Difference Raising Operators for Kirillov-Reshetikhin Characters and Parabolic Jing Operators

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(ABSTRACT)

In this paper, we use the techniques of plethystic substitution to reformulate the difference raising operators presented by Di Francesco and Kedem in [1]. A connection between these operators and Shimozono and Zabrocki's parabolic Jing operators is presented. In particular, we find that these operators are a renormalization of a particular case of the parabolic Jing operators.

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(GENERAL AUDIENCE ABSTRACT)

In response to an open problem in Physics, an idea is presented by Di Francesco and Kedem in [1]. A connection between this idea and a Math idea presented by Shimozono and Zabrocki in [9] is presented. It is common that unknown overlap exists when authors from different fields work on similar problems. This connection is seen once the techniques used by Di Francesco and Kedem are interpreted in the language used by Shimozono and Zabrocki. In particular, we find that the idea in [1] is a specialization of that in [9].

Contents

Chapter 1

Introduction

In [1], the authors consider the tensor product of $\mathfrak{sl}_n$-modules

$$\mathcal{M}_{\mathbf{n}} = \bigotimes_{1\le\alpha\le n-1}\bigotimes_{1\le l\le k} V(l\omega_\alpha)^{\otimes n_l^{(\alpha)}},$$

where $\mathbf{n} = \{n_l^{(\alpha)} : 1 \le l \le k, \alpha \in [1, n-1]\}$ is a set of nonnegative integers, the ω_α are the fundamental weights, and $V(\lambda)$ is the irreducible $\mathfrak{sl}_n$-module with highest weight λ. It is shown in [2] that such modules, which are tensor products of so called Kirillov-Reshetikhin modules, can be given a $\mathfrak{sl}_n[t]$-module structure as well as a $\mathfrak{sl}_n$-equivariant grading. The graded components $\mathcal{M}_{\mathbf{n}}[j]$ are $\mathfrak{sl}_n$-modules themselves, and so one can define graded characters of $\mathcal{M}_{\mathbf{n}}$ by

$$\chi_{\mathbf{n}}(q, z_1, \dots, z_n) = \sum_{j\ge 0} q^j \mathrm{ch}_{z_1,\dots,z_n}\mathcal{M}_{\mathbf{n}}[j],$$

where $\mathrm{ch}_{z_1,\dots,z_n}$ is the honest character of $\mathcal{M}_{\mathbf{n}}[j]$ as an $\mathfrak{sl}_n$-module. Theorem 5.7 of [1] gives an expression for these graded characters via the iterated action of q-difference operators on the constant function 1. These q-difference operators are raising operators for symmetric functions, i.e. they increase the total degree of the input, and are closely related to Macdonald operators.

In [3], Jing introduces a family of vertex operators acting on symmetric functions, which he uses to give a construction of the same kind as above for Hall-Littlewood polynomials. The authors in [9] generalize these vertex operators and give their generalization as plethystic type formulas.

In this paper, we establish a connection between the q-difference operators of [1] and the vertex operators of [9] by rendering the work of [1] as plethystic type formulas. We begin with an overview of symmetric functions, plethysm, and some symmetric difference operators. We conclude with our plethystic calculation and some discussion of the relationship between the two operators. Our main result regarding this relationship can be found at the beginning of Chapter 5.

Chapter 2

Symmetric Functions

Throughout this chapter, we closely follow [8].

2.1 The Ring of Symmetric Functions

We begin with an introduction to symmetric functions.

Take $\{z_i : i \in \mathbb{Z}^+\}$ to be a set of algebraically independent indeterminates. We can define an action of the symmetric group on n elements, S_n, on the polynomial ring $\mathbb{Z}[z_1, \ldots, z_n]$ by

$$w(f(z_1, \ldots, z_n)) = f(z_{w(1)}, \ldots, z_{w(n)}), \quad w \in S_n, f \in \mathbb{Z}[z_1, \ldots, z_n].$$

A polynomial $f \in \mathbb{Z}[z_1, \ldots, z_n]$ is called a symmetric polynomial if $w(f) = f$ for all $\omega \in S_n$. The set of such polynomials form a subring of $\mathbb{Z}[z_1, \ldots, z_n]$. We will denote this subring by

$$\Lambda_n = \mathbb{Z}[z_1, \ldots, z_n]^{S_n}.$$

If f is a monomial, i.e. it is an integer multiple of the product of indeterminates, we define the degree of f to be the sum of the powers of those indeterminents. We say that a polynomial in $\mathbb{Z}[z_1, \ldots, z_n]$ is homogeneous of degree k if it is the sum of degree k monomials. For example, the monomial $z_1z_2z_3 \in \Lambda_3$ has degree 3 and $(z_1z_2 + z_2z_3 + z_1z_3) \in \Lambda_3$ is homogeneous of degree 2.

Since each $f \in \Lambda_n$ can be written as the sum of homogeneous symmetric polynomials, the degrees form a grading on Λ_n, i.e.

$$\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k,$$

where Λ_n^k is the subgroup of Λ_n consisting of polynomials which are homogeneous of degree k. We take as a convention that 0 is homogeneous of every degree.

We can define a surjective homomorphism of graded rings $\Lambda_{n+1} \to \Lambda_n$ by setting $z_{n+1} = 0$. Note that the induced map $\Lambda^k_{n+1} \to \Lambda^k_n$ is surjective for all $k \geq 0$ and bijective if and only if $k \leq n$.

The above is enough to establish that we may consider, for each $k \geq 0$, the inverse limit

$$\Lambda^k = \varprojlim_n \Lambda^k_n.$$

We do this so that we may pass to the much more convenient setting of

$$\Lambda = \bigoplus_{k \geq 0} \Lambda^k,$$

the ring of symmetric functions in $\{z_i : i \in \mathbb{Z}^+\}$ with coefficients in $\mathbb{Z}$. The convenience here is that, given an $f_n \in \Lambda_n$, we can always find a $f \in \Lambda$ such that f_n is equal to f after setting all $\{z_{n+1}, z_{n+2}, \dots\}$ equal to 0. In this way, Λ allows us to work with all of the symmetric polynomials at once. With this in mind, we define the notation $f(z_1, \dots, z_n) = f_n$.

We can upgrade Λ to have scalars in any commutative ring R simply by considering $\Lambda \otimes_{\mathbb{Z}} R$.

It is well known that there are many useful bases for Λ. It turns out to be convenient to index these bases by sequences of integers known as partitions.

2.2 Partitions

In this section, we will review the definition of a partition, as well as establish some notation for describing them.

A partition is a sequence of integers

$$\lambda = (\lambda_1, \lambda_2, \dots)$$

such that $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$ and

$$|\lambda| := \sum_{i=1}^{\infty} \lambda_i < \infty.$$

This second criterion says that we may always consider λ to be a finite string by regarding two sequences as the same if they only differ by some amount of zeros to the right. For example, we regard $(1), (1, 0)$, and $(1, 0, \dots)$ as the same partition.

Each nonzero λ_i is called a part of λ, and the length of λ, $l(\lambda)$, is defined to be the number of parts. We will let $\mathbb{Y}$ denote the set of all partitions, and $\mathbb{Y}_\alpha$ denote the set of partitions with length at most α. Denote $(k, \dots, k)$, the partition of length α, by (k^α).

2.3 Schur Functions

In this section, we will review the construction of Schur functions.

Given any string $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers, let $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. Define

$$a_\alpha = \det(z_i^{\alpha_j})_{1 \leq i,j \leq n} = \sum_{w \in S_n} \varepsilon(w) w(z^\alpha),$$

where $\varepsilon(w)$ is the sign of w.

Noting that interchanging any two of the α_i will only change the sign of a_α, we see that $a_\alpha = 0$ if $\alpha_i = \alpha_j$ for some $i \neq j$. Thus we assume that $\alpha_1 > \cdots > \alpha_n \geq 0$, which allows us to write $\alpha = \lambda + \delta$, were λ is some partition of length at most n, $\delta = (n-1, n-2, \ldots, 1, 0)$, and the addition is done component-wise, where we may need to elongate one of the vectors with zeros to the right. This can be done since the entries of δ are seen to be minimal if we want $a_\delta \neq 0$. It is well known that $a_\delta = \prod_{1 \leq i < j \leq n}(z_i - z_j)$, the Vandermonde determinant. This uses the well known fact that any $f \in \mathbb{Z}[z_1, \ldots, z_n]$ with the property that exchanging any pair of indeterminants only flips the sign is divisible by the Vandermonde in $\mathbb{Z}[z_1, \ldots, z_n]$, which is a simple consequence of the root theorem for polynomial rings.

Define

$$s_\lambda(z_1 \ldots, z_n) = \frac{a_{\lambda+\delta}}{a_\delta}.$$

The comments above give that $s_\lambda \in \mathbb{Z}[z_1, \ldots, z_n]$, but more is true.

Proposition 2.3.1 $s_\lambda(z_1, \ldots, z_n) \in \Lambda_n^{|\lambda|}$

Proof. $s_\lambda(z_1, \ldots, z_n)$ is symmetric as it is the ratio of alternating polynomials. Its degree is $\deg(a_{\lambda+\delta}) - \deg(a_\delta) = |\lambda|$. □

Since $s_\lambda(z_1 \ldots, z_n, 0) = s_\lambda(z_1 \ldots, z_n)$, where $s_\lambda(z_1, \ldots, z_n, 0)$ is the image of $s_\lambda(z_1, \ldots, z_{n+1})$ under the map which sets $z_{n+1} = 0$, our statements in §2.1 give that there is a uniquely defined element $s_\lambda \in \Lambda$ such that, for any $n \geq l(\lambda)$, $s_\lambda(z_1, \ldots, z_n)$ equals s_λ after setting all $\{z_{n+1}, z_{n+2}, \ldots\}$ equal to 0. s_λ is the Schur function corresponding to λ. Note that this says that there is no ambiguity in the use of the notation $s_\lambda(z_1 \ldots, z_n)$ as the polynomial or the Schur function under our evaluation type map.

It is well known that the set $\{s_\lambda(z_1, \ldots, z_n) : \lambda \in \mathbb{Y}\} = \{s_\lambda(z_1, \ldots, z_n) : \lambda \in \mathbb{Y}_n\}$ form a $\mathbb{Z}$-basis for Λ_n, and so the set $\{s_\lambda : \lambda \in \mathbb{Y}\}$ form a $\mathbb{Z}$-basis for Λ.

2.4 Power Sums

In this section, we make note of another basis for Λ which will be used in the establishment of plethystic substitution in the following chapter.

Define the symmetric functions

$$p_k = \sum_{i \geq 1} z_i^k \quad k \geq 1.$$

It can be shown that $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[p_1, p_2, \ldots]$ with the p_k algebraically independent by expanding integral multiples of a known basis for Λ in terms of the p_k. This implies that the products of these power sum functions form a $\mathbb{Q}$-basis of $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. Moreover, there is a $\mathbb{Q}$-valued bilinear form $\langle f, g \rangle$, the so called Hall inner product, for which the products of power sum functions are an orthogonal basis and the Schur functions are an orthonormal basis. [6]

Chapter 3

Plethysm

Throughout this chapter, we will denote $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ by simply Λ.

3.1 Plethystic Substitution

In this section, we discuss the aspects of plethystic substitution that we will be using in what follows. Here we closely follow [6], while the standard background references are [4], [5], and [7].

For $f, g \in \Lambda$, define $f[g]$, read "plethystic substitution of g into f," to be an operation which satisfies the following properties:

(i) For all $m, n \geq 1$, $p_m[p_n] = p_{mn}$.

(ii) For all $m \geq 1$, let $L_m : \Lambda \to \Lambda$ be the map $L_m(g) = p_m[g]$. Then L_m is a $\mathbb{Q}$-algebra homomorphism.

(iii) For all $g \in \Lambda$, let $R_g : \Lambda \to \Lambda$ be the map $R_g(f) = f[g]$. Then R_g is a $\mathbb{Q}$-algebra homomorphism.

It is proved in [6] that there exists a unique binary operation on Λ which satisfies the above properties. This is a direct consequence of the universal mapping property for polynomial algebras.

To justify the name of plethystic substitution, we note Theorem 2 of [6], which says that for all $g \in \Lambda$ and $m \geq 1$,

$$(p_m[g])(z_1, \ldots, z_n) = g(z_1^m, \ldots, z_n^m),$$

where $g(z_1^m, \ldots, z_n^m)$ is the image of the homomorphism which replaces z_i in $g(z_1, \ldots, z_n)$ with z_i^m. This is also a direct consequence of the universal mapping property for polynomial algebras.

Putting this together with the fact that the power sum functions freely generate Λ, we see that computing $(f[g])(z_1, \ldots, z_n)$ is just a matter of writing f in terms of that basis and substituting into g appropriately.

Notice that for all $f \in \Lambda$, $f[p_1] = f$. It is common practice to denote p_1 by Z and $p_1(z_1, \ldots, z_n)$ by Z_n.

Our needs for plethysm will require a standard, minor generalization. We take as given a $\mathbb{Q}$-algebra A and, for each integer $m \geq 1$, a $\mathbb{Q}$-algebra homomorphism $L_m : A \to A$ given by $L_m(g) = p_m[g]$. The universal mapping property of Λ gives for each $g \in A$ a $\mathbb{Q}$-algebra homomorphism $R_g : \Lambda \to A$ by $R_g(p_m) = p_m[g]$. Then this defines a binary operation $\cdot[\cdot] : \Lambda \times A \to A$ by $f[g] = R_g(f)$.

To see that this is generalization is quite small, we present two examples (a minor change to Example 1 in [9] and Example 2 in [6]) where we display plethystic substitution at this level of generality as being the same type of monomial substitution that we saw before.

Example 3.1.1 Let $f = \sum z_i z_j$ where the sum is taken over all positive integer pairs (i, j) with $i \neq j$. Then $f = \frac{1}{2}(p_1^2 - p_2)$. Let $A = \Lambda(q)$. Then

$$f\left[\frac{Z}{1-q}\right] = \frac{1}{2}\left(p_1\left[\frac{Z}{1-q}\right]\right)^2 - \frac{1}{2}p_2\left[\frac{Z}{1-q}\right] = \frac{p_1[Z]^2}{2(1-q)^2} - \frac{p_1[Z]^2}{2(1-q^2)} = \frac{qp_1[Z]^2}{(1-q)(1-q^2)}.$$

Example 3.1.2 Let $f \in \Lambda$ and let $A = \mathbb{Q}(q, t)$ with q, t some algebraically independent elements transcendental over $\mathbb{Q}$. Then

$$f[2q + qt + 3t^4] = f[q + q + qt + t^4 + t^4 + t^4] = f(q, q, qt, t^4, t^4, t^4),$$

where $f(q, q, qt, t^4, t^4, t^4)$ is the image of the homomorphism which replaces z_1 in $f(z_1, \ldots, z_6)$ with q, z_3 with qt and so on. Theorem 7 in [6] says that this is what happens in general.

3.2 The Cauchy Kernel

In this section, we make note of a distinguished element of a completion of Λ which dramatically simplifies what follows. Here we closely follow [9].

Let $\hat{\Lambda} = \mathbb{Q}[[p_1, p_2, \ldots]]$, the $\mathbb{Q}$-algebra of formal power series in the power sum functions. Define $\Omega \in \hat{\Lambda}$ by

$$\Omega = \exp\left(\sum_{r \geq 1} \frac{p_r}{r}\right).$$

Let $Y = \sum_{i\geq 1} y_i$, where the y_i are algebraically independent indeterminants. We present three important consequences of the definition of plethystic substitution:

$$\Omega[Z+Y] = \Omega[Z]\Omega[Y]$$

$$\Omega[Z-Y] = \Omega[Z]\Omega[-Y] = \frac{\Omega[Z]}{\Omega[Y]}$$

$$\Omega[ZY] = \sum_{\lambda\in\mathbb{Y}} s_\lambda[Z]s_\lambda[Y] \tag{3.1}$$

where (3.1) is the Cauchy identity, since

$$\Omega[ZY] = \prod_{\substack{i\geq 1\\ j\geq 1}} \frac{1}{1-z_iy_j}.$$

Since we know that the Schur functions form a basis for Λ, we see that if we understand how an operator acts on $\Omega[ZY]$, then we can recover its action on any element of Λ.

3.3 Skewing

Here we record some notation used throughout [9].

For each $f \in \Lambda$, define the operator $\mathcal{L}_f : \Lambda \to \Lambda$ by $\mathcal{L}_f(g) = fg$ for all $g \in \Lambda$. Let $f^\perp$, read "f-perp" or "skewing by f," be the linear operator on Λ that is adjoint to $\mathcal{L}_f$ with respect to the Hall inner product, i.e. for all $f, g, h \in \Lambda$,

$$\langle f^\perp g, h\rangle = \langle g, fh\rangle = \langle g, \mathcal{L}_f(h)\rangle.$$

Then one can show

$$g[Z]^\perp \cdot \Omega[ZY] = g[Y]\Omega[ZY] \quad g \in \Lambda.$$

One can similarly define $g[Z_n]^\perp$.

Chapter 4

Symmetric Difference Operators

4.1 Macdonald Operators

In this section, we review the definition of the difference Macdonald operators introduced in [7]. We use the notation from [1], as we will be looking at their generalization in what follows.

Let q, t be invertible, central elements of $\mathbb{Z}[z_1, \ldots, z_n](q, t)$ that are transcendental over $\mathbb{Q}$. Define the operator

$$\Gamma_i(f(z_1, \ldots, z_n)) = f(z_1, \ldots, z_{i-1}, qz_i, z_{i+1}, \ldots, z_n), \quad f \in \mathbb{Z}[z_1, \ldots, z_n]$$

and extend this operator to $\mathbb{Z}[z_1, \ldots, z_n](q, t)$. Let $\Gamma_I = \prod_{i \in I} \Gamma_i$. Note that $\Gamma_i \Gamma_j = \Gamma_j \Gamma_i$ for all $i, j \in \{1, \ldots, n\}$. Now define

$$M_\alpha^{q,t} = \sum_{\substack{I \subseteq [n] \\ |I| = \alpha}} \prod_{\substack{i \in I \\ j \notin I}} \frac{tz_i - z_j}{z_i - z_j} \Gamma_I \quad \alpha \in [n].$$

Example 4.1.1 Let $n = 2$ and $\alpha = 1$.

$$M_1^{q,t} \cdot (z_1 + z_2 + z_1 z_2) = \left(\left(\frac{tz_1 - z_2}{z_1 - z_2} + \frac{tz_2 - z_1}{z_2 - z_1} \right) \Gamma_1 \Gamma_2 \right) \cdot (z_1 + z_2 + z_1 z_2)$$

$$= \left(\frac{(t+1)(z_1 - z_2)}{z_1 - z_2} \right) (qz_1 + qz_2 + q^2 z_1 z_2) = q(t+1)(z_1 + z_2 + qz_1 z_2)$$

Extending the S_n action defined above to $\mathbb{Z}[z_1, \ldots, z_n](q, t)$, we see that the result is also a symmetric polynomial. It is proved in [7] that these operators have many remarkable properties, including that they take symmetric polynomials to symmetric polynomials.

4.2 Generalized Macdonald Operators

In this section, we present the generalization of the Macdonald operators defined in [1]. Let v be such that $q = v^{-n}$ and define the operator

$$D_i(f(z_1, \ldots, z_n)) = f(vz_1, \ldots, vz_{i-1}, qvz_i, vz_{i+1}, \ldots, vz_n), \quad f \in \mathbb{Z}[z_1, \ldots, z_n]$$

and extend this operator to $\mathbb{Z}[z_1, \ldots, z_n](q, t)$.
Define the constants

$$C_{\alpha,\beta} = \min(\alpha, \beta)(n - \max(\alpha, \beta)) \quad \alpha, \beta \in \{0, \ldots, n\}.$$

For $\alpha \in \{0, \ldots, n\}, k \in \mathbb{Z}$, define the following difference operators:

$$\mathcal{D}_{\alpha,k} = v^{-\frac{C_{\alpha,\alpha}}{2}k - \sum_{\beta=1}^{n-1} C_{\alpha,\beta}} \sum_{\substack{I \subseteq [n] \\ |I| = \alpha}} \left(\left(\prod_{i \in I} z_i \right)^k \left(\prod_{\substack{i \in I \\ j \notin I}} \frac{z_i}{z_i - z_j} \right) \left(\prod_{i \in I} D_i \right) \right).$$

These operators are generalized Macdonald operators in the sense that, if we define

$$\Delta(f(z_1, \ldots, z_n)) = f(vz_1, \ldots, vz_n), \quad f \in \mathbb{Z}[z_1, \ldots, z_n]$$

and extend this operator to $\mathbb{Z}[z_1, \ldots, z_n](q, t)$, then a simple calculation shows

$$\mathcal{D}_{\alpha,0} = v^{-\sum_{\beta=1}^{n-1} C_{\alpha,\beta}} \left(\lim_{t \to \infty} t^{-\alpha(n-\alpha)} M_\alpha^{q,t} \right) \Delta^\alpha.$$

Chapter 5

A Plethystic Realization of $\mathcal{D}_{\alpha,k}$

Theorem 5.1 Let $C = -\frac{C_{\alpha,\alpha}}{2}k - \sum_{\beta=1}^{n-1} C_{\alpha,\beta}$. In the notation of §4.2,

$$\mathcal{D}_{\alpha,k} = v^C \sum_{\lambda \in \mathbb{Y}_\alpha} s_{\lambda+(k^\alpha)}[Z_n] s_\lambda[(q-1)Z_n]^\perp \Delta^\alpha$$

as operators on Λ_n.

Proof. We begin by, for each pair $\alpha \in \{0, \ldots, n\}$ and $k \in \mathbb{Z}$, defining the operator

$$\mathcal{D}'_{\alpha,k} = v^{-\frac{C_{\alpha,\alpha}}{2}k - \sum_{\beta=1}^{n-1} C_{\alpha,\beta}} \sum_{\substack{I \subseteq [n] \\ |I| = \alpha}} \left(\left(\prod_{i \in I} z_i \right)^k \left(\prod_{\substack{i \in I \\ j \notin I}} \frac{z_i}{z_i - z_j} \right) \left(\prod_{i \in I} \Gamma_i \right) \right).$$

Note

$$\Gamma_i \cdot \Omega[Z_n] = \Omega[Z_n + (q-1)z_i] = \Omega[(q-1)z_i]. \tag{5.1}$$

A slight generalization of a result proved in [DFK] gives

$$\begin{aligned}
\mathcal{D}'_{\alpha,k} &= v^C \sum_{\substack{I \subseteq [n] \\ |I| = \alpha}} \left(\left(\prod_{i \in I} z_i \right)^k \left(\prod_{\substack{i \in I \\ j \notin I}} \frac{z_i}{z_i - z_j} \right) \left(\prod_{i \in I} \Gamma_i \right) \right) \\
&= \frac{v^C}{\alpha!(n-\alpha)!} \sum_{w \in S_n} w \left(\left(\prod_{i \in I_0} z_i \right)^k \left(\prod_{\substack{i \in I_0 \\ j \notin I_0}} \frac{z_i}{z_i - z_j} \right) \left(\prod_{i \in I_0} \Gamma_i \right) \right) w^{-1}
\end{aligned}$$

where $I_0 = \{1, \ldots, \alpha\}$ and $w(D_i) = D_{w(i)}$. To check how the operator acts on the Schur basis, we apply the operator to $\Omega[Z_n Y]$ and simplify.

$$\left(\frac{v^C}{\alpha!(n-\alpha)!}\sum_{w\in S_n} w\left(\left(\prod_{i\in I_0} z_i\right)^k \left(\prod_{\substack{i\in I_0\\ j\notin I_0}} \frac{z_i}{z_i - z_j}\right)\left(\prod_{i\in I_0}\Gamma_i\right)\right) w^{-1}\right)\cdot \Omega[Z_n Y]$$

$$= \frac{v^C}{\alpha!(n-\alpha)!}\sum_{w\in S_n} w\left(\left(\prod_{i\in I_0} z_i\right)^k \left(\prod_{\substack{i\in I_0\\ j\notin I_0}} \frac{z_i}{z_i - z_j}\right)\left(\prod_{i\in I_0}\Gamma_i\right)\right)\Omega[Z_n Y]$$

since $\Omega[Z_n Y]$ is symmetric in the z_i. Then (5.1) gives

$$= \frac{v^C}{\alpha!(n-\alpha)!}\sum_{w\in S_n} w\left(\left(\prod_{i\in I_0} z_i\right)^k \left(\prod_{\substack{i\in I_0\\ j\notin I_0}} \frac{z_i}{z_i - z_j}\right)\left(\prod_{i\in I_0}\Omega[(q-1)z_i Y]\right)\right)$$

$$= \frac{v^C}{\alpha!(n-\alpha)!}\Omega[Z_n Y]\sum_{w\in S_n} w\left(\left(\prod_{i\in I_0} z_i\right)^k \left(\prod_{\substack{i\in I_0\\ j\notin I_0}} \frac{z_i}{z_i - z_j}\right)\left(\prod_{i\in I_0}\Omega[(q-1)z_i Y]\right)\right).$$

Thus we have

$$\frac{\mathcal{D}'_{\alpha,k}\cdot\Omega[Z_n Y]}{\Omega[Z_n Y]} = \frac{v^C}{\alpha!(n-\alpha)!}\sum_{w\in S_n} w\left(\left(\prod_{i\in I_0} z_i\right)^k \left(\prod_{\substack{i\in I_0\\ j\notin I_0}} \frac{z_i}{z_i - z_j}\right)\left(\prod_{i\in I_0}\Omega[(q-1)z_i Y]\right)\right)$$

$$= \frac{v^C}{\alpha!(n-\alpha)!}\sum_{w\in S_n} w\left(\left(\prod_{i\in I_0} z_i\right)^k \left(\prod_{\substack{i\in I_0\\ j\notin I_0}} \frac{z_i}{z_i - z_j}\right)\left(\Omega\left[(q-1)Y\sum_{i\in I_0} z_i\right]\right)\right)$$

$$= \frac{v^C}{\alpha!(n-\alpha)!}\sum_{w\in S_n} w\left(\left(\prod_{i\in I_0} z_i\right)^k \left(\prod_{\substack{i\in I_0\\ j\notin I_0}} \frac{z_i}{z_i - z_j}\right)\left(\sum_{\lambda\in\mathbb{Y}} s_\lambda[(q-1)Y]s_\lambda\left[\sum_{i\in I_0} z_i\right]\right)\right)$$

$$= \frac{v^C}{\alpha!(n-\alpha)!}\sum_{\lambda\in\mathbb{Y}_\alpha}\left(s_\lambda[(q-1)Y]\sum_{w\in S_n} w\left(\left(\prod_{i\in I_0} z_i\right)^k \left(\prod_{\substack{i\in I_0\\ j\notin I_0}} \frac{z_i}{z_i - z_j}\right)\left(s_\lambda\left[\sum_{i\in I_0} z_i\right]\right)\right)\right).$$

Now we focus on just

$$\frac{1}{\alpha!(n-\alpha)!}\sum_{w\in S_n} w\left(\left(\prod_{i\in I_0} z_i\right)^k \left(\prod_{\substack{i\in I_0\\ j\notin I_0}} \frac{z_i}{z_i-z_j}\right)\left(s_\lambda\left[\sum_{i\in I_0} z_i\right]\right)\right)$$
$$=\frac{1}{\alpha!(n-\alpha)!}\sum_{w\in S_n} w\left(\left(\prod_{i\in I_0} z_i\right)^k \left(\prod_{\substack{i\in I_0\\ j\notin I_0}} \frac{z_i}{z_i-z_j}\right)(s_\lambda(z_1,\ldots,z_\alpha))\right). \tag{5.2}$$

Note

$$\sum_{w\in S_{(n-\alpha)}} w\left(\prod_{\alpha+1\le i<j\le n} \frac{z_i}{z_i-z_j}\right)=1$$

where $S_{(n-\alpha)}$ denotes the symmetric group which acts on $\{z_{\alpha+1},\ldots,z_n\}$ by permuting the indices. Thus

$$\left(\prod_{i\in I_0} z_i\right)^k \left(\prod_{\substack{i\in I_0\\ j\notin I_0}} \frac{z_i}{z_i-z_j}\right)(s_\lambda(z_1,\ldots,z_\alpha))$$
$$=\left(\sum_{\pi\in S_{(n-\alpha)}} \pi\left(\prod_{\alpha+1\le i<j\le n} \frac{z_i}{z_i-z_j}\right)\right)\left(\prod_{i\in I_0} z_i\right)^k \left(\prod_{\substack{i\in I_0\\ j\notin I_0}} \frac{z_i}{z_i-z_j}\right)(s_\lambda(z_1,\ldots,z_\alpha))$$
$$=\sum_{\pi\in S_{(n-\alpha)}} \pi\left(\left(\prod_{\alpha+1\le i<j\le n} \frac{z_i}{z_i-z_j}\right)\left(\prod_{i\in I_0} z_i\right)^k \left(\prod_{\substack{i\in I_0\\ j\notin I_0}} \frac{z_i}{z_i-z_j}\right)(s_\lambda(z_1,\ldots,z_\alpha))\right)$$
$$=(n-\alpha)!\left(\left(\prod_{\alpha+1\le i<j\le n} \frac{z_i}{z_i-z_j}\right)\left(\prod_{i\in I_0} z_i\right)^k \left(\prod_{\substack{i\in I_0\\ j\notin I_0}} \frac{z_i}{z_i-z_j}\right)(s_\lambda(z_1,\ldots,z_\alpha))\right),$$

since π acts as the identity on each of the other factors.

Putting this back into (5.2),

$$\frac{1}{\alpha!}\sum_{w\in S_n} w\left(\left(\prod_{\alpha+1\leq i<j\leq n}\frac{z_i}{z_i-z_j}\right)\left(\prod_{i\in I_0} z_i\right)^k\left(\prod_{\substack{i\in I_0\\ j\notin I_0}}\frac{z_i}{z_i-z_j}\right)(s_\lambda(z_1,\ldots,z_\alpha))\right)$$
$$=\frac{1}{\alpha!}\sum_{w\in S_n} w\left(\left(\prod_{\alpha+1\leq i<j\leq n}\frac{z_i}{z_i-z_j}\right)\left(\prod_{i\in I_0} z_i\right)^k\left(\prod_{\substack{i\in I_0\\ j\notin I_0}}\frac{z_i}{z_i-z_j}\right)\left(\sum_{\sigma\in S_\alpha}\sigma\left(\frac{z^{\lambda+\delta}}{\prod_{1\leq i<j\leq\alpha}(z_i-z_j)}\right)\right)\right)$$
$$=\frac{1}{\alpha!}\sum_{w\in S_n} w\left(\sum_{\sigma\in S_\alpha}\sigma\left(\left(\prod_{\alpha+1\leq i<j\leq n}\frac{z_i}{z_i-z_j}\right)\left(\prod_{i\in I_0} z_i\right)^k\left(\prod_{\substack{i\in I_0\\ j\notin I_0}}\frac{z_i}{z_i-z_j}\right)\left(\frac{z^{\lambda+\delta}}{\prod_{1\leq i<j\leq\alpha}(z_i-z_j)}\right)\right)\right)$$
$$=\sum_{w\in S_n} w\left(\left(\prod_{\alpha+1\leq i<j\leq n}\frac{z_i}{z_i-z_j}\right)\left(\prod_{i\in I_0} z_i\right)^k\left(\prod_{\substack{i\in I_0\\ j\notin I_0}}\frac{z_i}{z_i-z_j}\right)\left(\frac{z^{\lambda+\delta}}{\prod_{1\leq i<j\leq\alpha}(z_i-z_j)}\right)\right)$$
$$=\sum_{w\in S_n} w\left(\left(\prod_{\alpha+1\leq i<j\leq n}\frac{z_i}{z_i-z_j}\right)\left(\prod_{\substack{i\in I_0\\ j\notin I_0}}\frac{z_i}{z_i-z_j}\right)\left(\frac{z^{\lambda+(k^\alpha)+\delta}}{\prod_{1\leq i<j\leq\alpha}(z_i-z_j)}\right)\right)$$
$$=s_{\lambda+(k^\alpha)}(z_1,\ldots,z_n).$$

Putting this all together, we have

$$\frac{\mathcal{D}'_{\alpha,k}\cdot\Omega[Z_nY]}{\Omega[Z_nY]}=v^C\sum_{\lambda\in\mathbb{Y}_\alpha}s_\lambda[(q-1)Y]s_{\lambda+(k^\alpha)}[Z_n].$$

This tells us that

$$\mathcal{D}'_{\alpha,k}\cdot\Omega[Z_nY]=v^C\sum_{\lambda\in\mathbb{Y}_\alpha}s_{\lambda+(k^\alpha)}[Z_n]s_\lambda[(q-1)Z_n]^\perp\cdot\Omega[Z_nY],$$

or equivalently

$$\mathcal{D}'_{\alpha,k}=v^C\sum_{\lambda\in\mathbb{Y}_\alpha}s_{\lambda+(k^\alpha)}[Z_n]s_\lambda[(q-1)Z_n]^\perp$$

as operators on Λ_n, which gives

$$\mathcal{D}_{\alpha,k}=v^C\sum_{\lambda\in\mathbb{Y}_\alpha}s_{\lambda+(k^\alpha)}[Z_n]s_\lambda[(q-1)Z_n]^\perp\Delta^\alpha$$

as operators on Λ_n. □

Chapter 6

The Connection to Parabolic Jing Operators

In [9], the authors define for each $\nu \in \mathbb{Y}_\alpha$ the operator

$$H^q_\nu = \sum_{\lambda,\mu \in \mathbb{Y}_\alpha} \bar{c}^\lambda_{\mu\nu} s_\lambda[Z] s_\mu[(q-1)Z]^\perp,$$

where $\bar{c}^\lambda_{\mu\nu}$ are the tensor multiplicities

$$s_\mu[Z] s_\nu[Z] = \sum_{\lambda \in \mathbb{Y}_\alpha} \bar{c}^\lambda_{\mu\nu} s_\lambda[Z].$$

This leads us to the following corollary to our main theorem:

Corollary 6.1 $v^{-C}\mathcal{D}'_{\alpha,k} = H^q_{(k^\alpha)}$ as operators on Λ_n.

Proof. By definition, we have

$$s_\mu[Z] s_{(k^\alpha)}[Z] = \sum_{\lambda \in \mathbb{Y}_\alpha} \bar{c}^\lambda_{\mu (k^\alpha)} s_\lambda[Z].$$

Calculating the left hand side directly, we have

$$\begin{aligned}
&\left(\sum_{w\in S_n} w\left(\frac{z^{\mu+\delta}}{\prod_{1\le i<j\le n}(z_i-z_j)}\right)\right)\left(\sum_{w\in S_n} w\left(\frac{z^{(k^\alpha)+\delta}}{\prod_{1\le i<j\le n}(z_i-z_j)}\right)\right)\\
&=\left(\sum_{w\in S_n} w\left(\frac{z^{\mu+\delta}}{\prod_{1\le i<j\le n}(z_i-z_j)}\right)\right)\left(\prod_{i=1}^{\alpha} z_i\right)^k\\
&=\sum_{w\in S_n} w\left(\frac{z^{\mu+(k^\alpha)+\delta}}{\prod_{1\le i<j\le n}(z_i-z_j)}\right)\\
&=s_{\mu+(k^\alpha)}(z_1,\ldots,z_n).
\end{aligned}$$

Thus the only nonzero contribution in the right hand sum occurs when $\lambda=\mu+(k^\alpha)$ and $\overline{c}^{\lambda}_{\mu(k^\alpha)}=1$. □

Chapter 7

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