Real-time Integration of Energy Storage

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Abstract

Increasing dynamics in power systems on account of renewable integration, electric vehicle penetration and rising demands have resulted in the exploration of energy storage for potential solutions. Recent technology- and industry-driven developments have led to a drastic decrease in costs of these storages, further advocating their usage. This thesis compiles the author's research on optimal integration of energy storage. Unpredictability is modelled using random variables favouring the need of stochastic optimization algorithms such as Lyapunov optimization and stochastic approximation. Moreover, consumer interactions in a competitive environment implore the need of topics from game theory. The concept of Nash equilibrium is introduced and methods to identify such equilibrium points are laid down. Utilizing these notions, two research contributions are made. Firstly, a strategy for controlling heterogeneous energy storage units operating at different timescales is put forth. They strategy is consequently employed optimally for arbitrage in an electricity market consisting of day-ahead and real-time pricing. Secondly, energy storages owned by consumers connected to different nodes of a power distribution grid are coordinated in a competitive market. A generalized Nash equilibrium problem is formulated for their participation in arbitrage and energy balancing, which is then solved using a novel *weighted* Lyapunov approach. In both cases, we design real-time algorithms with provable suboptimality guarantees in terms of the original centralized and equilibrium problems. The algorithms are tested on realistic scenarios comprising of actual data from electricity markets corroborating the analytical findings.

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General Audience Abstract

Modern power system, which is responsible for generation and transport of electricity, is witnessing a lot of changes such as the increased adoption of wind and solar energy, promotion of electric vehicles, and ever increasing consumer demands. Amidst such developments, energy storage devices like batteries are being propagated as a necessary addition to the power system for its safe operation. This has been further supported by the decrease in prices of these devices over time. An effective assimilation of energy storage however, requires extensive mathematical studies on account of unpredictable renewable generation and consumer demands.This thesis focuses upon the preceding concern. To this note, two novel research contributions are made. In the first, an individual consumer is modeled who wishes to reduce his/her energy costs by simultaneously employing energy storages belonging to different technologies. In the latter, a more challenging multi-consumer interaction is reviewed where multiple end consumers wish to reduce costs while competing against each other over limited resources. In either of the cases, efficient algorithms are designed that are shown to produce desirable results over real-life data and have mathematically provable performance guarantees.

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Chapter 1

Introduction

Electricity is an integral and indispensable part of the present world and seems to remain as crucial for any foreseeable future. From providing light to generating heat, from enabling cellular devices to powering supercomputers, from running electric vehicles to energizing trains, it surely has affected all spheres of human life. It is therefore imperative that the power system infrastructure that deals with the entirety of electricity production and transmission to the end consumer, adapts to any concerns that may arise, by evolving with modern day advancements in technology.

While nuclear and hydroelectric power have gained importance as sources of generation, the majority of electricity around world still comes from carbon fossil fuels such as coal, petroleum and natural gas [3]. This has led to an increase in greenhouse emission leading to environmental concerns such as global warming and ozone layer depletion. Further, escalating demand for electricity on account of rising per capita consumption leads to new peak consumptions that puts pressure on power systems to scale accordingly [4]. Yet another concern is the introduction of loads such as electric vehicles which are dynamic both in spatial and temporal terms, and therefore contribute to uncertainty in load forecasts. This can lead to voltage deviations at distribution buses violating the grid regulation standards [5].

Energy storage, invented in the 18th century, have been revived recently as integral part of solutions to the above concerns. Increased renewable penetration of sources such as solar and wind has been presented as a logical step to handle the environmental issues. However, their intermittency on account of dependency on weather conditions is detrimental to their adaptation. Energy storage come handy in tackling this intermittency by storing electricity when in excess and releasing it during deficits. Utilities have also identified storage as a method to tackle peak demands by employing them to level out loads, both at transmission and distribution levels. The latter has also been encouraged by ISOs across the US through demand response programs [6]. Finally, the voltage deviations at the distribution buses can also be tackled by integrating storage to locally mitigate the energy imbalance. Additionally, they can also be employed at individual customer's end for economic benefits by participating in arbitrage, peak shaving,balance and reserve [7].

Scheduling energy storage is challenging due to inherent coupling of decisions across time and the uncertainty involved in generation and costs. Storage scheduling solutions can be broadly classified into three groups. Approximate dynamic programming solvers typically incur high computational complexity and require the joint probability distribution function (pdf) of the related random processes to be known in advance; see e.g., [8], [9]. The second group comprises model predictive control (MPC)-based schemes, where battery charging is tackled in a deterministic or stochastic fashion over a finite horizon that is progressively shifted as time advances [10], [11]; yet there are no performance guarantees. The third group involves real-time solutions stemming from Lyapunov optimization with relatively mild assumptions. Leveraging tools from stochastic networking [12], methods in this group relax the time-coupling constraints and apply a modified greedy policy attaining feasible solutions with bounded suboptimality. In the smart grid context, the Lyapunov technique was first applied to harvest price differentials in [13], and to integrate energy storage in data centers [14]. A distributed implementation of online Lyapunov policies is derived in [15] as the coordination protocol between an energy aggregator and multiple storage devices. Reference [16] presents a method for energy management in micro grids under network constraints. Coupling storage with load shedding, [17] puts forth a stochastic approximation view of Lyapunov minimization. The Lyapunov technique is modified in [18] to account for battery leakage and charging inefficiencies. A Lyapunov-based scheme for managing energy storage but over a finite time horizon is suggested in [19]. Built on competitive analysis, finite time-horizon algorithms have been adopted for arbitrage using energy storage [20], and peak-shaving during electric vehicle charging [21].

In all previous schemes, battery decisions are synchronized with the control period of the energy system. However, storage technologies operating at slower timescales may have to be employed to lower investment costs.

Slower control rules could also be enforced by batteries having high degradation costs [22]. Hence, coordinating batteries at multiple timescales is practically relevant.

The aforementioned works also presume non-competitive setups therefore lacking the game-theoretic perspective. They usually solve a single optimization problem that models the costs for a user (sum of costs for multiple user scenario) and solve it. The need for a game-theoretic framework however is essential while studying a scenario where decisions of individual customers are coupled amongst themselves either due to competitive pricing or capacity constraints. This instead results in multiple mutually coupled optimization problems whose solution set is the equilibria for the game. Reference [23] solves a single time-instant generalized Stackelberg game for electric vehicle charging under an energy capacity constraint using variational inequalities. The competitive scenario where multiple users aim to minimize their day-ahead cost for running distributed generation and storage is analyzed in [24]. Sharing energy storage resources among users has been shown to be beneficial for arbitrage gains [25], while schemes for sharing storage and renewable generation resources have been analyzed as coalition games in [26]. Using a Stackelberg game formulation, Nash equilibria for charging electric vehicles under demand uncertainty for both (non-)cooperative setups are found in [27].

This thesis compiles the author's research in the two domains listed above. Firstly, the work presented in Chapter 4 develops a real-time control scheme for coordinating batteries at two timescales. The scheme entails Lyapunov optimization at the fast timescale and stochastic approximation at the slow timescale to yield feasible solutions with bounded suboptimality gap. Numerical tests on real data show the advantage of heterogeneous storage.

Secondly, the research in Chapter 6 explores a competitive scenario that coordinates user-owned ES units in a distribution grid. First a dynamic market is modelled that allows the customers to benefit from energy arbitrage and frequency regulation by optimally (dis)charging their respective ES units. While the two-feature cost function is similar to that in [15], the work additionally encourages competition as the energy prices are dependent upon total supply and demand of energy. Strict voltage regulation constraints are formulated that require coordination amongst energy injection/extraction across various nodes of the distribution grids. Secondly, different from [23]-[27], an infinite time horizon Generalized Nash Equilibrium Game (GNEP) is formulated which cannot be solved using traditional methods. This problem is tackled by formulating a centralized optimization problem around a *potential* function which is then solved using a *weighted* Lyapunov optimization method, for which theoretical sub-optimality bounds are proven to exist even under voltage regulation constraints.

The content in this thesis has been organized as follows. Chapter 2 presents a background on various energy storage technologies, their modelling and possible applications in power systems. Since, the work presented studies these applications in a non-deterministic environment, two stochastic optimization techniques: Lyapunov optimization and Stochastic approximation have been employed. The theory and application notes on these make up Chapter 3. Chapter 4 then uses them to present author's first research contribution towards developing a two time-scale algorithm for controlling heterogeneous energy storage in real time. This is followed by Chapter 5 which compiles essential concepts from game theory. Chapter 6 builds upon all these tools and Lyapunov optimization to study a network-constrained multiple users scenario, where users compete with each other for obtaining energy at minimum costs. The thesis is concluded in Chapter 7.

Chapter 2

Energy Storage and its Applications

Energy storage present solutions to many problems in power systems. They have the capacity to shift energy over time, are scalable and becoming more economical with time. Integrating energy storage in optimization problems is a complicated task as they couple decisions over time.

This chapter introduces various types of energy storages and presents a generic model for them. Next, methods to incorporate inefficiencies and degradation cost for the same are discussed. Finally, some potential applications are formulated in the form of optimization problems.

2.1 Types of Energy Storage

2.1.1 Pumped Hydro

Pumped hydro or pumped hydroelectric is the most commonly used method of storing energy across the world. It also the simplest as it is based upon the concept of gravity. Pumped hydro is a storage capacity that is built within hydroelectric dams, which operate on the principle of converting the potential energy of water into kinetic energy. In order to build this capacity, smaller pools are constructed behind the dam into which water is pumped back. These pools can be emptied on demand to power turbines and create electricity. Pumped hydro useful for large scale storage. However this implies it requires high capital investments. Moreover, due to large no of mechanical components, it also has a slower time response.

2.1.2 Flywheels

A flywheel is a mechanical device that stores energy in the form of rotational kinetic energy. It consists of a spinning mass that is connected to a motor and a generator circuit. When the amount of energy stored needs to be increased, electricity powers the motor to speed up the flywheel. On the other hand when energy needs to be released, the flywheel drives a generator to regenerate electricity. Flywheels have faster response dynamics than pumped hydro and are also scalable. They have found extensive applications in wind energy integration where they are connected in series with turbine rotor.

2.1.3 Batteries

Batteries are electrochemical devices that convert electricity to chemical energy and vice versa. For the purpose of this work, only the rechargeable technologies are considered. Although the earliest versions of these were invented as early as the 18th century, it has been the recent advances in their chemistries and corresponding decrease in their prices that has led to their employment in power systems applications . Different chemistries for the same include lithium-ion, lead acid, nickel-cadmium, nickel metal hydride, etc. While Lead acid remain the most prevalently technology due to lower costs (Figure 2.1), Lithium-ion batteries fueled by growing sales in gadgets, mobile devices and electric vehicles are also catching up. Battery based solutions are both fast and scalable. Their large-scale deployment in power systems on the other hand has been hindered by high costs and lower life cycles. However economies of scale are leading to lower costs as shown in Figure 2.2 and therefore indicating strongly their widespread presence in the future power systems.

2.1.4 Compressed Air Energy Storage

Compressed air energy storages are yet another mechanical solution that convert electricity into the potential energy of compressed air. They consist of a large reinforced tank with an air compressor at one end and a turbine at the other. When storing energy, the compressor is turned on and it compresses the air from atmosphere into the tank. This air is used to power a turbine

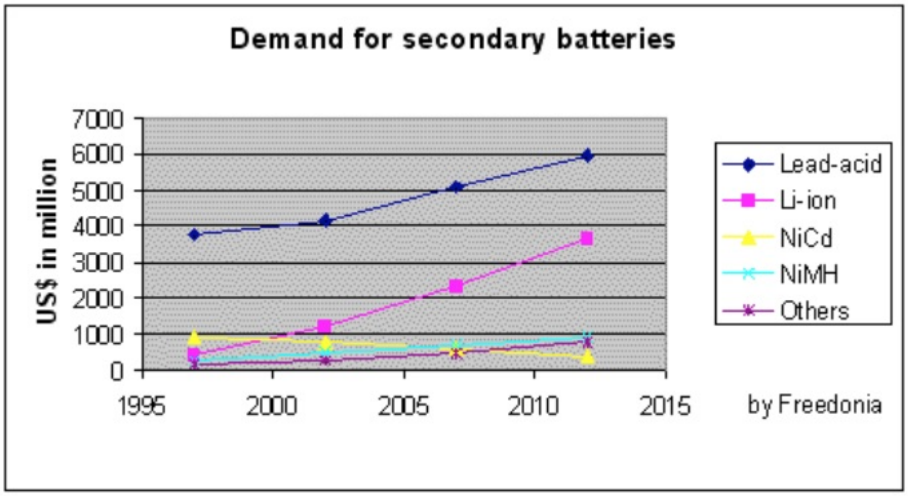


Figure 2.1: Chemistry wise market demand in the US. [1]

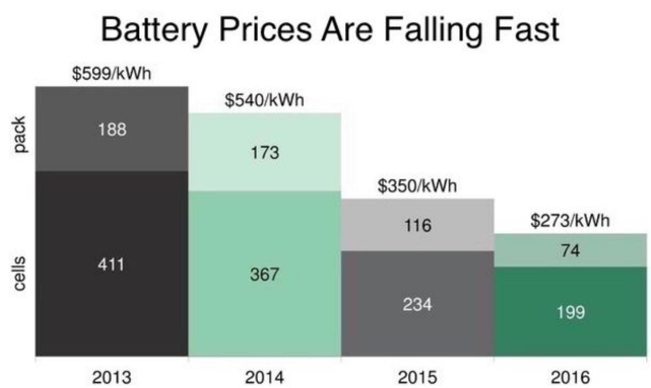


Figure 2.2: Decline in battery prices [2].

and generate back the electricity in the reverse process of energy release. Although first employed in the 19th century, they have been recently revived due to advent in newer technologies which offer higher conversion efficiency and low cost storages.

2.2 Modeling Energy Storage

Modeling energy storages is essential from the point of view of their mathematical analysis. To this end, it is important to incorporate all the dynamics and inefficiencies for maximum accuracy. Let us start with the generic state evolution based formulation that assumes ideal performance and a lossless scenario. Consider the energy storage schematic in Figure 2.3. Here, $s(t)$ represents the state of charge of the storage which models the amount of energy stored at the instant t and $b(t)$ represents the charge/energy being added to the existing reserve in this instant. Note that $b(t)$ can also take a negative value representing the discharge operation where energy is extracted from the storage. Upon addition/subtraction of this energy, the state of charge of

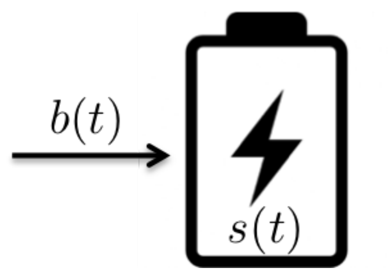


Figure 2.3: Energy Storage unit.

the battery evolves to a new value $s(t+1)$:

$$s(t+1) = s(t) + b(t) \tag{2.1}$$

Since any practical energy storage will have a finite size, there is an upper limit s_{max} on the amount of energy it can store. Similarly, a lower limit s_{min} also exists which may either be set to 0 to represent an empty storage or some other preset value as desired. This is modelled by the following inequality:

$$s_{min} \leq s(t) \leq s_{max} \tag{2.2}$$

Similar limits exists on $b(t)$ on account of maximum rate of power transfer in and out of the energy storage. These are implicitly imposed by the limit on the rate of energy conversion, battery chemistry, power limits on energy transferring media,etc. This is modelled as :

$$b_{min} \leq b(t) \leq b_{max} \tag{2.3}$$

Relations (2.1), (2.2) and (2.3) together formulate an ideal model for energy storages that can represent any of technologies previously discussed. This model will be employed in the research formulations in Chapter 4 and 6.

Various inefficiencies and additional cost dynamics can now be modelled into this basic set of relations.

2.2.1 Inefficient Energy Conversion

This models the loss of energy while charging or discharging the energy storage. Again this loss might be a result of the conversion process as well as the power transfer circuit. Building upon the formulation in [18], let $h^c(b(t))$ an $h^d(b(t))$ represent the charging and discharging conversion function. This is to say that $h^c(b(t))$ amount of energy needs to obtained in order to increase the energy levels by $b(t)$. Similarly, $h^d(b(t))$ is released when lowering the energy levels by $b(t)$. Further note that $h^c(b(t)) = 0$ when $b(t) < 0$ and $h^d(b(t)) = 0$ when $b(t) > 0$. The total amount of energy $h(b_t)$ required to add/remove $b(t)$ from the energy storage is given by

$$h\left(b(t)\right) = h^c\left(b(t)\right) - h^d\left(b(t)\right). \tag{2.4}$$

An example of such a formulation is adopted in [15] where the net energy $e(t)$ is given by :

$$e(t) = \eta_c \max\{b(t), 0\} - \eta_d \max\{-b(t), 0\} \tag{2.5}$$

where $\eta_c \in [1, \infty)$ and $\eta_d \in (0, 1]$.

2.2.2 Energy Leakage

Practical energy storage have the limitation that their energy levels or state of charge decreases over the time even if no energy is extracted from them. This can be simply modeled by changing (2.1) to

$$s(t+1) = \lambda s(t) + b(t) \tag{2.6}$$

where $\lambda \in (0, 1]$ models the fraction of energy retained after one time instant. Note that a low value of λ implies high leakage and vice versa.

2.2.3 Operating Costs

Energy storage, be of any kind, have a finite life of operation. This is due to the fact that they suffer wear and tear of mechanical parts as in flywheels and compressed-air based storage and deterioration in chemistry as in the case of electrochemical batteries. This means that the total cost invested on purchase and maintenance has to be modeled into the operation. Authors in [13] approach this through fixed costs C_c and C_d which are activated every time the battery is storage and discharged respectively. The resulting operating overhead is given by:

$$C_{op} = \mathbf{1}^b(t)C_c + (1 - \mathbf{1}^b(t))C_d \tag{2.7}$$

where $1^b(t)$ is an indicator function that takes values 1 and 0 when the storage is charging and discharging, respectively. A different model is employed in [16] which adds a variable cost component dependent upon the charging/discharging rate

$$C_{op}\left(b(t)\right) = \alpha_b b^2(t) + \chi_b \tag{2.8}$$

for constants α_b and χ_b are constants. The modeled inefficiencies through conversion and leakage losses as well as operating costs of storage are additional details to include in any sort of mathematical analysis thats being carried out. For example, in case an optimization problem is formulated that minimizes the total costs of energy transaction, it is important to add the operating costs of the storage into the objective. Similarly, (2.6) and (2.5) will be modified dynamics that will appear as constraints in the optimization problem.

2.3 Applications

2.3.1 Arbitrage

The deregulated energy market in the US operates in two timescales – day-ahead and real-time [28], [29], [30]. In the day ahead market bids are invited from all producers (generators) and consumers (utilities) and security constrained optimization programs (eg. unit commitment, optimal power flow, etc) are run to determine the prices and amount of energy purchase and sale for the next day. In real-time, any imbalance in energy from the prepared scheduled is handled by solving a similar optimization problem. Either of the markets are governed by dynamics of demand and supply and therefore the energy prices vary across different times of the day. This presents the opportunity for arbitrage whereby an energy storage owner can buy energy at cheap rates from the market, store it and then sell it back when the prices are higher, therefore making financial gains. NYISO encourages such participations through recently launched programs for demand response and distributed resources integration [6].

While the energy markets are not available for small consumers to participate in, utilities across the nation have introduced *time of use pricing* that charges electricity rates according to the time of day [31]. These prices rise during peak hours of consumption and go low in non-peak hours. A potential for arbitrage presents itself in a slightly different form where the customer can buy the energy it needs during peak hours in the previous non peak hours and store it in an energy storage device. Either way the goal is to solve an optimization problem of the form

$$\begin{aligned} &\min_{\{b(t)\}} \sum_{t=1}^{T} C_e(t)\,(b(t)+\ell(t)) \\ &\text{s. to } (2.1), (2.2), (2.3) \end{aligned} \tag{2.9}$$

where $C_e(t)$ represents the time varying energy price, ℓ_t is the customer's load and T is the period of optimization.

2.3.2 Peak Demand Shifting

While the previous subsection models the energy charge component of a customer's electricity bill, utilities are also known to charge additional *demand charge*. This is decided by calculating the peak consumption of the customer in kW for every interval of time, each lasting either 15 or 30 minutes [32]. Then this peak consumption is multiplied by the demand rate during that interval of time. Energy storage can help save money in such a scenario by levelling out the peaks during any one interval. They can also shift the peak from a highly priced interval to a lower priced one. This can be formulated in the following optimization problem :

$$\begin{aligned} &\min_{\{b(t)\}} \sum_{k=1}^{K} C_d(k) \max_{t\in k}\{b(t)+\ell(t)\} \\ &\text{s. to } (2.1), (2.2), (2.3) \end{aligned} \tag{2.10}$$

where $C_d(k)$ is demand rate for the interval k and the max operator calculates peak consumption in the same interval, assuming $b(t)$ and $\ell(t)$ are in kW.

2.3.3 Regulation

Regulation market aims to balance out the difference between the sources and users of power in an electric grid. Utilities now have recognized the important role that fast acting energy storages can play in such markets. PJM for example has come with a new regulation setup whereby energy storage owners can participate [33]. This requires them to first submit a bid in the day ahead market, which if cleared results in the transmission of a regulation signal called *RegD* in real time to owners. The owners then aim to operate their storages in correlation with this signal to maximize the received benefits. NYISO through its demand response programs also enables similar ancillary support from individual customers through aggregators [6]. In a simplified model, let $r(t)$ represents the regulations signal that coordinates charging and discharging of energy storage, such $r(t) > 0$ demands charging of batteries and discharging otherwise. Let $C_r(t)$ represent the regulation benefits received for the support. In such a scenario, a suitable optimization problem that maximizes returns would be

$$\min_{\{b(t)\}} \quad -\sum_{t=1}^{T} r(t) C_r(t) b(t) \tag{2.11}$$

$$\text{s. to } (2.1), (2.2), (2.3)$$

$$\operatorname{sign}(r(t)) = \operatorname{sign}(b(t)) \tag{2.12}$$

2.3.4 Renewable Energy Integration

While the growing environmental concerns call for integration of additional renewable sources of energy such as wind and solar, such integration raise reliability issues for the power system. These sources are intermittent in nature and therefore do not fit effortlessly into the traditional power system dynamics where there is a desired balance of generation and consumption. Even though forecasts for renewable generation are available they are highly sensitive to weather conditions. Energy storage with their ability to shift power over time are highly useful in this scenario as they can store in them energy whenever their is an excess generation and release it back into the grid during a deficit.

Finally, it is imperative to note that it is possible to formulate an all inclusive optimization problem that enables renewable energy integration while maximizing benefits from arbitrage, peak shifting, regulation and other ancillary support services. In Chapter 3, one such problem that integrates renewable integration with a two-scale arbitrage is formulated and solved. In Chapter 6, a different problem which co-optimizes regulation and arbitrage benefits is tackled. Both of the above approaches involve stochastic environment where prices, renewable generation and loads are modelled as random variables. A peak demand shifting problem, while being easy to solve in a deterministic framework, gets highly complicated in a stochastic environment as it introduces coupling with future, unknown scenarios.

2.4 Concluding Remarks

The chapter formulated a basic model for energy storage while providing modifications to incorporate inefficiencies. It was seen how these models couple decisions over time therefore complicating optimization formulation. The ensuing chapter presents a way to handle this coupling in a stochastic environment. Possible applications for storages were also discussed. Of these, arbitrage in a two-timescale environment will be further explored in Chapter 4 and a combined arbitrage plus regulation scenario will be reviewed from a game-theoretic perspective in Chapter 6.

Chapter 3

Stochastic Optimization

Randomness is a law of nature. Random variables are therefore of common occurrence in mathematical problems. Power system formulations are not free from them. Various quantities such as market prices, loads and measurement noise. can be modeled as random variables. Therefore, optimization problems that deal with such power system applications should be versatile to this randomness.

This chapter presents two stochastic optimization methods - Lyapunov optimization and stochastic approximation. The strength of the former method is its ability to handle time coupling introduced by energy storage. The latter is more generic and can be applied to a variety of power system applications.

3.1 Lyapunov Optimization

3.1.1 Queueing Concept

Lyapunov Optimization is a framework developed by Dr. Michael J. Neely of the University of Southern California [12].The framework was originally developed for applications in communication and queuing systems whereby data is being sequentially stored in queues/buffers and operated/processed upon. This section will first introduce the original intended application to get a perspective into the same. Consider the setup as demonstrated in Figure 3.1. It consists of a server that processes upon the requests $b(t)$ that it receives at time instant t. Let $C\left(b(t), t\right)$ represent the cost of processing

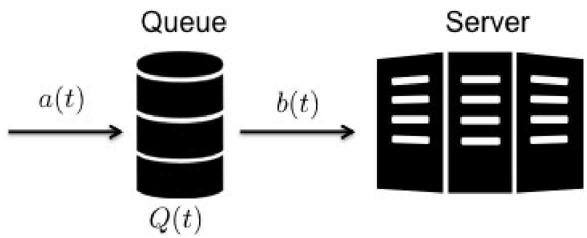


Figure 3.1: Schematic diagram for a queue based server operation.

upon such requests by the server. Note that this cost is dependent upon both $b(t)$ and the time instant t. This implies that serving the same amount of requests at two different instants might lead to different costs. The server is equipped with a buffering system which acts as an interface to it. Any serving requests $a(t)$ are first stored in this buffer and then depending upon the processing availability $b(t)$ requests are passed onto the server. Let the size dynamics of this buffering system be represented by $Q(t)$ such that:

$$Q(t+1) = Q(t) + a(t) - b(t) \tag{3.1}$$

$Q(t)$ therefore represents the requests that arrived up till instant t but have not been served yet. As it can be seen from (3.1), the buffer size $Q(t)$ will increase if $a(t) > b(t)$, decrease if $a(t) < b(t)$ and remain the same otherwise. The objective in such a scenario is to

- Serve the requests at minimum cost.
- Ensure that the queues do not grow *very large*.

The latter constraint might be induced because of finite queue capacities, controlling the delay in serving requests or higher overhead costs with increase in buffer size. In Queuing theory, a formal manner of ensuring constraint on the size of the queues is through the concept of *mean rate stability*. Specifically a queue $Q(t)$ is said to be mean rate stable if :

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}|Q(T)| = 0$$

which is to say that the queue does not grow faster than linearly with time. Here, $\mathbb{E}$ is the expectation operator over the randomness in $\{a(t), C(b(t), t)\}$ In this respect, an optimization strategy which minimizes the cost $C(b(t), t)$

over time while ensuring that the queues are mean rate is suitable for the problem at hand.

3.1.2 Infinite Horizon Optimization Problem

One way to capture the cost of serving requests is through an *infinite horizon expected average* of the same. Specifically, let $\bar{C}$ represent this quantity and be given by:

$$\bar{C}\left(\{b(t)\}\right) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\big[C\left(b(t), t\right)\big] \tag{3.2}$$

$\bar{C}$ therefore incorporates averaging both over time up till infinity through the $\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T}$ operation and over the randomness in $\{a(t), C\left(b(t), t\right)\}$ through the $\mathbb{E}$ operation. Let us now represent our objectives of cost minimization and queue stabilization in terms of an optimization problem

$$\min_{\{b(t)\}} \; \bar{C}\left(\{b(t)\}\right) \tag{3.3}$$

$$\text{s. to}$$

$$Q(t+1) = Q(t) + a(t) - b(t) \tag{3.4}$$

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}|Q(T)| = 0. \tag{3.5}$$

The above formulation has the following two interesting features. Firstly, it seeks decisions for the entire time horizon up till infinity. Secondly, it involves randomness through $C(b(t), t)$. In the later subsections it will be seen that many energy storage integration problems can be formulated in an analogous manner. Therefore, an optimization strategy that provides a solution to such a problem is required. Lyapunov optimization is one such strategy.

3.1.3 Drift Plus Penalty

The Lyapunov optimization strategy entails formulating "drift plus penalty" term which in turn yields a real time algorithm that solves the infinite horizon optimization problem previously introduced. To this end, consider the following definitions

$$L\left(Q(t)\right) := \frac{1}{2}(Q(t))^2 \tag{3.6}$$

$$\Delta\left(Q(t)\right) := \mathbb{E}\big[L\left(Q(t+1)\right) - L\left(Q(t)\right)|Q(t)\big] \tag{3.7}$$

where $L\left(Q(t)\right)$ and $\Delta\left(Q(t)\right)$ are the Lyapunov function and the conditional Lyapunov drift over $Q(t)$ respectively. Then the following theorem holds true

Theorem 1.

$$\begin{aligned}\Delta\left(Q(t)\right) + &V\mathbb{E}\big[C\left(b(t),t\right)|Q(t)\big] \leq \\ &B + Q(t)\mathbb{E}\big[a(t)-b(t)|Q(t)\big] + V\mathbb{E}\big[C\left(b(t),t\right)|Q(t)\big]\end{aligned} \tag{3.8}$$

where $V > 0$ is a parameter of choice and B is a constant that can be calculated as

$$B = \frac{\big(\max_t\left(|a(t)-b(t)|\right)\big)^2}{2}.$$

The term on the left hand of the above inequality is referred to as the "drift plus penalty" term as it combines the Lyapunov drift previously introduced with a penalty depending upon the instantaneous cost $C\left(b(t),t\right)$.

Proof. The proof follows simply from the definitions in (3.6) as will be shown below. First, from (3.1)

$$(Q(t+1))^2 = (Q(t))^2 + 2Q(t)\left(a(t)-b(t)\right) + \left(a(t)-b(t)\right)^2.$$

Employing this in (3.6) yields

$$\Delta\left(Q(t)\right) = \mathbb{E}\left[Q(t)\left(a(t)-b(t)\right) + \frac{\left(a(t)-b(t)\right)^2}{2}\middle|Q(t)\right]. \tag{3.9}$$

Adding $V\mathbb{E}\big[C\left(b(t),t\right)|Q(t)\big]$ on both sides of the (3.9), the result of Theorem 1 follows. $\square$

In the following subsection, it will be shown that greedily minimizing the right hand side of the inequality in Theorem 1 provides a bounded sub-optimal solution to the optimization problem in (3.3)-(3.5).

3.1.4 Real-Time Algorithm Formulation

The main contribution of Lyapunov optimization framework is that it solves the infinite horizon problem in (3.3)-(3.5) sub-optimally by the means of a real-time algorithm. Before discussing this further, it is required to state an important result from [12]

Proposition 1. *Under the assumption that variables $\{a(t), C(b(t),t)\}$ are randomly iid over time, there exists a (randomized) stationary policy that observes the their instantaneous values and decides $\tilde{b}(t)$ such that :*

- $\mathbb{E}\big[a(t) - \tilde{b}(t)\big] = 0$
- $\mathbb{E}\Big[C\left(\tilde{b}(t), t\right)\Big] = \bar{C}^*$

Where $= \bar{C}^*$ *is cost achieved by* $\bar{C}(\{b(t)\})$ *under a policy that solves* (3.3)-(3.5) *optimally.*

The proof of the above proposition has been omitted in this work for brevity. The next theorem yields a real-time algorithm that solves (3.3)-(3.5) sub-optimally.

Theorem 2. *Let $\hat{b}(t)$ be the solution to the optimization problem*

$$\min_{b(t)} \; Q(t)\left(a(t) - b(t)\right) + VC\left(b(t), t\right) \tag{3.10}$$

If $L(Q(0))$ is finite, it holds that

1. $\lim_{T\to\infty} \frac{1}{T}\mathbb{E}|Q(T)| = 0$
2. $\bar{C}^* \le \bar{C}\left(\{\hat{b}(t)\}\right) \le \bar{C}^* + B/V.$

Proof. First, it is straightforward to see that the objective in (6.23) is the greedy version of the right-hand side inequality in Theorem 1. This means that $\hat{b}(t)$ minimizes this right hand side over all feasible policies including the one in Proposition 1. Hence, from (6.37) it follows that

$$\Delta\left(Q(t)\right) + V\mathbb{E}\big[C\left(b(t), t\right)|Q(t)\big] \le B + V\bar{C}^*. \tag{3.11}$$

Taking expectations on both side of the above inequality wrt $Q(t)$, using the definition (3.6) and the law of total expectations, it is seen that

$$\mathbb{E}\big[L\left(Q(t+1)\right)-L\left(Q(t)\right)\big]+V\mathbb{E}\big[C\left(b(t),t\right)\big]\leq B+V\bar{C}^{*}> \qquad (3.12)$$

Taking summation over $t=0,1\ldots T-1$, yields

$$\mathbb{E}\big[L\left(Q(T)\right)-L\left(Q(0)\right)\big]+V\sum_{t=0}^{T-1}\mathbb{E}\big[C\left(b(t),t\right)\big]\leq TB+TV\bar{C}^{*}. \qquad (3.13)$$

To prove the first part of the theorem, let C_{min} denote the minimum expected cost of $C\left(b(t),t\right)$, i.e.

$$\mathbb{E}\big[C\left(b(t),t\right)\big]\geq C_{min}.$$

Then transferring terms in (3.13) gives:

$$\mathbb{E}\big[L\left(Q(T)\right)\big]\leq\mathbb{E}\big[L\left(Q(0)\right)\big]+T\big(B+V(\bar{C}^{*}-C_{min})\big).$$

Using (3.6), and taking square root on both sides of the previous inequality, dividing by T and taking its limit to infinity

$$\lim_{T\to\infty}\frac{\mathbb{E}\big[|Q(T)|\big]}{T}\leq\sqrt{\frac{\mathbb{E}\big[L\left(Q(0)\right)\big]}{T^{2}}+\frac{\big(B+V(\bar{C}^{*}-C_{min})\big)}{T}}$$

The first part of Theorem follows by observing that the term $\big(B+V(\bar{C}^{*}-C_{min})\big)\geq 0$ and $L\left(Q(0)\right)$ is finite. This also proves that $\hat{b}(t)$ is feasible with respect to the optimization problem in (3.3)-(3.5). Note that the inequality remains true even after the square root operation because of the definition in (3.6). To prove the second part, dividing both the sides of (3.13) by TV provides

$$\mathbb{E}\left[\frac{1}{VT}L\left(Q(T)\right)-\frac{1}{VT}L\left(Q(0)\right)\right]+\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\big[C\left(b(t),t\right)\big]\leq\frac{B}{V}+\bar{C}^{*} \qquad (3.14)$$

The right-hand side of the second part of the theorem then follows by taking limit of T to infinity and observing that $L\left(Q(0)\right)$ is finite whereas $L\left(Q(T)\right)>0$. $\square$

It is worth emphasizing the implications of Theorem 2

1. Instead of solving the infinite-horizon problem in (3.3)-(3.5), one can solve the real-time problem in (6.23) greedily.

2. To do so, only the current instances of the random variables $\{a(t), C(b(t), t)\}$ are used and no knowledge of future states or their pdf is required

3. Solving the real-time problem produces a policy $\hat{b}(t)$ that is feasible with respect to the original problem and produces a cost within a B/V bound of the optimal cost.

The strength of the Lyapunov optimization lies in the above summary. While, in its current form it is possible to asymptotically achieve optimal solution by taking $V \to \infty$, it will be seen that such a scenario is not feasible with energy storage control as it inherently restricts the maximum value for V.

3.1.5 Application to Energy Storage

To apply Lyapunov optimization to energy storage, it is first important to observe the analogy between them and previously introduced queues. This was first utilized in [13] where a real-time algorithm was developed to minimize the electricity costs in a data center with energy storage units. Consider the schematic in Figure 3.2. Here, $S(t)$, $R(t)$ and $D(t)$ represents the state of charge, the recharge amount and the discharge amount respectively of the energy storage.

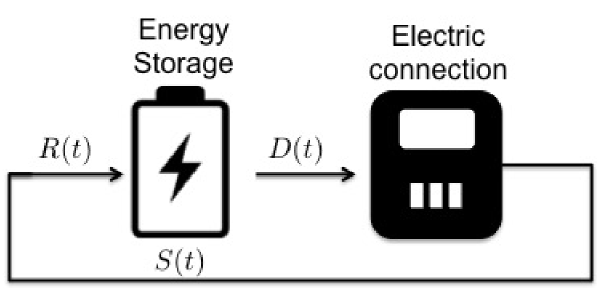


Figure 3.2: Schematic diagram for an energy storage with an electric connection.

The following equation represents the dynamics between these quantities

$$S(t+1) = S(t) + R(t) - D(t). \tag{3.15}$$

It is easy to see the similarity between Figures 3.2 and 3.1 and relations (3.15) and (3.1) respectively. Similarly, the cost of energy purchase, which is represented by $P\left(R(t), D(t), t\right)$ is analogous to the server cost. $C\left(b(t), t\right)$. Building upon this analogy further, one might want to minimize the cost of energy purchase while keeping the sate of charge $S(t)$ within the desirable levels. This analogy has been summarized in Table 3.1.

Data Serving	$Q(t)$	$a(t)$	$b(t)$	$C\left(b(t), t\right)$
Energy storage	$S(t)$	$R(t)$	$D(t)$	$P\left(R(t), D(t), t\right)$

Table 3.1: Analogous quantities between two applications.

The motivation for applying Lyapunov optimization to the energy storage scenario becomes quite clear from the comparison drawn. An infinite-horizon problem involving energy storages with the infinite horizon cost given as

$$\bar{P}\left(\{R(t), D(t)\}\right) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\big[P\left(R(t), D(t), t\right)\big] \tag{3.16}$$

might have the following structure

$$\begin{aligned}
\min_{\{R(t),D(t)\}} \quad & \bar{P}\left(\{R(t), D(t)\}\right) && (3.17)\\
& \text{s. to} \\
& 0 \le R(t) \le R_{max} && (3.18)\\
& 0 \le D(t) \le D_{max} && (3.19)\\
& 1_R(t) + 1_D(t) = 1 && (3.20)\\
& S(t+1) = S(t) + R(t) - D(t) && (3.21)\\
& S_{min} \le S(t) \le S_{max} && (3.22)
\end{aligned}$$

where

$$1_R(t) = 1 \text{ if } R(t) > 0,\ 0 \text{ otherwise} \tag{3.23}$$

$$1_D(t) = 1 \text{ if } D(t) > 0,\ 0 \text{ otherwise} \tag{3.24}$$

(3.20) with (3.23) and (3.24) ensure that at any time instant energy storage is either being charged or discharged, which makes sense for a battery. While constraints (3.18) and (3.19) do not add any complexity to the scenario, (3.22) is a hard constraint which was absent in the formulation (3.3)-(3.5). Next subsection will cover the steps for applying Lyapunov optimization to an energy storage scenario while taking care of these hard constraints.

3.1.6 Steps for Optimization of Energy Storage

For formulations analogous to (3.17)-(3.22) the following series of steps can be applied to obtain a real-time algorithm

1. Relax the problem by replacing the hard constraint by a relaxed average constraint.
2. Design virtual queues corresponding to each of the batteries.
3. Identify the real-time algorithm which greedily minimizes the right hand side *drift plus penalty* term.
4. Characterize the solutions of the real-time algorithm.
5. Obtain the range of values for various parameters to make the solution feasible wrt the original problem.

where Steps 1,4 and 5 are additionally required in order to take care of the constraint (3.22). Step 1 allows us to obtain a relaxed problem that is suitable for the direct application of the Lyapunov optimization framework. Let $\bar{P}'$ represent the optimal cost to this relaxed problem. Then we know that using Lyapunov optimization we can achieve a set of policies $\{\hat{R}(t), \hat{D}(t)\}$ that solve a real-time algorithm an obtain a cost $\bar{P}\{\hat{R}(t), \hat{D}(t)\}$ such that:

$$\bar{P}' \leq \bar{P}\{\hat{R}(t), \hat{D}(t)\} \leq \bar{P}' + B/V. \tag{3.25}$$

For some parameters B and V that will be later identified. However, such $\{\hat{R}(t), \hat{D}(t)\}$ might be infeasible with respect to the exact problem in (3.17)-(3.22). Steps 4 and 5 ensure such feasibility through a careful selection of parameters. In such a case if $\bar{P}^*$ represents the optimal cost of the exact problem, (3.25) changes to:

$$\bar{P}^* \leq \bar{P}\{\hat{R}(t), \hat{D}(t)\} \leq \bar{P}^* + B/V.$$

Therefore, the solutions of the real-time optimization provide a bounded sub-optimality with respect to the original problem as well. These series of steps will next be applied to an illustrative scenario.

3.1.7 Illustrative Scenario

In this subsection an optimization problem quite similar to that in (3.17)-(3.22) is solved using the steps defined previously. The problem is same as that in [13] however only the simple case is considered where the cost $P(t)$ is a linear function of energy purchased/sold. Consider the case where a customer connected to the grid has an access to an energy storage unit whose state charge is given by $s(t)$ and (discharging)charging rate by (-)$b(t)$ governed by following:

$$s(t+1) = s(t) + b(t) \tag{3.26}$$

$$s_{min} \leq s(t) \leq s_{max} \tag{3.27}$$

$$b_{min} \leq b(t) \leq b_{max} \tag{3.28}$$

where (3.26) captures the battery dynamics and 3.27 and 3.28 establish the limits on $s(t)$ and $b(t)$ respectively. Note that the variables $R(t)$ and $D(t)$ from previous scenario have been combined into one variable $b(t)$ for simpler analysis. The customer also has to serve a flexible primary load demand given by $\ell(t)$, which is assumed to be iid random in nature with unknown pdf. This scenario is represented in Figure 3.3. Let the price charged for this

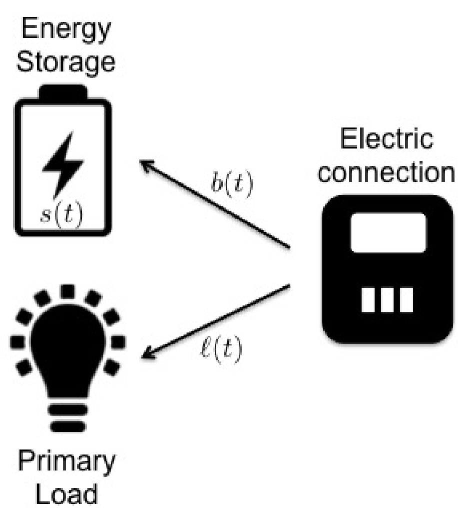


Figure 3.3: Customer with primary load and energy storage.

energy transaction be given by $p\left(b(t),t\right)$ such that

$$p\left(b(t),t\right)=c(t)\left(b(t)+\ell(t)\right) \tag{3.29}$$

where $c(t)$ is again assumed to be an iid random variable with unknown pdf. The customer desires to minimize its infinite-horizon expected time average cost giving rise to the following optimization problem

$$\mathbf{P1}:\min_{\{b(t)\}}\ \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\big[p\left(b(t),t\right)\big] \tag{3.30}$$

$$\text{s.to } (3.26),(3.27),(3.28) \tag{3.31}$$

The steps introduced previously will now by applied.

1. Relax the problem by replacing the hard constraint with a softer average constraint.

$$\mathbf{P2}:\min_{\{b(t)\}}\ \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\big[p\left(b(t),t\right)\big]$$

$$\text{s.to } (3.28)$$

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}b(t)=0 \tag{3.32}$$

   where (3.32) is an average queueing constraint. It is easy to see that the **P2** is a relaxed version of **P1**. To this end, summing up both sides of (3.26) from $t=0,1,\ldots,T-1$ yields

$$s(T)=s(0)+\sum_{t=0}^{T-1}b(t)$$

   (3.32) then follows by dividing both sides by T while talking its limit to infinity and observing that both $s(T)$ and $s(0)$ are finite because of (3.27) and a feasible starting assumption respectively.

2. Design virtual queues corresponding to each of the batteries.

$$q(t)=s(t)+\gamma \tag{3.33}$$

Note that the virtual queue $q(t)$ is a shifted version of $s(t)$ by a constant parameter γ. This means that it obeys the following dynamics

$$q(t+1) = q(t) + b(t)$$

Summing the above equation over $t = 0, 1, \ldots, T-1$, dividing by T with infinity as its limiting value and using the fact that $q(0)$ is finite, it can be seen that (3.32) is equivalent to (3.5). Therefore, **P2** is in a form solvable by Lyapunov optimization framework.

3. Identify the real-time algorithm which greedily minimizes the right hand side *drift plus penalty* term.

$$\mathbf{P3} : \min_{b(t)} \; q(t)b(t) + Vc(t)\left(b(t) + \ell(t)\right) \tag{3.34}$$

$$\text{s. to (3.28)} \tag{3.35}$$

where the above minimization problem is obtained as a simple result of this step. As shown previously in Theorem 2, solving the obtained real-time problem will ensure a bounded sub-optimal solution with mean rate stability for the queue $q(t)$.

4. Characterize the solutions of the real-time algorithm.

   (a) When $q(t) + Vc_{\min} > 0$ then $\hat{b}(t) = b_{min}$

   (b) When $q(t) + Vc_{\max} > 0$ then $\hat{b}(t) > b_{max}$

   where $c(t) \in [c_{min}, c_{max}]$.The above characterization is easy to follow from the structure of the real time problem.

5. Obtain the range of values for various parameters to make the solution feasible wrt original problem – Finally it can be shown through induction that be selecting the parameters V and γ as given below, the solution $\hat{b}_t$ to **P3** will always be feasible with respect to **P1** [13].

$$0 < V \leq \frac{s_{max} - s_{min} + b_{min} - b_{max}}{c_{max} - c_{min}} \tag{3.36}$$

$$-Vc_{min} + b_{max} - s_{max} \leq \gamma \leq -Vc_{max} + b_{min} - s_{min}. \tag{3.37}$$

3.2 Stochastic Approximation

Stochastic approximation is a class of recursive algorithms that are used to solve optimization problems with a certain degree of randomness. This randomness might occur in the form of noise in some parameter of the cost function. A similar problem has been studied previously where a customer tries to minimize its electricity cost under random pricing and load scenario.The general structure for such problem is given by

$$\min_{\theta \in \Theta} \mathbb{E}\big[F(\theta, \omega)\big] \tag{3.38}$$

where θ is the parameter of optimization and ω is a random variable over which $\mathbb{E}$ expectation operates. Therefore the objective is to optimize the expected value of the function $F()$. The concept of stochastic approximation is easily understood from the problem of estimating the expected value of a random variable ω. Then, the *law of large numbers* dictates that

$$\lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} \omega_k \to \mathbb{E}[\omega] \tag{3.39}$$

The above relation states that if ω is sampled large enough times, the sample mean tends to the required expectation. To arrive at an iterative way to solve this problem, first note that it can be translated into that of finding a root to $f(x)$ as defined below.

$$f(x) = x - \mathbb{E}(\omega) \tag{3.40}$$

The next iteration then solves for this root.

$$x_{k+1} = x_k - \mu_k(x_k - \omega_k) \tag{3.41}$$

where μ_k is the step size, and as previously stated ω_k is an observed sample.If $x_1 = \omega_1$ then it follows that

$$x_K = \frac{1}{K} \sum_{k=1}^{K} \omega_k. \tag{3.42}$$

And therefore if a large no of iterations are carried out with new samples for ω, the iterations converge to the desired expectation from (3.39). Building upon this an iterative stochastic subgradient method can be formulated that solves an optimization problem as in (3.38) which is the subject of the next subsections.

3.2.1 Subgradients

Before discussing the method it is important to lay down the concept of subgradients. Subgradients are the extension of derivatives (or gradients) to nondifferentiable functions. The subgradient method as discussed later therefore enables us to optimize nondifferentiable convex functions just as gradient methods do for differentiable counterparts. Formally a vector $g \in \mathbb{R}^n$ is a subgradient of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a point x in its domain if for every point z also in its domain [34]:

$$f(z) \geq f(x) + g^\top (z - x) \tag{3.43}$$

Note that the subgradient coincides with the derivative at the point x if the function is differentiable there. Further, the set of subgradients at a point x is known as the subdifferentiable of f at x and is represented as $\partial f(x)$. The result next from [34] establishes the existence of subgradients.

Lemma 1. *If f is convex and x is in the interior of the domain of f, then $\partial f(x)$ is nonempty and bounded.*

Now that the existence of subgradients has been established the next section presents the stochastic subgradient method.

3.2.2 Stochastic Subgradient Method

The stochastic subgradient method is an extension of the stochastic gradient method to case where $F(\theta, \omega)$ in (3.38) is convex. Focussing on latter, we know that for an unconstrained convex optimization problem the optimal solution can be found by setting its gradient to zero. If $F(\theta, \omega)$ is assumed to be convex differentiable in one dimensional θ then this equates to

$$\frac{d}{d\theta}\mathbb{E}\big[F(\theta,\omega)\big] = 0 \tag{3.44}$$

$$\implies \mathbb{E}\bigg[\frac{dF(\theta,\omega)}{d\theta}\bigg] = 0. \tag{3.45}$$

where the linear operators $\mathbb{E}$ and $\frac{d}{d\theta}$ have been interchanged in the last inequality. Therefore, the problem translates to identifying the root of the function $G(\theta) = \mathbb{E}\big[dF(\theta,\omega)/d\theta\big]$. Corresponding to (3.41), the iterative procedure below will solve for this root.

$$\theta_{k+1} = \theta_k - \mu_k \frac{dF(\theta,\omega_k)}{d\theta}\Big|_{\theta=\theta_k} \tag{3.46}$$

which is to say, obtain a sample for the random variable ω find the gradient at current iterate θ_k and obtain the next iterate. Extending this to to the subgradients, consider the following iteration

$$\theta_{k+1} = \theta_k - \mu_k g(\theta_k,\omega_k) \tag{3.47}$$

where, $g \in \partial F(\theta,\omega)$ at $\theta = \theta_k$. Then, if the step size is selected as per the following conditions, it can be shown that the method converges to the optimal solution.

- $\mu_k > 0 \;\forall\; k$
- $\lim_{K\to\infty}\sum_{k=1}^{K}\mu_k = \infty$
- $\lim_{K\to\infty}\sum_{k=1}^{K}\mu_k^2 < \infty$

One selection that satisfies the above constraints is that of the diminishing step size where $\mu_k = \mu_0/k$.

3.3 Concluding Remarks

Two stochastic optimization methods have been discussed. Lyapunov optimization has the ability to solve infinite horizon time coupled formulations involving energy storages. One of its strengths lies in a guaranteed mathematical bound on performance adn it will be adapted in Chapter 4 and

Chapter 6. The tool of Stochastic approximation will be coupled with Lyapunov optmization to yield a two-timescale optimization problem in 4.

Chapter 4

Real-time Control of Heterogeneous Energy Storage Units

Energy storage systems are becoming a key component in smart grids with increasing renewable penetration. Storage technologies feature diverse capacity, charging, and response specifications. Investment and degradation costs may require charging batteries at multiple timescales, potentially matching the control periods at which grids are dispatched.

To this end, a microgrid equipped with slow- and fast-responding batteries is considered here. Energy management decisions are taken at two stages. Slow-responding batteries are dispatched at an hourly resolution with decisions remaining invariant over multiple fast control slots. Building on Lyapunov optimization, slow- and fast-responding batteries are charged based on real-time and data-dependent with quantifiable suboptimality bounds. Numerical tests using real data demonstrate the advantage of operating heterogeneous batteries.

4.1 Microgrid Modeling

Consider a microgrid consisting of a photovoltaic, a variable load, and two energy storage units, which is coordinated by a controller as shown in Fig. 4.1. Due to heterogeneous storage technologies and the manner energy is exchanged between the microgrid and the main grid, control operations evolve in two timescales: The control horizon at the fast timescale is discretized into slots of equal duration indexed by t. A sequence of T consecutive fast-timescale slots comprises a control period for the slow timescale indexed by $n = \lfloor t/T \rfloor$. Time t can be then expressed as $t = nT + \tau$ to indicate the slow control period it belongs to, and the related offset τ.

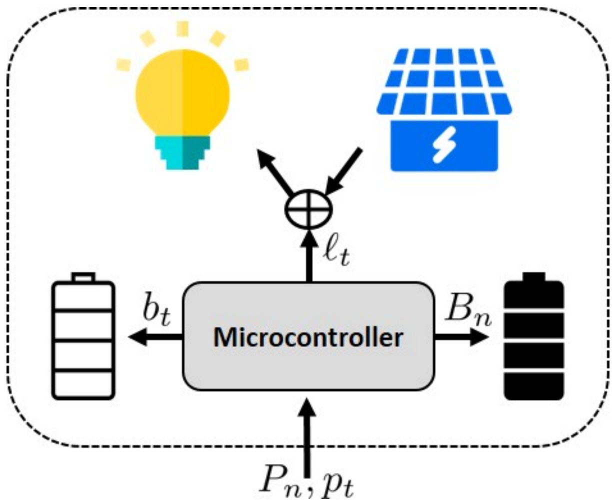


Figure 4.1: Charging batteries at different timescales.

At the slow timescale, the microgrid exchanges energy with the main grid through a time-ahead energy market and it operates the slower battery. The control decisions taken for slow period n remain unchanged over the next T fast control slots. If S_n represent the state of charge (SoC) for the slow battery at the beginning of slow period n, and B_n the amount by which the same unit is charged over period n, it holds

$$S_{n+1} = S_n + B_n \tag{4.1a}$$
$$\underline{S} \le S_n \le \overline{S} \tag{4.1b}$$
$$\underline{B} \le B_n \le \overline{B} \tag{4.1c}$$

where (4.1a) captures battery dynamics; (4.1b) preserves the SoC within the capacity $(\underline{S}, \overline{S})$; and (4.1c) enforces charging rates $(\underline{B}, \overline{B})$. The microgrid buys energy P_n from the main grid to charge the slow battery by B_n, while the remaining energy

$$E_n = P_n - B_n \tag{4.2}$$

serves the load over the next T fast control slots, and it is bounded as $E_n \in [\underline{E}, \overline{E}]$.

At the fast timescale, the controller collects information on solar generation and load demand, operates the fast battery, and exchanges energy with the main grid through a real-time market. Let ℓ_t denote the difference between demand and solar generation over fast period t. Similarly to (4.1), the SoC s_t and the charge b_t for the fast battery at time t satisfy

$$s_{t+1} = s_t + b_t \tag{4.3a}$$
$$\underline{s} \leq s_t \leq \overline{s} \tag{4.3b}$$
$$\underline{b} \leq b_t \leq \overline{b}. \tag{4.3c}$$

If p_t is the energy bought from the real-time market at period t and the amount of energy E_n is delivered uniformly across the next T fast times slots, then energy balance implies

$$p_t = \ell_t + b_t - \frac{E_n}{T}. \tag{4.4}$$

Energy costs are modeled as convex increasing functions $C_n(P_n)$ and $c_t(p_t)$ for the time-ahead and the real-time market, respectively. Two functions are assumed random. If $G_n = \partial C_n(P_n)$ and $g_t = \partial c_t(p_t)$ denote the cost subgradients, define their extreme values as $\underline{G} := \min_n\{G_n\}$, $\overline{G} := \max_n\{G_n\}$, $\underline{g} := \min_t\{g_t\}$, and $\overline{g} := \max_t\{g_t\}$.

4.2 Algorithm Development

Given that the time-ahead cost $C_n(P_n)$ occurs once every T control periods and the real-time cost $c_t(p_t)$ at each control period t, the energy management task can be posed as

$$\begin{aligned} \phi_1^* = \min \lim_{N\to\infty} \frac{1}{NT} \sum_{n=0}^{N-1} \sum_{\tau=0}^{T-1} \mathbb{E}\left[\frac{C_n(P_n)}{T} + c_t(p_t)\right] \quad & (4.5) \\ \text{over} \quad \{P_n, B_n, S_n, E_n\}, \{p_t, s_t, b_t\} \quad & \\ \text{s.to} \quad (4.1) - (4.4) \quad & \end{aligned}$$

where the expectation $\mathbb{E}$ is with respect to (wrt) $\{C_n, c_t, \ell_t\}$. Solving (4.5) is challenged by the randomness and the coupling across successive periods in (4.1a) and (6.2a). Conventional solutions based on approximate dynamic programming suffer from the curse of dimensionality and presume the joint pdf to be known [35]. Alternatively, Lyapunov optimization can approximately tackle infinite-horizon problems of particular structure by solving a sequence of relatively simple problems as time proceeds [12].

To transform the energy management task in (4.5) into the Lyapunov optimization framework, consider the problem

$$\begin{aligned}
\phi_2^* = \min \lim_{N\to\infty} \frac{1}{NT} \sum_{n=0}^{N-1} \sum_{\tau=0}^{T-1} \mathbb{E}\left[\frac{C_n(P_n)}{T} + c_t(p_t)\right] && \text{(4.6a)}\\
\text{over} \quad \{P_n, B_n, E_n\}, \{p_t, b_t\} && \\
\text{s.to} \quad (4.1\text{c}), (4.2), (6.2\text{c}), (4.4) && \text{(4.6b)}\\
\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} B_n = 0 && \text{(4.6c)}\\
\lim_{N\to\infty} \frac{1}{NT} \sum_{n=0}^{N-1} \sum_{\tau=0}^{T-1} b_t = 0 && \text{(4.6d)}
\end{aligned}$$

where constraints (4.1a)–(4.1b) and (6.2a)–(6.2b) appearing in (4.5) have been replaced by the time-averaged constraints (4.6c)–(4.6d) and variables $\{S_n, s_t\}$ have been eliminated. Problem (4.6) constitutes a relaxation of (4.5); see also [14], [17]. To see this, consider sequences $\{P_n, B_n, S_n, E_n\}$ and $\{p_t, s_t, b_t\}$ that are feasible for (4.5). Unfolding the dynamics (4.1a) and (6.2a) yields $S_N = S_0 + \sum_{n=0}^{N-1} B_n$ and $s_{NT} = s_0 + \sum_{n=0}^{N-1} \sum_{\tau=0}^{T-1} b_t$. Due to (4.1b) and (6.2b), the states S_N and s_{NT} are finite at all times. Dividing the previous equations by N and NT respectively, and sending N to infinity provides (4.6c) and (4.6d). Therefore, the sequence $\{P_n, B_n, E_n\}$ and $\{p_t, b_t\}$ that are feasible for (4.5) are also feasible for (4.6).

Since (4.6) is a relaxation of (4.5), it follows that $\phi_2^* \leq \phi_1^*$. If an algorithm solves (4.6) with a suboptimality gap of ϵ, and yields $\{P_n, B_n, E_n, p_t, b_t\}$ that are feasible for (4.5), then this algorithm attains an optimal value ϕ_2' for which $\phi_1^* \leq \phi_2' \leq \phi_2^* + \epsilon$. Combining the latter with $\phi_2^* \leq \phi_1^*$ proves that the algorithm would be ϵ-suboptimal for (4.5) too, i.e., $\phi_1^* \leq \phi_2' \leq \phi_1^* + \epsilon$. Since (4.6) does not involve time-coupling constraints, it is easier to solve than (4.5). An online approximate solver for (4.6) yielding control decisions feasible for (4.5) is derived next.

The Lyapunov technique introduces two queues X_n and x_t and minimizes the *drift plus penalty cost* for all n [36]:

$$\begin{aligned} \min \quad & X_n B_n + V C_n(P_n) + \sum_{\tau=0}^{T-1} \mathbb{E}\left[x_t b_t + V c_t(p_t)\right] \\ \text{over} \quad & P_n, B_n, E_n, \{p_t, b_t\} \\ \text{s.to} \quad & (4.1\text{c}), (4.2), (6.2\text{c}), (4.4) \end{aligned} \tag{4.7}$$

where the virtual queues relate to the SoCs as

$$X_n := S_n + \Gamma, \quad x_t := s_t + \gamma \tag{4.8}$$

for constants Γ and γ to be specified later. Heed that the expectation in the cost of (4.7) is now only wrt (c_t, ℓ_t). Nonetheless, problem (4.7) is still challenging since it involves expectations over the future values of the queue parameter x_t. Similarly to [37], to overcome this difficulty, the queue values $\{x_t\}_{\tau=0}^{T-1}$ are replaced by x_n and the resultant problem is handled using stochastic approximation.

Upon the aforesaid simplification and if $\{c_t, \ell_t\}_\tau$ are iid, problem (4.7) is substituted by

$$\min_{\underline{E} \leq E_n \leq \overline{E}} H(E_n; X_n) + T\mathbb{E}\left[F_t(E_n; x_n)\right] \tag{4.9}$$

where the function $H(E_n; X_n)$ is defined as

$$H(E_n; X_n) := \min_{\underline{B} \leq B_n \leq \overline{B}} X_n B_n + V C_n(B_n + E_n) \tag{4.10}$$

and each $F_t(E_n; x_n)$ relies on a single realization (c_t, ℓ_t) as

$$F_t(E_n; x_n) := \min_{\underline{b} \leq b_t \leq \overline{b}} x_n b_t + V c_t(\ell_t + b_t - E_n/T). \tag{4.11}$$

Because functions $H(E_n; X_n)$ and $\{F_t(E_n; x_n)\}$ are convex, the minimization in (4.9) is convex too.

Minimizing (4.9) over E_n could be solved via a projected subgradient scheme. A subgradient of $H(E_n; X_n)$ is $VG_n(B_n^j + E_n^j)$ with B_n^j being a min-

Algorithm 1 Two Timescale Storage Management Scheme

for $t = 0, 1, 2, \ldots$ **do**
  if t/T is integer **then**
    Set $n = t/T$ and observe cost C_n.
    Set $E_n^0 = E_{n-1}^*$ and $X_n = S_n + \Gamma$.
    for $j = 0, 1, 2, \ldots$ **do**
      Draw sample (ℓ_t, c_t) and solve (4.11) for E_n^j.
      Update E_n^{j+1} from (4.12).
    end for
    Find B_n^* by solving (4.10) for E_n^*.
    Buy energy $P_n^* = B_n^* + E_n^*$ from main grid.
  end if
  Observe (c_t, ℓ_t) and set $x_t = s_t + \gamma$.
  Find b_t^* from (4.11) for E_n^*.
  Buy energy $p_t^* = \ell_t + b_t^* - E_n^*/T$ from main grid.
end for

imizer of (4.10) attaining $H(E_n^j; X_n)$. Likewise, a subgradient of $F_t(E_n; x_n)$ is $-\frac{V}{T} g_t(\ell_t + b_t^j - E_n^j/T)$, where b_t^j is a minimizer of (4.11) attaining $F_t(E_n^j; x_n)$. The j-th subgradient update reads

$$E_n^{j+1} = \left[E_n^j - \mu_j V(G_n(B_n^j + E_n^j) - \tfrac{1}{T}\mathbb{E}[g_t(\ell_t + b_t^* - \tfrac{E_n^j}{T})])\right]_{\underline{E}}^{\overline{E}}$$

for $\mu_j > 0$. Observe that updating E_n requires solving (4.10) once, but also infinitely many problems of the form in (4.11).

To avoid the computational burden, stochastic approximation surrogates the previous update with a single evaluation of the related stochastic subgradient $F_t(E_n; x_n)$, that is

$$\begin{aligned} E_n^{j+1} := \Big[& E_n^j - \mu_j V G_n(B_n^j + E_n^j) \\ & + \frac{\mu_j V}{T} g_t(\ell_t + b_t^j - \tfrac{E_n^j}{T})\Big]_{\underline{E}}^{\overline{E}}. \end{aligned} \tag{4.12}$$

For $\mu_j = \mu/j$ with $\mu > 0$, the stochastic subgradient update of (4.12) is guaranteed to converge to a minimizer of (4.7). The charge B_n^* can now be found as the minimizer of (4.10) for E_n^*.

Having found (E_n^*, B_n^*), the real-time control decisions (p_t^*, b_t^*) for the next T fast time slots can be found by solving (4.11) for E_n^*. Note that (4.11) has a closed form solution for differentiable and convex c_t. Steps 2–11 of Alg. 1 precede the slow control period n, and Steps 12–14 correspond to fast control slots.

4.3 Algorithm Performance

We first provide the conditions under which the charging decisions of Alg. 1 are feasible for problem (4.5) [14], [37]

Proposition 2. *Under the mild assumptions that* $\overline{S} - \underline{S} > \overline{B} - \underline{B}$ *and* $\overline{s} - \underline{s} > T(\overline{b} - \underline{b})$, *the control decisions* $\{B_n^*\}$ *and* $\{b_t^*\}$ *found by Alg. 1 are feasible for problem* (4.5) *if the parameters* (Γ, γ, V) *satisfy:*

$$-V\underline{G} + \overline{B} - \overline{S} \leq \Gamma \leq -V\overline{G} + \underline{B} - \underline{S} \tag{4.13a}$$

$$-V\underline{g} + T\overline{b} - \overline{s} \leq \gamma \leq -V\overline{g} + T\underline{b} - \underline{s} \tag{4.13b}$$

$$0 < V \leq \overline{V} \tag{4.13c}$$

where $\overline{V} = \min\left\{\frac{\overline{S} - \underline{S} + \underline{B} - \overline{B}}{\overline{G} - \underline{G}}, \frac{\overline{s} - \underline{s} + T(\underline{b} - \overline{b})}{\overline{g} - \underline{g}}\right\}$.

Proof of Prop. 2. Proving (4.13a) by mathematical induction, it is shown next that if $S_n \in [\underline{S}, \overline{S}]$, the same holds for S_{n+1}. From (4.1a) and (4.8), it follows that $S_{n+1} = S_n + B_n = X_n - \Gamma + B_n$, where B_n is the minimizer of (4.10). Depending on the queue value X_n, three cases can be considered:

(C1) If $X_n \leq -V\overline{G}$, it is easy to see that $B_n = \overline{B}$, and hence, S_{n+1} can only increase compared to S_n. To ensure $S_{n+1} = X_n - \Gamma + \overline{B} \leq \overline{S}$, it suffices that $\Gamma \geq -V\overline{G} + \overline{B} - \overline{S}$.

(C2) If $X_n \geq -V\underline{G}$, it holds that $B_n = \underline{B}$, and hence, S_{n+1} can only decrease compared to S_n. To ensure $S_{n+1} = X_n - \Gamma + \underline{B} \geq \underline{S}$, it suffices that $\Gamma \leq -V\underline{G} + \underline{B} - \underline{S}$.

(C3) When $-V\overline{G} \leq X_n \leq -V\underline{G}$, the minimizer has to be feasible $B_n \in [\underline{B}, \overline{B}]$. The previous limits on X_n are valid since $\underline{G} \leq \overline{G}$ and $V > 0$. A sufficient condition ensuring $S_{n+1} \leq \overline{S}$ is $\Gamma \geq -V\underline{G} + \overline{B} - \overline{S}$ and a sufficient condition ensuring $S_{n+1} \geq \underline{S}$ is $\Gamma \leq -V\overline{G} + \underline{B} - \underline{S}$. Claim (4.13a) follows since the limits under case (C3) are tighter than the respective limits under (C1) and (C2).

Claim (6.27b) is shown likewise. From (6.2a) and (4.8), it holds that $s_{n+1} = s_n + \sum_{\tau=0}^{T-1} b_t = x_n - \gamma + \sum_{\tau=0}^{T-1} b_t$, where b_t is a minimizer of (4.11). Based on x_n, three cases are distinguished:

(c1) If $x_n \leq -V\overline{g}$, then $\{b_t = \overline{b}\}_{\tau=0}^{T-1}$. Thus, s_{n+1} is larger than s_n and $s_{n+1} \leq \overline{s}$ is ensured if $\gamma \geq -V\overline{g} + T\overline{b} - \overline{s}$.

(c2) If $x_n \geq -V\underline{g}$, then $\{b_t = \underline{b}\}_{\tau=0}^{T-1}$. Thus, s_{n+1} is smaller than s_n and $s_{n+1} \geq \underline{s}$ is ensured if $\gamma \leq -V\underline{g} + T\underline{b} - \underline{s}$.

(c3) When $-V\overline{g} \leq x_n \leq -V\underline{g}$, the minimizers $\{b_t\}_{\tau=0}^{T-1}$ lie in $[\underline{b}, \overline{b}]$. Guaranteeing $s_{n+1} \in [\underline{s}, \overline{s}]$ is assured if $\gamma \geq -V\underline{g} + T\overline{b} - \overline{s}$ and $\gamma \leq -V\overline{g} + T\underline{b} - \underline{s}$. These two bounds are tighter than those obtained under (c1)–(c2), and (6.27b) follows. Ensuring $s_n \in [\underline{s}, \overline{s}]$ implies that $s_t \in [\underline{s}, \overline{s}]$ at all times.

Bounding V by $\overline{V}$ in (4.13c) assures that the upper bounds in (4.13a)–(6.27b) are larger than the related lower bounds, and hence, Γ and γ are implementable. $\square$

Lemma 2 ([12]). *Let $\{P_n^{st}, B_n^{st}, E_n^{st}, p_t^{st}, b_t^{st}\}$ be the decisions under a policy that selects them based solely on the current realization $\{C_n, c_t, \ell_t\}$. If states are iid over time, there exists one such policy satisfying* (4.1c), (6.2c), (4.2), *and* (4.4), *for which:*

$$\mathbb{E}[B_n^{st}] = 0, \mathbb{E}[b_t^{st}] = 0, \mathbb{E}\left[C_n(P_n^{st})/T + c_t(p_t^{st})\right] = \phi_1^*. \tag{4.14}$$

The next result upper bounds $\hat{\phi}_1$ and asserts that $V = \overline{V}$ yields the tightest bound while maintaining feasibility.

Proposition 3. *If $\{\zeta_t\}$ are iid over time, then $\hat{\phi}_1 \leq \phi_1^* + \frac{K_B}{2TV} + \frac{TK_b}{2V}$, where $K_B := \max\{\underline{B}^2, \overline{B}^2\}$ and $K_b := \max\{\underline{b}^2, \overline{b}^2\}$.*

Proof of Prop. 3. Define the Lyapunov function $L_t := \frac{1}{2}(X_t^2 + x_t^2)$ and the T-slot Lyapunov drift $\Delta_t := \mathbb{E}[L_{t+T} - L_t | X_t, x_t]$ for $t = nT$. Using (4.1a), (6.2a), and (4.8) yields

$$\begin{aligned}
\Delta_{nT} &= \tfrac{1}{2}\mathbb{E}[X_{n+1}^2 - X_n^2 + x_{n+1}^2 - x_n^2 | X_n, x_n] \\
&= \frac{1}{2}\mathbb{E}\left[2X_nB_n + B_n^2 + 2x_n\sum_{\tau=0}^{T-1} b_t + \left(\sum_{\tau=0}^{T-1} b_t\right)^2 | X_n, x_n\right] \\
&\leq \mathbb{E}\left[X_nB_n + x_n\sum_{\tau=0}^{T-1} b_t | X_n, x_n\right] + \frac{K_B + T^2K_b}{2}
\end{aligned} \tag{4.15}$$

Define $\phi_n^t = C_n(P_n)/T + c_t(p_t)$; add $V\sum_{\tau=0}^{T-1}\mathbb{E}[\phi_n^t|X_n,x_n]$ on both sides of (4.15); and rearrange to get

$$\Delta_{nT} + V\sum_{\tau=0}^{T-1}\mathbb{E}[\phi_n^t|X_n,x_n] \leq \mathbb{E}[X_nB_n + x_n\sum_{\tau=0}^{T-1}b_t|X_n,x_n] + V\sum_{\tau=0}^{T-1}\mathbb{E}[\phi_n^t|X_n,x_n] + \frac{K_B+T^2K_b}{2}. \tag{4.16}$$

Notice that the minimization of the right-hand side of (6.22) coincides with (4.7). Hence, the value of (6.22) attained by Alg. 1 would be the minimum over all feasible policies, including the one of Lemma 2. From (4.14) if follows:

$$\Delta_{nT} + \sum_{\tau=0}^{T-1}\mathbb{E}[\phi_n^t|X_n,x_n] \leq VT\phi_1^* + \tfrac{K_B+T^2K_b}{2}. \tag{4.17}$$

Taking expectations on both sides of (4.17) wrt (X_n, x_n); applying the law of total expectation; and summing over N consecutive slow intervals yields $V\sum_{n=0}^{N-1}\sum_{\tau=0}^{T-1}\mathbb{E}[\phi_n^t] \leq VNT\phi_1^* + \frac{N(K_B+T^2K_b)}{2} - \mathbb{E}[L_{NT} - L_0]$. Prop. 3 is obtained by dividing both sides of the last inequality by VNT; taking N to infinity; and noting $\mathbb{E}[L_0]$ is finite and $\mathbb{E}[L_{NT}] \geq 0$. □

4.4 Numerical Tests

Algorithm 1 was tested using 5-min load data from Home C of the Smart* project [38], scaled up by a factor of 10 and repeated to yield 20 weeks of load ℓ_t [38]. Costs $\{C_n, c_t\}$ were modeled as convex piecewise linear with different selling and buying prices. Buying prices for C_n were taken as the hourly day-ahead prices for the Michigan hub in the MISO market over April, 2015, repeated to match the duration of ℓ_t. Buying prices for c_t were simulated as uniformly distributed having the related C_n as mean and \$10/MWh as variance. Selling prices for C_n and c_t were set to 0.9 times the buying prices. The faster timescale had $T = 12$ fast intervals. Battery parameters were set to $\overline{S} = 1$ MWh, $\overline{s} = 84$ kWh, $\underline{S} = \underline{s} = 0$ kWh, and $\overline{B} = -\underline{B} = \overline{b} = -\underline{b} = 10$ kW, and $\overline{E} = -\underline{E} = 30$ kWh. Figure 4.2 depicts the time-averaged operational costs for three microgrid scenarios. The curves demonstrate that

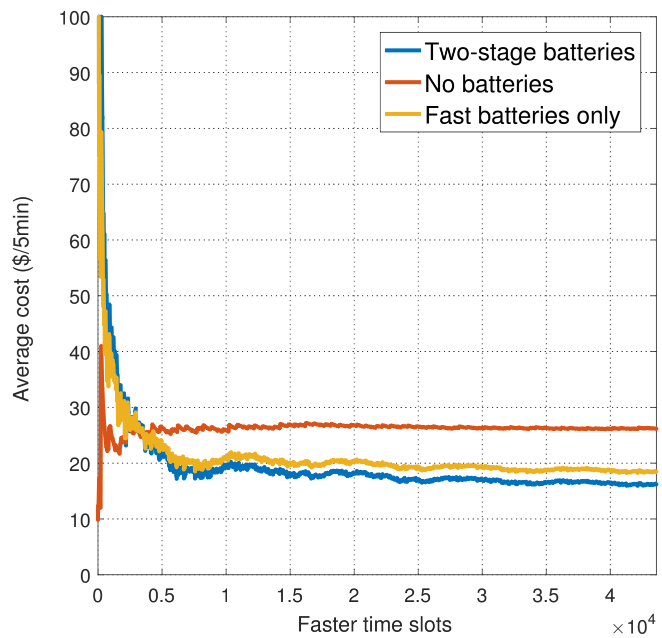


Figure 4.2: Time-averaged microgrid operation cost.

adding heterogeneous batteries lowers the operational cost compared to the other two cases.

4.5 Concluding Remarks

Heterogeneous energy storages with different time response characteristics were jointly controlled in an optimal manner. This was achieved with a help of a two-timescale algorithm that combines Lyapunov optimization with stochastic approximation. It was seen that such an integration allows more economic benefits than controlling individual storages. Sub-optimality bounds show that performance improves with larger sizes of storage, smaller charging rates, lower variability in prices and lastly lesser difference amongst the two-timescales.

Chapter 5

Game Theory Concepts

Game theory is a branch of mathematics that studies competition. When multiple agents are involved in a decision making process with each agent motivated by its own selfish gains, the outcome is often different from the case where a single agent decides for overall welfare. Such scenarios can arise in power systems when multiple entities are competing for a single limited resource eg. energy markets. Game theory concepts provide useful insights in these cases.

This chapter lays down some fundamental concepts that will be later on employed in Chapter 6. First, the formal definition of a game is presented and various categories of games are discussed. Next, Nash equilibrium is introduced as a solution concept. Finally, methods to formulate an aggregated optimization problem that identify its Nash equilibrium are summarised.

5.1 Definition of a Game

To define a game in its *strategic form* the following three elements are required

- A set of players given by $\mathcal{N} = \{1, 2, \ldots, N\}$
- A set of strategies/actions for every player $i \in \mathcal{N}$ given by $\mathcal{S}_i$
- Utility/cost/payoff functions for every player $i \in \mathcal{N}$ given by u_i corresponding to all possible combination of strategies amongst the players

To illustrate this, consider the scenario famously known as the *prisoner's dilemma* in Table 5.1. The setup for it is as follows: Two prisoners held in separate confessions rooms by the police are being interrogated for a crime they carried out together. Each of them has two possible strategies to select from – {Confess, Not Confess} to the crime. The numbers in the cells are the sentences in year they get (with negative signs) corresponding to the strategies selected by both of them. For this game,

	P2 Confess	P2 Not Confess
P1 Confess	(-4,-4)	(0,-5)
P1 Not Confess	(-5,0)	(-2,-2)

Table 5.1: Prisoner's dilemma

- $\mathcal{N} = \{1, 2\}$
- $\mathcal{S}_1 = \mathcal{S}_2 =$ {Confess,Not Confess}
- u_1 and u_2 can be obtained be selecting the first and the second values in the cells respectively.

5.2 Types of Games

While there are several ways to classify games, following bifurcations are most commonly found in literature

5.2.1 Non-Cooperative vs Cooperative Games

This classification is based upon the freedom of players to coordinate (cooperate) amongst themselves. Non-cooperative games model a scenario where the utility of one player is partially or completely conflicted with that of others while making decisions independent from other players [39]. The last part of the previous sentence is important as it emphasises that it is essential that the players act individually without coordinating with others. If the game formulation in such a scenario leads to a self-enforced cooperation, so be it. As an example for a non-cooperative game, consider a scenario where multiple electric vehicle owners are trying to charge their vehicles from a bus in the distribution grid. Assume that the price of electricity increases with the total energy demand and an increased priced decreases the utility for the owners. To simplify the model, further assume that the utility of each of them increases with increased charged in their batteries and decreases with an increase in price per unit energy. This formulates a non-cooperative game whereby each of the electric vehicle owner competes with everyone else for maximum energy at the best possible price.

In contrast to Non-cooperative games, under a cooperative setup players are allowed to coordinate amongst themselves to in order to arrive at set of strategies that is jointly beneficial to all of them. This involves concepts such as coalition formation and bargaining. The simplest (although violent) example of a cooperative game, is that of coalition formations amongst the various countries during the world wars. It is safe to assume that such coalitions were formed after the leaders of the fighting countries agreed upon respective benefits once their side won the war. Interested reader may refer to [39] for further details on cooperative games.

5.2.2 Dynamic vs Static Games

While the previous bifurcation focused upon the ability of the players to coordinate amongst themselves, the current classification is based upon the evolution of various components of the game with time. In a dynamic game, the sequence of decisions taken by the players as well as the information gathered by players about past decisions of other players effects the outcome of the game [39]. This might be possible through a different utility depending upon sequence of such decisions. Chess is one such dynamic game.

On the other hand, in static game, players do not account for past information while acting. This means the structure of the game doesn't change depending upon what sequence of actions were taken. No additional information is revealed to the players as they act simultaneously. Note that either of the cooperative or non-cooperative games can be static or dynamic in nature. The example of prisoner's dilemma is a static game as both the prisoners act at once.

5.3 Nash Equilibrium

Once a game has been defined a logical next step is to identify the possible *solutions* for the game. The term *solutions* in itself is quite abstract and can be used to refer to several ways of decision making. Extending this abstractness further, a *good* solution is a set of strategies amongst the players that will make them all *happy*. Since game theory formulations involve multiple utilities in contrast to an optimization formulation that enjoys the simplicity of a single utility to be maximized, the *goodness* of the solution is not always that straightforward to measure.

John Nash in the year 1951 introduced the solution concept in his article titled *Non-Cooperative Games* [40] which has been since then referred to as Nash equilibrium. It represents a set of strategies for all players such that each of their utility is maximized while the everybody else's strategy is fixed at the Nash equilibrium. Before formally stating this definition it is useful to bifurcate into two categories of Nash equilibria – Pure strategy and Mixed strategy.

5.3.1 Pure Strategy Nash Equilibrium

A pure strategy Nash equilibrium is a set of strategies $\mathbf{s}^* = \{s_1^*, s_2^*, \ldots, s_N^*\}$ of all players with the property that no player i can do better by choosing a strategy different from s_i^* while every other player $j \neq i$ stays at s_j^*. Introducing the notation $(s_i, \mathbf{s}_{-i})$ where $\mathbf{s}_{-i}$ is the collection of strategies for all players but i, if $\mathbf{s}_{-i}^*$ is the Nash equilibrium of the game, then :

$$u_i(s_i^*, \mathbf{s}_{-i}^*) \geq u_i(s_i, \mathbf{s}_{-i}^*), \ \forall s_i \in \mathcal{S}_i \tag{5.1}$$

In this sense, the Nash equilibrium is a *no regret* policy since none of the players will any regrets selecting it upon finding out how the rest of the players acted. It is also straightforward to see that $\mathbf{s}^*$ simultaneously solves the following optimization problem $\forall i \in \mathcal{N}$:

$$\max_{s_i \in \mathcal{S}_i} \ u_i(s_i, \mathbf{s}_{-i}) \tag{5.2}$$

One can observe that {Confess, Confess} is the pure strategy Nash equilibrium for the prisoner's dilemma introduced previously as none of the prisoners can reduce their sentence by changing to {Not confess} while the other stays at {Confess}.

Note: The existence and uniqueness of a pure strategy Nash equilibrium for a game is not always guaranteed.

5.3.2 Mixed Strategy Nash Equilibrium

Mixed strategy Nash equilibrium, a generalization of the pure strategy Nash equilibrium, was the actual subject of Nash's original paper. Specifically, Nash proved that such an equilibrium always exits for a any game with finite no of players and finite set of actions for each of those players.

Unlike pure strategy case where every player selects its strategy deterministically, a mixed strategy allows the players to perform probabilistically. To this end, let $\sigma_i(s_i)$ be the probability distribution over the strategies $\mathcal{S}_i$ for the player i. For a finite set, this distribution reduces to a probability mass function. If each of the players are supposed to pick such probability distribution/mass functions for their strategies, then it results in a mixed strategy game. If $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \ldots, \sigma_N\}$ and $\mathbf{s} = \{s_1, s_2, \ldots, s_N\}$, then the *expected* pay off of player i is given by:

$$u_i(\boldsymbol{\sigma}) = \sum_{\mathbf{s}\in\mathcal{S}} \left(\prod_{j=1}^{N} \sigma_j(s_j) \right) u_i(s_i, \mathbf{s}_{-i}) \tag{5.3}$$

Similar to (5.1), if $\boldsymbol{\sigma}^*$ represents the mixed strategy Nash equilibrium of a game then the following holds true

$$u_i(\sigma_i^*, \boldsymbol{\sigma}_{-i}^*) \geq u_i(\sigma_i^*, \boldsymbol{\sigma}_{-i}^*),\ \forall \sigma_i \in \Sigma_i \tag{5.4}$$

Where Σ_i is set of all feasible probability distribution functions for player i. Similarly, it is easy to follow that $\boldsymbol{\sigma}^*$ simultaneously solves the following optimization problem for $\forall i \in \mathcal{N}$:

$$\max_{\sigma_i\in\Sigma_i}\ u_i(\sigma_i, \boldsymbol{\sigma}_{-i}) \tag{5.5}$$

While the Nash equilibria as defined are *stable* because of the *no regret* property, they might not be the most efficient solutions. To this end let a *pareto optimal* outcome be defined as one where there is no other possible outcome that makes every other player at least as well off and at least one player strictly better off. Pareto optimality in this sense is a metric for efficiency rather than stability. For the prisoner's dilemma, it is easy to see that the strategy {Not confess, Not confess} is pareto optimal and the Nash equilibrium {Confess, Confess} is less efficient than this strategy.

5.3.3 Generalized Nash Equilibrium

Both the problem formulations in (5.2) and (5.5) have one element in comment. The strategy spaces $\mathcal{S}_i$ in the former and Σ_i in the latter for player i were independent of the actions $\mathbf{s}_{-i}$ and $\boldsymbol{\sigma}_{-i}$ respectively. This is to say that possible set of strategies for a player did not depend upon other players. Of course, the dependence of the utilities would require a player to be considerate about other players to maximize its payoff, the strategy space in itself remains unchanged. The problems (5.2) and (5.5) are also referred to as Nash Equilibrium Problems (NEP) as they lead us to the Nash Equilibriums (NE) of the game. While they cover a lot of competitive scenario, there is a possible generalization that extends their applications further.

Consider previously discussed example of electric vehicle owners trying to charge their batteries from a distribution grid. Also assume that they are connected to different nodes of the distribution grids and that their energy demands are sufficiently large. In such a scenario it is possible that combined demand for energy exceeds the available energy at the feeder bus. Even prior to that, it is possible that these energy demands might affect the voltage levels of the buses in the distribution grid. Therefore, not only will the electric vehicle owners will have to keep in consideration how everybody else's demand affects the energy price, they will also have to be mindful that together they do not demand more energy than available. Similarly, they will need to ensure that their demands do not change the voltage levels to unacceptable values.

Scenarios such as above give birth to the Generalized Nash Equilibrium Problems (GNEP) [41] where the feasible strategy spaces change with decisions of other players. In such a case, a pure strategy $\mathbf{s}^*$ that solves the optimization problem as in (5.6) $\forall i \in \mathcal{N}$ is referred to as a Generalized Nash Equilibrium (GNE):

$$\max_{s_i \in \mathcal{S}_i(\mathbf{s}_{-i})} \; u_i(s_i, \mathbf{s}_{-i}) \tag{5.6}$$

Note that the term $s_i \in \mathcal{S}_i(\mathbf{s}_{-i})$ establishes the dependence of strategy space for player i on other players' actions. It is also logical to say that this dependence creates additional complexity to the already non-trivial problem of identifying NEs.

5.4 Aggregated Problem Formulation

In the previous sections coupled optimization problems were introduced to identify the NEs in (5.2) and (5.5) and to GNEs in (5.6) respectively. While they provide a mathematical formulation to the problems at hand, solving them is not a trivial task. First of all there is no guarantee towards the existence of pure strategy NEs and GNEs. Secondly, their existence does not imply uniqueness. Further, GNEPs have additional constraints in terms of dependent strategy space.

One approach towards solving such problems is through the route of *learning* algorithms. In such algorithms, the players reach NE/GNE of the game iteratively by taking actions, observing the outcomes and then improving upon the actions in next iteration. For example, algorithms such as best response dynamics and fictitious play guarantee convergence to the NE under certain conditions [39]. Similarly the regret matching algorithm converges to the coarse correlated equilibrium, another generalization of equilibria.

Yet another approach is to arrive at an aggregated formulation, the solution to which coincides with the sought NE/GNE.This approach is advantageous in the sense that it often leads to a single optimization problem which can be then be solved using the extensive literature of optimization algorithms. Two such methods that result in aggregate problem formulation are stated next.

5.4.1 Variational Inequalities

The concept of Variational Inequalities (VI) can be used to solve for both NE and GNE. Consider a vector valued function $\mathbf{F}(\mathbf{s})$ and a corresponding vector space defining it's domain given by $\mathbf{S}$. Then the variational inequality problem $\mathrm{VI}(\mathbf{S}, \mathbf{F}(\mathbf{s}))$ consists of finding a vector $\hat{\mathbf{s}} \in \mathbf{S}$ such that:

$$(\mathbf{y} - \hat{\mathbf{s}})^\top \mathbf{F}(\hat{\mathbf{s}}) \geq 0, \ \forall \ \mathbf{y} \in \mathbf{S} \tag{5.7}$$

Focussing on the pure strategy equilibria, let $\mathbf{F}(\mathbf{s})$ be defined as follows:

$$\mathbf{F}(\mathbf{s}) := -\left[\frac{\partial u_1(\mathbf{s})}{\partial s_1} \ \frac{\partial u_2(\mathbf{s})}{\partial s_2} \ \ldots \frac{\partial u_N(\mathbf{s})}{\partial s_N}\right]^\top \tag{5.8}$$

Where u_i is the utility of player i as previously discussed. Next consider the following assumptions.

Assumption 1. *For every player i the utility $u_i(, \mathbf{s}_{-i})$ is convex for every $\mathbf{s}_{-i}$ and the set $\mathcal{S}_i$ is convex.*

Assumption 2. *Utilities u_i are C^1 $\forall$ $i \in \mathcal{N}$, where C^1 stands for the first order continuity.*

Then the following result from [41] enables us to identify NEs by solving a VI.

Theorem 3. *If Assumptions 1 and 2 hold true, then a strategy $\mathbf{s}^*$ is an NE if and only if it is a solution to the variational inequality VI$(\mathbf{S}, \mathbf{F}(\mathbf{s}))$, where $\mathbf{F}(\mathbf{s})$ has been defined in* (5.8) *and*

$$\mathbf{S} = \prod_{i=1}^{N} \mathcal{S}_i \tag{5.9}$$

A similar result can be stated for GNEs with an additional requirement. Consider the following definition from [41]

Definition 1. *A GNEP is said to be jointly convex if it satisfies Assumption 1 and if for some closed convex $\mathbf{S} \subseteq \mathbb{R}^n$ and $\forall$ $i \in \mathcal{N}$, there exists*

$$\mathcal{S}_i(\mathbf{s}_{-i}) = \{s_i \in \mathbb{R}^{n_i} : (s_i, \mathbf{s}_{-i}) \in \mathbf{S}\} \tag{5.10}$$

Where, n_i is dimension of the strategy space $\mathcal{S}_i$ and $n = \sum_{i=1}^{N} n_i$.

The following result also from [41] identifies a GNE by solving a VI.

Theorem 4. *For a jointly convex GNEP satisfying Assumption 2, every solution to variational inequality VI$(\mathbf{S}, \mathbf{F}(\mathbf{s}))$ is also a solution to the GNEP, where $\mathbf{S}$ has been defined in Definition 1.*

Thus the problem of identifying NEs and GNEs has been transferred to the problem of solving variational inequality. It is important ot note that while Theorem 3 is two-way mapping stating that every NE solves the VI and vice versa, 4 only maps it from VI to GNE. This means that there might exists GNEs which may not be the solution to the VI as defined previously. From this point onwards the following two approaches can be used to solve the obtained VI.

1. Formulating a centralized optimization problem whose first order optimality condition coincides with (5.7).
2. Through KKT conditions as done in [23]

Either of the approach allows an aggregated analysis to otherwise coupled individual optimization problems.

5.4.2 Potential Functions

The concept of Potential Functions as introduced by Monderer and Shapley [42] is utilized to formulate a single optimization problem that solves for the pure strategy NE of the game. Furthermore, a game which posses such a function is known as a potential game. Similarly, a GNEP that possess a corresponding function is referred to as a generalized potential game [43]. Further, this work will be restricted to a specialized class of games known as the *exact* potential game which has been defined next :

Definition 2. *Let* $\mathbf{S}$ *be a set defined by the Cartesian product of the strategy spaces for an NEP as in* (5.9). *Then the NEP is said to be an exact potenial game if there is a function* $\mathbf{\Phi} : \mathbf{S} \to \mathbb{R}$ *such that* $\forall\ \mathbf{s}_{-i} \in \mathbf{S}_{-i}$ *and* $\forall\ s_i^{'}, s_i^{''} \in \mathcal{S}_i$*:*

$$\mathbf{\Phi}(s_i^{'}, \mathbf{s}_{-i}) - \mathbf{\Phi}(s_i^{''}, \mathbf{s}_{-i}) = u_i(s_i^{'}, \mathbf{s}_{-i}) - u_i(s_i^{''}, \mathbf{s}_{-i}) \tag{5.11}$$

That is a change in action for a player i changes its utility by the same amount as the potential function. The following definition from [43] extends this definition to GNEPs with an additional caveat that the strategy spaces amongst the players are dependent upon each other:

Definition 3. *A GNEP is an exact generalized potential game if:*

a. *There exists a nonempty closed set* $\mathbf{S} \subseteq \mathbb{R}^n$ *such that* $\forall\ i \in \mathcal{N}$

$$\mathcal{S}_i(\mathbf{s}_{-i}) \equiv \{s_i \in \mathcal{D}_i : (s_i, \mathbf{s}_{-i}) \in \mathbf{S}\} \tag{5.12}$$

where n *is as defined in Definition 1 and* $\mathcal{D}_i \subseteq \mathbb{R}^{n_i}$ *are nonempty closed sets such that* $\prod_{i=1}^{N} \mathcal{D}_i \cap \mathbf{S} \neq \emptyset$.

b. *There is a function* $\mathbf{\Phi} : \mathbf{S} \to \mathbb{R}$ *such that* $\forall\ \mathbf{s}_{-i} \in \mathbf{S}_{-i}$ *and* $\forall\ s_i^{'}, s_i^{''} \in \mathcal{S}_i(\ \mathbf{s}_{-i})$*:*

$$\mathbf{\Phi}(s_i^{'}, \mathbf{s}_{-i}) - \mathbf{\Phi}(s_i^{''}, \mathbf{s}_{-i}) = u_i(s_i^{'}, \mathbf{s}_{-i}) - u_i(s_i^{''}, \mathbf{s}_{-i}) \tag{5.13}$$

Then the following theorem yields a method to arrive at a NE/GNE

Theorem 5. *With the above definitions:*

1. *The global maximizer of the following optimization problem is an NEP*

$$\max_{\mathbf{s}\in\mathbf{S}} \ \mathbf{\Phi}(\mathbf{s}) \tag{5.14}$$

2. *Similarity the global maximizer of the optimization problem below is a GNEP*

$$\max_{\mathbf{s}\in\mathbf{S}} \ \mathbf{\Phi}(\mathbf{s}) \tag{5.15}$$

$$s.\ to\ \mathbf{s_i} \in \mathcal{D}_\mathbf{i}, \forall\ i \in \mathcal{N} \tag{5.16}$$

Therefore, single optimization problems that solve for NEP/GNEP are obtained. Additionally, if the corresponding functions $\mathbf{\Phi}$ are also concave and the constraints in (5.16) are convex, the optimization problem is concave and therefore there exists a unique minimizer which is also the global minimizer of the problem.

5.5 Concluding Remarks

This chapter presented some fundamental concepts from game theory which enables the modelling of competition. Solution concepts in the form of Nash equilibrium were also presented and it was seen that solving for them is a non-trivial task involving multiple coupled optimization problems. To this end two approaches - variational inequalities and Potential functions, were introduced to formulate an aggregated optimization problem that solves for these equilibrium points. The latter will be employed in the next chapter in a power system setup.

Chapter 6

Optimal Real-Time Coordination of Energy Storage Units as a Feeder-Constrained Game

6.1 Problem Setup

Consider a power distribution system serving $n = 1, \ldots, N$ electricity users. The system operation has been discretized into periods indexed by t. Let ℓ_n^t and q_n^t denote respectively the active and reactive load for user n during period t. For a compact representation, the loads at period t can be collected into the N-dimensional vector $\boldsymbol{\ell}^t$ and $\mathbf{q}^t$. User loads are assumed inelastic and bounded within the given intervals.

$$\underline{\boldsymbol{\ell}} \leq \boldsymbol{\ell}^t \leq \overline{\boldsymbol{\ell}} \tag{6.1a}$$

$$\underline{\mathbf{q}} \leq \mathbf{q}^t \leq \overline{\mathbf{q}}. \tag{6.1b}$$

Every user owns an energy storage unit. Regarding the storage unit of user n, its state of charge (SoC) at the beginning of slot t is denoted by s_n^t, while b_n^t is the charging amount over period t. The latter is positive when battery n is being charged over period t, and negative otherwise. For simplicity, energy storage units are considered ideal and operated at unit power factor. The latter assumption is practical in distribution grids where currently customers are charged only for active power. Upon stacking $\{b_n^t, s_n^t\}_{n=1}^N$ accordingly in vectors $(\mathbf{b}^t, \mathbf{s}^t)$, the battery dynamics are described as

$$\mathbf{s}^{t+1} = \mathbf{s}^t + \mathbf{b}^t \tag{6.2a}$$
$$\underline{\mathbf{s}} \leq \mathbf{s}^t \leq \overline{\mathbf{s}} \tag{6.2b}$$
$$\underline{\mathbf{b}} \leq \mathbf{b}^t \leq \overline{\mathbf{b}} \tag{6.2c}$$

where (6.2b) preserves the SoCs within $[\underline{\mathbf{s}}, \overline{\mathbf{s}}]$, and (6.2c) imposes limits on the charging amount. Customers can reduce their consumption costs by altering their net active power demand p_n^t using batteries as

$$\mathbf{p}^t = \boldsymbol{\ell}^t + \mathbf{b}^t \tag{6.3}$$

where vector $\mathbf{p}^t$ collects $\{p_n^t\}_{n=1}^N$.

The underlying power distribution grid is modelled as a radial single-phase system represented by the graph $\mathcal{G} = (\{0, \mathcal{N}\}, \mathcal{E})$. The substation bus is indexed by 0 and the remaining buses correspond to the node set $\mathcal{N} := \{0, 1, \ldots, N\}$. Each bus is assumed to host an energy storage unit. The edge set $\mathcal{E}$ models distribution lines. Adopting the linear distribution flow model [44], the vector of squared voltage magnitudes in all but the substation buses can be approximated as [45]

$$\mathbf{v}^t = -\mathbf{R}\mathbf{p}^t - \mathbf{X}\mathbf{q}^t + v_0\mathbf{1}_N$$

where v_0 is the squared voltage magnitude at the substation, $\mathbf{1}_N$ is the all-ones vector of size N, and matrices $(\mathbf{R}, \mathbf{X})$ depend on distribution line parameters. Because the entries of $\mathbf{R}$ and $\mathbf{X}$ are non-negative for overhead lines, the above model implies that voltage magnitudes decrease with increasing $(\mathbf{p}^t, \mathbf{q}^t)$. This implication is critical for the analysis presented in Section 6.4 as it enables solution characterization and feasibility claims. Grid regulation standards confine nodal voltage magnitudes to lie close to the substation voltage v_0 [5, 46]. If α is the maximum allowable voltage deviation, nodal power demands are constrained within

$$-\alpha\mathbf{1}_N \leq \mathbf{R}\mathbf{p}^t + \mathbf{X}\mathbf{q}^t \leq \alpha\mathbf{1}_N \tag{6.4}$$

thus coupling charging decisions across customers.

The cost of electricity is varying across time and consists of two components: (i) an energy charge related to real-time energy prices; and (ii) a balancing charge compensating users for participating in frequency regulation. In detail, the cost of electricity for user n at time t is

$$f_n^t(b_n^t, \mathbf{b}_{-n}^t) = \frac{1}{N} c_p^t \left(\sum_{i=1}^{N} p_i^t \right) p_n^t + c_0^t p_n^t - r^t c_r^t b_n^t \tag{6.5}$$

where the notation $\mathbf{b}_{-n}^t$ signifies a vector containing the charging decisions for all but the n-th user. The energy charge is given by the first two summands in (6.5) where c_p^t models its competieqtive component and c_0^t represents the base cost. Given that energy charges increase with increasing total demand, a competitive environment is introduced.

The last summand in (6.5) is the balancing charge defined as the product between the regulation signal r^t, the related price c_r^t and the battery charge b_n^t. The regulation command r^t is issued by the operator and takes the values ± 1. When there is energy surplus, $r^t = +1$ and energy storage owners can only charge. At times of energy deficit, $r^t = -1$ and batteries can only discharge. In other words, the signal r^t determines the sign of charging decisions for all n

$$\text{sign}(b_n^t) = \text{sign}(r^t). \tag{6.6}$$

Due to (6.6), the regulation benefit $r^t c_r^t b_n^t = c_r^t |b_n^t|$ is always positive, and it can therefore reduce the total electricity cost for user n in (6.5). Although purchase and regulation prices can vary in an arbitrary manner, they are assumed bounded within $0 \leq \underline{c}_p \leq c_p^t \leq \overline{c}_p$, $0 \leq \underline{c}_0 \leq c_0^t \leq \overline{c}_0$ and $0 \leq \underline{c}_t \leq c_r^t \leq \overline{c}_r$ at all times.

6.2 A Game-Theoretic Perspective

Minimizing the electricity costs for all users constitutes a network-constrained non-cooperative game. User n intends to minimize its time-averaged expected electricity cost

$$C_n(\{b_n^t, \mathbf{b}_{-n}^t\}) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[f_n^t(b_n^t, \mathbf{b}_{-n}^t)]. \tag{6.7}$$

where $\mathbb{E}$ is the expectation operator over the involved random variables $\{r^t, c_p^t, c_r^t, \boldsymbol{\ell}^t, \mathbf{q}^t\}_{t=1}^T$. Then, user n would like to solve the infinite-horizon minimization

$$\begin{aligned}
\min_{\{b_n^t, s_n^t\}} \quad & C_n(\{b_n^t, \mathbf{b}_{-n}^t\}) && \text{(6.8a)}\\
\text{s.to} \quad & p_n^t = \ell_n^t + b_n^t && \text{(6.8b)}\\
& s_n^{t+1} = s_n^t + b_n^t && \text{(6.8c)}\\
& \underline{s}_n \leq s_n^t \leq \overline{s}_n && \text{(6.8d)}\\
& \underline{b}_n \leq b_n^t \leq \overline{b}_n && \text{(6.8e)}\\
& (6.6), (6.4). && \text{(6.8f)}
\end{aligned}$$

Since the instantaneous per-user cost f_n^t depends on the total demand as per (6.5), the average per-user costs C_n are coupled across users. The optimal charging decisions in (6.8) are further coupled through the voltage regulation constraints in (6.4), thus yielding rise to *generalized Nash equilibrium problem* [41].

The coupling across user charging decisions implies a game, which in its strategic form is described by

- the set of N users $\mathcal{N}$ connected to the grid;
- the user costs C_n's defined in (6.7); and
- the space of feasible strategies $\mathcal{B}$, i.e., the strategies $\mathbf{b}^t$ satisfying (6.4).

Let us also introduce the space $\mathcal{B}_n$ of feasible strategies particularly for user n as $\mathcal{B}_n(\{\mathbf{b}_{-n}^t\}) := \{\{b_n^t\} : \{b_n^t, \mathbf{b}_{-n}^t\} \in \mathcal{B}\}$.

A sequence of charging decisions $\{\tilde{\mathbf{b}}^t\}$ is termed a *generalized Nash equilibrium* (GNE) if it solves simultaneously the N coupled minimizations in (6.8). In other words, a GNE is a feasible strategy minimizing the per-user cost as long as the remaining users maintain their strategies fixed, that is

$$C_n(\{\tilde{b}_n^t, \tilde{\mathbf{b}}_{-n}^t\}) \leq C_n(\{\breve{b}_n^t, \tilde{\mathbf{b}}_{-n}^t\}) \tag{6.9}$$

for all $\breve{b}_n^t \in \mathcal{B}_n(\{\tilde{\mathbf{b}}_{-n}^t\})$. A GNE may not necessarily exist; and even if it exists, finding it is not always a tractable computational task. Fortunately, the problem structure of (6.8) features a GNE that is also computationally tractable as elucidated next.

To this end, let us define the time-averaged aggregate cost.

$$F(\{\mathbf{b}^t\}) := \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[f^t(\mathbf{b}^t)] \tag{6.10}$$

and the instantaneous aggregate cost $f^t(\mathbf{b}^t)$ is

$$\begin{aligned} f^t(\mathbf{b}^t) &:= \frac{c_p^t}{2N}\left(\sum_{n=1}^{N} p_n^t\right)^2 + \frac{c_p^t}{2N}\sum_{n=1}^{N}\left(p_n^t\right)^2 + c_0^t \sum_{n=1}^{N} p_n^t - r^t c_r^t \sum_{n=1}^{N} b_n^t && (6.11)\\ &= \frac{c_p^t}{2N}(\mathbf{b}^t + \boldsymbol{\ell}^t)^\top \left(\mathbf{I} + \mathbf{1}\mathbf{1}^\top\right)(\mathbf{b}^t + \boldsymbol{\ell}^t) + c_0^t \mathbf{1}^\top(\mathbf{b}^t + \boldsymbol{\ell}^t) - r^t c_r^t \mathbf{1}^\top \mathbf{b}^t. && (6.12)\end{aligned}$$

Consequently, consider the time-averaged aggregate minimization problem:

$$\begin{aligned} \tilde{\phi} := \min_{\{\mathbf{b}^t\}\in\mathcal{B}} \quad & F(\{\mathbf{b}^t\}) && (6.13a)\\ \text{s.to} \quad & (6.2), (6.3), (6.6), (6.4). && (6.13b)\end{aligned}$$

The next result establishes the link between the aggregate problem of (6.13) and the original problem of (6.8).

Proposition 4. *The minimizer* $\{\tilde{\mathbf{b}}^t\}$ *of* (6.13) *is a GNE for* (6.8).

Proof. Note that the constraints in (6.13) are linear, while function $f^t(\mathbf{b}^t)$ is quadratic in terms of $\mathbf{b}^t$ with a positive definite Hessian matrix $\mathbf{I}+\mathbf{1}\mathbf{1}^\top$. Then, the function $f^t(\mathbf{b}^t)$ is strictly convex and so is $F(\{\mathbf{b}^t\})$ over the charging sequence $\{\mathbf{b}^t\}$. Therefore, problem (6.13) enjoys a unique minimizer and so it follows that:

$$F(\{\tilde{b}_n^t, \tilde{\mathbf{b}}_{-n}^t\}) < F(\{\breve{b}_n^t, \tilde{\mathbf{b}}_{-n}^t\}) \tag{6.14}$$

for all $\breve{b}_n^t \in \mathcal{B}_n(\{\tilde{\mathbf{b}}_{-n}^t\})$ with $\breve{b}_n^t \neq \tilde{b}_n^t$ and $n \in \mathcal{N}$.

It can be readily shown that:

$$F(\{\tilde{\mathbf{b}}^t\}) - F(\{\breve{b}_n^t, \tilde{\mathbf{b}}_{-n}^t\}) = C_n(\{\tilde{\mathbf{b}}^t\}) - C_n(\{\breve{b}_n^t, \tilde{\mathbf{b}}_{-n}^t\}) \tag{6.15}$$

for all $n \in \mathcal{N}$. Note that $\{\breve{b}_n^t\}$ is feasible for (6.8) and spans $\mathcal{B}_n(\{\mathbf{b}_{-n}^t\})$ entirely. Combining (6.14) with (6.15) therefore provides (6.9) with strict inequality, thus proving the claim. □

Remark 1. Readers acquainted with game theory may note that (6.15) establishes $F(\{\mathbf{b}^t\})$ as the exact potential function for (6.8). This in return classifies (6.8) as a Generalized Potential Game with the property that the minimizer(s) of its potential function is(are) its GNE(s) [43].

Remark 2. In the less generalized case where per-user instantaneous cost f_n^t is independent of other users, $F(\{\mathbf{b}^t\})$ can be formulated as the sum of average costs over all users, i.e.:

$$F(\{\mathbf{b}^t\}) = \sum_{n=1}^{\mathcal{N}} C_n(\{b_n^t\})$$

In such a scenario, (6.13) will produce a GNE which is the *social welfare* solution for the game in (6.8) as it minimizes the net average costs over all users.

Proposition 4 asserts that identifying a GNE amounts to solving the aggregate convex problem in (6.13). Solving (6.13) however is technically challenging: Decisions are coupled over the infinite time horizon via (6.2a)–(6.2b) as well as spatially across the power system through (6.4). Moreover, coping with the expected electricity cost requires knowing the joint probability density function of the random processes $\{c_p^t, c_0^t, c_r^t, \ell^t, \mathbf{q}^t\}^{\mathcal{T}}$. Similar problems are traditionally tackled through approximate dynamic programming schemes, which are computationally intense [35]. Alternatively, stochastic techniques building on Lyapunov optimization [**?**] and dual decomposition updates can find near-optimal minimizers in real-time as delineated next.

6.3 A Real-Time Solver

6.3.1 Relaxed Problem

To develop a real-time algorithm for solving (6.13), consider the minimization problem:

$$\begin{aligned}
\phi' := \min_{\{\mathbf{b}^t\}\in\mathcal{B}} \quad & F(\{\mathbf{b}^t\}) && (6.16a)\\
\text{s.to} \quad & (6.2c), (6.3), (6.6), (6.4) && (6.16b)\\
& \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}\big[b_n^t\big] = 0,\ \forall n \in \mathcal{N}. && (6.16c)
\end{aligned}$$

The latter has been derived from (6.13) by simply replacing constraints (6.2a)–(6.2b) by the constraint (6.16c) on the expected time-averaged battery charging.

Problem (6.16) is actually a relaxation of (6.13). To see that, note that the battery state dynamics in (6.2a) yield that $s_n^T = s_n^0 + \sum_{t=0}^{T-1} b_n^t$. Applying the statistical expectation operator, dividing by T, and taking the limit of T to infinity provides:

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}\big[s_n^T - s_n^0\big] = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}\big[b_n^t\big] \tag{6.17}$$

for all $n \in \mathcal{N}$. Because s_n^T and s_n^0 are finite by (6.2b), the right-hand side (RHS) of (6.17) goes to zero, thus implying (6.16c) [17]. Since constraints (6.2a)–(6.2b) ensure (6.16c) is satisfied, problem (6.16) is a relaxation of (6.13) and hence $\phi' \leq \phi^*$.

We subsequently leverage Lyapunov optimization [] and devise a real-time approximate solver for the relaxed problem (6.16). This solver outputs charging decisions $\{\hat{\mathbf{b}}^t\}$ attaining the objective value $\hat{\phi} = F(\{\hat{\mathbf{b}}^t\})$. However, the attained $\hat{\phi}$ will be provably ϵ-suboptimal for the relaxed problem in (6.16), that is $\phi' \leq \hat{\phi} \leq \phi' + \epsilon$ for some $\epsilon \geq 0$. If it is additionally ensured that the sequence $\{\hat{\mathbf{b}}^t\}$ is feasible for the non-relaxed problem (6.13) as well, then it follows that

$$\tilde{\phi} \leq \hat{\phi} \leq \phi' + \epsilon \leq \tilde{\phi} + \epsilon. \tag{6.18}$$

In other words, the real-time solver achieves bounded sub-optimality with respect to the centralized formulation in (6.13). Despite $\tilde{\phi}$ in (6.18) corresponds to that of the centralized problem in (6.13) and not the individual costs in (6.7) or the GNEs, it will be shown later that (6.18) ensures that the GNE can be reached within a bounded average Euclidean distance.

6.3.2 Greedy Problem Formulation

The Lyapunov optimization approach entails formulating virtual queues and stabilizing them to ensure (6.16c) is satisfied [12]. For each user $n \in \mathcal{N}$, introduce a parameter γ_n and define the virtual queue:

$$x_n^t := s_n^t + \gamma_n \tag{6.19}$$

that is a shifted version of s_n^t and thus evolves similar to (6.2a):

$$x_n^{t+1} = x_n^t + b_n^t. \tag{6.20}$$

Define also the *weighted* Lyapunov function as

$$L^t := \frac{1}{2}\sum_{n=1}^{N} w_n (x_n^t)^2 \tag{6.21}$$

where $\{w_n\}_{n\in\mathcal{N}}$ are positive weights introduced to handle the heterogeneous capacities and charging rates across energy storage units. The following lemma then upper bounds the expected difference of weighted Lyapunov function given the values of virtual queues.

Lemma 3. *The drift function* $\Delta^t := \mathbb{E}\left[L^{t+1} - L^t | \mathbf{x}^t\right]$ *is upper bounded by:*

$$\Delta^t \leq \mathbb{E}\left[\sum_{n=1}^{N} w_n x_n^t b_n^t | \mathbf{x}^t\right] + \frac{1}{2}\sum_{n=1}^{N} w_n \max\{\bar{b}_n^2, \underline{b}_n^2\}. \tag{6.22}$$

Proof. Plugging (6.20) into the definition of the drift function provides:

$$\begin{aligned}
\Delta^t &= \mathbb{E}\left[\frac{1}{2}\sum_{n=1}^{N} w_n \left(\left(x_n^{t+1}\right)^2 - \left(x_n^t\right)^2\right) | \mathbf{x}^t\right] \\
&= \mathbb{E}\left[\frac{1}{2}\sum_{n=1}^{N} w_n \left(2x_n^t b_n^t + \left(b_n^t\right)^2\right) | \mathbf{x}^t\right]
\end{aligned}$$

The claim follows since constraint (6.2c) guarantees $\left(b_n^t\right)^2 \leq \max\{\bar{b}_n^2, \underline{b}_n^2\}$ for all n. $\square$

Adopting the toolbox of Lyapunov optimization, a real-time solver can be now devised [12]. The instantaneous charging decisions $\hat{\mathbf{b}}^t$ are found by minimizing the current cost $f^t(\mathbf{b}^t)$ scaled by a $\mu > 0$ plus the upper bound on the drift term

$$\begin{aligned}
\hat{\mathbf{b}}^t := \arg\min_{\mathbf{b}^t} \quad & (\mathbf{x}^t)^\top \operatorname{diag}(\mathbf{w})\mathbf{b}^t + \mu f^t(\mathbf{b}^t) \\
\text{s.to} \quad & (6.2c), (6.3), (6.6), (6.4)
\end{aligned} \tag{6.23}$$

where $\mathbf{x}^t$ is the vector collecting the virtual queues $\{x_n^t\}_{n\in\mathcal{N}}$ and $\text{diag}(\mathbf{w})$ is a diagonal matrix having the weights $\{w_n\}_{n\in\mathcal{N}}$ on its main diagonal. Although the average constraint (6.16c) does not appear in (6.23), it is implicitly enforced upon convergence [12]. It is worth stressing that (6.23) depends solely on the current realization of $\mathbf{x}^t$ and $\{c_p^t, c_0^t, c_r^t, \ell_n^t, q_n^t\}$ to find the charging decisions $\mathbf{b}^t$. Observe that the real-time solver of (6.23) depends on the design parameters $\{\gamma_n, w_n\}_{n\in\mathcal{N}}$ and μ that will are tuned in the next section to optimize the performance of the algorithm.

6.4 Feasibility and Performance Analysis

This section provides suboptimality bounds for the charging decisions $\hat{\mathbf{b}}^t$ and guarantees their feasibility for the aggregate problem in (6.13). Our results rely on the ensuing assumptions:

Assumption 3. *In the absence of energy storage, the (re)active loads* $(\boldsymbol{\ell}^t, \mathbf{q}^t)$ *do not violate voltage regulation limits, that is* $-\alpha\mathbf{1}_N \leq \mathbf{R}\boldsymbol{\ell}^t + \mathbf{X}\mathbf{q}^t \leq \alpha\mathbf{1}_N$.

Assumption 4. *For every energy storage unit* $n \in \mathcal{N}$*, its capacity and charge limits satisfy* $\overline{s}_n - \underline{s}_n > \overline{b}_n - \underline{b}_n$.

Assumption 3 entails that the distribution grid does not face any voltage regulation issues in the absence of energy storage. This can be met by providing reactive power support by installed solar generation on distributed grid. Assumption 4 is typically met under typical operating conditions. For the special case where $\underline{s}_n = 0$ and $\underline{b}_n = -\overline{b}_n$, the assumption simply requires that $\overline{b}_n < \overline{s}_n/2$.

The rest of this section has the following structure. First, the above assumptions are utilized to characterize the solution of (6.23) and claim its feasibility with respect to (6.13). Then, the suboptimality bounds are quantified both in terms of the optimal cost for (6.13) and the GNE of the game in (6.8). Finally, an optimal selection weights $\mathbf{w}$ is presented that minimizes these suboptimality bounds.

6.4.1 Solution Characterization and Feasibility

Proposition 5. *Under Assumption 3, the minimizer* $\hat{\mathbf{b}}^t$ *of* (6.23) *satisfies for all* $n \in \mathcal{N}$

(a) If $x_n^t + \frac{\mu}{w_n}\underline{g}_n \geq 0$ *and* $r^t > 0$*, then* $\hat{b}_n^t = 0$*;*

(b) If $x_n^t + \frac{\mu}{w_n}\overline{g}_n \leq 0$ *and* $r^t < 0$*, then* $\hat{b}_n^t = 0$*;*

for $\underline{g}_n := \frac{\underline{c}_p}{N}\underline{\boldsymbol{\ell}}^\top \mathbf{1} + \frac{\underline{c}_p}{N}\underline{\ell}_n + \underline{c}_0 - \overline{c}_r$ *and* $\overline{g}_n := \frac{\overline{c}_p}{N}\overline{\boldsymbol{\ell}}^\top \mathbf{1} + \frac{\overline{c}_p}{N}\overline{\ell}_n + \overline{c}_0 + \overline{c}_r$*.*

Proof. Proving by contradiction, assume that hypothesis (a) holds for user n yet $\hat{b}_n^t > 0$; the case $\hat{b}_n^t < 0$ is excluded because $r^t > 0$ has been assumed. Construct vector $\check{\mathbf{b}}^t$ with $\check{b}_n^t = 0$ and $\check{b}_i^t = \hat{b}_i^t$ for all $i \neq n$. Under Assumption 3 and because matrix $\mathbf{R}$ has non-negative entries, if $\hat{\mathbf{b}}^t$ is feasible for (6.23), then $\check{\mathbf{b}}^t$ is feasible too.

Next, it will be shown that $\check{\mathbf{b}}^t$ attains an objective value for (6.23) smaller than or equal to the one attained by $\hat{\mathbf{b}}^t$, that is

$$(\mathbf{x}^t)^\top \operatorname{diag}(\mathbf{w})\hat{\mathbf{b}}^t + \mu f^t(\hat{\mathbf{b}}^t) \geq (\mathbf{x}^t)^\top \operatorname{diag}(\mathbf{w})\check{\mathbf{b}}^t + \mu f^t(\check{\mathbf{b}}^t). \tag{6.24}$$

To that end, the difference of the upper bounds is $(\mathbf{x}^t)^\top \operatorname{diag}(\mathbf{w})(\hat{\mathbf{b}}^t - \check{\mathbf{b}}^t) = \hat{b}_n^t w_n x_n^t$, while the difference of the instantaneous costs can be readily shown to be

$$f^t(\hat{\mathbf{b}}^t) - f^t(\check{\mathbf{b}}^t) = \hat{b}_n^t \left[\frac{c_p^t}{N}\sum_{i=1}^N (\hat{b}_i^t + \ell_i^t) + \frac{c_p^t}{N}\ell_n^t + c_0^t - c_r^t\right]. \tag{6.25}$$

From the two differences and granted $\hat{b}_n^t > 0$, the inequality in (6.24) holds only if

$$\frac{w_n}{\mu}x_n^t + \frac{c_p^t}{N}\sum_{i=1}^N (\hat{b}_i^t + \ell_i^t) + \frac{c_p^t}{N}\ell_n^t + c_0^t - c_r^t \geq 0. \tag{6.26}$$

Observe that $\hat{\mathbf{b}}^t \geq \mathbf{0}$ since $r^t > 0$; loads are lower bounded as $\boldsymbol{\ell}^t \geq \underline{\boldsymbol{\ell}}$ from (6.1a); while the prices c_p^t, c_0^t and c_r^t are also bounded. Then, the minimum value for $\frac{c_p^t}{N}\sum_{i=1}^N(\hat{b}_i^t + \ell_i^t) + \frac{c_p^t}{N}\ell_n^t + c_0^t - c_r^t$ is $\underline{g}_n$ by definition, and the hypothesis in (a) implies that (6.26) holds true. It has been shown that $\check{\mathbf{b}}^t$ yields the same or a lower cost than the unique minimizer $\hat{\mathbf{b}}^t$ of (6.23) does. The latter is a contradiction and proves the claim. Claim (b) can be shown in a similar fashion. $\square$

Heed that the charging decisions obtained from (6.23) are selected without considering the capacity limits on the corresponding states of charge, i.e.,

$\hat{\mathbf{s}}^t \in [\underline{\mathbf{s}}, \overline{\mathbf{s}}]$ with $\hat{\mathbf{s}}^{t+1} = \hat{\mathbf{s}}^t + \hat{\mathbf{b}}^t$. However, using the properties of $\hat{\mathbf{b}}^t$ shown in Prop. 5, it is next shown that through proper parameter tuning, the decisions $\{\hat{\mathbf{b}}^t\}$ yield feasible states $\{\hat{\mathbf{s}}^t\}$ and are hence feasible for the non-relaxed problem (6.13) too.

Proposition 6. *Under Assumption 4, the minimizer* $\hat{\mathbf{b}}^t$ *of* (6.23) *is also feasible for* (6.13) *when the design parameters* $\{\gamma_n, w_n\}_{n\in\mathcal{N}}$ *and* μ *are selected as*

$$0 < \frac{\mu}{w_n} \leq \delta_n \tag{6.27a}$$

$$-\frac{\mu}{w_n}\underline{g}_n + \overline{b}_n - \overline{s}_n \leq \gamma_n \leq -\frac{\mu}{w_n}\overline{g}_n + \underline{b}_n - \underline{s}_n \tag{6.27b}$$

where $\delta_n := (\overline{s}_n - \underline{s}_n + \underline{b}_n - \overline{b}_n)/(\overline{g}_n - \underline{g}_n) > 0$ *for all* $n \in \mathcal{N}$.

Proof. Proving by induction across time t, the base case holds true since $\hat{\mathbf{s}}^0 \in [\underline{\mathbf{s}}, \overline{\mathbf{s}}]$. Assuming $\hat{\mathbf{s}}^t \in [\underline{\mathbf{s}}, \overline{\mathbf{s}}]$, it will be ensured that $\hat{\mathbf{s}}^{t+1} \in [\underline{\mathbf{s}}, \overline{\mathbf{s}}]$ as well. The analysis is performed on a per-user basis and three cases are identified:

Case 1: $x_n^t + \frac{\mu}{w_n}\underline{g}_n \geq 0$. Depending on the regulation signal r^t, two subcases are considered. If $r^t > 0$, then Prop. 5 asserts that $\hat{b}_n^t = 0$ and $\hat{s}_n^{t+1} = \hat{s}_n^t$ so that the state remains feasible.

If $r^t < 0$, then $\hat{b}_n^t \in [\underline{b}_n, 0]$ and only the lower limit on $\hat{s}_n^{t+1}$ has to be ensured. To that end, substitute (6.19) in the assumption for Case 1 to get $\hat{s}_n^t + \gamma_n + \frac{\mu}{w_n}\underline{g}_n \geq 0$. Combining the last inequality with the state update $\hat{s}_n^{t+1} = \hat{s}_n^t + \hat{b}_n^t$ yields

$$\hat{s}_n^{t+1} \geq -\gamma_n - \frac{\mu}{w_n}\underline{g}_n + \hat{b}_n^t. \tag{6.28}$$

The lower limit on $\hat{s}_n^{t+1}$ will be satisfied if the minimum value of the RHS in (6.28) is larger or equal to $\underline{s}_n$. Since $\hat{b}_n^t \in [\underline{b}_n, 0]$, the latter is guaranteed if

$$\gamma_n \leq -\frac{\mu}{w_n}\underline{g}_n + \underline{b}_n - \underline{s}_n. \tag{6.29}$$

Case 2: $x_n^t + \frac{\mu}{w_n}\overline{g}_n \leq 0$. If $r^t < 0$, then Prop. 5 guarantees $\hat{b}_n^t = 0$ and the updated state $\hat{s}_n^{t+1} = \hat{s}_n^t$ remains feasible.

If $r^t > 0$, then $\hat{b}_n^t \in [0, \overline{b}_n]$ and only the upper limit on $\hat{s}_n^{t+1}$ needs to be ensured. From (6.19) and the assumption of Case 2, it follows that $\hat{s}_n^t + \gamma_n +$

$\frac{\mu}{w_n}\overline{g}_n \leq 0$. Combining the last inequality with the state update $\hat{s}_n^{t+1} = \hat{s}_n^t + \hat{b}_n^t$ yields:

$$\hat{s}_n^{t+1} \leq -\gamma_n - \frac{\mu}{w_n}\overline{g}_n + \hat{b}_n^t. \tag{6.30}$$

The upper limit on $\hat{s}_n^{t+1}$ will be satisfied if the maximum value of the RHS in (6.30) is smaller or equal to $\overline{s}_n$. Since $\hat{b}_n^t \in [0, \overline{b}_n]$, the latter is guaranteed if

$$\gamma_n \geq -\frac{\mu}{w_n}\overline{g}_n + \overline{b}_n - \overline{s}_n. \tag{6.31}$$

Case 3: $-\frac{\mu}{w_n}\overline{g}_n \leq x_n^t \leq -\frac{\mu}{w_n}\underline{g}_n$. If $r^t > 0$ then $\hat{b}_n^t \in [0, \overline{b}_n]$ and only the upper limit on $\hat{s}_n^{t+1}$ needs to be maintained. Substituting (6.19) in the second inequality of Case 3 provides $\hat{s}_n^t + \gamma_n \leq -\frac{\mu}{w_n}\underline{g}_n$. Plugging the state transition into the last inequality yields:

$$\hat{s}_n^{t+1} \leq -\gamma_n - \frac{\mu}{w_n}\underline{g}_n + \hat{b}_n^t \tag{6.32}$$

The upper limit on $\hat{s}_n^{t+1}$ is respected if the maximum value of the RHS in (6.32) is smaller or equal to $\overline{s}_n$. Since $\hat{b}_n^t \in [0, \overline{b}_n]$, the latter is guaranteed if

$$\gamma_n \geq -\frac{\mu}{w_n}\underline{g}_n + \overline{b}_n - \overline{s}_n. \tag{6.33}$$

Because $\overline{g}_n > \underline{g}_n$, the bound of (6.33) is tighter than the one in (6.31), and therefore (6.33) provides the lower bound on γ_n in (6.27b).

If $r^t < 0$, then $\hat{b}_n^t \in [\underline{b}_n, 0]$ and only the lower limit on $\hat{s}_n^{t+1}$ needs to be maintained. Substituting (6.19) in the first inequality of Case 3 provides $-\frac{\mu}{w_n}\overline{g}_n \leq \hat{s}_n^t + \gamma_n$. Plugging the state transition into the last inequality yields:

$$\hat{s}_n^{t+1} \geq -\gamma_n - \frac{\mu}{w_n}\overline{g}_n + \hat{b}_n^t. \tag{6.34}$$

Symmetrically to (6.32), the lower limit on $\hat{s}_n^{t+1}$ is respected if the minimum value of the RHS in (6.34) is larger or equal to $\underline{s}_n$. Since $\hat{b}_n^t \in [\underline{b}_n, 0]$, the latter is guaranteed if

$$\gamma_n \leq -\frac{\mu}{w_n}\overline{g}_n + \underline{b}_n - \underline{s}_n. \tag{6.35}$$

Since the bound of (6.35) is tighter than the one in (6.29), it is the former that determines the upper bound on γ_n in (6.27b).

Finally, it is not hard to see that the condition in (6.27a) guarantees that the bounds on γ_n in (6.27b) yield a non-empty interval for all $n \in \mathcal{N}$. $\square$

6.4.2 Sub-optimality Bound

Lemma 4 ([12]). *If $\{c_p^t, c_0^t, c_r^t, \boldsymbol{\ell}^t, \mathbf{q}^t\}$ are independent and identically distributed (iid) over time, there exists a stationary policy, that is a policy selecting $\grave{\mathbf{b}}^t$ based only on the current realizations of the random variables involved. Moreover, this policy satisfies* (6.2c), (6.3), (6.6), (6.4), $\mathbb{E}[\grave{\mathbf{b}}^t] = \mathbf{0}$, *and* $\mathbb{E}[f^t(\grave{\mathbf{b}}^t)] = \tilde{\phi}$.

Finally, if $\hat{\phi}$ is average cost over infinite time horizon achieved by plugging $\hat{\mathbf{b}}^t$ into the cost in (6.13), the following sub-optimality claim can be made

Proposition 7. *If $\{c_p^t, c_0^t, c_r^t, \boldsymbol{\ell}^t, \mathbf{q}^t\}$ are iid, it holds that*

$$\tilde{\phi} \leq \hat{\phi} \leq \tilde{\phi} + \sum_{n=1}^{N} \frac{w_n \max\{\bar{b}_n^2, \underline{b}_n^2\}}{2\mu}. \tag{6.36}$$

Proof. Consider the "drift plus penalty term" produced below and its upper bound as obtained from (6.22).

$$\begin{aligned} \Delta^t + \mu\mathbb{E}\left[h^t(\mathbf{b}^t)|\mathbf{x}^t\right] \leq \mathbb{E}&\left[\frac{1}{2}\sum_{n=1}^{N} w_n \left(2x_n^t b_n^t + (b_n^t)^2\right)|\mathbf{x}^t\right] \\ &+ \mu\mathbb{E}\left[h^t(\mathbf{b}^t)|\mathbf{x}_t\right] \end{aligned} \tag{6.37}$$

Note that $\hat{\mathbf{b}}^t$ which solves (6.23) essentially minimizes the RHS of (6.37). Hence, the value obtained for this RHS would be minimum over all feasible polices including $\grave{\mathbf{b}}^t$ in Lemma 4. It therefore follows that

$$\Delta^t + \mu\mathbb{E}\left[h^t(\hat{\mathbf{b}}^t)|x_n^t, \forall n \in \mathcal{N}\right] \leq \mu\phi^* + \sum_{n=1}^{N} w_n \frac{\max\{\bar{b}_n^2, \underline{b}_n^2\}}{2}$$

where the RHS was obtained by substituting $\mathbf{b}^t = \grave{\mathbf{b}}^t$ in the RHS of (6.37). Taking expectations on both sides of the previous inequality wrt $\mathbf{x}_t$ and applying the law of total expectation yields

$$\mathbb{E}\left[L^{t+1} - L^t\right] + \mu\mathbb{E}\left[h^t(\hat{\mathbf{b}}^t)\right] \leq \mu\phi^* + \sum_{n=1}^{N} w_n \frac{\max\{\bar{b}_n^2, \underline{b}_n^2\}}{2}$$

Summing the last inequality over time instants $t = [0, 1 \ldots T]$ gives

$$\mathbb{E}\left[L^T - L^0\right] + \mu\sum_{t=0}^{T-1}\mathbb{E}\left[h^t(\hat{\mathbf{b}}^t)\right] \leq \mu T\phi^* + T\sum_{n=1}^{N} w_n \frac{\max\{\bar{b}_n^2, \underline{b}_n^2\}}{2}$$

which upon being divided by μT on both the sides and taking the limit of T to infinity produces

$$\lim_{T\to\infty}\left(\frac{1}{\mu T}\mathbb{E}\left[L^T - L^0\right] + \frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\left[h^t(\hat{\mathbf{b}}^t)\right]\right) \leq \phi^* + \sum_{n=1}^{N}\frac{w_n}{\mu}\frac{\max\{\bar{b}_n^2, \underline{b}_n^2\}}{2}$$

Note that $\mathbb{E}[L^0]$ is finite and $\mathbb{E}[L^T] \geq 0$. This reduces the above inequality to

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\left[h^t(\hat{\mathbf{b}}^t)\right] \leq \phi^* + \sum_{n=1}^{N}\frac{w_n}{\mu}\frac{\max\{\bar{b}_n^2, \underline{b}_n^2\}}{2} \tag{6.38}$$

which proves the claim. □

The following proposition translates the sub-optimality gap in the average expected costs to a bounded ℓ_2-norm distance between the real-time solutions and the GNE.

Proposition 8. *Let* $\tilde{\mathbf{b}}$ *be the optimal solution to* (6.13) *and therefore the required GNE. Then,*

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}[\|\hat{\mathbf{b}}^t - \tilde{\mathbf{b}}^t\|_2^2] \leq \frac{2KN}{\underline{c}_p} \tag{6.39}$$

where $K = \sum_{n=1}^{N} \frac{w_n}{\mu} \frac{\max\{\bar{b}_n^2, \underline{b}_n^2\}}{2}$.

Proof. Consider the following Taylor series expansion of strictly convex $f^t(\mathbf{b}^t)$ around a point $\mathbf{b}_0^t$

$$f^t(\mathbf{b}^t) > f^t(\mathbf{b}_0^t) + \left(\nabla f^t(\mathbf{b}_0^t)\right)^\top (\mathbf{b}^t - \mathbf{b}_0^t) + \frac{\underline{\lambda}^t}{2}\|\mathbf{b}^t - \mathbf{b}_0^t\|_2^2 \tag{6.40}$$

where the inequality follows from Rayleigh quotient and $\underline{\lambda}^t$ is the smallest eigenvalue of $\nabla^2 f^t$. Substituting $\mathbf{b}^t = \hat{\mathbf{b}}^t$ and $\mathbf{b}_0^t = \tilde{\mathbf{b}}^t$, taking expectations on with respect to the randomness in $\{c_p^t, c_r^t, \boldsymbol{\ell}^t, \mathbf{q}^t\}$ and summing over all t with the limit of T to infinity yields

$$\begin{aligned}\lim_{T\to\infty}\sum_{t=0}^{T-1}\mathbb{E}\left[f^t(\hat{\mathbf{b}}^t)\right] > &\lim_{T\to\infty}\sum_{t=0}^{T-1}\mathbb{E}\left[f^t(\tilde{\mathbf{b}}^t)\right] \\ &+ \lim_{T\to\infty}\sum_{t=0}^{T-1}\mathbb{E}\left[\left(\nabla f^t(\tilde{\mathbf{b}}^t)\right)^\top (\hat{\mathbf{b}}^t - \tilde{\mathbf{b}}^t)\right] \\ &+ \frac{\underline{\lambda}}{2}\lim_{T\to\infty}\sum_{t=0}^{T-1}\mathbb{E}\left[\|\hat{\mathbf{b}}^t - \tilde{\mathbf{b}}^t\|_2^2\right]\end{aligned}$$

where $\underline{\lambda} = \min_t \underline{\lambda}^t$. Note that the first-order optimality conditions for the global minimizers $\tilde{\mathbf{b}}^t$ ensures

$$\lim_{T\to\infty}\sum_{t=0}^{T-1}\mathbb{E}\left[\left(\nabla f^t(\mathbf{b}^{*t})\right)^\top (\hat{\mathbf{b}}^t - \tilde{\mathbf{b}}^t)\right] \geq 0. \tag{6.41}$$

Using (6.41) and the definitions for $\tilde{\phi}$ and $\hat{\phi}$ in (6.40) provides

$$\lim_{T\to\infty}\sum_{t=0}^{T-1}\mathbb{E}\left[\|\hat{\mathbf{b}}^t - \tilde{\mathbf{b}}^t\|_2^2\right] \leq \frac{2}{\underline{\lambda}}(\hat{\phi} - \tilde{\phi}). \tag{6.42}$$

It is easy to see that $\nabla^2 f^t = \frac{1}{N} c_t^p (\mathbf{I} + \mathbf{1}\mathbf{1}^\top)$ and therefore it follows that $\underline{\lambda} = \frac{c_p}{N}$. Using this result and (6.36) in (6.42) proves the proposition. □

From Propositions 7 and 8 it is evident that the minimum value of the sub-optimality bounds is attained when $\frac{\mu}{w_n}$ achieve the maximum values while keeping the problem feasible. Next proposition finds optimal weights w_ns that achieve such bounds.

6.4.3 Selecting Optimal Weights

Proposition 9. *If the weights w_n in* (6.21) *are set such that $w_n\delta_n = \mu$ for all $n \in \mathcal{N}$, the minimizers of* (6.23) *achieve the smallest average sub-optimality bound both on cost and decisions given by*

$$\tilde{\phi} \le \hat{\phi} \le \tilde{\phi} + \sum_{n=1}^{N} \alpha_n \tag{6.43a}$$

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\hat{\mathbf{b}}^t - \tilde{\mathbf{b}}^t\|_2^2] \le \frac{2N\sum_{n=1}^{N}\alpha_n}{\underline{c}_p} \tag{6.43b}$$

where $\alpha_n = \max(\overline{b}_n^2, \underline{b}_n^2)/(2\delta_n)$ for all n.

Proof. Consider the following optimization problem

$$\min_{\beta_n} \sum_{n=1}^{N} \frac{\max\{\overline{b}_n^2, \underline{b}_n^2\}}{2\beta_n} \tag{6.44a}$$

s.to.

$$0 < \beta_n \le \delta_n, \quad \forall n \in \mathcal{N} \tag{6.44b}$$

where the constraint (6.44b) is derived from (6.27a). Note that (6.44) minimizes the sub-optimality bound in (6.36) with the auxiliary variable β_ns replacing μ/w_n. It can then directly be seen that $\beta_n = \delta_n$ minimizes (6.44) and required bound can be found by substituting this value of β_n, hence proving the claim. $\square$

Remark 3. The *non-weighted* Lypaunov optimization employed in [14]- [47] would result in the following sub-optimality bound for the problem at hand

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\hat{\mathbf{b}}^t - \tilde{\mathbf{b}}^t\|_2^2] \le \frac{2N\sum_{n=1}^{N}\hat{\alpha}_n}{\underline{c}_p}$$

where $\hat{\alpha_n} = \max(\overline{b}_n^2, \underline{b}_n^2)/(2\delta_{min})$ for all n and $\delta_{min} := \min_n\{\delta_n\}$. Therefore, the user with minimum δ_n will control the performance of the algorithm.

This makes the *weighted* method in Proposition 9 is superior as it decouples the sub-optimality bound amongst the users such that it depends upon their individual specifications.

6.5 Numerical Tests

6.5.1 Case 1

First, the performance is compared with a greedy algorithm that minimizes $f^t(\mathbf{b}^t)$ in (6.11) at every instant of time subject to (6.2)–(6.4). Intuitively, the issue with this approach is that its agnostic to future scenarios and therefore prone to saturate the energy storage to their maximum or minimum capacity prematurely. To demonstrate this, an IEEE 13-bus feeder is selected as in Figure 6.1 ,where loads and energy storage are connected to each of the non-feeder buses . Without the loss of generalization, $\underline{s}_n = 0$ and $\underline{b}_n = -\overline{b}_n$ are set $\forall\ n \in \mathcal{N}$. The values for $\overline{s}_n$ and $\overline{b}_n$ are presented in Table 6.1.

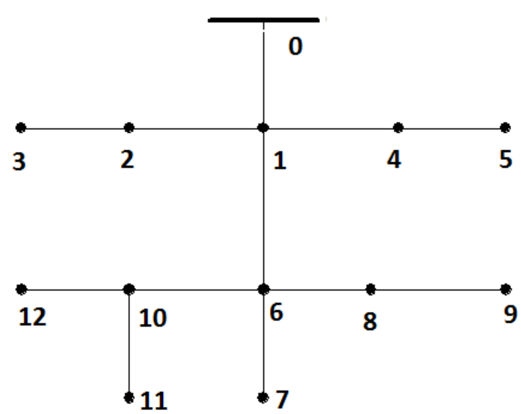
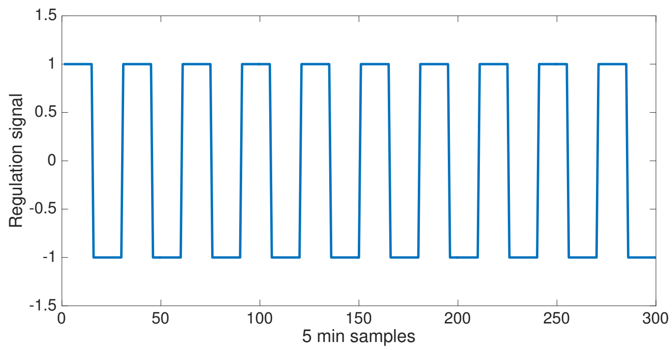


Figure 6.1: IEEE 13-bus feeder configuration.

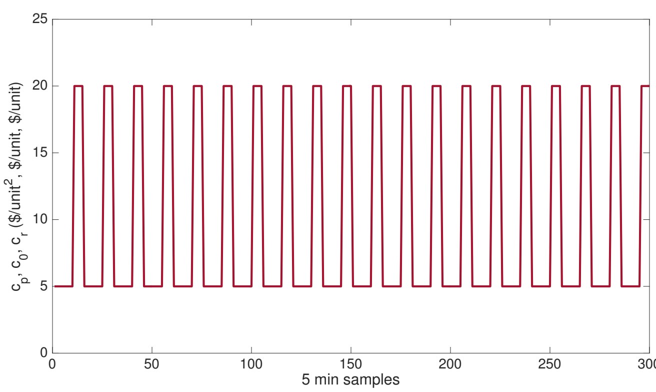
Bus no.	1	2	3	4	5	6	7	8	9	10	11	12
$\overline{s}_n$ (10^{-1})	9.95	2.57	2.62	2.50	3.60	0.16	2.65	2.78	2.09	2.04	3.71	2.00
$\overline{b}_n$ (10^{-2})	16.6	4.29	4.38	4.16	5.99	0.27	4.42	4.63	3.49	3.40	6.18	3.34

Table 6.1: Energy storage parameters for Case 1 (in p.u.).

The load data was obtained from the “Pecan Street” project [48], averaged over 5 minutes and scaled as per Assumption 3. The value for the maximum voltage deviation was set to $\alpha = 0.03$. Finally the regulation signal r^t and prices c_r^t, c_p^t and c_0^t were generated as in Figures 6.2a and 6.2b respectively. As it can be seen r^t was made to oscillate between $\{+1, -1\}$ after



(a)



(b)

Figure 6.2: Simulation setup for Case 1 (a) r^t and (b) c_r^t, c_0^t, c_r^t.

15 time samples alternatively . Similarly, c_p^t, c_0^t and c_r^t shifted between the values $\{5, 20\}$\$/unit with the former lasting for 12 samples and latter for 3. The results comparing the two algorithms have been presented in Figure 6.3 The proposed *weighted* Lyapunov algorithm clearly outperforms the greedy algorithm. This is because the greedy algorithm charges/discharges the energy storages myopically to their maximum capacities during the low price of 5\$unit rather than waiting to reap maximum rewards at 20\$/unit. The

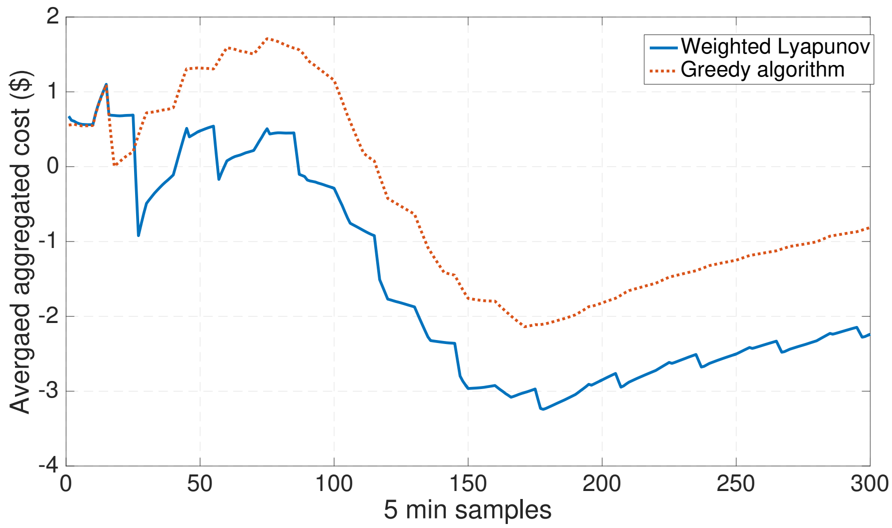


Figure 6.3: Comparison of average costs for Case 1.

weighted Lyapunov algorithm is considerate enough to leave some room for the latter scenario.

6.5.2 Case 2

Here, the algorithms are compared under a more realistic scenario where c_p^t and c_0^t are taken from the LMP rates for the "PJM-RTO" node and c_r^t are set to the ancillary market rates for the same node. Further, r^t signal is modelled as a random variable with a Gaussian distribution to simulate *Reg D* signal issued by PJM [33]. The source for load data remains the same as previous case, $\underline{s}_n = 0$ and $\underline{b}_n = -\overline{b}_n \; \forall \; n \in \mathcal{N}$. The values for $\overline{s}_n$ are also same as that in Table 6.1 however new values for $\overline{b}_n$ given in Table 6.2 are selected, Performance of three different algorithms is compared – *weighted* Lyapunov,

Bus no.	1	2	3	4	5	6	7	8	9	10	11	12
$\overline{b}_n$ (10^{-2})	0.99	0.26	0.26	0.25	0.36	0.02	0.27	0.28	0.21	0.20	0.37	0.20

Table 6.2: Energy storage parameters for Case 2 (in p.u.).

non-weighted Lyapunov with the best case performance as in (3) and the greedy algorithm. The results comparing the average cost are presented in

Figure 6.4. From the results the *weighted* Lyapunov performs better than

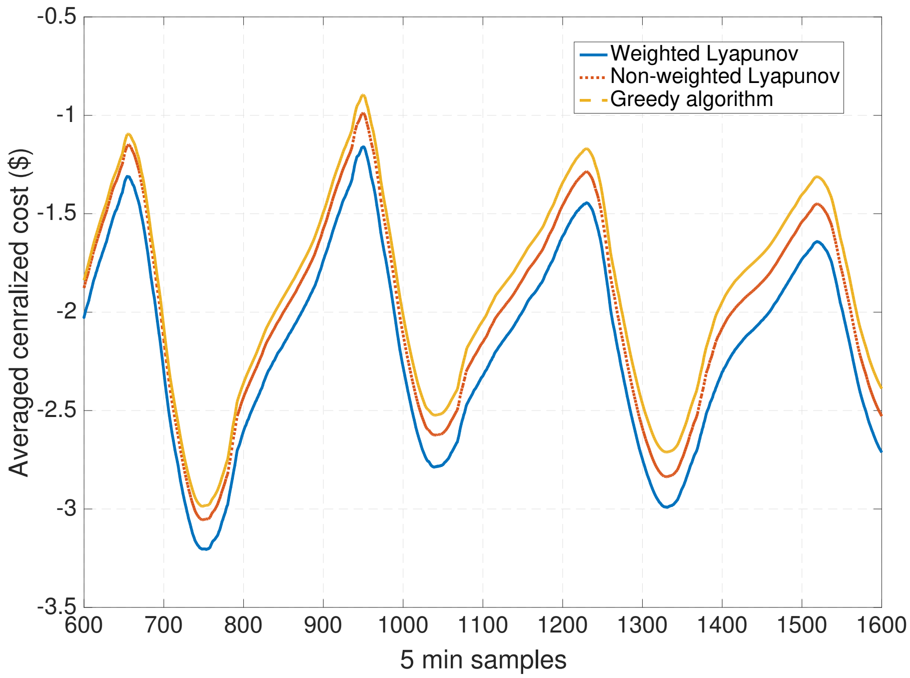


Figure 6.4: Comparison of average costs for Case 2.

both the other algorithms.

6.6 Concluding Remarks

This chapter modeled a game theoretic scenario where consumers compete for energy in a market while ensuring voltage regulation standards. To this end a generalized Nash equilibrium formulation was formulated which was then converted to an aggregated optimization problem via the potential function route. *Weighted* Lyapunov optimization was then proposed to solve this problem via a real time algorithm. Mathematical bound on the performance of this real time algorithm in a network constrained environment was calculated and shown to be superior than the *non-weighted* formulation. This was also confirmed through numerical tests on an actual dataset.

Chapter 7

Conclusion

The potential of energy storages in yielding economic benefits is established. This was achieved on account of participation in arbitrage and energy balancing where both the prices and loads were modelled as random variables with unknown probability distributions. Chapter 2 formulates the basic model for an energy storage and presents some power system applications in the form of optimization problems. Techniques of Lypaunov optimization and stochastic approximation were explored in Chapter 3. Chapter 5 comprised of concepts form game theory such as Nash equilibrium, variational inequalities and potential functions which help model competitive behaviours. The two highlights of this thesis are Chapters 4 and Chapter 6 which incorporate the novelties of author's research. A real-time algorithm based on two-scale lyapunov optimization is employed to control heterogeneous energy storage in Chapter 4. It is shown through mathematical derivation and numerical tests that the proposed method is economically beneficial as it enables arbitrage in both the time ahead and real time markets. Further, the performance is proven to be sub-optimally bounded. Chapter 6 studies a much more complicated scenario where multiple energy storages interact in a competitive pricing setup. Moreover, their power consumptions are coupled through nodal voltage regulation constraints. An aggregated optimization problem is developed after identifying the scenario as a generalized potential game which is solved through novel *weighted* Lyapunov optimization. Once again, a real-time algorithm is formulated that provides decisions which are within a bounded ℓ_2 norm distance from the Nash equilibrium of the game. The efficacy of the algorithm is proven through numerical results that compare it with the standard non-weighted Lyapunov approach and a greedy algorithm.

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