Noncommutative Kernels

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(ABSTRACT)

Positive kernels and their associated reproducing kernel Hilbert spaces have played a key role in the development of complex analysis and Hilbert-space operator theory, and they have recently been extended to the setting of free noncommutative function theory. In this paper, we develop the subject further in a number of directions. We give a characterization of completely positive noncommutative kernels in the setting of Hilbert C^*-modules and Hilbert W^*-modules. We prove an Arveson-type extension theorem for completely positive noncommutative kernels, and we show that a uniformly bounded noncommutative kernel can be decomposed into a linear combination of completely positive noncommutative kernels.

Noncommutative Kernels

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(GENERAL AUDIENCE ABSTRACT)

Over the last several decades, positive kernels and their associated reproducing kernel Hilbert spaces have played a key role in the development of complex analysis and Hilbert-space operator theory. Recently, they have been extended to the setting of free noncommutative function theory which is an active area of research with motivation from several different sources including free probability and noncommutative real semialgebraic geometry. In this paper, we develop further the theory of positive kernels in the noncommutative setting.

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Chapter 1

Introduction

The goal of the present paper is to expand on the theory of noncommutative kernels and noncommutative reproducing kernel Hilbert spaces as presented in [9]. The main result of [9] says that a completely positive noncommutative kernel has an associated noncommutative reproducing kernel Hilbert space. In Chapters 3 and 4, we extend this result to the setting of Hilbert C^*-modules and Hilbert W^*-modules. In Chapter 5, we develop an Arveson-type extension theorem for completely positive noncommutative kernels. In Chapter 6, we introduce uniformly bounded noncommutative kernels and prove extension and decomposition theorems in this setting.

One can view **free noncommutative function theory** as an extension of the theory of holomorphic functions of several complex variables $z = (z_1, \ldots, z_d)$ to a theory of functions on matrix tuples $Z = (Z_1, \ldots, Z_d)$ where Z consists of freely noncommutating $n \times n$ matrices such that $n \in \mathbb{N}$ is allowed to vary with Z. A noncommutative function is a mapping on these matrix tuples Z which is graded, respects direct sums, and respects similarities (see Section 2.1 for details). This characterization of free noncommutative function theory originates in research by Taylor into developing a functional calculus for functions of several noncommuting variables [34, 35]. The recent monograph by Kaliuzhnyi-Verbovetskyi and Vinnikov [17] adds completeness and insight to the work of Taylor. There the authors develop the theory of free noncommutative functions from first principles and find that a weak local boundedness condition implies analyticity for noncommutative functions.

Motivation for the study of free noncommutative function theory comes from several different sources. We point to the work of Voiculescu [36, 37] in free probability, the work of Helton-Klep-McCullough [15, 16] in noncommutative real semialgebraic geometry, and the work of Muhley-Solel [22] and Popescu [26] in generalized Hardy algebras. Furthermore, there is an active industry in which classical results in Analysis are generalized to the setting of free noncommutative function theory, e.g., Banach fixed point theorem [2], Oka-Weil theorem [4], implicit and inverse function theorems [1]. The author's work with Ball and Vinnikov on reproducing kernel Hilbert spaces [9] and interpolation problems [10] are more examples of this trend.

Hilbert C^*-modules can be thought of as a generalization of Hilbert spaces where the inner product has values in a C^*-algebra rather than complex numbers. There are several excellent sources on this subject, e.g., [19, 20, 27, 14]. As noted in [19], applications of the theory include induced representations and Morita equivalence [28, 29], KK-theory [18], and C^*-algebraic quantum group theory [40]. There has also been work on completely positive maps and positive kernels in this setting [18, 23, 33], and the content of Chapter 3 builds on these results. We also take up the development of noncommutative kernels in the setting of **Hilbert W^*-modules** which are a subclass of Hilbert C^*-modules which act more like Hilbert spaces. Much of the development of the theory of Hilbert W^*-modules is due to Paschke [24] and Rieffel [28, 29] and applications include the work of Muhly and Solel on interpolation and Hardy algebras [22] and the work of Skeide on von Neumann modules [31]

To set the stage for our discussion of kernels, we first present a classical result due to Aronszajn [7] and Moore [21] which characterizes positive kernels.

Theorem 1.1. *Consider the function $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{Y})$ (i.e., a kernel) where Ω is a set and $\mathcal{L}(\mathcal{Y})$ denotes bounded operators on the Hilbert space $\mathcal{Y}$. The following are equivalent:*

*(1) K is a **positive kernel**:*

$$\sum_{i,j=1}^{N} \langle K(\omega_i, \omega_j) y_j, y_i \rangle_{\mathcal{Y}} \geq 0 \tag{1.1}$$

for all $\omega_1, \dots, \omega_N \in \Omega$, $y_1, \dots, y_N \in \mathcal{Y}$, $N \in \mathbb{N}$

*(2) There exists a Hilbert space $\mathcal{H}(K)$ (called a **reproducing kernel Hilbert space**) consisting of functions $f : \Omega \to \mathcal{Y}$ with the following properties:*

(a) For any $\omega \in \Omega$ and $y \in \mathcal{Y}$, the function $K_{\omega,y}$ given by $K_{\omega,y}(\omega') = K(\omega', \omega)y$ belongs to $\mathcal{H}(K)$

*(b) for all $f \in \mathcal{H}(K)$ and $y \in \mathcal{Y}$, the **reproducing property***

$$\langle f, K_{\omega,y} \rangle_{\mathcal{H}(K)} = \langle f(\omega), y \rangle_{\mathcal{Y}} \tag{1.2}$$

holds.

*(3) There exists a Hilbert space $\mathcal{X}$ and a function $H : \Omega \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$ so that the **Kolmogorov decomposition** holds:*

$$K(\omega', \omega) = H(\omega') H(\omega)^*. \tag{1.3}$$

In the past 25 years, Theorem 1.1 has been extended by several authors. The work of Murphy [23] and Szafraniec [33] generalizes Theorem 1.1 to the setting where $\mathcal{L}(\mathcal{Y})$ is replaced by the set of adjointable operators on a Hilbert C^*-module and recent work from

Barreto-Bhat-Liebscher-Skeide [11] establishes the analogue of (1) $\Leftrightarrow$ (3) for the setting where $\mathcal{L}(\mathcal{Y})$ is replaced by $\mathcal{L}(\mathcal{A},\mathcal{B})$ where $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras.

In [9], we extend the theory of reproducing kernel Hilbert spaces to the setting of free noncommutative function theory. Let Ω be a nc set, $\mathcal{A}$ a C^*-algebra, and $\mathcal{L}(\mathcal{Y})$ bounded operators on the Hilbert space $\mathcal{Y}$. Recall that a nc function is graded, respects direct sums, and respects similarities. We say that $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{\rm nc}$ is a **noncommutative kernel** if it satisfies the appropriate 2-variable version of these properties. The main theorem of [9] is obtained by considering the following modifications of parts (1), (2), and (3) of Theorem 1.1:

(1) $\Rightarrow$ (1') K is a **completely positive noncommutative kernel**, i.e.,

$$Z \in \Omega_n,\ P \succeq 0 \text{ in } \mathcal{A}^{n\times n} \Rightarrow K(Z,Z)(P) \succeq 0 \text{ in } \mathcal{L}(\mathcal{Y})^{n\times n} \tag{1.4}$$

for any $n \in \mathbb{N}$.

(2) $\Rightarrow$ (2') There exists a reproducing kernel Hilbert space $\mathcal{H}(K)$ consisting of noncommutative functions $f : \Omega \to \mathcal{L}(\mathcal{A},\mathcal{Y})_{\rm nc}$ with more elaborate versions of properties (a) and (b). Furthermore, $\mathcal{H}(K)$ is equipped with a unital $*$-representation σ mapping $\mathcal{A}$ to $\mathcal{L}(\mathcal{H}(K))$.

(3) $\Rightarrow$ (3') Kolmogorov decomposition: There is a Hilbert space $\mathcal{X}$ equipped with a unital $*$-representation $\sigma\colon \mathcal{A} \to \mathcal{L}(\mathcal{X})$ together with a nc function $H\colon \Omega \to \mathcal{L}(\mathcal{X},\mathcal{Y})_{\rm nc}$ so that

$$K(Z,W)(P) = H(Z)(\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \sigma)(P)H(W)^* \tag{1.5}$$

for all $Z \in \Omega_n$, $W \in \Omega_m$, $P \in \mathcal{A}^{n\times m}$.

The result above contains Theorem 1.1 and the Barreto-Bhat-Liebsher-Skeida results [11] as special cases. Furthermore, the well-known Stinespring dilation theorem for completely positive maps is a special case of (1') $\Leftrightarrow$ (3'). However, the result does not generalize the work of Murphy [23] or the Hilbert C^*-module version of the Stinespring dilation theorem due to Kasparov [18]. There are two principle motivations for extending the result above to the setting of Hilbert C^*-modules. Firstly, we would like to have a general theorem which contains the mentioned cases. Secondly, the generalization of statements (1'), (2'), (3') to the Hilbert C^*-module setting proves to be very natural. This fact is readily seen in the proof of Theorem 3.6.

To motivate the theory of completely positive noncommutative kernels in the setting of Hilbert W^*-modules, we quote another theorem from the classical theory of reproducing kernel Hilbert spaces due to Aronszajn [7] and Moore [21].

Theorem 1.2. *Let $\mathcal{H}$ be a Hilbert space whose elements are functions $f : \Omega \to \mathcal{Y}$ where Ω is a set and $\mathcal{Y}$ is a Hilbert space. If the evaluation maps $\mathbf{ev}_x : f \mapsto f(x)$ are bounded linear functionals, then there exists a positive kernel K such that $\mathcal{H} = \mathcal{H}(K)$ (as in Theorem 1.1, (2)).*

Theorem 1.2 tells us when a Hilbert space of functions comes from a completely positive kernel. In [9], we see that this result has a natural generalization to the setting of noncommutative reproducing kernel Hilbert spaces. However, in the setting of Hilbert C^*-modules taken up in Chapter 3, the analogue of this result does not hold. In particular, the proof of Theorem 1.2 relies on the Riesz-Frechet theorem for Hilbert spaces which does not extend to the setting of Hilbert C^*-modules. This deficiency motivates the development of noncommutative kernels in the setting of Hilbert W^*-modules. We see that the proofs in this setting are more cumbersome as the natural topology in this setting is the w^*-topology in contrast to the norm topology used in Hilbert C^*-module setting. The payoff is that Hilbert W^*-modules are similar enough to Hilbert spaces to prove an analogue of Theorem 1.2.

In Chapter 5, we generalize Arveson's extension theorem [8] to the setting of noncommutative kernels. We quote the original result below.

Theorem 1.3. *(Arveson's extension theorem) Let $\mathcal{D}$ be an operator system contained in a C^*-algebra $\mathcal{A}$ with $I_{\mathcal{D}} = I_{\mathcal{A}}$ and let $\phi : \mathcal{D} \to \mathcal{L}(\mathcal{Y})$ be a completely positive map. Then there exists a completely positive map $\Phi : \mathcal{A} \to \mathcal{L}(\mathcal{Y})$ which extends ϕ.*

We prove that a completely positive noncommutative kernel $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{D}, \mathcal{L}(\mathcal{Y}))_{\text{nc}}$ extends to a completely positive noncommutative kernel $\widetilde{K} : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{\text{nc}}$ which extends K. The proof of our result is broken down into two cases. For the first case, we assume Ω is a finite set. In this way, we can reduce our extension problem to one that can be solved with Arveson's extension theorem. A key part of the argument is that for finite Ω a completely positive noncommutative kernel can be identified with a completely positive map with a special bimodule structure and that this structure is maintained after applying Arveson's extension theorem. In the second case, we consider general Ω and use a theorem due to Kurosh (see [6]) to reduce the general case to the finite case.

In Chapter 6, we introduce uniformly bounded noncommutative kernels. We see that completely bounded maps are a special case, and we generalize several important results from completely bounded maps to our setting. In particular, Wittstock's decomposition theorem [38] says that a completely bounded map ϕ can be decomposed as completely positive maps, i.e., $\phi = \xi_1 - \xi_2 + i(\xi_3 - \xi_4)$ where the ξ_i's are completely positive maps (see [25]). We show that a uniformly bounded noncommutative kernel can be similarly decomposed by uniformly bounded completely positive kernels. This work is inspired by the work of Bhattacharyya-Dritschel-Todd [12] where the authors prove an analogue of Wittstock's decompostion theorem for a subclass of noncommutative kernels under a weaker assumption than uniform boundedness.

Chapter 2

Preliminaries

In this chapter, we present noncommutative functions and completely positive noncommutative kernels. Our treatment of free noncommutative function theory and noncommutative functions in particular is based on the monograph [17]. Our presentation of noncommutative kernels is based on [9] where noncommutative kernels are first introduced.

2.1 Noncommutative spaces and functions

Let $\mathcal{V}$ be a vector space over $\mathbb{C}$. We define a **noncommutative space** $\mathcal{V}_{\text{nc}}$ to be the disjoint union of $n \times n$ matrices over $\mathcal{V}$, i.e., $\mathcal{V}_{\text{nc}} = \amalg_{n=1}^{\infty} \mathcal{V}^{n\times n}$. We will often refer to the n-th level subset of $\mathcal{V}_{\text{nc}}$ which we denote by $\mathcal{V}_n$, i.e., $\mathcal{V}_n = \mathcal{V}_{\text{nc}} \cap \mathcal{V}^{n\times n}$. We say that a subset Ω of $\mathcal{V}_{\text{nc}}$ is a **noncommutative set** if it is closed under direct sums, i.e. ,

$$Z \in \Omega_n, W \in \Omega_m \Rightarrow \left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right] \in \Omega_{n+m}. \tag{2.1}$$

We define multiplication between matrices over $\mathcal{V}$ and matrices over $\mathbb{C}$ by the standard matrix product. Thus for $\alpha \in \mathbb{C}^{k\times n}$, $Z \in \mathcal{V}^{n\times m}$, $\beta \in \mathbb{C}^{m\times \ell}$, we have that $\alpha Z \beta \in \mathcal{V}^{k\times \ell}$.

Example 2.1. Consider the case where $\mathcal{V} = \mathbb{C}^d$ and $(\mathbb{C}^d)_{\text{nc}} = \amalg_{n=1}^{\infty} (\mathbb{C}^d)^{n\times n}$. We identify matrices over $\mathbb{C}^d$ with d-tuples of matrices over $\mathbb{C}$, i.e., $(\mathbb{C}^d)^{n\times n} \simeq (\mathbb{C}^{n\times n})^d$. Thus for every $Z \in (\mathbb{C}^d)_n$, we have

$$Z = (Z_1, Z_2, \ldots, Z_d)$$

where $Z_i \in \mathbb{C}^{n\times n}$ for $i = 1, 2, \ldots, d$. Note that under this identification multiplication by matrices over $\mathbb{C}$ (as described above) is given by

$$\alpha Z \beta = (\alpha Z_1 \beta, \alpha Z_2 \beta, \ldots, \alpha Z_d \beta)$$

where $\alpha \in \mathbb{C}^{k\times n}$ and $\beta \in \mathbb{C}^{n\times \ell}$.

Suppose that $\mathcal{V}$ and $\mathcal{V}_0$ are vector spaces over $\mathbb{C}$ and Ω is a subset of $\mathcal{V}_{\text{nc}}$. Given a function $f\colon \Omega \to \mathcal{V}_{0,\text{nc}}$, we say that f is a **noncommutative (nc) function** if

- f is **graded**, i.e., $f\colon \Omega_n \to \mathcal{V}_{0,n}$,

  and

- f **respects intertwinings**:

$$Z \in \Omega_n,\ \widetilde{Z} \in \Omega_m,\ \alpha \in \mathbb{C}^{m\times n} \text{ with } \alpha Z = \widetilde{Z}\alpha \Rightarrow \alpha f(Z) = f(\widetilde{Z})\alpha. \tag{2.2}$$

As noted in [17], this definition of nc function is analogous to the following: The map f is a nc function if

- f is **graded**,
- f **respects direct sums**, i.e.,

$$Z, W, \left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right] \in \Omega \Rightarrow f(\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right]) = \begin{bmatrix} f(Z) & 0 \\ 0 & f(W) \end{bmatrix}, \tag{2.3}$$

  and

- f **respects similarities**, i.e., for invertible $\alpha \in \mathbb{C}^{n\times n}$,

$$Z, \alpha Z\alpha^{-1} \in \Omega_n \Rightarrow f(\alpha Z\alpha^{-1}) = \alpha f(Z)\alpha^{-1}. \tag{2.4}$$

We note that the definitions above are not identical to those presented in [17]. There the sets Ω are assumed to be nc sets (2.1). Here we place no such restriction on Ω.

Example 2.2. Given a polynomial with freely noncommuting indeterminates and complex coefficients, e.g.,

$$p(z_1, z_2, z_3) = z_1z_2z_3 + 2z_1z_2 - 3z_2z_1,$$

we may define a nc function from $\Omega = (\mathbb{C}^3)_{\text{nc}}$ (see Example 2.1) to $\mathbb{C}_{\text{nc}}$ by

$$p(Z) = p(Z_1, Z_2, Z_3) = Z_1Z_2Z_3 + 2Z_1Z_2 - 3Z_2Z_1$$

where $Z = (Z_1, Z_2, Z_3) \in (\mathbb{C}^{n\times n})^3$. We check that p is a nc function. Clearly, the function is graded as d-tuples of $n \times n$ matrices are mapped to a single $n \times n$ matrix. To check that p respects intertwinings (2.2), we consider $Z \in \Omega_n, \widetilde{Z} \in \Omega_m, \alpha \in \mathbb{C}^{m\times n}$ such that $\alpha Z = \widetilde{Z}\alpha$ (or equivalently $\alpha Z_i = \widetilde{Z}_i\alpha$ for $i = 1, 2, 3$). We have

$$\alpha p(Z) = \alpha Z_1Z_2Z_3 + \alpha 2Z_1Z_2 - \alpha 3Z_2Z_1 = \widetilde{Z}_1\widetilde{Z}_2\widetilde{Z}_3\alpha + 2\widetilde{Z}_1\widetilde{Z}_2\alpha - 3\widetilde{Z}_2\widetilde{Z}_1\alpha = p(\widetilde{Z})\alpha$$

which establishes that p respects intertwinings.

2.2 Noncommutative kernels

As before, let $\mathcal{V}$ be a vector space and let Ω be a subset of $\mathcal{V}_{\mathrm{nc}}$. Consider two additional vector spaces $\mathcal{V}_0$ and $\mathcal{V}_1$ and the function $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{V}_1, \mathcal{V}_0)_{\mathrm{nc}}$ where

$$\mathcal{L}(\mathcal{V}_1, \mathcal{V}_0)_{\mathrm{nc}} := \amalg_{n,m=1}^{\infty} \mathcal{L}(\mathcal{V}_1^{n\times m}, \mathcal{V}_0^{n\times m}).$$

We say that K is a **noncommutative kernel** if

- K is **graded**, i.e.,

$$Z \in \Omega_n,\ W \in \Omega_m \Rightarrow K(Z, W) \in \mathcal{L}(\mathcal{V}_1^{n\times m}, \mathcal{V}_0^{n\times m}), \tag{2.5}$$

  and

- K **respects intertwinings** in the following sense:

$$\begin{aligned} &Z \in \Omega_n,\ \widetilde{Z} \in \Omega_{\widetilde{n}},\ \alpha \in \mathbb{C}^{\widetilde{n}\times n} \text{ such that } \alpha Z = \widetilde{Z}\alpha, \\ &W \in \Omega_m,\ \widetilde{W} \in \Omega_{\widetilde{m}},\ \beta \in \mathbb{C}^{\widetilde{m}\times m} \text{ such that } \beta W = \widetilde{W}\beta, \\ &P \in \mathcal{V}_1^{n\times m} \Rightarrow \alpha\, K(Z, W)(P)\, \beta^* = K(\widetilde{Z}, \widetilde{W})(\alpha P \beta^*). \end{aligned} \tag{2.6}$$

As in the case of nc functions, we have an equivalent definition for nc kernels:

- K is **graded**,

- K **respects direct sums**, i.e.,

$$K\left(\begin{bmatrix} Z & 0 \\ 0 & \widetilde{Z} \end{bmatrix}, \begin{bmatrix} W & 0 \\ 0 & \widetilde{W} \end{bmatrix}\right)\left(\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\right) = \begin{bmatrix} K(Z,W)(P_{11}) & K(Z,\widetilde{W})(P_{12}) \\ K(\widetilde{Z},W)(P_{21}) & K(\widetilde{Z},\widetilde{W})(P_{22}) \end{bmatrix}. \tag{2.7}$$

  and

- K **respects similarities**:

$$\begin{aligned} &Z, \widetilde{Z} \in \Omega_n,\ \alpha \in \mathbb{C}^{n\times n} \text{ invertible such that } \widetilde{Z} = \alpha Z \alpha^{-1}, \\ &W, \widetilde{W} \in \Omega_m,\ \beta \in \mathbb{C}^{m\times m} \text{ invertible such that } \widetilde{W} = \beta W \beta^{-1}, \\ &P \in \mathcal{V}_1^{n\times m} \Rightarrow K(\widetilde{Z}, \widetilde{W})(P) = \alpha K(Z, W)(\alpha^{-1} P \beta^{-1*})\beta^* \end{aligned} \tag{2.8}$$

Example 2.3. Consider the nc functions $f, g : \Omega \subset \mathcal{V}_{\mathrm{nc}} \to \mathbb{C}_{\mathrm{nc}}$ and let $K : \Omega\times\Omega \to \mathcal{L}(\mathbb{C}, \mathbb{C})_{\mathrm{nc}}$ be given by $K(Z, W)(P) = f(Z)Pg(W)^*$ where $Z \in \Omega_n$, $W \in \Omega_m$, and $P \in \mathbb{C}^{n\times m}$. Then K is clearly graded (2.5). To show that K respects intertwinings, we assume the setup from (2.6) and calculate

$$\alpha K(Z, W)(P)\beta^* = \alpha f(Z)Pg(W)^*\beta^* = f(\widetilde{Z})(\alpha P\beta^*)g(\widetilde{W})^* = K(\widetilde{Z}, \widetilde{W})(\alpha P\beta^*).$$

Example 2.4. Let $\phi : \mathcal{V}_1 \to \mathcal{V}_0$ be a linear map and consider the inflation maps $\phi_{n,m} : \mathcal{V}_1^{n\times m} \to \mathcal{V}_0^{n\times m}$ given by $\phi_{n,m}([v_{ij}]) = [\phi(v_{ij})]$ for $[v_{ij}] \in \mathcal{V}_1^{n\times m}$. We show that such maps define a nc kernel. Let Ω be the nc subset of the nc space $\mathbb{C}_{\text{nc}}$ consisting of the $n \times n$ matrix identities, i.e., $\Omega_n = \{I_{\mathbb{C}^{n\times n}}\}$. We construct a map $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{V}_1, \mathcal{V}_0)_{\text{nc}}$ via the identification $K(I_{\mathbb{C}^{n\times n}}, I_{\mathbb{C}^{m\times m}})([v_{ij}]) = \phi_{n,m}([v_{ij}])$ where in particular $K(1,1) = \phi$. We proceed by verifying that our map K is an nc kernel. By construction, the map is graded (2.5). To show that K respects intertwinings, we assume the setup from (2.6) and calculate

$$\alpha K(Z,W)(P)\beta^* = \alpha\phi_{n,m}(P)\beta^* = \phi_{\widetilde{n},\widetilde{m}}(\alpha P\beta^*) = K(\widetilde{Z},\widetilde{W})(\alpha P\beta^*). \tag{2.9}$$

Example 2.5. Let X be a set. We show that kernels of the form $k : X \times X \to \mathcal{L}(\mathcal{V}_1, \mathcal{V}_0)$ define a nc kernel. We construct a vector space $\mathcal{V}$ such that the elements of its basis are given by the elements of X. Let Ω be the nc subset of the nc space $\mathcal{V}_{\text{nc}}$ consisting of those diagonal matrices containing only basis elements, i.e., any $Z \in \Omega_n$ is given by

$$Z = \begin{bmatrix} z_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & z_n \end{bmatrix}$$

for some $z_1, \ldots, z_n \in X$. We define a map $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{V}_1, \mathcal{V}_0)_{\text{nc}}$ by $K(Z,W)(P) = K(Z,W)([p_{ij}]) = [K(z_i, w_j)(p_{ij})]$ for $Z \in \Omega_n$, $W \in \Omega_m$, and $P = [p_{ij}] \in \mathcal{V}_1^{n\times m}$. We note that K and k agree over X, i.e., $K(z,w) = k(z,w)$ for any $z, w \in X$. We proceed by checking that K is a nc kernel. As the map is graded (2.5) and respects direct sums (2.7) by construction, it remains to check that K respects similarities (2.8). Consider $Z, \widetilde{Z} \in \Omega_n$ and invertible $\alpha \in \mathbb{C}^{n\times n}$ such that $\widetilde{Z} = \alpha Z \alpha^{-1}$. We observe that the equality only holds if α is a permutation matrix, so we have $\alpha = \sum_{i=1}^n \mathbf{e_i}\mathbf{e}^*_{\pi(\mathbf{i})}$ where $\mathbf{e_1}, \ldots, \mathbf{e_n}$ denote the standard basis elements of $\mathbb{C}^n$ and $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ denotes a permutation on $\{1, \ldots, n\}$. We assume the setup of (2.8) with $\alpha = \sum_{i=1}^n \mathbf{e_i}\mathbf{e}^*_{\pi(\mathbf{i})}$ and $\beta = \sum_{i=1}^n \mathbf{e_i}\mathbf{e}^*_{\widehat{\pi}(\mathbf{i})}$ and calculate

$$\begin{aligned}
\alpha K(Z,W)(P)\beta^* &= \left(\sum_{i=1}^n \mathbf{e_i}\mathbf{e}^*_{\pi(\mathbf{i})} \otimes I_{\mathcal{V}_0}\right)\left(\sum_{k,\ell=1}^n \mathbf{e_k}\mathbf{e}^*_{\ell} \otimes K(z_k, w_\ell)([p_{k,\ell}])\right)\left(\sum_{j=1}^n \mathbf{e}_{\widehat{\pi}(\mathbf{j})}\mathbf{e}^*_{\mathbf{j}} \otimes I_{\mathcal{V}_0}\right) \\
&= \sum_{i,k,\ell,j=1}^n \mathbf{e_i}\mathbf{e}^*_{\pi(\mathbf{i})}\mathbf{e_k}\mathbf{e}^*_{\ell}\mathbf{e}_{\widehat{\pi}(\mathbf{j})}\mathbf{e}^*_{\mathbf{j}} \otimes K(z_k, w_\ell)([p_{k,\ell}]) \\
&= \sum_{i,j=1}^n \mathbf{e_i}\mathbf{e}^*_{\mathbf{j}} \otimes K(z_{\pi(i)}, w_{\widehat{\pi}(j)})([p_{\pi(i),\widehat{\pi}(j)}]) \\
&= K(\widetilde{Z},\widetilde{W})(\alpha P\beta^*).
\end{aligned}$$

We conclude that $\alpha K(Z,W)(\alpha^{-1}P\beta^{-1*})\beta^* = K(\widetilde{Z},\widetilde{W})(P)$.

We now recall some preliminary results on C^*-algebras. A ***C^**-algebra** $\mathcal{A}$ is a norm-closed, $*$-closed subalgebra of bounded operators on a Hilbert space $\mathcal{Y}$. In the sequel, we assume that all C^*-algebras are unital, i.e., a C^*-algebra containing the identity. An element $a \in \mathcal{A}$ is said to be **positive** if it has the form $a = b^*b$ for some $b \in \mathcal{A}$, and equivalently, an element $a \in \mathcal{A}$ is positive if $\langle y, ay \rangle \geq 0$ for any $y \in \mathcal{Y}$. A subspace $\mathcal{M}$ of a C^*-algebra is called an **operator space**. A subspace $\mathcal{S}$ of a unital C^*-algebra which is self-adjoint (i.e. $a \in \mathcal{S} \Rightarrow a^* \in \mathcal{S}$) and contains the identity is called an **operator system**.

We note that we have presented the *concrete* definitions of a C^*-algebra, operator space, and operator system. For the abstract constructions and more details, see [25].

We now consider nc kernels of the form $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{S}_1, \mathcal{S}_0)_{\rm nc}$ where $\mathcal{S}_1$ and $\mathcal{S}_0$ are operator systems. We say that such a K is **completely positive** if

$$[P_{ij}] \succeq 0 \Rightarrow [K(Z^{(i)}, Z^{(j)})(P_{ij})] \succeq 0 \tag{2.10}$$

for any $Z^{(i)} \in \Omega_{n_i}$, $P_{ij} \in \mathcal{S}_1^{n_i \times n_j}$, and $i = 1, 2, \ldots, n$.

Proposition 2.6. *Let Ω be a nc subset of $\mathcal{V}_{\rm nc}$. Then a nc kernel $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{S}_1, \mathcal{S}_0)_{\rm nc}$ is completely positive if and only if*

$$P \succeq 0 \Rightarrow K(Z, Z)(P) \succeq 0 \tag{2.11}$$

where $P \in \mathcal{S}_1^{N \times N}$, $Z \in \Omega_N$, and $N \in \mathbb{N}$

Proof. If K is a cp nc kernel, then the implication (2.11) follows by definition. Assume now that (2.11) holds and consider any finite set $Z^{(1)}, Z^{(2)}, \ldots, Z^{(n)} \in \Omega$. As Ω is assumed to be a nc set, we can construct a single element $Z \in \Omega$ given by

$$Z = \begin{bmatrix} Z^{(1)} & & \\ & \ddots & \\ & & Z^{(n)} \end{bmatrix}.$$

A consequence of the fact that a nc kernel respects direct sums (2.7) is that condition (2.11) with this choice of $Z \in \Omega$ is equivalent to (2.10). □

Example 2.7. Let Ω be a nc set. Given a nc function $f : \Omega \to \mathbb{C}_{\rm nc}$, we consider the map $K : \Omega \times \Omega \to \mathcal{L}(\mathbb{C}, \mathbb{C})_{\rm nc}$ given by $K(Z, W)(P) = f(Z)Pf(W)^*$ where $Z \in \Omega_n$, $W \in \Omega_m$, and $P \in \mathbb{C}^{n \times m}$. Recall from Example 2.3 that such a map is a nc kernel. We now establish that this special case is a cp nc kernel. Consider $Z \in \Omega_n$ and positive $P \in \mathbb{C}^{n \times n}$, then we have

$$K(Z, Z)(P) = K(Z, Z)(QQ^*) = f(Z)QQ^*f(Z)^* = (f(Z)Q)(f(Z)Q)^*$$

which shows that $K(Z, Z)(P)$ is positive and by Proposition 2.6 K is completely positive.

Example 2.8. Let $\mathcal{S}_1$ and $\mathcal{S}_0$ be operator spaces and let $\phi : \mathcal{S}_1 \to \mathcal{S}_0$ be a linear map. We say that ϕ is a **completely positive map** if the inflation maps $\phi_{n,n}$ are positive maps for each $n \in \mathbb{N}$, i.e.

$$[p_{ij}] \succeq 0 \Rightarrow \phi_{n,n}([p_ij]) = [\phi(p_{ij})] \succeq 0.$$

for $[p_{ij}] \in \mathcal{S}_1^{n\times m}$. From Example 2.4, we see that ϕ and induces a nc kernel K, and it follows from (2.10) and the discussion in Example 2.4 that ϕ is a completely positive map if and only if the induced nc kernel K is completely positive.

Again, considering a nc kernel of the form $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{S}_1, \mathcal{S}_0)_{\mathrm{nc}}$. We define the kernel adjoint K^* by $K^*(Z,W)(P) = K(W,Z)(P^*)^*$, and we say that a nc kernel is **hermitian** if

$$K^*(Z,W)(P) = K(W,Z)(P^*)^* = K(Z,W)(P). \tag{2.12}$$

A positive element of a C^*-algebra is self-adjoint, and we can prove an analogous result holds for cp nc kernels. To this end, we need the following lemma from [25].

Lemma 2.9. *Let $\mathcal{S}$ be an operator system. For $P \in \mathcal{S}^{n\times m}$, the matrix*

$$\begin{bmatrix} I_{\mathcal{S}^{n\times n}} & P \\ P^* & I_{\mathcal{S}^{m\times m}} \end{bmatrix}$$

is positive in $\mathcal{S}^{(n+m)\times(n+m)}$ if and only if $||P|| \leq 1$.

Proof. We note that $||P|| \leq 1$ if and only if $I_{\mathcal{S}^{m\times m}} - P^*P \succeq 0$. The result then follows from the following equality:

$$\begin{bmatrix} I_{\mathcal{S}^{n\times n}} & P \\ P^* & I_{\mathcal{S}^{m\times m}} \end{bmatrix} = \begin{bmatrix} I_{\mathcal{S}^{n\times n}} & 0 \\ P^* & I_{\mathcal{S}^{m\times m}} \end{bmatrix} \begin{bmatrix} I_{\mathcal{S}^{n\times n}} & 0 \\ 0 & I_{\mathcal{S}^{m\times m}} - P^*P \end{bmatrix} \begin{bmatrix} I_{\mathcal{S}^{n\times n}} & P \\ 0 & I_{\mathcal{S}^{m\times m}} \end{bmatrix}.$$

□

Proposition 2.10. *A cp nc kernel $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{S}_1, \mathcal{S}_0)_{\mathrm{nc}}$ is hermitian.*

Without loss of generality, we consider $P \in \mathcal{S}_1^{n\times m}$ with $||P|| = 1$. By Lemma 2.9, we have

$$0 \preceq K\left(\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right], \left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right]\right)\left(\left[\begin{smallmatrix} I & P \\ P^* & I \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} K(Z,Z)(I) & K(Z,W)(P) \\ K(W,Z)(P^*) & K(W,W)(I) \end{smallmatrix}\right].$$

for $Z \in \Omega_n$ and $W \in \Omega_m$. We conclude that $K(Z,W)(P) = K(W,Z)(P^*)^*$ and so K is hermitian.

2.3 Noncommutative envelopes

The following extensions of subsets Ω of $\mathcal{V}_{\mathrm{nc}}$ are first considered in [17, 10]:

1. We say that $[\Omega]_{\mathrm{nc}}$ is the **noncommutative envelope of** Ω if it is the smallest subset of $\mathcal{V}_{\mathrm{nc}}$ which is closed under direct sums, i.e.,

$$Z \in \Omega_n,\ W \in \Omega_m \Rightarrow \left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right] \in \Omega_{n+m}. \tag{2.13}$$

2. We say that $[\Omega]_{\text{sim}}$ is the **similarity envelope of** Ω if it is the smallest subset of $\mathcal{V}_{\text{nc}}$ which is closed under direct sums (2.13) and under similarity transforms, i.e.,

$$Z \in \Omega_n,\ \alpha \in \mathbb{C}^{n\times n} \text{ invertible } \Rightarrow \alpha Z \alpha^{-1} \in \Omega_n. \tag{2.14}$$

3. We say that $[\Omega]_{\text{full}}$ is the **full noncommutative envelope of** Ω if it is the smallest subset of $\mathcal{V}_{\text{nc}}$ which is closed under direct sums and left injective intertwinings, i.e.,

$$Z \in \Omega_n,\ \widetilde{Z} \in \mathcal{V}_{\text{nc},m} \text{ such that } \mathcal{I}\widetilde{Z} = Z\mathcal{I} \text{ for an injective } \mathcal{I} \in \mathbb{C}^{n\times m} \Rightarrow \widetilde{Z} \in \Omega_m. \tag{2.15}$$

Alternatively, we can characterize these extensions of Ω more constructively as noted in the following result from [10].

Proposition 2.11. *Let Ω be a subset of $\mathcal{V}_{\text{nc}}$. Then*

1. *The nc envelope $[\Omega]_{\text{nc}}$ consists of those elements of $\mathcal{V}_{\text{nc}}$ which are constructed by taking the direct sums of elements of Ω, i.e. every $Z \in [\Omega]_{\text{nc}}$ has the form*

$$Z = \begin{bmatrix} Z^{(1)} & & \\ & \ddots & \\ & & Z^{(N)} \end{bmatrix}$$

where $Z^{(1)}, Z^{(2)}, \dots, Z^{(N)} \in \Omega$.

2. *The nc similarity envelope $[\Omega]_{\text{sim}}$ consists of those elements of $\mathcal{V}_{\text{nc}}$ which are of the form $\alpha\widetilde{Z}\alpha^{-1}$ where α is an invertible matrix over $\mathbb{C}$ and $\widetilde{Z} \in [\Omega]_{\text{nc}}$.*

3. *The full nc envelope $[\Omega]_{\text{full}}$ consists of those $Z \in \mathcal{V}_{\text{nc}}$ which satisfy the intertwining relation $\mathcal{I}Z = \widetilde{Z}\mathcal{I}$ for a left injective intertwiner $\mathcal{I}$ and $\widetilde{Z} \in [\Omega]_{\text{nc}}$.*

Furthermore, we observe the following chain of containments:

$$\Omega \subset [\Omega]_{\text{nc}} \subset [\Omega]_{\text{sim}} \subset [\Omega]_{\text{full}}. \tag{2.16}$$

2.4 Extensions of noncommutative functions and kernels

In some cases, we can uniquely extend a nc function, nc kernel, or cp nc kernel to be defined on a larger domain. The following results are from [17, 10].

Proposition 2.12. *Suppose that Ω is a subset (not necessarily a nc subset) of $\mathcal{V}_{\text{nc}}$.*

1. *Any nc function* $f\colon \Omega \to \mathcal{V}_{0,\mathrm{nc}}$ *extends uniquely to a nc function* $\widetilde{f}$ *on the nc envelope* $[\Omega]_{\mathrm{nc}}$ *and on the nc similarity envelope* $[\Omega]_{\mathrm{nc,sim}}$ *but not necessarily on the full nc envelope* $[\Omega]_{\mathrm{full}}$.

2. *Any nc kernel* $K\colon \Omega \to \mathcal{L}(\mathcal{V}_1, \mathcal{V}_0)_{\mathrm{nc}}$ *extends uniquely to a nc kernel* $\widetilde{K}$ *on the nc envelope* $[\Omega]_{\mathrm{nc}}$ *and on the nc similarity envelope* $[\Omega]_{\mathrm{nc,sim}}$*, but not necessarily on the full envelope* $[\Omega]_{\mathrm{full}}$.

3. *Any cp nc kernel* $K\colon \Omega \times \Omega \to \mathcal{L}(\mathcal{V}_1, \mathcal{V}_0)_{\mathrm{nc}}$ *extends uniquely to a cp nc kernel on the nc envelope* $[\Omega]_{\mathrm{nc}}$ *and on the nc similarity envelope* $[\Omega]_{\mathrm{nc,sim}}$*, but not necessarily on the full nc envelope* $[\Omega]_{\mathrm{full}}$.

2.5 Tests for complete positivity of nc kernels

An interesting fact is that, at least in some special cases, there is finite test for complete positivity of a nc kernel with domain equal to a finitely generated nc set. The following results first appear in [10].

Proposition 2.13. *Let* Ω *be the full nc envelope* $[\Omega_F]_{\mathrm{full}}$ *of the finite subset* $\Omega_F = \{Z^{(1)}, \dots, Z^{(N)}\}$. *Suppose that* $K\colon \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{\mathrm{nc}}$ *is a nc kernel. Then* K *is a cp nc kernel if and only if*

$$K\left(\bigoplus_{i=1}^{N} Z^{(i)}, \bigoplus_{i=1}^{N} Z^{(i)}\right)$$

is a completely positive map.

Proof. Necessity follows from the definition of a cp nc kernel.

For sufficiency, proceed as follows. Let n_i denote the size of $Z^{(i)}$ (so $Z^{(i)} \in \mathcal{V}^{n_i \times n_i}$). For $1 \le i_0 \le N$, let $E^{(i_0)}$ be the $(\sum_{i=1}^{N} n_i) \times n_{i_0}$ matrix of column-block structure with i-th block column having size $n_i \times n_{i_0}$ such that the i_0-block is equal to the $i_0 \times i_0$ identity matrix I_{i_0} and all other blocks are equal to 0:

$$E^{(i_0)} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{i_0} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

From the intertwining relation $\left(\bigoplus_{i=1}^{N} Z^{(i)}\right) E^{(i_0)} = E^{(i_0)} Z^{(i_0)}$ and the respects intertwinings property (2.6), we see that

$$E^{(i_0)*} K\left(\bigoplus_{i=1}^{N} Z^{(i)}, \bigoplus_{i=1}^{N} Z^{(i)}\right)([P_{ij}])\, E^{(i_0)} = K(Z^{(i_0)}, Z^{(i_0)})(P_{i_0 i_0}).$$

We conclude that the map $K\left(\bigoplus_{i=1}^N Z^{(i)}, \bigoplus_{i=1}^N Z^{(i)}\right)$ being positive implies that the map $K(Z^{(i_0)}, Z^{(i_0)})$ is a positive map for each i_0, $1 \le i_0 \le N$. A similar argument gives that $K\left(\bigoplus_{i=1}^N Z^{(i)}, \bigoplus_{i=1}^N Z^{(i)}\right)$ being completely positive implies that $K(Z^{(i_0)}, Z^{(i_0)})$ is completely positive. More generally, $K\left(\bigoplus_{i=1}^N Z^{(i)}, \bigoplus_{i=1}^N Z^{(i)}\right)$ being completely positive implies that $K\left(\bigoplus_1^M \left(\bigoplus_{i=1}^N Z^{(i)}\right), \bigoplus_1^M \left(\bigoplus_{i=1}^N Z^{(i)}\right)\right)$ is completely positive for any $M \in \mathbb{N}$. Invoking a variant of the intertwining argument once again, we see that $K(\bigoplus_{j=1}^L Z^{(i_j)}, \bigoplus_{j=1}^L Z^{(i_j)})$ is a completely positive map, where here $\{Z^{i_1}, \ldots, Z^{i_L}\}$ is any subcollection of the set of points $\{Z^{(1)}, \ldots, Z^{(N)}\}$ with each allowed to be repeated any number of times up to M times. We conclude that $K(Z,Z)$ is completely positive for any Z in the nc envelope $(\Omega_F)_{\rm nc}$ of Ω_F.

It remains to check that $K(\widetilde{Z}, \widetilde{Z})$ is positive for any $\widetilde{Z} \in [\Omega_F]_{\rm full}$. Suppose that such a $\widetilde{Z}$ is in Ω_m. By definition there is an injective matrix $\mathcal{I} \in \mathbb{C}^{n\times m}$ and a $Z \in [\Omega_F]_{\rm nc}$ of size $n \times n$ so that $\mathcal{I}\widetilde{Z} = Z\mathcal{I}$. We use the respects intertwinings property (2.6) of the nc kernel K to see that

$$\mathcal{I}K(\widetilde{Z}, \widetilde{Z})(P)\mathcal{I}^* = K(Z,Z)(\mathcal{I}P\mathcal{I}^*) \succeq 0$$

for any $P \succeq 0$ in $\mathcal{A}^{m\times m}$. As $\mathcal{I}$ is injective, we conclude that $K(\widetilde{Z}, \widetilde{Z})(P) \succeq 0$ and $K(\widetilde{Z}, \widetilde{Z})$ is a positive map.

□

Corollary 2.14. *Let Ω and K be as in Proposition 2.13 and consider the special case where $\mathcal{A} = \mathbb{C}$. Then K is a cp nc kernel if and only if*

$$K\left(\bigoplus_{j=1}^{N'}\bigoplus_{i=1}^{N} Z^{(i)}, \bigoplus_{j=1}^{N'}\bigoplus_{i=1}^{N} Z^{(i)}\right)(\mathfrak{C}_{N'}) \tag{2.17}$$

*is positive where N' is set equal to the level of Ω containing $\bigoplus_{i=1}^N Z^{(i)}$ (i.e., $\bigoplus_{i=1}^N Z^{(i)} \in \Omega_{N'}$) and where $\mathfrak{C}_{N'}$ (the **Choi matrix** at level N') is the $(N')^2 \times (N')^2$ matrix written out as a block $N' \times N'$ matrix with $N' \times N'$ matrix entries given by*

$$\mathfrak{C}_{N'} = [E_{i,j}^{N'}]_{i,j=1,\ldots,N'}$$

where $E_{i,j}^{N'}$ is the $N' \times N'$ matrix with (i,j)-entry equal to 1 and all other entries equal to 0.

Proof. Necessity follows from the definition of a cp nc kernel and the fact that $\mathfrak{C}_{N'}$ is a positive map.

For sufficiency, we assume that (2.17) is positive. Since an nc kernel respects direct sums, we have that

$$K\left(\bigoplus_{j=1}^{N'}\bigoplus_{i=1}^{N} Z^{(i)}, \bigoplus_{j=1}^{N'}\bigoplus_{i=1}^{N} Z^{(i)}\right)(\mathfrak{C}_{N'}) = \left[K\left(\bigoplus_{i=1}^{N} Z^{(i)}, \bigoplus_{i=1}^{N} Z^{(i)}\right)\left(E_{i,j}^{N'}\right)\right]_{i',j'}$$

where the right-hand side is a block $N' \times N'$ matrix ($1 \leq i', j' \leq N'$). By [25, Theorem 3.14], the map $K\left(\bigoplus_{i=1}^{N} Z^{(i)}, \bigoplus_{i=1}^{N} Z^{(i)}\right)$ is completely positive, and the result follows from Proposition 2.13. $\square$

Chapter 3

Noncommutative reproducing kernel C^*-correspondences

3.1 Hilbert C^*-modules and correspondences

The content of this section has its roots in the seminal papers of Rieffel [28, 29] and Paschke [24]. All of the results can be found in any of the many references on the subject [19, 20, 27, 14].

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. We say that $\mathcal{E}$ is an **inner product $\mathcal{B}$-module** if $\mathcal{E}$ is a right module over $\mathcal{B}$ with a $\mathcal{B}$-valued inner product $\langle e, e' \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{B}$ such that

$$(1)\ \langle e, e'b \rangle = \langle e, e' \rangle b \tag{3.1}$$

$$(2)\ \langle e, e' \rangle^* = \langle e', e \rangle \tag{3.2}$$

$$(3)\ \langle e, e \rangle \succeq 0 \text{ and } \langle e, e \rangle = 0 \text{ if and only if } e = 0 \tag{3.3}$$

for $e, e' \in \mathcal{E}$ and $b \in \mathcal{B}$. If $\mathcal{E}$ satisfies all of the conditions above except the second part of (3.3), we say that $\mathcal{E}$ is a **semi-inner product $\mathcal{B}$-module**.

In this setting, we have the following version of the Cauchy-Schwarz inequality (see [27, Lemma 2.5] for details).

Proposition 3.1. *If $\mathcal{E}$ is a semi-inner product $\mathcal{B}$-module and $e, e' \in \mathcal{E}$, then*

$$\langle e', e \rangle \langle e, e' \rangle \preceq ||\langle e, e \rangle|| \langle e', e' \rangle. \tag{3.4}$$

For a semi-inner product $\mathcal{B}$-module $\mathcal{E}$, we define the map $||\cdot||_{\mathcal{E}} : \mathcal{E} \to \mathbb{C}$ by

$$||e||_{\mathcal{E}} := ||\langle e, e \rangle||_{\mathcal{B}}^{1/2}. \tag{3.5}$$

We see that (3.4) gives $||\langle e, e'\rangle||_{\mathcal{B}} \leq ||e||_{\mathcal{E}}||e'||_{\mathcal{E}}$, and with this fact, it is straightforward to check that (3.5) defines a semi-norm on a semi-inner product $\mathcal{B}$-module and a norm on an inner product $\mathcal{B}$-module by the usual arguments from the Hilbert-space case. We call an inner product $\mathcal{B}$-module which is complete in this norm a **Hilbert C^*-module over $\mathcal{B}$** or a **Hilbert $\mathcal{B}$-module**.

Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert $\mathcal{B}$-modules. We let $\mathcal{L}(\mathcal{E}, \mathcal{F})$ denote the set of bounded linear operators from $\mathcal{E}$ to $\mathcal{F}$. We say that $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ is a **$\mathcal{B}$-module map** if $T(eb) = T(e)b$ for $e \in \mathcal{E}$ and $b \in \mathcal{B}$. An **adjointable map** from $\mathcal{E}$ to $\mathcal{F}$ is a linear map $T : \mathcal{E} \to \mathcal{F}$ with corresponding linear map $S : \mathcal{F} \to \mathcal{E}$ such that

$$\langle Te, f\rangle_{\mathcal{F}} = \langle e, Sf\rangle_{\mathcal{E}}.$$

Proposition 3.2. *Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert $\mathcal{B}$-modules. Every adjointable map $T : \mathcal{E} \to \mathcal{F}$ is a bounded $\mathcal{B}$-module map.*

We denote the space of adjointable maps from $\mathcal{E}$ to $\mathcal{F}$ by $\mathcal{L}_a(\mathcal{E}, \mathcal{F})$ and make the abbreviation $\mathcal{L}_a(\mathcal{E}) := \mathcal{L}_a(\mathcal{E}, \mathcal{E})$.

Proposition 3.3. *Let $\mathcal{E}$ be a Hilbert $\mathcal{B}$-module. The set of adjointable maps $\mathcal{L}_a(\mathcal{E})$ is a C^*-algebra.*

We say that $\mathcal{E}$ is a **C^*-correspondence** from $\mathcal{A}$ to $\mathcal{B}$ (or $\mathcal{E}$ is an $(\mathcal{A}, \mathcal{B})$-correspondence) if $\mathcal{E}$ is a Hilbert $\mathcal{B}$-module with a left module action from $\mathcal{A}$ given by a unital $*$-homomorphism $\sigma : \mathcal{A} \to \mathcal{L}_a(\mathcal{E})$.

Example 3.4. We consider the special case where $\mathcal{E} = \mathcal{B}$ where the $\mathcal{B}$-valued inner product is given by

$$\langle b, b'\rangle = b^* b'$$

for $b, b' \in \mathcal{B}$. Properties (3.1), (3.2), and (3.3) follow directly from fact that $\mathcal{B}$ is a C^*-algebra. We note that $|| \cdot ||_{\mathcal{E}}$ is equal to the norm on the C^*-algebra $\mathcal{B}$, i.e.,

$$||b||_{\mathcal{E}} = ||\langle b, b\rangle||_{\mathcal{B}}^{1/2} = ||b^* b||_{\mathcal{B}}^{1/2} = ||b||_{\mathcal{B}}.$$

Since C^*-algebras are complete in their norm, it follows that $\mathcal{E} = \mathcal{B}$ is a Hilbert $\mathcal{B}$-module.

In this case, we have a characterization of all adjointable maps $\mathcal{L}_a(\mathcal{B})$. For $t \in \mathcal{B}$, consider the map $L_t : b \mapsto tb$. We see that

$$\langle L_t(b'), b\rangle = \langle tb', b\rangle = b'^* t^* b = b'^* L_{t^*}(b) = \langle b', L_{t^*}(b)\rangle$$

We conclude that L_t is adjointable with $L_t^* = L_{t^*}$. For any $T \in \mathcal{L}_a(\mathcal{B})$, we have

$$\begin{aligned}
\langle T(b'), b\rangle &= \langle b', T^*(b)\rangle \\
&= \langle T^*(b), b'\rangle^* \\
&= (\langle T^*(b), 1_{\mathcal{B}}\rangle b')^* \\
&= (\langle b, T(1_{\mathcal{B}})\rangle b')^* \\
&= (\langle b, T(1_{\mathcal{B}}) b'\rangle)^* \\
&= \langle T(1_{\mathcal{B}}) b', b\rangle \\
&= \langle L_{T(1_{\mathcal{B}})} b', b\rangle,
\end{aligned}$$

and we conclude that every adjointable map in $\mathcal{L}_a(\mathcal{B})$ is given by a left multiplication map L_t for some $t \in \mathcal{B}$.

We may now construct an example of a bounded linear map on a Hilbert $\mathcal{B}$-module which is not adjointable. We consider the same setting as before in the special case where $\mathcal{B} = \mathbb{C}^{2\times 2}$. Consider the map $R : \mathbb{C}^{2\times 2} \to \mathbb{C}^{2\times 2}$ given by right multiplication by the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. From the discussion above, we observe that to be adjointable the map must be given by left multiplication by a matrix, i.e., for all $b \in \mathbb{C}^{2\times 2}$, $R(b) = b\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right] b$ for some $\left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right] \in \mathbb{C}^{2\times 2}$. We see that this is impossible by setting $b = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ as

$$\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right].$$

We note that our characterization of $\mathcal{L}_a(\mathcal{B})$ is false if $\mathcal{B}$ is not unital. Let $\mathcal{B}$ be the C^*-algebra of compact operators on an infinite dimensional Hilbert space $\mathcal{Y}$. Clearly, the map L_I is an element of $\mathcal{L}_a(\mathcal{B})$. For any $B \in \mathcal{B}$, we have that $I \neq B$ as the identity operator is not compact. Thus, we can find $y \in \mathcal{Y}$ such that $I(y) = y \neq B(y)$. If we let P be the compact operator which projects onto the subspace spanned by y, we have that $L_I P = P \neq BP = L_B P$ since $P(y) = y \neq B(y) = BP(y)$. We conclude that there exist elements of $\mathcal{L}_a(\mathcal{B})$ which are not given by L_B for $B \in B$.

Example 3.5. Let $\mathcal{Y}$ be a Hilbert space with inner product $\langle y', y\rangle$ which is linear in the second argument. Then $\mathcal{Y}$ is a Hilbert C^*-module over $\mathbb{C}$ where we make the following identification:

$$\langle y', y\rangle = y'^* y.$$

We view $y \in \mathcal{Y}$ as an operator $y \in \mathcal{L}(\mathbb{C}, \mathcal{Y})$. Then $y'^* \colon \mathcal{Y} \to \mathbb{C}$ is the adjoint of the operator $y' \colon \mathbb{C} \to \mathcal{Y}$ given by

$$y'^* \colon y \mapsto y'^* y := \langle y', y\rangle.$$

Note that the expression $y'^* y$ is suggestive of the order used for a Hilbert C^*-module inner product $\langle y', y\rangle$ (linear in the second rather in the first argument).

3.2 Main theorem

In this section, we state and prove the main result of the chapter. The proof is similar to that seen in [9] where we consider the special case $\mathcal{L}_a(\mathcal{E}) = \mathcal{L}(\mathcal{Y})$.

Theorem 3.6. *Suppose that Ω is a nc subset of $\mathcal{V}_{nc}$ for some vector space $\mathcal{V}$, $\mathcal{E}$ is a Hilbert $\mathcal{B}$-module, $\mathcal{A}$ is a unital C^*-algebra, and $K\colon \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}_a(\mathcal{E}))_{\mathrm{nc}}$ is a given function. Then the following are equivalent.*

1. *K is a cp nc kernel.*
2. *There is a $(\mathcal{A}, \mathcal{B})$-correspondence $\mathcal{H}(K)$ whose elements are nc functions $f\colon \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{E})_{\mathrm{nc}}$ such that:*
   (a) *For each $W \in \Omega_m$, $v \in \mathcal{A}^{1\times m}$, and $e \in \mathcal{E}^m$, the function*
   $$K_{W,v,e}\colon \Omega_n \to \mathcal{L}(\mathcal{A}, \mathcal{E})^{n\times n} \cong \mathcal{L}(\mathcal{A}^n, \mathcal{E}^n)$$
   defined by
   $$K_{W,v,e}(Z)u = K(Z, W)(uv)e \tag{3.6}$$
   for $Z \in \Omega_n$, $u \in \mathcal{A}^n$ belongs to $\mathcal{H}(K)$.
   (b) *The kernel elements $K_{W,v,e}$ as in (3.6) have the reproducing property: For $f \in \mathcal{H}(K)$, $W \in \Omega_m$, $v \in \mathcal{A}^{1\times m}$, $e \in \mathcal{E}^m$,*
   $$\langle e, f(W)(v^*)\rangle_{\mathcal{E}^m} = \langle K_{W,v,e}, f\rangle_{\mathcal{H}(K)}. \tag{3.7}$$
   (c) *$\mathcal{H}(K)$ is equipped with a unital $*$-representation σ mapping $\mathcal{A}$ to $\mathcal{L}_a(\mathcal{H}(K))$ such that*
   $$(\sigma(a)f)(W)(v^*) = f(W)(v^*a) \tag{3.8}$$
   for $a \in \mathcal{A}$, $W \in \Omega_m$, $v \in \mathcal{A}^{1\times m}$, with action on kernel elements $K_{W,v,e}$ given by
   $$\sigma(a)\colon K_{W,v,e} = K_{W,av,e}. \tag{3.9}$$
3. *K has a Kolmogorov decomposition: There is an $(\mathcal{A}, \mathcal{B})$-correspondence $\mathcal{X}$ together with a nc function $H\colon \Omega \to \mathcal{L}_a(\mathcal{X}, \mathcal{E})_{\mathrm{nc}}$ so that*
   $$K(Z, W)(P) = H(Z)(\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \sigma)(P)H(W)^* \tag{3.10}$$
   for all $Z \in \Omega_n$, $W \in \Omega_m$, $P \in \mathcal{A}^{n\times m}$.

Proof of (1) $\Rightarrow$ (2). We assume that K is a cp nc kernel and build the $(\mathcal{A}, \mathcal{B})$-correspondence $\mathcal{H}(K)$ from kernel elements $K_{W,v,e}$.

Kernel elements are nc functions. Assume the setup as stated in part (2) of the theorem. We have

$$\begin{aligned}\|K_{W,v,e}(Z)u\|_{\mathcal{E}^n} &= \|K(Z,W)(uv)e\|_{\mathcal{E}^n} \\ &\leq \|K(Z,W)(uv)\|_{\mathcal{L}_a(\mathcal{E}^m,\mathcal{E}^n)}\|e\|_{\mathcal{E}^m} \\ &\leq \|K(Z,W)\|_{\mathcal{L}(\mathcal{A}^{n\times m},\mathcal{L}_a(\mathcal{E}^m,\mathcal{E}^n))}\|uv\|_{\mathcal{A}^{n\times m}}\|e\|_{\mathcal{E}^m} \\ &\leq \left(\|K(Z,W)\|_{\mathcal{L}(\mathcal{A}^{n\times m},\mathcal{L}_a(\mathcal{E}^m,\mathcal{E}^n))}\|v\|_{\mathcal{A}^{1\times m}}\|e\|_{\mathcal{E}^m}\right)\|u\|_{\mathcal{A}^n}\end{aligned}$$

which shows that $K_{W,v,e}(Z)$ is a bounded operator from $\mathcal{A}^n$ to $\mathcal{E}^n$ for $Z \in \Omega_n$. Thus, the kernel elements $K_{W,v,e}$ are graded functions mapping Ω to $\mathcal{L}(\mathcal{A},\mathcal{E})_{\mathrm{nc}}$. It remains to check that the kernel elements respect intertwinings. To this end, suppose that $\alpha Z = \widetilde{Z}\alpha$ where $Z \in \Omega_n$, $\widetilde{Z} \in \Omega_m$, and $\alpha \in \mathbb{C}^{m\times n}$. By (3.6) and the fact that K is a nc kernel, we have

$$\begin{aligned}\alpha K_{W,v,e}(Z)(u) &= \alpha K(Z,W)(uv)e \\ &= K(\widetilde{Z},W)(\alpha uv)e \\ &= K_{W,v,e}(\widetilde{Z})(\alpha u)\end{aligned}$$

which confirms that kernel elements respect intertwinings.

Construction of $\mathcal{H}(K)$. Let $\mathcal{H}^\circ(K)$ be the vector space given by the linear span of the kernel elements:

$$\mathcal{H}^\circ(K) = \operatorname{span}\{K_{W,v,e}\colon W \in \Omega_m,\ v \in \mathcal{A}^{1\times m},\ e \in \mathcal{E}^m,\ m = 1,2,\dots\}.$$

The following lemma shows that the kernel elements are closed under addition and scalar multiplication. Thus, the vector space $\mathcal{H}^\circ(K)$ is really just the set of kernel elements.

Lemma 3.7. *Consider the nc functions $K_{W,v,e}$ given by (3.6) for a nc kernel K. For $W^{(1)} \in \Omega_m$, $W^{(2)} \in \Omega_{\widetilde{m}}$, $v^{(1)} \in \mathcal{A}^{m\times 1}$, $v^{(2)} \in \mathcal{A}^{\widetilde{m}\times 1}$, $e^{(1)} \in \mathcal{E}^m$, $e^{(2)} \in \mathcal{E}^{\widetilde{m}}$, and $a,b \in \mathbb{C}$, we have the identity*

$$K_{\left[\begin{smallmatrix} W^{(1)} & 0 \\ 0 & W^{(2)}\end{smallmatrix}\right],\left[\begin{smallmatrix} v^{(1)} & v^{(2)}\end{smallmatrix}\right],\left[\begin{smallmatrix} ae^{(1)} \\ be^{(2)}\end{smallmatrix}\right]} = aK_{W^{(1)},v^{(1)},e^{(1)}} + bK_{W^{(2)},v^{(2)},e^{(2)}} \tag{3.11}$$

Proof. The equality (3.11) follows from the fact that nc kernels respect direct sums (2.7). In detail for $Z \in \Omega_n$ and $u \in \mathcal{A}^n$,

$$\begin{aligned}K_{\left[\begin{smallmatrix} W^{(1)} & 0 \\ 0 & W^{(2)}\end{smallmatrix}\right],\left[\begin{smallmatrix} v^{(1)} & v^{(2)}\end{smallmatrix}\right],\left[\begin{smallmatrix} ae^{(1)} \\ be^{(2)}\end{smallmatrix}\right]}(Z)(u) &= K\left(Z,\left[\begin{smallmatrix} W^{(1)} & 0 \\ 0 & W^{(2)}\end{smallmatrix}\right]\right)\left(u\left[\begin{smallmatrix} v^{(1)} & v^{(2)}\end{smallmatrix}\right]\right)\left(\left[\begin{smallmatrix} ae^{(1)} \\ be^{(2)}\end{smallmatrix}\right]\right) \\ &= \left[\begin{smallmatrix} K\left(Z,W^{(1)}\right)(uv^{(1)}) & K\left(Z,W^{(2)}\right)(uv^{(2)})\end{smallmatrix}\right]\left[\begin{smallmatrix} ae^{(1)} \\ be^{(2)}\end{smallmatrix}\right] \\ &= K\left(Z,W^{(1)}\right)(uv^{(1)})(ae^{(1)}) + K\left(Z,W^{(2)}\right)(uv^{(2)})(be^{(2)}) \\ &= aK\left(Z,W^{(1)}\right)(uv^{(1)})(e^{(1)}) + bK\left(Z,W^{(2)}\right)(uv^{(2)})(e^{(2)}) \\ &= aK_{W^{(1)},v^{(1)},e^{(1)}}(Z)(u) + bK_{W^{(2)},v^{(2)},e^{(2)}}(Z)(u).\end{aligned}$$

□

As $\mathcal{H}^\circ(K)$ consists of functions, the zero element of the space is the zero function, i.e., $f \in \mathcal{H}^\circ(K)$ such that $f(Z) = 0$ for all $Z \in \Omega$. We define a $\mathcal{B}$-valued inner product on kernel elements by

$$\langle K_{W,v,e},\, K_{W',v',e'}\rangle_{\mathcal{H}^\circ(K)} := \langle e, K(W,W')(v^*v')e'\rangle_{\mathcal{E}^m} = \langle e, K_{W',v',e'}(W)(v^*)\rangle_{\mathcal{E}^m}. \tag{3.12}$$

Thus, we have by construction that the kernel elements satisfy the reproducing property (3.7) with this inner product.

We proceed by establishing that the inner product above gives an inner product $\mathcal{B}$-module. We define a right module action on $\mathcal{H}^\circ(K)$ by $K_{W,v,e} \cdot b = K_{W,v,eb}$ and check (3.1) for kernel elements:

$$\begin{aligned}\langle K_{W,v,e},\, K_{W',v',e'} \cdot b\rangle_{\mathcal{H}^\circ(K)} &= \langle K_{W,v,e},\, K_{W',v',e'b}\rangle_{\mathcal{H}^\circ(K)}\\ &= \langle e, K(W,W')(v^*v')e'b\rangle_{\mathcal{E}^m}\\ &= \langle e, K(W,W')(v^*v')e'\rangle_{\mathcal{E}^m} b\\ &= \langle K_{W,v,e},\, K_{W',v',e'}\rangle_{\mathcal{H}^\circ(K)} b.\end{aligned}$$

Likewise, we check that (3.2) holds:

$$\begin{aligned}\langle K_{W,v,e},\, K_{W',v',e'}\rangle^*_{\mathcal{H}^\circ(K)} &= \langle e, K(W,W')(v^*v')e'\rangle^*_{\mathcal{E}^m}\\ &= \langle K(W,W')(v^*v')e', e\rangle_{\mathcal{E}^m}\\ &= \langle e', K(W,W')(v^*v')^*e\rangle_{\mathcal{E}^m}\\ &= \langle e', K(W',W)(v'^*v)e\rangle_{\mathcal{E}^m}\\ &= \langle K_{W',v',e'},\, K_{W,v,e}\rangle_{\mathcal{H}^\circ(K)}.\end{aligned}$$

It remains to check (3.3). Since K is a cp kernel, the inner product is positive semidefinite, i.e., $\langle f, f\rangle \succeq 0$ for all $f \in \mathcal{H}^\circ(K)$. To check the second part of (3.3), we use the Cauchy-Schwarz inequality (Proposition 3.1). For any kernel element $K_{W,v,e}$ such that $\langle K_{W,v,e}, K_{W,v,e}\rangle = 0$, we have

$$\begin{aligned}&\langle e', K_{W,v,e}(Z)(u)\rangle_{\mathcal{E}^n}\langle K_{W,v,e}(Z)(u), e'\rangle_{\mathcal{E}^n}\\ =&\langle K_{Z,u^*,e'}, K_{W,v,e}\rangle_{\mathcal{H}^\circ(K)}\langle K_{W,v,e}, K_{Z,u^*,e'}\rangle_{\mathcal{H}^\circ(K)}\\ \preceq&||K_{W,v,e}, K_{W,v,e}\rangle||\langle K_{Z,u^*,e'}, K_{Z,u^*,e'}\rangle_{\mathcal{H}^\circ(K)} \quad \text{(by Proposition 3.1)}\\ =&0.\end{aligned}$$

for all $Z \in \Omega_n$, $u \in \mathcal{A}^n$, and $e' \in \mathcal{E}^n$ which establishes that $K_{W,v,e}$ is the zero function. We conclude that (3.12) is an inner product $\mathcal{B}$-module.

We define a norm on $\mathcal{H}^\circ(K)$ given by $||f||_{\mathcal{H}^\circ(K)} = ||\langle f, f\rangle||^{1/2}_{\mathcal{B}}$. By [27, Lemma 2.16], we can extend $\mathcal{H}^\circ(K)$ to a Hilbert $\mathcal{B}$-module $\mathcal{H}(K)$ such that $\mathcal{H}^\circ(K)$ is dense in $\mathcal{H}(K)$, i.e., we take the completion of $\mathcal{H}^\circ(K)$ such that the space remains an inner product $\mathcal{B}$-module.

We show that $\mathcal{H}(\mathcal{K})$ consists of nc functions on Ω. We let $\mathrm{ev}_Z : \mathcal{H}^\circ(K) \to \mathcal{L}(\mathcal{A}^n, \mathcal{E}^n)$ be the linear map given by evaluating a kernel element at $Z \in \Omega_n$. We can extend the map to all of $\mathcal{H}(K)$ by continuity provided $\mathrm{ev_Z}$ is a bounded operator which in turn is a consequence of the following calculation:

$$\begin{aligned}
||\mathrm{ev}_Z(f)|| &= ||f(Z)|| \\
&= \sup_{||u||=1} ||f(Z)u|| \\
&= \sup_{||u||=||e||=1} ||\langle e, f(Z)u\rangle|| \\
&= \sup_{||u||=||e||=1} ||\langle K_{Z,u^*,e}, f\rangle|| \\
&\leq \sup_{||u||=||e||=1} ||K_{Z,u^*,e}||\,||f|| \\
&= \sup_{||u||=||e||=1} ||\langle K_{Z,u^*,e}, K_{Z,u^*,e}\rangle||^{1/2}\,||f|| \\
&= \sup_{||u||=||e||=1} ||\langle e, K(Z,Z)(uu^*)e\rangle||^{1/2}\,||f|| \\
&\leq \sup_{||u||=||e||=1} ||e||^{1/2}||K(Z,Z)(uu^*)e||^{1/2}\,||f|| \\
&\leq \sup_{||u||=||e||=1} ||K(Z,Z)||^{1/2}\,||e||\,||u||\,||f|| \\
&= ||K(Z,Z)||^{1/2}||f||.
\end{aligned} \tag{3.13}$$

Thus, we can extend $\mathrm{ev_Z}$ to a continuous map on all of $\mathcal{H}(K)$. We define point evaluations on $f \in \mathcal{H}(K)$ by $f(Z) = \mathrm{ev_Z}(f)$. To check that the elements of $\mathcal{H}(K)$ are nc functions, it remains to check intertwinings. Let (f_n) be a sequence of kernel elements which converges in norm to $f \in \mathcal{H}(K)$. For $\alpha Z = \widetilde{Z}\alpha$ where $Z \in \Omega_n$, $\widetilde{Z} \in \Omega_m$, and $\alpha \in \mathbb{C}^{m\times n}$, we have

$$\begin{aligned}
\alpha f(Z) &= \alpha \mathrm{ev}_Z(f) \\
&= \alpha \mathrm{ev}_Z(\lim_{n\to\infty} f_n) \\
&= \alpha \lim_{n\to\infty} f_n(Z) \\
&= \lim_{n\to\infty} f_n(\widetilde{Z})\alpha \\
&= \mathrm{ev}_{\widetilde{Z}}(\lim_{n\to\infty} f_n)\alpha \\
&= f(\widetilde{Z})\alpha.
\end{aligned}$$

We can verify in a similar manner that the reproducing property (3.7) holds and that the zero element is equal to the zero function in $\mathcal{H}(K)$. The following calculation shows the right $\mathcal{B}$-module action on the functions $f \in \mathcal{H}(K)$ is given by $(fb)(W)(v^*) = f(W)(v^*)b$:

$$\begin{aligned} \langle e, (fb)(W)(v^*)\rangle_{\mathcal{E}} &= \langle K_{W,v,e}, fb\rangle_{\mathcal{H}(K)} \\ &= \langle K_{W,v,e}, f\rangle_{\mathcal{H}(K)} b \\ &= \langle e, f(W)(v^*)\rangle_{\mathcal{E}} b \\ &= \langle e, f(W)(v^*)b\rangle_{\mathcal{E}}. \end{aligned} \tag{3.14}$$

We show that $\mathcal{H}(K)$ is an $(\mathcal{A}, \mathcal{B})$-correspondence. For $a \in \mathcal{A}$, we define a left module action on kernel elements $K_{W,v,e}$ by

$$\sigma(a) : K_{W,v,e} \mapsto K_{W,av,e}.$$

We check that σ is additive $\sigma(a_1 + a_2) = \sigma(a_1) + \sigma(a_2)$:

$$\begin{aligned} \sigma(a_1 + a_2)K_{W,v,e}(Z)(u) &= K_{W,a_1v+a_2v,e}(Z)(u) \\ &= K(Z, W)(ua_1v + ua_2v) \\ &= K(Z, W)(ua_1v) + K(Z, W)(ua_2v) \\ &= K_{W,a_1v,e}(Z)(u) + K_{W,a_2v,e}(Z)(u) \\ &= \sigma(a_1)K_{W,v,e}(Z)(u) + \sigma(a_2)K_{W,v,e}(Z)(u) \end{aligned}$$

multiplicative $\sigma(a_1a_2) = \sigma(a_1) \circ \sigma(a_2)$:

$$\begin{aligned} \sigma(a_1a_2)K_{W,v,e} &= K_{W,a_1a_2v,e} \\ &= \sigma(a_1)K_{W,a_2v,e} \\ &= \sigma(a_1)\sigma(a_2)K_{W,v,e} \end{aligned}$$

unital $\sigma(I_{\mathcal{A}}) = I_{\mathcal{H}(K)}$:

$$\sigma(I_{\mathcal{A}})K_{W,v,e} = K_{W,v,e}$$

and respects adjoints $\sigma(a^*) = \sigma(a)^*$:

$$\begin{aligned} \langle K_{W',v',e'}, \sigma(a)K_{W,v,e}\rangle_{\mathcal{H}(K)} &= \langle K_{W',v',e'}, K_{W,av,e}\rangle_{\mathcal{H}(K)} \\ &= \langle e', K(W', W)(v'^*av)e\rangle_{\mathcal{E}^{m'}} \\ &= \langle e', K(W', W)((a^*v')^*v)e\rangle_{\mathcal{E}^{m'}} \\ &= \langle K_{W',a^*v',e'}, K_{W,v,e}\rangle_{\mathcal{H}(K)} \\ &= \langle \sigma(a^*)K_{W',v',e'}, K_{W,v,e}\rangle_{\mathcal{H}(K)}. \end{aligned}$$

We may extend the map to all of $\mathcal{H}(K)$ by continuity provided $\sigma(a)$ is a bounded operator. As $\|a\|^2 I_{\mathcal{A}} - a^*a$ is a positive element of $\mathcal{A}$, we have that $\sigma(\|a\|^2 I_{\mathcal{A}} - a^*a)$ is a positive element of $\mathcal{H}(K)$ and so $\|a\|^2 I_{\mathcal{H}(K)} \succeq \sigma(a)^*\sigma(a)$. We conclude that $\|\sigma(a)\| \leq \|a\|$ and σ extends to a unital $*$-representation $\sigma : \mathcal{A} \mapsto \mathcal{L}_a(\mathcal{H}(K))$.

Finally, we check formula (3.8)

$$\begin{aligned}\langle e, (\sigma(a)f)(W)(v^*)\rangle_{\mathcal{E}^m} &= \langle K_{W,v,e}, \sigma(a)f\rangle_{\mathcal{H}(K)} \\ &= \langle K_{W,a^*v,e}, f\rangle_{\mathcal{H}(K)} \\ &= \langle e, f(W)(v^*a)\rangle_{\mathcal{E}^m}\end{aligned}$$

and that concludes (1) $\Rightarrow$ (2).

□

Proof of (2) $\Rightarrow$ (3). Let the $(\mathcal{A},\mathcal{B})$-correspondence $\mathcal{H}(K)$ be $\mathcal{X}$. We proceed by defining a nc function $H : \Omega \to \mathcal{L}_a(\mathcal{H}(K),\mathcal{E})_{\text{nc}}$ so that

$$K(Z,W)(P) = H(Z)(\text{id}_{\mathbb{C}^{n\times m}} \otimes \sigma)(P)H(W)^* \tag{3.15}$$

for all $Z \in \Omega_n$, $W \in \Omega_m$, $P \in \mathcal{A}^{n\times m}$.

Let $Z \in \Omega_m$ and $f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix} \in \mathcal{H}(K)^m$. We define the map $H(W) : \mathcal{H}(K)^m \to \mathcal{E}^m$ by

$$H(W)(f) = \sum_{i=1}^{m} f_i(W)(\mathbf{e_i^m} \otimes 1_{\mathcal{A}}) \tag{3.16}$$

where $\mathbf{e_1^m}, \mathbf{e_2^m}, ..., \mathbf{e_m^m}$ are the standard basis vectors in $\mathbb{C}^m$.

We check that the map is bounded:

$$\begin{aligned}||H(W)(f)||_{\mathcal{E}^m} &\le \sum_{i=1}^{m} ||f_i(W)(\mathbf{e_i^m} \otimes 1_{\mathcal{A}})|| \\ &\le \sum_{i=1}^{m} ||f_i(W)|| \\ &\le \sum_{i=1}^{m} ||\text{ev}_W||\,||f_i|| \quad \text{(see (3.13))} \\ &= \sqrt{m}||f||||\text{ev}_W||\end{aligned}$$

and a $\mathcal{B}$-module map:

$$\begin{aligned}H(W)(fb) &= \sum_{i}^{m}(f_i b)(\mathbf{e_i^m} \otimes \mathbf{1_A}) \\ &= \sum_{i}^{m} f_i(\mathbf{e_i^m} \otimes \mathbf{1_A})b \quad \text{(see (3.14))} \\ &= H(W)(f)b.\end{aligned}$$

We define extended kernel elements to be the elements of $\mathcal{H}(K)^m$ given by $K_{W,U,e} = \begin{bmatrix} K_{W,u_1,e} \\ K_{W,u_2,e} \\ \vdots \\ K_{W,u_m,e} \end{bmatrix}$ for $e \in \mathcal{E}^m$ and $U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$, $u_i \in \mathcal{A}^{1\times m}$. We have

$$\begin{aligned} \langle e, H(W)(f)\rangle_{\mathcal{E}^m} &= \langle e, \sum_{i=1}^m f_i(W)(\mathbf{e_i^m} \otimes 1_{\mathcal{A}})\rangle_{\mathcal{E}^m} \\ &= \sum_{i=1}^m \langle K_{W,\mathbf{e_i^m}^*\otimes 1_{\mathcal{A}},e}, f_i\rangle_{\mathcal{H}(K)} \\ &= \langle K_{W,1_{\mathcal{A}^{m\times m}},e}, f\rangle_{\mathcal{H}(K)^m} \end{aligned}$$

and so

$$H(W)^*(e) = K_{W,1_{\mathcal{A}^{m\times m}},e} = \begin{bmatrix} K_{W,\mathrm{e}_1^{\mathrm{m}*}\otimes 1_{\mathcal{A}},e} \\ \vdots \\ K_{W,\mathrm{e}_{\mathrm{m}}^{\mathrm{m}*}\otimes 1_{\mathcal{A}},e} \end{bmatrix}. \tag{3.17}$$

With $H(W)$ and $H(W)^*$ worked out, we proceed by constructing an analogue of σ from (3.8) for the current setting and check that the Kolmogorov decomposition (3.10) holds. For $P = [P_{ij}] \in \mathcal{A}^{n\times m}$, we define $(\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \sigma) : \mathcal{A}^{n\times m} \to \mathcal{L}_a(\mathcal{H}(K))^{n\times m}$ by $(\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \sigma)([P_{ij}]) = [\sigma(P_{ij})]$ and note its action on the extended kernel elements of $\mathcal{H}(K)^m$:

$$\begin{aligned} (\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \sigma)(P)K_{W,U,e} &= [\sigma(P_{ij}] \begin{bmatrix} K_{W,u_1,e} \\ \vdots \\ K_{W,u_m,e} \end{bmatrix} \\ &= \mathrm{col}_i[\sum_j \sigma(P_{ij})K_{W,u_j,e}] \\ &= \mathrm{col}_i[\sum_j K_{W,P_{ij}u_j,e}] \\ &= K_{W,PU,e}. \end{aligned}$$

For $Z \in \Omega_n$, $W \in \Omega_m$, and $P = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \in \mathcal{A}^{n\times m}$, we have

$$\begin{aligned} H(Z)(\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \sigma)(P)H(W)^*e &= H(Z)(\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \sigma)(P)K_{W,1_{\mathcal{A}^{m\times m}},e} \\ = H(Z)K_{W,P,e} = H(Z) \begin{bmatrix} K_{W,p_1,e} \\ K_{W,p_2,e} \\ \vdots \\ K_{W,p_n,e} \end{bmatrix} &= \sum_{i=1}^n K_{W,p_i,e}(Z)(\mathbf{e_i^m} \otimes 1_{\mathcal{A}}) \\ = \sum_{i=1}^n K(Z,W)(\mathbf{e_i^m} \otimes p_i)e &= K(Z,W)(P)e \end{aligned}$$

which provides the required Kolmogorov decomposition.

We check that H is a nc function. To this end, we suppose that $Z \in \Omega_n$, $\widetilde{Z} \in \Omega_m$ and $\alpha \in \mathbb{C}^{m\times n}$ are such that $\alpha Z = \widetilde{Z}\alpha$, and that $f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \in \mathcal{H}(K)^n$. The following computation

$$\begin{aligned}
\alpha H(Z)(f) &= \alpha\left(\sum_{j=1}^{n} f_j(Z)(\mathbf{e_j^{(n)}} \otimes 1_{\mathcal{A}})\right) = \sum_{j=1}^{n} f_j(\widetilde{Z})(\alpha\mathbf{e_j^{(n)}} \otimes 1_{\mathcal{A}}) \\
&= \sum_{j=1}^{n} f_j(\widetilde{Z})\left(\sum_{i=1}^{m} \alpha_{ij}\mathbf{e_i^{(m)}} \otimes 1_{\mathcal{A}}\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} f_j(\widetilde{Z})(\alpha_{ij}\mathbf{e_i^{(m)}} \otimes 1_{\mathcal{A}}) \\
&= \sum_{i=1}^{m}\left(\sum_{j=1}^{n} \alpha_{ij} f_j(\widetilde{Z})(\mathbf{e_i^{(m)}} \otimes 1_{\mathcal{A}})\right) = H(\widetilde{Z})(\alpha f).
\end{aligned} \tag{3.18}$$

verifies the desired result. This completes the proof of (2) ⇒ (3) in Theorem 3.6. □

Proof of (3) ⇒ (1). We assume that K has a Kolmogorov decomposition

$$K(Z,W)(P) = H(Z)(\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \sigma)(P)H(W)^*.$$

For any $Z \in \Omega_n$ and $0 \preceq P \in A^{n\times n}$, we factor P as R^*R and write out the decomposition

$$\begin{aligned}
K(Z,Z)(P) &= H(Z)(\mathrm{id}_{\mathbb{C}^{n\times n}} \otimes \sigma)(P)H(Z)^* \\
&= H(Z)(\mathrm{id}_{\mathbb{C}^{n\times n}} \otimes \sigma)(R^*R)H(Z)^* \\
&= H(Z)((\mathrm{id}_{\mathbb{C}^{n\times n}} \otimes \sigma)(R))^*((\mathrm{id}_{\mathbb{C}^{n\times n}} \otimes \sigma)(R))H(Z)^*.
\end{aligned}$$

We conclude that the kernel K is positive.

We check that K respects intertwining. Suppose that we are given $Z \in \Omega_n$, $\widetilde{Z} \in \Omega_{\tilde{n}}$, $\alpha \in \mathbb{C}^{\tilde{n}\times n}$ such that $\alpha Z = \widetilde{Z}\alpha$ and $W \in \Omega_m$, $\widetilde{W} \in \Omega_{\tilde{m}}$, $\beta \in \mathbb{C}^{\tilde{m}\times m}$ with $\beta W = \widetilde{W}\beta$. We express the kernel K by its Kolmogorov decomposition and utilize the fact that H respects intertwinings as a nc function:

$$\begin{aligned}
\alpha\, K(Z,W)(P)\,\beta^* &= \alpha H(Z)(\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \sigma)(P)H(W)^*\beta^* \\
&= H(\widetilde{Z})\alpha\,(\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \sigma)(P))\,\beta^* H(\widetilde{W})^* \\
&= H(\widetilde{Z})(\mathrm{id}_{\mathbb{C}^{\tilde{n}\times\tilde{m}}} \otimes \sigma)(\alpha P\beta^*)H(\widetilde{W})^* \\
&= K(\widetilde{Z},\widetilde{W})(\alpha P\beta^*).
\end{aligned}$$

We conclude that K is a nc kernel. □

3.3 Special cases and examples

In this section, we identify some special cases of Theorem 3.6 appearing in the literature.

In [9], completely positive nc kernels are first introduced, and we realize the main result there by considering those nc kernels of the form $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{\text{nc}}$. Other examples come from kernels without any special noncommutative structure. In Example 2.5, we see how to identify a kernel $k : X \times X \to \mathcal{L}(\mathcal{V}_1, \mathcal{V}_0)$ with a nc kernel $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{V}_1, \mathcal{V}_0)_{\text{nc}}$. Under this identification, the characterization of classical kernels given in Theorem 1.1 is the special case of Theorem 3.6 where $\mathcal{V}_1 = \mathbb{C}$ and $\mathcal{V}_0 = \mathcal{L}(\mathcal{Y})$. Likewise, we obtain the more general results of Barreto-Bhat-Liebscher-Skeide [11] by letting $\mathcal{V}_1$ and $\mathcal{V}_0$ be C^*-algebras, and we obtain the results of Murphy [23] and Szafraniec [33] by letting $\mathcal{V}_1 = \mathbb{C}$ and $\mathcal{V}_0 = \mathcal{L}_a(\mathcal{E})$.

Other examples from the theory of completely positive maps. We recall the well-known Stinespring dilation theorem [32] below.

Theorem 3.8. *Consider a linear map $\phi : \mathcal{A} \to \mathcal{L}(\mathcal{Y})$. Then the following are equivalent.*

1. *ϕ is a completely positive map.*
2. *There is a Hilbert space $\mathcal{X}$ equipped with a unital $*$-representation $\sigma : \mathcal{A} \to \mathcal{L}(\mathcal{X})$ along with a bounded operator $H : \mathcal{X} \to \mathcal{Y}$ such that*

$$\phi(a) = H\sigma(a)H^*. \tag{3.19}$$

Following the discussion in Examples 2.4 and 2.8, we may identify a completely positive map $\phi : \mathcal{A} \to \mathcal{L}(\mathcal{Y})$ with a completely positive kernel $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{\text{nc}}$ such that $K(1,1) = \phi$. We see that Theorem 3.8 follows from the equivalence (1) $\Leftrightarrow$ (3) in Theorem 3.6. In particular, the representation of ϕ given by (3.19) is the same as the Kolmogorov decomposition (3.10) when restricted to level 1:

$$\phi(a) = K(1,1)(a) = H(1)\phi(a)H(1)^*.$$

There is an analogous result for completely positive maps of the form $\phi : \mathcal{A} \to \mathcal{L}_a(\mathcal{E})$ due to Kasparov [18], and this result follows from Theorem 3.6 similarly.

Chapter 4

Noncommutative reproducing kernel W^*-correspondences

4.1 Hilbert W*-modules and correspondences

As in the previous chapter, most of the content of this section can be attributed to the work of Rieffel [28, 29] and Paschke [24]. For more recent references, see [20, 14].

We begin by listing off the definitions that we need for this section. Most of the terms have analogues already described in Section 3.1. The main difference is that in this section our objects of study should be self-dual in the appropriate sense.

Recall that a C^*-algebra (concrete) is a norm-closed, $*$-closed subalgebra of $\mathcal{L}(\mathcal{Y})$ where $\mathcal{Y}$ is a Hilbert space. A ***W**-algebra** (concrete) is a C^*-algebra which is closed in the weak-$*$ topology of $\mathcal{L}(\mathcal{Y})$. Throughout this chapter, $\mathcal{A}$ and $\mathcal{B}$ are W^*-algebras. For an inner product $\mathcal{B}$-module $\mathcal{E}$, we denote by $\mathcal{E}'$ the collection of all bounded module maps from $\mathcal{E}$ to $\mathcal{B}$. We note that every $e \in \mathcal{E}$ induces such a map $\hat{e} \in \mathcal{E}'$ given by $\hat{e} : e' \to \langle e, e' \rangle$. We say that $\mathcal{E}$ is **self-dual** if for every element $f \in \mathcal{E}'$ there is a corresponding element $e \in \mathcal{E}$ such that $f(e') = \langle e, e' \rangle$.

A **Hilbert *W**-module over $\mathcal{B}$** is a Hilbert C^*-module over $\mathcal{B}$ which is self-dual. We say that a Hilbert W^*-module over $\mathcal{B}$ is a ***W**-correspondence from $\mathcal{A}$ to $\mathcal{B}$** (or an $(\mathcal{A}, \mathcal{B})$ W^*-correspondence) if it has a left module action from $\mathcal{A}$ given by a normal (i.e. weak-$*$ continuous) unital $*$-homomorphism $\sigma : \mathcal{A} \to \mathcal{L}_a(\mathcal{E})$.

Remark 4.1. We note that a unital $*$-homomorphism $\sigma : \mathcal{A} \to \mathcal{L}_a(\mathcal{E})$ need not be normal. For instance, we can construct a non-normal representation of $\mathcal{A}$ by implementing the GNS construction with a non-normal state. For an example of a non-normal state, consider the special case $\mathcal{A} = \mathcal{L}(\mathcal{Y})$ where $\mathcal{Y}$ is a Hilbert space. Let ϕ be a state on $\mathcal{L}(\mathcal{Y})$ which annihilates the compacts and let (p_λ) be a net of finite rank projections which is weak-$*$ convergent to the identity on $\mathcal{L}(\mathcal{Y})$. We see that $\lim_\lambda \phi(p_\lambda) = 0 \neq 1 = \phi(I)$ and conclude that ϕ is non-normal. For more on this topic, see [13].

Hilbert W^*-modules have much in common with Hilbert spaces. In particular, bounded module maps on Hilbert W^*-modules are adjointable, and as subspaces, they are orthogonally complementable. We make these facts precise in the following proposition.

Proposition 4.2. *Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert W^*-modules over $\mathcal{B}$.*

1. *Every bounded module map $T : \mathcal{E} \to \mathcal{F}$ has an adjoint. Moreover, the result holds under the weaker assumption that $\mathcal{F}$ is an inner product $\mathcal{B}$-module.*
2. *([20], Proposition 2.5.4) Let $\mathcal{E} \subset \mathcal{F}$. Then $\mathcal{F} = \mathcal{E} \oplus \mathcal{E}^\perp$.*

Hilbert W^*-modules have w^*-topologies. Below, we present several important results which we use when working with this topology.

Proposition 4.3. *Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert W^*-modules over $\mathcal{B}$.*

1. *$\mathcal{E}$ has a unique Banach space predual such that the inner product on $\mathcal{E}$ is separately w^*-continuous. A net (e_λ) converges to e in the w^*-topology of $\mathcal{E}$ if and only if for each $e' \in \mathcal{E}$, the net $\langle e', e_\lambda \rangle_\mathcal{E}$ converges to $\langle e', e \rangle_\mathcal{E}$ in the w^*-topology of $\mathcal{B}$.*
2. *$\mathcal{L}_a(\mathcal{E}, \mathcal{F})$ has a Banach space predual. A net (T_λ) is w^*-convergent to $T \in \mathcal{L}_a(\mathcal{E}, \mathcal{F})$ if and only if for every $e \in \mathcal{E}$ and $f \in \mathcal{F}$, the net $\langle f, T_\lambda e \rangle_\mathcal{E}$ converges to $\langle f, Te \rangle_\mathcal{E}$ in the w^*-topology of $\mathcal{B}$.*
3. *([14], Theorem A.2.5) If $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$, then T is weak-$*$ continuous if and only if for every convergent bounded net $e_\lambda \xrightarrow{w^*} e$ in $\mathcal{E}$, we have $T(e_\lambda) \xrightarrow{w^*} T(e)$. Moreover, the result holds with $\mathcal{E}, \mathcal{F}$ being arbitrary dual Banach spaces.*
4. *([14], Corollary 8.5.8) If $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$, then T is w^*-continuous.*

The final preliminary result is a very impressive extension theorem due to W.L. Paschke [24].

Theorem 4.4. *An inner product $\mathcal{B}$-module $\mathcal{E}$ extends to a Hilbert W^*-module $\mathcal{F}$ such that*

1. *The elements of $\mathcal{F}$ coincide with $\mathcal{E}'$.*
2. *$\mathcal{E}$ is w^*-dense in $\mathcal{F}$.*
3. *For $f \in \mathcal{F}$, the norm on f as an element of $\mathcal{F}$ agrees with the norm on f as an operator $e \mapsto \langle f, e \rangle_\mathcal{F}$ on $\mathcal{E}$, i.e., $||f||_\mathcal{F} = ||f||_{\mathcal{L}(\mathcal{E}, \mathcal{B})}$.*

4.2 Main theorems

The following theorem is analogous to Theorem 3.6. The theorem statements are more or less identical aside from the fact that we have traded Hilbert C^*-modules for Hilbert W^*-modules. Some parts of the proof here are identical to Theorem 3.6 as well, but many aspects of the proof are more technical as we work with w^*-topologies in place of the norm topologies used in the previous result. The payoff is that the pleasant properties of Hilbert W^*-modules laid out in the previous section allow us to prove an analogue of Theorem 1.2.

Theorem 4.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be W^*-algebras. Suppose that Ω is a nc subset of $\mathcal{V}_{nc}$, $\mathcal{E}$ is a Hilbert W^*-module over $\mathcal{B}$, and $K\colon \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}_a(\mathcal{E}))_{\rm nc}$ is a given function. Then the following are equivalent.*

1. *K is a cp nc kernel such that the maps $K(Z, W) \in \mathcal{L}(\mathcal{A}^{n\times m}, \mathcal{L}_a(\mathcal{E})^{n\times m})$ are weak-$*$ continuous.*
2. *There is an $(\mathcal{A}, \mathcal{B})$ W^*-correspondence $\mathcal{H}(K)$ whose elements are nc functions $f\colon \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{E})_{\rm nc}$ such that:*
   (a) *For each $W \in \Omega_m$, $v \in \mathcal{A}^{1\times m}$, and $e \in \mathcal{E}^m$, the function*
   $$K_{W,v,e}\colon \Omega_n \to \mathcal{L}(\mathcal{A}, \mathcal{E})^{n\times n} \cong \mathcal{L}(\mathcal{A}^n, \mathcal{E}^n)$$
   defined by
   $$K_{W,v,e}(Z)u = K(Z, W)(uv)e \tag{4.1}$$
   for $Z \in \Omega_n$, $u \in \mathcal{A}^n$ belongs to $\mathcal{H}(K)$.
   (b) *The kernel elements $K_{W,v,e}$ as in (4.1) have the reproducing property: For $f \in \mathcal{H}(K)$, $W \in \Omega_m$, $v \in \mathcal{A}^{1\times m}$, $e \in \mathcal{E}^m$,*
   $$\langle e, f(W)(v^*)\rangle_{\mathcal{E}^m} = \langle K_{W,v,e}, f\rangle_{\mathcal{H}(K)}. \tag{4.2}$$
   (c) *$\mathcal{H}(K)$ is equipped with a normal unital $*$-representation σ mapping $\mathcal{A}$ to $\mathcal{L}_a(\mathcal{H}(K))$ such that*
   $$(\sigma(a)f)(W)(v^*) = f(W)(v^*a) \tag{4.3}$$
   for $a \in \mathcal{A}$, $W \in \Omega_m$, $v \in \mathcal{A}^{1\times m}$, with action on kernel elements $K_{W,v,e}$ given by
   $$\sigma(a)\colon K_{W,v,e} = K_{W,av,e}. \tag{4.4}$$
3. *K has a Kolmogorov decomposition: There is an $(\mathcal{A}, \mathcal{B})$ W^*-correspondence $\mathcal{X}$ together with a nc function $H\colon \Omega \to \mathcal{L}_a(\mathcal{X}, \mathcal{E})_{\rm nc}$ so that*
   $$K(Z, W)(P) = H(Z)(\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \sigma)(P)H(W)^* \tag{4.5}$$
   for all $Z \in \Omega_n$, $W \in \Omega_m$, $P \in \mathcal{A}^{n\times m}$.

Proof. (1) $\Rightarrow$ (2):

In the proof of Theorem 3.6, we introduced $\mathcal{H}^\circ(K)$ which we define again

$$\mathcal{H}^\circ(K) = \operatorname{span}\{K_{W,v,e}\colon W \in \Omega_m,\ v \in \mathcal{A}^{1\times m},\ e \in \mathcal{E}^m,\ m = 1, 2, \dots\},$$

and as before, we see by Lemma 3.7 that $\mathcal{H}^\circ(K)$ is really just the set of kernel elements. The space is equipped with a $\mathcal{B}$-valued inner product defined by

$$\langle K_{W,v,e},\ K_{W',v',e'}\rangle_{\mathcal{H}^\circ(K)} = \langle e, K(W,W')(v^*v')e'\rangle_{\mathcal{E}^m} = \langle e, K_{W',v',e'}(W)(v^*)\rangle_{\mathcal{E}^m}. \tag{4.6}$$

We show in the proof of Theorem 3.6 that $\mathcal{H}^\circ(K)$ is an inner product $\mathcal{B}$-module. We quote Theorem 4.4 to extend $\mathcal{H}^\circ(K)$ to a self-dual Hilbert $\mathcal{B}$-module $\mathcal{H}(K)$ which equals $\mathcal{H}^\circ(K)'$.

We show that $\mathcal{H}(K)$ consists of nc functions which have the reproducing property (4.2). For $W \in \Omega_m$, $v \in \mathcal{A}^{1\times m}$, and $m \in \mathbb{N}$, we define a $\mathcal{B}$-module map $T_{W,v} : \mathcal{E}^m \to \mathcal{H}(K)$ given by $T_{W,v}(e) = K_{W,v,e}$. We see that this map is bounded

$$\begin{aligned}
||T_{W,v}(e)||^2_{\mathcal{H}(K)} &= ||K_{W,v,e}||^2_{\mathcal{H}(K)} \\
&= ||\langle K_{W,v,e}, K_{W,v,e}\rangle_{\mathcal{H}(K)}||_{\mathcal{E}^m} \\
&= ||\langle e, K(W,W)(v^*v)(e)\rangle_{\mathcal{E}^m}||_{\mathcal{E}^m} \\
&\le ||e||_{\mathcal{E}^m}||K(W,W)(v^*v)(e)||_{\mathcal{E}^m} \\
&\le ||e||^2_{\mathcal{E}^m}||K(W,W)(v^*v)||_{\mathcal{L}_a(\mathcal{E}^m)},
\end{aligned}$$

and conclude that

$$||T_{W,v}||_{\mathcal{L}_a(\mathcal{E}^m,\mathcal{H}(K))} \le ||K(W,W)(v^*v)||^{1/2}_{\mathcal{L}_a(\mathcal{E}^m)}. \tag{4.7}$$

By Proposition 4.2, it has an adjoint T^*. We make the identification $f(W)(v^*) = T^*_{W,v}(f)$ and show that with this identification f is a nc function. We check that $f(W)$ is a linear map. For $a, b \in \mathbb{C}$, we have

$$\begin{aligned}
\langle e, f(W)(av^* + bu^*)\rangle_{\mathcal{E}^m} &= \langle e, T^*_{W,\bar{a}v+\bar{b}u}(f)\rangle_{\mathcal{E}^m} \\
&= \langle T_{W,\bar{a}v+\bar{b}u}e, f\rangle_{\mathcal{H}(K)} \\
&= \langle K_{W,\bar{a}v+\bar{b}u,e}, f\rangle_{\mathcal{H}(K)} \\
&= \langle \bar{a}K_{W,v,e} + \bar{b}K_{W,u,e}, f\rangle_{\mathcal{H}(K)} \\
&= \langle e, af(W)(v^*) + bf(W)(u^*)\rangle_{\mathcal{H}(K)}.
\end{aligned}$$

We make use of (4.7) to check that $f(W)$ is bounded

$$\begin{aligned}
||f(W)(v^*)||_{\mathcal{E}^m} &= ||T^*_{W,v}(f)||_{\mathcal{E}^m} \\
&\le ||T^*_{W,v}||_{\mathcal{L}_a(\mathcal{H}(K),\mathcal{E}^m)}||f||_{\mathcal{H}(K)} \\
&= ||T_{W,v}||_{\mathcal{L}_a(\mathcal{E}^m,\mathcal{H}(K))}||f||_{\mathcal{H}(K)} \\
&\le ||K(W,W)(v^*v)||^{1/2}_{\mathcal{L}_a(\mathcal{E}^m)}||f||_{\mathcal{H}(K)} \\
&\le ||K(W,W)||^{1/2}_{\mathcal{L}(\mathcal{A}^{m\times m},\mathcal{L}_a(\mathcal{E})^{m\times m})}||v^*||_{\mathcal{A}^m}||f||_{\mathcal{H}(K)}
\end{aligned}$$

We see that f is graded by inspection, so it remains to show that f respects intertwinings. To do so, we first present a helpful lemma.

Lemma 4.6. *Given a nc kernel K as in Theorem 4.5, consider the nc functions $K_{W,v,e}$ given by* (4.1)*. For $W \in \Omega_m$, $\widetilde{W} \in \Omega_{\widetilde{m}}$, and $\alpha \in \mathbb{C}^{\widetilde{m}\times m}$ such that $\alpha W = \widetilde{W}\alpha$, we have the identity*

$$K_{W,v,\alpha^* e} = K_{\widetilde{W},v\alpha^*,e}.$$

Proof. For any $Z \in \Omega_n$, $u \in \mathcal{A}^n$, and $n \in \mathbb{N}$, the identity follows from the fact that the nc kernel K respects intertwinings (2.6):

$$\begin{aligned} K_{W,v,\alpha^* e}(Z)(u) &= K(Z,W)(uv)(\alpha^* e) \\ &= K(Z,\widetilde{W})(uv\alpha^*)(e) \\ &= K_{\widetilde{W},v\alpha^*,e}(Z)(u). \end{aligned}$$

□

With Lemma 4.6, we continue by showing that f respects intertwinings (2.2):

$$\begin{aligned} \langle e, \alpha f(W)(v^*)\rangle_{\mathcal{E}^{\widetilde{m}}} &= \langle \alpha^* e, T^*_{W,v^*} f\rangle_{\mathcal{E}^{\widetilde{m}}} \\ &= \langle K_{W,v,\alpha^* e}, f\rangle_{\mathcal{H}(K)} \\ &= \langle K_{\widetilde{W},v\alpha^*,e}, f\rangle_{\mathcal{H}(K)} \\ &= \langle e, T^*_{\widetilde{W},\alpha v^*} f\rangle_{\mathcal{E}^{\widetilde{m}}} \\ &= \langle e, f(\widetilde{W})(\alpha v^*)\rangle_{\mathcal{E}^{\widetilde{m}}}. \end{aligned}$$

We conclude that f is a nc function.

Proposition 4.7. *An element $f \in \mathcal{H}(K)$ is the zero function if and only if $||f||_{\mathcal{H}(K)} = 0$.*

Proof. Assume that $||f||_{\mathcal{H}(K)} = 0$. For any $W \in \Omega_m$, $v \in \mathcal{A}^{1\times m}$, $e \in \mathcal{E}^M$, we have

$$||\langle e, f(W)(v^*)\rangle_{\mathcal{E}^m}|| = ||\langle K_{W,v,e}, f\rangle_{\mathcal{H}(K)}|| \leq ||K_{W,v,e}||_{\mathcal{H}(K)}||f||_{\mathcal{H}(K)} = 0 \tag{4.8}$$

which implies that f is the zero function.

Assume now that f is the zero function. We use the characterization of the $\mathcal{H}(K)$ norm presented in Theorem 4.4 to show that $||f||_{\mathcal{H}(K)} = 0$:

$$\begin{aligned} ||f||_{\mathcal{H}(K)} &= \sup_{||K_{W,v,e}||=1} ||f(K_{W,v,e})||_{\mathcal{B}} \\ &= \sup_{||K_{W,v,e}||=1} ||\langle f, K_{W,v,e}\rangle||_{\mathcal{B}} \\ &= \sup_{||K_{W,v,e}||=1} ||\langle f(W)(v^*), e\rangle||_{\mathcal{B}} \\ &= 0. \end{aligned} \tag{4.9}$$

□

Corollary 4.8. *For $f \in \mathcal{H}(K)$, the element $f = 0$ if and only if $f(K_{W,v,e}) = \langle f, K_{W,v,e}\rangle_{\mathcal{H}(K)} = 0$ for all $K_{W,v,e} \in \mathcal{H}^\circ(K)$.*

The final element of (1) ⇒ (2) is the construction of a unital *-representation σ mapping $\mathcal{A}$ to $\mathcal{L}_a(\mathcal{H}(K))$ which satisfies (4.3) and (4.4).

For $f \in \mathcal{H}(K)$, consider the linear map $\sigma(a)f : \mathcal{H}^\circ(K) \to \mathcal{B}$ given by

$$(\sigma(a)f)(K_{W,v,e}) := \langle f, K_{W,a^*v,e}\rangle_{\mathcal{H}(K)} = \langle f(W)(v^*a), e\rangle_{\mathcal{E}^m}. \tag{4.10}$$

We show that $\sigma(a)f$ is an element of $\mathcal{H}^\circ(K)'$. Indeed, it is a module map:

$$\begin{aligned}(\sigma(a)f)(K_{W,v,e})b &= \langle f, K_{W,a^*v,e}\rangle_{\mathcal{H}(K)}b \\ &= \langle f, K_{W,a^*v,eb}\rangle_{\mathcal{H}(K)} \\ &= (\sigma(a)f)(K_{W,v,eb}) \\ &= (\sigma(a)f)(K_{W,v,e}b)\end{aligned}$$

and bounded:

$$\begin{aligned}||(\sigma(a)f)(K_{W,v,e})||_{\mathcal{B}} &= ||\langle f, K_{W,a^*v,e}\rangle_{\mathcal{H}(K)}||_{\mathcal{B}} \\ &\leq ||f||_{\mathcal{H}(K)}||K_{W,a^*v,e}||_{\mathcal{H}(K)} \\ &= ||f||_{\mathcal{H}(K)}||\langle K_{W,a^*v,e}, K_{W,a^*v,e}\rangle||_{\mathcal{B}}^{1/2} \\ &= ||f||_{\mathcal{H}(K)}||\langle e, K(W,W)(v^*aa^*v)e\rangle||_{\mathcal{B}}^{1/2} \\ &= ||f||_{\mathcal{H}(K)}||\langle e, K(W,W)(v^*\frac{aa^*}{||a||_{\mathcal{A}}^2}v)e\rangle||_{\mathcal{B}}^{1/2}||a||_{\mathcal{A}} \\ &\leq ||f||_{\mathcal{H}(K)}||\langle e, K(W,W)(v^*v)e\rangle||_{\mathcal{B}}^{1/2}||a||_{\mathcal{A}} \\ &= ||f||_{\mathcal{H}(K)}||\langle K_{W,v,e}, K_{W,v,e}\rangle||_{\mathcal{B}}^{1/2}||a||_{\mathcal{A}} \\ &= ||f||_{\mathcal{H}(K)}||K_{W,v,e}||_{\mathcal{H}(K)}||a||_{\mathcal{A}}.\end{aligned} \tag{4.11}$$

We conclude that $\sigma(a)f$ is an element of $\mathcal{H}^\circ(K)'$ and therefore an element of $\mathcal{H}(\mathcal{K})$. Furthermore, the function $\sigma(a)f$ satisfies the identity (4.3). We calculate the action of $\sigma(a)$ on kernel elements

$$\begin{aligned}\langle\sigma(a)K_{W',v',e'}(W)v^*, e\rangle_{\mathcal{E}^m} &= \langle K_{W',v',e'}(W)(v^*a), e\rangle_{\mathcal{E}^m} \\ &= \langle K(W',W)(v^*av')(e'), e\rangle_{\mathcal{E}^m} \\ &= \langle K_{W',av',e'}(W)v^*, e\rangle_{\mathcal{E}^m}\end{aligned}$$

and observe that (4.4) holds as well.

Next, we prove that $\sigma(a)$ is an element of $\mathcal{L}_a(\mathcal{H}(K))$. It follows from (4.11) and the characterization of the $\mathcal{H}(K)$ norm presented in Theorem 4.4 that $\sigma(a)$ is bounded and

$||\sigma(a)|| \leq ||a||$. To prove that $\sigma(a)$ is a module map, we show that $(\sigma(a)f)b - \sigma(a)(fb)$ is the zero function in $\mathcal{H}(K)$ for any f:

$$\begin{aligned}\langle((\sigma(a)f)b - \sigma(a)(fb))(W)(v^*), e\rangle_{\mathcal{E}^m} &= \langle(\sigma(a)f)b - \sigma(a)(fb), K_{W,v,e}\rangle_{\mathcal{H}(K)} \\ &= \langle(\sigma(a)f)b, K_{W,v,e}\rangle_{\mathcal{H}(K)} - \langle\sigma(a)(fb), K_{W,v,e}\rangle_{\mathcal{H}(K)} \\ &= b^*\langle\sigma(a)f, K_{W,v,e}\rangle_{\mathcal{H}(K)} - \langle fb, K_{W,a^*v,e}\rangle_{\mathcal{H}(K)} \\ &= b^*\langle f, K_{W,a^*v,e}\rangle_{\mathcal{H}(K)} - b^*\langle f, K_{W,a^*v,e}\rangle_{\mathcal{H}(K)} \\ &= 0.\end{aligned}$$

The existence of the adjoint comes for free by Proposition 4.2.

It remains to show that σ is a normal unital $*$-representation. We check that the map is additive $\sigma(a+b) = \sigma(a) + \sigma(b)$:

$$\begin{aligned}\langle(\sigma(a+b)f)(W)v^*, e\rangle_{\mathcal{E}^m} &= \langle f(W)(v^*a + v^*b), e\rangle_{\mathcal{E}^m} \\ &= \langle f(W)(v^*a) + f(W)(v^*b), e\rangle_{\mathcal{E}^m} \\ &= \langle(\sigma(a) + \sigma(b))f)(W)v^*, e\rangle_{\mathcal{E}^m}\end{aligned}$$

multiplicative $\sigma(ab) = \sigma(a)\sigma(b)$:

$$\begin{aligned}\langle(\sigma(ab)f)(W)v^*, e\rangle_{\mathcal{E}^m} &= \langle f(W)(v^*ab), e\rangle_{\mathcal{E}^m} \\ &= \langle(\sigma(b)f)(W)(v^*a), e\rangle_{\mathcal{E}^m} \\ &= \langle(\sigma(a)\sigma(b)f)(W)(v^*), e\rangle_{\mathcal{E}^m}\end{aligned}$$

unital $\sigma(I_{\mathcal{A}}) = I_{\mathcal{H}(K)}$:

$$\langle(\sigma(I_{\mathcal{A}})f)(W)v^*, e\rangle_{\mathcal{E}^m} = \langle f(W)v^*, e\rangle_{\mathcal{E}^m}$$

and respects adjoints $\sigma(a)^* = \sigma(a^*)$:

$$\begin{aligned}\langle(\sigma(a)^* - \sigma(a^*))f(W)(v^*), e\rangle_{\mathcal{E}^m} &= \langle(\sigma(a)^* - \sigma(a^*))f, K_{W,v,e}\rangle_{\mathcal{H}(K)} \\ &= \langle\sigma(a)^*f, K_{W,v,e}\rangle_{\mathcal{H}(K)} - \langle\sigma(a^*)f, K_{W,v,e}\rangle_{\mathcal{H}(K)} \\ &= \langle f, \sigma(a)K_{W,v,e}\rangle_{\mathcal{H}(K)} - \langle f, K_{W,av,e}\rangle_{\mathcal{H}(K)} \\ &= \langle f, K_{W,av,e}\rangle_{\mathcal{H}(K)} - \langle f, K_{W,av,e}\rangle_{\mathcal{H}(K)} \\ &= 0.\end{aligned}$$

We prove that σ is normal (i.e. w^*-continuous). In the following arguments, we make frequent use of the w^*-convergence and w^*-continuity results presented in Proposition 4.3. Let (a_λ) be a bounded net which converges to a in the weak-$*$ topology of $\mathcal{A}$. As a consequence of $K(Z,W)$ being w^*-continuous, we have

$$\begin{aligned}w^*\text{-}\lim_\lambda\langle K_{W,a_\lambda v,e}, K_{Z,u^*,e'}\rangle_{\mathcal{H}(K)} &= w^*\text{-}\lim_\lambda\langle e, K(Z,W)(ua_\lambda v)e'\rangle_{\mathcal{E}^m} \\ &= \langle e, K(Z,W)(uav)e'\rangle_{\mathcal{E}^m} \\ &= \langle K_{W,av,e}, K_{Z,u^*,e'}\rangle_{\mathcal{H}(K)}\end{aligned}$$

from which we conclude that $\langle K_{W,a_\lambda v,e} - K_{W,av,e}, g\rangle \xrightarrow{w^*} 0$ for $g \in \mathcal{H}^\circ(K)$. As (a_λ) is a bounded net, the net $K_\lambda = (K_{W,a_\lambda v,e} - K_{W,av,e})$ in $\mathcal{H}(K)$ is bounded as well. Consider any subnet of (K_λ) denoted by $(K_{\lambda'})$. By the Banach-Alaoglŭ theorem, there is a w^*-convergent subnet $(K_{\lambda''})$ of $(K_{\lambda'})$ converging to some $K \in \mathcal{H}(K)$. Thus for $f \in \mathcal{H}(K)$, we have

$$w^*\text{-}\lim_{\lambda}\langle K_{\lambda''}, f\rangle_{\mathcal{H}(K)} = \langle K, f\rangle_{\mathcal{H}(K)}.$$

From the preceding discussion, we know that this limit is zero when $f \in \mathcal{H}(K)^\circ$ and from Corollary 4.8, we see that $K = 0$. Since any subnet of (K_λ) has a subnet converging to 0, we conclude that $K_{W,a_\lambda v,e} \xrightarrow{w^*} K_{W,av,e}$. Thus, we have that σ is weak-$*$ continuous over the kernel elements $\mathcal{H}^\circ(K)$.

It remains to show that σ is normal over all of $\mathcal{H}(K)$. Again, we let (a_λ) be a bounded net which converges to a in the weak-$*$ topology of $\mathcal{A}$. From the previous discussion, we have

$$\begin{aligned} w^*\text{-}\lim_{\lambda}\langle \sigma(a_\lambda)f, K_{W,v,e}\rangle_{\mathcal{H}(K)} &= w^*\text{-}\lim_{\lambda}\langle f, \sigma(a_\lambda^*)K_{W,v,e}\rangle_{\mathcal{H}(K)} \\ &= \langle f, \sigma(a^*)K_{W,v,e}\rangle_{\mathcal{H}(K)} \\ &= \langle \sigma(a)f, K_{W,v,e}\rangle_{\mathcal{H}(K)} \end{aligned}$$

from which we conclude that $\langle \sigma(a_\lambda)f - \sigma(a)f, g\rangle \xrightarrow{w^*} 0$ for $g \in \mathcal{H}^\circ(K)$. By repeating the arguments made in the previous paragraph, we see that $\sigma(a_\lambda)f \xrightarrow{w^*} \sigma(a)f$ and conclude that σ is weak-$*$ continuous over all of $\mathcal{H}(K)$.

(2) $\Rightarrow$ (3): The proof is identical to the proof of (2) $\Rightarrow$ (3) presented in Theorem 3.6.

(3) $\Rightarrow$ (1): Again, the proof is the same as the proof (3) $\Rightarrow$ (1) presented in Theorem 3.6 except in this case we must establish that the maps $K(Z,W) \in \mathcal{L}(\mathcal{A}^{n\times m}, \mathcal{L}_a(\mathcal{E})^{n\times m})$ are weak-* continuous. The fact that $\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \sigma(P)$ is w^*-continuous follows from the fact that σ is normal, and by Proposition 4.3 item (4), we see that $H(Z)$ is w^*-continuous as well. Thus for a bounded net $P_\lambda \in \mathcal{A}^{n\times m}$ which is w^*-convergent to P, we have

$$\begin{aligned} w^*\text{-}\lim_{\lambda}\langle K(Z,W)(P_\lambda)e, e'\rangle &= w^*\text{-}\lim_{\lambda}\langle H(Z)(\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \sigma)(P_\lambda)H(W)^*e, e'\rangle \\ &= \langle H(Z)w^*\text{-}\lim_{\lambda}(\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \sigma)(P_\lambda)H(W)^*e, e'\rangle \\ &= \langle H(Z)(\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \sigma)(P)H(W)^*e, e'\rangle \\ &= \langle K(Z,W)(P)e, e'\rangle. \end{aligned}$$

□

Theorem 4.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be W^*-algebras. Suppose that Ω is a nc subset of $\mathcal{V}_{\mathrm{nc}}$, $\mathcal{E}$ is a Hilbert W^*-module over $\mathcal{B}$, and $\mathcal{H}$ is a W^*-correspondence from $\mathcal{A}$ to $\mathcal{B}$ whose elements are nc functions $f : \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{E})_{\mathrm{nc}}$ such that*

1. *For each* $W \in \Omega_m$*, the map* ev_W *given by* $f \mapsto f(W)$ *is bounded as an operator from* $\mathcal{H}$ *to* $\mathcal{L}(\mathcal{A},\mathcal{E})^{m\times m} \cong \mathcal{L}(\mathcal{A}^m,\mathcal{E}^m)$*,*

2. *The right module action of* $\mathcal{B}$ *is given by*

$$(fb)(W)(u) = f(W)(u)b \tag{4.12}$$

for $f \in \mathcal{H}$*,* $W \in \Omega_m$*,* $u \in \mathcal{A}^m$*, and*

3. *The left module action of* $\mathcal{A}$ *on* $\mathcal{H}$ *is given by*

$$(\sigma(a)f)(W)(u) = f(W)(ua). \tag{4.13}$$

Then there is a cp nc kernel K *which is weak-$*$ continuous such that* $\mathcal{H}$ *is isometrically equal to* $\mathcal{H}(K)$ *defined in statement (2) of Theorem 4.5.*

Proof. For $W \in \Omega_m$, $v \in \mathcal{A}^{1\times m}$, and $e \in \mathcal{E}^m$, we define the map $\widehat{K}_{W,v,e} : \mathcal{H} \to \mathcal{B}$ by

$$\widehat{K}_{W,v,e}(f) = \langle e, f(W)(v^*)\rangle_{\mathcal{E}^m}.$$

We see that it is a $\mathcal{B}$-module map:

$$\begin{aligned}\widehat{K}_{W,v,e}(f)b &= \langle e, f(W)(v^*)\rangle_{\mathcal{E}^m} b\\ &= \langle e, f(W)(v^*)b\rangle_{\mathcal{E}^m}\\ &= \langle e, (fb)(W)(v^*)\rangle_{\mathcal{E}^m}\\ &= \widehat{K}_{W,v,e}(fb)\end{aligned}$$

and bounded:

$$\begin{aligned}||\widehat{K}_{w,v,e}(f)||_{\mathcal{B}} &= ||\langle e, f(W)(v^*)\rangle_{\mathcal{E}^m}||_{\mathcal{B}}\\ &\le ||e||_{\mathcal{E}^m}||f(W)||_{\mathcal{L}(\mathcal{A}^m,\mathcal{E}^m)}||v^*||_{\mathcal{A}^m}\\ &\le ||e||_{\mathcal{E}^m}||\mathrm{ev}_W||_{\mathcal{L}(\mathcal{H}(K),\mathcal{L}(\mathcal{A}^m,\mathcal{E}^m))}||f||_{\mathcal{H}(K)}||v^*||_{\mathcal{A}^m}.\end{aligned}$$

Thus, $\widehat{K}_{w,v,e} \in \mathcal{H}(K)'$ and as $\mathcal{H}(K)$ is self-dual, we may view $\widehat{K}_{W,v,e}$ as an element of $\mathcal{H}(K)$ such that

$$\langle \widehat{K}_{w,v,e}, f\rangle_{\mathcal{H}(K)} = \langle e, f(W)(v^*)\rangle_{\mathcal{E}^m}. \tag{4.14}$$

We call these functions the kernel elements of $\mathcal{H}$.

For kernel elements, we have that $\sigma(a)\widehat{K}_{W,v,e} = \widehat{K}_{W,av,e}$ by the following calculation: For $Z \in \Omega_n$, $u \in \mathcal{A}^n$, and $e' \in \mathcal{E}^n$,

$$\begin{aligned} \langle e', (\sigma(a)\widehat{K}_{W,v,e})(Z)u\rangle_{\mathcal{E}^n} &= \langle \widehat{K}_{Z,u^*,e'}, \sigma(a)\widehat{K}_{W,v,e}\rangle_{\mathcal{H}} \\ &= \langle \sigma(a^*)\widehat{K}_{Z,u^*,e'}, \widehat{K}_{W,v,e}\rangle_{\mathcal{H}} \\ &= \langle (\sigma(a^*)\widehat{K}_{Z,u^*,e'}(W)(v^*), e\rangle_{\mathcal{E}^m} \\ &= \langle \widehat{K}_{Z,u^*,e'}(W)(v^*a^*), e\rangle_{\mathcal{E}^m} \\ &= \langle \widehat{K}_{Z,u^*,e'}, K_{W,av,e}\rangle_{\mathcal{H}} \\ &= \langle e', \widehat{K}_{W,av,e}(Z)u\rangle_{\mathcal{E}^n}. \end{aligned} \tag{4.15}$$

We also establish that $\widehat{K}_{W,v,e} + \widehat{K}_{W,v',e} = \widehat{K}_{W,v+v',e}$:

$$\begin{aligned} \langle e', (\widehat{K}_{W,v,e} + \widehat{K}_{W,v',e})(Z)u\rangle_{\mathcal{E}^n} &= \langle \widehat{K}_{Z,u^*,e'}, \widehat{K}_{W,v,e} + K_{W,v',e}\rangle_{\mathcal{H}} \\ &= \langle \widehat{K}_{Z,u^*,e'}(W)v^*, e\rangle_{\mathcal{E}^m} + \langle \widehat{K}_{Z,u^*,e'}(W)v'^*, e\rangle_{\mathcal{E}^m} \\ &= \langle \widehat{K}_{Z,u^*,e'}(W)(v^* + v'^*), e\rangle_{\mathcal{E}^m} \\ &= \langle \widehat{K}_{Z,u^*,e'}, \widehat{K}_{W,v+v',e}\rangle_{\mathcal{H}} \\ &= \langle e', \widehat{K}_{W,v+v',e}(Z)u\rangle_{\mathcal{E}^n}. \end{aligned} \tag{4.16}$$

We proceed as in the proof of $(ii) \Rightarrow (iii)$ of Theorem 3.6. For $W \in \Omega_m$ and $f = \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix} \in \mathcal{H}^m$, we define the map $H(W) : \mathcal{H}^m \to \mathcal{E}^m$ by

$$H(W)(f) = \sum_{i=1}^{m} f_i(W)(\mathbf{e_i^m} \otimes 1_{\mathcal{A}})$$

where $\mathbf{e_1^m}, \mathbf{e_2^m}, ..., \mathbf{e_m^m}$ are the standard basis vectors in $\mathbb{C}^m$. As in the proof of Theorem 3.6, we see that H is a nc function $H : \Omega \to \mathcal{L}_a(\mathcal{H}, \mathcal{E})_{\text{nc}}$ such that the adjoint of $H(Z)$ is given by

$$H(W)^*(e) = \begin{bmatrix} \widehat{K}_{W,\mathrm{e}_1^{\mathrm{m}*}\otimes 1_{\mathcal{A}},e} \\ \vdots \\ \widehat{K}_{W,\mathrm{e}_{\mathrm{m}}^{\mathrm{m}*}\otimes 1_{\mathcal{A}},e} \end{bmatrix}.$$

We define a function $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}_a(\mathcal{E}))_{\text{nc}}$ by $K(Z,W) = H(Z)(\text{id}_{\mathbb{C}^{n\times m}} \otimes \sigma(P)H(W)^*$. By Theorem 4.5, we see that K is a cp nc kernel such that the maps $K(Z,W) \in \mathcal{L}(\mathcal{A}^{n\times m}, \mathcal{L}_a(\mathcal{E})^{n\times m})$ are weak-$*$ continuous.

It remains to show that $\mathcal{H}$ is isometrically equal to the $(\mathcal{A},\mathcal{B})$ W^*-correspondence $\mathcal{H}(K)$ described in Theorem 4.5. To this end, we show that the kernel elements of $\mathcal{H}$ and the kernel elements of $\mathcal{H}(K)$ are identical as nc functions from Ω to $\mathcal{L}(\mathcal{A},\mathcal{E})_{\rm nc}$ and that they agree with respect to their $\mathcal{B}$-valued inner products:

$$\begin{aligned}
\langle e', K_{W,v,e}(Z)u\rangle_{\mathcal{E}^n} &= \langle K_{Z,u^*,e'}, K_{W,v,e}\rangle_{\mathcal{H}(K)} \\
&= \langle e', K(Z,W)(uv)e\rangle_{\mathcal{E}^n} \\
&= \langle e', H(Z)(\mathrm{id}\otimes\sigma)(uv)H(W)^*e\rangle_{\mathcal{E}^n} \\
&= \langle H(Z)^*e', (\mathrm{id}\otimes\sigma)(uv)H(W)^*e\rangle_{\mathcal{H}^n} \\
&= \langle H(Z)^*e', \begin{bmatrix}\sigma(u_1)\\ \vdots \\ \sigma(u_n)\end{bmatrix}\begin{bmatrix}\sigma(v_1) & \cdots & \sigma(v_m)\end{bmatrix} H(W)^*e\rangle_{\mathcal{H}^n} \\
&= \langle \begin{bmatrix}\sigma(u_1^*) & \cdots & \sigma(u_n^*)\end{bmatrix} H(Z)^*e', \begin{bmatrix}\sigma(v_1) & \cdots & \sigma(v_m)\end{bmatrix} H(W)^*e\rangle_{\mathcal{H}} \\
&= \langle [\sigma(u_1^*)\cdots\sigma(u_n^*)]\begin{bmatrix}\widehat{K}_{Z,\mathbf{e_1^n}^*,e'}\\ \vdots \\ \widehat{K}_{Z,\mathbf{e_n^n}^*,e'}\end{bmatrix}, [\sigma(v_1)\cdots\sigma(v_m)]\begin{bmatrix}\widehat{K}_{W,\mathbf{e_1^m}^*,e}\\ \vdots \\ \widehat{K}_{W,\mathbf{e_m^m}^*,e}\end{bmatrix}\rangle_{\mathcal{H}} \\
&= \langle \widehat{K}_{Z,u^*,e'}, \widehat{K}_{W,v,e}\rangle_{\mathcal{H}} \quad \text{(see (4.15) and (4.16))} \\
&= \langle e', \widehat{K}_{W,v,e}(Z)(u)\rangle_{\mathcal{E}^n}.
\end{aligned}$$

Thus, the kernel elements of $\mathcal{H}$ and $\mathcal{H}(K)$ are isometrically identical, and the set of all kernel elements which we denote by $\mathcal{H}^\circ(K)$ is a submodule of both $\mathcal{H}$ and $\mathcal{H}(K)$. As both $\mathcal{H}$ and $\mathcal{H}(K)$ are w^*-closed, we see that $\overline{\mathcal{H}^\circ(K)}^{w^*} = \mathcal{H}(K)$ (see Theorem 4.4) and $\overline{\mathcal{H}^\circ(K)}^{w^*} \subseteq \mathcal{H}$ and so $\mathcal{H}(K)\subseteq\mathcal{H}$. By (4.14), we see that $\mathcal{H}^\circ(K)^\perp = 0$ (as a subspace of $\mathcal{H}$) which implies that $\mathcal{H}(K)^\perp = 0$ (again as a subspace of $\mathcal{H}$). We conclude by Proposition 4.2 that $\mathcal{H} = \mathcal{H}(K)\oplus 0$ and so $\mathcal{H} = \mathcal{H}(K)$. □

Remark 4.10. We see that Theorem 4.9 fails under the weaker assumption that $\mathcal{H}$ is a C^*-correspondence from $\mathcal{A}$ to $\mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, i.e., the setting of Chapter 3. Let $\widehat{\mathcal{A}}$ be a unital C^*-subalgebra of a strictly larger unital C^*-algebra $\widehat{\mathcal{B}}$ such that $1_{\widehat{\mathcal{A}}} = 1_{\widehat{\mathcal{B}}}$. We set $\mathcal{A} = \mathcal{B} = \widehat{\mathcal{A}}$, $\mathcal{E} = \widehat{\mathcal{B}}$ (as in Example 3.4), and $\Omega = \widehat{\mathcal{B}}_{\rm nc}$. We identify $\widehat{\mathcal{A}}$ with the space of nc functions $\mathcal{H}$ as follows: For $f\in\widehat{\mathcal{A}}$, we define a nc function $f:\Omega\to\mathcal{L}(\mathcal{A},\mathcal{E})_{\rm nc}$ by $f(Z)u = Zuf$ for $Z\in\Omega_n$ and $u\in\mathcal{A}^n$. The nc function space $\mathcal{H}$ is an $(\mathcal{A},\mathcal{B})$-correspondence with right and left module action given by left and right multiplication within the C^*-algebra $\widehat{\mathcal{A}}$ and $\mathcal{B}$-valued inner product given by $\langle f, f'\rangle = f^*f'$. With this setup, it is easily checked that the functions in $\mathcal{H}$ satisfy conditions (1),(2), and (3) in Theorem 4.9.

We now show that the function space $\mathcal{H}$ cannot come from a kernel K as described in Theorem 3.6. Assume the opposite. By (3.7), we have

$$\begin{aligned}
K^*_{W,v,e}f &= \langle K_{W,v,e}, f\rangle_{\mathcal{H}} \\
&= \langle e, f(W)(v^*)\rangle_{\mathcal{E}^m} \\
&= \langle e, Wv^*f\rangle_{\mathcal{E}^m} \\
&= e^*Wv^*f.
\end{aligned}$$

By setting $f = 1_{\widehat{\mathcal{A}}} = 1_{\widehat{\mathcal{B}}}$, we conclude that the kernel elements have the form $K_{W,v,e} = vW^*e$. Since $W \in \Omega = \widehat{\mathcal{B}}_{\text{nc}}$, we can construct kernel elements equal to any element of $\widehat{\mathcal{B}}$. By assumption, the kernel elements are contained in $\mathcal{H}$, but this results in a contradiction as $\mathcal{H} = \widehat{\mathcal{A}}$ which is strictly smaller than $\widehat{\mathcal{B}}$.

Chapter 5

Arveson's extension theorem for completely positive kernels

We begin this chapter by introducing Arveson's extension theorem [8]:

Theorem 5.1. *(Arveson's extension theorem) Let $\mathcal{D}$ be an operator system contained in a C^*-algebra $\mathcal{A}$ with $1_{\mathcal{A}} = 1_{\mathcal{D}}$ and let $\phi : \mathcal{D} \to \mathcal{L}(\mathcal{Y})$ be a completely positive map. Then there exists a completely positive map $\Phi : \mathcal{A} \to \mathcal{L}(\mathcal{Y})$ which extends ϕ.*

The main result of this chapter is Theorem 5.2 which generalizes Arveson's extension theorem to the setting of cp nc kernels.

Theorem 5.2. *Let $\mathcal{D}$ be an operator system contained in a C^*-algebra $\mathcal{A}$ with $1_{\mathcal{A}} = 1_{\mathcal{D}}$ and let $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{D}, \mathcal{L}(\mathcal{Y}))_{\mathrm{nc}}$ be a completely positive nc kernel. Then there exists a completely positive nc kernel $\widetilde{K} : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{\mathrm{nc}}$ which extends K.*

Remark 5.3. We say that a unital C^*-algebra $\mathcal{M}$ is **injective** if Arveson's extension theorem holds with $\mathcal{L}(\mathcal{Y})$ replaced by $\mathcal{M}$. We note that the proof of Theorem 5.2 remains the same with $\mathcal{L}(\mathcal{Y})$ replaced by an injective C^*-algebra.

Before proceeding with the proof of Theorem 5.2, we introduce some results about completely positive maps. Let $\mathcal{B}, \mathcal{C}$ be subalgebras of $\mathbb{C}^{n\times n}$ containing the identity $I_{\mathbb{C}^{n\times n}}$ and consider the linear map $\phi : \mathcal{D}^{n\times n} \to \mathcal{L}(\mathcal{Y})^{n\times n}$. We say that ϕ is a **left $\mathcal{B}$-module map** if for all $P \in \mathcal{D}^{n\times n}$ and $\alpha \in \mathcal{B}$

$$\alpha\phi(P) = \phi(\alpha P).$$

where we identify $\mathcal{B}$ as a unital subalgebra of both $\mathcal{D}^{n\times n}$ and $\mathcal{L}(\mathcal{Y})^{n\times n}$. A **right $\mathcal{C}$-module map** is defined analogously. If the map ϕ is both a left $\mathcal{B}$-module map and a right $\mathcal{C}$-module map, we say that ϕ is a **$(\mathcal{B}, \mathcal{C})$-bimodule map**.

Lemma 5.4. *Let $\phi : \mathcal{D}^{n\times n} \to \mathcal{L}(\mathcal{Y})^{n\times n}$ be a completely positive map. Then the following are equivalent.*

1. ϕ *is a* $(\mathcal{B}, \mathcal{B}^*)$*-bimodule map.*
2. ϕ *is a left* $\mathcal{B}$*-module map.*
3. ϕ *is a right* $\mathcal{B}^*$*-module map.*

Proof. It suffices to show that (2) $\Leftrightarrow$ (3). Assume (2). It is a well-known fact that if ϕ is positive, then ϕ is self-adjoint, i.e., $\phi(P)^* = \phi(P^*)$ (see [25]). For all $\alpha \in \mathcal{B}$, we have

$$\phi(P)\alpha^* = (\alpha\phi(P^*))^* = \phi(\alpha P^*)^* = \phi(P\alpha^*)$$

which gives (3). The reverse direction follows similarly. □

Lemma 5.5. *Let* $\phi : \mathcal{D}^{n\times n} \subseteq \mathcal{A}^{n\times n} \to \mathcal{L}(\mathcal{Y})^{n\times n}$ *be a completely positive map and* $\mathcal{B}$ *be a subalgebra of* $\mathbb{C}^{n\times n}$ *containing the identity* $I_{\mathbb{C}^{n\times n}}$ *. Then the following are equivalent:*

1. $\alpha\phi(I)\beta^* = \phi(\alpha\beta^*)$ *for all* $\alpha, \beta \in \mathcal{B}$*.*
2. ϕ *is a* $(\mathcal{B}, \mathcal{B}^*)$*-bimodule map.*

Proof. (1) $\Rightarrow$ (2). Let $P \in \mathcal{D}^{n\times n}$, $\alpha, \beta \in \mathcal{B}$, and $y \in \mathcal{Y}^n$. We extend ϕ to a completely positive map $\Phi : \mathcal{A}^{n\times n} \to \mathcal{L}(\mathcal{Y})^{n\times n}$ by Theorem 5.1, and we consider the cp nc kernel induced by Φ (see Examples 2.4, 2.8). In particular, we have that

$$\phi(P) = \Phi(P) = K(1,1)(P) = H(1)\sigma(P)H(1)^*$$

where the last equality is given by the Kolmogorov decomposition (3.10).

As a first step, we prove that $K_{1,\alpha^*,y} = K_{1,I,\alpha^* y}$ as elements of $\mathcal{H}(K)$:

$$\begin{aligned}
||K_{1,\alpha^*,y} - K_{1,I,\alpha^* y}||^2 =& \langle K_{1,\alpha^*,y}, K_{1,\alpha^*,y}\rangle - \langle K_{1,\alpha^*,y}, K_{1,I,\alpha^* y}\rangle \\
&- \langle K_{1,I,\alpha^* y}, K_{1,\alpha^*,y}\rangle + \langle K_{1,I,\alpha^* y}, K_{1,I,\alpha^* y}\rangle \\
=& \langle y, K(1,1)(\alpha\alpha^*)y\rangle - \langle y, K(1,1)(\alpha)\alpha^* y\rangle \\
&- \langle \alpha^* y, K(1,1)(\alpha^*)y\rangle + \langle \alpha^* y, K(1,1)(I)\alpha^* y\rangle \\
=& \langle y, \alpha K(1,1)(I)\alpha^* y\rangle - \langle y, \alpha K(1,1)(I)\alpha^* y\rangle \\
&- \langle y, \alpha K(1,1)(I)\alpha^* y\rangle + \langle y, \alpha K(1,1)(I)\alpha^* y\rangle \\
&= 0.
\end{aligned}$$

Without loss of generality, we define $H(1)^*$ by (3.17). We see that $\sigma(\alpha^*)H(1)^* = H(1)^*\alpha^*$:

$$\sigma(\alpha^*)H(1)^* y = \sigma(\alpha^*)K_{1,I,y} = K_{1,\alpha^*,y} = K_{1,I,\alpha^* y} = H(1)^*\alpha^* y.$$

and $H(1)\sigma(\alpha) = \alpha H(1)$ follows by taking the adjoint. The final result now follows:

$$\begin{aligned}\phi(\alpha P\beta^*) &= K(1,1)(\alpha P\beta^*)\\ &= H(1)\sigma(\alpha P\beta^*)H(1)^*\\ &= H(1)\sigma(\alpha)\sigma(P)\sigma(\beta^*)H(1)^*\\ &= \alpha H(1)\sigma(P)H(1)^*\beta^*\\ &= \alpha K(1,1)(P)\beta^*\\ &= \alpha\phi(P)\beta^*.\end{aligned}$$

(2) $\Rightarrow$ (1). The result follows directly from the definition of a $(\mathcal{B}, \mathcal{B}^*)$-bimodule map.

□

Lemma 5.6. *Let $\phi : \mathcal{D}^{n\times n} \to \mathcal{L}(\mathcal{Y})^{n\times n}$ be a completely positive $(\mathcal{B}, \mathcal{B}^*)$-bimodule map and let $\Phi : \mathcal{A}^{n\times n} \to \mathcal{L}(\mathcal{Y})^{n\times n}$ be a completely positive extension of ϕ. Then Φ is a $(\mathcal{B}, \mathcal{B}^*)$-bimodule map.*

Proof. Let $\alpha, \beta \in \mathcal{B}$. The result follows from Lemma 5.5 and the following calculation

$$\alpha\Phi(I)\beta^* = \alpha\phi(I)\beta^* = \phi(\alpha\beta^*) = \Phi(\alpha\beta^*).$$

□

Proof of Theorem 5.2. We subdivide the proof into two cases.

Case 1: Ω is finite. Let $\Omega = \{Z^{(1)}, Z^{(2)}, \ldots, Z^{(m)}\}$ where $Z^{(i)} \in \Omega_{n_i}$. Consider the direct sum $Z = \oplus_{i=1}^m Z^{(i)} \in \Omega_n$ where $n = \sum_{i=1}^m n_i$. For $1 \le i_0 \le m$, let E_{i_0} be the $n \times n_{i_0}$ matrix over $\mathbb{C}$ with block-column structure such that the i-th block in the column (where $i = 1, \ldots, m$) has size $n_i \times n_{i_0}$, the i_0 block is equal to the $i_0 \times i_0$ identity matrix, and all other blocks are equal to 0, i.e.,

$$E_{i_0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{\mathbb{C}^{i_0\times i_0}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

We define a linear map $\phi = K(Z, Z) : \mathcal{D}^{n\times n} \to \mathcal{L}(\mathcal{Y})^{n\times n}$. Since K is a completely positive nc kernel, it follows that ϕ is a completely positive map. We see that $K(Z^{(i)}, Z^{(j)})(P) = E_i^*\phi(E_iPE_j^*)E_j$ for $P \in \mathcal{D}^{n_i\times n_j}$:

$$K(Z^{(i)}, Z^{(j)})(P) = E_i^*K(Z, Z)(E_iPE_j^*)E_j = E_i^*\phi(E_iPE_j^*)E_j \tag{5.1}$$

where the first equality follows from the fact that K respects direct sums.

For $Z^{(i)}, Z^{(j)} \in \Omega$ and $\alpha \in \mathbb{C}^{n_j \times n_i}$ such that $\alpha Z^{(i)} = Z^{(j)}\alpha$, we have

$$(E_j \alpha E_i^*)Z = E_j \alpha Z^{(i)} E_i^* = E_j Z^{(j)} \alpha E_i^* = Z(E_j \alpha E_i^*).$$

As K is a nc kernel, we see that $(E_j\alpha E_i^*)K(Z,Z)(P) = K(Z,Z)((E_j\alpha E_i^*)P)$. By Lemma 5.4, the completely positive map $\phi = K(Z,Z)$ is a $(\mathcal{B}, \mathcal{B}^*)$-bimodule map where $\mathcal{B}$ contains all matrices of the form $(E_j\alpha E_i^*)$ which are derived by an intertwining relation.

By Theorem 5.1, there exists a completely positive map $\Phi : \mathcal{A}^{n\times n} \to \mathcal{L}(\mathcal{Y})^{n\times n}$ which extends ϕ. We use Φ to construct a completely positive kernel $\widetilde{K} : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{\mathrm{nc}}$ which extends K. We define $\widetilde{K}$ by

$$\widetilde{K}(Z^{(i)}, Z^{(j)})(P) = E_i^* \Phi(E_i P E_j^*) E_j \tag{5.2}$$

where $P \in \mathcal{A}^{n_i \times n_j}$. By Lemma 5.6, the completely positive map Φ is a $(\mathcal{B}, \mathcal{B}^*)$-bimodule map.

We now show that $\widetilde{K}$ defined above is a nc kernel which extends K. By (5.1) and (5.2), we see that $\widetilde{K}(Z^{(i)}, Z^{(j)})(P) = K(Z^{(i)}, Z^{(j)})(P)$ for $P \in \mathcal{D}^{n_i\times n_j}$. We check that $\widetilde{K}$ respects intertwinings. Let $\alpha Z^{(i)} = Z^{(i')}\alpha$ and $\beta Z^{(j)} = Z^{(j')}\beta$ where $\alpha \in \mathbb{C}^{n_{i'}\times n_i}$ and $\beta \in \mathbb{C}^{n_{j'}\times n_j}$. For $P \in \mathcal{A}^{n_i\times n_j}$, we have

$$\begin{aligned}
\alpha\widetilde{K}(Z^{(i)}, Z^{(j)})(P)\beta^* &= \alpha E_i^*\Phi(E_iPE_j^*)E_j\beta^* \\
&= E_{i'}^*(E_{i'}\alpha E_i^*)\Phi(E_iPE_j^*)(E_j\beta^*E_{j'}^*)E_{j'} \\
&= E_{i'}^*\Phi((E_{i'}\alpha E_i^*)E_iPE_j^*(E_j\beta^*E_{j'}^*))E_{j'} \\
&= E_{i'}^*\Phi(E_{i'}\alpha P\beta^*E_{j'}^*)E_{j'} \\
&= \widetilde{K}(Z^{(i')}, Z^{(j')})(\alpha P\beta^*)
\end{aligned}$$

which is the desired property. Note that the third equality follows from the fact that Φ is a $(\mathcal{B}, \mathcal{B}^*)$-bimodule map.

To establish that $\widetilde{K}$ is a completely positive kernel, we first show that $\widetilde{K}(Z,Z) = \Phi$: For $P = [P_{ij}] \in \mathcal{D}^{n\times n}$ where $P_{ij} \in \mathcal{D}^{n_i\times n_j}$, we have

$$\begin{aligned}
K(Z,Z)(P) &= [K(Z^{(i)}, Z^{(j)})(P_{ij})] \\
&= \sum_{ij} E_i[K(Z^{(i)}, Z^{(j)})(P_{ij})]E_j^* \\
&= \sum_{ij} E_iE_i^*\Phi(E_iP_{ij}E_j^*)E_jE_j^* \\
&= \sum_{ij} \Phi(E_iP_{ij}E_j^*) \\
&= \Phi(P).
\end{aligned}$$

As Φ is a completely positive map, we see by Theorem 2.13 that $\widetilde{K}$ is a completely positive kernel.

Case 2: The general case. The general case can be reduced to the special situation of Case 1 through an application of a theorem of Kurosh which says that the limit of an inverse spectrum of nonempty compacta is a nonempty compactum (see [[3], Theorem 2.56]). In the following, we go through the proof presented in [[6], pages 73-75] adapted to the special case here.

The directed set $\mathfrak{A}$ Let $\mathfrak{A}$ be the set of all μ where μ identifies a finite subset of Ω which we denote Ω_μ. We define a partial order on $\mathfrak{A}$ as follows: Given two elements μ and ν of $\mathfrak{A}$, we say that $\mu \preceq \nu$ if $\Omega_\mu \subseteq \Omega_\nu$ (as finite subsets of Ω). Then $\preceq$ satisfies reflexivity and transitivity.

We establish that $\mathfrak{A}$ is a directed set, i.e., given any $\mu, \nu \in \mathfrak{A}$, one can always find a $\gamma \in \mathfrak{A}$ such that both $\mu \preceq \gamma$ and $\nu \preceq \gamma$. Let $\Omega_\gamma = \Omega_\mu \cup \Omega_\nu$, then Ω_γ is a finite subset of Ω such that γ has the property that both $\mu \preceq \gamma$ and $\nu \preceq \gamma$.

The compact set $\mathbb{K}_\mu$ for $\mu \in \mathfrak{A}$.

Let $\mathfrak{X}_\mu$ be the linear space of all kernels $\Gamma : \Omega_\mu \times \Omega_\mu \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{\text{nc}}$. For $Z \in \Omega_{\mu,n_Z}$, $W \in \Omega_{\mu,n_W}$ and $\Gamma \in \mathfrak{X}_\mu$, we have that $\Gamma(Z,W)$ is an element of the Banach space $\mathcal{L}(\mathcal{A}^{n_Z \times n_W}, \mathcal{L}(\mathcal{Y})^{n_Z \times n_W})$. Thus, we have

$$\mathfrak{X}_\mu = \bigoplus_{Z,W \in \Omega_\mu} \mathcal{L}(\mathcal{A}^{n_Z \times n_W}, \mathcal{L}(\mathcal{Y})^{n_Z \times n_W}).$$

If we endow $\mathfrak{X}_\mu$ with the supremum norm

$$||\Gamma|| = \sup_{Z,W \in \Omega_\mu} ||\Gamma(Z,W)||, \tag{5.3}$$

then $\mathfrak{X}_\mu$ becomes a Banach space.

Remark 5.7. Banach spaces of the form $\mathcal{L}(\mathcal{A}^{n\times m}, \mathcal{L}(\mathcal{Y})^{n\times m})$ have a predual $\mathcal{L}(\mathcal{A}^{n\times m}, \mathcal{L}(\mathcal{Y})^{n\times m})_*$ such that on bounded sets the weak-$*$ topology on $\mathcal{L}(\mathcal{A}^{n\times m}, \mathcal{L}(\mathcal{Y})^{n\times m})$ is the same as the pointwise weak-$*$ topology, i.e., for a net $\phi_\lambda \in \mathcal{L}(\mathcal{A}^{n\times m}, \mathcal{L}(\mathcal{Y})^{n\times m})$, we have that $\phi_\lambda \to \phi$ in the weak-$*$ topology on $\mathcal{L}(\mathcal{A}^{n\times m}, \mathcal{L}(\mathcal{Y})^{n\times m})$ if and only if for each $T \in \mathcal{A}^{n\times m}$, we have $\phi_\lambda(T) \to \phi(T)$ in the weak-$*$ topology on $\mathcal{L}(\mathcal{Y})^{n\times m}$. As the weak and weak-$*$ topologies agree on bounded sets, this topology is sometimes called the bounded-weak topology or BW-topology (see [25] for more details).

Since the direct summands of $\mathfrak{X}_\mu$ have a predual $\mathcal{L}(\mathcal{A}^{n\times m}, \mathcal{L}(\mathcal{Y})^{n\times m})_*$, we have a predual on $\mathfrak{X}_\mu$:

$$(\mathfrak{X}_\mu)_* = \bigoplus_{Z,W \in \Omega_\mu} \mathcal{L}(\mathcal{A}^{n_Z \times m_W}, \mathcal{L}(\mathcal{Y})^{n_Z \times m_W})_*$$

with pairing

$$\langle \Gamma, \tau \rangle = \sum_{Z,W \in \Omega_\mu} \langle \Gamma(Z,W), \tau(Z,W) \rangle$$

for $\Gamma = \bigoplus_{Z,W\in\Omega_\mu} \Gamma(Z,W) \in \mathfrak{X}_\mu$ and $\tau = \bigoplus_{Z,W\in\Omega_\mu} \tau(Z,W) \in (\mathfrak{X}_\mu)_*$. Thus, the space $\mathfrak{X}_\mu$ has a weak-$*$ topology given by $(\mathfrak{X}_\mu)_*$

For each $\mu \in \mathfrak{A}$, we define the cp nc kernel $K_\mu : \Omega_\mu \times \Omega_\mu \to \mathcal{L}(\mathcal{D}, \mathcal{L}(\mathcal{Y}))_{\rm nc}$ by

$$K_\mu(Z,W)(P) = K(Z,W)(P) \tag{5.4}$$

for all $Z \in \Omega_{\mu,n}$, $W \in \Omega_{\mu,m}$, and $P \in \mathcal{D}^{n\times m}$. We let $\mathbb{K}_\mu$ be the subset of $\mathfrak{X}_\mu$ consisting of all cp nc kernels $\Gamma : \Omega_\mu \times \Omega_\mu \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{\rm nc}$ which extend K_μ (5.4). The accomplishment of Case 1 of the proof is to show that $\mathbb{K}_\mu$ is nonempty for all $\mu \in \mathfrak{A}$.

Our aim now is to show that $\mathbb{K}_\mu$ is compact in the weak-$*$ topology described above. By the Banach-Alaoglŭ Theorem, norm closed and bounded sets are pre-compact in the weak-$*$ topology. Thus, we show that $\mathbb{K}_\mu$ is compact by demonstrating that the set is weak-$*$ closed (which implies norm closed) and bounded.

For boundedness, it suffices to show that $||\Gamma(Z,W)||$ is uniformly bounded for $Z, W \in \Omega_\mu$ and $\Gamma \in \mathbb{K}_\mu$. Let $\Omega_\mu = \{Z^{(1)}, Z^{(2)}, \ldots, Z^{(m)}\}$ where $Z^{(i)} \in \Omega_{n_i}$. Consider the direct sum $Z = \oplus_{i=1}^m Z^{(i)} \in \Omega_n$ where $n = \sum_{i=1}^m n_i$. We have

$$\begin{aligned} ||\Gamma(Z^{(i)}, Z^{(j)})|| &\le ||\Gamma\left(\begin{bmatrix} Z^{(i)} & 0 \\ 0 & Z^{(j)} \end{bmatrix}, \begin{bmatrix} Z^{(i)} & 0 \\ 0 & Z^{(j)} \end{bmatrix}\right)|| \\ &\le ||\Gamma(Z,Z)|| \\ &= ||\Gamma(Z,Z)(I)|| \\ &= ||K(Z,Z)(I)|| \end{aligned}$$

where the first two inequalities follow from the fact that Γ respects direct sums and the third line is a consequence of $\Gamma(Z,Z)$ being a completely positive map.

We recall that $\mathbb{K}_\mu$ is the subset of $\mathfrak{X}_\mu$ whose elements (i) extend K over Ω_μ and (ii) satisfy the properties of a nc cp kernel over Ω_μ. Let (Γ_λ) be a net in $\mathbb{K}_\mu$ which is weak-$*$ convergent to $\Gamma \in \mathfrak{X}_\mu$. To establish the weak-$*$ closure of $\mathbb{K}_\mu$, we must show that Γ satisfies properties (i) and (ii) noted above. By Remark 5.7, it suffices to work with the weak topology on $\mathcal{L}(\mathcal{Y})$, i.e. $K_\lambda(Z,W) \xrightarrow{w^*} K(Z,W)$ if and only if $\langle y, K_\lambda(Z,W)(P)y'\rangle \to \langle y, K(Z,W)(P)y'\rangle$ for every $P \in \mathcal{A}^{n\times m}$, $y \in \mathcal{Y}^n$, and $y' \in \mathcal{Y}^m$.

(i) We have that $\Gamma_\lambda(Z,W)(P) = K_\mu(Z,W)(P)$ for every $Z \in \Omega_{\mu,n}$, $W \in \Omega_{\mu,m}$, and $P \in \mathcal{D}^{n\times m}$ and so $\Gamma(Z,W)(P) = K_\mu(Z,W)(P)$ which establishes property (i).

(ii) Γ is graded by construction, so it remains to check that Γ respects intertwinings and is completely positive. Let $Z \in \Omega_{\mu,n}$, $\widetilde{Z} \in \Omega_{\mu,\widetilde{n}}$, $W \in \Omega_{\mu,m}$, $\widetilde{W} \in \Omega_{\mu,\widetilde{W}}$ such that $\alpha Z = \widetilde{Z}\alpha$

and $\beta W = \widetilde{W}\beta$ where $\alpha \in \mathbb{C}^{\tilde{n}\times n}$ and $\beta \in \mathbb{C}^{\tilde{m}\times m}$. For $P \in \mathcal{A}^{n\times m}$, $y \in \mathcal{Y}^{\tilde{n}}$, $y' \in \mathcal{Y}^{\tilde{m}}$, we have

$$\langle y, (\alpha\Gamma_\lambda(Z,W)(P)\beta^*)y'\rangle = \langle y, \Gamma_\lambda(\widetilde{Z},\widetilde{W})(\alpha P\beta^*)y'\rangle.$$

Taking the limit, we see that $\alpha\Gamma(Z,W)(P)\beta^* = \Gamma(\widetilde{Z},\widetilde{W})(\alpha P\beta^*)$ which establishes that Γ respects intertwinings.

To show that Γ is completely positive, we consider $Z \in [\Omega_\mu]_{\mathrm{nc},n}$. For $0 \preceq P \in \mathcal{A}^{n\times n}$ and $y \in \mathcal{Y}^n$, we have

$$\langle y, \Gamma_\lambda(Z,Z)(P)y\rangle \geq 0$$

Again by taking the limit, we see that $\Gamma(Z,Z)(P)$ is positive and conclude that Γ is a completely positive kernel.

The restriction maps π^ν_μ. Let μ and ν be elements of $\mathfrak{A}$ such that $\mu \preceq \nu$. We define a map $\pi^\nu_\mu : \mathbb{K}_\nu \to \mathbb{K}_\mu$ as follows: For $\Gamma \in \mathbb{K}_\nu$, define $\pi^\nu_\mu\Gamma \in \mathbb{K}_\mu$ by

$$(\pi^\nu_\mu\Gamma)(Z,W)(P) = \Gamma(Z,W)(P) \tag{5.5}$$

for all $Z \in \Omega_{\mu,n}$, $W \in \Omega_{\mu,m}$, and $P \in \mathcal{A}^{n\times m}$. It is straightforward to check that the restriction maps are continuous.

We observe that the collection of maps $\{\pi^\nu_\mu\}$ has the following properties:

$$\pi^\mu_\mu = \mathrm{id}_{\mathbb{K}_\mu} \tag{5.6}$$

$$\gamma \preceq \mu,\ \mu \preceq \nu \Rightarrow \pi^\mu_\gamma \circ \pi^\nu_\mu = \pi^\nu_\gamma \tag{5.7}$$

The set $\mathbb{K}$. We let $\mathbb{K}$ be the Cartesian-product space

$$\mathbb{K} = \Pi_{\mu\in\mathfrak{A}}\,\mathbb{K}_\mu.$$

We denote an element of $\mathbb{K}$ by $\mathbf{\Gamma} = \{\Gamma_\mu\}_{\mu\in\mathfrak{A}}$ where each $\Gamma_\mu \in \mathbb{K}_\mu$. We endow $\mathbb{K}$ with the standard Cartesian-product topology. Since each fiber $\mathbb{K}_\mu$ is nonempty and compact by the preceding discussion, it is a consequence of Tikhonov's Theorem that $\mathbb{K}$ is nonempty and compact.

The set $\mathbb{K}^{\mu\nu}$. Consider $\mu,\nu \in \mathfrak{A}$ such that $\mu \preceq \nu$. We define $\mathbb{K}^{\mu\nu}$ to be the subset of $\mathbb{K}$ consisting of those $\mathbf{\Gamma} = \{\Gamma_\gamma\}_{\gamma\in\mathfrak{A}}$ such that $\pi^\nu_\mu\Gamma_\nu = \Gamma_\mu$.

We show that the sets $\mathbb{K}^{\mu\nu}$ are nonempty and compact. To this end, we construct a continuous surjective map $P_{\mu\nu} : \mathbb{K} \to \mathbb{K}^{\mu\nu}$ as follows: For any element $\mathbf{\Gamma} = \{\Gamma_\gamma\}_{\gamma\in\mathfrak{A}}$, let $P_{\mu\nu}(\mathbf{\Gamma}) = \{\Gamma'_\gamma\}_{\gamma\in\mathfrak{A}}$ such that

$$\Gamma'_\gamma = \begin{cases} \pi^\nu_\mu(\Gamma_\nu) & \text{if } \gamma = \mu \\ \Gamma_\gamma & \text{otherwise.} \end{cases}$$

The map $P_{\mu\nu}$ is continuous since π_μ^ν is continuous. As $\mathbb{K}$ is compact and $P_{\mu\nu}(\mathbb{K}) = \mathbb{K}^{\mu\nu}$, we see that $\mathbb{K}^{\mu\nu}$ is compact as well. Furthermore, we can use the map $P_{\mu\nu}$ to establish that $\mathbb{K}^{\mu\nu}$ is nonempty.

The set $\mathbb{K}^\nu$. For $\nu \in \mathfrak{A}$, we define $\mathbb{K}^\nu$ by

$$\mathbb{K}^\nu = \bigcap_{\mu \in \mathfrak{A}: \mu \preceq \nu} \mathbb{K}^{\mu\nu}. \tag{5.8}$$

We first show that the set $\mathbb{K}^\nu$ is nonempty. To this end, pick any $\Gamma_\nu \in \mathbb{K}_\nu$ and let $\mathbf{\Gamma} = \{\Gamma_\gamma\}_{\gamma \in \mathfrak{A}}$ be defined by

$$\Gamma_\gamma = \begin{cases} \pi_\gamma^\nu(\Gamma_\nu) & \text{if } \gamma \preceq \nu \\ \text{any element of } \mathbb{K}_\gamma & \text{otherwise.} \end{cases}$$

We observe that $\mathbf{\Gamma} \in \mathbb{K}^\nu$ and conclude that the set is nonempty.

We now show that the set $\cap_{\nu \in \mathfrak{A}} \mathbb{K}^\nu$ is nonempty. A consequence of $\mathbb{K}$ being compact is that any collection of closed subsets of $\mathbb{K}$ with the Finite Intersection Property has nonempty intersection. We have already established that $\mathbb{K}^{\mu\nu}$ is compact and therefore closed. By (5.8), $\mathbb{K}^\nu$ is closed as well. Therefore, it remains to show that the collection of sets $\{\mathbb{K}^\nu : \nu \in \mathfrak{A}\}$ has the Finite Intersection Property, i.e., given $\nu^{(1)}, \ldots, \nu^{(N)} \in \mathfrak{A}$, we have that $K^{\nu^{(1)}} \cap \cdots \cap K^{\nu^{(N)}}$ is nonempty. As $\mathfrak{A}$ is a directed set, there exists a $\nu \in \mathfrak{A}$ such that $\nu^{(i)} \preceq \nu$ for $i = 1, \ldots, N$. We check that $\mathbb{K}^\nu \subset \mathbb{K}^{\nu^{(i)}}$ for each $i = 1, \ldots, N$. To this end, we consider $\mu \in \mathfrak{A}$ such that $\mu \preceq \nu^{(i)}$. Then for $\mathbf{\Gamma} = \{\Gamma_{\nu'}\}_{\nu' \in \mathfrak{A}} \in \mathbb{K}^\nu$, we have

$$\pi_\mu^{\nu^{(i)}} \Gamma_{\nu^{(i)}} = \pi_\mu^{\nu^{(i)}} \pi_{\nu^{(i)}}^\nu \Gamma_\nu = \pi_\mu^\nu \Gamma_\nu = \Gamma_\mu$$

and conclude that $\mathbf{\Gamma} \in \mathbb{K}^{\nu^{(i)}}$ and $\mathbb{K}^\nu \subset \mathbb{K}^{\nu^{(i)}}$ for $i = 1, \ldots, N$. As $\mathbb{K}^\nu$ is nonempty and $\mathbb{K}^\nu \subset \cap_{i=1}^N \mathbb{K}^{\nu^{(i)}}$, we have established that $\cap_{\nu \in \mathfrak{A}} \mathbb{K}^\nu$ has the Finite Intersection Property, and we conclude that the set is nonempty.

Construction of the extension $\widetilde{K}$. We wish to define a nc kernel $\widetilde{K} : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))$ such that

$$\widetilde{K}(Z, W)(P) = K(Z, W)(P)$$

for all $Z \in \Omega_n$, $W \in \Omega_m$, and $P \in \mathcal{D}^{n \times m}$.

From the previous discussion, we have established the existence of $\mathbf{\Gamma} = \{\Gamma_{\nu'}\}_{\nu' \in \mathfrak{A}} \in \cap_{\nu \in \mathfrak{A}} \mathbb{K}^\nu$, i.e., we have $\mathbf{\Gamma} \in \mathbb{K}$ such that

$$\mu \preceq \nu \Rightarrow \pi_\mu^\nu \Gamma_\nu = \Gamma_\mu. \tag{5.9}$$

Given $Z \in \Omega_n$ and $W \in \Omega_m$, choose $\mu \in \mathfrak{A}$ such that $Z, W \in \Omega_\mu$. We define the extension $\widetilde{K}$ by

$$\widetilde{K}(Z, W) = \Gamma_\mu(Z, W).$$

By construction the maps agree on $\mathcal{D}^{n\times m}$, and the right-hand side is independent of the choice of μ. Indeed, let $\mu' \in \mathfrak{A}$ be another index such that $Z, W \in \Omega_{\mu'}$. Then there exists $\nu \in \mathfrak{A}$ such that $\mu \preceq \nu$ and $\mu' \preceq \nu$. By (5.9), we have

$$\Gamma_\mu(Z, W) = \Gamma_\nu(Z, W) = \Gamma_{\mu'}(Z, W).$$

As the properties of cp nc kernels involve only finitely many points Z, W in Ω and each Γ_μ is a cp nc kernel on Ω_μ, it is straightforward to show that $\widetilde{K}$ is a nc cp kernel.

□

Chapter 6

Uniformly bounded kernels

We begin our discussion of completely bounded kernels by first introducing completely bounded maps. An **operator space** $\mathcal{M}$ is a subspace of a C^*-algebra $\mathcal{A}$. We consider linear maps of the form $\phi : \mathcal{M} \to \mathcal{B}$ where $\mathcal{B}$ is a C^*-algebra. We say that ϕ is **completely bounded** if $||\phi||_{\mathrm{cb}} := \sup_n ||\phi_{n,n}||$ is finite where $\phi_{n,n}$ are the inflation maps $\phi_{n,n} : \mathcal{M}^{n\times n} \to \mathcal{B}^{n\times n}$ (see Example 2.4 for more on inflation maps). As suggested by the notation, completely bounded maps form a normed space with norm $|| \cdot ||_{\mathrm{cb}}$.

The following well-known theorems due to Wittstock [39, 38] are core results in operator space theory.

Theorem 6.1. *(Wittstock's extension theorem) Let $\mathcal{M}$ be an operator space contained in the C^*-algebra $\mathcal{A}$ and let $\mathcal{B}$ be an injective C^*-algebra. A completely bounded map $\phi : \mathcal{M} \to \mathcal{B}$ has a bound-preserving extension to a completely bounded map $\Phi : \mathcal{A} \to \mathcal{B}$.*

Theorem 6.2. *(Wittstock's decomposition theorem) Let $\mathcal{A}$ be a C^*-algebra and $\psi : \mathcal{A} \to \mathcal{L}(\mathcal{Y})$ be a completely bounded map. Then there exist completely positive maps $\phi^{(i)}$, $i = 1, 2, 3, 4$ such that*

$$\psi = \phi^{(1)} - \phi^{(2)} + i(\phi^{(3)} - \phi^{(4)}).$$

The purpose of this chapter is to establish the analogue of these theorems for the noncommutative kernel setting. All of the results below have analogues in the setting of completely bounded maps which can be found in Chapter 8 of [25].

In this section, we assume that Ω is closed under direct sums and similarities, i.e., $\Omega = [\Omega]_{\mathrm{sim}}$ (see (2.14)). We consider noncommutative kernels of the form $K : \Omega\times\Omega \to \mathcal{L}(\mathcal{M}, \mathcal{B})_{\mathrm{nc}}$. We say that such a kernel is **uniformly bounded** if

$$||K(Z, W)|| \leq M_K \tag{6.1}$$

for a fixed M_K where $Z, W \in \Omega$. We say that K is **uniformly contractive** if $M_K = 1$.

Lemma 6.3. *A nc kernel $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{M}, \mathcal{B})_{\rm nc}$ is uniformly bounded by M if and only if $K(Z, Z)$ is a bounded map with $||K(Z, Z)|| \le M$ for every $Z \in \Omega$.*

Proof. Assume that K is a uniformly bounded nc kernel. By definition, $||K(Z, Z)|| \le M$ for any $Z \in \Omega$.

For the reverse direction, let $\widetilde{Z}, \widetilde{W} \in \Omega$. Since $\left[\begin{smallmatrix} \widetilde{Z} & 0 \\ 0 & \widetilde{W} \end{smallmatrix}\right] \in \Omega$, we have

$$||K(\widetilde{Z}, \widetilde{W})(P)|| = ||K(\left[\begin{smallmatrix} \widetilde{Z} & 0 \\ 0 & \widetilde{W} \end{smallmatrix}\right], \left[\begin{smallmatrix} \widetilde{Z} & 0 \\ 0 & \widetilde{W} \end{smallmatrix}\right])(\left[\begin{smallmatrix} 0 & P \\ 0 & 0 \end{smallmatrix}\right])|| \le M||P||$$

which establishes that K is a uniformly bounded kernel with bound M. □

For an operator space $\mathcal{M}$ in a C^*-algebra $\mathcal{A}$, we let $\mathcal{F} \subset \mathcal{A}^{2\times 2}$ denote the operator system defined by

$$\mathcal{F} = \left\{ \left[\begin{smallmatrix} \lambda I_{\mathcal{A}} & d \\ d'^* & \mu I_{\mathcal{A}} \end{smallmatrix}\right] : d, d' \in \mathcal{M} \text{ and } \lambda, \mu \in \mathbb{C} \right\}.$$

An element of $F \in \mathcal{F}^{n\times m}$ has the block structure $F = [F_{i,j}]$ where

$$F_{i,j} = \begin{bmatrix} \lambda_{i,j} I_{\mathcal{A}} & d_{i,j} \\ d'^*_{i,j} & \mu_{i,j} I_{\mathcal{A}} \end{bmatrix}.$$

We perform the *canonical shuffle* to make the identification $(\mathcal{A}^{2\times 2})^{n\times m} \simeq (\mathcal{A}^{n\times m})^{2\times 2}$, and in this way, we identify $F \in \mathcal{F}^{n\times m}$ as

$$F = \begin{bmatrix} A \otimes I_{\mathcal{A}} & P \\ Q^* & B \otimes I_{\mathcal{A}} \end{bmatrix} \tag{6.2}$$

where $A = [\lambda_{ij}] \in \mathbb{C}^{n\times m}$, $B = [\mu_{ij}] \in \mathbb{C}^{n\times m}$, $P = [d_{ij}] \in \mathcal{M}^{n\times m}$ and $Q = [d'_{ji}] \in \mathcal{M}^{m\times n}$. We note that the *canonical shuffle* preserves positivity and norm.

Lemma 6.4. *Assume the set up above and let $\mathcal{B}$ be a C^*-algebra. Let $K : \Omega\times\Omega \to \mathcal{L}(\mathcal{M}, \mathcal{B})_{\rm nc}$ be a uniformly contractive nc kernel, and let $\Gamma : \Omega \times \Omega \to \mathcal{L}(\mathbb{C} \otimes I_{\mathcal{A}}, \mathcal{B})_{\rm nc}$ be the completely positive nc kernel given by*

$$\Gamma(Z, W)(A \otimes I_{\mathcal{A}}) = A \otimes I_{\mathcal{B}}$$

for $Z \in \Omega_n$, $W \in \Omega_m$, and $A \in \mathbb{C}^{n\times m}$. Then the map $\widehat{K} : \Omega \times \Omega \to \mathcal{L}(\mathcal{F}, \mathcal{B}^{2\times 2})_{\rm nc}$ given by

$$\begin{aligned} \widehat{K}(Z, W)\left(\begin{bmatrix} A \otimes I_{\mathcal{A}} & P \\ Q^* & B \otimes I_{\mathcal{A}} \end{bmatrix}\right) &= \left(\begin{bmatrix} \Gamma(Z, W)(A \otimes I_{\mathcal{A}}) & K(Z, W)(P) \\ K(W, Z)(Q)^* & \Gamma(Z, W)(B \otimes I_{\mathcal{A}}) \end{bmatrix}\right) \\ &= \left(\begin{bmatrix} A \otimes I_{\mathcal{B}} & K(Z, W)(P) \\ K(W, Z)(Q)^* & B \otimes I_{\mathcal{B}} \end{bmatrix}\right) \end{aligned}$$

is a uniformly contractive completely positive nc kernel.

Proof. We first establish that $\widehat{K}$ is a nc kernel. To this end, we check that $\widehat{K}$ respects intertwinings: Let $Z \in \Omega_n$, $\widetilde{Z} \in \Omega_{\widetilde{n}}$, $W \in \Omega_m$, $\widetilde{W} \in \Omega_{\widetilde{W}}$ such that $\alpha Z = \widetilde{Z}\alpha$ and $\beta W = \widetilde{W}\beta$ where $\alpha \in \mathbb{C}^{\widetilde{n}\times n}$ and $\beta \in \mathbb{C}^{\widetilde{m}\times m}$. For $F \in \mathcal{F}^{n\times m}$, we have

$$\begin{aligned}
\alpha\widehat{K}(Z,W)(F)\beta^* &= \alpha \begin{bmatrix} A\otimes I_{\mathcal{B}} & K(Z,W)(P) \\ K(W,Z)(Q)^* & B\otimes I_{\mathcal{B}} \end{bmatrix} \beta^* \\
&= \begin{bmatrix} \alpha A\beta^* \otimes I_{\mathcal{B}} & \alpha K(Z,W)(P)\beta^* \\ \alpha K(W,Z)(Q)^*\beta^* & \alpha B\beta^* \otimes I_{\mathcal{B}} \end{bmatrix} \\
&= \begin{bmatrix} \alpha A\beta^* \otimes I_{\mathcal{B}} & K(\widetilde{Z},\widetilde{W})(\alpha P\beta^*) \\ K(\widetilde{W},\widetilde{Z})(\beta Q\alpha^*)^* & \alpha B\beta^* \otimes I_{\mathcal{B}} \end{bmatrix} \\
&= \widehat{K}(\widetilde{Z},\widetilde{W})(\alpha F\beta^*)
\end{aligned}$$

which establishes that $\widehat{K}$ is a nc kernel.

We now check that $\widehat{K}$ is completely positive. In the following, we omit the tensors and identities $I_{\mathcal{A}}, I_{\mathcal{B}}$ which clutter the calculations. Consider $F = \left[\begin{smallmatrix} A & P \\ Q^* & B\end{smallmatrix}\right] \in \mathcal{F}^{n\times n}$ such that $F \succeq 0$. A consequence of $F \succeq 0$ is that A and B are positive matrices and $Q = P$, i.e.,

$$F = \begin{bmatrix} A & P \\ P^* & B \end{bmatrix}.$$

Without loss of generality, we assume that A and B are positive and invertible, i.e. for $\epsilon > 0$, identify A with $A+\epsilon$ and B with $B+\epsilon$ in the following and conclude the proof by taking the limit as $\epsilon \to 0$. Thus, we have

$$\begin{bmatrix} I & A^{-1/2}PB^{-1/2} \\ B^{-1/2}P^*A^{-1/2} & I \end{bmatrix} = \begin{bmatrix} A^{-1/2} & 0 \\ 0 & B^{-1/2} \end{bmatrix} \begin{bmatrix} A & P \\ P^* & B \end{bmatrix} \begin{bmatrix} A^{-1/2} & 0 \\ 0 & B^{-1/2} \end{bmatrix},$$

and from Lemma 2.9 and the fact that $F \succeq 0$, we have that $||A^{-1/2}PB^{-1/2}|| \leq 1$.

To establish that $\widehat{K}$ is completely positive, we show that $\widehat{K}(Z,Z)(F) \succeq 0$ for $Z \in \Omega_n$. Let $Z_A = A^{-1/2}ZA^{1/2}$ and $Z_B = B^{-1/2}ZB^{1/2}$ which we rewrite as the intertwinings $A^{-1/2}Z = Z_AA^{-1/2}$ and $B^{-1/2}Z = Z_BB^{-1/2}$ to facilitate the following calculation:

$$\begin{aligned}
\widehat{K}(Z,Z)(F) &= \begin{bmatrix} A & K(Z,Z)(P) \\ K(Z,Z)(P)^* & B \end{bmatrix} \\
&= \begin{bmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{bmatrix} \begin{bmatrix} I & A^{-1/2}K(Z,Z)(P)B^{-1/2} \\ B^{-1/2}K(Z,Z)(P)^*A^{-1/2} & I \end{bmatrix} \begin{bmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{bmatrix} \\
&= \begin{bmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{bmatrix} \begin{bmatrix} I & K(Z_A,Z_B)(A^{-1/2}PB^{-1/2}) \\ K(Z_A,Z_B)(A^{-1/2}PB^{-1/2})^* & I \end{bmatrix} \begin{bmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{bmatrix}.
\end{aligned}$$

As K is a completely contractive kernel, we have that

$$||K(Z_A,Z_B)(A^{-1/2}PB^{-1/2})|| \leq ||A^{-1/2}PB^{-1/2}|| \leq 1.$$

Thus, the last line of the string of equalities above is positive by Lemma 2.9. We conclude that $\widehat{K}(Z,Z)(F)$ is positive which implies that $\widehat{K}$ is completely positive.

The final step is to show that $\widehat{K}$ is uniformly contractive. We observe that $\widehat{K}(Z,Z)(I) = I$ for any $Z \in \Omega$. As $\widehat{K}$ is a cp nc kernel, it follows that

$$||\widehat{K}(Z,Z)|| = ||\widehat{K}(Z,Z)(I)|| = 1 \tag{6.3}$$

for any $Z \in \Omega$. By Lemma 6.3, we conclude that $\widehat{K}$ is completely contractive. □

We now prove the analogue of Theorem 6.1 for the noncommutative kernel setting.

Theorem 6.5. *Let $\mathcal{M}$ be an operator space contained in the C^*-algebra $\mathcal{A}$ and let $\mathcal{B}$ be an injective C^*-algebra. A uniformly bounded nc kernel $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{M},\mathcal{B})_{\mathrm{nc}}$ has a bound-preserving extension to a uniformly bounded nc kernel $\widetilde{K} : \Omega \times \Omega \to \mathcal{L}(\mathcal{A},\mathcal{B})_{\mathrm{nc}}$.*

Proof. Without loss of generality, we assume that K is uniformly contractive. From Lemma 6.4, we obtain a uniformly contractive completely positive nc kernel $\widehat{K}_0 : \Omega\times\Omega \to \mathcal{L}(\mathcal{F},\mathcal{B}^{2\times 2})$ where $\mathcal{F}$ is an operator system contained in $\mathcal{A}^{2\times 2}$.

We apply Theorem 5.2 to extend $\widehat{K}_0$ to a completely positive nc kernel $\widehat{K} : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}^{2\times 2},\mathcal{B}^{2\times 2})$. We note that this extension is also uniformly contractive in light of (6.3) and the fact that $\widehat{K}_0$ and $\widehat{K}$ agree when acting on the identity.

For $Z \in \Omega_n$, $W \in \Omega_m$, and $P \in \mathcal{A}^{n\times m}$, we define $\widetilde{K}$ via the identification

$$\widehat{K}(Z,W)(\left[\begin{smallmatrix} 0 & P \\ 0 & 0 \end{smallmatrix}\right]) = \left[\begin{smallmatrix} * & \widetilde{K}(Z,W)(P) \\ * & * \end{smallmatrix}\right].$$

Clearly, $\widetilde{K}$ is an extension of K under this identification, and it is straightforward to check that it remains an nc kernel. Let $Z \in \Omega_n$, $\widetilde{Z} \in \Omega_{\widetilde{n}}$, $W \in \Omega_m$, $\widetilde{W} \in \Omega_{\widetilde{m}}$ such that $\alpha Z = \widetilde{Z}\alpha$ and $\beta W = \widetilde{W}\beta$ where $\alpha \in \mathbb{C}^{\widetilde{n}\times n}$ and $\beta \in \mathbb{C}^{\widetilde{m}\times m}$. Then

$$\begin{aligned}
\left[\begin{smallmatrix} * & \alpha K(Z,W)(P)\beta^* \\ * & * \end{smallmatrix}\right] &= \alpha \left[\begin{smallmatrix} * & \widetilde{K}(Z,W)(P) \\ * & * \end{smallmatrix}\right] \beta^* \\
&= \alpha\widehat{K}(Z,W)(\left[\begin{smallmatrix} 0 & P \\ 0 & 0 \end{smallmatrix}\right])\beta^* \\
&= \widehat{K}(\widetilde{Z},\widetilde{W})(\alpha \left[\begin{smallmatrix} 0 & P \\ 0 & 0 \end{smallmatrix}\right] \beta^*) \\
&= \widehat{K}(\widetilde{Z},\widetilde{W})(\left[\begin{smallmatrix} 0 & \alpha P\beta^* \\ 0 & 0 \end{smallmatrix}\right]) \\
&= \left[\begin{smallmatrix} * & \widetilde{K}(\widetilde{Z},\widetilde{W})(\alpha P\beta^*) \\ * & * \end{smallmatrix}\right]
\end{aligned}$$

which establishes that $\widetilde{K}$ is an nc kernel.

It remains to show that $\widetilde{K}$ is uniformly contractive. For $Z \in \Omega_n$, $W \in \Omega_m$, and $P \in \mathcal{A}^{n\times m}$, we have

$$||\widetilde{K}(Z,W)(P)|| \leq ||\widehat{K}(Z,W)(\left[\begin{smallmatrix} 0 & P \\ 0 & 0 \end{smallmatrix}\right])|| \leq ||\widehat{K}(Z,W)|| \cdot ||P|| \leq ||P||$$

where the last inequality follows from the fact that $\widehat{K}$ is uniformly contractive. □

The following lemma is necessary to establish our decomposition theorem for uniformly bounded kernels.

Lemma 6.6. *Let $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{B})_{\mathrm{nc}}$ be a uniformly bounded nc kernel where $\mathcal{A}$ is a C^*-algebra and $\mathcal{B}$ is an injective C^*-algebra. Then there exist completely positive nc kernels $\Gamma_i : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{B})_{\mathrm{nc}}$, $i = 1, 2$ such that the nc kernel $\widehat{K} : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}^{2\times 2}, \mathcal{B}^{2\times 2})_{\mathrm{nc}}$ given by*

$$\widehat{K}(Z,W)(\left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{smallmatrix}\right]) = \begin{bmatrix} \Gamma_1(Z,W)(P_{11}) & K(Z,W)(P_{12}) \\ K^*(Z,W)(P_{21}) & \Gamma_2(Z,W)(P_{22}) \end{bmatrix} \tag{6.4}$$

is completely positive. Moreover, $\widehat{K}$, Γ_1, and Γ_2 are uniformly bounded by M_K, the uniform bound of K.

Proof. Without loss of generality, we assume that K is uniformly contractive. We proceed as in the proof of Theorem 6.5. We apply Lemma 6.4 to obtain a uniformly contractive completely positive nc kernel $\widehat{K}_0 : \Omega \times \Omega \to \mathcal{L}(\mathcal{F}, \mathcal{B}^{2\times 2})$ and apply Theorem 5.2 to extend this kernel to a uniformly contractive completely positive nc kernel $\widehat{K} : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}^{2\times 2}, \mathcal{B}^{2\times 2})$.

We cannot be sure that the structure given in Lemma 6.4 holds after applying Theorem 5.2, so we must check that $\widehat{K}$ has the form (6.4). We note that $P = \left[\begin{smallmatrix} 0 & P_{12} \\ P_{21} & 0 \end{smallmatrix}\right] \in (\mathcal{A}^{n\times m})^{2\times 2}$ is an element of $\mathcal{F}^{n\times m}$, so the action of $\widehat{K}$ on P agrees with that of $\widehat{K}_0$. Thus for $Z \in \Omega_n$ and $W \in \Omega_m$, we have

$$\widehat{K}(Z,W)(\left[\begin{smallmatrix} 0 & P_{12} \\ P_{21} & 0 \end{smallmatrix}\right]) = \begin{bmatrix} 0 & K(Z,W)(P_{12}) \\ K(W,Z)(P_{21}^*)^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & K(Z,W)(P_{12}) \\ K^*(Z,W)(P_{21}) & 0 \end{bmatrix}.$$

We establish an analogous result for the diagonal blocks. Let $P = \left[\begin{smallmatrix} P_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right] \in (\mathcal{A}^{n\times n})^{2\times 2}$ where $0 \prec P_{11} \preceq I$. For $Z \in \Omega_n$, We have

$$\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] \preceq \widehat{K}(Z,Z)(\left[\begin{smallmatrix} P_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right]) \preceq \widehat{K}(Z,Z)(\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]) = \widehat{K}_0(Z,Z)(\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]) = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]. \tag{6.5}$$

As $0 \preceq \widehat{K}(Z,Z)(\left[\begin{smallmatrix} P_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right])$, we have that $\widehat{K}(Z,Z)(\left[\begin{smallmatrix} P_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right]) = \left[\begin{smallmatrix} A & C \\ C^* & D \end{smallmatrix}\right]$ where $A, C, D \in \mathcal{A}^{n\times n}$ and A, D are positive. A consequence of (6.5) is that $C = 0$ and $D = 0$. Thus, we conclude that

$$\widehat{K}(Z,Z)(\left[\begin{smallmatrix} P_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right]) = \left[\begin{smallmatrix} * & 0 \\ 0 & 0 \end{smallmatrix}\right].$$

We observe that the equality above holds for any $P_{11} \in \mathcal{A}^{n\times n}$ as an element of $\mathcal{A}^{n\times n}$ is contained in the span of its positive elements.

We can now show that $\widehat{K}(Z,W)(\left[\begin{smallmatrix} P_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right]) = \left[\begin{smallmatrix} * & 0 \\ 0 & 0 \end{smallmatrix}\right]$ for $Z \in \Omega_n$, $W \in \Omega_m$, and $P_{11} \in \mathcal{A}^{n\times m}$. By the intertwining relations $\left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right] = Z\left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right] = W\left[\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right]$, we have

$$\begin{aligned} \widehat{K}(Z,W)(\left[\begin{smallmatrix} P_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right]) &= \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right] K(\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right], \left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right]) \left(\begin{bmatrix} \left[\begin{smallmatrix} 0 & P_{11} \\ 0 & 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] \end{bmatrix} \right) \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \\ &= \begin{bmatrix} \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right] * \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

We define $\Gamma_1(Z,W)(P_{11})$ by

$$\widehat{K}(Z,W)(\left[\begin{smallmatrix} P_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right]) = \left[\begin{smallmatrix} \Gamma_1(Z,W)(P_{11}) & 0 \\ 0 & 0 \end{smallmatrix}\right].$$

We check that Γ_1 respects intertwinings: Let $Z \in \Omega_n$, $\widetilde{Z} \in \Omega_{\widetilde{n}}$, $W \in \Omega_m$, $\widetilde{W} \in \Omega_{\widetilde{W}}$ such that $\alpha Z = \widetilde{Z}\alpha$ and $\beta W = \widetilde{W}\beta$ where $\alpha \in \mathbb{C}^{\widetilde{n}\times n}$ and $\beta \in \mathbb{C}^{\widetilde{m}\times m}$. Then for $P_{11} \in \mathcal{A}^{n\times m}$

$$\begin{aligned}
\left[\begin{smallmatrix} \alpha\Gamma_1(Z,W)(P_{11})\beta^* & 0 \\ 0 & 0 \end{smallmatrix}\right] &= \alpha \left[\begin{smallmatrix} \Gamma_1(Z,W)(P_{11}) & 0 \\ 0 & 0 \end{smallmatrix}\right] \beta^* \\
&= \alpha K(Z,W)(\left[\begin{smallmatrix} P_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right])\beta^* \\
&= K(\widetilde{Z},\widetilde{W})(\left[\begin{smallmatrix} \alpha P_{11}\beta^* & 0 \\ 0 & 0 \end{smallmatrix}\right]) \\
&= \left[\begin{smallmatrix} \Gamma_1(\widetilde{Z},\widetilde{W})(\alpha P_{11}\beta^*) & 0 \\ 0 & 0 \end{smallmatrix}\right].
\end{aligned}$$

We construct a nc kernel Γ_2 in the same way and conclude that $\widehat{K}$ has the desired form:

$$\widehat{K}(Z,W)(\left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{smallmatrix}\right]) = \begin{bmatrix} \Gamma_1(Z,W)(P_{11}) & K(Z,W)(P_{12}) \\ K^*(Z,W)(P_{21}) & \Gamma_2(Z,W)(P_{22}) \end{bmatrix}.$$

As noted at the outset, $\widehat{K}$ is uniformly contractive which guarantees that Γ_1 and Γ_2 are as well. □

We can now give our version of Wittstock's decomposition theorem. The proof is analogous to that given in [12].

Theorem 6.7. *Let $K : \Omega\times\Omega \to \mathcal{L}(\mathcal{A},\mathcal{B})_{\mathrm{nc}}$ be a uniformly bounded nc kernel where $\mathcal{A}$ is a C^*-algebra and $\mathcal{B}$ is an injective C^*-algebra. Then there exist completely positive nc kernels $K_i : \Omega\times\Omega \to \mathcal{L}(\mathcal{A},\mathcal{B})_{\mathrm{nc}}$, $i = 1,2,3,4$ such that*

$$K = K_1 - K_2 + i(K_3 - K_4).$$

Proof. We apply Lemma 6.6 to form the uniformly bounded nc kernel

$$\widehat{K}(Z,W)(P) = \widehat{K}(Z,W)(\left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{smallmatrix}\right]) = \begin{bmatrix} \Gamma_1(Z,W)(P_{11}) & K(Z,W)(P_{12}) \\ K^*(Z,W)(P_{21}) & \Gamma_2(Z,W)(P_{22}) \end{bmatrix}.$$

We now consider only P of the form $P = \left[\begin{smallmatrix} P' & P' \\ P' & P' \end{smallmatrix}\right]$, and for the following calculations, we suppress the arguments of the kernels and assume the notation convenience: For $a,b,c,d \in \mathbb{C}$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \mathrm{id}_{\mathbb{C}^{n\times m}} \otimes aI_{\mathcal{B}} & \mathrm{id}_{\mathbb{C}^{n\times m}} \otimes bI_{\mathcal{B}} \\ \mathrm{id}_{\mathbb{C}^{n\times m}} \otimes cI_{\mathcal{B}} & \mathrm{id}_{\mathbb{C}^{n\times m}} \otimes dI_{\mathcal{B}} \end{bmatrix}.$$

We consider the following positivity preserving conjugations:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \Gamma_1 & K \\ K^* & \Gamma_2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \Gamma_1+\Gamma_2+K+K^* & \Gamma_1-\Gamma_2-(K-K^*) \\ \Gamma_1-\Gamma_2+(K-K^*) & \Gamma_1+\Gamma_2-(K+K^*) \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix} \begin{bmatrix} \Gamma_1 & K \\ K^* & \Gamma_2 \end{bmatrix} \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix} = \begin{bmatrix} \Gamma_1 + \Gamma_2 - i(K - K^*) & i(\Gamma_1 - \Gamma_2) - (K + K^*) \\ -i(\Gamma_1 - \Gamma_2) - (K + K^*) & \Gamma_1 + \Gamma_2 + i(K - K^*) \end{bmatrix}.$$

We note that $0 \preceq P'$ implies $0 \preceq \left[\begin{smallmatrix} P' & P' \\ P' & P' \end{smallmatrix}\right]$. As $\widehat{K}$ is a completely positive map, we conclude that the kernels given along the diagonals of the equalities above are completely positive kernels. We define K_i for $i = 1, 2, 3, 4$ by

$$\begin{aligned} K_1 &= (1/4)(\Gamma_1 + \Gamma_2 + K + K^*) \\ K_2 &= (1/4)(\Gamma_1 + \Gamma_2 - (K + K^*)) \\ K_3 &= (1/4)(\Gamma_1 + \Gamma_2 - i(K - K^*)) \\ K_4 &= (1/4)(\Gamma_1 + \Gamma_2 + i(K - K^*)), \end{aligned}$$

and the decomposition follows. □

Remark 6.8. Much of the inspiration for this section comes from the work of Bhattacharyya-Dritschel-Todd in [12]. There the authors consider the special class of nc kernels described in Example 2.5. They show that those kernels which are appropriately bounded under a given regularization scheme are identical to the span of completely positive kernels. That is, they prove an "if and only if" version of Theorem 6.7 for their setting. At the moment, it is unclear how to generalize this stronger result as the regularization scheme used does not readily extend to the setting of uniformly bounded noncommutative kernels.

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