Strategic Resource Dependence and Adoption of a Substitute under Learning-by-Doing

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Abstract

There is ample evidence that the production of alternative sources of energy is subject to learning-by-doing. The present paper examines the implications of learning-by-doing in the bilateral resource monopoly studied by Gerlagh and Liski (2011). We derive the socially optimal use of both oil and the substitute, as well as the Markov-perfect equilibrium. Our results are qualitatively different from those of Gerlagh and Liski. We show that it may be socially efficient to discard part of the cheap oil stock. Interestingly, we find that, in the Markov-perfect equilibrium, the buyer curbs his consumption to conserve the oil stock owned by the seller.

JEL classification: C73, Q30, O30

Keywords: learning-by-doing, bilateral monopoly, Markov-perfect equilibrium, oil, alternative sources of energy

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1 Introduction

It is well understood that the availability of alternative energy sources limits the ability of oil producers to adopt an ever-rising price path. Building on the framework of Hotelling (1931), many authors have examined the effects of the availability of a backstop technology on oil extraction and pricing —see for example Nordhaus (1973) Dasgupta *et al.* (1983), Hoel (1983). Following this line of idea, Gerlagh and Liski (2011) consider a bilateral oil monopoly where there is a time-to-build period for the buyer to implement the backstop once the investment decision has been made. Their results suggest that, unlike in the standard Hotelling (1931) framework, oil supply increases as the stock is being depleted. The seller's increasing supply path is chosen to keep the buyer indifferent between (a) immediately investing in the substitute and (b) purchasing oil while postponing the investment decision.

The present paper examines the implications of learning-by-doing —which occurs in the backstop production [see Chakravorty *et al.* (2011)]— within the model of strategic resource dependence of Gerlagh and Liski (2011). We consider a bilateral monopoly where one side (the buyer) imports a nonrenewable resource —**oil**, for short— produced by the other side (the seller). There is a substitute to oil available and, although it is initially higher than the cost of extracting a barrel of oil, the unit cost of the substitute decreases as its cumulative use increases (thus making it more economically viable). As in Michielsen (2012), we assume that the substitute is immediately available to the buyer when the decision is made to cut oil consumption. Our model differs from Michielsen's (2012) —which features an invariable unit cost for the substitute— due to the fact that learning-by-doing makes the production of the substitute more efficient as its cumulative use increases.

Note that we do not model the R&D process that makes the substitute available to the buyer. Many papers have examined the optimal effort that oil-importing countries should invest in R&D (whether success in developing a substitute to oil is deterministic or random); see for example Dasgupta *et al.* (1983), Harris and Vickers (1995), Bahel (2011). Like Gerlagh and Liski's (2011) and Michielsen's (2012), our analysis rather focuses on strategic considerations after the backstop becomes available to the buyer side, that is, after a successful R&D process.

Firstly, we determine the socially efficient use of both oil and the substitute. The curve representing energy consumption is typically U-shaped. Interestingly, we find that part of the oil stock may be discarded, even with a substitute that is costlier than oil. Secondly, focusing on the dynamic game that takes place between the buyer and the seller, we derive the Markov-perfect equilibrium. The equilibrium time path typically exhibits three stages: I- the price of oil increases —and extraction therefore decreases; II- the oil price and supply then stabilize until exhaustion occurs; III- the substitute kicks in and the buyer's energy consumption increases (due to learning-by-doing) towards its asymptotic value.

Like Gerlagh and Liski (2011), we find that the seller compensates the buyer for delaying the implementation of the substitute. However, within our framework, the seller may not be able to postpone the switch to the substitute until after oil exhaustion. In fact, like under the social optimum, the oil stock may be left untouched even if its unit extraction cost is lower than the marginal cost of the substitute. Another noticeable finding is the fact that, on the equilibrium path, during the stage I described above, the buyer typically draws a positive surplus from oil consumption. As a result, unlike in Gerlagh and Liski (2011) and Michielsen (2012), the buyer is concerned with the depletion of the oil stock and, therefore, has incentives to curb his oil consumption in order to maintain the seller's ability to price below the reservation price (for as long as possible). After stage I, the buyer typically purchases oil at his reservation price (until exhaustion occurs).

The paper is structured as follows. Section 2 presents the framework. In Section 3, we solve for the socially optimal use of the two sources of energy. Section 4 presents the differential game between buyer and seller, as well as the Markov-perfect equilibrium. We conclude in Section 5. All proofs are relegated to the Appendix.

2 The model

Let the buyer's instantaneous surplus associated with the consumption of q units of energy be $u(q)$, where the function $u : \mathbb{R}_+ \to \mathbb{R}_+$ is assumed to be increasing, concave and continuously differentiable in q. We also make the usual assumption that $u'(0) = +\infty$, which guarantees that the buyer will consume a positive amount no matter what the unit cost of energy.

In a static environment, if faced with the price p, the buyer would demand the amount $D(p) = u'^{-1}(p)$ of oil. Put differently, the (static) profit function of the seller is given by $\pi(q) = (u'(q) - c_0)q$, where c_o is the seller's unit cost of extraction. Let us assume that π is strictly concave and continuously differentiable in q. In addition, let p^m denote the seller's monopoly price associated with the (static) demand function $D(p)$.[1]

Our analysis involves a dynamic framework with continuous time. Let p_t be the price set by the seller at date t. In case oil is the buyer's only source of energy at date t, the buyer's instantaneous surplus is $u(q_t) - p_t q_t$; and the seller's profit is $\pi_t = p_t q_t - c_o q_t$. However, the buyer has access to a renewable

[1] That is to say, p^m solves the problem $\max_{p \geq 0} (p - c_o) u'^{-1}(p)$.

source of energy (solar energy, for short) whose unit cost, $c_s \equiv c_s(L_t)$, decreases over time as its cumulative production L_t increases (i.e., $c_s'(L_t) < 0$).

The state variable L_t can be seen as the know-how accumulated by the buyer in producing solar energy. We will assume throughout the paper that $c_s(0) > c_o$, which means that the unit cost of the solar is initially higher than that of oil. An interesting family of learning processes is described by: $c_s(L_t) = \theta e^{-\delta L_t} + \beta$, where the parameters θ, β, δ are all positive and satisfy $c_s(0) = \theta + \beta > c_o$. For these processes, the difference between the unit cost and (its limit) β decreases at the constant rate $\delta > 0$.

In the more general case where an amount $q_t^s \geq 0$ of solar energy is consumed at date t, we will write the buyer's overall energy consumption as $q_t = q_t^o + q_t^s$,[2] where q_t^o (q_t^s) stands for oil consumption (solar-energy consumption). The buyer's surplus is then $u(q_t) - p_t q_t^o - c_s(L_t) q_t^s$. On the other hand, the producer's profit is then $\pi_t = p_t q_t^o - c_o q_t^o$.

Let us denote by $V(S_0, L_0)$ the discounted sum of the seller's instantaneous profits from date t_0 on, given the initial stocks of oil (S_0) and know-how (L_0). We can write:

$$V(S_0, L_0) = \int_{t_0}^{+\infty} e^{-rt} \pi_t dt, \tag{1}$$

where r is the discount rate. Likewise, the intertemporal utility of the buyer, $W(S_0, L_0)$, can be written as:

$$W(S_0, L_0) = \int_{t_0}^{+\infty} e^{-rt} [u(q_t) - p_t q_t^o - c_s(L_t) q_t^s] dt. \tag{2}$$

[2]This expression of overall energy consumption means that the amount of solar energy is expressed in barrel-of-oil equivalent. This choice of unit is convenient and obviously does not affect the results.

3 The socially efficient extraction and substitute development

To solve for the socially efficient oil extraction and solar energy development, one may think of a consumer who owns the oil stock. From date t_0 on, the problem of such a consumer would be to choose the consumption paths $\{(\hat{q}_t^o, \hat{q}_t^s)\}_{t \geq t_0}$ that solve

$$\widehat{W}(S_0, L_0) = \max_{\{(q_t^o, q_t^s)\}_{t \geq t_0}} \int_{t_0}^{+\infty} e^{-r(t-t_0)} \left[u(q_t^o + q_t^s) - c_o q_t^o - c_s(L_t) q_t^s \right] dt \qquad (3)$$

subject to:

$$\dot{S}_t = -q_t^o \qquad (\text{with } S_{t_0} = S_0 \geq 0) \qquad (4)$$

$$\dot{L}_t = q_t^s \qquad (\text{with } L_{t_0} = L_0 \geq 0) \qquad (5)$$

$$q_t^o, q_t^s \geq 0 \qquad \text{and } \lim_{t \to \infty} S_t \geq 0. \qquad (6)$$

Note that (4) and (5) are the laws of motion for the state variables: the stock S_t is depleted at the rate of oil consumption; and the know-how L_t increases at the rate of solar-energy consumption. At date t_0, we have the (initial) stock of oil S_0 and level of know-how L_0, which corresponds to the unit cost $c_s(L_0)$ for solar energy.

Since the above problem is autonomous, the initial date can be set (without loss of generality) to $t_0 = 0$; and the Hamiltonian can be written as

$$H(q_t^o, q_t^s, S_t, L_t, \lambda_t^o, \lambda_t^s, t) = e^{-rt} \left[u(q_t^o + q_t^s) - c_o q_t^o - c_s(L_t) q_t^s \right] - \lambda^o q_t^o + \lambda^s q_t^s, \quad (7)$$

where λ_t^o and λ_t^s are the respective shadow values associated with the stocks of oil (S_t) and know-how (L_t). Using Pontryagin's maximum principle, one can then state the necessary conditions for optimality:

$$u'(q_t^o + q_t^s) - c_o - \lambda_t^o e^{rt} \leq 0 \text{ (with = if } q_t^o > 0); \tag{8}$$

$$u'(q_t^o + q_t^s) - c_s(L_t) + \lambda_t^s e^{rt} \leq 0 \text{ (with = if } q_t^s > 0); \tag{9}$$

$$\dot{\lambda}_t^o = 0 \text{ and } \dot{\lambda}_t^s = e^{-rt} c_s'(L_t) q_t^s; \tag{10}$$

$$\dot{S}_t = -q_t^o \text{ and } \dot{L}_t = q_t^s; \tag{11}$$

$$\lim_{t\to\infty} \lambda_t^o S_t = 0 \text{ and } \lim_{t\to\infty} \lambda_t^s = 0. \tag{12}$$

In what follows we study the conditions (8)-(12) and describe the solution to the problem posed in (3).

Let us first state the following preliminary result.

Lemma 1 *The solution $(q_t^o, q_t^s)_{t\geq 0}$ to (3) is such that: for any $t_1 \geq 0$,*

$$q_{t_1}^s > 0 \Rightarrow (q_t^o = 0, \text{ for all } t > t_1).$$

Using the result of Lemma 1 allows to prove the proposition below, which pertains to the consumption paths associated with the respective sources of energy.

Proposition 1 *On the socially optimal path, we have the following.*
(i) Given $c_s(\cdot)$ —the learning process, there exists a unique threshold $\alpha \in (0, c_s(0))$ such that: a. *if $c_s(0) - c_o < \alpha$ then $q_t^o = 0$, for any $t \geq 0$; and* b. *$q_{t=0}^o > 0$ if $c_s(0) - c_o > \alpha$.*
(ii) There exists a date $T \geq 0$ such that $q_t^s = 0$, $q_t^s > 0$, for $t \geq T$; and $q_t^s > 0$, $q_t^o = 0$, for $t > T$.

The statement (i) of Proposition 1 shows that oil will be used only if solar energy is relatively costly ($c_s(0) - c_o > \alpha$) at the outset —the parameter $c_s(0)$, recall, is the initial unit cost of the substitute. If instead the initial difference between the unit costs is too small ($c_s(0) - c_o < \alpha$), then oil is not used. Note in particular that, in the case where $c_o - \alpha < c_s(0) < c_o$, the oil stock will be left untouched although solar energy is initially costlier. It is shown in the Appendix that the threshold α can be uniquely determined as the shadow value of solar energy at date $t = 0$.

As for statement (*ii*), it points out that there necessarily is a date T such that only solar energy is consumed from T on. Observe from statement (*i*) that $T = 0$ if and only if $c_s(0) - c_o > \alpha$. Figure 1 depicts the case where $T > 0$; oil and solar energy are both used (with oil being used first). Our next result relates to the shape of the energy consumption path.

Proposition 2 *The socially optimal path for oil consumption, $(q_t^o)_{t \geq 0}$, is non-increasing over time; whereas that of solar energy consumption, $(q_t^s)_{t \geq 0}$, is non-decreasing. In addition, we have $\lim_{t \to \infty} q_t^s = \bar{q}$, where $\bar{q} = u'^{-1}\left(\lim_{L \to \infty} c_s(L)\right)$.*

Proposition 2 says that the socially efficient oil consumption typically[3] decreases prior to T and falls down to zero thereafter; whereas solar energy consumption is zero prior to T and increasing afterwards. As a consequence, the socially optimal path of (total) energy consumption is typically U-shaped. See Figure 1, which also illustrates the limit $\bar{q}$ of energy consumption.

As argued in the Appendix, it is easy to see that the following system of two equations with two unknowns (λ, T)

$$\begin{cases} (\alpha + \lambda)e^{rT} = c_s(0) - c_o \\ \int_0^T u'^{-1}\left(c_o + \lambda e^{rt}\right) dt = S_0 \end{cases} \tag{13}$$

exhibits a unique solution that we denote by $(\bar{\lambda}, \overline{T})$. Note that this solution depends (in particular) on α and the initial stock of oil S_0.

[3] As will be seen further on, oil consumption may be constant prior to T.

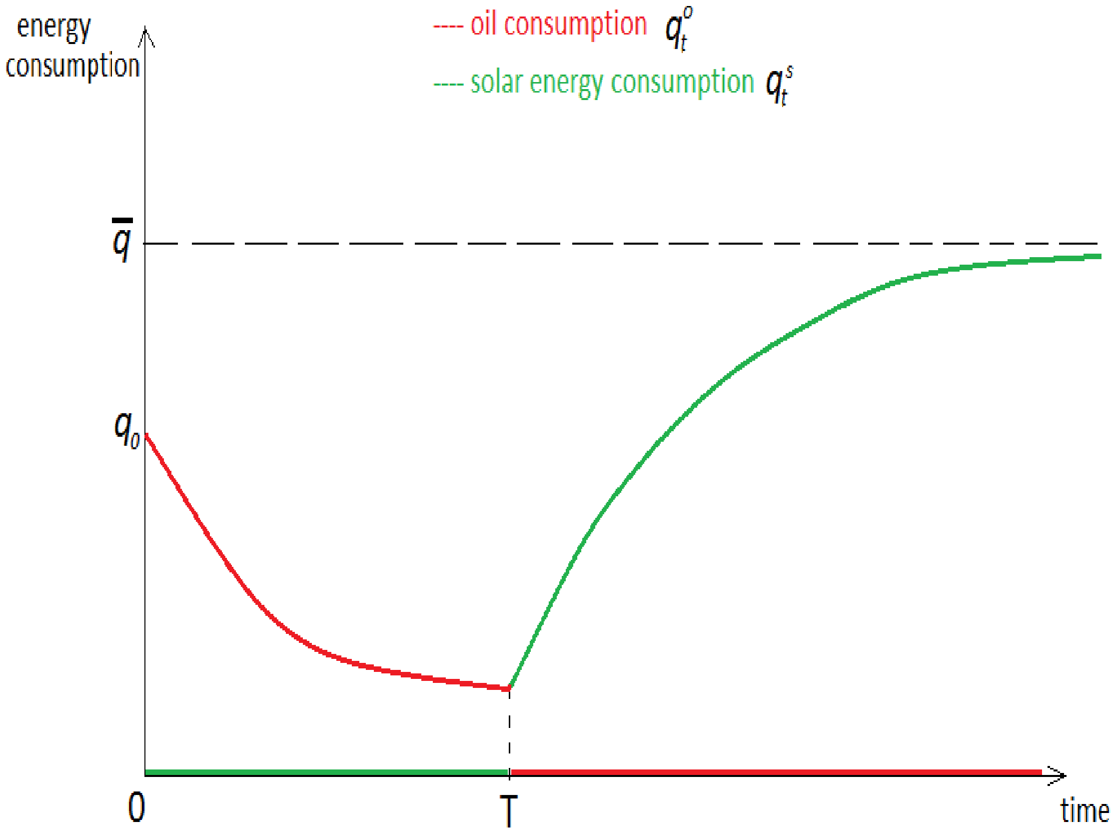

Figure 1: The socially optimal use of the two sources of energy

Next, let us denote by L_t^* the unique solution to the second-order differential equation (SODE)

$$u''(\dot{L}_t)\ddot{L}_t - r\left[u'(\dot{L}_t) - c_s(L_t)\right] = 0 \tag{14}$$

that satisfies the initial and terminal conditions: $L_{[t=0]} = 0$ and $\lim_{t\to\infty} L_t = +\infty$. In addition, for any $t \geq 0$, let

$$\lambda_t^* \equiv u''(\dot{L}_t^*)\ddot{L}_t^*. \tag{15}$$

Note that $(\lambda_t^*)_{t\geq 0}$ is uniquely determined by $r, u(.), c_s(.)$, which are primitives of the model. The upcoming result fully characterizes the socially efficient path.

Proposition 3 *The socially efficient path can be expressed (depending on the relevant case) as follows.*
(i) If $\underline{c_s(0) - c_o < \alpha}$*:* $T = 0$*; for any* $t \geq 0$*,* $q_t^o = 0$ *and* $q_t = q_t^s = u'^{-1}\left(c_s(L_t^*) - \lambda_t^* e^{rt}\right)$*, where* L_t^* *and* λ_t^* *are respectively given by (14) and (15).*
(ii) If $\underline{c_s(0) - c_o \geq \alpha}$*: then there exists a threshold* $\bar{S}$ *such that*

a- *whenever* $S_0 \geq \bar{S}$*, we have* $T = \frac{1}{r}\ln\left(\frac{c_s(0)-c_o}{\alpha}\right)$ *and*

$$\begin{cases} q_t = q_t^o = u'^{-1}(c_o), & \textit{for } t \leq T \\ q_t = q_t^s = u'^{-1}\left(c_s(L_{t-T}^*) - \lambda_{t-T}^* e^{rt}\right) & \textit{for } t > T; \end{cases}$$

b- *whenever* $S_0 < \bar{S}$*, we have* $T = \overline{T}$ *and*

$$\begin{cases} q_t = q_t^o = u'^{-1}(c_o + \bar{\lambda} e^{rt}), & \textit{for } t \leq T \\ q_t = q_t^s = u'^{-1}\left(c_s(L_{t-T}^*) - \lambda_{t-T}^* e^{rt}\right) & \textit{for } t > T, \end{cases}$$

where $(\bar{\lambda}, \overline{T})$ *is the solution to (13).*

The above Proposition 3 describes the socially optimal use of the two sources of energy in all possible cases. If the initial gap between the two unit costs is not high enough [case (i)], then it is efficient to produce solar energy from the outset (and thus disregard fossil fuel). In the more plausible case where the cost of oil is much lower than that of solar energy, oil will be used first; but even then, some of the oil stock may be discarded [case (ii)-a]. If oil is not excessively abundant [case (ii)-b], its consumption decreases over time and the stock is eventually exhausted; solar energy takes over from T onwards. In all cases, the consumption of solar energy increases after T and goes asymptotically to $\bar{q}$ (see Figure 1).

The socially efficient use of the two sources of energy described in this section is important in at least two regards. Firstly, it helps us understand how a country owning oil reserves should deplete them when a substitute (subject to learning-by-doing) is available. Secondly, as will be seen in the following section, knowing the social optimum is crucial to understanding the equilibrium of the strategic game between the buyer and the seller of the resource.

4 Strategic interaction between buyer and seller and Markov-perfect equilibrium

Let us now examine the differential game that takes place between the buyer and the seller. Time is continuous and, considering small time intervals of the form $[t, t + dt]$, we let the agents make their decisions at the beginning and commit to these choices for the entire period (of length dt). In each interval $[t, t + dt]$ such that the remaining oil stock is positive, the timing is as follows.

(1) The seller sets a unit price p_t for oil.[4]

(2) The buyer then chooses q_t^o and q_t^s, his respective consumptions of oil and solar energy.

In case the oil stock is depleted at date t, the buyer relies only on solar energy and his (undiscounted) continuation value is therefore $\widehat{w}(L_t) \equiv \widehat{W}(0, L_t)$ —where $\widehat{W}$ is defined by (3). In what follows, we determine the Markov-perfect equilibrium of the dynamic bilateral monopoly described above when dt approaches zero. A strategy of the seller assumes the form $\tilde{p} : \mathbb{R}_+^2 \to \mathbb{R}_+$; and, given the oil stock S_t and level of know-how L_t, the seller chooses the oil price $p_t \equiv \tilde{p}(S_t, L_t)$.

On the other hand, a strategy of the seller is of the form $\tilde{q} : \mathbb{R}_+^3 \to \mathbb{R}_+^2$.

[4] Our price-setting approach is equivalent to that of Gerlagh and Liski (2011) who consider a quantity-setting seller. With learning by doing, it is convenient to let the seller set a price, and the buyer choose quantities.

That is to say, given S_t, L_t and the oil price p_t, the buyer chooses the respective consumptions of oil and solar energy $(q_t^o, q_t^s) \equiv \tilde{q}(S_t, L_t, p_t)$.

We will assume throughout this section that $c_o < c_s(0) - \alpha$, which means that some strategic interaction indeed takes place between the buyer and the seller.[5] The parameter α, recall, was introduced in Proposition 1 as the initial shadow value of the know-how accumulated in producing solar energy.

4.1 The buyer's energy consumption

The buyer determines his optimal choices depending on the pricing strategy of the seller. Let us then assume an arbitrary (continuous) strategy $\tilde{p}$ of the seller. At any date t_0 such that the initial oil stock S_0 is positive,[6] the buyer's problem can be written as:

$$W(S_0, L_0) = \max_{\{q_t^o, q_t^s\}_{t \geq t_0}} \int_{t_0}^{+\infty} e^{-r(t-t_0)} [u(q_t^o + q_t^s) - \underbrace{\tilde{p}(S_t, L_t)}_{p_t} q_t^o - c_s(L_t) q_t^s] dt \quad (16)$$

subject to:

$$\dot{S}_t = -q_t^o \qquad (\text{with } S_{t_0} = S_0 \geq 0) \quad (17)$$

$$\dot{L}_t = q_t^s \qquad (\text{with } L_{t_0} = L_0 \geq 0) \quad (18)$$

$$q_t^o, q_t^s \geq 0.$$

In (16), unlike in the problem studied in Section 3, the buyer acquires barrels of oil at the seller's unit price p_t, which is obviously higher than c_o, the unit cost of extraction. On the other hand, the laws describing the evolution of the state variables S_t and L_t remain the same. It thus follows that the purchase strategy of the buyer is qualitatively similar to the oil consumption policy described by

[5] If we have $c_o \geq c_s(0) - \alpha$ instead, then the buyer will adopt solar energy from the outset because the seller side cannot profitably price below its unit cost c_o.

[6] Recall that the buyer's continuation value is $\widehat{w}(L_t)$ when oil gets exhausted.

Proposition 3, with the precision that c_o is now replaced with p_t in the expression of q_t^o.

Proposition 4 *There exists a Markov-perfect strategy for the buyer; it is characterized by two functions* $\tilde{S} \equiv \tilde{S}(L_t)$ *and* $\mu \equiv \mu(S_t, L_t)$ *s.t.*

- ***a-*** *whenever* $S_t \leq \tilde{S}$*:* $q_t = q_t^o = u'^{-1}(p_t)$ *if* $c_s(L_t) - \alpha \geq p_t$ *(with* $q_t^o = 0$ *otherwise);*

- ***b-*** *whenever* $S_t > \tilde{S}$*:* $q_t = q_t^o = u'^{-1}(p_t + \mu)$ *if* $c_s(L_t) - \alpha \geq p_t$ *(with* $q_t^o = 0$ *otherwise).*

In essence, Proposition 4 states that the buyer will consume oil as long as the price p_t remains below $c_s(L_t) - \alpha$, his full marginal cost in producing solar energy. It is interesting to note from the proposition that, if the oil stock is high enough (case **b**) and $c_s(L_t) - \alpha > p_t$, then the buyer will curb his oil demand (in comparison with a myopic buyer) in order to extend the period of positive surplus as much as possible. Indeed, the buyer gets a positive surplus during that period —since he pays less than his reservation price $c_s(L_t) - \alpha$ for each barrel.

4.2 The seller's pricing strategy

The seller's objective is to maximize the discounted sum of his profits, which can be written as $\pi_t = (p_t - c_o)q^o(S_t, L_t, p_t)$, given the initial state (S_0, L_0) and the optimal purchase strategy of the buyer:

$$V(S_0, L_0) = \max_{\{p_t\}_{t \geq t_0}} \int_{t_0}^{+\infty} e^{-r(t-t_0)} (p_t - c_o) \tilde{q}^o(S_t, L_t, p_t) dt \qquad (19)$$

subject to:

$$\dot{S}_t = -\tilde{q}^o(S_t, L_t, p_t) \qquad \text{(with } S_{t_0} = S_0\text{)}$$
$$p_t \leq c_s(L_t) - \alpha.$$

Recall that the buyer does not purchase oil when $p_t > c_s(L_t) - \alpha$. In addition, given the demand function $D(p)$ introduced earlier, let p^m denote the (static) monopoly price of the seller. It is not difficult to show the following.

Lemma 2 *At any date t such that $c_s(L_t) - \alpha > c_o$, the seller's optimal price $p(S_t, L_t)$ satisfies:*

$$\begin{cases} p(S_t, L_t) \in [p^m, c_s(L_t) - \alpha], & \textit{if } p^m \leq c_s(L_t) - \alpha; \\ p(S_t, L_t) = c_s(L_t) - \alpha, & \textit{otherwise.} \end{cases}$$

The above result stems in particular from the fact that, on the optimal path, the seller will never price above the full marginal cost at which the buyer can produce the substitute. Indeed, if $p_t > c_s(L_t) - \alpha$, the buyer will exclusively consume solar energy (see Proposition 4); and the seller thus makes zero profits at date t. In addition, the production of solar energy at date t reduces the future costs of the buyer (due to learning-by-doing), which further harms the seller by reducing his continuation value.

Let $\bar{p} \equiv c_s(L_t) - \alpha$ and $\bar{q} \equiv u'^{-1}(\bar{p})$. Using Lemma 2 and recalling the values $\tilde{S}$ and μ (introduced in Proposition 4), we can describe the seller's strategy as follows.

Proposition 5 *There exists a Markov-perfect pricing strategy of the seller s.t.*
(i). $p(S_t, L_t) = c_s(L_t) - \alpha$ if $p^m \geq c_s(L_t) - \alpha \geq c_o$;
(ii). if instead $p^m < c_s(L_t) - \alpha$ then

a- *$p(S_t, L_t) = c_s(L_t) - \alpha$ for $S_t \leq \tilde{S}$;*

b- *$p(S_t, L_t) = u'(\pi'^{-1}(e^{r(t-T_1)}\pi'(\bar{q}))) - \mu$ for $S_t > \tilde{S}$, where $T_1 \equiv T_1(L_t, S_t)$.*

It can be shown that the Markov-perfect equilibrium described by Propositions 4 and 5 is unique if we impose the following conditions: (a) the pricing strategy of the seller $\tilde{p}(S, L)$ is a nondecreasing function of the oil stock S for any fixed L; (b) the purchasing strategy of the buyer $\tilde{q^o}(S, L, p)$ is a decreasing function of the price p, for any fixed S and L.

Observe from Proposition 5 that, if oil is sufficiently abundant, the seller will initially price below the buyer's reservation price, which leaves the latter with a positive surplus. As soon as the oil stock falls below the threshold $\tilde{S}(L_t)$, the seller adopts a constant price until oil is exhausted, with the price chosen so as to leave the buyer exactly indifferent between immediately adopting the substitute and consuming oil (while postponing the adoption of the substitute). On the optimal path, solar energy is not adopted by the buyer as long as $p(S_t, L_t) \geq c_s(L_t) - \alpha$. Figure 2 depicts the price and consumption paths that arise when the players use their optimal strategies. The equilibrium path exhibits three stages. In the first stage —see the area I in Figure 2, the price of oil increases, that is, extraction decreases. The main feature of stage I is that the seller prices below the reservation price, which leaves the buyer with a positive surplus. It is interesting to note from Proposition 4 that the buyer optimally curbs oil consumption during stage I, which is due to the fact that he would like to extend this period of positive surplus as much as possible. As a result, unlike in Gerlagh and Liski (2011) and Michielsen (2012), the buyer is concerned with the depletion of the oil stock and, therefore, has incentives to reduce his oil consumption in order to conserve oil and maintain the seller's ability to price below the reservation price (for as long as possible).

Stage II begins as the stock falls below the threshold $\tilde{S}$. The seller's price (supply) stabilizes and remains constant; and the buyer, no longer concerned by the remaining stock, behaves myopically until the stock is depleted. Note that in the very unlikely case where $p^m > c_s(L_t) - \alpha$, stage I does not take place and the equilibrium path starts with stage II.

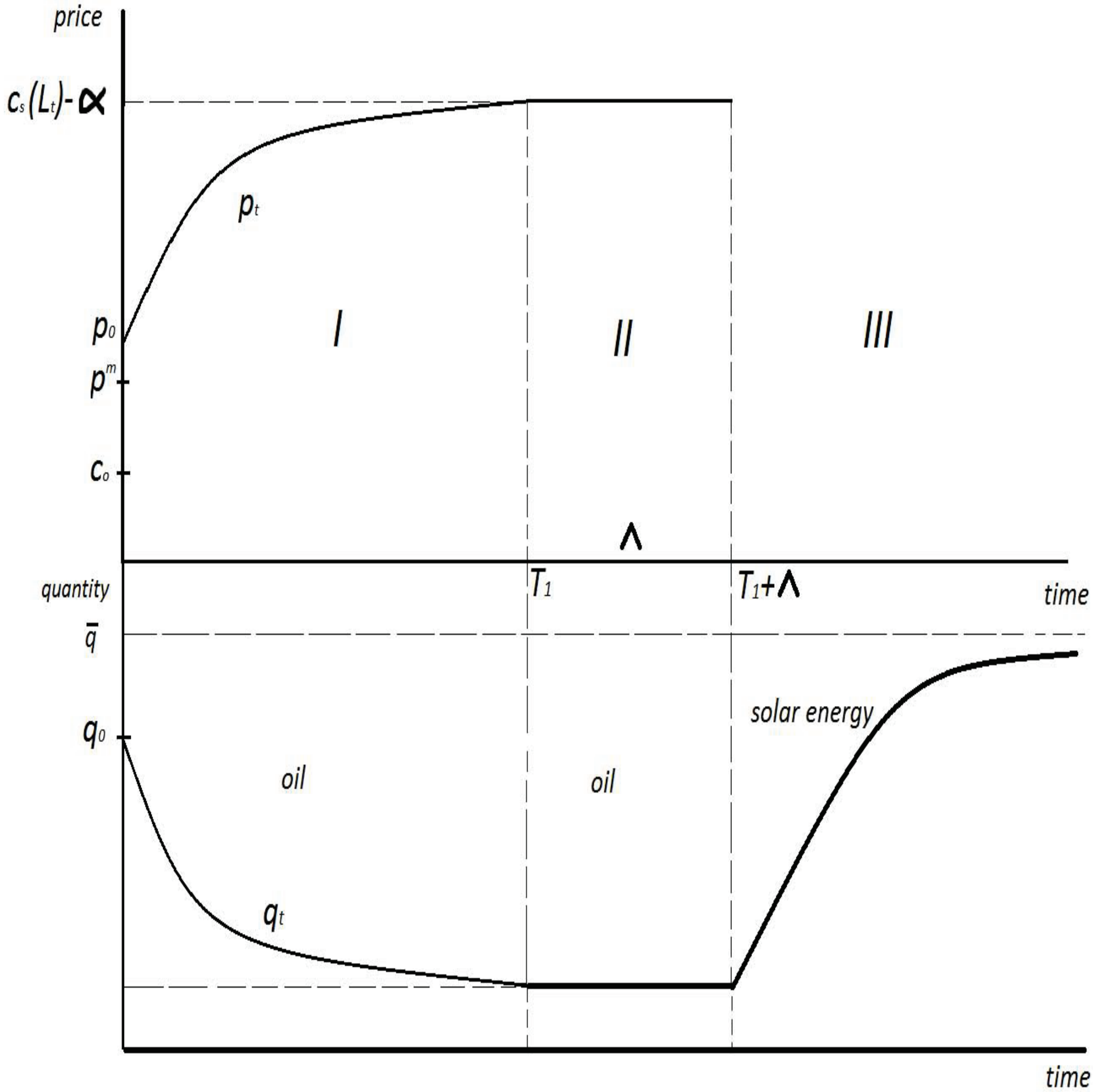

Figure 2: Markov-perfect price and energy-consumption paths

In stage III, after oil is depleted, the buyer switches to solar energy and consumption therefore increases to its asymptotic limit due to learning-by-doing. Note that if the learning process is extremely fast, it will drastically reduce the (future) costs of solar energy production; and α gets very large as a result. In particular, in the case where $c_s(0) - \alpha < c_0 < c_s(0)$, the buyer adopts solar energy from the outset, even if its cost is much higher than that of oil extraction.

5 Time-to-build delay

In this section we discuss how the assumption of a time-to-build delay would affect the socially optimal path and the Markov equilibrium. Suppose that, once the decision to adopt solar energy has been made, the buyer needs an exogenous period of length k for its production to begin. This is the key assumption of Gerlagh and Liski (2011). We argue that, under this assumption, our solutions described respectively in Sections 3 and 4 can be adjusted to describe the corresponding social optimum and Markov-perfect equilibrium.

5.1 Effect on the socially optimal path

Recall the socially optimal path described by Proposition 3 and Figure 1, where the adoption of the substitute occurs at date $T \geq 0$. Suppose first that $T < k$. If there is no time-to-build delay, it is optimal to adopt solar energy at T. With a time to build delay, the buyer should make the decision to switch to solar energy at the outset ($t = 0$), but consume oil during the time interval $[0, k]$ —the substitute becoming available only after date k. If we have $T \geq k$ instead, then it is not difficult to see that the decision to switch to solar energy should optimally be made at date $T - k \geq 0$, with the substitute becoming available at date T exactly when it is needed.

5.2 Effect on the Markov-perfect equilibrium

With a time-to-build delay, the buyer cannot instantly switch to solar energy and the reserve price $c_s(L_t) - \alpha$ no longer applies. In the Markov-perfect outcome, the seller increases the price up to the point where the buyer is indifferent between adopting solar energy and postponing the decision to switch (while consuming oil). From that moment on, the price decreases to keep the buyer indifferent while oil is being depleted (only to rise again during the time-to-build period k). As pointed out by Gerlagh and Liski (2011), the seller is always able to delay the consumption of solar energy up until oil exhaustion occurs. In our case learning-by-doing raises the continuation value of the buyer, which means that, prior to the decision to adopt solar energy, the oil price will peak earlier than without learning-by-doing.

6 Discussion

Our analysis incorporates learning-by-doing into the strategic resource dependence model. We have considered a nonrenewable resource bilateral monopoly where the buyer has the option to adopt a substitute whose marginal cost decreases as cumulative use increases. The results exhibits interesting qualitative differences in comparison with the findings of Gerlagh and Liski (2011). In particular, it is shown that the oil stock may be discarded even with a substitute that is much costlier than oil.

In addition, we show that the Markov-perfect outcome exhibits a stage where the seller prices below the reservation price; and the buyer curbs oil consumption in order to enjoy a positive surplus for as long as possible. Indeed, the buyer conserves oil because, on the equilibrium path, the seller sets the price equal to the buyer's reservation price as soon as the stock falls below a given threshold. As far as we know, this "non-myopic" behavior of the buyer (who is concerned with the depletion of the stock) is new to the literature on strategic resource dependence. In the Markov perfect equilibria of Gerlagh and Liski (2011) and Michielsen (2012), the buyer is essentially indifferent between (a) investing immediately and (b) consuming oil while postponing the adoption of the substitute.

For simplicity and comparison purposes,[7] we have considered a unit cost for the substitute that is constant at any fixed date. As a consequence, the importing country consumes either oil or solar energy (not both) at any given time. We leave for future research the case of a learning-by-doing process with a nonlinear cost function for the substitute production.[8] We believe such a framework would yield much of the same results as our present paper, while allowing the simultaneous use of the different sources of energy.

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[7]Note that both Gerlagh and Liski (2011) and Michielsen (2012) assume a constant unit cost for the substitute.

[8]Chakravorty, Leach and Moreaux (2012) examine such a learning process with nonlinear costs. They show that the combined effects of learning-by-doing and environmental regulation may lead to cyclical oil prices.

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A Proofs

A.1 Lemma 1

Suppose that $q^s_{t_1} > 0$ for some $t_1 \geq 0$. Then the necessary condition (9) implies that

$$u'(q^o_{t_1} + q^s_{t_1}) = c_s(L_{t_1}) - \lambda^s_{t_1} e^{rt_1}.$$

Combining the above with (8), one can write

$$c_s(L_{t_1}) - \lambda^s_{t_1} e^{rt_1} = u'(q^o_{t_1} + q^s_{t_1}) \leq c_o + \lambda^o_{t_1} e^{rt_1}.$$

In other words, at date $t = t_1$, (the current value of) the full marginal cost of solar energy , $c_s(L_t) - \lambda^s_t e^{rt}$, is no higher than that of oil consumption, $c_o + \lambda^o_t e^{rt}$. Note in addition that, since $\dot{\lambda}^o_t = 0$ [by condition (10)], the full marginal cost of oil ($c_o + \lambda^o_t e^{rt}$) is nondecreasing over time.

In order to conclude the proof, it is thus sufficient to show that $c_s(L_t) - \lambda^s_t e^{rt}$ is decreasing (over time) on the optimal path. To that end, let us observe that:

$$\begin{aligned} \frac{d\left[c_s(L_t) - \lambda^s_t e^{rt}\right]}{dt} &= c'_s(L_t)\dot{L}_t - \dot{\lambda}^s_t e^{rt} - r\lambda^s_t e^{rt} \\ &= c'_s(L_t) q^s_t - \dot{\lambda}^s_t e^{rt} - r\lambda^s_t e^{rt} \qquad \text{by the law of motion (5).} \end{aligned}$$

Using the necessary condition (10), we can write $c'_s(L_t) q^s_t = \dot{\lambda}^s_t e^{rt}$. This, plugged into the above time derivative, gives:[9]

$$\frac{d\left[c_s(L_t) - \lambda^s_t e^{rt}\right]}{dt} = \dot{\lambda}^s_t e^{rt} - \dot{\lambda}^s_t e^{rt} - r\lambda^s_t e^{rt} = -r\lambda^s_t e^{rt} < 0.$$

This shows that $c_s(L_t) - \lambda^s_t e^{rt} < c_o + \lambda^o_t e^{rt}$ for any $t > t_1$. That is to say, if solar energy is used at date t_1, oil will not be used from t_1 onwards (due to its higher full marginal cost).□

[9] Notice that the shadow value λ^s_t is positive (and decreasing) by condition (10), since $c'(L_t) < 0$.

A.2 Proposition 1

By (10), we have $\lambda_t^o = \bar{\lambda}^o \geq 0$, for any $t \geq 0$. Let then $\lambda_{t=0}^s \equiv \alpha$ be the optimal shadow value of solar energy at date $t = 0$. At $t = 0$, the respective full marginal costs for solar energy and oil are $c_s(0) - \alpha$ and $c_o + \bar{\lambda}^o$.

(i).a If $\underline{c_s(0) - c_o < \alpha}$, then we have $c_s(0) - \alpha < c_o + \bar{\lambda}^o$. In this case, combining the conditions (9) and (10), shows that $q_{t=0}^s > 0$ and $q_{t=0}^o = 0$. This, given the result of Lemma 1, proves the first claim (a).

(i).b If instead $\underline{c_s(0) - c_o > \alpha}$: by way of contradiction, suppose that $q_{t=0}^o = 0$ —this means that, $c_s(0) - \alpha < c_o + \bar{\lambda}^o$ and $q_{t=0}^s > 0$. It then follows from Lemma 1 that $q_t^o = 0$, $\forall t > 0$ (thus, oil is never used and $S_t = S_0$ for any t). In addition, the transversality condition ($\lim_{\infty} \lambda_t^o S_0 = 0$) implies that λ_t^o=0. Hence, $c_s(0) - \alpha > c_o + \bar{\lambda}^o = c_o$, which is a contradiction. Therefore, we must have $q_{t=0}^o > 0$ if $c_s(0) - c_o > \alpha$.

(ii) If $c_s(0) - c_o < \alpha$ then it suffices to take $T = 0$ [by (i).a]. Otherwise, it follows from the case (i).b above that $c_s(0) - \alpha > c_o + \bar{\lambda}^o$ and $q_{t=0}^o > 0$ (while $q_{t=0}^s = 0$). For any T satisfying $q_t^s = 0$ $\forall t \in [0, T]$, we get from the (5) that $L_t = 0$ $\forall t \in [0, T]$. It then follows from (10) that $\lambda_t^s = \lambda_{t=0}^s = \alpha$ $\forall t \in [0, T]$.

Thus, as long as $q_t^s = 0$, the full marginal cost of solar energy (in current value) is $c_s(0) - \alpha e^{rt}$ and decreases over time. On the other hand, the full marginal cost of oil is given by $c_o + \bar{\lambda}^o e^{rt}$ and increases over time. It is easy to see that $c_o + \bar{\lambda}^o e^{rt} = c_s(0) - \alpha e^{rt}$ for $t = T = \ln\left(\frac{c_s(0) - c_o}{\bar{\lambda}^o + \alpha}\right)$; and $c_o + \bar{\lambda}^o e^{rt} > c_s(0) - \alpha e^{rt}$ for $t > T$. Therefore, solar energy is used after T (due to its lower full marginal cost) and, from Lemma 1, oil is not consumed from T on (that is, $q_{T+t'}^o = 0$).$\square$

A.3 Proposition 2

The desired result is easily obtained by combining Proposition 1-(ii) and the facts that the full marginal cost of oil ($c_o + \bar{\lambda}^o e^{rt}$) is increasing whereas that of solar energy $(c_s(0) - \lambda_t^s e^{rt})$ is decreasing —as shown in the proof of Lemma 1. Hence, solar energy consumption increases after date T and, given that $\lim_{t\to\infty} L_t = +\infty$, condition (9) —with equality— gives $\lim_{t\to\infty} q_t^s = u'^{-1}\left(\lim_{L\to\infty} c_s(L)\right) \equiv \bar{q}.\square$

A.4 Proposition 3

Let us first determine the optimal solar energy consumption path after the switch to solar energy at date T.[10] The necessary conditions (9)-(11) come down to:

$$u'(q_t^s) = c_s(L_t) - \lambda_t^s e^{rt} \tag{20}$$

$$\dot{\lambda}_t^s = e^{-rt} c_s'(L_t) q_t^s \tag{21}$$

$$\dot{L}_t = q_t^s \tag{22}$$

Combining (20)-(22), we obtain the following second-order differential equation:

$$u''(\dot{L}_t)\ddot{L}_t - r\left[u'(\dot{L}_t) - c_s(L_t)\right] = 0. \tag{23}$$

Let then L_t^* be the unique solution to (23) that satisfies the initial and terminal conditions $L^*_{[t=0]} = 0$ and $\lim_{t\to\infty} L_t^* = +\infty$.[11] The optimal shadow value of solar energy at date $t \geq T$ is then

$$\lambda_t^* = u''(\dot{L}_t)\ddot{L}_t, \tag{24}$$

as specified in (15). Recall that we defined α as the shadow value of solar energy at $t = 0$ —and as long as $t \leq T$, due to (10). It follows from what precedes that:

$$\lambda_t^s = \begin{cases} \alpha, & \text{if } t \leq T \\ \lambda_{t-T}^*, & \text{if } t > T, \end{cases} \tag{25}$$

where λ_t^* is given by (24). It is easy to see that λ_t^s is nonincreasing and continuous. We now discuss the different cases of Proposition 3.

[10]It is known from Proposition 1 that oil is no longer consumed after the switch.

[11]Note that L_t^* is uniquely determined given the primitives of the model: u, c_s, r.

(i) Suppose that $\underline{c_s(0) - c_o < \alpha}$: then the desired result follows from the combination of Proposition 1-(i), (20) and (25) [where $T = 0$].

(ii) If $\underline{c_s(0) - c_o > \alpha}$: then it follows from Proposition 1-(i) that $q_t^o > 0$ (oil is used at the outset). As seen in the proof of Proposition 1, the switch to solar energy is made *by the latest* at $T^* = \ln\left(\frac{c_s(0)-c_o}{\alpha}\right)$,[12] regardless of the remaining oil stock S_{T^*}.

$a-$ In the subcase where $S_0 \geq \bar{S} \equiv u'^{-1}(c_o)T^*$, the optimal oil consumption *up until* T^* is clearly constant and given by $q_t^o = u'^{-1}(c_o)$ (that is, oil is not scarce and $\bar{\lambda}^o = 0$). Combining (20) and (25) then gives $q_t^s = u'^{-1}\left(c_s(L_{t-T}^*) - \lambda_{t-T}^* e^{rt}\right)$ for $t > T$.

$b-$ When $S_0 < \bar{S} \equiv u'^{-1}(c_o)T^*$, the constant oil consumption path above is not feasible (i.e., oil is scarce and $\bar{\lambda}^o > 0$). Oil consumption is then given by $q_t^o = u'^{-1}(c_o + \bar{\lambda}^o e^{rt})$, for any $t \leq T$. And the optimal date T of the switch to solar energy is such that the $c_s(0) - \alpha e^{rT} = c_o - \bar{\lambda}^o e^{rT}$, that is to say, the (current) full marginal costs are equal. It also follows from (12) that $S_T = 0$, which gives: $\int_0^T u'^{-1}\left(c_o + \lambda e^{rt}\right) dt = S_0$.

Combining the two conditions above, one gets the system introduced in (13):

$$\begin{cases} (\alpha+\lambda)e^{rT} = c_s(0) - c_o \quad \text{[i.e., } T(\lambda) = \ln\left((c_s(0)-c_o)/(\alpha+\lambda)\right)\text{]} \\ \int_0^T u'^{-1}\left(c_o + \lambda e^{rt}\right) dt = S_0. \end{cases}$$

Using the intermediate value theorem, it is easy to see that there exists a unique $\bar{\lambda}$ that solves $\int_0^{T(\lambda)} u'^{-1}\left(c_o + \lambda e^{rt}\right) dt = S_0$. Letting $\bar{T} \equiv T(\bar{\lambda})$, we can then write:

$$\begin{cases} q_t = q_t^o = u'^{-1}(c_o + \bar{\lambda} e^{rt}), & \text{for } t \leq \bar{T} \\ q_t = q_t^s = u'^{-1}\left(c_s(L_{t-\bar{T}}^*) - \lambda_{t-\bar{T}}^* e^{rt}\right) & \text{for } t > \bar{T}. \square \end{cases}$$

[12]This is because $\bar{\lambda}^o \geq 0$.

A.5 Proposition 4

For the buyer, the cost of acquiring a barrel of oil at any date t is p_t, and the full marginal cost of the substitute is $c_s(L_t) - \alpha$. It is straightforward to see from Section 3 that the buyer optimally consumes solar energy (that is, $q_o(t) = 0$) at any time t such that $p_t > c_s(L_t) - \alpha$.

In the case where $p_t \leq c_s(L_t) - \alpha$, we have $q_s(t) = 0$ and $q_o(t) > 0$ instead. It will be shown in the proof of Proposition 5 that (in the Markov-perfect equilibrium) the buyer prices above the seller's reservation price $c_s(L_t) - \alpha$ whenever the oil stock S_t is above some threshold $\tilde{S}(L_t)$. And for any $S_t \leq \tilde{S}(L_t)$, the seller will choose $p_t = \bar{p} = c_s(L_t) - \alpha$. Thus, the oil stock has no value for the buyer as soon as $S_t \leq \tilde{S}(L_t)$ —since he is charged the same price from then on; and he behaves in a myopic way by consuming $u'^{-1}(p_t)$. On the other hand, the buyer receives a positive surplus [above the reservation payoff $\bar{u} \equiv u\left(u'^{-1}(\bar{p})\right)$] as long as $S_t > \tilde{S}(L_t)$. Thus, given the Markov strategy of the seller, $p(S_t)$, the buyer has to solve the following problem at any date t [s.t. the oil stock is $S_t > \tilde{S}(L_t)$]:

$$\max_{\{q_\tau\}_{t \leq \tau \leq \bar{T}}} \int_t^{\bar{T}} e^{-r\tau} \left[u(q_\tau) - p(S_\tau)q_\tau - \bar{u}\right] d\tau \tag{26}$$

subject to:

$$\dot{S}_\tau = -q_t \text{ and } S_{\bar{T}} = \tilde{S}(L_t).$$

Writing the Hamiltonian and the optimality conditions for this problem, it is easy to see that the buyer's optimal oil consumption at date t assumes the form $q_t^o = u'^{-1}(p_t + \mu)$, where $p_t \equiv p(S_t)$ and $\mu \equiv \mu(S_t, L_t)$ is the oil scarcity value [associated with the problem (26)] at date $\tau = t$.$\square$

A.6 Lemma 2

At any date t, a price $p_t > c_s(L_t) - \alpha$ would trigger the buyer's use of the substitute (instead of oil) at date t, that is to say, $q_t^s > 0 = q_t^o$. This would have two negative effects on the seller: (a) his profit at date t would be zero; (b) his continuation value $W(S_t, L_t)$ would decrease due to a higher L after t.[13] It is clearly better for the seller to choose

$$p_t \leq c_s(L_t) - \alpha \tag{27}$$

and make positive profits at date t while preventing the know-how of the buyer (L) from increasing. Therefore, in a Markov-perfect equilibrium, we will always have $p_t \leq c_s(L_t) - \alpha$. Recall that p^m is the optimal price of the seller for the (static) demand function $D(p)$. One can see that the instantaneous profit $\pi_t = (p_t - c_o)D(p_t)$ is increasing in p_t as long as $p_t < p^m$. Let us now discuss the following two cases.

• Suppose that $p^m \leq c_s(L_t) - \alpha$ at date t. By choosing a price $p_t < p^m$, the seller earns lower profits at t (than with a price of p^m) and is left with a lower oil stock due to higher sales at t; this is clearly not optimal. Thus, in a Markov-perfect equilibrium, we have $p^m \leq p_t \leq c_s(L_t) - \alpha$.

• If $p^m > c_s(L_t) - \alpha$, choosing $p_t < c_s(L_t) - \alpha$ would give lower instantaneous profits while depleting the stock faster (which is not optimal). This shows that we must have $p_t \geq c_s(L_t) - \alpha$. Recalling (27) then gives the desired result: $p_t = c_s(L_t) - \alpha$.

[13] Note that L increases because $q_t^s > 0$ and $\dot{L}_t = q_t^s$. Furthermore, recall that the cost $c_s(L)$ of producing solar energy is a decreasing function of L.

A.7 Proposition 5

We show that the pricing strategy described in Proposition 5 is the seller's best response against the buyer's strategy of Proposition 4 (and vice-versa).

We first show that, against any strategy of the buyer which can be written in the form $\tilde{q}^o(S_t, L_t, p_t) = u'^{-1}(p_t + \mu)$ [where $\mu \equiv \mu(S_t, L_t)$],[14] the seller's best response involves a threshold $\tilde{S}$ such that: $p_t = \bar{p} \equiv c_s(L_t) - \alpha$ if $S_t < \tilde{S}$; and $p_t < \bar{p}$ otherwise. Indeed, for any such strategy of the buyer, the problem (19) of the seller at date t_0 can be written as:[15]

$$\max_{T_1, \{q_t^o\}_{t \in [t_0, T_1]}} \int_{t_0}^{T_1} e^{-r(t-t_0)} \left(p_t - c_o - \mu\right) q_t^o dt + e^{-r(T_1-t_0)} \left[1 - e^{-r\frac{S_{T_1}}{\bar{q}}}\right] (\bar{p} - c_o)\bar{q}/r \quad (28)$$

subject to:

$$\dot{S}_t = -q_t^o \text{ (with } S_{t_0} = S_0)$$
$$q_{T_1}^o = \bar{q} \equiv u'^{-1}(c_s(L_{t_0}) - \alpha).$$

Recalling the function π and letting $\tilde{\mu} \equiv e^{-r(t-t_0)}\mu$, one can rewrite the objective:

$$\max_{T_1, \{q_t^o\}_{t \in [t_0, T_1]}} \int_{t_0}^{T_1} \left[e^{-r(t-t_0)}\pi(q_t^o) - \tilde{\mu} q_t^o\right] dt + e^{-r(T_1-t_0)} \underbrace{\left[1 - e^{-r\frac{S_{T_1}}{\bar{q}}}\right] \pi(\bar{q})/r}_{B(S_{T_1})}.$$

Thus, the Hamiltonian is: $H = e^{-r(t-t_0)}\pi(q_t^o) - [\tilde{\mu} + \lambda]\, q_t^o$, where λ is the seller's scarcity value for the oil stock. Using Pontryagin's maximum principle, we

[14] Note that the corresponding inverse demand function is $p_t = u'(q_t^o) - \mu$ when the seller chooses oil extraction q_t^o.

[15] Note that we are implicitly using Lemma 2 to write the second term of the objective (28): when the price reaches $\bar{p}$ at date T_1, it can no longer increase and will remain constant at $\bar{p}$ for the time period $[T, T + \frac{S_T}{\bar{q}}]$. Equivalently, oil supply increases up until T and then remains constant at $\bar{q}$ during the period $\bar{q}$ during the period $[T_1, T_1 + \frac{S_{T_1}}{\bar{q}}]$.

obtain the optimality conditions:

$$\pi'(q_t^o) = e^{r(t-t_0)}\left[\tilde{\mu} + \lambda\right] \tag{29}$$

$$\dot{\lambda} = \frac{\partial \tilde{\mu}}{\partial S} q_t^o \tag{30}$$

$$\dot{S}_t = -q_t^o \tag{31}$$

$$H(T_1) = -\frac{\partial\left[e^{-rT_1} B(S_{T_1})\right]}{\partial T_1} \tag{32}$$

$$\lambda(T_1) = -\frac{\partial\left[e^{-rT_1} B(S_{T_1})\right]}{\partial S} \tag{33}$$

Note that (32) is the transversality condition relating to the free terminal time T_1, whereas (33) pertains to the free terminal stock S_{T_1}. We combine these conditions to find $\tilde{S} \equiv S_{T_1}$.

First, observe from (30) that $\dot{\tilde{\mu}} + \dot{\lambda} = \dot{\tilde{\mu}} + \frac{\partial \tilde{\mu}}{\partial S} q_t^o = \frac{\partial \tilde{\mu}}{\partial S} \dot{S}_t + \frac{\partial \tilde{\mu}}{\partial S} q_t^o = 0$ —given that $\dot{S}_t = -q_t^o$ by (31). That is to say, $\tilde{\mu} + \lambda$ is constant over time. Plugging this into (32) then gives:

Next, note from (32) that:

$$e^{-r(T_1-t_0)}\pi(q_{T_1}^o) - \left[\tilde{\mu} + \lambda\right] q_{T_1}^o = e^{-r(T_1-t_0)}\left[1 - e^{-r\frac{S_{T_1}}{\bar{q}}}\right]\pi(\bar{q}).$$

Recalling that $q_{T_1}^o = \bar{q}$ and simplifying, we obtain

$$\left[\tilde{\mu} + \lambda\right]\bar{q} = e^{-r(T_1-t_0)} e^{-r\frac{S_{T_1}}{\bar{q}}}\pi(\bar{q}). \tag{34}$$

Taking $t = T_1$ in (29), one can write $\tilde{\mu} + \lambda = e^{-r(T_1-t_0)}\pi'(\bar{q})$. Plugging this last equality into (34), we get $e^{-r\frac{S_{T_1}}{\bar{q}}} = \frac{\bar{q}\pi'(\bar{q})}{\pi(\bar{q})}$. It thus follows that

$$\tilde{S} \equiv S_{T_1} = \frac{\bar{q}}{r}\ln\left(\frac{\pi(\bar{q})}{\bar{q}\pi'(\bar{q})}\right).$$

Note that $\tilde{S}$ is a function of L_{t_0} —just as $\bar{q}$— but does not depend on the initial stock, S_{t_0}. Also remark that the stock S_{T_1} above always exists given our assumption that π is concave (i.e., $\pi(\bar{q}) > \bar{q}\pi'(\bar{q})$). From what precedes, we conclude that $p_t = c_s(L_t) - \alpha$ whenever $S_t \leq \tilde{S}$ (that is, $T_1 = 0$).

In the case where $S_t > \tilde{S}$, it follows that $T_1 > 0$. Recalling from above that $\tilde{\mu} + \lambda = e^{-r(T_1 - t_0)}\pi'(\bar{q})$, one can use (29) to characterize q_t^o:

$$\pi'(q_t^o) = e^{r(t-t_0)}e^{-r(T_1-t_0)}\pi'(\bar{q}) = e^{r(t-T_1)}\pi'(\bar{q}).$$

It follows that $q_t^o = \pi'^{-1}(e^{r(t-T_1)}\pi'(\bar{q}))$. As stated by Proposition 5, we thus have $p_t = u'(q_t^o) - \mu = u'(e^{r(t-T_1)}\pi'(\bar{q})) - \mu$, where T_1 is determined by the condition:[16]

$$\int_0^{T_1} \underbrace{\pi'^{-1}(e^{r(t-T_1)}\pi'(\bar{q}))}_{q_t^o} dt = S_0 - \tilde{S} \tag{35}$$

and μ is determined by the combination of (29) and the transversality condition (33).$\square$

[16]Equation (35) states the fact that the oil stock is eventually exhausted.