Chapter 2 Effect of Elliptical Geometry on Cylinder Response

This chapter addresses the influences of ellipticity by using the semi-analytical scheme described in the previous chapter.

2.1 Numerical Values of Problem Parameters

Though ultimate interest with elliptical cylinders is for application to aircraft fuselage structures, initial experimental work will take place with small scale cylinders. The displacement, strain, and stress response of these smaller cylinders must be understood before studies of scaled-up cylinders can commence. To that end, in the present study numerical results will be shown for eight and nine layer graphite-epoxy cylinders with semi-major diameters of 5 in., ellipticities of 0.7, and lengths of 12.5 in. The material and geometric properties of a layer of graphite-epoxy are taken to be

$$E_1 = 18.85 \text{ Msi} \qquad E_2 = 1.407 \text{ Msi}$$
$$G_{12} = 0.725 \text{ Msi} \qquad \nu_{12} = 0.300 \qquad h = 0.0055 \text{ in.} \tag{2.1}$$

where h is the thickness of a single layer. The laminates considered are: quasi-isotropic, $[\pm 45/0/90]_S$; axially-stiff, $[\pm 45/0_2/90_{½}]_S$; circumferentially-stiff, $[\pm 45/90_2/0_{½}]_S$, where 0 degrees is along the axial direction. These lay-ups were selected because each has at least one layer with its fibers in the axial direction, at least one layer with its fibers in the circumferential direction, and ±45 degree layers. Eight or nine layers is a reasonable number from the point of view of manufacturing the cylinders by hand on elliptical mandrels.

The inverse radius of curvature of eq. 1.10 requires *I*=7 for convergence of the cosine series with the exact solution, as shown in fig. 2-1. In fig. 2-1 the vertical axis represents the error when using the series of eq. 1.10, and it is seen that I=7 results in minimal error at all circumferential locations.

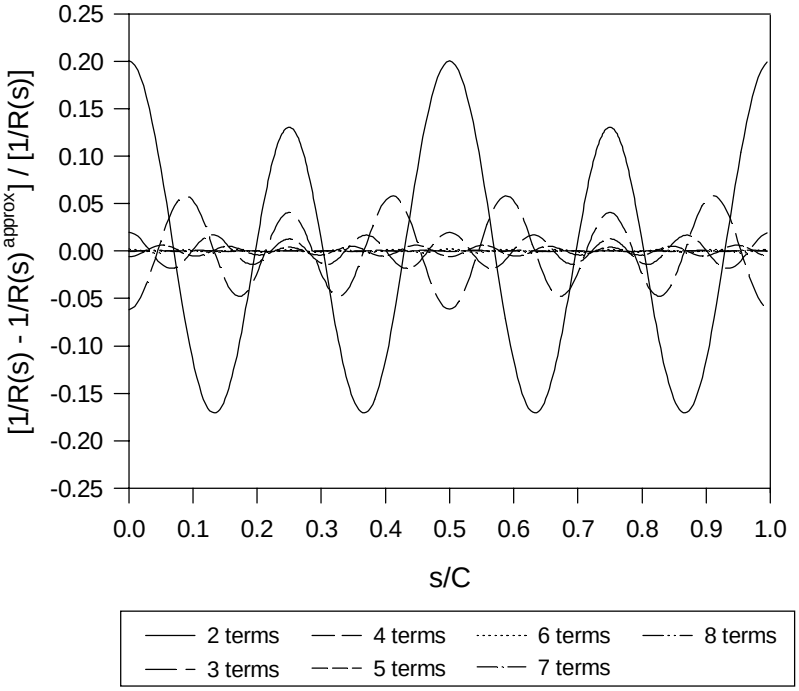

Figure 2-1. Convergence study for the inverse radius of curvature.

Accordingly, the three displacement series of eq. 1.11 require *N*=7 and *M*=4 for convergence of the displacements and force resultants. The displacement series is expanded as follows

$$u^o(x,s) = u^o_0(x) + u^o_1(x)cos(4\pi s/C) + u^o_2(x)cos(8\pi s/C) + u^o_3(x)cos(12\pi s/C) + u^o_4(x)cos(16\pi s/C)$$
$$+ u^o_5(x)cos(20\pi s/C) + u^o_6(x)cos(24\pi s/C) + u^o_7(x)cos(28\pi s/C)$$
$$+ u^o_8(x)sin(4\pi s/C) + u^o_9(x)sin(8\pi s/C) + u^o_{10}(x)sin(12\pi s/C) + u^o_{11}(x)sin(16\pi s/C)$$

$$v^o(x,s) = v^o_0(x) + v^o_1(x)cos(4\pi s/C) + v^o_2(x)cos(8\pi s/C) + v^o_3(x)cos(12\pi s/C) + v^o_4(x)cos(16\pi s/C)$$
$$+ v^o_5(x)sin(4\pi s/C) + v^o_6(x)sin(8\pi s/C) + v^o_7(x)sin(12\pi s/C) \qquad (2.2)$$
$$+ v^o_8(x)sin(16\pi s/C) + v^o_9(x)sin(20\pi s/C) + v^o_{10}(x)sin(24\pi s/C) + v^o_{11}(x)sin(28\pi s/C)$$

$$w^o(x,s) = w^o_0(x) + w^o_1(x)cos(4\pi s/C) + w^o_2(x)cos(8\pi s/C) + w^o_3(x)cos(12\pi s/C) + w^o_4(x)cos(16\pi s/C)$$
$$+ w^o_5(x)cos(20\pi s/C) + w^o_6(x)cos(24\pi s/C) + w^o_7(x)cos(28\pi s/C)$$
$$+ w^o_8(x)sin(4\pi s/C) + w^o_9(x)sin(8\pi s/C) + w^o_{10}(x)sin(12\pi s/C) + w^o_{11}(x)sin(16\pi s/C).$$

Further details regarding convergence can be found in ref. 1.

2.2 Displacements

In order to demonstrate the basic responses of an elliptical cylinder subjected to internal pressure, a comparison is made with circular cylinders. For this comparison a quasi-isotropic laminate is chosen and a geometrically linear analysis is used. The basic cylinder responses considered are reference surface displacements, reference surface strains and curvatures, and force and moment resultants. Figure 2-2a-f illustrates axial, circumferential, and normal displacements as a function of the axial and circumferential coordinates. The displacements have been normalized by the laminate thickness H. An internal pressure of p_o=100 psi is used to compute the results in these figures. The format of the fig. 2-2a-f illustrates the response of one-eighth of the cylinder. The coordinate locations have been normalized and, referring to fig. 1-2, the range of $0 \le x/L \le 0.5$ and $0 \le s/C \le 0.25$ is considered. Due to the presence of D_{16} and D_{26}, the problem does not exhibit octal symmetry. However looking at only one eighth of the cylinder provides a fairly accurate detailing of the response, and simplifies displaying the results. Implementing symmetry and

antisymmetry arguments for various responses, the response for the remainder of the cylinder can be envisioned.

Regarding the axial displacement of fig. 2-2a-b, for an internally pressurized cylinder the axial displacement is the net result of the pressure forcing the end enclosures apart and the Poisson effect due to circumferential expansion pulling them together. For a circular cylinder, this results in a nearly linear axial displacement with the axial coordinate. Since the internal pressure problem for a circular cylinder is axisymmetric, the axial displacement does not vary with s. Recall from the boundary conditions of eq. 1.3 that the axial displacement is zero at x/L = -0.5, and at x/L = 0.5 the axial displacement is determined by eq. 1.4. Because of the nearly linear variation with x, the axial displacement at x/L = 0.5 is approximately twice the value at x/L = 0. As for the elliptical cylinder, the internal pressure problem is not axisymmetric, and the axial displacement is far from being linear with x. For the elliptical cylinder notice that along the crown of the cylinder, s/C=0, the axial displacement is positive, while along the side of the elliptical cylinder the axial displacement is actually negative at certain axial locations. Since the axial displacement changes signs with spatial location, there are some locations besides x=-L/2 where the axial displacement is zero. This is not a situation that appears in the circular case. It should be noted, however, that the axial displacement at x/L=0 is practically independent of s, as it is at x/L=0.5, and the axial displacements at these locations differ by a factor of 2, as they do for the circular case.

Figure 2-2c-d illustrates the circumferential displacement, a response that clearly distinguishes an elliptical cylinder from a circular cylinder. An internally pressurized circular cylinder has no circumferential displacement response for balanced symmetric laminates. However, the elliptical case shows circumferential movement away from the sides and toward the crown and keel of the cylinder. Figure 2-2e-f illustrates another distinguishing difference between a circular

and elliptical cylinder, as was mentioned in connection with fig. 1-1. The normal displacement of a circular cylinder is uniformly outward. In contrast, for the elliptical case the cylinder tends to become more circular. The elliptical cylinder under internal pressure moves outward at the crown and keel, but moves inward at the sides. As will be seen, this has important consequences at the ends of the cylinder.

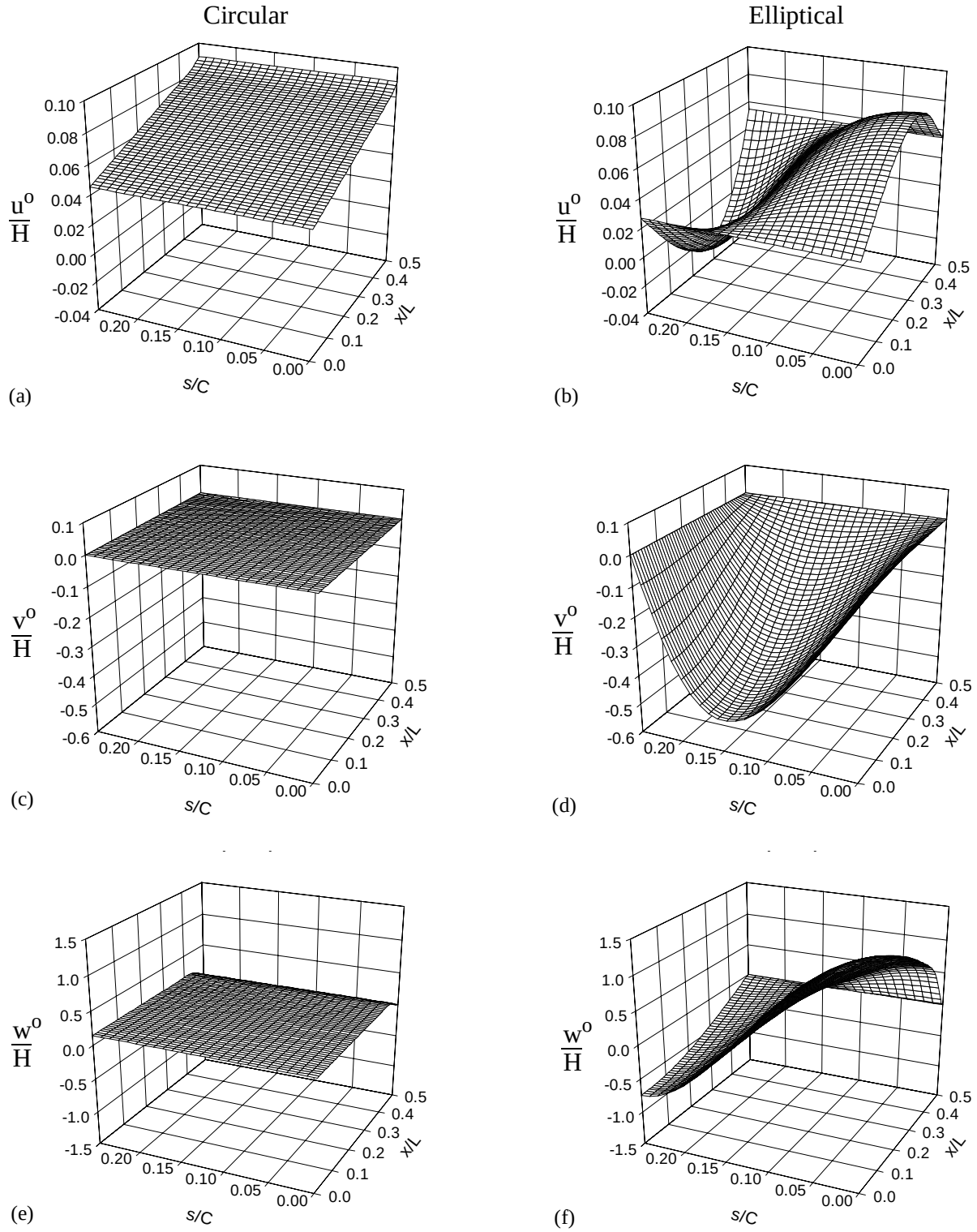

Figure 2-2. Influence of ellipticity on the displacements of a quasi-isotropic cylinder.

2.3 Strains and Curvatures

A comparison of reference surface strains in circular and elliptical cylinders provides a further demonstration on the influence of geometry on responses. In fig. 2-3a-f normalized reference surface strains are compared. Note that from here forward all normalized terms are denoted by an overbar. The normalization factor for the strains is the circumferential reference surface midspan strain in an internally pressurized quasi-isotropic circular cylinder, namely,

$$\left(\frac{A_{11}-\frac{1}{2}A_{12}}{A_{11}A_{22}-A_{12}^{2}}\right)p_o R, \tag{2.3}$$

where the A_{ij}s for a quasi-isotropic laminate are used. As a result of this normalization, the pressure level used in the calculations does not influence the magnitudes of the responses shown in the figures.

The circumferential strain, $\varepsilon^o{}_s$, shown in fig. 2-3a-b, varies considerably with both axial and circumferential locations for the elliptical case, whereas, the circumferential strains for the circular cylinder vary only with axial location and only near the ends. This behavior for all ellipses can be explained by studying the relationship between circumferential strain and displacements in eq. 1.7b. As seen in that equation, the inverse radius of curvature and the change in v^o with respect to circumferential location determine the behavior of circumferential strain. In the circular case v^o is zero, and therefore does not change with respect to circumferential or axial location, and the inverse radius of curvature is constant with s. The circumferential strain is determined solely by the inverse radius of curvature term. In the elliptical case, v^o and the inverse radius of curvature change significantly with respect to circumferential location and result in the behavior in fig. 2-3b. For the elliptical case, except for the cylinder ends, there is no location

where the circumferential strain is zero, and it changes sign and magnitude with location The circumferential strain is zero at the ends due to boundary condition on v^o and w^o given in eq. 1.3.

The axial strain, ε^o_x, shown in fig. 2-3c-d, shows behavior similar to the circumferential strain. For instance, in the circular case the midspan regions exhibit uniform strains, and in the elliptical case the strains vary with both axial and circumferential locations, and, in fact, change sign. However, the driving force behind these similar behaviors is due to a different displacement. The relationship between axial strain and displacements is shown in eq. 1.7a as the change in u^o with respect to axial location. In the circular case, u^o does change with respect to axial location, but the change is nearly linear, resulting in a uniform axial strain in the midspan region. For the elliptical case, u^o also changes with respect to axial location, but the change is nowhere near linear and therefore the axial strain, shown in fig. 2-3d, is not uniform.

The shear strain, γ^o_{xs}, shown in fig. 2-3e-f, varies considerably with the axial and circumferential location for the elliptical case, while the shear strain for the circular case is zero. The relationship between shear strain and displacements is shown in eq. 1.7c to be dependent on the change in u^o with respect to circumferential location and the change in v^o with respect to axial location. As a result, the shear strain for elliptical cylinder varies significantly with both circumferential and axial location. The presence of shear strain is another distinguishing feature of the elliptical cylinder. Note also that the shear strain in an ellipse is as large, or larger, than the other two strain components.

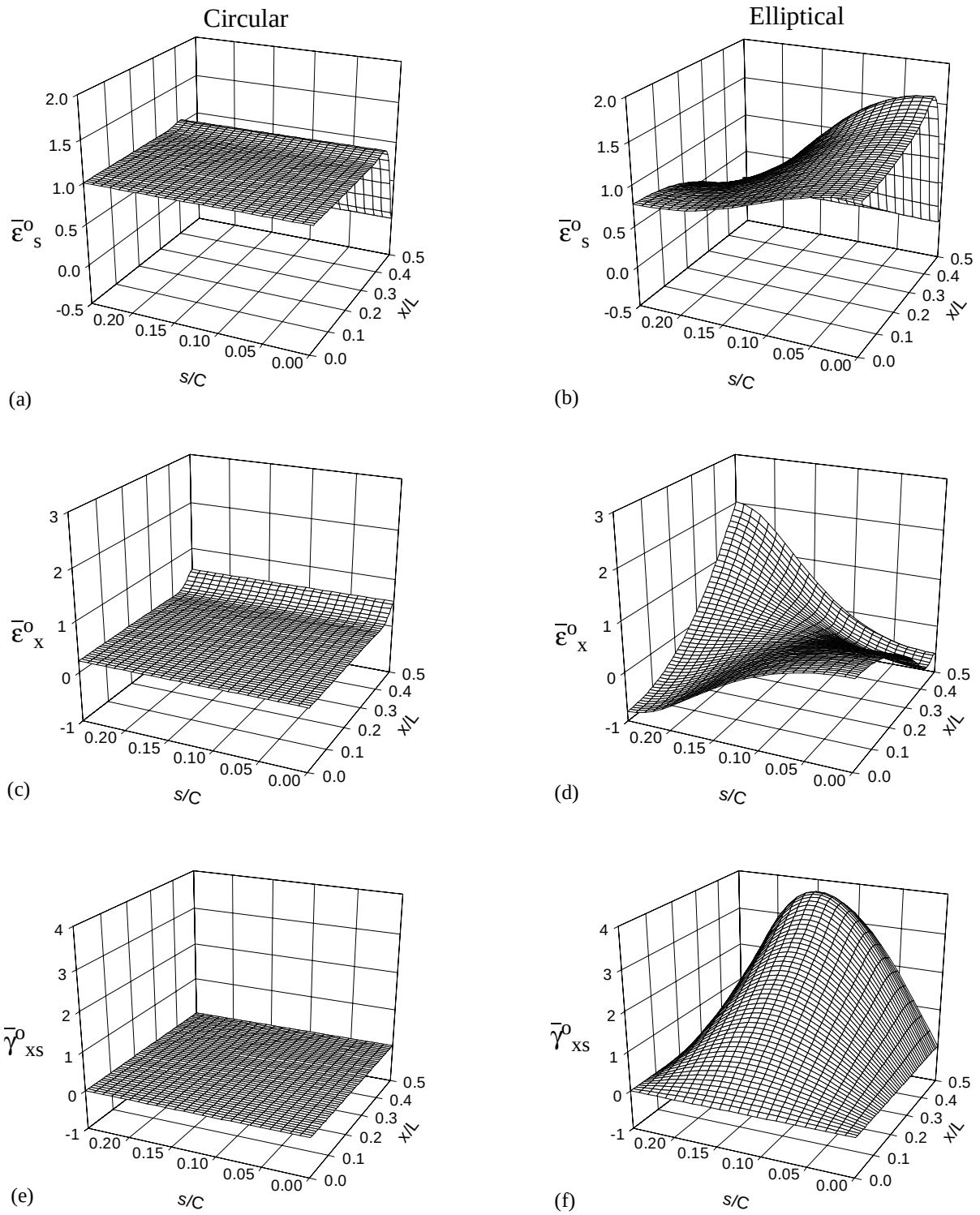

Figure 2-3. Influence of ellipticity on the strains of a quasi-isotropic cylinder.

Comparisons of circumferential, axial, and twist curvatures, $\overline{\kappa}^o_s$, $\overline{\kappa}^o_x$, and $\overline{\kappa}^o_{xs}$ respectively, for circular and elliptical cylinders are also necessary to demonstrate the influence of ellipticity on responses. To make the curvatures comparable to the previous figures involving the strains, the curvatures are converted to normalized strain measures by multiplying them by $H/2$ and then dividing this result by the quantity in eq. 2.3. The result is the normalized strain that would occur at the outer surface of the cylinder due to the curvature. (Note that by multiplying the curvature by $-H/2$, the strain that would occur at the inner surface of the cylinder due to the curvature can be computed.)

The curvatures, shown in fig. 2-4a-f, are strictly a function of normal displacement, w^o, and how it varies with the x and s coordinates, as given by the last three expressions of eq. 1.7. The magnitudes of the curvatures are notable because the axial curvature is an order of magnitude greater than the circumferential or twist curvatures. Due to the uniform outward normal expansion and a lack of variation with the s coordinate, the circular cylinder has zero circumferential and twist curvatures. The boundary conditions on w^o, from eq. 1.3, causes w^o to have a gradient in the x direction. Thus the axial curvature shows a variation with x for the circular cylinder.

In contrast, the elliptical cylinder does not have uniform outward normal expansion, rather it varies both with the x and s coordinates. As a result, the circumferential curvature varies with both x and s in the midspan region, but goes to zero at the boundary. This behavior at the boundary is caused by the boundary conditions which force w^o to be independent of s there. The axial curvature is zero in the midspan region, but varies in the boundary region. Focusing on the behavior at the boundary, recall the normal displacement at the boundary is forced to zero for an elliptical cylinder, while away from the boundary, as shown in fig. 1-1, the normal displacement is outward at the crown and keel and inward at the sides. This situation creates axial curvature which

is positive on the sides and negative on the crown and keel. The twist curvature is due to a variation in w^o with both the x and s coordinates. At the boundary the twist curvature, like the circumferential curvature, is zero. However, beyond the boundary, the twist curvature varies with both x and s.

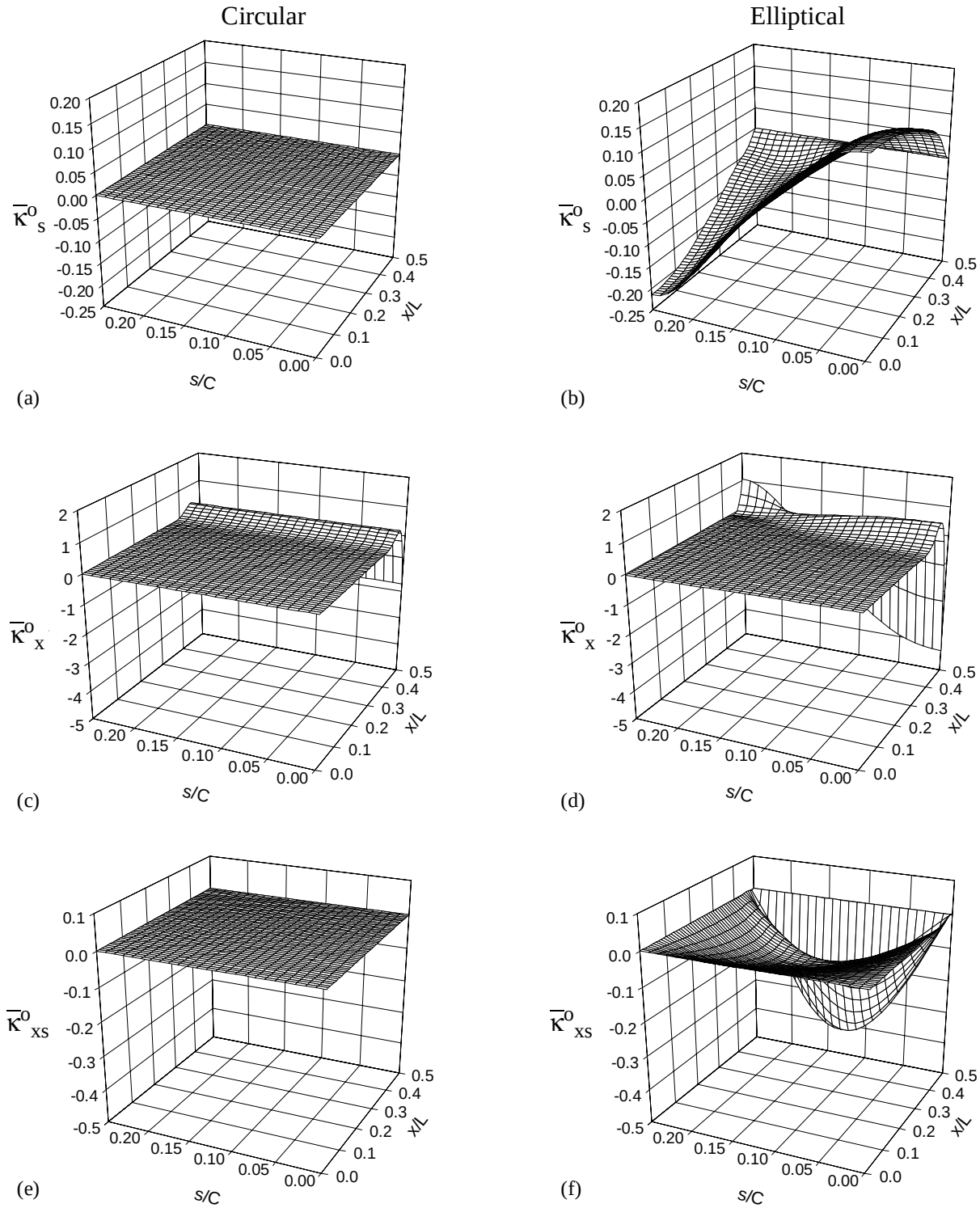

Figure 2-4. Influence of ellipticity on the curvatures of a quasi-isotropic cylinder.

2.4 Force and Moment Resultants

The circumferential, axial, and inplane shear force resultants, $\overline{N}_s$, $\overline{N}_x$, and $\overline{N}_{xs}$, respectively, seen in Figure 2-5a-f, are normalized by the midspan hoop force resultant for a circular cylinder, namely,

$$p_o R. \tag{2.4}$$

The force resultants are based upon a combination of the strains seen in Figure 2-3a-f, each multiplied by a constant, A_{ij}, as given in eq. 1.9a-c. Since A_{16} and A_{26} are zero due to a balanced laminate scheme, the circumferential and axial force resultants are a combination of only the reference surface circumferential and axial strains, and the shear force resultant is proportional only to the reference surface shear strain.

Because of the normalization given in eq. 2.4, the normalized circumferential force resultant for a circular cylinder is unity at the midspan, but deviates from unity at the boundary due to end effects there. The circumferential force resultant for the elliptical case varies with the x and s coordinates, but, for a given s, behaves similarly to the circular cylinder case in the x direction. Of note is that at the midspan the average of the circumferential force resultant for the elliptical cylinder case is approximately equal to the circumferential force resultant for the circular cylinder case, namely unity.

Again because of the normalization given in eq. 2.4, for the circular cylinder case the normalized axial force resultant is 1/2 and is spatially uniform. In contrast, the axial force resultant for the elliptical cylinder case varies with the x and s coordinate. Concentrating on the midspan of the elliptical cylinder, it is seen that the sides of the cylinder are in axial compression and the crown and keel are in tension. Concentrating on the boundary, it is seen that the sides of the cylinder are in tension and the crown and keel are in compression, just the opposite of the midspan. As

with the axial displacement for an elliptical cylinder, the axial force resultant changes signs with spatial location, causing some locations to be zero. This is not a situation that appears in the circular case.

The inplane shear force resultant for the circular case is zero due to the shear strain being zero. For the elliptical case, the shear force resultant is nonzero varies considerably with both *x* and *s*, and the magnitude is comparable to that of the circumferential and axial force resultants. Though it cannot be seen in the figure, $\overline{N}_{xs}$ is not zero at x/L=0.5. Since $\frac{\partial v^o}{\partial x}$ is nonzero there, $\overline{N}_{xs}$ is nonzero there.

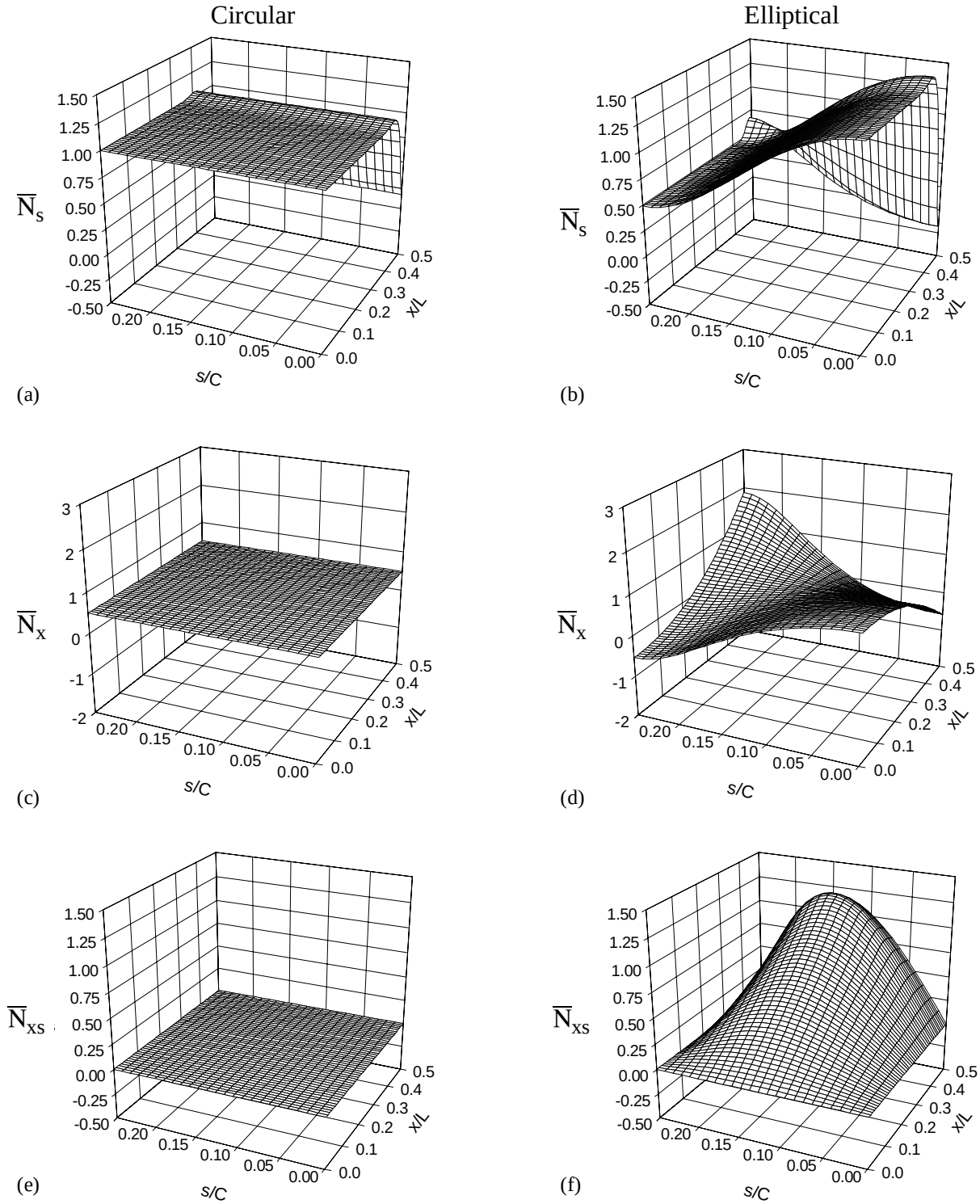

Figure 2-5. Influence of ellipticity on the force resultants of a quasi-isotropic cylinder.

The circumferential, axial, and twist moment resultants, $\overline{M}_s$, $\overline{M}_x$, and $\overline{M}_{xs}$, respectively, are normalized by the factor H/2 times the midspan hoop force resultant in a circular cylinder, namely,

$$p_o R \frac{H}{2} \quad . \tag{2.5}$$

The moment resultants are based upon a combination of the curvatures seen in fig. 2-4a-f, each multiplied by a bending stiffness, D_{ij}, as given in eq. 1.9d-f. Recall from fig. 2-4 that the axial curvature is an order of magnitude greater than the circumferential or twist curvatures. However, the bending stiffnesses, D_{ij}, control the degree to which the curvature influences the moment resultants. For the quasi-isotropic layup used, D_{16} and D_{26} are an order of magnitude smaller than D_{11}, D_{12}, D_{22}, and D_{66}. As a result, the circumferential and axial moment resultants are dominated by $\overline{\kappa}^o_x$, as can be seen by examining the character of $\overline{\kappa}^o_x$ vs. x and s and $\overline{M}_s$ and $\overline{M}_x$ vs. x and s, particularly near the ends. On the other hand, for the twist moment $\overline{M}_{xs}$, $D_{66}\,\overline{\kappa}^o_{xs}$ is similar in magnitude to $D_{16}\,\overline{\kappa}^o_x$ both of which are larger than $D_{26}\,\overline{\kappa}^o_s$. Therefore, $\overline{M}_{xs}$ is controlled by both $\overline{\kappa}^o_{xs}$ and $\overline{\kappa}^o_x$. Note that $\overline{M}_{xs}$ is the smallest of the three moment resultants.

Continuing with the discussion of the moment resultants: the most significant portion of circumferential, axial, and twist moment resultants is at the boundary. There the moment resultants are simply a response to the clamped boundary condition from eq. 1.3. The circular cylinder response to internal pressure is a uniform outward normal expansion which is restricted to be zero at the boundary, independent of s. The moment resultants at the boundary of fig. 2-6a,c,e show a response not dependent on the s coordinate. However, because the elliptical cylinder response to internal pressure is inward normal displacement on the sides and outward normal displacement on the crown and keel, the moment resultants on the boundary are dependent on s. This is reflected in

fig. 2-6b,d,f with a sign reversal in the moment resultants at the boundary between s/C=0 and s/C=0.25.

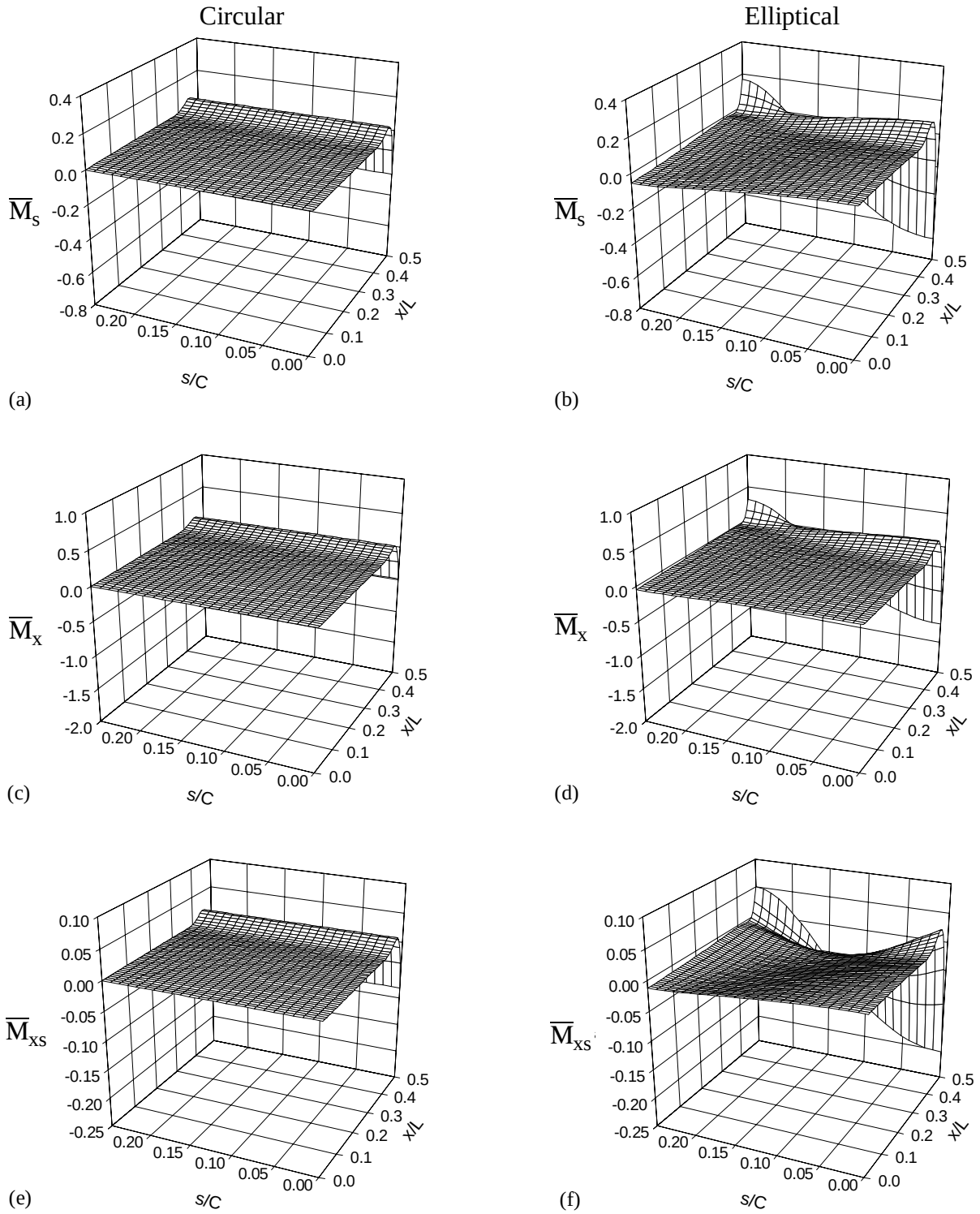

Figure 2-6. Influence of ellipticity on the moment resultants of a quasi-isotropic cylinder.

The transverse shear stresses $\tau_{x\zeta}$ and $\tau_{s\zeta}$ are not directly included in the energy expression of eq. 1.5. Therefore, there are no transverse shear force resultants associated with these stresses in the theory presented. However, Newtonian equilibrium approaches indicate that transverse shear force resultants are necessary for enforcing equilibrium. Moment equilibrium of a differential element of the cylinder wall dictates the following relation between transverse shear force resultants, Q_x and Q_s, and moment resultants:

$$\begin{aligned} Q_x &= \frac{\partial M_x}{\partial x} + \frac{\partial M_{xs}}{\partial s} \\ Q_s &= \frac{\partial M_{xs}}{\partial x} + \frac{\partial M_s}{\partial s}. \end{aligned} \tag{2.6}$$

In terms of transverse shear stresses

$$\begin{aligned} Q_x &= \int_{-\frac{H}{2}}^{+\frac{H}{2}} \tau_{x\zeta} d\zeta \\ Q_s &= \int_{-\frac{H}{2}}^{+\frac{H}{2}} \tau_{s\zeta} d\zeta. \end{aligned} \tag{2.7}$$

Since the present theory involves the moment resultants that appear on the right side of the equations in eq. 2.6 as explicit functions of x and s, Q_x and Q_s can be computed from eq. 2.6.

The circumferential and axial transverse shear force resultants, $\overline{Q}_s$ and $\overline{Q}_x$, respectively, are illustrated in fig. 2-7a-d, and they are normalized by the same factor used for the force resultants, namely eq. 2.4. Similar to the moment responses, the significant transverse shear force resultants are restricted to the boundary. It is essentially these force resultants that enforce the w^o=0 condition at the boundary. The uniformity of the circular case in the s direction results in a reaction at the boundary that is independent of s. However, with the elliptical cylinder, the values of $\overline{Q}_s$ and $\overline{Q}_x$ change sign at the boundary. This is reflected in fig. 2-6b,d with a sign reversal at the

boundary between *s/C*=0 and *s/C*=0.25. Note that the magnitudes of the transverse shear force resultants are much less than the magnitudes of the inplane force resultants.

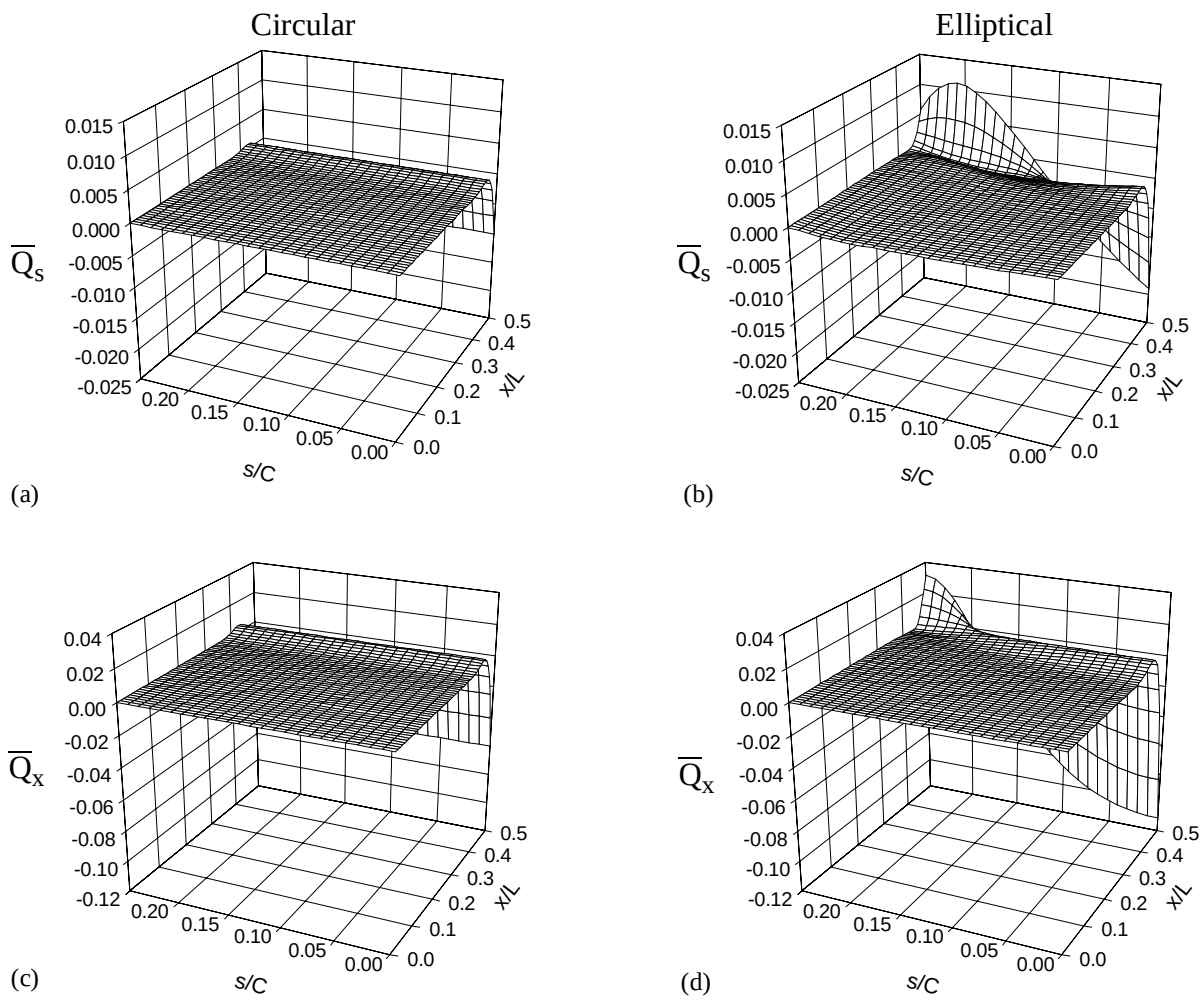

Figure 2-7. Influence of ellipticity on the transverse shear force resultants of a quasi-isotropic cylinder.

2.5 Summary of the Effects of Ellipticity

The effects of ellipticity seen in this chapter included several key issues. For instance, responses for the elliptical case varied with both the *x* and *s* coordinate. This variation was seen in every elliptical response, either over the entire domain, or at the boundary. Also, axial responses for the elliptical case were compressive for axial displacement, axial strain, and axial force result-

ant. For the elliptical cylinder, the circumferential displacement and shear force resultant were not zero, whereas, both of these responses were zero for the circular cylinder. Finally, an ellipticity of 0.7 caused a change in sign of the response at the boundary for axial curvature, all moment resultants, and the shear force resultants as s varies from s/C=0 to s/C=0.25. It is felt less severe ellipses, e.g., e=0.90, may not experience these sign reversals. To this point the effects of ellipticity have been evaluated using a geometrically linear analysis for both circular and elliptical cylinders constructed with a quasi-isotropic laminate. In the next chapter the focus will be shifted from comparing the response of elliptical cylinders with circular cylinders to comparing the responses of elliptical cylinders as predicted by the geometrically nonlinear theory with the responses as predicted by the linear theory.