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Symmetry in Domination for Hypergraphs with Choice

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Academic Editor: Angel Garrido

Received: 7 January 2017; Accepted: 14 March 2017; Published: 22 March 2017

Abstract: In this paper, we introduce the concept of (pair-wise) domination graphs for hypergraphs endowed with a choice function on edges. We are interested, for instance, in minimal numbers of edges for associated domination graphs. Theorems regarding the existence of balanced (zero-edge) domination graphs are presented. Several open questions are posed.

Keywords: hypergraph; choice; symmetry; hypertournament; domination

1. Introduction

In this paper, we introduce the concept of pair-wise domination for hypergraphs endowed with a choice function on edges. A hypergraph is a pair $\mathcal{H} = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ is a set of n vertices (or nodes) and $E = \{e_1, e_2, \ldots, e_m\}$ is a set of m non-empty subsets of V called hyperedges or edges (see for instance [1]). A k-hypertournament is a complete k-hypergraph $H = (V, E)$ (i.e., E consists of all the $\binom{n}{k}$ possible k-subsets), with each k-edge endowed with an orientation. Here we are interested in hypergraphs where each edge, e, has a chosen element $C(e) \in e$ (in place of a complete orientation). We will refer to the pair (H, C) as a (complete) k-hypergraph with choice (or an (n, k) choice-hypergraph). For various considerations of choice functions, see for instance [2,3]. For some recent work related to choice in the context of Cayley graphs, see [4].

In the case $k = 2$, both k-hypertournaments and hypergraphs with choice reduce to standard tournaments. For discussion of tournaments, see for instance [5–7]. The following is an example of a standard tournament with $n = 5$ vertices.

Example 1. *Consider the complete* 2*-hypertournament with five nodes and ten edges, depicted via the table in Figure 1a. Here, for instance, vertex 1 is chosen in the presence of vertices 2 and 4 (the first and third lines in the table). In fact, in this particular instance, each vertex is chosen for exactly two of the* $\binom{5}{2} = 10$ *edges. Figure 1b gives a graphical display with a directed edge from vertex* v *to vertex* w *whenever vertex* v *is chosen in the presence of* w.

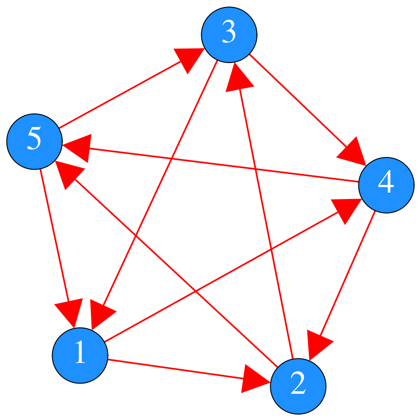
In the next example, we consider a complete 3-hypergraph with choice.

Example 2. *Consider the complete* 3*-hypergraph with choice with five vertices and 10 edges, depicted in the table in Figure 2a. Here, vertex 1 is chosen once in the presence of vertices 2 and 4 (for edges* $\{1, 2, 3\}$ *and* $\{1, 3, 4\}$*, respectively), twice in the presence of vertex 3 (again edges* $\{1, 2, 3\}$ *and* $\{1, 3, 4\}$*) and never in the presence of vertex 5. Note that again, as in Example 1, each node is chosen in the case of exactly two edges.*

In considering possible analogues to Figure 1b, summarizing domination, one might include a directed edge from vertex v *to vertex* w*, if and only if, for edges that include both* v *and* w*, the tally of wins for* v *exceeds that* *of w. The resulting graph is depicted in Figure 2b. Note that there is a directed edge from vertex 1 to vertex 2, since vertex 1 is chosen for edge $\{1,2,3\}$, while there is no edge, e, where vertex 2 is chosen when $1 \in e$. It may be noted that in this case the associated graph is path connected.*

	e	orientation	C(e)
1	{1, 2}	(1,2)	1
2	{1, 3}	(3,1)	3
3	{1, 4}	(1,4)	1
4	{1, 5}	(5,1)	5
5	{2, 3}	(2,3)	2
6	{2, 4}	(4,2)	4
7	{2, 5}	(2,5)	2
8	{3, 4}	(3,4)	3
9	{3, 5}	(5,3)	5
10	{4, 5}	(4,5)	4

(**a**)

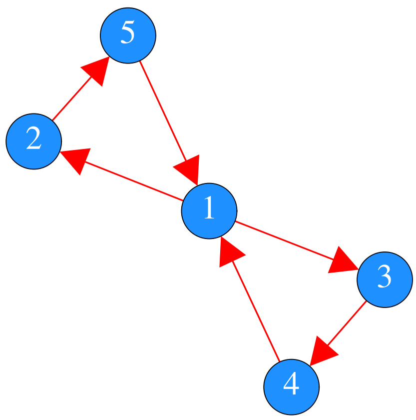


(**b**)

Figure 1. (**a**) An example of a standard tournament with five vertices; and (**b**) an associated graphical display with a directed edge from vertex v to vertex w whenever vertex v is chosen in the presence of w.

	e	$C_0(e)$
1	{1, 2, 3}	1
2	{1, 2, 4}	4
3	{1, 2, 5}	5
4	{1, 3, 4}	1
5	{1, 3, 5}	3
6	{1, 4, 5}	4
7	{2, 3, 4}	3
8	{2, 3, 5}	2
9	{2, 4, 5}	2
10	{3, 4, 5}	5

(**a**)



(**b**)

Figure 2. (**a**) An example of a possible (5,3) choice-hypergraph; (**b**) with a possible graph summarizing pair-wise domination.

Now, define the function $\tau : V \times V \to \mathbb{Z}^+$ via:

$$\tau(v,w) = |\{e \in E : v,w \in e \text{ and } C(e) = v\}|, \tag{1}$$

i.e., $\tau(v,w)$ is the number of edges for which v is chosen in the presence of w.

Example 2 leads to consideration of potential appropriate graphs on n vertices reflecting domination properties among vertices. Here, we mention three possibilities.

(i) There is a directed edge from vertex v to vertex w if:

$$\tau(w,v) = 0, \tag{2}$$

i.e., if vertex w is never chosen in the presence of vertex v.

(ii) There is a directed edge from vertex v to vertex w if:

$$\tau(v,w) > \tau(w,v), \tag{3}$$

i.e., among the edges containing both v and w, v is chosen with greater frequency.

(iii) There is a directed edge from vertex v to vertex w if:

$$\tau(v,w) > \frac{\binom{n-2}{k-2}}{2}, \tag{4}$$

i.e., v is chosen for a majority of the edges containing both v and w.

We will restrict attention henceforth to Option (ii), above, unless stated otherwise. It should be noted that for a standard tournament graph (i.e., $k = 2$) all three formulations are equivalent; furthermore, Option (iii) is a stricter requirement than Option (ii). For discussion of ranking for vertices in hypertournaments, see for instance [8].

We refer to graphs as in Figures 2b and 3 (below) as (n,k)-*choice-domination graphs* or simply (n,k)-*domination graphs*. When n and k are clear from context, we will at times simply refer to these as domination graphs. As with tournaments, domination graphs could be valuable in considerations of individual dominance in competitive settings, as may arise for instance in biology, game theory or decision analysis. Note that hypergraphs with choice allow for analysis of scenarios wherein selection (but not full orientation) information is available.

Example 3. *Table 1 gives a* 3*-hypergraph with* $n = 5$ *vertices and* $m = 10$ *edges, along with four possible choice functions,* C_0 *(from Example 2),* C_1*,* C_2 *and* C_3 *on E, while Figure 3 includes the associated domination graphs, for comparison.*

For fixed n and k, many natural questions arise as to the properties of the resulting domination graphs; for instance:

1. What are the maximal and minimal number of edges possible for an (n,k)-domination graph?
2. What proportion of (n,k)-domination graphs are strongly path connected (for example, Figure 3a,b?
3. What is the distribution of the number of edges in the domination graph for a uniformly selected choice function on the edges of a k-hypergraph on n vertices?
4. What is the number of non-isomorphic (n,k)-domination graphs?

Table 1. Four possible $(5,3)$ choice-hypergraphs.

	e	$C_0(e)$	$C_1(e)$	$C_2(e)$	$C_3(e)$
1	{1, 2, 3}	1	1	1	1
2	{1, 2, 4}	4	1	2	2
3	{1, 2, 5}	5	5	5	5
4	{1, 3, 4}	1	3	4	4
5	{1, 3, 5}	3	5	3	3
6	{1, 4, 5}	4	4	1	1
7	{2, 3, 4}	3	2	3	4
8	{2, 3, 5}	2	2	2	2
9	{2, 4, 5}	2	4	4	4
10	{3, 4, 5}	5	3	5	3

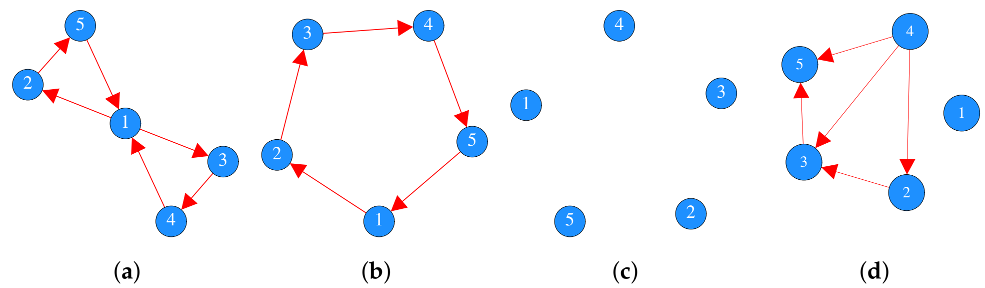


Figure 3. Domination graphs arising from choice-hypergraphs with choice functions (**a**) C_0; (**b**) C_1; (**c**) C_2; and (**d**) C_3 as in Table 1.

Example 4. *($(5,3)$ domination graphs.) For $(n,k) = (5,3)$, we have that $|E| = 10$ and the number of distinct choice functions on E is $3^{10} = 59049$. Tables 2 and 3 give frequency tables for the number of choice functions leading to domination graphs with a given number of edges, and a given number of strongly connected components, respectively. Note that 3348 choice functions result in strongly connected domination graphs. There are 225 non-isomorphic $(5,3)$-domination graphs (of which 21 are strongly connected); plots of these are provided in the Supplementary Materials; Table 4 gives the frequencies for these graphs. The two most frequent domination graphs (each occurring for 1560 distinct choice functions, C), are given in Figure 4a,b respectively; the most frequently occurring strongly connected domination graph is given in Figure 4c.*

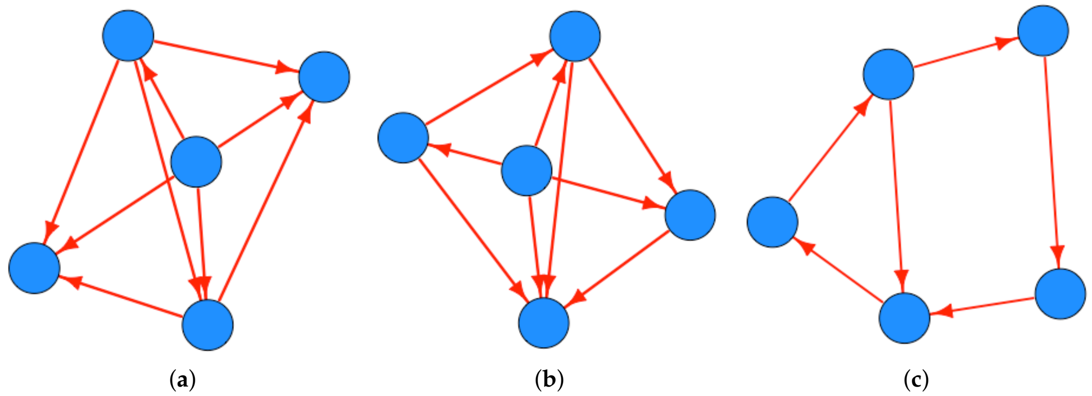


Figure 4. Three frequently occurring $(5,3)$-domination graphs. The two most frequent domination graphs are given in (**a**,**b**). The most frequently occurring strongly connected domination graph is given in (**c**).

Table 2. Edge distribution for (5,3)-domination graphs.

Edges	0	2	3	4	5	6	7	8	9	10
Frequency	6	60	120	1035	3324	10080	15180	16920	9180	3144

Table 3. Component distribution for (5,3)-domination graphs.

Components	1	2	3	5
Frequency	3348	6630	11760	37311

Table 4. Frequency distribution for non-isomorphic $(5,3)$-domination graphs.

Ind.	Freq.	Ind.	Freq.	Ind.	Freq.	Ind.	Freq.	Ind.	Freq.	Ind.	Freq.	Ind.	Freq.
1	1560	2	510	3	360	4	300	5	840	6	480	7	240
8	480	9	120	10	600	11	840	12	240	13	960	14	120
15	720	16	120	17	480	18	360	19	240	20	120	21	420
22	840	23	240	24	120	25	120	26	960	27	180	28	180
29	180	30	120	31	120	32	240	33	120	34	840	35	240
36	480	37	240	38	120	39	600	40	120	41	240	42	360
43	600	44	180	45	360	46	300	47	15	48	180	49	1200
50	720	51	1560	52	240	53	960	54	480	55	120	56	360
57	600	58	240	59	120	60	240	61	240	62	840	63	360
64	840	65	480	66	840	67	360	68	120	69	240	70	720
71	480	72	360	73	240	74	240	75	480	76	240	77	480
78	360	79	120	80	240	81	120	82	240	83	120	84	120
85	240	86	240	87	120	88	120	89	240	90	360	91	480
92	360	93	480	94	120	95	240	96	240	97	120	98	120
99	120	100	240	101	360	102	360	103	120	104	120	105	720
106	240	107	120	108	210	109	120	110	240	111	120	112	120
113	120	114	600	115	120	116	360	117	360	118	120	119	240
120	360	121	120	122	120	123	120	124	120	125	240	126	120
127	360	128	240	129	240	130	120	131	120	132	120	133	360
134	120	135	240	136	300	137	120	138	120	139	120	140	60
141	360	142	120	143	120	144	120	145	60	146	120	147	300
148	240	149	120	150	120	151	360	152	240	153	240	154	120
155	120	156	120	157	120	158	120	159	120	160	120	161	120
162	120	163	240	164	120	165	480	166	120	167	240	168	240
169	120	170	240	171	120	172	120	173	240	174	120	175	120
176	120	177	240	178	120	179	120	180	120	181	240	182	120
183	120	184	120	185	240	186	120	187	120	188	120	189	120
190	120	191	120	192	120	193	120	194	120	195	120	196	120
197	120	198	120	199	120	200	120	201	120	202	120	203	120
204	120	205	120	206	120	207	120	208	120	209	120	210	60
211	360	212	120	213	120	214	120	215	120	216	24	217	120
218	120	219	120	220	120	221	120	222	240	223	120	224	24
225	6												

In reference to Question 1 above, in Section 2 below, we will prove the following two results.

Theorem 1. *Suppose $n, k \geq 1$ and $H = (V, E)$ is a complete k-hypergraph on n vertices. If k is odd and $gcd(n, k) = 1$, then there exists a choice function, C, on E resulting in a zero-edge domination graph.*

Theorem 2. *If (H, C) is a choice-hypergraph with a zero-edge domination graph, then for all $v \in V$:*

$$|\{e \in E : C(e) = v\}| = \frac{1}{n} \cdot \binom{n}{k} \tag{5}$$

that is, each vertex is chosen for an equal number of edges, in E.

The question of minimal edges in associated domination graphs may be of interest in instances where notions of "fairness" and equitable distribution are of importance, such as in resource allocation, decision theory, data and network processing, and clinical trials. Fairness and choice have been considered in the past, notably in the context of social welfare and information processing. The interested reader might like to consult, for instance [2–4,9–13].

Figure 5 provides an example of a $(9,5)$ choice-hypergraph with vertex set $V = \{1, 2, \ldots, 9\}$, possessing a zero edge domination graph (employing the construction in the proof of Theorem 1). Note that $|E| = \binom{9}{5} = 126$, and $\tau(1,2) = 8 = \tau(2,1)$ (as highlighted in red; $C(e)$ for $e \in E$ satisfying $\{1,2\} \in e$ are indicated in bold). It may also be verified that $|\{e \in E : C(e) = 1\}| = 126/9 = 14$, as required by Theorem 2.

1 2 3 4 5	**3**	1 2 4 6 9	**2**	1 3 4 7 8	1	1 4 6 8 9	8	2 3 5 7 9	7	3 4 5 6 7	5
1 2 3 4 6	**3**	1 2 4 7 8	**1**	1 3 4 7 9	1	1 4 7 8 9	4	2 3 5 8 9	2	3 4 5 6 8	5
1 2 3 4 7	**7**	1 2 4 7 9	**1**	1 3 4 8 9	1	1 5 6 7 8	7	2 3 6 7 8	7	3 4 5 6 9	9
1 2 3 4 8	**2**	1 2 4 8 9	**1**	1 3 5 6 7	5	1 5 6 7 9	7	2 3 6 7 9	9	3 4 5 7 8	5
1 2 3 4 9	**2**	1 2 5 6 7	**6**	1 3 5 6 8	1	1 5 6 8 9	8	2 3 6 8 9	9	3 4 5 7 9	5
1 2 3 5 6	**3**	1 2 5 6 8	**8**	1 3 5 6 9	3	1 5 7 8 9	8	2 3 7 8 9	9	3 4 5 8 9	4
1 2 3 5 7	**3**	1 2 5 6 9	**1**	1 3 5 7 8	3	1 6 7 8 9	8	2 4 5 6 7	5	3 4 6 7 8	6
1 2 3 5 8	**2**	1 2 5 7 8	**8**	1 3 5 7 9	5	2 3 4 5 6	4	2 4 5 6 8	5	3 4 6 7 9	6
1 2 3 5 9	**2**	1 2 5 7 9	**9**	1 3 5 8 9	1	2 3 4 5 7	4	2 4 5 6 9	4	3 4 6 8 9	6
1 2 3 6 7	**2**	1 2 5 8 9	**5**	1 3 6 7 8	8	2 3 4 5 8	8	2 4 5 7 8	5	3 4 7 8 9	8
1 2 3 6 8	**1**	1 2 6 7 8	**8**	1 3 6 7 9	9	2 3 4 5 9	3	2 4 5 7 9	9	3 5 6 7 8	6
1 2 3 6 9	**6**	1 2 6 7 9	**9**	1 3 6 8 9	9	2 3 4 6 7	4	2 4 5 8 9	2	3 5 6 7 9	6
1 2 3 7 8	**1**	1 2 6 8 9	**9**	1 3 7 8 9	9	2 3 4 6 8	4	2 4 6 7 8	6	3 5 6 8 9	6
1 2 3 7 9	**1**	1 2 7 8 9	**9**	1 4 5 6 7	1	2 3 4 6 9	3	2 4 6 7 9	2	3 5 7 8 9	7
1 2 3 8 9	**1**	1 3 4 5 6	4	1 4 5 6 8	6	2 3 4 7 8	3	2 4 6 8 9	4	3 6 7 8 9	3
1 2 4 5 6	**4**	1 3 4 5 7	4	1 4 5 6 9	5	2 3 4 7 9	2	2 4 7 8 9	9	4 5 6 7 8	6
1 2 4 5 7	**4**	1 3 4 5 8	3	1 4 5 7 8	7	2 3 4 8 9	2	2 5 6 7 8	2	4 5 6 7 9	6
1 2 4 5 8	**2**	1 3 4 5 9	3	1 4 5 7 9	7	2 3 5 6 7	5	2 5 6 7 9	7	4 5 6 8 9	6
1 2 4 5 9	**2**	1 3 4 6 7	4	1 4 5 8 9	9	2 3 5 6 8	5	2 5 6 8 9	8	4 5 7 8 9	7
1 2 4 6 7	**4**	1 3 4 6 8	8	1 4 6 7 8	7	2 3 5 6 9	3	2 5 7 8 9	8	4 6 7 8 9	7
1 2 4 6 8	**6**	1 3 4 6 9	3	1 4 6 7 9	7	2 3 5 7 8	5	2 6 7 8 9	8	5 6 7 8 9	7

Figure 5. A (9,5)-choice hypergraph resulting in a zero-edge domination graph. For a given $e \in E$, the value of $C(e)$ is listed to the right of the five elements of e in a demarcated column.

Before turning to the proofs of Theorems 1 and 2, we will briefly mention some recent related work on hypertournaments, which carry over to choice-hypergraphs. Recall that a k-hypertournament is a complete k-hypergraph $H = (V, E)$, with each edge endowed with an orientation. We will refer to the oriented edges as arcs.

One concept considered extensively in the literature is score sequences (see for instance [6,7,14–18]). In particular, for a given $1 \leq i \leq n$ define the *score*, s_i of a vertex v_i of a k-hypertournament on $H = (V, E)$ as the number of arcs containing v_i in which v_i is not the last element (this is with a complete orientation on the edges, rather than a choice function solely selecting a single element). Similarly, define the *losing score*, r_i as the number of arcs containing v_i in which v_i is the last element. The *total score*, t_i, is then given by $t_i = s_i - r_i$. Finally, we obtain the *score sequences* $(s_1, \ldots, s_n)$, $(r_1, \ldots, r_n)$ and $(t_1, \ldots, t_n)$. Guofei et al. proved the following results regarding the existence of score sequences (see also [16,19]).

Theorem 3. *(Guofei et al. [14]) Given two non-negative integers n and k with $n \geq k > 1$, a non-decreasing sequence $R = (r_1, r_2, \ldots, r_n)$ of non-negative integers is a losing score sequence of some k-hypertournament if and only if for each j ($k \leq j \leq n$):*

$$\sum_{i=1}^{j} r_i \geq \binom{j}{k}, \tag{6}$$

with equality when $j = n$.

Theorem 4. *(Guofei et al. [14]) Given two non-negative integers n and k with $n \geq k > 1$, a non-decreasing sequence $S = (s_1, s_2, \ldots, s_n)$ of non-negative integers is a score-sequence of some k-hypertournament if and only if for each j ($k \leq j \leq n$):*

$$\sum_{i=1}^{j} s_i \geq j\binom{n-1}{k-1} + \binom{n-j}{k} - \binom{n}{k}, \tag{7}$$

with equality when $j = n$.

A k-hypertournament is said to be *regular* if for each vertex, v, the tally of arcs containing v as the last element is $\binom{n}{k}/n$. Koh and Ree [16] proved the following.

Theorem 5. *(Koh and Ree, [16]) A regular (n, k) hypertournament exists if and only if $n|\binom{n}{k}$.*

For alternative considerations of regularity, see [8,20].

Compare Theorem 5 with Theorems 1 and 2, above. Note that symmetry in domination (i.e., the existence of choice functions resulting in zero-edge domination graphs) is a stronger requirement than regularity. To see this, simply note that all standard tournaments have $\binom{n}{k}$-edge domination graphs.

For further work on hypertournaments or score sequences, see for instance Pirizda et al. [15], Landau [18], Marshall [8], Khan et al. [6], Guofei et al. [14], Gunderson et al. [21], Li et al. [22], Guo and Surmacs [23], and Chou and Guofei [24].

In the next section, we prove Theorems 1 and 2.

2. Proof of Theorems 1 and 2

Before moving on to the proofs of Theorems 1 and 2, we introduce some preliminary notation. First, suppose $n \geq k \geq 1$ are fixed and (H, C) is an (n, k)-hypergraph with choice, where $H = (V, E)$ is a complete k-hypergraph on n vertices. Without loss of generality, we assume that $V = \{0, 1, \ldots, n-1\}$. Similar to in [16], define the rotation operator $P : E \to E$, via $P(e) = e + 1 \pmod{n}$, i.e., P acts on k-subsets of V by shifting the elements (cyclically) to the right by one. Here, $e + 1$ indicates $\{v + 1 : v \in e\}$.g For $e \in E$ and $j \geq 0$, define P^j as the j-fold iteration of P and the *order* of e, α_e, via:

$$\alpha_e \stackrel{def}{=} \min\{\gamma \geq 1 : P^\gamma(e) = e\}. \tag{8}$$

We will denote the set of equivalence classes under successive application of P by $\mathcal{R} = \{R_1, \ldots, R_q\}$. Note that for $R \in \mathcal{R}$, $|R|$ is the order of each $e \in R$. We will refer to elements of $\mathcal{R}$ as *rotation classes* of H.

In general, addition of the form $S + c$ for $S \subseteq \mathbb{Z}$ and $c \in \mathbb{Z}$, will be modulo n, unless stated otherwise.

The set $e \in E$ is said to be *symmetric* if $-e = e \pmod{n}$, and more generally $R \in \mathcal{R}$ is symmetric if, for all $e \in R$, $-e \in R$. Note that if for some $e \in R$, $e = -e$, then for $1 \leq i \leq n$, $-(e + i) = -e - i = e - i \in R$, and hence if $e \in R$ is symmetric, then R is symmetric. If R is not symmetric, then there exists an $R' \in \mathcal{R} \setminus R$ such that $e \in R$ implies $-e \in R'$.

Let $\mathcal{H}_{n,k}$ be the set of all (n, k)-choice hypergraphs for fixed $n, k \in \mathbb{Z}^+$ and $\mathcal{G}_n$ be the set of all directed graphs on n vertices. Suppose a domination scheme, D is fixed (see (i)–(iii), above, for examples) and define $G_D : \mathcal{H}_{n,k} \to \mathcal{G}_n$, where $G_D(T)$ is the domination graph for choice-hypergraph $T = (H, C)$ under domination scheme D.

We have the following elementary lemma:

Lemma 1. *Suppose H is a k-hypertournament on n vertices with $gcd(n, k) = 1$. Then, for any $R \in \mathcal{R}$, $|R| = n$.*

Proof. Suppose $gcd(n, k) = 1$, and $\mathcal{R} = \{R_1, R_2, \ldots, R_q\}$ for some $q \geq 1$. For any $R \in \mathcal{R}$ and any $e \in R$, set $\alpha = \alpha_e$ and represent e as a binary vector $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n)$. We then have, with $a = \sum_{1 \leq i \leq \alpha} \zeta_i$ and $b = n/\alpha$:

$$k = ba, \qquad n = b\alpha. \tag{9}$$

Now, suppose $n = \alpha p + r$, with $p \geq 0$ and $0 \leq r \leq \alpha - 1$. Then:

$$e = P^{\alpha(p+1)}(e) = P^{n-r+\alpha}(e) = P^{\alpha-r}(e). \tag{10}$$

Since α is minimal, we have $r = 0$ and hence $b = n/\alpha \in \mathbb{Z}$. Thus, since $gcd(n,k) = 1$, (9) gives that $b = 1$, $\alpha = n$ and finally $|R| = n$. $\square$

For convenience of notation, as in (1), define the function $\tau_R : V \times V \to \mathbb{Z}^+$ via:

$$\tau_R(v,w) = |\{e \in R : v, w \in e \text{ and } C(e) = v\}|, \tag{11}$$

i.e., $\tau_R(v,w)$ is the number of edges in the rotation class $R \in \mathcal{R}$ for which v is chosen in the presence of w. Note that

$$\tau(v,w) = \sum_{R \in \mathcal{R}} \tau_R(v,w). \tag{12}$$

Lemma 2. *Suppose $n, k \geq 1$ with k odd, $H = (V,E)$ is a complete k-hypergraph on n vertices, and $R \in \mathcal{R}$. If R is symmetric and $e \in R$, then there exists a $\xi \in V$ and $\mathcal{T} = \{\delta_1, \delta_2, \ldots, \delta_k\}$, such that $e = \xi + \mathcal{T}$, and $\delta \in \mathcal{T}$ implies $-\delta \in \mathcal{T}$.*

Proof. Suppose that R is symmetric and $e = \{x_1, \ldots, x_k\} \in R$. Then, there exists an i such that $e + i = -e$, and hence a permutation $(j_1, j_2, \ldots, j_k)$ of $(1, 2, \ldots, k)$ such that:

$$x_{j_q} + i = -x_q, \quad 1 \leq q \leq k. \tag{13}$$

Since k is odd, there exists a Q such that:

$$x_Q + i = -x_Q, \tag{14}$$

and taking differences, (13) and (14) imply:

$$x_{j_q} - x_Q = x_Q - x_q = -(x_q - x_Q), \quad 1 \leq q \leq k. \tag{15}$$

The result follows upon setting $\xi = x_Q$, and $\mathcal{T} = \{x_q - x_Q : 1 \leq q \leq k\}$. $\square$

We will now prove Theorem 1 regarding the existence of choice functions with symmetry in domination.

Proof of Theorem 1. Suppose $gcd(n,k) = 1$, k is odd, and (H,C) is an (n,k)-choice hypergraph. Consider E, the set of all edges in H, and let $\mathcal{R} = \{R_1, R_2, \ldots, R_q\}$ be the set of all rotation classes of H, where, by Lemma 1, for $R \in \mathcal{R}$, $|R| = n$, and $|\mathcal{R}| = \binom{n}{k}/n$. Fix $v, w \in V$, with $v \neq w$, and choose some $R \in \mathcal{R}$.

Suppose R is symmetric and fix an $e \in R$. Then, by Lemma 2, $e = \xi + \mathcal{T}$, where $\xi \in V$ and $\mathcal{T}$ is closed under additive inverses. For $u = e + i \in R$, set $C(u) = \xi + i$. Suppose $v = \xi + j_v$ and $w = \xi + j_w$ for $j_v, j_w \in \{0, 1, ..., n-1\}$. For $f \in R$, $C(f) = v$ implies $f = e + j_v$ and $C(f) = w$ implies $f = e + j_w$. Note that $w \in e + j_v$ if and only if $j_w - j_v \in \mathcal{T}$. Similarly, $v \in e + j_w$ if and only if $j_v - j_w \in \mathcal{T}$. Since $\mathcal{T}$ is closed under inverses, we have:

$$\begin{aligned} \tau_R(v,w) &= |\{e \in R : v, w \in e \text{ and } C(e) = v\}| \\ &= |\{e \in R : v, w \in e \text{ and } C(e) = w\}| \\ &= \tau_R(w,v). \end{aligned} \tag{16}$$

Suppose R is not symmetric, and consider $R' = \{-e : e \in R\}$, and note that $R \cap R' = \emptyset$. Fix an $e \in R$ and $\xi \in e$, and write $e = \xi + \mathcal{T}$ (note that $\mathcal{T}$ is not closed under inverses). For $f \in R \cup R'$, set:

$$C(f) = \begin{cases} \xi + i & \text{if } f = e + i = \xi + i + \mathcal{T} \in R \\ -\xi + i & \text{if } f = -e + i = -\xi + i - \mathcal{T} \in R'. \end{cases} \tag{17}$$

Now, suppose $v = \xi + j_v$ and $w = \xi + j_w$. For $f \in R \cup R'$, $C(f) = v$ implies $f = e + j_v$ or $f = -e + j_v$, and $C(f) = w$ implies $f = e + j_w$ or $f = -e + j_w$. Note that $w \in e + j_v$ if and only if $j_w - j_v \in \mathcal{T}$, and $w \in -e + j_v$ if and only if $j_w - j_v \in -\mathcal{T}$ (i.e., $j_v - j_w \in \mathcal{T}$). Similarly, $v \in e + j_w$ if and only if $j_v - j_w \in \mathcal{T}$, and $v \in -e + j_w$ if and only if $j_v - j_w \in -\mathcal{T}$ (i.e., $j_w - j_v \in -\mathcal{T}$). Thus:

$$\begin{aligned} \tau_{R \cup R'}(v, w) &= |\{f \in R \cup R' : v, w \in f \text{ and } C(f) = v\}| \\ &= |\{f \in R \cup R' : v, w \in f \text{ and } C(f) = w\}| \\ &= \tau_{R \cup R'}(w, v). \end{aligned} \tag{18}$$

Employing (16), (18) and (12), the result follows. □

We will now prove Theorem 2.

Proof of Theorem 2. Suppose (H, C) is a choice-hypergraph with a zero-edge domination graph, where $H = (V, E)$ is a complete k-hypergraph on n vertices. For a fixed vertex $v \in V$, define $\sigma_v = \{e \in E : v \in e\}$ and $\omega_v = \{e \in \sigma_v : C(e) = v\}$, i.e., σ_v is the set of edges to which v belongs and ω_v is the set of edges for which v is selected. Note that:

$$|\omega_v| \leq |\sigma_v| = \binom{n-1}{k-1}. \tag{19}$$

Assume $|\omega_v| < \binom{n}{k}/n$. Since (H, C) has a zero-edge domination graph, $\tau(w, v) = \tau(v, w)$ for all $w \in V$ and hence:

$$\binom{n-1}{k-1} = |\sigma_v| = (k-1)|\omega_v| + |\omega_v| \tag{20}$$

$$< k\frac{1}{n}\binom{n}{k} = \binom{n-1}{k-1}. \tag{21}$$

Thus, $|\omega_v| \geq \binom{n}{k}/n$ for all $v \in V$. The result follows upon noting that:

$$\sum_{v \in V} \omega_v = |E| = \binom{n}{k}. \tag{22}$$

□

Supplementary Materials: The following are available online at http://www.mdpi.com/2073-8994/9/3/46/s1, File S1: Plots of the 225 distinct non-isomorphic (5,3) domination graphs. File S2: Plots of the 21 distinct strongly connected, non-isomorphic (5,3) domination graphs

Author Contributions: Kenneth S. Berenhaut, Brendan P. Lidral-Porter, Theodore H. Schoen and Kyle P. Webb contributed to the results; Kenneth S. Berenhaut and Brendan P. Lidral-Porter wrote the paper.

Conflicts of Interest: The authors declare no conflict of interest.

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