2/X + \mu\tilde{X} - q_1/X\right) \\ \partial_Y \left(d\tilde{X} + cX^2/\tilde{X} + \nu X - q_2/\tilde{X}\right) & \partial_{\tilde{Y}} \left(d\tilde{X} + cX^2/\tilde{X} + \nu X - q_2/\tilde{X}\right) \end{bmatrix}$$
$$= \quad \det \begin{bmatrix} \partial_\psi \det(A\psi + B\tilde{\psi}) & \partial_{\tilde{\psi}} \det(A\psi + B\tilde{\psi}) \\ \partial_\psi \det(C\psi + D\tilde{\psi}) & \partial_{\tilde{\psi}} \det(C\psi + D\tilde{\psi}) \end{bmatrix},$$

where $X = \exp(Y)$, $\tilde{X} = \exp(\tilde{Y})$, after identifying X with ψ and $\tilde{X}$ with $\tilde{\psi}$, which is straightforward to verify. Thus, all genus zero correlation functions of our B/2 Landau-Ginzburg model, the proposed dual to $\mathbb{P}^1 \times \mathbb{P}^1$, do indeed match the correlation functions of the (0,2) theory on $\mathbb{P}^1 \times \mathbb{P}^1$.

We will apply a more general version of this argument when checking genus zero correlation functions of the proposed B/2 Landau-Ginzburg dual to A/2 theories on $\mathbb{P}^n \times \mathbb{P}^n$ in section 3.3.2.

3.2.4 Correlation functions for Toda-like dual to $\mathbb{P}^1 \times \mathbb{P}^1$

In this appendix we list the two-point and four-point correlation functions for our proposed Toda-like dual (0,2) Landau-Ginzburg model, with superpotential of the form

$$J = aX + b\frac{\tilde{X}^2}{X} + \mu\tilde{X} - \frac{q_1}{X},$$
$$\tilde{J} = d\tilde{X} + c\frac{X^2}{\tilde{X}} + \nu X - \frac{q_2}{\tilde{X}}.$$

The two-point correlation functions in this (0,2) Landau-Ginzburg model can be shown to be

$$\begin{aligned} \langle XX \rangle &= \gamma^{-1}(b\nu - d\mu), \\ \langle X\tilde{X} \rangle &= \gamma^{-1}(ad - bc), \\ \langle \tilde{X}\tilde{X} \rangle &= \gamma^{-1}(c\mu - a\nu), \end{aligned}$$

where $\gamma = b^2c^2 - 2abcd + a^2d^2 + cd\mu^2 - (bc+ad)\mu\nu + ab\nu^2$.

The four-point correlation functions in this (0,2) Landau-Ginzburg model can be shown to be

$$\begin{aligned}
\langle XXXX\rangle_{10} &= \gamma^{-2}(-(d\mu - b\nu)(d(2ad-\mu\nu)+b(-2cd+\nu^2))),\\
\langle XXXX\rangle_{01} &= \gamma^{-2}(-d\mu + b\nu)(2b^2c + d\mu^2 - b(2ad+\mu\nu)),\\
\langle XXX\tilde{X}\rangle_{10} &= \gamma^{-2}(d((bc-ad)^2 - cd\mu^2) + 2bcd\mu\nu - b^2c\nu^2),\\
\langle XXX\tilde{X}\rangle_{01} &= \gamma^{-2}(-b^3c^2 + ad^2\mu^2 - adb(ad+2\mu\nu) + ab^2(2cd+\nu^2)),\\
\langle XX\tilde{X}\tilde{X}\rangle_{10} &= \gamma^{-2}(bc-ad)(-2cd\mu + bc\nu + ad\nu),\\
\langle XX\tilde{X}\tilde{X}\rangle_{01} &= \gamma^{-2}(bc-ad)(bc\mu + ad\mu - 2ab\nu),\\
\langle X\tilde{X}\tilde{X}\tilde{X}\rangle_{10} &= \gamma^{-2}(c(-(bc-ad)^2 + cd\mu^2) - 2acd\mu\nu + a^2d\nu^2),\\
\langle X\tilde{X}\tilde{X}\tilde{X}\rangle_{01} &= \gamma^{-2}(a^3d^2 - bc^2\mu^2 + abc(bc+2\mu\nu) - a^2b(2cd+\nu^2)),\\
\langle \tilde{X}\tilde{X}\tilde{X}\tilde{X}\rangle_{10} &= \gamma^{-2}(c\mu - a\nu)(2bc^2 - c\mu\nu + a(-2cd+\nu^2)),\\
\langle \tilde{X}\tilde{X}\tilde{X}\tilde{X}\rangle_{01} &= \gamma^{-2}(c\mu - a\nu)(2a^2d + c\mu^2 - a(2bc+\mu\nu)),
\end{aligned}$$

where the 10 and 01 subscripts indicate the coefficients of q_1, q_2, as in the previous subsection.

As remarked in section 3.2.3, if we identify the parameters above with matrix determinants as

$$\begin{aligned}
a &= \det A, \quad b = \det B, \quad c = \det C, \quad d = \det D,\\
\mu &= \det(A+B) - \det A - \det B,\\
\nu &= \det(C+D) - \det C - \det D,
\end{aligned}$$

for A, B, C, D the matrices appearing in the A/2-twisted (0,2) model on $\mathbb{P}^1 \times \mathbb{P}^1$, the correlation functions in the Landau-Ginzburg model above match those of the A/2 model.

3.3 Generalization to $\mathbb{P}^n \times \mathbb{P}^m$

This section was previously published in [84].

3.3.1 The A/2-twisted nonlinear sigma model

Let us begin by briefly reviewing pertinent properties of the (0,2) nonlinear sigma model on $\mathbb{P}^n \times \mathbb{P}^m$, whose dual we shall describe. First, the gauge bundle in this (0,2) theory is a deformation $\mathcal{E}$ of the tangent bundle of $\mathbb{P}^n \times \mathbb{P}^m$, which can be described as a cokernel

$$0 \longrightarrow \mathcal{O}^2 \xrightarrow{E} \mathcal{O}(1,0)^{n+1} \oplus \mathcal{O}(0,1)^{m+1} \longrightarrow \mathcal{E} \longrightarrow 0,$$

where

$$E = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

in which A, B are $(n+1)\times(n+1)$ matrices and C, D are $(m+1)\times(m+1)$ matrices. The quantum sheaf cohomology ring of an A/2-twisted nonlinear sigma model on $\mathbb{P}^n \times \mathbb{P}^m$ with the bundle above takes the form [53, 54, 67, 70]

$$\det(A\psi + B\tilde{\psi}) \;=\; q_1, \quad \det(C\psi + D\tilde{\psi}) \;=\; q_2,$$

and for later use, we expand the determinants as follows:

$$\det(A\psi + B\tilde{\psi}) = a\psi^{n+1} + b\tilde{\psi}^{n+1} + \sum_{i=1}^{n} \mu_i \psi^i \tilde{\psi}^{n+1-i}, \tag{3.9}$$

$$\det(C\psi + D\tilde{\psi}) = c\psi^{m+1} + d\tilde{\psi}^{m+1} + \sum_{k=1}^{m} \nu_k \psi^k \tilde{\psi}^{m+1-k}, \tag{3.10}$$

where

$$a = \det A, \quad b = \det B, \quad c = \det C, \quad d = \det D,$$

μ_i is a sum of determinants of matrices, each of which is formed by taking i rows of A and $n+1-i$ rows of B, and ν_i is formed similarly from C, D.

3.3.2 The Toda-like mirror theory

We claim the (0,2) superpotential of the (0,2) Landau-Ginzburg Toda-like mirror to $\mathbb{P}^n \times \mathbb{P}^m$ is

$$W = \sum_{i=1}^{n} F_i J_i + \sum_{k=1}^{m} \tilde{F}_k \tilde{J}_k, \tag{3.11}$$

where

$$J_i = a^{(1-n)/n} \left(aX_i - \frac{q_1}{X_1 \cdots X_n} + b\frac{\tilde{X}_1^{n+1}}{X_1^n} + \sum_{i=1}^{n} \mu_{n+1-i} \frac{\tilde{X}_1^i}{X_1^{i-1}} \right), \tag{3.12}$$

$$\tilde{J}_k = d^{(1-m)/m} \left(d\tilde{X}_k - \frac{q_2}{\tilde{X}_1 \cdots \tilde{X}_m} + c\frac{X_1^{m+1}}{\tilde{X}_1^m} + \sum_{k=1}^{m} \nu_k \frac{X_1^k}{\tilde{X}_1^{k-1}} \right), \tag{3.13}$$

which clearly generalizes the dual to $\mathbb{P}^1 \times \mathbb{P}^1$ discussed in section 3.2.3.

First, note that if the parameters a, b, c, d, and the μ_i, ν_k are related to the matrices A, B, C, D of the A/2 model as above, then the vacua of this theory, defined by $J_i = 0 = \tilde{J}_k$, are the solutions of

$$X_1 \;=\; X_2 \;=\; \cdots \;=\; X_n \;\equiv\; X, \quad \tilde{X}_1 \;=\; \tilde{X}_2 \;=\; \cdots \;=\; \tilde{X}_m \;\equiv\; \tilde{X},$$

$$\det(AX + B\tilde{X}) \;=\; q_1, \quad \det(CX + D\tilde{X}) \;=\; q_2,$$

identical to the solutions of the quantum sheaf cohomology relations, as one would expect for a sensible Toda-like dual.

One can show that the correlation functions of this B/2-twisted Landau-Ginzburg model computed by equation (3.8) equal the correlation functions of A/2-twisted model on $\mathbb{P}^n \times \mathbb{P}^m$ [70]:

$$\langle \sigma_{a_1} \cdots \sigma_{a_l} \rangle = \sum_{\sigma | \mathcal{J}=0} \sigma_{a_1} \cdots \sigma_{a_l} \left[\det_{a,b} \mathcal{J}_{a,b} \prod_{\alpha} \det M_{(\alpha)} \right]^{-1} \tag{3.14}$$

with

$$\mathcal{J}_a = \ln \left[q_a^{-1} \prod_{\alpha} \det M_{(\alpha)}^{Q^a_{(\alpha)}} \right]. \tag{3.15}$$

In the present case, for $\mathbb{P}^n \times \mathbb{P}^m$, there are only two σ's, which we label σ_1, σ_2, and

$$\begin{aligned} \mathcal{J}_1 &= \ln \left[q_1^{-1} \det(A\sigma_1 + B\sigma_2) \right], \\ \mathcal{J}_2 &= \ln \left[q_2^{-1} \det(C\sigma_1 + D\sigma_2) \right]. \end{aligned}$$

To show that the two expressions for correlation functions match, it suffices to show that

$$\det |J_{i,j}| = \det_{a,b} |\mathcal{J}_{a,b}| \prod_{\alpha} \det M_{(\alpha)}, \tag{3.16}$$

by identifying X_i with σ_1 and $\tilde{X}_k$ with σ_2 on the space of vacua, since

$$\det_{a,b} |\mathcal{J}_{a,b}| \prod_{\alpha} \det M_{(\alpha)} = \det \begin{bmatrix} \partial_{\sigma_1} \det(A\sigma_1 + B\sigma_2) & \partial_{\sigma_2} \det(A\sigma_1 + B\sigma_2) \\ \partial_{\sigma_1} \det(C\sigma_1 + D\sigma_2) & \partial_{\sigma_2} \det(C\sigma_1 + D\sigma_2) \end{bmatrix} \tag{3.17}$$

on the classical vacua $\mathcal{J}_a(\sigma) = 0$.

In order to show (3.16), we will need a minor linear algebra result. For an $(n+m) \times (n+m)$ matrix of the form

$$\begin{matrix} n\left\{ \begin{matrix} \\ \\ \\ \\ \\ \end{matrix} \right. \\ m\left\{ \begin{matrix} \\ \\ \\ \\ \\ \end{matrix} \right. \end{matrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & \beta & 0 & & \cdots & 0 \\ -\alpha & \alpha & 0 & \cdots & 0 & 0 & 0 & & \cdots & 0 \\ -\alpha & 0 & \alpha & \cdots & 0 & 0 & & & & 0 \\ \vdots & & & \ddots & \vdots & \vdots & & & \ddots & \vdots \\ -\alpha & 0 & \cdots & 0 & \alpha & 0 & & & \cdots & 0 \\ \rho & 0 & & \cdots & 0 & d_{11} & d_{12} & d_{13} & \cdots & d_{1m} \\ 0 & 0 & & \cdots & 0 & -\delta & \delta & 0 & \cdots & 0 \\ 0 & & & & 0 & -\delta & 0 & \delta & \cdots & 0 \\ \vdots & & & \ddots & \vdots & \vdots & & & \ddots & \vdots \\ 0 & & & \cdots & 0 & -\delta & 0 & \cdots & 0 & \delta \end{bmatrix}, \tag{3.18}$$

its determinant has the form

$$(\det \zeta)(\det \eta) - \beta\rho\alpha^{n-1}\delta^{m-1}, \tag{3.19}$$

where $\det \zeta$ is the determinant of the upper-left $n \times n$ submatrix and $\det \eta$ is the determinant of the lower-right $m \times m$ submatrix, given by

$$\det \zeta = \alpha^{n-1} \sum_{i=1}^{n} a_{1i}, \tag{3.20}$$

$$\det \eta = \delta^{m-1} \sum_{k=1}^{m} d_{1k}. \tag{3.21}$$

Next, we need to compute

$$\det |J_{i,j}| = \det \begin{bmatrix} \partial_{Y_1} J_1 & \cdots & \partial_{Y_n} J_1 & \partial_{\tilde{Y}_1} J_1 & \cdots & \partial_{\tilde{Y}_m} J_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \partial_{Y_1} J_n & \cdots & \partial_{Y_n} J_n & \partial_{\tilde{Y}_1} J_n & \cdots & \partial_{\tilde{Y}_m} J_n \\ \partial_{Y_1} \tilde{J}_1 & \cdots & \partial_{Y_n} \tilde{J}_1 & \partial_{\tilde{Y}_1} \tilde{J}_1 & \cdots & \partial_{\tilde{Y}_m} \tilde{J}_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \partial_{Y_1} \tilde{J}_m & \cdots & \partial_{Y_n} \tilde{J}_m & \partial_{\tilde{Y}_1} \tilde{J}_m & \cdots & \partial_{\tilde{Y}_m} \tilde{J}_m \end{bmatrix},$$

where $X_i = \exp(Y_i)$ and $\tilde{X}_i = \exp(\tilde{Y}_i)$. By taking suitable linear combinations, one can rewrite the matrix above in the form of the matrix (3.18), with the following identifications:

$$\begin{aligned} a_{11} &= a^{(1-n)/n} \left(aX_1 + \frac{q_1}{X_1 \cdots X_n} - nb\frac{\tilde{X}_1^{n+1}}{X_1^n} + \sum_{i=1}^{n} (1-i)\mu_{n+1-i} \frac{\tilde{X}_1^i}{X_1^{i-1}} \right), \\ &= a^{(1-n)/n} \left(2aX + (1-n)b\frac{\tilde{X}^{n+1}}{X^n} + \sum_{i=1}^{n} (2-i)\mu_{n+1-i} \frac{\tilde{X}^i}{X^{i-1}} \right), \\ a_{12} = a_{13} = \cdots = a_{1n} &= a^{(1-n)/n} \left(+ \frac{q_1}{X_1 \cdots X_n} \right), \\ &= a^{(1-n)/n} \left(aX + b\frac{\tilde{X}^{n+1}}{X^n} + \sum_{i=1}^{n} \mu_{n+1-i} \frac{\tilde{X}^i}{X^{i-1}} \right), \end{aligned}$$

$$\begin{aligned} \alpha &= a^{(1-n)/n}(aX_1), \\ &= a^{1/n}X, \\ \beta &= a^{(1-n)/n} \left((n+1)b\frac{\tilde{X}_1^{n+1}}{X_1^n} + \sum_{i=1}^{n} i\mu_{n+1-i} \frac{\tilde{X}_1^i}{X_1^{i-1}} \right), \\ &= a^{(1-n)/n} \left((n+1)b\frac{\tilde{X}^{n+1}}{X^n} + \sum_{i=1}^{n} i\mu_{n+1-i} \frac{\tilde{X}^i}{X^{i-1}} \right), \end{aligned}$$

$$
\begin{aligned}
d_{11} &= d^{(1-m)/m}\left(d\tilde{X}_1 + \frac{q_2}{\tilde{X}_1\cdots\tilde{X}_m} - cm\frac{X_1^{m+1}}{\tilde{X}_1^m} + \sum_{k=1}^m (1-k)\nu_k\frac{X_1^k}{\tilde{X}_1^{k-1}}\right),\\
&= d^{(1-m)/m}\left(2d\tilde{X} + (1-m)c\frac{X^{m+1}}{\tilde{X}^m} + \sum_{k=1}^m (2-k)\nu_k\frac{X^k}{\tilde{X}^{k-1}}\right),\\
d_{12} = d_{13} = \cdots = d_{1n} &= d^{(1-m)/m}\left(+\frac{q_2}{\tilde{X}_1\cdots\tilde{X}_m}\right),\\
&= d^{(1-m)/m}\left(d\tilde{X} + c\frac{X^{m+1}}{\tilde{X}^m} + \sum_{k=1}^m \nu_k\frac{X^k}{\tilde{X}^{k-1}}\right),
\end{aligned}
$$

$$
\begin{aligned}
\delta &= d^{(1-m)/m}(d\tilde{X}_1),\\
&= d^{1/m}\tilde{X},\\
\rho &= d^{(1-m)/m}\left((m+1)c\frac{X_1^{m+1}}{\tilde{X}_1^m} + \sum_{k=1}^m k\nu_k\frac{X_1^k}{\tilde{X}_1^{k-1}}\right),\\
&= d^{(1-m)/m}\left((m+1)\frac{X^{m+1}}{\tilde{X}^m} + \sum_{k=1}^m k\nu_k\frac{X^k}{\tilde{X}^{k-1}}\right).
\end{aligned}
$$

(In the expressions above, the second line is obtained by evaluation on vacua.)

Putting this together, we can write

$$
\begin{aligned}
\det|J_{i,j}| &= \det\begin{bmatrix} \det\zeta & \beta\alpha^{n-1} \\ \rho\delta^{m-1} & \det\eta \end{bmatrix}\\
&= \det\begin{bmatrix} \alpha^{n-1}(a_{11}+(n-1)a_{12}) & \alpha^{n-1}\beta \\ \delta^{m-1}\rho & \delta^{m-1}(d_{11}+(n-1)d_{12}) \end{bmatrix}
\end{aligned}
$$

which is easily checked to be the determinant of

$$
\begin{bmatrix}
(n+1)aX^n + \sum_{i=1}^n (n+1-i)\mu_{n+1-i}\tilde{X}^i X^{n-i} & (n+1)b\tilde{X}^{n+1}X^{-1} + \sum_{i=1}^n i\mu_{n+1-i}\tilde{X}^i X^{n-i} \\
(m-1)cX^{m+1}\tilde{X}^{-1} + \sum_{k=1}^m k\nu_k X^k\tilde{X}^{m-k} & (m-1)d\tilde{X}^m + \sum_{k=1}^m (m+1-k)\nu_k X^k\tilde{X}^{m-k}
\end{bmatrix}.
$$

By identifying X_i with σ_1 and $\tilde{X}_k$ with σ_2, we see that the determinant above matches (3.17).

Thus, all genus-zero correlation functions in our proposed Toda dual match those of the (0,2) theory on $\mathbb{P}^n \times \mathbb{P}^m$ with a deformation of the tangent bundle. In addition to constructing a general argument that correlation functions should match, we have also compared correlation functions in special cases, as we shall outline next.

3.3.3 Example: $\mathbb{P}^1 \times \mathbb{P}^2$

As a consistency check, as we have already studied the dual to $\mathbb{P}^1 \times \mathbb{P}^1$, we next consider the special case $\mathbb{P}^1 \times \mathbb{P}^2$. Specializing the results for $\mathbb{P}^n \times \mathbb{P}^m$, the mirror (0,2) Landau-Ginzburg model is defined by the superpotential

$$W = FJ + \widetilde{F_1}\widetilde{J_1} + \widetilde{F_2}\widetilde{J_2}$$

with

$$J = aX - \frac{q_1}{X} + b\frac{\tilde{X}_1^2}{X} + \mu\tilde{X}_1, \tag{3.22}$$

$$\widetilde{J_1} = d^{-\frac{1}{2}}\left(d\tilde{X}_1 - \frac{q_2}{\tilde{X}_1\tilde{X}_2} + c\frac{X^3}{\tilde{X}_1^2} + fX + g\frac{X^2}{\tilde{X}_1}\right), \tag{3.23}$$

$$\widetilde{J_2} = d^{-\frac{1}{2}}\left(d\tilde{X}_2 - \frac{q_2}{\tilde{X}_1\tilde{X}_2} + c\frac{X^3}{\tilde{X}_1^2} + fX + g\frac{X^2}{\tilde{X}_1}\right). \tag{3.24}$$

In the expression above,

$$\begin{aligned} a &= \det A, \quad b = \det B, \quad c = \det C, \quad d = \det D, \\ \mu &= \det(A+B) - \det A - \det B, \end{aligned}$$

for the matrices A, B, C, D defining the gauge bundle deformation in the A/2-twisted nonlinear sigma model, and where g is a sum of determinants of three matrices, each of which is formed from taking two rows of C and one row of D, and f is similarly a sum of three determinants, involving matrices formed as two rows of D and one row of C.

We have directly computed correlation functions in the proposed dual Landau-Ginzburg theory above, in the special case $c = f = g = 0$. On the vacua, $\tilde{X}_1 = \tilde{X}_2$, so in computing correlation functions, we will use $\tilde{X}$ to denote either $\tilde{X}_1$ or $\tilde{X}_2$. In any event, the three-point correlation functions in this case are given by

$$\begin{aligned} \langle XXX \rangle &= -(ab - \mu^2)(a^3 d)^{-1}, \\ \langle XX\tilde{X} \rangle &= -\mu(a^2 d)^{-1}, \\ \langle X\tilde{X}\tilde{X} \rangle &= (ad)^{-1}, \\ \langle \tilde{X}\tilde{X}\tilde{X} \rangle &= 0. \end{aligned}$$

The five-point correlation functions are given by

$$\begin{aligned} \langle X^5 \rangle &= -(a^4 d)^{-1}(2ab - 3\mu^2)q_1, \\ \langle X^4\tilde{X} \rangle &= -2(a^3 d)^{-1}\mu q_1, \\ \langle X^3\tilde{X}^2 \rangle &= (a^2 d)^{-1} q_1, \\ \langle X^2\tilde{X}^3 \rangle &= 0, \\ \langle X\tilde{X}^4 \rangle &= 0, \\ \langle \tilde{X}^5 \rangle &= 0. \end{aligned}$$

The six-point correlation functions are given by

$$\begin{aligned}
\langle X^6 \rangle &= -(a^6 d^2)^{-1}\mu(3a^2b^2 - 4ab\mu^2 + \mu^4)q_2, \\
\langle X^5 \tilde{X} \rangle &= (a^5 d^2)^{-1}(a^2b^2 - 3ab\mu^2 + \mu^4)q_2, \\
\langle X^4 \tilde{X}^2 \rangle &= -(a^4 d^2)^{-1}\mu(-2ab + \mu^2)q_2, \\
\langle X^3 \tilde{X}^3 \rangle &= -(a^3 d^2)^{-1}(ab - \mu^2)q_2, \\
\langle X^2 \tilde{X}^4 \rangle &= -(a^2 d^2)^{-1}\mu q_2, \\
\langle X \tilde{X}^5 \rangle &= (ad^2)^{-1} q_2, \\
\langle \tilde{X}^6 \rangle &= 0.
\end{aligned}$$

If we identify X with ψ and $\tilde{X}$ with $\tilde{\psi}$, then these correlation functions match those of the corresponding A/2 model, for this case ($c = f = g = 0$). We have listed the A/2 model correlation functions (for the general case) in section 3.3.4.

3.3.4 Appendix: A/2 correlation functions on $\mathbb{P}^1 \times \mathbb{P}^2$

In this appendix we list the three-point, five-point and six-point correlation functions for A/2 twisted nonlinear sigma models on $\mathbb{P}^1 \times \mathbb{P}^2$ with a deformation $\mathcal{E}$ of the tangent bundle, defined by

$$0 \longrightarrow \mathcal{O}^2 \xrightarrow{E} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^3 \longrightarrow \mathcal{E} \longrightarrow 0,$$

with A, B 2×2 matrices and C, D 3×3 matrices.

Correlation functions in this theory can be computed in a variety of methods, such as *e.g.* residues [67]. In writing the correlation functions, we use the following notation:

$$a = \det A, \quad b = \det B, \quad c = \det C, \quad d = \det D,$$

$$\mu = \det(A + B) - \det A - \det B,$$

g is a sum of determinants of three matrices, each formed from two rows of C and one row of D, and f is similarly a sum of three determinants, each having two rows of D and one row of C.

Three-point functions in the A/2 theory are given by

$$\begin{aligned}
\langle \psi^3 \rangle &= \Delta^{-1}(-abd + b^2 g - bf\mu + d\mu^2), \\
\langle \psi^2 \tilde{\psi} \rangle &= \Delta^{-1}(-b^2 c + abf - ad\mu), \\
\langle \psi \tilde{\psi}^2 \rangle &= \Delta^{-1}(a^2 d - abg + bc\mu), \\
\langle \tilde{\psi}^3 \rangle &= \Delta^{-1}(abc - a^2 f + ag\mu - c\mu^2),
\end{aligned}$$

where

$$\begin{aligned}
\Delta = a^3 d^2 + b\left((bc - af)^2 - 2a^2 dg + abg^2\right) + (bcf + adg)\mu^2 - cd\mu^3 \\
- (ad(-3bc + af) + b(bc + af)g)\mu.
\end{aligned}$$

Five-point correlation functions in the A/2 theory are given by

$$\begin{aligned}\langle\psi^5\rangle = {} & q_1\Delta^{-2}\big(b^4(c^2d-2cfg+g^3)+d^2\mu^2(3a^2d-2af\mu+g\mu^2)\\ & +2b^3(ag(f^2-2dg)+(cf^2+cdg-fg^2)\mu)\\ & +b^2(a^2d(-f^2+5dg)-2af(f^2-dg)\mu+(-4cdf+g(f^2+2dg))\mu^2)\\ & -2bd(a^3d^2+a^2df\mu-2a(f^2-dg)\mu^2+(-cd+fg)\mu^3)\big)\,,\end{aligned}$$

$$\begin{aligned}\langle\psi^4\tilde{\psi}\rangle = {} & q_1\Delta^{-2}\big(2a^3d^2(bf-d\mu)+2ab^2c(-bf^2+2bdg+df\mu)\\ & +a^2(b^2(-3cd^2+f^3-2dfg)+2bd(-f^2+dg)\mu+d^2f\mu^2)\\ & +c(b^4(cf-g^2)-b^2(f^2+2dg)\mu^2+2bdf\mu^3-d^2\mu^4\\ & +b^3(-2cd\mu+2fg\mu))\big)\,,\end{aligned}$$

$$\begin{aligned}\langle\psi^3\tilde{\psi}^2\rangle = {} & q_1\Delta^{-2}\big(a^4d^3-2a^3bd^2g+b^2c^2(b^2g-2bf\mu+3d\mu^2)\\ & +a^2(b^2(2cdf-f^2g+dg^2)+2bdfg\mu-d^2g\mu^2)\\ & -2ac(b^3cd+2bdf\mu^2-d^2\mu^3+b^2(-f^2\mu+dg\mu))\big)\,,\end{aligned}$$

$$\begin{aligned}\langle\psi^2\tilde{\psi}^3\rangle = {} & q_1\Delta^{-2}\big(-b^4c^3+2ab^3c^2f-a^2d^2(a^2f-2ag\mu+3c\mu^2)\\ & +b^2(a^2(fg^2-c(f^2+2dg))-2acfg\mu+c^2f\mu^2)\\ & +2bd(a^3cd+a^2(cf-g^2)\mu+2acg\mu^2-c^2\mu^3)\big)\,,\end{aligned}$$

$$\begin{aligned}\langle\psi\tilde{\psi}^4\rangle = {} & q_1\Delta^{-2}\big(a^4d(f^2-dg)+2a^3d(-2bcf+bg^2+cd\mu-fg\mu)\\ & +a^2(b^2(3c^2d+2cfg-g^3)-2bcdg\mu+d(2cf+g^2)\mu^2)\\ & +c^2\mu(2b^3c-b^2g\mu+d\mu^3)-2ac(b^3cg+b^2(cf-g^2)\mu+dg\mu^3)\big)\,,\end{aligned}$$

$$\begin{aligned}\langle\tilde{\psi}^5\rangle = {} & q_1\Delta^{-2}\big(-a^4(cd^2+f^3-2dfg)-c^2\mu^2(3b^2c-2bg\mu+f\mu^2)\\ & +2a^3(bf(2cf-g^2)-(cdf-f^2g+dg^2)\mu)\\ & +a^2(b^2c(-5cf+g^2)+2bg(-cf+g^2)\mu-(2cf^2-4cdg+fg^2)\mu^2)\\ & +2ac(b^3c^2+b^2cg\mu+2b(cf-g^2)\mu^2+(-cd+fg)\mu^3)\big)\,.\end{aligned}$$

Six-point correlation functions in the A/2 theory are given by

$$\begin{aligned}
\langle\psi^6\rangle &= q_2\Delta^{-2}\left((abd - b^2g + bf\mu - d\mu^2)(-2b^3c + d\mu^3 - b\mu(3ad + f\mu)\right.\\
&\left.+b^2(2af + g\mu))\right),
\end{aligned}$$

$$\begin{aligned}
\langle\psi^5\tilde{\psi}\rangle &= q_2\Delta^{-2}\left(-b^5c^2 + ab^4(2cf + g^2) + ad^2\mu^4 - abd\mu^2(3ad + 2f\mu)\right.\\
&- ab^3(a(f^2 + 2dg) + 2(cd + fg)\mu) + ab^2(a^2d^2 + 4adf\mu\\
&\left.+(f^2 + 2dg)\mu^2)\right),
\end{aligned}$$

$$\langle\psi^4\tilde{\psi}^2\rangle = q_2\Delta^{-2}\left((b^2c - abf + ad\mu)(2a^2bd + b^2c\mu + a(-2b^2g + bf\mu - d\mu^2))\right),$$

$$\begin{aligned}
\langle\psi^3\tilde{\psi}^3\rangle &= q_2\Delta^{-2}\left(-a^4bd^2 - a^2b^3(2cf + g^2) - b^3c^2\mu^2 + ab^3c(bc + 2g\mu)\right.\\
&\left.+a^3(b^2(f^2 + 2dg) - 2bdf\mu + d^2\mu^2)\right),
\end{aligned}$$

$$\begin{aligned}
\langle\psi^2\tilde{\psi}^4\rangle &= q_2\Delta^{-2}\left(-(a^2d - abg + bc\mu)(-bc\mu^2 + a^2(-2bf + d\mu)\right.\\
&\left.+ab(2bc + g\mu))\right),
\end{aligned}$$

$$\begin{aligned}
\langle\psi\tilde{\psi}^5\rangle &= q_2\Delta^{-2}\left(a^5d^2 - a^4b(f^2 + 2dg) - bc^2\mu^4 + abc\mu^2(3bc + 2g\mu)\right.\\
&+ a^3b(b(2cf + g^2) + 2(cd + fg)\mu)\\
&\left.-a^2b(b^2c^2 + 4bcg\mu + (2cf + g^2)\mu^2)\right),
\end{aligned}$$

$$\begin{aligned}
\langle\tilde{\psi}^6\rangle &= q_2\Delta^{-2}\left(-(a^2f + c\mu^2 - a(bc + g\mu))(2a^3d - c\mu^3 - a^2(2bg + f\mu)\right.\\
&\left.+a\mu(3bc + g\mu))\right).
\end{aligned}$$

Chapter 4

Toda-like mirrors on del Pezzo surfaces

The work in this chapter is based on results that will be published in [85], and represents our own work. In this chapter, we extend our work to del Pezzo surfaces with degree 2 and 3. A del Pezzo surface is a two complex dimensional manifold with positive first Chern class. There are ten different del Pezzo surfaces. One is $F_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$, the zeroth Hirzebruch surface. The others are $\mathbb{CP}^2$ and $\mathbb{CP}^2$ with up to 8 points blown up. These are denoted dP_n, where n counts the number of blowups of $\mathbb{CP}^2$.

The Hodge number of the del Pezzo surfaces are

$$h^\bullet(\mathbb{CP}^1 \times \mathbb{CP}^1) = \{h^{0,0}, h^{0,1}, h^{0,2}, h^{2,2}\} = \{1, 0, 0, 2\},$$

and

$$h^\bullet(dP_n) = \{h^{0,0}, h^{0,1}, h^{0,2}, h^{2,2}\} = \{1, 0, 0, 1 + n\}.$$

The $h^{1,1} = 2$ Kähler moduli of $\mathbb{CP}^1 \times \mathbb{CP}^1$ correspond to the volumes of the two $\mathbb{CP}^1$ factors, and the $h^{1,1}(dP_n) = n + 1$ Kahler moduli of dP_n correspond to the volumes of $\mathbb{CP}^2$ and n exceptional divisors.

This chapter begins with the construction of dP_2. The del Pezzo dP_1 will be discussed in [85], but is not discussed here, since in this thesis we focus on our own contributions. Since dP_4 and higher surfaces are not toric, they are not considered.

4.1 The del Pezzo dP_2

We give two equivalent descriptions of Toda-dual superpotential to NLSM on dP_2, and generalize both to (0,2) Toda-like mirror models.

4.1.1 Description with no Lagrange multiplier

Review of (2,2)

The first del Pezzo surface we consider, dP_2, is $\mathbb{P}^2$ blown up at two points, which can be described as a toric variety by a fan with edges $(1,0)$, $(0,1)$, $(-1,-1)$, $(0,-1)$, $(-1,0)$. The gauged linear sigma model has five chiral superfields $\phi_i, i = 1\ldots5$ which are charged under the gauge group $U(1)\times U(1)\times U(1)$ as follows:

(1,0)	(-1,-1)	(0,1)	(0,-1)	(-1,0)
1	1	1	0	0
0	0	1	1	0
1	0	0	0	1

The superpotential of the corresponding Toda dual is given by,

$$W = \sum_{i=1}^{5} \exp(-Y_i)$$

with constraints:

$$Y_1 + Y_2 + Y_3 = r_1, \quad Y_3 + Y_4 = r_2, \quad Y_1 + Y_5 = r_3.$$

For convenience, let us define $X_4 = \exp(-Y_4)$ and $X_5 = \exp(-Y_5)$ since the identifications of these two are simply $X_4 \sim \psi_2$ and $X_5 \sim \psi_3$. Then, the superpotential can be rewritten as

$$W = X_4 + X_5 + \frac{q_3}{X_5} + \frac{q_1}{q_2 q_3} X_4 X_5 + \frac{q_2}{X_4}. \tag{4.1}$$

The quantum cohomology relations are:

$$\begin{aligned} q_1 &= (\psi_1+\psi_3)\psi_1(\psi_1+\psi_2), \\ q_2 &= (\psi_1+\psi_2)\psi_2, \\ q_3 &= (\psi_1+\psi_3)\psi_3. \end{aligned}$$

These quantum cohomology relations are satisfied on the space of vacua of the dual theory after identifying $X_4 \sim \psi_2$ and $X_5 \sim \psi_3$.

Construction of (0,2) dual model

The $(0,2)$ deformation of dP_2 is defined by fifteen complex numbers α_i, β_j, γ_k, δ_m, ϵ_n, with $i,j,k,m,n = 1,2,3$, which define a deformation $\mathcal{E}$ of the tangent bundle as follows:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}^3 \xrightarrow{E} \mathcal{O}(1,0,1)\oplus\mathcal{O}(1,0,0)\oplus\mathcal{O}(1,1,0)\oplus\mathcal{O}(0,1,0)\oplus\mathcal{O}(0,0,1) \\ \longrightarrow \mathcal{E} \longrightarrow 0, \end{aligned}$$

where

$$E = \begin{bmatrix} \alpha_1 s_1 & \alpha_2 s_1 & \alpha_3 s_1 \\ \beta_1 s_2 & \beta_2 s_2 & \beta_3 s_2 \\ \gamma_1 s_3 & \gamma_2 s_3 & \gamma_3 s_3 \\ \delta_1 s_4 & \delta_2 s_4 & \delta_3 s_4 \\ \epsilon_1 s_5 & \epsilon_2 s_5 & \epsilon_3 s_5 \end{bmatrix}.$$

$\mathcal{E}$ reduces to the tangent bundle when

$$\begin{aligned} &\alpha_1 = 1, \quad \alpha_2 = 0, \quad \alpha_3 = 1, \\ &\beta_1 = 1, \quad \beta_2 = \beta_3 = 0, \\ &\gamma_1 = \gamma_2 = 1, \quad \gamma_3 = 0, \\ &\delta_1 = 0, \quad \delta_2 = 1, \quad \delta_3 = 0, \\ &\epsilon_1 = \epsilon_2 = 0, \quad \epsilon_3 = 1. \end{aligned}$$

The quantum sheaf cohomology relations are

$$q_1 = Q_{(1)} Q_{(2)} Q_{(3)}, \tag{4.2}$$
$$q_2 = Q_{(3)} Q_{(4)}, \tag{4.3}$$
$$q_3 = Q_{(1)} Q_{(5)}, \tag{4.4}$$

where

$$Q_{(1)} = \sum_{i=1}^{3} \alpha_i \psi_i, \quad Q_{(2)} = \sum_{i=1}^{3} \beta_i \psi_i, \quad Q_{(3)} = \sum_{i=1}^{3} \gamma_i \psi_i,$$
$$Q_{(4)} = \sum_{i=1}^{3} \delta_i \psi_i, \quad Q_{(5)} = \sum_{i=1}^{3} \epsilon_i \psi_i,$$

The superpotential is the following:

$$\begin{aligned} J_1 &= \frac{q_1}{q_2 q_3} Q'_{(4)} Q'_{(5)} - Q'_{(2)} + \frac{Q'_{(3)} Q'_{(4)}}{X_4} - \frac{q_2}{X_4}, \\ J_2 &= \frac{q_1}{q_2 q_3} Q'_{(4)} Q'_{(5)} - Q'_{(2)} + \frac{Q'_{(1)} Q'_{(5)}}{X_5} - \frac{q_3}{X_5}, \\ J_3 &= Q'_{(3)} Q'_{(4)} - q_2, \end{aligned}$$

where

$$
\begin{aligned}
Q'_{(1)} &= \alpha_1 X_1 + \alpha_2 X_4 + \alpha_3 X_5, \\
Q'_{(2)} &= \beta_1 X_1 + \beta_2 X_4 + \beta_3 X_5, \\
Q'_{(3)} &= \gamma_1 X_1 + \gamma_2 X_4 + \gamma_3 X_5, \\
Q'_{(4)} &= \delta_1 X_1 + \delta_2 X_4 + \delta_3 X_5, \\
Q'_{(5)} &= \epsilon_1 X_1 + \epsilon_2 X_4 + \epsilon_3 X_5.
\end{aligned}
$$

The superpotential reduces to (2,2) case when the parameters α, β, $\ldots$, take the right value. One can check that the quantum ring relations (4.2), (4.3), (4.4) are satisfied. The correlation functions are checked on a general ground. It suffices to compute both sides of Eq. (3.16) to show the equality. The results are lengthy:

$$
\begin{aligned}
\text{l.h.s.} = \text{r.h.s} =& X_4^2(\alpha_3(2\beta_1\gamma_2\delta_2\epsilon_2 + \beta_2(\gamma_1\delta_2\epsilon_2 - \gamma_2(2\delta_2\epsilon_1 + \delta_1\epsilon_2)))\\
&+ \alpha_2(2\beta_3(-\gamma_2\delta_2\epsilon_1 + \gamma_2\delta_1\epsilon_2 + \gamma_1\delta_2\epsilon_2)\\
&+ \beta_2(\gamma_3\delta_2\epsilon_1 + 3\gamma_2\delta_3\epsilon_1 + 2\gamma_3\delta_1\epsilon_2 - 2\gamma_1\delta_3\epsilon_2 - 3\gamma_2\delta_1\epsilon_3 - \gamma_1\delta_2\epsilon_3)\\
&- 2\beta_1(\gamma_3\delta_2\epsilon_2 + \gamma_2\delta_3\epsilon_2 - \gamma_2\delta_2\epsilon_3))\\
&+ \alpha_1(-2\beta_3\gamma_2\delta_2\epsilon_2 + \beta_2(-\gamma_3\delta_2\epsilon_2 + \gamma_2\delta_3\epsilon_2 + 2\gamma_2\delta_2\epsilon_3)))\\
&+ X_4X_5(\alpha_1(-\beta_3(3\gamma_3\delta_2 + \gamma_2\delta_3)\epsilon_2 + \beta_2(\gamma_3\delta_2 + 3\gamma_2\delta_3)\epsilon_3)\\
&+ \alpha_3(\beta_3(-4\gamma_2\delta_2\epsilon_1 + \gamma_2\delta_1\epsilon_2 + 3\gamma_1\delta_2\epsilon_2) + 4\beta_1\gamma_2\delta_2\epsilon_3\\
&+ \beta_2(-\gamma_3\delta_2\epsilon_1 + \gamma_2\delta_3\epsilon_1 + \gamma_3\delta_1\epsilon_2 - \gamma_1\delta_3\epsilon_2 - 4\gamma_2\delta_1\epsilon_3))\\
&- \alpha_2(4\beta_1\gamma_3\delta_3\epsilon_2 + \beta_3(\gamma_3\delta_2\epsilon_1 - \gamma_2\delta_3\epsilon_1 - 4\gamma_3\delta_1\epsilon_2 + \gamma_2\delta_1\epsilon_3 - \gamma_1\delta_2\epsilon_3)\\
&+ \beta_2(-4\gamma_3\delta_3\epsilon_1 + \gamma_3\delta_1\epsilon_3 + 3\gamma_1\delta_3\epsilon_3)))\\
&+ X_5^2(\alpha_2(2\beta_3\gamma_3\delta_3\epsilon_1 + \beta_3\gamma_3\delta_1\epsilon_3 - \beta_3\gamma_1\delta_3\epsilon_3 - 2\beta_1\gamma_3\delta_3\epsilon_3)\\
&+ \alpha_1(-2\beta_3\gamma_3\delta_3\epsilon_2 - \beta_3\gamma_3\delta_2\epsilon_3 + \beta_3\gamma_2\delta_3\epsilon_3 + 2\beta_2\gamma_3\delta_3\epsilon_3)\\
&+ \alpha_3(\beta_3(-3\gamma_3\delta_2\epsilon_1 - \gamma_2\delta_3\epsilon_1 + 3\gamma_3\delta_1\epsilon_2 + \gamma_1\delta_3\epsilon_2 - 2\gamma_2\delta_1\epsilon_3 + 2\gamma_1\delta_2\epsilon_3)\\
&+ 2(\beta_2\gamma_3\delta_3\epsilon_1 - \beta_1\gamma_3\delta_3\epsilon_2 - \beta_2\gamma_3\delta_1\epsilon_3 + \beta_1\gamma_3\delta_2\epsilon_3 - \beta_2\gamma_1\delta_3\epsilon_3 + \beta_1\gamma_2\delta_3\epsilon_3)))\\
&+ X_1^2(\alpha_3(-2\beta_2\gamma_1\delta_1\epsilon_1 + \beta_1(-\gamma_2\delta_1\epsilon_1 + \gamma_1\delta_2\epsilon_1 + 2\gamma_1\delta_1\epsilon_2))\\
&+ \alpha_1(-2\beta_3(\gamma_2\delta_1\epsilon_1 + \gamma_1\delta_2\epsilon_1 - \gamma_1\delta_1\epsilon_2) + 2\beta_2(\gamma_3\delta_1\epsilon_1 + \gamma_1\delta_3\epsilon_1 - \gamma_1\delta_1\epsilon_3)\\
&+ \beta_1(-2\gamma_3\delta_2\epsilon_1 + 2\gamma_2\delta_3\epsilon_1 - \gamma_3\delta_1\epsilon_2 - 3\gamma_1\delta_3\epsilon_2 + \gamma_2\delta_1\epsilon_3 + 3\gamma_1\delta_2\epsilon_3))\\
&+ \alpha_2(2\beta_3\gamma_1\delta_1\epsilon_1 + \beta_1(\gamma_3\delta_1\epsilon_1 - \gamma_1(\delta_3\epsilon_1 + 2\delta_1\epsilon_3))))\\
&+ X_1(X_4(\alpha_3(-\beta_2(3\gamma_2\delta_1 + \gamma_1\delta_2)\epsilon_1 + \beta_1(\gamma_2\delta_1 + 3\gamma_1\delta_2)\epsilon_2)\\
&+ \alpha_2(4\beta_3\gamma_1\delta_1\epsilon_2 + \beta_2(3\gamma_3\delta_1\epsilon_1 + \gamma_1\delta_3\epsilon_1 - 4\gamma_1\delta_1\epsilon_3)\\
&+ \beta_1(-\gamma_3\delta_2\epsilon_1 + \gamma_2\delta_3\epsilon_1 - 4\gamma_1\delta_3\epsilon_2 - \gamma_2\delta_1\epsilon_3 + \gamma_1\delta_2\epsilon_3))\\
&- \alpha_1(4\beta_3\gamma_2\delta_2\epsilon_1 + \beta_2(-4\gamma_2\delta_3\epsilon_1 - \gamma_3\delta_1\epsilon_2 + \gamma_1\delta_3\epsilon_2 + \gamma_2\delta_1\epsilon_3 - \gamma_1\delta_2\epsilon_3)\\
&+ \beta_1(3\gamma_3\delta_2\epsilon_2 + \gamma_2\delta_3\epsilon_2 - 4\gamma_2\delta_2\epsilon_3)))\\
&+ X_5(\alpha_2(\beta_3(3\gamma_3\delta_1 + \gamma_1\delta_3)\epsilon_1 - \beta_1(\gamma_3\delta_1 + 3\gamma_1\delta_3)\epsilon_3)\\
&- \alpha_3(\beta_3(3\gamma_2\delta_1\epsilon_1 + \gamma_1\delta_2\epsilon_1 - 4\gamma_1\delta_1\epsilon_2) + 4\beta_2\gamma_1\delta_1\epsilon_3\\
&+ \beta_1(\gamma_3\delta_2\epsilon_1 - \gamma_2\delta_3\epsilon_1 - \gamma_3\delta_1\epsilon_2 + \gamma_1\delta_3\epsilon_2 - 4\gamma_1\delta_2\epsilon_3))\\
&+ \alpha_1(4\beta_2\gamma_3\delta_3\epsilon_1 + \beta_3(-4\gamma_3\delta_2\epsilon_1 + \gamma_3\delta_1\epsilon_2 - \gamma_1\delta_3\epsilon_2 - \gamma_2\delta_1\epsilon_3 + \gamma_1\delta_2\epsilon_3)\\
&+ \beta_1(-4\gamma_3\delta_3\epsilon_2 + \gamma_3\delta_2\epsilon_3 + 3\gamma_2\delta_3\epsilon_3)))),
\end{aligned}
$$

where we have identified $X_1 \sim \psi_1$, $X_4 \sim \psi_2$ and $X_5 \sim \psi_3$ on the right hand side.

4.1.2 Description with Lagrange multiplier

Review of (2,2)

Start again with the superpotential $W = \sum_{i=1}^{5} \exp(-Y_i)$ with three constraints

$$Y_1 + Y_2 + Y_3 = r_1, \quad Y_3 + Y_4 = r_2, \quad Y_1 + Y_5 = r_3.$$

This time instead of using all three constraints and leaving only two fundamental fields, we will only use two constraints and leave three fundamental fields. For the third constraint, we will introduce a Lagrange multiplier, so that the constraint naturally embeds into the superpotential,

$$W = X_1 + X_2 + X_3 + \frac{q_2}{X_3} + \frac{q_3}{X_1} + Z\left(1 - \frac{q_1}{X_1 X_2 X_3}\right), \tag{4.5}$$

where $X_i = \exp(-Y_i)$.

The vacua are given by

$$\begin{aligned} -X_1\partial_1 W &= -X_1 + \frac{q_3}{X_1} - Z\frac{q_1}{X_1X_2X_3} = 0, \\ -X_2\partial_2 W &= -X_2 - Z\frac{q_1}{X_1X_2X_3} = 0, \\ -X_3\partial_3 W &= -X_3 + \frac{q_2}{X_3} - Z\frac{q_1}{X_1X_2X_3} = 0, \\ \partial_Z W &= 1 - \frac{q_1}{X_1X_2X_3} = 0. \end{aligned}$$

One can show that the vacua satisfy the quantum cohomology relations after identifying $X_1 \sim \psi_1 + \psi_3$, $X_2 \sim \psi_1$, and $X_3 \sim \psi_1 + \psi_2$, and all correlation functions will match those of A twisted theory. The advantage of this alternative description is that the number of fundamental fields is the same in the A/2 theory and its mirror (without considering the Lagrange multiplier Z) and the generalizations to the $(0,2)$ mirror will be more natural.

Construction of (0,2) Toda-like dual

As in section (4.1.2), we proposed a new construction of $(2,2)$ Toda mirror by introducing Lagrange multiplier. Here, we will propose a $(0,2)$ Toda-like mirror by generalizing the alternative superpotential (4.5),

$$J_1 = \frac{(\beta \cdot X)}{X_2} - \frac{q_1}{X_2(\alpha \cdot X)(\gamma \cdot X)}, \tag{4.6}$$

$$J_2 = \frac{q_3}{X_1} - \frac{(\alpha \cdot X)(\epsilon \cdot X)}{X_1} - (\beta \cdot X) - \frac{Zq_1}{(\alpha \cdot X)(\beta \cdot X)(\gamma \cdot X)}, \tag{4.7}$$

$$J_3 = \frac{q_2}{X_3} - \frac{(\gamma \cdot X)(\delta \cdot X)}{X_3} - (\beta \cdot X) - \frac{Zq_1}{(\alpha \cdot X)(\beta \cdot X)(\gamma \cdot X)}, \tag{4.8}$$

$$J_4 = -(\beta \cdot X) - \frac{Zq_1}{(\alpha \cdot X)(\beta \cdot X)(\gamma \cdot X)}, \tag{4.9}$$

where

$$(\alpha \cdot X) = \alpha_1 X_1 + \alpha_2 (X_3 - X_2) + \alpha_3 (X_1 - X_2),$$
$$\vdots$$
$$(\epsilon \cdot X) = \epsilon_1 X_1 + \epsilon_2 (X_3 - X_2) + \epsilon_3 (X_1 - X_2),$$

and Z is the Lagrange multiplier.

With the appropriate correspondence,

$$X_1 = \psi_1 + \psi_2,\ X_2 = \psi_1,\ X_3 = \psi_1 + \psi_3,$$

one can check that the quantum sheaf cohomology ring relations are satisfied on the vacua.

Again, we use Eq. (3.16) to show that the correlation functions match:

$$
\begin{aligned}
\text{r.h.s} = \text{l.h.s} =& \psi_1^2(\alpha_3(2\beta_2\gamma_1\delta_1\epsilon_1 + \beta_1(\gamma_2\delta_1\epsilon_1 - \gamma_1(\delta_2\epsilon_1 + 2\delta_1\epsilon_2)))\\
&+ \alpha_2(-2\beta_3\gamma_1\delta_1\epsilon_1 + \beta_1(-\gamma_3\delta_1\epsilon_1 + \gamma_1\delta_3\epsilon_1 + 2\gamma_1\delta_1\epsilon_3))\\
&+ \alpha_1(2\beta_3(\gamma_2\delta_1\epsilon_1 + \gamma_1\delta_2\epsilon_1 - \gamma_1\delta_1\epsilon_2) - 2\beta_2(\gamma_3\delta_1\epsilon_1 + \gamma_1\delta_3\epsilon_1 - \gamma_1\delta_1\epsilon_3)\\
&+ \beta_1(2\gamma_3\delta_2\epsilon_1 - 2\gamma_2\delta_3\epsilon_1 + \gamma_3\delta_1\epsilon_2 + 3\gamma_1\delta_3\epsilon_2 - \gamma_2\delta_1\epsilon_3 - 3\gamma_1\delta_2\epsilon_3)))\\
&+ \psi_2\psi_3(\alpha_1(\beta_3(3\gamma_3\delta_2 + \gamma_2\delta_3)\epsilon_2 - \beta_2(\gamma_3\delta_2 + 3\gamma_2\delta_3)\epsilon_3)\\
&+ \alpha_3(\beta_3(4\gamma_2\delta_2\epsilon_1 - \gamma_2\delta_1\epsilon_2 - 3\gamma_1\delta_2\epsilon_2) - 4\beta_1\gamma_2\delta_2\epsilon_3\\
&+ \beta_2(\gamma_3\delta_2\epsilon_1 - \gamma_2\delta_3\epsilon_1 - \gamma_3\delta_1\epsilon_2 + \gamma_1\delta_3\epsilon_2 + 4\gamma_2\delta_1\epsilon_3))\\
&+ \alpha_2(4\beta_1\gamma_3\delta_3\epsilon_2 + \beta_3(\gamma_3\delta_2\epsilon_1 - \gamma_2\delta_3\epsilon_1 - 4\gamma_3\delta_1\epsilon_2 + \gamma_2\delta_1\epsilon_3 - \gamma_1\delta_2\epsilon_3)\\
&+ \beta_2(-4\gamma_3\delta_3\epsilon_1 + \gamma_3\delta_1\epsilon_3 + 3\gamma_1\delta_3\epsilon_3)))\\
&+ \psi_3^2(\alpha_2(-2\beta_3\gamma_3\delta_3\epsilon_1 - \beta_3\gamma_3\delta_1\epsilon_3 + \beta_3\gamma_1\ \delta_3\epsilon_3 + 2\beta_1\gamma_3\delta_3\epsilon_3)\\
&+ \alpha_1(2\beta_3\gamma_3\delta_3\epsilon_2 + \beta_3\gamma_3\delta_2\epsilon_3 - \beta_3\gamma_2\delta_3\epsilon_3 - 2\beta_2\gamma_3\delta_3\epsilon_3)\\
&+ \alpha_3(\beta_3(3\gamma_3\delta_2\epsilon_1 + \gamma_2\delta_3\epsilon_1 - 3\gamma_3\delta_1\epsilon_2 - \gamma_1\delta_3\ \epsilon_2 + 2\gamma_2\delta_1\epsilon_3 - 2\gamma_1\delta_2\epsilon_3)\\
&+ 2(-\beta_2\gamma_3\delta_3\epsilon_1 + \beta_1\gamma_3\delta_3\epsilon_2 + \beta_2\gamma_3\delta_1\epsilon_3 - \beta_1\gamma_3\delta_2\epsilon_3 + \beta_2\gamma_1\delta_3\epsilon_3 - \beta_1\gamma_2\delta_3\ \epsilon_3)))\\
&+ \psi_2^2(\alpha_3(-2\beta_1\gamma_2\delta_2\epsilon_2 + \beta_2(2\gamma_2\delta_2\epsilon_1 + \gamma_2\delta_1\epsilon_2 - \gamma_1\delta_2\epsilon_2))\\
&+ \alpha_2(2\beta_3(\gamma_2\delta_2\epsilon_1 - \gamma_2\delta_1\epsilon_2 - \gamma_1\delta_2\epsilon_2)\\
&+ \beta_2(-\gamma_3\delta_2\epsilon_1 - 3\gamma_2\delta_3\epsilon_1 - 2\gamma_3\delta_1\epsilon_2 + 2\gamma_1\delta_3\epsilon_2 + 3\gamma_2\delta_1\epsilon_3 + \gamma_1\delta_2\epsilon_3)\\
&+ 2\beta_1(\gamma_3\delta_2\epsilon_2 + \gamma_2\delta_3\epsilon_2 - \gamma_2\delta_2\epsilon_3))\\
&+ \alpha_1(2\beta_3\gamma_2\delta_2\epsilon_2 + \beta_2(\gamma_3\delta_2\epsilon_2 - \gamma_2(\delta_3\epsilon_2 + 2\delta_2\epsilon_3))))\\
&+ \psi_1(\psi 2(\alpha_3(\beta_2(3\gamma_2\delta_1 + \gamma_1\delta_2)\epsilon_1 - \beta_1(\gamma_2\delta_1 + 3\gamma_1\delta_2)\epsilon_2)\\
&- \alpha_2(4\beta_3\gamma_1\delta_1\epsilon_2 + \beta_2(3\gamma_3\delta_1\epsilon_1 + \gamma_1\delta_3\epsilon_1 - 4\gamma_1\delta_1\epsilon_3)\\
&+ \beta_1(-\gamma_3\delta_2\epsilon_1 + \gamma_2\delta_3\epsilon_1 - 4\gamma_1\delta_3\epsilon_2 - \gamma_2\delta_1\epsilon_3 + \gamma_1\delta_2\epsilon_3))\\
&+ \alpha_1(4\beta_3\gamma_2\delta_2\epsilon_1 + \beta_2(-4\gamma_2\delta_3\epsilon_1 - \gamma_3\delta_1\epsilon_2 + \gamma_1\delta_3\epsilon_2 + \gamma_2\delta_1\epsilon_3 - \gamma_1\delta_2\epsilon_3)\\
&+ \beta_1(3\gamma_3\delta_2\epsilon_2 + \gamma_2\delta_3\epsilon_2 - 4\gamma_2\delta_2\epsilon_3)))\\
&+ \psi_3(\alpha_2(-\beta_3(3\gamma_3\delta_1 + \gamma_1\delta_3)\epsilon_1 + \beta_1(\gamma_3\delta_1 + 3\gamma_1\delta_3)\epsilon_3)\\
&+ \alpha_3(\beta_3(3\gamma_2\delta_1\epsilon_1 + \gamma_1\delta_2\epsilon_1 - 4\gamma_1\delta_1\epsilon_2) + 4\beta_2\gamma_1\delta_1\epsilon_3\\
&+ \beta_1(\gamma_3\delta_2\epsilon_1 - \gamma_2\delta_3\epsilon_1 - \gamma_3\delta_1\epsilon_2 + \gamma_1\delta_3\epsilon_2 - 4\gamma_1\delta_2\epsilon_3))\\
&+ \alpha_1(-4\beta_2\gamma_3\delta_3\epsilon_1 + \beta_3(4\gamma_3\delta_2\epsilon_1 - \gamma_3\delta_1\epsilon_2 + \gamma_1\delta_3\epsilon_2 + \gamma_2\delta_1\epsilon_3 - \gamma_1\delta_2\epsilon_3)\\
&+ \beta_1(4\gamma_3\delta_3\epsilon_2 - \gamma_3\delta_2\epsilon_3 - 3\gamma_2\delta_3\epsilon_3)))).
\end{aligned}
$$

On the left hand side, we identify

$$X_1 = \psi_1 + \psi_2,\ X_2 = \psi_1,\ X_3 = \psi_1 + \psi_3$$

to get the result.

4.2 The del Pezzo dP_3

4.2.1 Review of $(2,2)$

Next we consider the toric del Pezzo dP_3. It can be described by a fan with edges

$$(1,0),\ (0,1),\ (-1,-1),\ (1,1),\ (-1,0),\ (0,-1).$$

The corresponding GLSM then can be constructed by six chiral superfields $\phi_i, i = 1, \ldots, 6$ which are under charged by gauge group $U(1) \times U(1) \times U(1) \times U(1)$ as follow,

(1,0)	(-1,-1)	(0,1)	(0,-1)	(-1,0)	(1, 1)
1	1	1	0	0	0
0	0	1	1	0	0
1	0	0	0	1	0
0	1	0	0	0	1

The superpotential of the corresponding Toda dual is given by

$$W = \sum_{i=1}^{6} \exp(-Y_i)$$

with constraints:

$$Y_1 + Y_2 + Y_3 = r_1, \quad Y_3 + Y_4 = r_2, \quad Y_1 + Y_5 = r_3, \quad Y_2 + Y_6 = r_4.$$

Then, the superpotential can be rewritten as

$$W = X_4 + X_5 + \frac{q_2}{X_4} + \frac{q_3}{X_5} + \frac{q_1}{q_2 q_3} X_4 X_5 + \frac{q_2 q_3 q_4}{q_1} \frac{1}{X_4 X_5}, \tag{4.10}$$

where $X_i = \exp(-Y_i)$.

The quantum cohomology- relations are

$$\begin{aligned} q_1 &= (\psi_1 + \psi_3)(\psi_1 + \psi_4)(\psi_1 + \psi_2), \\ q_2 &= (\psi_1 + \psi_2)\psi_2, \\ q_3 &= (\psi_1 + \psi_3)\psi_3, \\ q_4 &= (\psi_1 + \psi_4)\psi_4. \end{aligned}$$

One can check that these quantum cohomology ring relations are satisfied on the space of vacua of the Toda dual theory after identifying $X_4 \sim \psi_2$ and $X_5 \sim \psi_3$.

4.2.2 $(0,2)$ deformations

To describe the $(0,2)$ deformation of dP_3, we will need 24 complex parameters

$$\alpha_i, \beta_j, \gamma_k, \delta_l, \epsilon_m, \zeta_n, i, j, k, l, m, n = 1, \ldots, 4.$$

$$\begin{aligned} 0 \longrightarrow \mathcal{O}^3 \xrightarrow{E} \mathcal{O}(1,0,1,0) \oplus & \mathcal{O}(1,0,0,1) \oplus \mathcal{O}(1,1,0,0) \oplus \mathcal{O}(0,1,0,0) \\ & \oplus \mathcal{O}(0,0,1,0) \oplus \mathcal{O}(0,0,0,1) \longrightarrow \mathcal{E} \longrightarrow 0, \end{aligned}$$

with E defined by:

$$E = \begin{bmatrix} \alpha_1 s_1 & \alpha_2 s_2 & \alpha_3 s_3 & \alpha_4 s_4 \\ \beta_1 s_1 & \beta_2 s_2 & \beta_3 s_3 & \beta_4 s_4 \\ \gamma_1 s_1 & \gamma_2 s_2 & \gamma_3 s_3 & \gamma_4 s_4 \\ \delta_1 s_1 & \delta_2 s_2 & \delta_3 s_3 & \delta_4 s_4 \\ \epsilon_1 s_1 & \epsilon_2 s_2 & \epsilon_3 s_3 & \epsilon_4 s_4 \\ \zeta_1 s_2 & \zeta_2 s_2 & \zeta_3 s_3 & \zeta_4 s_4 \end{bmatrix}.$$

$\mathcal{E}$ reduces to the tangent bundle and thus the corresponding theory on $(2,2)$ locus when we take

$$\begin{aligned} &\alpha_1 = 1, \alpha_2 = 0, \alpha_3 = 1, \alpha_4 = 0, \\ &\beta_1 = 1, \beta_2 = \beta_3 = 0, \beta_4 = 1, \\ &\gamma_1 = \gamma_2 = 1, \gamma_3 = \gamma_4 = 0, \\ &\delta_1 = 0, \delta_2 = 1, \delta_3 = \delta_4 = 0, \\ &\epsilon_1 = \epsilon_2 = 0, \epsilon_3 = 1, \epsilon_4 = 0, \\ &\zeta_1 = \zeta_2 = \zeta_3 = 0, \zeta_4 = 1. \end{aligned}$$

If we define:

$$Q_{(1)} = \sum_{i=1}^{4} \alpha_i \psi_i, \quad Q_{(2)} = \sum_{i=1}^{4} \beta_i \psi_i, \quad Q_{(3)} = \sum_{i=1}^{4} \gamma_i \psi_i,$$
$$Q_{(4)} = \sum_{i=1}^{4} \delta_i \psi_i, \quad Q_{(5)} = \sum_{i=1}^{4} \epsilon_i \psi_i, \quad Q_{(6)} = \sum_{i=1}^{4} \zeta_i \psi_i,$$

Then the quantum sheaf cohomology ring relations are

$$\begin{aligned} q_1 &= Q_{(1)} Q_{(2)} Q_{(3)}, \\ q_2 &= Q_{(3)} Q_{(4)}, \\ q_3 &= Q_{(1)} Q_{(5)}, \\ q_4 &= Q_{(2)} Q_{(6)}, \end{aligned}$$

which reduce to quantum cohomology ring relations when take the above special E deformation.

Note there are correspondence relations between ψ's and X's on the space of vacua:

$$X_3 \sim \psi_1 + \psi_2, \tag{4.11}$$
$$X_4 \sim \psi_2, \tag{4.12}$$
$$X_5 \sim \psi_3, \tag{4.13}$$
$$X_6 \sim \psi_4, \tag{4.14}$$

so we define Q''s in the same fashion as Q's:

$$\begin{aligned}
Q'_{(1)} &= \alpha_1(X_3 - X_4) + \alpha_2 X_4 + \alpha_3 X_5 + \alpha_4 X_6, \\
Q'_{(2)} &= \beta_1(X_3 - X_4) + \beta_2 X_4 + \beta_3 X_5 + \beta_4 X_6, \\
Q'_{(3)} &= \gamma_1(X_3 - X_4) + \gamma_2 X_4 + \gamma_3 X_5 + \gamma_4 X_6, \\
Q'_{(4)} &= \delta_1(X_3 - X_4) + \delta_2 X_4 + \delta_3 X_5 + \delta_4 X_6, \\
Q'_{(5)} &= \epsilon_1(X_3 - X_4) + \epsilon_2 X_4 + \epsilon_3 X_5 + \epsilon_4 X_6, \\
Q'_{(6)} &= \zeta_1(X_3 - X_4) + \zeta_2 X_4 + \zeta_3 X_5 + \zeta_4 X_6.
\end{aligned}$$

The proposal to the dual to three blow-ups are following:

$$\begin{aligned}
J_1 &= -\frac{q_1}{q_2 q_3} Q'_{(4)} Q'_{(5)} + \frac{q_2 q_3 q_4}{q_1} \frac{1}{Q'_{(4)} Q'_{(5)}} + \frac{q_2}{X_4} - \frac{Q'_{(3)} Q'_{(4)} - Q'_{(2)} + Q'_{(6)}}{X_4}, \\
J_2 &= -\frac{q_1}{q_2 q_3} Q'_{(4)} Q'_{(5)} + \frac{q_2 q_3 q_4}{q_1} \frac{1}{Q'_{(4)} Q'_{(5)}} + \frac{q_3}{X_5} - \frac{Q'_{(1)} Q'_{(5)} - Q'_{(2)} + Q'_{(6)}}{X_5}, \\
J_3 &= \frac{q_3 - Q'_{(1)} Q'_{(5)}}{X_3 X_5}, \\
J_4 &= \frac{q_4 - Q'_{(2)} Q'_{(6)}}{X_6 (Q'_{(2)} + Q'_{(6)})}.
\end{aligned}$$

The ring relations match those of the A model. We also compared some (though not all) of the correlation functions, to verify that normalizations also match. Note that on the first column, those parameters are on (2,2) locus.

Table 4.1: Cases with correlation functions checked

α_1	1	1	1	1	1	1	1
α_2	0	1	0	0	0	0	0
α_3	1	1	1	1	1	1	1
α_4	0	1	0	0	0	0	0
β_1	1	1	1	1	1	1	1
β_2	0	0	1	0	0	0	0
β_3	0	0	1	0	0	0	0
β_4	1	1	1	1	1	1	1
γ_1	1	1	1	1	1	1	1
γ_2	1	1	1	1	1	1	1
γ_3	0	0	0	1	0	0	0
γ_4	0	0	0	1	0	0	0
δ_1	0	0	0	0	1	0	0
δ_2	1	1	1	1	1	1	1
δ_3	0	0	0	0	1	0	0
δ_4	0	0	0	0	1	0	0
ϵ_1	0	0	0	0	0	1	0
ϵ_2	0	0	0	0	0	1	0
ϵ_3	1	1	1	1	1	1	1
ϵ_4	0	0	0	0	0	1	0
ζ_1	0	0	0	0	0	0	1
ζ_2	0	0	0	0	0	0	1
ζ_3	0	0	0	0	0	0	1
ζ_4	1	1	1	1	1	1	1

Chapter 5

Conclusion

In this thesis, we discussed various models with (2,2) supersymmetry and their generalizations with (0,2) supersymmetry. We started with a brief introduction to supersymmetry in two dimensions. This includes (2,2) supersymmetry and its generalization (0,2) supersymmetry, and GLSMs with those supersymmetries. In order to test some dualities, we constructed anomaly-free examples of (0,2) GLSMs for ordinary Grassmannians, hypersurfaces in Grassmannians, complete intersections in Grassmannians, and Pfaffian varieties.

Then we moved to mirror symmetry. We reviewed (2,2) mirror symmetry and (0,2) mirror symmetry of which the latter is not well understood. As an attempt to build examples of (0,2) mirrors in special cases, we described the example of the Toda-like dual to the A/2 NLSM on $\mathbb{P}^1 \times \mathbb{P}^1$, namely a B/2 Landau-Ginzburg model with appropriate superpotential. We then constructed Toda-like mirrors to NLSMs on $\mathbb{P}^n \times \mathbb{P}^m$. We also constructed Toda-like mirrors on the blow-ups of del Pezzo surface, dP_2 and dP_3. In those cases, we use the method of Lagrange multipliers.

So far, some progress has been made on two-dimensional $\mathcal{N} = (0,2)$ theories, but there is much more that needs to be done. For example, are there any non-complete intersections other than Pfaffians which can be realized as GLSMs? If so, what are (0,2) generalizations? Can we find another way other than LG models to construct Toda-like mirror to the product of projective spaces? Is there a general way to build mirror model dual to toric varieties? These interesting questions are left to future study.

Appendix A

Representations of $U(k)$

This section was previously published in [90]. The representation theory of $SU(k)$ is certainly well-known; however, representations of $U(k)$ can be more complicated, because of the possibility of tensoring in powers of the determinant. In this appendix, we give our conventions for describing representations of $U(k)$.

Any irreducible unitary representation of $U(k)$ is given by a k-tuple of ordered integers [41][sections 19-22]

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k), \ \ \lambda_i \in \mathbb{Z}, \ \ \forall i. \tag{A.1}$$

This is the highest weight of the corresponding representation. For completeness, here are a few examples [41][sections 19-22]:

- The defining fundamental representation of $U(k)$ has highest weight $(1, 0, \cdots, 0)$, while its conjugate, the antifundamental representation, has highest weight $(0, \cdots, 0, -1)$.
- The exterior product representation on $\wedge^\ell \mathbb{C}^k$ has highest weight $(1, 1, \cdots, 1, 0, 0, \cdots, 0)$ (ℓ 1's). In particular, the determinant representation has highest weight $(1, 1, \cdots, 1)$.
- The adjoint representation of $U(k)$ is reducible: $ad = (1, 0, \cdots, 0) \otimes (0, 0, \cdots, -1) = (1, 0, \cdots, 0, -1) \oplus (0, 0, \cdots, 0)$.

Below are some frequently used formulas for $U(k)$ representations [42][chapter 5]:

- The dimension of λ is given by [42][equ'n (4.56)]

$$d_\lambda = \prod_{i<j} \frac{l_i - l_j}{l_i^0 - l_j^0} = \prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j - i}, \tag{A.2}$$

  where $l_i^0 = k - i$, and $l_i = \lambda_i + k - i$, with $i, j = 1, \cdots, k$.

- The eigenvalue of the first Casimir operator on λ is [42][equ'n (5.24), table 5.1]

$$\mathrm{Cas}_1(\lambda) = \sum_i \lambda_i, \tag{A.3}$$

- The eigenvalue of the second Casimir operator on λ is [42][equ'n (5.24), table 5.1]

$$\mathrm{Cas}_2(\lambda) = \sum_i \lambda_i(\lambda_i + k + 1 - 2i). \tag{A.4}$$

In terms of bundles on $G(k,n)$ of the form $\mathcal{O}(\lambda)$ for some representation λ, it is straightforward to show that

$$c_1(\mathcal{O}(\lambda)) \;=\; \frac{d_\lambda \mathrm{Cas}_1(\lambda)}{k} \sigma_{\Box}, \tag{A.5}$$

where $\sigma_{\Box}$ denotes the Schubert cycle generating $H^2(G(k,n),\mathbb{Z})$, which is one-dimensional, and

$$\begin{aligned} \mathrm{ch}_2(\mathcal{O}(\lambda)) &= (1/2)c_1(\mathcal{O}(\lambda))^2 \;-\; c_2(\mathcal{O}(\lambda)), \\ &= d_\lambda \mathrm{Cas}_2(\lambda) \left[-\frac{1}{k^2-1}\sigma_{\substack{\Box\\\Box}} \;+\; \frac{1}{2k(k+1)}\sigma_{\Box}^2 \right] \\ &\qquad + d_\lambda \mathrm{Cas}_1(\lambda)^2 \left[\frac{1}{k(k^2-1)}\sigma_{\substack{\Box\\\Box}} \;+\; \frac{1}{2k(k+1)}\sigma_{\Box}^2 \right], \end{aligned} \tag{A.6}$$

where $\sigma_{\substack{\Box\\\Box}}$ and $\sigma_{\Box\Box}$ generate

$$H^4(G(k,n),\mathbb{Z}) \;=\; \mathbb{Z}^2$$

and

$$\sigma_{\Box}^2 \;=\; \sigma_{\substack{\Box\\\Box}} \;+\; \sigma_{\Box\Box}.$$

As a consistency check, recall that the bundle $\wedge^p S^* \to G(k,n)$ has rank

$$\left(\begin{array}{c} k \\ p \end{array} \right)$$

and

$$c_1\left(\wedge^p S^*\right) \;=\; \left(\begin{array}{c} k-1 \\ p-1 \end{array} \right) \sigma_{\Box},$$

and these are both consistent with the formulas above for the representation

$$(1,1,\cdots,1,0,\cdots,0)$$

(p 1's) of $U(k)$, which defines the bundle $\wedge^p S^*$. We list here results for a few other cases, which can also be used to check the general formulas above. For $p=1$ [43], [44][prop. 3.5.5],

$$c_2(S^*) \;=\; \sigma_{\substack{\Box\\\Box}}, \quad \mathrm{ch}_2(S^*) \;=\; (1/2)\sigma_{\Box}^2 \;-\; \sigma_{\substack{\Box\\\Box}},$$

and in fact $c_i(S^*)$ is given by the Schubert cycle associated to the Young diagram with i vertical boxes. In the special case $p = 2$,

$$c_2(\wedge^2 S^*) \;=\; \begin{pmatrix} k-1 \\ 2 \end{pmatrix} \sigma_{\square}^2 \;+\; (k-2)\sigma_{\substack{\square\\\square}},$$

which one can use to show that for the representation $(2, 0, \cdots, 0)$,

$$\mathrm{rk}\,\mathrm{Sym}^2 S^* \;=\; \frac{k(k+1)}{2}, \quad c_1(\mathrm{Sym}^2 S^*) \;=\; (k+1)\sigma_{\square},$$

$$\mathrm{ch}_2(\mathrm{Sym}^2 S^*) \;=\; \frac{k+3}{2}\sigma_{\square}^2 \;-\; (k+2)\sigma_{\substack{\square\\\square}},$$

and similarly

$$c_2(\wedge^3 S^*) \;=\; \frac{k(k-1)(k-2)(k-3)}{8}\sigma_{\square}^2 \;+\; \frac{(k-2)(k-3)}{2}\sigma_{\substack{\square\\\square}},$$

$$\mathrm{ch}_2(\wedge^3 S^*) \;=\; \frac{(k-1)(k-2)}{4}\sigma_{\square}^2 \;-\; \frac{(k-2)(k-3)}{2}\sigma_{\substack{\square\\\square}}.$$

At the level of Lie algebras, $u(k) \cong su(k) \oplus u(1)$. Therefore, given a representation λ of $u(k)$, we can get an irreducible representation of $su(k) \oplus u(1)$: the representation of $su(k)$ is given by the Young diagram $(\lambda_1 - \lambda_k \geq \lambda_2 - \lambda_k \geq \cdots \geq 0)$, and the representation of $u(1)$ is given by the integer $\mathrm{Cas}_1(\lambda)$.

For completeness, the eigenvalue of an $su(k)$ second Casimir operator on the $su(k)$ representation $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0)$ is given by [42][equ'n (5.24), table 5.1]

$$\mathrm{Cas}_2(\lambda) = \sum_i \left(\lambda_i - \frac{\sum_i \lambda_i}{k} \right) \left(\lambda_i - \frac{\sum_i \lambda_i}{k} + 2k - 2i \right). \tag{A.7}$$

For example:

$$\begin{aligned} \mathrm{Cas}_2(ad) &= \quad 2k, \\ \mathrm{Cas}_2(1, 0, \cdots, 0) &= \quad (k^2-1)/k. \end{aligned}$$

As a consistency check, [45][equ'n (2.18)] lists an index for $su(2)$ representations defined by Young diagrams with n boxes:

$$I_2(n) \;=\; \frac{1}{6}n(n+1)(n+2),$$

where

$$\mathrm{Tr}\left(T_R^a T_R^b\right) \;=\; I_2(R)\delta^{ab}.$$

It is straightforward to check that

$$I_2(n) \;=\; \frac{d_{(n,0)}\mathrm{Cas}_2(n,0)}{\dim su(2)},$$

where $d_{(n,0)} = n+1$ and

$$\mathrm{Cas}_2(n,0) \;=\; (n/2)(n/2+4-2) + (-n/2)(-n/2+4-4) \;=\; (1/2)n(n+2).$$

Appendix B

Some algebra on Pfaffians

Lemma 1. *(We will assume that $A(\phi)$ is an $n \times n$ matrix in this appendix.) Let x be an $n \times k_d$ matrix with rank $k_d = n - k$, then if $x \in \ker A(\phi)$, i.e., $A(\phi)x = 0$ then* $\operatorname{rank} A(\phi) \leq k$.

Proof. The rank of x is k_d means that there are k_d linearly independent row vectors of x. This implies the dimension of $\ker(A)$ is greater or equal to k_d. $A(\phi) : F^n \to F^n$, where F^n is a n dimensional vector space, then

$$\begin{aligned} \operatorname{rank}(A) &= n - \dim(\ker A), \\ &\leq n - k_d, \\ &= k. \end{aligned}$$

□

Theorem 1. *If*

$$pp^\dagger - x^\dagger x = r_{U(k_d)} \mathbb{1}_{k_d},$$

with $r_{U(k_d)} < 0$ then $\operatorname{rank}(x) = k_d$.

Proof. Since $pp^\dagger$ is hermitean, we can diagonalise it by a unitary matrix U. We have

$$Upp^\dagger U^\dagger - r\mathbb{1} = Ux^\dagger x U^\dagger.$$

$$\Rightarrow \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{k_d} \end{pmatrix} - r \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} = Ux^\dagger x U^\dagger.$$

$$\Rightarrow \begin{pmatrix} a_1 - r & & \\ & \ddots & \\ & & a_{k_d} - r \end{pmatrix} = Ux^\dagger x U^\dagger.$$

$$\Rightarrow \det(Ux^\dagger x U^\dagger) = \det(U)\det(x^\dagger x)\det(U^\dagger) > 0.$$

$$\Rightarrow \det(x^\dagger x) > 0 \text{ (U is unitary.)}$$

So we have

$$\det(x^\dagger x) \neq 0,$$

i.e., $x^\dagger x$ is full rank. We can treat $x^\dagger x$ and x as linear maps:

$$x^\dagger x : V \to W,$$

with $\dim(V) = \dim(W) = k_d$;

$$x : V \to U,$$

with $\dim(U) = n$. So

$$\dim(\ker(x^\dagger x)) = k_d - \text{rank}(x^\dagger x) = 0,$$

i.e.,

$$\ker(x^\dagger x) = 0.$$

For a vector v in a k_d dimensional vector space, we have

$$xv = 0 \Rightarrow x^\dagger x v = 0,$$

i.e., $v \in \ker(x)$, which implies $v \in \ker(x^\dagger x)$, so

$$\ker(x) \subseteq \ker(x^\dagger x) = 0.$$

Hence,

$$\text{rank}(x) = k_d - \dim(\ker(x)) = k_d.$$

□

Theorem 2. *If*

$$A(\phi)x = 0, \qquad pA(\phi) = 0, \qquad \text{tr}(p\partial_a A(\phi)x) = 0$$

with x being $n \times k_d$ matrix, p being $k_d \times n$ matrix, then at the smooth points, $p = 0$.

Proof. Define $X_A = \{(\phi, x) \in V_{\epsilon,k_d} \mid A(\phi)x = 0\}$, where V_{ϵ,k_d} is a Grassmanian $G(k_d, n)$ fibered over the vector space V with coordinate ϕ. The codimension of X_A in V is $(n-k)^2$. V_{ϵ,k_d} locally looks like $V \times G(k_d, n)$. So the codimension of X_A in V_{ϵ,k_d} is that:

$$\begin{aligned}\text{codim}(X_A) \text{ in } V_{\epsilon,k_d} &= \text{codim}(X_A) \text{ in } V + \dim(G(k_d, n)),\\ &= (n-k)^2 + k_d(n-k_d),\\ &= n(n-k).\end{aligned}$$

Let $E_{i\alpha} = A_{ij}(\phi)x_{j\alpha}$, $i = 1, ..., n$, $\alpha = 1, ..., n-k$. Then $\{E = 0\}$ is the defining function of X_A. So the normal bundle of X_A is spanned by

$$\nabla E_{i\alpha}|_{X_A} = \partial_{(\phi,x)} E_{i\alpha}|_{X_A},$$

which have total $nk_d = n(n-k)$ vectors. The smoothness is defined by the rank of normal bundle being equal to codimension: it tells us that $\{\partial_{(\phi,x)} E_{i\alpha}|_{X_A}\}$ are linearly independent vectors. We can rewrite the condition

$$pA(\phi) = 0, \qquad \text{tr}(p\partial_a A(\phi)x) = 0$$

as

$$p_{\alpha i}\frac{\partial E_{i\alpha}}{\partial(\phi_a, x_{j\beta})} = 0.$$

As $\frac{\partial E_{i\alpha}}{\partial(\phi_a, x_{j\beta})}$ are linearly independent, we have that:

$$p_{\alpha i} = 0 \ (\alpha = 1, ..., n-k, i = 1, ..., n).$$

□

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