ON A METHOD OF MULTIPLE-PRESENTATION SCALING OF SUCCESSIVE INTERVALS

by

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I. INTRODUCTION

The construction of scales for the purpose of expressing qualitative phenomena in numerical terms is probably as old as science itself, and many reports on this topic have been produced for almost every field of human endeavor. The interested reader will find reviews of the type of scaling methods to which the present study belongs in [4] and [7]. We are here concerned with a technique known as the "Method of Successive Intervals" [8,9,10]. More specifically, we are dealing with those cases where the objects are rated two or more times [1,2,6].

The experimental situation for which a particular analysis will be studied is as follows: A number of "stimuli" is presented to an "individual". The "stimuli" may be term papers which the "individual", a teacher, has to grade; or they may be different kinds of meat presented for grading, different brands of merchandise presented for judgment of quality, etc. In each case the "individual" is asked to place the "stimulus" at a point of a given scale (as A,B,C,D,F or 0-100 for the grading of term papers; "prime","choice","good", etc., for the grading of meat; "excellent","good","fair", etc., for the quality of merchandise). We will assume that there are k+1 such points on a "scale of successive intervals", numbered 0,1,2,...,k. We will further assume that the same set of "stimuli" is presented to the same "individual" on two, three or four successive occasions, and that the interval between these presentations is short enough so that no change of the individual's "attitude" toward the stimuli occurs.

The classical question, which has been approached by a variety of methods, is, "What is the distance between the points on the scale?" It is in most circumstances wholly inadequate to treat these points as equidistant, especially when we want to perform a regression or correlation study comparing different sets of stimuli. The time-honored tradition of averaging a grade of A and another of C to a total grade of B is probably the main source of unreliability in forecasting or assessing progress of a student. "Ranks", "percentile ranks" and, especially, "normalized percentile ranks" have been used by more careful educators since the turn of the century, but, except in the case of "normalized percentile ranks" there is little, if any, justification for treating these points as equidistant or for averaging ratings over such points. In probably every science, the use of the normal curve for constructing useful scales has been advocated. Under different names, and with convenient locations of zero-points and units of measurement (both usually arbitrary), they are known to many scientists (e.g., "normalized quantiles", "normalized proportions", "normalized standard scores", "Deviation I.Q.", "probits", etc.). Techniques for the construction of such scales and estimation of points on the scale have been developed by many scientists, and an enormous amount of duplication occurred during the past century. Reports on some of the most extensive, but by no means exhaustive, developments in this field are presented in [3], [5], and [7]. The whole areas of "Psychophysical Scaling" and "Statistics of Biological Assay" are essentially concerned with the development, utilization, and interpretation of such scales. It is regrettable, and illustrative of the long history of the methods, that educators and scientists are still "averaging" grades and points even though the fallacy or, at best, unreliability of such techniques has been clearly recognized for almost a century.

We base our scale construction on the uncertainty of the individual in rating the same stimulus on two or more different occasions. This approach is by no means new. The concept of "discriminal dispersion" [9] is directly related to this "interval of uncertainty". But a direct measurement for the same individual of this "discriminal dispersion" is seldom performed. Instead, assumptions are made regarding the distribution of such responses or, more often, responses given by groups of individuals are treated as if they were multiple responses of one individual (Transition from Thurstone's "Case I" to "Case II"). This latter procedure may be quite appropiate, especially where a "true" response can be assigned to a given stimulus, as for instance in the judgment of weights of different objects, or where the "group" can be considered as homogeneous in its rating of responses, for instance, we hope, in the evaluation of the performance of boxers by several judges.

As soon as we present the same set of stimuli to the same person on two different occasions, we are faced with another dilemma. He may remember a rating formerly given if the time interval is too short, or he may, in the meantime, change his attitude toward some stimuli if the interval is too long. Suppose, for instance, that we let a person assess the quality of ten different cigarettes on a ten-point scale, on the Monday of each of four consecutive weeks. Between these administrations he may have developed a particular like or dislike toward some brand. Such "change of attitude" is assumed to be non-existent in our approach (and, for that matter, in most double or multiple presentation scaling techniques). An example of a situation where this influence is probably negligible is the following: Suppose we present ten time-signals of varying duration, in random order, and with adequate intervals between two successive signals, and ask the individual to estimate the duration. If we repeat this experiment a week later, there is no reason to assume that the individual's attitude to a "length of time" has changed. In the evaluation of term papers, especially if they are anonymous, and in the expression of attitudes to ideologies, such "change of attitude" can be considered fairly small unless the time interval between two administrations is unusually long. We are thus restricting our attention to those cases where two different ratings given to the same stimulus or object by the same individual represent evidence of uncertainty of the individual, and that all "changes of attitude" are of negligible order of magnitude compared with this "uncertainty".

The double-presentation method described in Chapter III of this report has been tried out for several studies at the Psychometric Laboratory of the University of North Carolina. Comparison with Thurstone's successive interval scaling method showed very close agreement in all cases studied so far. It is surmised, however, that agreement between the two methods will be poorer when we are dealing with stimuli showing bimodal responses as, for instance, appreciation of modern art. In that case, the unweighted and normalized mean response for a group (as used in Thurstone's method) may differ considerably from the weighted least-squares estimate of the mean response. The use of the present method is recommended in those cases where people differ considerably in judgment ability, for in that event it would seem inappropriate to attach the same weight to a response given by an individual who is almost completely consistent and that given by another individual who may assign very different ratings to the same object on two successive occasions.

II. DEFINITIONS AND NOTATION

Given the scale $(0,1,\ldots,k)$, and a set of s stimuli to rate on this scale, an individual may, from trial to trial, differ on some or all of his ratings of the same stimulus. By attaching two or more different ratings to one and the same stimulus, if there is no reason to assume a change of attitude toward that stimulus, he gives evidence of his inability to discriminate between these points on the rating scale.

We propose a model in which this uncertainty is of the same magnitude regardless of the position of a point on the scale. This naturally gives rise to unequally spaced points. In our model we assume that a person's "interval of uncertainty" is a constant c.

Figure I.

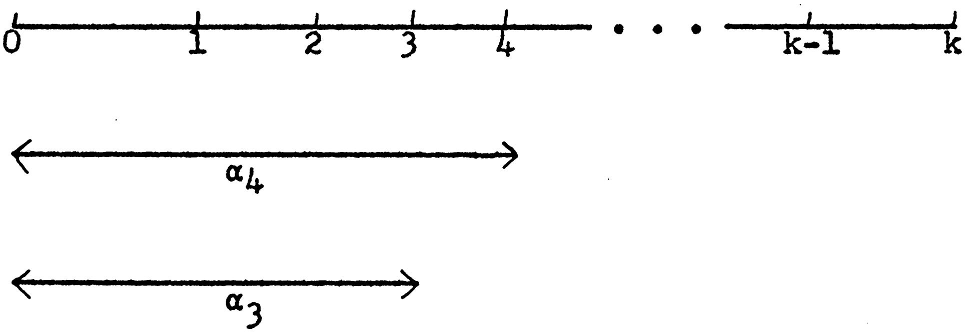


Figure I shows the configuration of such a scale. Suppose an individual gave responses 3 and 4 to the same stimulus. In accordance with our model, we assume that $\alpha_4 - \alpha_3 = c + \epsilon$, where ϵ is an error with expectation 0 and variance σ^2.

If the individual is given s stimuli to be rated on a k+1 point scale, and repeats this rating once, we may order all pairs of responses in such a way that the first is less than or equal to the second. Our complete observation equation would then be

$$X^* \underline{\alpha}^* = c\underline{j} + \underline{\epsilon} \tag{2.1}$$

where X^* has elements

$x_{\gamma i} = -1$, if response (i,j) was given to stimulus $\gamma (i<j)$
$x_{\gamma j} = \ \ 1$, if response (i,j) was given to stimulus $\gamma (i<j)$
$x_{\gamma i} = \ \ 0$, if response (i,i) was given to stimulus γ
$x_{\gamma \beta} = \ \ 0$, all other elements in γ'th row,
and is a $(s \times \overline{k+1})$ matrix.

Explanation of other notation:

$\underline{\alpha}^{*\prime} = [\alpha_0, \alpha_1, \ldots, \alpha_k]$ is a row vector whose elements denote the distance of each scale point from the origin (α_0).

$\underline{j}$, in the usual notation, is a vector which has all elements equal to unity.

$\underline{\epsilon}$ is the error vector.

For instance, if ordered responses to ten stimuli on a six point scale were

(0,1) (1,2) (2,2) (3,4) (3,3) (4,4) (2,5) (1,5) (0,3) (4,4),

the model would be

$$
\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5
\end{bmatrix}
=
\begin{bmatrix}
c \\ c \\ c \\ c \\ c \\ c \\ c \\ c \\ c \\ c
\end{bmatrix}
+
\begin{bmatrix}
\epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \\ \epsilon_7 \\ \epsilon_8 \\ \epsilon_9 \\ \epsilon_{10}
\end{bmatrix}
\qquad (2.2)
$$

For three and four administrations, the entries for the matrix X are given in Chapters IV and V.

The techniques for the construction of such scales are presented in Chapter III for two administrations, in Chapter IV for three administrations, and in Chapter V for four administrations. The extension to any number of administrations is obvious; the weights appearing in the X matrix for m administrations are the linear contrasts of the m'th order orthogonal polynomials. However, since the construction of $X^{*\prime}X^{*}$ from the tables of frequencies is rather cumbersome for $m > 4$, and since these cases occur very rarely in practice, we refrained from dealing with the general case and extended the present study only up to and including $m = 4$.

If all responses are perfectly consistent, i.e., if the rating given to a stimulus in the first administration is identical with the rating given the stimulus in subsequent administrations, for all stimuli, scale construction will not be possible. The interval of uncertainty would, in that case, be of zero length and thus all scale points would be indeterminate. Except for this situation, the procedures outlined in Chapters III, IV and V will lead to valid scales. However, unless there is some connection between all points on the scale, the solution will be zero and one for the scale points, and thus fairly meaningless. The term "connection" in this case indicates that each rating occurs at least once with each other rating either directly or indirectly. A response (3,4) would establish a direct connection between scale points 3 and 4; an example of indirect connection between scale points 3 and 4 is the occurrence of (2,3) and (2,4), or (3,5) and (4,5), etc. For example, connection would not exist if a particular rating, m, say, never occurred with any other rating,i.e., if an individual were perfectly consistent in one scale point. (In this case, the particular scale point would be indeterminate. If connection exists for all other points in such a case, the proper procedure would be to treat the point m as if it were non-existent and perform scaling on a k point scale (instead of a k + 1 point scale). The point m would then lie somewhere between (m - 1) and (m + 1), but its proper location cannot be ascertained.)

III DOUBLE PRESENTATION SCALING

3.1 Model Given Two Responses Per Stimulus

As shown in Chapter II, the model for this case is

$$X^*\underline{\alpha}^* = c\underline{j} + \underline{\varepsilon}$$

subject to two constraints

a) $\alpha_0 = 0$

b) either: $c = 1$ (i.e., we make the interval of uncertainty our unit of measurement)

or: $\alpha_k = 1$ (i.e., we standardize the overall scale length); this case can be easily reduced to the above, as shown in the sequel.

Denote by $\underline{\alpha}$ the vector $\underline{\alpha}^*$ omitting α_0 and by X the matrix X^* omitting the first column. Then, by least squares, the estimates of $\underline{\alpha}$ are given by

$$X'X\hat{\underline{\alpha}} = cX'\underline{j}. \tag{3.1}$$

Let $X'\underline{j}$ be denoted by $\underline{g}$; equation (3.1) can be stated as

$$X'X\hat{\underline{\alpha}} = c\underline{g}.$$

Since the sum of the errors must equal zero:

$$\underline{j}'X\underline{\hat{\alpha}} = c\underline{j}'\underline{j} = cs$$

therefore $$c = \frac{1}{s}\,(\underline{j}'X\underline{\hat{\alpha}}) = \frac{1}{s}\,(\underline{g}'\underline{\hat{\alpha}}) \qquad (3.2)$$

were s is the number of stimuli.

Substituting into (3.1) we obtain the homogeneous equations

$$\left[X'X - \frac{1}{s}\,(X'\ \underline{jj}'X)\right]\underline{\hat{\alpha}} = 0$$

or $$\left[X'X - \frac{1}{s}\,\underline{gg}'\right]\underline{\hat{\alpha}} = 0. \qquad (3.3)$$

We now make the constraint $\alpha_k = 1$, which implies that we drop the last row of $(X'X - \frac{1}{s}\,\underline{gg}')$ and transfer the last column of this matrix to the right hand side with opposite sign. We thus obtain a system of k - 1 non-homogeneous linear equations in $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$. With the assumed constraints, $\alpha_0 = 0$ and $\alpha_k = 1$, the solution of this system gives us the complete scale.

If we prefer the constraint c = 1, we evaluate, from (3.2), $\hat{c} = \frac{1}{s}\,(\underline{j}'X\underline{\hat{\alpha}}) = \frac{1}{s}\,(\underline{g}'\underline{\hat{\alpha}})$, including $\alpha_k = 1$, and multiply each value of the obtained $\hat{\alpha}_i$'s by $\frac{1}{\hat{c}}$, yielding $\hat{\beta}_i$ the points on the new scale. The resulting set of $\hat{\beta}_i$'s will now have c as the unit of measurement, i.e., c = 1, and scale length:

$$\hat{\beta}_k = \frac{s}{\underline{g}'\underline{\hat{\alpha}}}, \tag{3.4}$$

where $\underline{\hat{\alpha}}$ are the points on the old scale (unit length) and the β_i's are the points on the new scale (c = 1).

3.2 Construction of $X^{*'}X^{*}$ and $X^{*'}\underline{j}$, for Two Presentations

The steps for the construction of the $X^{*'}\underline{j}$ and $X^{*'}X^{*}$ matrices from observed data, given two responses to each of s stimuli $(1,2,\ldots,\gamma,\ldots s)$ on a k + 1 point scale $(0,1,\ldots k)$ are as follows:

Step 1. First order each of the paired responses from low to high, thus constructing an (s x 2) matrix whose γ'th row is the ordered double response to the γ'th stimulus. This makes the elements of the first column of the matrix equal to the lower response of each pair, and the corresponding elements in the second column the higher of the two responses.

Step 2 Next make the frequency table (F) of paired responses. The table will be of the form shown in 3.5, where all entries will be on or above the main diagonal. The number of times the i'th and j'th scale points are given as responses to the same stimulus is represented by f_{ij}.

$$F = \begin{array}{c|cccccc|c}
 & 0 & 1 & \cdot & j & \cdot & k & \\ \hline
0 & f_{00} & f_{01} & \cdot & \cdot & \cdot & f_{0k} & f_{0\cdot} \\
1 & & f_{11} & \cdot & f_{1j} & \cdot & f_{1k} & f_{1\cdot} \\
\cdot & & & \cdot & \cdot & \cdot & & \\
i & \text{blank} & & & f_{ij} & & & f_{i\cdot} \\
k & & & & & & f_{kk} & f_{k\cdot} \\ \hline
 & f_{\cdot 0} & & & f_{\cdot j} & & f_{\cdot k} &
\end{array} \qquad (3.5)$$

Step 3. Next find the row and column sums of F:

$f_{i\cdot} = \sum_{j=i}^{k} f_{ij}$. This is the sum of the i'th row of F (terms to the right of and including main diagonal of the i'th row) and represents the number of times rating (i) was given as the lower response or as both responses.

$f_{\cdot j} = \sum_{i=0}^{j} f_{ij}$. This is the sum of the j'th column of F (terms on and above main diagonal in j'th column). It represents the number of times rating (j) was the higher of the two, or both ratings were (j).

Of course, $$\sum_j f_{\cdot j} = \sum_i f_{i\cdot} = s . \qquad (3.6)$$

Step 4. Construction of $X^{*'}\underline{j}$: This will be a $(\overline{k+1} \times 1)$ column vector. The i'th term is given by $(f_{.j} - f_{i.})$ $(i = 0,1,\ldots,k)$. It is of the form

$$\begin{bmatrix} f_{.0} - f_{0.} \\ \\ f_{.i} - f_{i.} \\ f_{.k} - f_{k.} \end{bmatrix} \tag{3.7}$$

and is denoted by $\underline{g}^*$.

Step 5. Construction of $X^{*'}X^*$ (a symetric k + 1 square matrix):

Cross product (off-diagonal) terms: Both (ij)'th and (ji)'th $(i < j)$ term given by $(-f_{ij})$.

Diagonal terms: The (ii) term is given by $(f_{i.} + f_{.i} - 2f_{ii})$.

Thus,

$$X^{*'}X^* = \begin{bmatrix} (f_{0.}+f_{.0}-2f_{00})\ldots(-f_{0j})\ldots & & (-f_{0k}) \\ & (-f_{ij}) & (-f_{ik}) \\ \\ (-f_{0k}) & & (f_{k.}+f_{.k}-2f_{kk}) \end{bmatrix} \tag{3.8}$$

Proof: Since the (i)'th term of $\underline{g}^*$ is the sum of the i'th column of X^*, this term is given by the frequency of times the i'th scale point was given as the high response minus the number of times it was given as the low response. So, from F, we have $(f_{.j} - f_{i.})$.

The (ij)'th term of $X^{*\prime}X^*$ is the sum of the cross products of the i'th and j'th row of X. This is the number of times the response $(i,j)(i \neq j)$ was given. Since there will be a minus sign in the i'th column of X when (i) was the low response, and correspondingly a positive sign when (j) was the high response, the contribution to the sum of cross products of each (i,j) response is (-1). Thus the (ij)'th term of $X^{*\prime}X^*$ is $(-f_{ij})$. Because of the symmetry of $X^{*\prime}X^*$ the (ji)'th term is also $(-f_{ij})$.

The (ii) element of $X^{*\prime}X^*$ is the sum of times the response (i) is listed as the low response only (since $(-1)^2 = 1$) plus the number of times it is listed as the high response only. Since both the row sum and column sum of F include the frequency of the double response (i,i), we have the (i,i) element of $X^{*\prime}X^*$ to be $(f_{i.} + f_{.i} - 2f_{ii})$, since the contribution of an (i,i) response to the sum of squares is zero.

3.3 Procedure for Solution of Scale Points

Step 1. To solve for the scale points $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$ subject to $\alpha_0 = 0$, $\alpha_k = 1$, first construct $X^{*\prime}\underline{j}$ and $X^{*\prime}X^*$ matrices as outlined in the previous section.

Step 2. Strike out first row and first column of $X^{*\prime}X^*$ (thus applying restriction $\alpha_0 = 0$). *The remaining* matrix is $X'X(k \times k)$. Also strike out first element of $X^{*\prime}\underline{j}$ (or $\underline{g}^*$), giving $X'\underline{j}$ (or $\underline{g}$), a $(k \times 1)$ vector.

Step 3. Construct $\underline{gg}'$. This is a $(k \times k)$ matrix whose terms are the squares and cross products of all elements in $\underline{g}$.

Thus,

$$\begin{bmatrix} f_{.1} - f_{1.} \\ \vdots \\ f_{.k} - f_{k.} \end{bmatrix} \begin{bmatrix} f_{.1} - f_{1.}, \ldots, f_{.k} - f_{k.} \end{bmatrix} = \underline{gg}'$$

$$= \begin{bmatrix} (f_{.1} - f_{1.})^2 & \ldots\ldots & (f_{.1} - f_{1.})(f_{.k} - f_{k.}) \\ \vdots & & \vdots \\ (f_{.1} - f_{1.})(f_{.k} - f_{k.}) & \ldots\ldots & (f_{.k} - f_{k.})^2 \end{bmatrix}. \qquad (3.9)$$

Step 4. Strike out last row of $X'X$ and $\underline{gg}'$.

Step 5. Multiply each term of reduced $\underline{gg}'$ matrix by $\frac{1}{s}$ and subtract from corresponding term of $X'X$, to give $\left[X'X - \frac{1}{s}\,\underline{gg}'\right]$.

Step 6. Each row of the new matrix $\left[X'X - \frac{1}{s}\,\underline{gg}'\right]$ is one of (k - 1) simultaneous equations. The last term of each (i.e., the last column) is the constant term and should be transferred to the right hand side and its sign changed. This procedure introduces the constraint $\alpha_k = 1$.

Step 7. Solve the (k - 1) equations to get $\underline{\hat{\alpha}}$.

Step 8. Calculate $\hat{c} = \frac{1}{s}\,\underline{g}'\underline{\hat{\alpha}}$ (including $\alpha_k = 1$ in $\underline{\hat{\alpha}}$).

Step 9. If it is desired to have $c = 1$, divide each point of $\underline{\hat{\alpha}}$ by $\hat{c}$, thus making the scale of length $\frac{s}{\underline{g}'\underline{\hat{\alpha}}} = \hat{\beta}_k$.

IV TRIPLE PRESENTATION SCALING

4.1 Model Given Three Responses Per Stimulus

In the case of three replications, as in the case of two, we wish to construct our model so that a person's "interval of uncertainty" is of constant magnitude over all the scale. When the subject gives three ratings to each stimulus, there will be $\binom{3}{2}$ or 3 possible combinations of his ordered responses to consider as estimates of this "c".

Suppose, for instance, the subject gives the ordered responses (3,4,5) to the same stimulus. Three combinations of α_5, α_4, α_3, will be estimates of c, i.e., the differences $(\alpha_5 - \alpha_4)$, $(\alpha_4 - \alpha_3)$ and $(\alpha_5 - \alpha_3)$. Further, since there is no change in attitude toward the stimulus and we do not assume any significance in the order in which the responses were given, the three are equally valid estimates.

Accordingly, the sum of these three estimates the quantity 3c, hence

$$(\alpha_5 - \alpha_4) + (\alpha_5 - \alpha_3) + (\alpha_4 - \alpha_3) = 3c + \epsilon,$$

where ϵ is an error with expectation 0, variance σ^2 (ϵ and σ^2 of different magnitude than in the two replication case).

Thus, summing, we have

$$2\alpha_5 - 2\alpha_3 = 3c + \epsilon$$

or
$$\alpha_5 - \alpha_3 = \frac{3}{2} c + \frac{\epsilon}{2} .$$

Thus,

$$E(X^*\underline{\alpha}^*) = \frac{3}{2} c\underline{j} \tag{4.1}$$

where X^* is defined below.

Given the ordered response $(i,j,l)(i \leq j \leq l)$ to the γ'th stimulus, the elements of $\underline{X}_\gamma'$ are

$x_{\gamma i} = -1$, if i is listed as lowest response, and $i<l$.

$x_{\gamma l} = 1$, if l is listed as highest response, and $i<l$.

$x_{\gamma j} = 0$, if j is listed as middle response.

$x_{\gamma i} = x_{\gamma l} = 0$, if $i=j=l$.

$x_{\gamma m} = 0$, if m is not given as response. (4.2)

Then X^* $(s \times \overline{k+1})$ is the matrix whose γ'th row is defined as

$$\underline{x}_\gamma' = [x_{\gamma 1}, x_{\gamma 2}, \ldots, x_{\gamma i}, \ldots, x_{\gamma j}, \ldots, x_{\gamma l}, \ldots, x_{\gamma m}]$$
$$= [\ 0\ ,\ 0\ , \ldots, -1, \ldots,\ 0\ , \ldots,\ 1\ , \ldots,\ 0\ ] \tag{4.3}$$

if response (i,j,l) given $(i \leq j \leq l$ subject to $i<l)$ to γ'th stimulus, and

$\underline{x}_\gamma' = \underline{0}$, if response (i,i,i) was given.

The two necessary constraints are the same as before, namely

1) $\alpha_0 = 0$

2) $\alpha_k = 1$, or $c = 1$.

The solution for the three replication case is identical in development with the two replication case with the exception that the coefficient of c is no longer unity, but is now three-halves.

Thus,

$$X'X\underline{\alpha} = \frac{3}{2}\, cX'\underline{j}$$

and
$$\hat{c} = \frac{2}{3}\frac{1}{s}\,(\underline{j}'X\underline{\hat{\alpha}}).$$

Substituting, we have

$$\left[\hat{X'X}\underline{\hat{\alpha}} - \frac{3}{2}\cdot\frac{2}{3}\cdot\frac{1}{s}\,(\underline{j}'X\underline{\hat{\alpha}})(X'\underline{j})\right]$$

or
$$\left[\hat{X'X}\underline{\hat{\alpha}} - \frac{1}{s}\,\underline{g}'\underline{g}\right], \text{ as before.}$$

It should be noted that

$$\hat{c} = \frac{2}{3}\cdot\frac{1}{s}\,(\underline{j}'X\underline{\hat{\alpha}}). \tag{4.4}$$

If we wish to make c = 1, we divide each scale point by c, thus making the scale of length

$$\hat{\beta}_k = \frac{3}{2}\,\frac{s}{\underline{g}'\underline{\hat{\alpha}}}\,. \tag{4.5}$$

4.2 Construction of $X^{*\prime}\underline{j}$ and $X^{*\prime}X^{*}$ Matrices for Three Presentations

Step 1. Order each of the triple responses from low to high. This gives a s x 3 matrix whose γ'th row contains the ordered responses to the γ'th stimulus, and the first, second, third columns are the low, middle and high resonses, respectively.

Step 2. Strike out middle column of the above matrix (i.e., the middle response to each stimulus), thus leaving only the first column (the low responses), and the last column (the high responses).

Step 3. Working with the s x 2 matrix of step 2, proceed as in Section 3.2.

Proof: Since the coefficients for the low, middle, and high responses are -1, 0, 1, respectively, we may treat the low and high responses as two replications for the purpose of construction of $X^{*\prime}\underline{j}$ and $X^{*\prime}X^{*}$.

The other procedures in the solution of the scale points are identical with those of Section 3.3, except that

$$\hat{c} = \frac{2}{3}\,\underline{g}'\hat{\underline{a}}.$$

V QUADRUPLE PRESENTATION SCALING

5.1 Model for Four Responses Per Stimulus

When four ratings are given to each stimulus, and the ratings are ordered from low to high, we have six possible estimates. We may take these as equally valid and thus sum over the six to give an estimate of 6c.

Suppose the γ'th response given, when ordered from low to high, was (2, 3, 4, 5). Then we would have

$(\alpha_5-\alpha_4)+(\alpha_5-\alpha_3)+(\alpha_5-\alpha_2)+(\alpha_4-\alpha_3)+(\alpha_4-\alpha_2)+(\alpha_3-\alpha_2) = 6c + \varepsilon$,

or $3\alpha_5 + \alpha_4 - \alpha_3 - 3\alpha_2 = 6c + \varepsilon$.

Our complete observation matrix would then be

$$X^{*\prime}\underline{\alpha}^* = 6c\underline{j} + \underline{\varepsilon} \tag{5.1}$$

where X^* is to be defined below.

The elements of X^* are as follows:

If the four responses (i,j,l,m) are distinct, i.e., $(i<j<l<m)$, then

$$x_{\gamma i} = -3$$
$$x_{\gamma j} = -1$$
$$x_{\gamma l} = 1$$
$$x_{\gamma m} = 3$$

where γ indicates γ'th stimulus, thus γ'th row.

If the four responses are not distinct, ratings for the repeated responses are

$x_{\gamma i} = -4$, if the response was $(i,i,l,m)(i<l\leq m)$,

$x_{\gamma i} = -3$, if the response was $(i,i,i,m)(i<m)$,

$x_{\gamma i} = 0$, if the response was (i,i,i,i),

$x_{\gamma j} = 0$, if the response was $(i,j,j,m)(i<j<m)$,

$x_{\gamma j} = 3$, if the response was $(i,j,j,j)(i<j)$,

$x_{\gamma l} = 4$, if the response was $(i,j,l,l)(i\leq j<l)$.

For instance, suppose that on a five point scale, nine stimuli received the following ordered ratings: (1,2,3,4), (1,2,2,3), (0,0,0,1), (1,1,1,1), (1,1,2,3), (1,2,2,2), (1,2,3,3), (0,0,1,1), (0,0,2,3). Then

$$X^* = \begin{bmatrix} 0 & -3 & -1 & 1 & 3 \\ 0 & -3 & 0 & 3 & 0 \\ -3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 1 & 3 & 0 \\ 0 & -3 & 3 & 0 & 0 \\ 0 & -3 & -1 & 4 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ -4 & 0 & 1 & 3 & 0 \end{bmatrix} \qquad (5.2)$$

Returning to our general formula, (5.1) reduces to $X'\underline{\alpha} = 6c\underline{j} + \underline{\varepsilon}$ when we set $\alpha_0 = o$.

Then by least squares

$$X'X\hat{\underline{\alpha}} = 6cX'\underline{j} \tag{5.3}$$

and since $\Sigma\varepsilon = 0$

$$\underline{j}'X\hat{\underline{\alpha}} = 6c\underline{j}'\underline{j} = 6cs$$

therefore $$\hat{c} = \frac{1}{6s}\, j'X\hat{\underline{\alpha}}. \tag{5.4}$$

Substituting (5.4) into (5.3) we have

$$X'X - \frac{1}{s}\,\underline{gg'}\ \hat{\underline{\alpha}} = 0.$$

The solution is thus the same as in the case of two, except $\hat{c} = \frac{1}{6s}\,\underline{g}'\hat{\underline{\alpha}}$, or if $c = 1$, scale length is

$$\hat{\beta}_k = \frac{6s}{\underline{g}'\hat{\underline{\alpha}}}\ .$$

5.2 Construction of $X^{*\prime}\underline{j}$ and $X^{*\prime}X^{*}$ for Four Presentations

Stepl. First order the four responses to each stimulus from low to high. List in matrix form. The γ'th row of this matrix will be the ordered response to the γ'th stimulus. The columns will be the low, second low, second high, and high responses, respectively.

Step 2. Construction of frequency tables. Six frequency tables are required, which contain the tallies of each pair of ratings in each pair of presentations.

The tables are:

Table one: the frequencies of paired ratings occurring as low and second low ratings, i.e., frequencies by pairs in first and second columns of the matrix of step 1.

Table two: frequencies by pairs in first and third columns.

Table three: frequencies by pairs in first and fourth columns.

Table four: frequencies by pairs in columns two and three.

Table five: frequencies by pairs in columns two and four.

Table six: frequencies by pairs in columns three and four.

All entries in the tables are on or above the main diagonal. The superscript (t) refers to the number of the table, e.g., $F^{(1)}$ is the first table, (frequencies in columns one and two).

$$F^{(t)} = \begin{array}{c|ccccc|c} & 0 & \cdots & j & \cdots & k & \\ \hline 0 & f_{00}^{(t)} & & f_{0j}^{(t)} & & f_{0k}^{(t)} & f_{0.}^{(t)} \\ 1 & & & f_{1j}^{(t)} & \cdots & f_{1k}^{(t)} & f_{1.}^{(t)} \\ \cdot & & & & & \cdot & \\ i & & & f_{ij}^{(t)} & & f_{ik}^{(t)} & f_{i.}^{(t)} \\ \cdot & & & \cdot & & \cdot & \\ \cdot & & & & & \cdot & \\ k & & & & & f_{kk}^{(t)} & f_{k.}^{(t)} \\ \hline & & & f_{.j}^{(t)} & & f_{.k}^{(t)} & \end{array}$$

Step 3. Find for each of the six tables the quantities:

$f_{.j}^{(t)}$ — the sum of the j'th column of $F^{(t)} = \sum_{i=0}^{j} f_{ij}^{(t)}$

$f_{i.}^{(t)}$ — the sum of the i'th row of $F^{(t)} = \sum_{j=i}^{k} f_{ij}^{(t)}$.

Check for accuracy by $\sum_j f_{.j}^{(t)} = \sum_i f_{i.}^{(t)} = s$

$f_{i.}^{(1)} = f_{i.}^{(2)} = f_{i.}^{(3)}$ $f_{.j}^{(2)} = f_{.j}^{(4)} = f_{i.}^{(6)}$

$f_{.j}^{(1)} = f_{.j}^{(4)} = f_{.j}^{(5)}$ $f_{.j}^{(3)} = f_{.j}^{(5)} = f_{.j}^{(6)}$

Step 4. The i'th element of $\underline{g}^*$ is then

$$-3f_{i.}^{(1)} - f_{.i}^{(1)} + f_{i.}^{(6)} + 3f_{.i}^{(6)} \qquad (i = 0,1,\ldots k)$$

Step 5. To find the cross product terms of $X^{*'}X^*$, calculate, for the $(ij)'$th $(i<j)$ term the quantity

$$3f_{ij}^{(1)} - 3f_{ij}^{(2)} - 9f_{ij}^{(3)} - f_{ij}^{(4)} - 3f_{ij}^{(5)} + 3f_{ij}^{(6)}.$$

Because of the symmetry of the matrix the $(ji)'$th $(i<j)$ element is identical with the $(ij)'$th element given above.

Step 6. To find diagonal terms of $X^{*'}X^*$, find, for the (ii) term

$$(9f_{i.}^{(1)} + f_{.i}^{(1)} + f_{i.}^{(6)} + 9f_{.i}^{(6)})$$
$$+ 2(3f_{ii}^{(1)} - 3f_{ii}^{(2)} - 9f_{ii}^{(3)} - f_{ii}^{(4)} - 3f_{ii}^{(5)}$$
$$+ 3f_{ii}^{(6)}) .$$

The remaining procedures in the solution of the scale points are identical with those of Section 3.3, except that

$$\hat{c} = \frac{1}{6s} \underline{g}'\underline{\hat{\alpha}} .$$

VI COMBINATION AND COMPARISON OF SCALES

In Chapter I, two alternative solutions were proposed for the construction of the scale:

A: Set $\alpha_0 = 0$ and $\alpha_k = 1$, i.e., make the length of the scale equal to unity.

B: Set $\beta_0 = 0$ and $c = 1$, thus obtain a scale $\beta_0 = 0$, $\hat{\beta}_i = \hat{\alpha}_i/\hat{c}$, where $\hat{c}$ is the estimate of the length of the interval of uncertainty given by (3.2) for two administrations, by (4.5) for three administrations, and by (5.4) for four administrations.

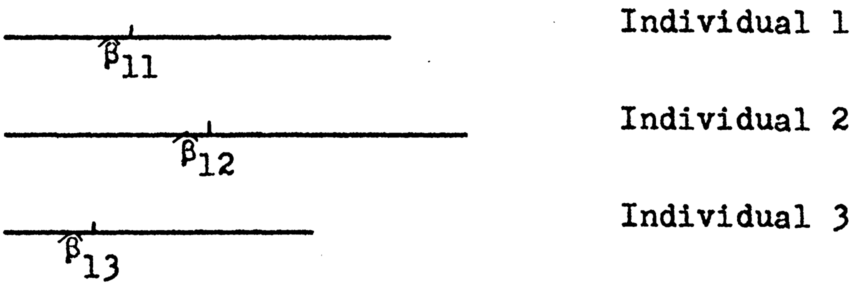
Suppose we have a scale of the type A for a group of n individuals. If, for a scale for the whole group, we took the unweighted mean of any given scale point over all individuals, we would attach the same weight to each person's responses. However, a small value of c for an individual indicates consistency and a high discriminatory ability of that individual, whereas a larger value of c indicates poor discrimination. It would thus seem reasonable to weigh each person's $\hat{\alpha}_i$ $(i = 1, 2, \ldots, k)$ by $1/\hat{c}$ before averaging.

Let $\hat{\alpha}_{ij}$ be the value obtained for the j'th individual on rating i. Then weighing by $1/\hat{c}_j$ just results in

$$1/c_j \, \alpha_{ij} = \beta_{ij} , \tag{6.1}$$

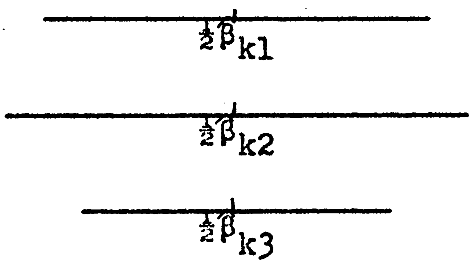
the corresponding point on the type B scale. There is one drawback to this weighing, for the scales would now be combined as indicated in the following graph:

Illustration 6.1



Thus, the scales would be compared at the lower end ($\beta_{0j} = 0$ for all j). A more reasonable way of comparison would be the following:

Illustration 6.2



For the purpose of scale combination, i.e., development of a group scale, the following alternative solution is recommended:

C: Define $\hat{\gamma}_i = \hat{\beta}_i - \frac{1}{2}\hat{\beta}_k$,

and obtain the i'th point on the group scale as

$$\bar{\gamma}_i = \frac{1}{n} \sum_{j=1}^{n} \hat{\gamma}_{ij}, \qquad (6.2)$$

where n = number of individuals.

This type C scale is, of course, equivalent to the type B scale for an individual, the only change being a new definition of the origin, which is arbitrary and, in the type C scale, is placed at the midpoint of the entire scale.

VII TESTS OF CONSISTENCY AND ELIMINATION OF BIAS

The scaling technique described in Chapter I presupposes the absence of bias (see Introduction) between administrations. Suppose object i was rated m in the first and n in the second administration. Then, we would assume

$$E(\hat{\beta}_m - \hat{\beta}_n) = 0 \tag{7.1}$$

Let

$$d_i = \hat{\beta}_m - \hat{\beta}_n \tag{7.2}$$

and

$$\delta_i = \beta_m - \beta_n + \epsilon_i \tag{7.3}$$

where ϵ_i is now assumed to be normally distributed with expectation 0 and variance σ^2. To test consistency, we would test the hypothesis:

$$H_o: \ \delta_i = 0. \tag{7.4}$$

The statistic for this test is, of course,

$$t = \frac{\bar{d}\sqrt{s}}{\left[\frac{1}{s-1}\sum_i (d_i - \bar{d})^2\right]^{\frac{1}{2}}}, \tag{7.5}$$

which, under H_o, is distributed as Student's t with s-1 d.f., where

$$\bar{d} = \frac{1}{s} \sum_{i=1}^{s} d_i, \qquad (7.6)$$

and s = number of stimuli.

For three replications, suppose that the i'th response was (m,n,o) for the first, second and third administrations, respectively. We can now form the two differences:

$$d_{1i} = \hat{\beta}_m - \hat{\beta}_n$$

and

$$d_{2i} = \hat{\beta}_m - \hat{\beta}_o.$$

We would then have to make the bivariate test

$$E(d_{1i}) = E(d_{2i}) = 0. \qquad (7.7)$$

The appropiate statistic here is Hotelling's T^2 which, as is well known, is equivalent to an F statistic. With a slight modification of Hotelling's T^2 we shall proceed directly to the F statistic and define:

$$\bar{d}_1 = \frac{1}{s} \sum_{i=1}^{s} d_{1i}$$

$$\bar{d}_2 = \frac{1}{s} \sum_{i=1}^{s} d_{2i}$$

$$\underline{\bar{d}}' = (\bar{d}_1, \bar{d}_2), \text{ i.e., a row vector,}$$

$$\underline{\bar{d}} = \text{the corresponding column vector}$$

$$v_1 = \sum_i d_{1i}^{\,2} - s\bar{d}_1^{\,2}$$

$$v_2 = \sum_i d_{2i}^{\,2} - s\bar{d}_2^{\,2}$$

$$c_{12} = \Sigma\, d_{1i}\, d_{2i} \,--s\bar{d}_1\, \bar{d}_2$$

and

$$S = \frac{1}{s-2}\begin{bmatrix} v_1 & c_{12} \\ c_{12} & v_2 \end{bmatrix} \tag{7.8}$$

(this latter S being different from Hotelling's definition, where the multiplier is $\frac{1}{(s-1)}$. With this definition:

$$F = \frac{s}{2}\, \underline{\bar{d}}'\, S^{-1}\, \underline{\bar{d}} \tag{7.9}$$

is distributed as F with (2,s-2) d.f. under the null hypothesis (7.7).

Finally, in four replications, let the i'th response be (m,n,o,p); then

$$d_{1i} = \hat{\beta}_m - \hat{\beta}_n$$

$$d_{2i} = \hat{\beta}_m - \hat{\beta}_o$$

$$d_{3i} = \hat{\beta}_m - \hat{\beta}_p \tag{7.10}$$

The null hypothesis of absence of bias:

$$E(d_{1i}) = E(d_{2i}) = E(d_{3i}) = 0 \tag{7.11}$$

can be tested analogously to (7.9), with the following changes:

$$\bar{d}_1 = \frac{1}{s}\sum_i d_{1i},\ \bar{d}_2 = \frac{1}{s}\sum_i d_{2i},\ \bar{d}_3 = \frac{1}{s}\sum_i d_{3i}$$

$$\underline{\bar{d}}' = (\bar{d}_1,\ \bar{d}_2,\ \bar{d}_3)$$

v_1, v_2, and c_{12} as in (7.8) and

$$c_{13} = \Sigma d_{1i}d_{3i} - s\bar{d}_1\bar{d}_3; c_{23} = \Sigma d_{2i}d_{3i} - s\bar{d}_2\bar{d}_3;$$

$$v_3 = \sum_j d_{3i}{}^2 - s\bar{d}_3{}^2$$

and

$$S = \frac{1}{s-3}\begin{bmatrix} v_1 & c_{12} & c_{13} \\ c_{12} & v_2 & c_{23} \\ c_{13} & c_{23} & v_3 \end{bmatrix}. \tag{7.12}$$

With this definition, under the null hypothesis,

$$F = \frac{s}{3}\,\underline{\bar{d}}'\ S^{-1}\underline{\bar{d}} \tag{7.13}$$

is distributed as F with (3,s-3) d.f. This testing procedure is illustrated in Numerical Example No. 3b.

If the test (7.5), (7.9), or (7.13) (whichever applies) shows significant departure from the null hypothesis, we cannot retain the assumption of no bias. The scaling up to this point could then be considered as preliminary only, and a correction for bias will have to be made before an appropiate scale can be constructed. The following correction for bias is recommended at this time:

Suppose the mean differences are $\bar{d}_1$, $\bar{d}_2$, and $\bar{d}_3$, then, in order to compare ratings in the second, third and fourth administrations with those of the first, the preliminary values $\hat{\beta}_m$ for scale points (m) must be replaced by $\hat{\beta}_m + \bar{d}_1$, for the second, $\hat{\beta}_m + \bar{d}_2$, for the third and $\hat{\beta}_m + \bar{d}_3$, for the fourth administration.

However, $\hat{\beta}_m + \bar{d}_j$ $(j=1,2,3)$ will usually correspond to a point between m and (m+1) or m and (m-1). Since the scaling technique applies to discrete points only, the following randomization technique is proposed:

Suppose $\hat{\beta}_m + \bar{d}_1$ $(m=0,1,2,\ldots,k-1)$ produces a value between $\hat{\beta}_m$ and $\hat{\beta}_{m+1}$. We may then compute

$$p_m = \frac{\bar{d}_1}{\hat{\beta}_{m+1} - \hat{\beta}_m} \quad (m=0,1,\ldots,k-1) \qquad (7.14)$$

and perform a random experiment with probability of success equal to p_m. For instance, if $p_m = .24$, we may take a table of random numbers, define the occurrence of a number between 42 and 65, say, as "success". We then assign to each response m in the second administration a number in a preassigned row or column of a table of two-digit random numbers. If any of these numbers is between 42 and 65, we will change that rating m to m+1; if not, we will leave it unchanged. After making these "randomized" corrections we may then construct a scale based upon corrected responses. This procedure is valid only if the bias is assumed to affect all responses and not if it affects only some stimuli. If bias is detected, rescaling can be performed by this method on the asumption that the individual tended to give consistently higher (or lower) ratings in successive administrations. It is illustrated in Numerical Example No. 3c.

VII NUMERICAL EXAMPLES

For purposes of illustration, examples are given for two, three and four presentations. The subjects were given a list of fifty foods and were asked to assign ratings to each of these stimuli. A seven point scale was used. Foods such as broiled steak were usually given a rating of six, denoting the subjects' fondness for the food. Liver was rated (0,0,0) by one subject who greatly disliked it, and was rated (2,3,2,3) by another subject, who was neither particularly fond of, nor who particularly disliked liver. Vegetables were usually assigned ratings near the middle of the scale, and more inconsistencies were noted in the rating of such foods than those which were assigned ratings at one extreme of the scale or the other.

Example 1. Double Presentation. The following double responses to the fifty stimuli were given. They are listed in order of presentation.

(6,6) (6,6) (0,0) (3,3) (1,1) (0,0) (2,2) (1,1) (3,3) (3,2)
(6,6) (4,4) (0,0) (6,5) (5,5) (0,1) (4,3) (3,3) (3,3) (6,6)
(6,6) (3,2) (6,6) (4,3) (1,2) (3,2) (6,6) (5,6) (4,3) (3,2)
(6,6) (3,3) (0,0) (4,4) (5,4) (3,3) (5,5) (5,5) (5,4) (4,4)
(5,5) (3,3) (6,6) (4,3) (3,3) (2,2) (2,2) (2,3) (3,3) (4,4)

In accordance with step one, the two responses to each stimulus are ordered from low to high. This facilitates the construction of F.

6,6	6,6	0,0	3,3	1,1	0,0	2,2	1,1	3,3	2,3
6,6	4,4	0,0	5,6	5,5	0,1	3,4	3,3	3,3	6,6
6,6	2,3	6,6	3,4	1,2	2,3	6,6	5,6	3,4	2,3
6,6	3,3	0,0	4,4	4,5	3,3	5,5	5,5	4,5	4,4
5,5	3,3	6,6	3,4	3,3	2,2	2,2	2,3	3,3	4,4

The frequency table of responses, F, (step 2) with row and column sums (step 3) is

	0	1	2	3	4	5	6	
0	4	1	0	0	0	0	0	5
1		2	1	0	0	0	0	3
2			3	5	0	0	0	8
3				9	4	0	0	13
4					4	2	0	6
5						4	2	6
6							9	9
	4	3	4	14	8	6	11	

= F.

We are now able to construct g^* (step 5). For instance, the last term is the sum of the last column minus the sum of the last row. This is (11-9)=2. The complete g^* vector is

$$g^* = [\,-1,\ 0,\ -4,\ 1,\ 2,\ 0,\ 2\,].$$

In the construction of $X^{*\prime}X^{*}$, the terms above the diagonal are the negatives of the corresponding entries in F, and those below the diagonal are symetric with those above; for instance, the (0,1) and (1,0) terms are both (—1), since the (0,1) term of F is one. The diagonal terms are equal to $[\ f_{\cdot i} + f_{j\cdot} - 2f_{ii}\ ]$. For element (0,0) this is [4+5-2(4)].

For this example the matrix is

$$X^{*\prime}X^{*} = \left|\begin{array}{c|cccccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 6 & -5 & 0 & 0 & 0 \\ 0 & 0 & -5 & 9 & -4 & 0 & 0 \\ 0 & 0 & 0 & -4 & 6 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & -2 & 2 \end{array}\right|$$

where the smaller matrix is $X'X$ (step 2 of Section3.3).
Also omit the first term of g^{*} to get

$g' = [\ 0,\ -4,\ 1,\ 2,\ 0,\ 2\ ]$.

The reduction in $X^{*\prime}X^{*}$ and g^{*} results in applying the constraint $\alpha_0 = 0$.

The next step gives

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 16 & -4 & -8 & 0 & -8 \\ 0 & -4 & 1 & 2 & 0 & 2 \\ 0 & -8 & 2 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -8 & 2 & 4 & 0 & 4 \end{bmatrix}$$

Since $s = 50$, we subtract from each element of $X'X$ (except in the last row, which we now omit altogether, according to step 4) the corresponding element of $\frac{1}{50}\underline{gg}'$ (step 5). Thus $[X'X - \frac{1}{50}\underline{gg}']$ is

$$\begin{bmatrix} 2 & -1.00 & 0 & 0 & 0 & 0 \\ -1 & 5.68 & -4.92 & .16 & 0 & .16 \\ 0 & -4.92 & 8.98 & -4.04 & 0 & -.04 \\ 0 & .16 & -4.04 & 5.92 & -2 & -.08 \\ 0 & 0 & 0 & -2.00 & 4 & -2.00 \end{bmatrix}$$

Transposing the last column of the above to the right hand side (which with the omission of the last row above, applies constraint $\alpha_k = 1$) results in the simultaneous equation (step 6) to be solved for $\hat{\underline{\alpha}}$. The set of equations is

$$
\begin{array}{lllllll}
2\hat{\alpha}_1 & -1.00\hat{\alpha}_2 & & & & = & .00 \\
-1\hat{\alpha}_1 & +5.68\hat{\alpha}_2 & -4.92\hat{\alpha}_3 & +.16\hat{\alpha}_4 & & = & -.16 \\
 & -4.92\hat{\alpha}_2 & +8.98\hat{\alpha}_3 & -4.04\hat{\alpha}_4 & & = & .04 \\
 & .16\hat{\alpha}_2 & -4.04\hat{\alpha}_3 & +5.92\hat{\alpha}_4 & -2\hat{\alpha}_5 & = & .08 \\
 & & & -2.00\hat{\alpha}_4 & +4\hat{\alpha}_5 & = & 2.00
\end{array}
$$

Solving, we find

$$\underline{\hat{\alpha}}^{*\prime} = [0, .26, .52, .61, .70, .85, 1.00]$$

($\alpha_0=0$ and $\alpha_k=1$ were the constraints applied in solving for the other points).

Now, we calculate $\hat{c}$ (step 8). This is found by

$$\frac{1}{50}[0, -4, 1, 2, 0, 2] \begin{vmatrix} .26 \\ .52 \\ .61 \\ .70 \\ .85 \\ 1.00 \end{vmatrix} = .05$$

or, if we desire $c = 1$ (step 9), devide $\underline{\hat{\alpha}}^{*\prime}$ by .05 to get

$$\underline{\hat{\beta}}^{*\prime} = [0, 5.20, 10.40, 12.20, 14.00, 17.00, 20.00]$$

Example 2. <u>Three presentation study</u>. Data for this example, with responses listed as given, is

(6,6,6) (4,3,4) (3,3,3) (4,4,3) (6,6,6) (4,3,1) (0,0,0)
(5,4,4) (4,4,4) (3,3,2) (4,3,4) (3,3,3) (1,1,0) (1,1,3)
(4,4,3) (3,2,2) (4,4,4) (1,1,1) (3,4,4) (5,5,6)
(4,4,3) (4,3,3) (4,4,3) (1,3,2) (4,4,4) (5,5,6)
(4,4,4) (4,3,4) (3,3,4) (2,4,4) (4,5,4) (1,2,2)
(4,3,3) (3,3,3) (0,3,1) (0,2,2) (4,4,3) (0,0,0)
(5,4,5) (6,5,6) (4,3,1) (3,3,3) (5,5,5) (5,4,4)
(5,4,5) (0,1,0) (3,4,4) (4,4,3) (4,5,4) (0,0,0)

Next we record the lowest and highest response given to each stimulus, in that order:

(6,6) (3,4) (2,3) (0,1) (3,4) (3,4) (3,3) (4,4) (1,4) (0,0)
(4,5) (4,5) (3,4) (3,3) (0,3) (1,1) (3,4) (4,5) (0,1) (4,5)
(3,4) (4,5) (3,4) (2,3) (1,4) (1,3) (6,6) (3,4) (5,6) (0,0)
(3,4) (3,4) (3,3) (4,4) (3,4) (2,4) (3,3) (5,5) (5,6) (0,0)
(4,4) (4,4) (5,6) (3,4) (3,4) (0,2) (3,4) (4,5) (1,2) (1,3)

The F table, with row and column totals, is

	0	1	2	3	4	5	6	
0	3	2	1	1	0	0	0	7
1		1	1	2	2	0	0	6
2			0	2	1	0	0	3
3				4	14	0	0	18
4					4	6	0	10
5						1	3	4
6							2	2
	3	3	2	9	21	7	5	

The procedures for construction of $g^{*\prime}$ and $X^{*\prime}X^{*}$ are identical with the two presentation case. For this example these are:

$g^{*\prime}$ = [-4, -3, -1, -9, 11, 3, 3], and

$$X^{*\prime}X^{*} = \begin{bmatrix} 4 & -2 & -1 & -1 & 0 & 0 & 0 \\ -2 & 7 & -1 & -2 & -2 & 0 & 0 \\ -1 & -1 & 5 & -2 & -1 & 0 & 0 \\ -1 & -2 & -2 & 19 & -14 & 0 & 0 \\ 0 & -2 & -1 & -14 & 23 & -6 & 0 \\ 0 & 0 & 0 & 0 & -6 & 9 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Subtracting $\frac{1}{50}\,\underline{g}\underline{g}'$ from $X'X$, we obtain the set of equations:

$$\begin{array}{rrrrrcr}
6.82\hat{\alpha}_1 & -1.09\hat{\alpha}_2 & -2.54\hat{\alpha}_3 & -1.34\hat{\alpha}_4 & +.18\hat{\alpha}_5 & = & -.18 \\
-1.09\hat{\alpha}_1 & +4.98\hat{\alpha}_2 & -2.18\hat{\alpha}_3 & -.78\hat{\alpha}_4 & +.06\hat{\alpha}_5 & = & -.06 \\
-2.54\hat{\alpha}_1 & -2.18\hat{\alpha}_2 & +17.38\hat{\alpha}_3 & -12.02\hat{\alpha}_4 & +.54\hat{\alpha}_5 & = & -.54 \\
-1.34\hat{\alpha}_1 & -.78\hat{\alpha}_2 & -12.02\hat{\alpha}_3 & +20.56\hat{\alpha}_4 & -6.66\hat{\alpha}_5 & = & +.66 \\
.18\hat{\alpha}_1 & +.06\hat{\alpha}_2 & +.54\hat{\alpha}_3 & -6.66\hat{\alpha}_4 & +8.82\hat{\alpha}_5 & = & +3.18
\end{array}$$

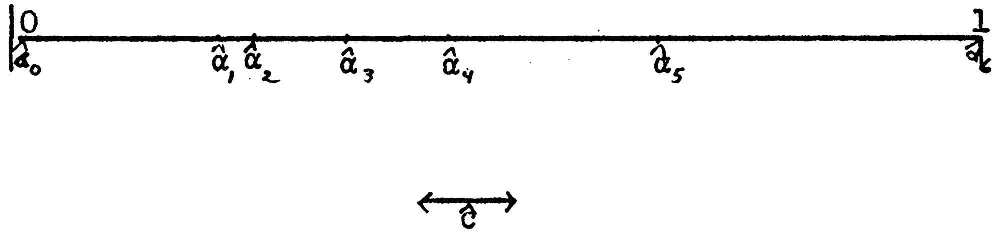
Solving, we find $\underline{\hat{\alpha}}^* = [0, .22, .26, .35, .49, .70, 1.00]$

and $\hat{c} = \frac{2}{3\times 50}[-3, -1, -9, 11, 3, 3] \begin{vmatrix} .22 \\ .26 \\ .35 \\ .49 \\ .70 \\ 1.00 \end{vmatrix} = .09$

or if we desire $c = 1$,

$$\underline{\hat{\alpha}}^* = [0, 2.44, 2.89, 3.89, 5.44, 7.78, 11.11].$$

The configuration of the type A scale is



Example 3a. Four presentation study--solution of $\underline{a}$. For purposes of comparison, two additional responses from the same individual were added to the two response study. These four responses to each ofthe fifty stimuli were

(6,6,6,6)	(0,0,0,0)	(1,1,0,0)	(2,2,1,2)	(3,3,3,3)
(6,6,5,6)	(0,0,0,0)	(5,5,4,5)	(4,3,4,2)	(3,3,2,3)
(6,6,5,5)	(6,6,6,6)	(1,2,2,3)	(6,6,5,6)	(4,3,4,4)
(6,6,6,6)	(0,0,0,0)	(5,4,4,4)	(5,5,5,5)	(5,4,4,4)
(5,5,4,5)	(6,6,5,5)	(3,3,3,4)	(2,2,2,3)	(3,3,3,2)
(6,6,4,5)	(3,3,4,5)	(0,0,0,0)	(1,1,1,1)	(3,2,2,2)
(4,4,4,4)	(6,5,5,6)	(0,1,1,1)	(3,3,2,3)	(6,6,6,6)
(3,2,3,2)	(4,3,3,4)	(3,2,2,1)	(5,6,4,5)	(3,2,3,2)
(3,3,2,3)	(4,4,4,4)	(3,3,3,3)	(5,5,3,5)	(4,4,3,3)
(3,3,2,3)	(4,3,3,3)	(2,2,2,2)	(2,3,2,3)	(4,4,4,3)

The ordered responses to each stimulus are

(6,6,6,6)	(0,0,0,0)	(0,0,1,1)	(1,2,2,2)	(3,3,3,3)
(5,6,6,6)	(0,0,0,0)	(4,5,5,5)	(2,3,4,4)	(2,3,3,3)
(5,5,6,6)	(6,6,6,6)	(1,2,2,3)	(5,6,6,6)	(3,4,4,4)
(6,6,6,6)	(0,0,0,0)	(4,4,4,5)	(5,5,5,5)	(4,4,4,5)
(4,5,5,5)	(5,5,6,6)	(3,3,3,4)	(2,2,2,3)	(2,3,3,3)
(4,5,5,6)	(3,3,4,5)	(0,0,0,0)	(1,1,1,1)	(2,2,2,3)
(4,4,4,4)	(5,5,6,6)	(0,1,1,1)	(2,3,3,3)	(6,6,6,6)
(2,2,3,3)	(3,3,4,4)	(1,2,2,3)	(4,5,5,6)	(2,2,3,3)

(2,3,3,3) (4,4,4,4) (3,3,3,3) (3,5,5,5) (3,3,4,4)

(2,3,3,3) (3,3,3,4) (2,2,2,2) (2,2,3,3) (3,4,4,4).

The six frequency tables (step 2) and their marginal totals (step 3) are now constructed. For instance, $F^{(1)}$ is the frequency table of the lowest responses.

$F^{(1)}$ =

	0	1	2	3	4	5	6	
0	5	1	0	0	0	0	0	6
1		1	3	0	0	0	0	4
2			6	6	0	0	0	12
3				7	2	1	0	10
4					4	4	0	8
5						4	2	6
6							4	4
	5	2	9	13	6	9	6	

$F^{(2)} =$

	0	1	2	3	4	5	6	
0	4	2	0	0	0	0	0	6
1		1	3	0	0	0	0	4
2			3	8	1	0	0	12
3				4	5	1	0	10
4					4	3	1	8
5						1	5	6
6							4	4
	4	3	6	12	10	5	10	

$F^{(3)} =$

	0	1	2	3	4	5	6	
0	4	2	0	0	0	0	0	6
1		1	1	2	0	0	0	4
2			1	10	1	0	0	12
3				2	6	2	0	10
4					2	4	2	8
5						1	5	6
6							4	4
	4	3	2	14	9	7	11	

$F^{(4)} =$

	0	1	2	3	4	5	6	
0	4	1	0	0	0	0	0	5
1		2	0	0	0	0	0	2
2			6	3	0	0	0	9
3				9	4	0	0	13
4					6	0	0	6
5						5	4	9
6							6	6
	4	3	6	12	10	5	10	

$F^{(5)} =$

	0	1	2	3	4	5	6	
0	4	1	0	0	0	0	0	5
1		2	0	0	0	0	0	2
2			2	7	0	0	0	9
3				7	5	1	0	13
4					4	2	0	6
5						4	5	9
6							6	6
	4	3	2	14	9	7	11	

$$F^{(6)} = \begin{array}{c|ccccccc|c}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \\
\hline
0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
1 & & 3 & 0 & 0 & 0 & 0 & 0 & 3 \\
2 & & & 2 & 4 & 0 & 0 & 0 & 6 \\
3 & & & & 10 & 2 & 0 & 0 & 12 \\
4 & & & & & 7 & 3 & 0 & 10 \\
5 & & & & & & 4 & 1 & 5 \\
6 & & & & & & & 10 & 10 \\
\hline
 & 4 & 3 & 2 & 14 & 9 & 7 & 11 &
\end{array}$$

Next we find the elements of $\underline{g}^{*\prime}$ by step 3. For example the last element is found by

$$-3f_{6.}^{(1)} \quad -f_{.6}^{(1)} \quad +f_{6.}^{(6)} \quad +3f_{.6}^{(6)}$$

$$= \quad -3(4) \quad -(6) \quad +(10) \quad +3(11) \quad = 25.$$

$$\underline{g}^{*\prime} = [-7, -2, -33, 11, 7, 1, 25]$$

Now proceding to step 5, we find the elements of $X'X$. For instance, the (0,1) element is

$$(X'X)_{01} = 3f_{01}^{(1)} \quad -3f_{01}^{(2)} \quad -9f_{01}^{(3)} \quad -f_{01}^{(4)} \quad -3f_{01}^{(5)} \quad +3f_{01}^{(6)}$$

$$= 3(1) \quad -3(2) \quad -9(2) \quad -(1) \quad -3(1) \quad +3(0)$$

$$= -25$$

The diagonal term (0,0) is found by

$$9f_{0.}^{(1)} + f_{.0}^{(1)} + f_{0.}^{(6)} + 9f_{.0}^{(6)}$$

$$+2[3f_{00}^{(1)} - 3f_{00}^{(2)} - 9f_{00}^{(3)} - f_{00}^{(4)} - 3f_{00}^{(5)} + 3f_{00}^{(6)}]$$

$$= 9(6) + (5) + (4) + 9(4)$$

$$+2[3(5) - 3(4) - 9(4) - (4) - 3(4) + 3(4)] = 25.$$

so

$$X^{*'}X^{*} = \left[\begin{array}{c|cccccc}
25 & -25 & 0 & 0 & 0 & 0 & 0 \\ \cline{2-7}
0 & 52 & -9 & -18 & 0 & 0 & 0 \\
0 & -9 & 129 & -108 & -12 & 0 & 0 \\
0 & -18 & -108 & 223 & -76 & -21 & 0 \\
0 & 0 & -12 & -76 & 139 & -30 & -21 \\
0 & 0 & 0 & -21 & -30 & 121 & -70 \\
0 & 0 & 0 & 0 & -21 & -70 & 91
\end{array}\right]$$

The reduced matrix is $X'X$ (step 2 in the procedure for solution of points).

Taking $[X'X - \frac{1}{s}\,\underline{g}\underline{g}']$, omitting last row and transposing last column to right side, thus imposing the restriction $\alpha_k = 1$, we get the simultaneous equations:

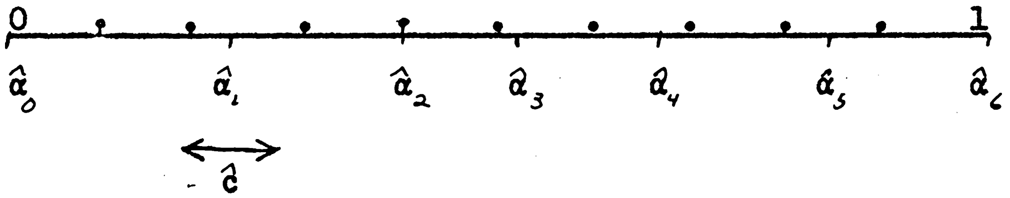
$$
\begin{aligned}
51.92\hat{\alpha}_1 - 10.32\hat{\alpha}_2 - 17.56\hat{\alpha}_3 + .28\hat{\alpha}_4 + .02\hat{\alpha}_5 &= -1.00\\
-10.32\hat{\alpha}_1 + 107.22\hat{\alpha}_2 - 100.74\hat{\alpha}_3 - 7.38\hat{\alpha}_4 + .66\hat{\alpha}_5 &= -16.50\\
-17.56\hat{\alpha}_1 - 100.74\hat{\alpha}_2 + 230.26\hat{\alpha}_3 - 77.54\hat{\alpha}_4 - 21.22\hat{\alpha}_5 &= 5.50\\
.28\hat{\alpha}_1 - 7.38\hat{\alpha}_2 - 77.54\hat{\alpha}_3 + 138.02\hat{\alpha}_4 - 30.07\hat{\alpha}_5 &= 24.50\\
.02\hat{\alpha}_1 + .66\hat{\alpha}_2 - 21.22\hat{\alpha}_3 - 30.07\hat{\alpha}_4 + 120.99\hat{\alpha}_5 &= 70.50
\end{aligned}
$$

Solving these, we find

$$\hat{\underline{\alpha}}^{*\prime} = [0, .23, .40, .52, .67, .84, 1.00]$$

and $\hat{c} = \frac{1}{6(50)} \underline{g}' \hat{\underline{\alpha}} = .08$

The configuration of the A scale is



Or if we desire c = 1, then

$$\hat{\underline{\beta}}^{*\prime} = [0, 3.06, 5.25, 6.68, 8.89, 11.09, 13.22].$$

Example 3b. Test for bias. In order to test for the presence of bias in the four presentation case we need, for each stimulus, the quantities $d_1 = \hat{\beta}_1 - \hat{\beta}_2$, $d_2 = \hat{\beta}_1 - \hat{\beta}_3$, $d_3 = \hat{\beta}_1 - \hat{\beta}_4$, where the subscripts (1,2,3,4) refer to the first, second, third and fourth presentations, respectively. The d_i's are given below.

s	d_1	d_2	d_3	s	d_1	d_2	d_3
1	0	0	0	16	0	-2.03	-4.23
2	0	2.13	0	17	2.13	2.13	0
3	0	2.13	2.13	18	2.03	2.03	0
4	0	0	0	19	0	0	0
5	0	2.20	0	20	2.03	2.03	2.03
6	0	4.33	2.13	21	0	3.06	3.06
7	0	0	0	22	0	2.20	0
8	1.61	0	1.61	23	-2.19	-2.19	-3.80
9	0	1.61	0	24	2.20	2.20	2.20
10	0	1.61	0	25	0	0	-2.03
11	0	0	0	26	0	0	0
12	0	0	0	27	3.06	-3.06	-3.06
13	0	0	0	28	1.61	1.61	3.80
14	0	0	0	29	0	0	0
15	0	2.13	2.13	30	0	0	0
				31	0	2.19	0

s	d_1	d_2	d_3
32	2.03	0	3.64
33	0	2.13	0
34	0	0	0
35	0	0	-1.61
36	0	0	0
37	0	1.61	0
38	-2.13	2.20	0
39	0	4.23	0
40	-1.61	0	-1.61

s	d_1	d_2	d_3
41	0	0	0
42	0	1.61	0
43	2.03	0	0
44	2.20	2.20	2.20
45	0	0	1.61
46	1.61	1.61	1.61
47	0	0	0
48	1.61	0	1.61
49	0	2.03	2.03
50	0	0	2.03

In order to test for bias we find the following, from 7.13.

$$\bar{d}_1 = .2420, \qquad \bar{d}_2 = .8786, \qquad \bar{d}_3 = .3496,$$

and
$$S = \frac{1}{47}\begin{bmatrix} 59.4294 & 26.4855 & 51.1319 \\ 26.4855 & 105.2850 & 65.2068 \\ 51.1319 & 65.2068 & 127.9656 \end{bmatrix}$$

where S is the variance-covariance matrix of d_1, d_2, d_3.

$$S^{-1} = \frac{47}{359,577.42}\begin{bmatrix} 9220.9414 & -55.0853 & -3656.3874 \\ -55.0853 & 4990.4476 & -2520.9471 \\ -3656.3874 & -2520.9471 & 5555.5427 \end{bmatrix}$$

$$F = \frac{50}{3}\begin{bmatrix} .2420, & .8786, & .3496 \end{bmatrix} \; S^{-1} \begin{bmatrix} .2420 \\ .8786 \\ .3496 \end{bmatrix} = \frac{47x50}{3} \times \frac{2880.5602}{359,57742}$$

$= 6.27$

Comparing with the tabular F value for (3,47) d.f., we have to reject the hypothesis of no bias at the .01 level.

Example 3c. Correction for bias. Since the F tests in 3b showed significance, we need to adjust the scale points to account for the bias. We therefore follow the randomization procedure discussed in (7.14).

For instance, to adjust for the bias which results in a mean difference between the first and second presentations of .2420, we preform a randomization procedure with chance of success $p_m = \frac{.2420}{\hat{\beta}_{m+1} - \hat{\beta}_m} = .08$; This should be the probability with which a response zero given on the second presentation must be changed to one. For the randomization procedure the table of random numbers in Dixon and Massey's " Introduction to Statistical Analysis" was used. We chose the first column of page 366, and defined as success annumber (00-07). Since the first success occured at the 31st number and there were fewer than 31 zeros given as responses in the second administration no zeros were changed into ones.

In the second presentation, there were no zeros, ones or twos changed, two threes (the eleventh and twelth threes given as responses), one four (the sixth one given), and one five(the fifth). The corresponding p_m's were .08, .11, .15, .12, .11, .11.

To adjust the responses given in the third presentation we take $p_m = \frac{.8786}{\hat{\beta}_{m+1}-\hat{\beta}_m}$. For instance, $\hat{\beta}_1-\hat{\beta}_0 = 3.06$, and $p_0 = .29$. We therefore take a row or column (here, the first row of page 370) and define as a success any number 10-38, say, which occured in that particular row. Since a success occured in the first, third, and fifth places the first, third, and fifth zeros were changed to ones.

Likewise, for entries in the fourth column, we define $p_m = \frac{.3496}{\hat{\beta}_{m+1}-\hat{\beta}_m}$. For instance, for $m=0$, $p_m = p_0 = \frac{.3496}{3.06}$.

After finding p_m for all m from zero to five and for all presentations, and correcting the data by the randomization procedure with proper p_m, we have the following data.

(6,6,6,6)	(0,0,1,0)	(1,1,0,0)	(2,2,1,2)	(3,4,3,4)
(6,6,5,6)	(0,0,0,1)	(5,5,4,5)	(4,3,4,2)	(3,4,3,4)
(6,6,5,5)	(6,6,6,6)	(1,2,2,3)	(6,6,5,6)	(4,3,4,4)
(6,6,6,6)	(0,0,1,0)	(5,4,4,4)	(5,5,5,5)	(5,4,5,4)
(5,5,4,5)	(6,6,5,6)	(5,3,4,4)	(2,2,2,3)	(3,3,3,2)
(6,6,5,5)	(3,3,4,5)	(0,0,1,0)	(1,1,1,1)	(3,2,3,2)
(4,4,5,4)	(6,5,6,6)	(0,1,1,1)	(3,3,3,3)	(6,6,6,6)
(3,2,3,2)	(4,3,3,4)	(3,2,3,2)	(5,6,5,5)	(3,2,4,2)
(3,3,2,3)	(4,4,4,4)	(3,3,3,3)	(5,6,3,6)	(4,4,4,3)
(3,3,3,3)	(4,3,4,3)	(2,2,3,2)	(2,3,3,3)	(4,5,4,3)

In the usual manner of solution of points, we find

$$\hat{\underline{\alpha}}^{*\prime} = [0,\ .16,\ .26,\ .52,\ .71,\ 1.00] \qquad \text{and } \hat{c} = .09$$

$$\hat{\underline{\beta}}^{*\prime} = [0,\ 1.78,\ 2.89,\ 4.22,\ 5.78,\ 7.89,\ 11.11]$$

and $\bar{d}_1 = .06$, $\bar{d}_2 = .28$, $\bar{d}_3 = .15$, a noticeable reduction from the orginial $\bar{d}_i$'s.

IX SUMMARY

The assumption is made that any differences between any two responses to the same stimulus by the same individual is due to an uncertainty on the part of the individual regarding the location of the scale points, and that this uncertainty is of uniform magnitude over all the scale.

The model and the procedures for derivation of scale points are given for 2,3, and 4 presentations in Chapters III, IV, and V.

Chapter VI deals with the combination of the scales of several people with different intervals of uncertainty.

Chapter VII discusses the tests for presence of bias between administrations, and a method of correcting for bias if the test shows significance.

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ABSTRACT

ON A METHOD OF MULTIPLE-PRESENTATION SCALING OF SUCCESSIVE INTERVALS

by

Mary Beth McGahey

Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in candidacy for the degree of

MASTER OF SCIENCE

in

STATISTICS

September 1958

ABSTRACT

This thesis discusses a multiple-presentation scaling technique, by a somewhat modified successive interval approach. If a subject gives two or more different responses to the same stimulus in repeated presentations this may be considered as an "uncertainty" or lack of the subject's ability to discriminate between adjacent points on the rating scale. This "interval of uncertainty" is assumed to be a constant, and the scale points are estimated, on the basis of this assumption, by a least-squares technique.

Analysis and explicit computational procedures have been developed for the case of two, three, and four presentations. Numerical illustrations have been added for each case.

The thesis also includes a discussion on the combination of scales for different subjects if their intervals of uncertainty are different. Finally, it proposes methods of testing for bias between presentations, and a randomized procedure to correct for such bias if its presence is indicated by the test.