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RIESZ BASES AND POSITIVE OPERATORS ON HILBERT SPACE

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It is shown that a normalized Riesz basis for a Hilbert space H (i.e., the isomorphic image of an orthonormal basis in H) induces in a natural way a new, but equivalent, inner product on H in which it is an orthonormal basis, thereby extending the sense in which Riesz bases and orthonormal bases are thought of as being the *same*. A consequence of the method of proof of this result yields a series representation for all positive isomorphisms on a Hilbert space.

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1. Introduction. Let H denote a Hilbert space (assumed real, for notational convenience) with inner product $(\cdot,\cdot)$ and let $\{x_i\}$ be a basis for H having coefficient functionals $\{f_i\}$ denoted by $\{x_i,f_i\}$. We say that $\{x_i,f_i\}$ is a *Riesz basis* for H if it has the property that $\sum a_i x_i$ converges in H if and only if $\{a_i\}$ is in the sequence space l^2. Equivalently, $\{x_i,f_i\}$ is a Riesz basis for H if and only if there is an isomorphism U on H and some orthonormal basis $\{\phi_i\}$ for H so that $U\phi_i = x_i$ for all i, implying that Riesz bases and orthonormal bases are the "same" in linear-topological terms, but differ in geometrical ones due to the additional orthogonality relations between basis vectors in an orthonormal basis that is lacking in a Riesz basis. The result below (Theorem 2.1) shows that this is, in a sense, an artificial distinction by showing that every Riesz basis, in fact, is an orthonormal basis for H under a different, but equivalent, inner product.

2. Main results

Theorem 2.1. *Let $\{x_i,f_i\}$ be a normalized Riesz basis for a Hilbert space H. Then there is an equivalent inner product on H in which $\{x_i\}$ is an orthonormal basis for H under the norm induced by this inner product.*

Proof. If x and y are any two vectors in H, then the sequences $\{(f_i,x)\}$ and $\{(f_i,y)\}$ are in l^2, implying that $\sum(f_i,x)(f_i,y)$ converges. Clearly, the bilinear form on $H\times H$, defined by $\langle x,y\rangle = \sum(f_i,x)(f_i,y)$, is then an inner product on H for which $\langle x_i,x_j\rangle = d_{ij}$ for all i and j, in which $\{x_i\}$ is an orthonormal set that is also complete, since if $\langle x_n,x\rangle = 0$ for all n, then $0 = \sum(f_i,x_n)(f_i,x) = (f_n,x)$ for all n; that is, $0 = \sum(f_i,x_n)(f_i,x)$ by definition of the new inner product for all n, implying that $(f_n,x) = 0$ for all n, and hence that $x = 0$.

As usual, the inner product $\langle\cdot,\cdot\rangle$ defines a norm $\|\cdot\|_1$ on H by $\|x\|_1^2 = \langle x,x\rangle = \sum|(f_i,x)|^2$. Since $\{x_i\}$ is a Riesz basis, there is an isomorphism U on H that maps each vector ϕ_i in an orthonormal basis $\{\phi_i\}$ for H to the vector x_i, implying that the isomorphism $V=(U^*)^{-1}U^{-1}$ on H maps x_i to f_i for all i. Since, for any x in H, $\langle x,x\rangle = \sum(f_i,x)(f_i,x) = (\sum(f_i,x)(Vx_i,x)) = \sum(f_i,x)(Vx_i,x) = (V[\sum(f_i,x)x_i],x) = (Vx,x)$, we see that $(Vx,x) = \sum|(f_i,x)|^2 = \|x\|_1^2$ for all X in H, so V is a positive operator. If we let W denote the positive square root of V, then W is also an isomorphism on H so that, for any x in H, we have $\|x\|_1^2 = (Vx,x) = (Wx,Wx) = \|Wx\|^2 \le \|w\|^2\|x\|^2$. In the same way, we see that $\|x\|_1^2 \le \|W^{-1}\|^2\|x\|^2$, and it follows that the new norm $\|\cdot\|_1$ is equivalent to the original norm on H. In particular, H is then complete under the new norm, hence a Hilbert space, in which $\{x_i\}$ is then an orthonormal basis, being an orthonormal set, that is complete in the new inner product. □

3. Positive operators. In the proof above we used the fact that if $\{x_i,f_i\}$ is a Riesz basis for a Hilbert space H, then the operator U on H, mapping x_i to f_i, is a positive isomorphism on H. It is interesting to note that, in fact, *every* positive isomorphism on H is such an operator for some Riesz basis in H, thereby providing a representation for all positive isomorphisms U on a Hilbert space.

Theorem 3.1. *An operator U on a Hilbert space on H is a positive isomorphism if and only if U is of the form $U=\sum f_i\otimes f_i$ for some Riesz basis $\{x_i,f_i\}$ for H (i.e., $Ux_i = f_i$ for all i).*

Proof. If $U=\sum f_i\otimes f_i$ for some Riesz basis $\{x_i,f_i\}$ for H, $\{\phi_i\}$ is an orthonormal basis for H, and T is the isomorphism on H mapping ϕ_i to f_i for all i, then $U=\sum T\phi_i\otimes T\phi_i = TT^*$, a positive isomorphism on H.

Conversely, if U is any positive isomorphism on H, then W, the positive square root of U, is also an isomorphism on H. If we set $f_i = W\phi_i$ for some orthonormal basis $\{\phi_i\}$, then $\{f_i\}$ is a Riesz basis for H so that, for any x in H, we have $Ux = W^2x = W[\sum(\phi_i,Wx)\phi_i] = W[\sum(W\phi_i,x)\phi_i] = \sum(f_i,x)W\phi_i = \sum(f_i,x)f_i$. That is, $U=\sum_i f_i\otimes f_i$ and the proof is complete. □

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