FUNCTIONS OF SUBNORMAL OPERATORS

by

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Introduction and Preliminaries

A subset L of the complex plane is understood to have the subspace topology inherited from C, the complex plane. The interior of L is denoted intL and its closure, L^-; we denote the boundary of L by ∂L. If $A \subset B$ then $\mathrm{int}_B A$ and $\partial_B A$ are respectively the interior and boundary of A as a subspace of B. For a function f defined on B, the symbol $f|_A$ denotes the restriction of f to A. If L is compact, $\hat{L}$, the polynomially convex hull of L, is defined to be the complement of the unbounded component of the complement of L. Equivalently, $\hat{L}$ is the set of all $\lambda \in C$ such that $|p(\lambda)| \leq \sup \{p(\omega)| \ \omega \in L\}$ (see [5], pg. 200).

All Hilbert spaces are assumed to be separable; that is, they have a countable basis. For H a Hilbert space, $B(H)$ denotes the Banach algebra of continuous linear maps from H into H, and if $T \in B(H)$ then $\sigma(T)$ is the compact set consisting of all $\lambda \in C$ such that $T-\lambda$ is not invertible. $C^*(T)$ denotes the smallest norm-closed self-adjoint subalgebra of $B(H)$ containing T and 1. A net $(T_\alpha)_{\alpha\in\Lambda}$ in $B(H)$ converges to T in the weak operator topology (abbreviated WOT) if for each x and y in H, $\langle T_\alpha x, y\rangle \to \langle Tx, y\rangle$. Let W(T) denote the smallest WOT-closed subalgebra of $B(H)$ containing T and 1. If X is a Banach space with dual X^*, then $x_\alpha^* \to x^*$ weak-star in X^* provided $x_\alpha^*(x) \to x^*(x)$ for each $x \in X$. $B(H)$ is the dual of the space of trace class operators; $T_\alpha \to T$ weak-star in $B(H)$ provided that for every pair of sequences (x_n) and (y_n) in H such that $\sum_n \|x_n\|^2 < \infty$ and $\sum_n \|y_n\|^2 < \infty$,

$$\sum_n \langle T_\alpha x_n, y_n\rangle \to \sum_n \langle T x_n, y_n\rangle .$$

$W^*(T)$ is the weak-star closure of $C^*(T)$ (see for example [6], pgs. 19-26).

Let N be a normal operator on the Hilbert space H. Then $C^*(N)$ is isometrically $*$-isomorphic to $C(\sigma(N))$, the algebra of continuous complex valued functions on $\sigma(N)$. This isomorphism is given by the Gelfand-Naimark map, $f \in C(\sigma(N)) \to f(N) \in C^*(N)$. Each pair of vectors x and y in H induces a continuous linear functional on $C(\sigma(N))$ via $f \to \langle f(N)x,y\rangle$, and thus defines a regular Borel measure $\mu_{x,y}$ supported on $\sigma(N)$. Moreover, for $\alpha,\beta \in C$,

$$\mu_{\alpha x_1+\beta x_2,y} = \alpha\mu_{x_1,y} + \beta\mu_{x_2,y} \quad ,$$

and

$$\mu_{x,\alpha y_1+\beta \mu_2} = \bar{\alpha}\mu_{x,y_1} + \bar{\beta}\mu_{x,y_2} \quad .$$

Also, $\mu_{x,y} = \bar{\mu}_{y,x}$ and $\|\mu_{x,y}\| \leq \|x\|\;\|y\|$. Fix f a bounded, Borel measurable function on $\sigma(N)$ and fix $x \in H$. Then the map from H into C defined by

$$y \to \int f \, d\mu_{y,x}$$

for each $y \in H$ is a linear functional, and so there exists $z \in H$

such that $\langle y,z\rangle = \int f \, d\mu_{y,x}$ for each $y \in \mathcal{H}$. The map $x \to z$ is a bounded linear transformation, and is denoted $f(N)$. Thus the Gelfand-Naimark map may be extended to a $*$-homomorphism from $B(N)$, the algebra of bounded, Borel measurable functions on $\sigma(N)$ into $\mathcal{B}(\mathcal{H})$. If χ_Δ is the characteristic function for a Borel set Δ then the projection $\chi_\Delta(N)$ is denoted $E(\Delta)$ and $\langle E(\Delta)x,y\rangle = \int \chi_\Delta d\mu_{x,y}$. The set function E is the spectral measure for N and

$$f(N) \equiv \int f \, dE$$

for each bounded Borel measurable function f. The map $f \to \int f \, dE = f(N)$ is a contraction; that is, $\|f(N)\| \leq \|f\|$, and maps $B(\sigma(N))$ onto $W^*(N)$.

$W^*(N)$ is abelian and thus has a separating vector x_0; that is, x_0 is such that if $A \in W^*(N)$ and $Ax_0 = 0$ then $A = 0$. The positive measure $\mu = \mu_{x_0,x_0}$ is called a scalar-valued spectral measure for N, and $L^\infty(\mu)$ is isometrically $*$-isomorphic to $W^*(N)$ via

$$f \to \int f \, dE \, .$$

Moreoever, this map is a weak-star, weak-star homeomorphism. If ν is a positive measure on $\sigma(N)$ then ν is absolutely continuous with respect to μ if and only if $\nu = \mu_{y,y}$ for some $y \in \mathcal{H}$. For a more systematic discussion of these facts, see [6], pgs. 67-72 and pgs. 91-95 and [8], pg. 112.

An operator S on the Hilbert space $\mathcal{H}$ is subnormal provided there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator N on $\mathcal{K}$ such that $N|_{\mathcal{H}} = S$. We say N is the minimal normal extension of S if $\mathcal{K}$ is the smallest reducing subspace for N which contains $\mathcal{H}$.

The question of whether a normal operator N on $\mathcal{H}$ is the minimal normal extension of any non-normal subnormal operator is equivalent to whether N has non-reducing invariant subspaces. Let $\mathcal{S}(N)$ be the set of non-normal subnormal operators with minimal extension N. If μ is a scalar-valued spectral measure for N, and if $P^{\infty}(\mu)$ denotes the weak-star closure in $L^{\infty}(\mu)$ of the set of polynomials, then $\mathcal{S}(N) \neq \emptyset$ if and only if $P^{\infty}(\mu) \neq L^{\infty}(\mu)$ (see [7], pg. 24).

In Sections 1 and 2 some criteria are developed which determine in the case that N is unitary and f an analytic function defined in a neighborhood of $\partial D = \{z \mid |z| = 1\}$ whether $\mathcal{S}(f(N)) \neq \emptyset$. Since $\sigma(f(N)) = f(\sigma(N))$ and $\nu = \mu \circ f^{-1}$ is a scalar valued spectral measure for f(N) if μ is a scalar valued spectral measure for N, $\mathcal{S}(f(N)) \neq \emptyset$ if, and only if $P^{\infty}(\nu) \neq L^{\infty}(\nu)$.

In order to determine $P^{\infty}(\nu)$, we employ the procedure due to D. Sarason [20], but before describing Sarason's process, additional preliminaries are needed.

If L is a compact subset of the plane, then C(L) is the algebra of continuous complex valued functions on L, furnished with the supremum norm. $C_r(L)$ is the real subspace of C(L) consisting of real valued functions on L, and $\mathcal{R}(L)$ denotes the closure in C(L) of the set of rational functions with poles off L.

$\mathcal{R}(L)$ is a Dirichlet algebra provided that the real linear manifold

$$\{Re(q)|_{\partial L} \mid q \in R(L)\}$$

is uniformly dense in $C_r(\partial L)$. If $R(L)$ is a Dirichlet algebra, the components of intL are simply connected (see [6], pg. 356).

Let G be a bounded open set in the plane and $u \in C_r(\partial G)$. The Perron family for u, $P(u, G)$, consists of all subharmonic functions ϕ on G such that $\limsup_{z\to a} \phi(z) \leq u(a)$ for all $a \in \partial G$. Then $\hat{u}(z) = \sup \{\phi(z) \mid \phi \in P(u, G)\}$ is harmonic on G. G is a Dirichlet set precisely when $\hat{u}$ extends to a continuous function on G^- such that $\hat{u}|_{\partial G} = u$; in particular, every simply connected region is a Dirichlet set (see [5], pgs. 268 and 276).

Let L be compact and G = intL. Then the map $C_r(\partial L) \to C_r(G)$ defined by $u \to (u|_{\partial G})^\wedge$ is positive, linear and a contraction. For each $a \in G$ therefore, there is a unique probability measure, ω_a, supported on ∂G such that for all $u \in C_r(\partial L)$,

$$\int u\,d\omega_a = (u|_{\partial G})^\wedge(a) \ .$$

The measure ω_a is called harmonic measure for L evaluated at a. If A is a uniformly closed subalgebra of C(L), a representing measure for a in L is a probability measure λ_a such that $f(a) = \int f d\lambda_a$ for all $f \in A$. If $R(L)$ is a Dirichlet algebra, then each point a in L has a unique representing measure supported on ∂L. In case $a \in \partial L$, λ_a is point mass measure at a, and if $a \in$ intL, then $\lambda_a = \omega_a$. If a and b are contained in the same component of intL, then ω_a and ω_b are boundedly mutually absolutely continuous; otherwise, if they are in different components, λ_a and λ_b are singular (see [6], pgs. 328-336).

Let L be compact and $U_1, U_2, \ldots$ the components of intL. If intL = $\emptyset$, define harmonic measure for L to be 0; otherwise, choose $a_n \in U_n$ and define harmonic measure for L to be

$$\omega = \sum_n \frac{1}{2^n} \omega_{a_n} \quad .$$

While ω depends on the choice of the sequence $\{a_n\}$, the sets of measure zero do not. If $\{a_n'\}$ is another sequence of points such that $a_n' \in U_n$, then ω_{a_n} and $\omega_{a'_n}$ are mutually absolutely continuous. Therefore $L^\infty(\partial L) = L^\infty(\omega)$ is well-defined. Also, by the Radon-Nikodyn theorem, $L'(\omega)$ may be identified with the space of Borel measures absolutely continuous with respect to ω, and is therefore dependent only on the sets of harmonic measure zero. Thus the weak-star topology on $L^\infty(\partial L)$ is well-defined. Let $H^\infty(\partial L)$ denote the weak-star closure of the polynomials in $L^\infty(\partial L)$, (see [6], pg. 383).

Theorem 1 (see [6], pg. 387): If $R(L)$ is a Dirichlet algebra and $U_1, U_2, \ldots$ the components of intL, then $H^\infty(\partial L)$ is isometrically isomorphic to $H^\infty(\text{intL})$ and thus to $\bigoplus_n H^\infty(U_n)$. The isomorphism is given by the map $f \to \hat{f}$ where

$$\hat{f}(z) = \int f \, d\omega_z \ ;$$

Moreover, $f \to \hat{f}$ is a weak-star, weak-star homeomorphism.

Let σ be a positive regular Borel measure with support contained in the compact set L. The L-admissable component of σ is the measure

$$(\sigma \mid_{\partial L})_a + \sigma_{intL}$$

where

$$\sigma|_{\partial L} = (\sigma \mid_{\partial L})_a + (\sigma|_{\partial L})_s$$

is the Lebesgue decomposition of $\sigma|_{\partial L}$ with respect to harmonic measure for L. For $R(L)$ a Dirichlet algebra and $q \in H^{\infty}(intL)$, there is, by Theorem 1, a unique $h \in H^{\infty}(\partial L)$ so that $\hat{h} = q$. Thus we may extend q to ∂L, and if $\sigma|_{\partial L}$ is absolutely continuous with respect to harmonic measure for L, then $q \in L^{\infty}(\sigma)$.

We can now describe Sarason's process to determine $P^{\infty}(\sigma)$. For each countable ordinal α, define a compact set L_α and a measure σ_α as follows:

(i) Let $L_1 = (\text{support } \sigma)\hat{}$. Since L_1 is polynomially convex, $R(L_1)$ is a Dirichlet algebra. Let σ_1 be the L_1-admissable component of σ.

(ii) If L_α is defined so that $R(L_\alpha)$ is a Dirichlet algebra, let σ_α be the L_α admissible component of σ. Define $L_{\alpha+1}$ by

$$L_{\alpha+1} = (\text{support } \sigma) \cup \{z \in intL_\alpha \mid |q(z)| \le \|q\|_{\sigma_\alpha} \text{ for all } q \in H^{\infty}(intL\ )\}.$$

Then $R(L_{\alpha+1})$ is a Dirichlet algebra.

(iii) If α is a limit ordinal and L_β is defined for each $\beta < \alpha$, let

$$L_\alpha = \bigcap_{\beta \varepsilon \alpha} L_\beta \ .$$

Then $R(L_\alpha)$ is a Dirichlet algebra provided $R(L_\alpha)$ is for each $\beta < \alpha$.

<u>Theorem 2</u> [20]: There is countable ordinal α_0 so that $L_\alpha = L_{\alpha_0}$ for each $\alpha \geq \alpha_0$. If $\mathrm{int}L_{\alpha_0} = \emptyset$ then $P^\infty(\sigma) = L^\infty(\sigma)$; otherwise, $\sigma - \sigma_{\alpha_0} \perp \sigma_{\alpha_0}$ and

$$P^\infty(\sigma) = P^\infty(\sigma_{\alpha_0}) \oplus L^\infty(\sigma - \sigma_{\alpha_0}) .$$

$P^\infty(\sigma_{\alpha_0})$ is isometrically isomorphic to $H^\infty(\partial L_{\alpha_0})$ and thus to $H^\infty(\mathrm{int}L_{\alpha_0})$.

$\tilde{L} = L_{\alpha_0}$ is the Sarason hull of σ, and if $\tilde{\sigma}$ is the $\tilde{L}$-admissable component of σ then $(\tilde{L},\tilde{\sigma})$ is called the Sarason pair for σ.

Let m denote nomalized Lebesgue measure on ∂D. The purpose of this paper is to show that for N unitary with scalar-valued spectral measure μ and f a function analytic in a neighborhood of $\partial D = \{z \mid |z| = 1\}$, $S(f(N))$ is non empty if, and only if there is a rectifiable Jordan curve Γ contained in the curve $f(\partial D)$ such that $m \circ f^{-1}|_\Gamma$ is absolutely continuous with respect to $\mu_a \circ f^{-1}|_\Gamma$. Here

$\mu = \mu_a + \mu_s$ is the Lebesgue decomposition of μ with respect to m.

Thus, letting $f(z) = z$, if μ is a positive regular Borel measure supported on ∂D, then $P^\infty(\mu) \neq L^\infty(\mu)$ if, and only if $m \ll \mu_a$.

Let P denote the Cantor ternary set and ψ Lebesgue's singular function (see [11], pg. 113). Define a measure μ_s on ∂D by
$\int q \, d\mu_s = \int_0^1 q(\exp(i(\Pi t - \frac{\Pi}{2})))d\psi(t)$ and $\mu_a = m|_{\{e^{i\Theta} \mid \frac{\Pi}{2} \leq |\Theta| \leq \pi\}}$.

Let $\mu = \mu_a + \mu_s$ and $f(z) = (z+1)^4$. (see Figure 1). Define ν on $f(\partial D)$ to be the measure $\nu = \mu \circ f^{-1}$. Then $P^\infty(\mu) = L^\infty(\mu)$. But $\Gamma = f(\{e^{i\Theta} | \frac{\Pi}{2} \leq |\Theta| \leq \Pi\})$ is a rectifiable Jordan curve on which

$$m \circ f^{-1}|_\Gamma = \mu_a \circ f^{-1}|_\Gamma$$

and therefore $P^\infty(\nu) \neq L^\infty(\nu)$; in fact, $P^\infty(\nu)$ is isometrically isomorphic to

$$H^\infty(\operatorname{int}\Gamma) \oplus L^\infty(\mu_s \circ f^{-1}) .$$

That $S(f(N))$ is independent of μ_s fails for f a disc algebra function rather than analytic. Define the singular measure μ on ∂D by $\int g \, d\mu = \int_0^1 g(e^{it}) d\psi(t)$ where again ψ is Lebesgue's singular function supported on P Gantor's ternary set. Let M_z be the unitary operator on $L^2(\mu)$ defined by $M_z\phi = z\phi$. There is a continuous function q from $P' = \{e^{i\Theta} | \Theta \in P\}$ onto D^-, and, since $m(P') = 0$, q extends to a Disc Algebra function f of norm 1 (see [13], pg. 81). Then $f(M_z) = M_f$ and $\sigma(M_f) = D^-$; in particular, $\operatorname{int}(\sigma(M_f)) \neq \emptyset$ and therefore M_f is non-reductive.

Throughout Sections 1 and 2, let f be analytic in a neighborhood of ∂D and μ a positive regular Borel measure with support contained in ∂D. Let $K = f(\partial D)$, and define the measure $\nu = \mu \circ f^{-1}$ on K. In order to apply Sarason's process to determine $P^\infty(\nu)$, we must first examine the geometry of K. This is the content of the first section.

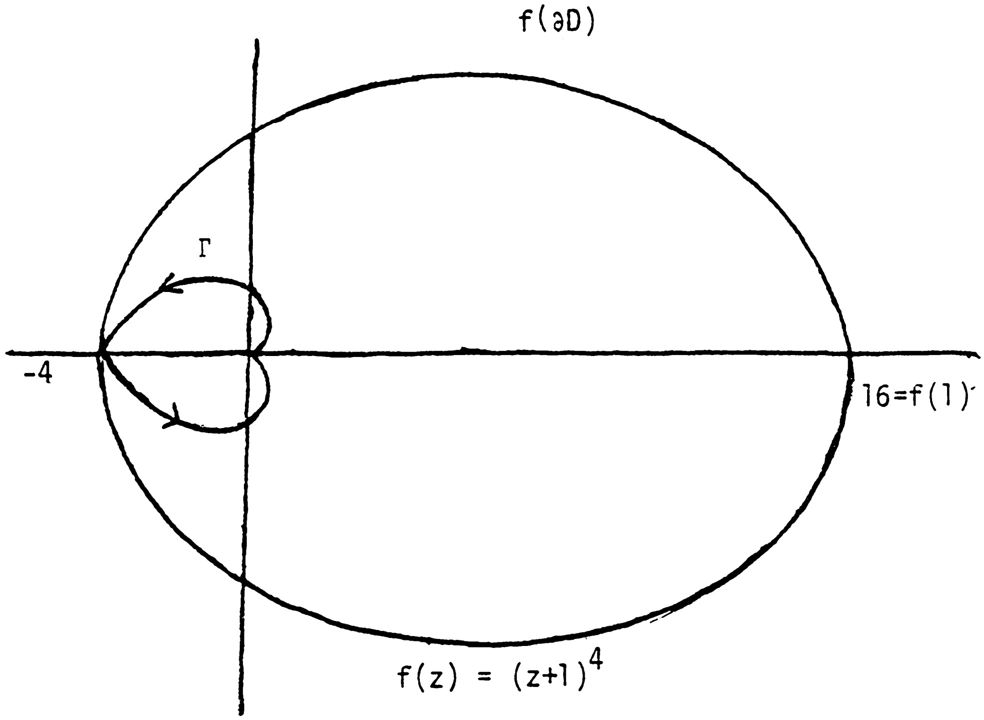

Figure 1.

In it we show that of a finite set S, the map

$$f|_{\partial D} : \partial D - S \to K$$

is an open map. Consequently, the complement of K has finitely many components. Moreover, if U is a simply connected, bounded region with ∂U contained in K then ∂U is the union of finitely many rectifiable Jordan curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ such that $\Gamma_i \cap \Gamma_j$ is finite whenever $i \neq j$.

In the second Section, we consider a compact set L such that ∂L is contained in K and $R(L)$ is a Dirichlet algebra. Let U be a component of intL. Since harmonic measure for $\bar{U}$ is defined only up to mutual absolute continuity, the results of Section 1 enable us to decompose harmonic measure for $\bar{U}$ as $\sum_{j=1}^{n} m \circ \phi_j^{-1}$ where $\partial U = \bigcup_{j=1}^{n} \Gamma_j$ and ϕ_j is an absolutely continuous, one-to-one function from ∂D onto Γ_j. This is turn allows us to show that harmonic measure for L and $m \circ f^{-1}|_{\partial(\mathrm{int}L)}$ are mutually absolutely continuous. The L-admissable component of ν is then

$$\nu_L = (\nu|_{\partial(\mathrm{int}L)})_a + \nu|_{\mathrm{int}L}$$

$$= \mu_a \circ f^{-1}|_{\partial(\mathrm{int}L)} + \mu \circ f^{-1}|_{\mathrm{int}L}$$

where $\nu|_{\partial(\mathrm{int}L)} = \mu_a \circ f^{-1}|_{\partial(\mathrm{int}L)} + \mu_s \circ f^{-1}|_{\partial(\mathrm{int}L)}$ is the Lebesgue decomposition of $\nu|_{\partial(\mathrm{int}L)}$ with respect to harmonic measure of L.

In the third section we consider another problem. Let G be a bounded region of the plane. If we put planar Lebesgue measure on G then $H^\infty(G)$ is a weak-star closed subspace of $L^\infty(G)$, the dual of the separable Banach space $L^1(G)$, (see for example [18]). Moreover, a sequence $\{f_n\}$ in $H^\infty(G)$ converges weak-star to zero if, and only if $\{\|f_n\|\}$ is bounded and $\{f_n(\lambda)\}$ converges to zero for each λ in G.

By a representation of $H^\infty(G)$ we mean a norm continuous homomorphism π from $H^\infty(G)$ into $B(H)$ for some separable Hilbert space H. π is unital provided $\pi(1) = 1_H$, where 1_H denotes the identity operator on H. In [2] B. Chevreau, C. Pearcy and A. Shields have shown the following:

Theorem: Let $\lambda_1,\ldots,\lambda_n$ be distinct points in D, and let $0 < \delta_j < \min(\{|\lambda_j - \lambda_k| \mid 1 \le k \le n, j \ne k\} \cup \{1 - |\lambda_j|\})$, and $B_j = \{\lambda \mid |\lambda - \lambda_j| < \delta_j\}$. Let $G = D$ or $G = D - \bigcup_{j=1}^{n} \bar{B}_j$. If $A \in B(H)$ is such that there exists an $M > 0$ for which $\|r(A)\| \le M \sup_{\lambda\in G^-} |r(\lambda)|$ for each rational function r with poles off G^-, and if the sequences $\{A^m\}_m$ and $\{(\delta_j(A-\lambda_j)^{-1})^m\}_m$ converge to zero sot for each j then there is a unique unital representation π of $H^\infty(G)$ so that $\pi(z) = A$. Moreover, π is weak-star, sot continuous. Here, a net $\{A_i\}$ in $B(H)$ converges to zero sot provided that for each $x \in H$ $\|A_i x\| \to 0$.

Chevreau, Pearcy and Shields have asked whether if $A \in \mathcal{B}(\mathcal{H})$ is such that $\|r(A)\| \leq \|r\|_{D^-}$ for each rational function r with poles off D^- and if A^k converges to zero wot implies a representation of $\mathcal{H}^\infty(D)$ sending z to A must be unique.

In [17], the authors have answered this question negatively. In fact, if μ is a positive regular Borel measure supported on ∂D then for each infinite Blaschke product ϕ whose zeroes accumulate at each point of the support of μ and for each function $f \in L^\infty(\mu)$ with $\|f\| \leq 1$ there is a unital, norm continuous representation π such that

$$\pi(z) = M_z$$

and

$$\pi(\phi) = M_f ,$$

where M_z and M_f are the normal operators on $L^2(\mu)$ multiplication by z and f respectively.

If, however, S is a pure subnormal contraction on $\mathcal{H}$ and N its minimal normal extension on $\mathcal{K}$, then we shall show that the natural representation of $\mathcal{H}^\infty(D)$ defined by $f \to f(N)|_{\mathcal{H}}$ is unique. The notation f(N) makes sense since it follows from the purity of S that E is absolutely continuous with respect to Lebesgue measure on ∂D (see [7], pg. 30). Since $\mathcal{H}^\infty(D)$ is the weak-star closure of the polynomials in $L^\infty(m)$, the operator f(N) leaves $\mathcal{H}$ invariant for each $f \in \mathcal{H}^\infty(D)$.

More generally, the third section establishes that if π is a unital representation of $H^{\infty}(G)$ such that $\pi(z) = S$ then π is weak-star, weak-star continuous. Thus the uniqueness of the representation of $H^{\infty}(D)$ mentioned above follows from the weak-star density of the polynomials in $H^{\infty}(D)$.

If $G^{*} = \{\lambda | \bar{\lambda} \in G\}$, and if for each $f \in H^{\infty}(G^{*})$, $\tilde{f}$ is defined on G by $\tilde{f}(z) = (f(\bar{z}))^{-}$, then the conjugate linear representation π^{*} defined on $H^{\infty}(G^{*})$ by

$$\pi^{*}(f) = (\pi(\tilde{f}))^{*}$$

is weak-star, sot continuous. In particular, if S is a pure subnormal contraction then S^{*n} converges to zero sot.

The problem of the existence of such subnormal representations is much harder. For instance, if G is the slit disc, that is $G = D - \{\lambda | \lambda \leq 0\}$, then it is unknown for which pure subnormal operators S there is a unital representation π of $H^{\infty}(G)$ so that $\pi(z) = S$.

The Geometry of $f(\partial D)$

Lemma 3: Let C be a circle in S^2, the Riemann sphere, and suppose that ϕ is analytic at $a \in C$. If $\{a_n\}_{n \in N}$ is an infinite sequence in C converging to a such that $\phi(a_n) \in C$ for all n, then there is an $\varepsilon > 0$ so that $\phi(B(a,\varepsilon) \cap C) \subset C$.

Proof: We may assume $C = \partial D$. If $\xi(z) = \frac{1}{\bar{z}}$ then ξ is a homeomorphism of $C - \{0\}$ onto $C - \{0\}$ and thus an open map. Suppose that ϕ is analytic on $B(a,r)$ for some r, $0 < r < 1$, and let $V = B(a,r) \cap \xi(B(a,r))$. Then V is connected and open about a and $\xi(V) \subset V$. Let

$$\psi(z) = (\phi(\xi(z)))^-$$

then ψ is analytic (see [5], pg. 215), and if $z \in V \cap \partial D$ then

$$\psi(z)\ \phi(z) = |\phi(z)|^2 .$$

Since $\psi(a_n)\ \phi(a_n) = 1$ for each n, $\psi\phi$ is identically 1 on V; in particular, if $z \in V \cap \partial D$, then $1 = \psi(z)\phi(z) = |\phi(z)|^2$ ∎

Recall that a map between topological spaces X and Y is open at $x \in X$ provided that $\{f(U_i)\}_{i \in I}$ is a neighborhood base for Y at f(x) whenever $\{U_i\}_{i \in I}$ is a neighborhood base for X at x. Thus if $f|_{\partial D} : \partial D \to K$ is not open at $a \in \partial D$ then for every $\varepsilon > 0$ there is a δ with $0 < \delta \leq \varepsilon$ and a sequence $\{b_n\}_{n \in N} \subset \partial D$ such that $f(b_n)$ converges to $f(a)$, but $f(b_n) \notin f(B(a,\delta) \cap \partial D)$. By considering a subsequence, we may assume that the sequence $\{b_n\}_{n\varepsilon N}$ converges to a point b on ∂D.

Lemma 4: $f|_{\partial D} : \partial D \to K$ is open at all but finitely many points.

Proof: Suppose that $\{a_n\}_{n\varepsilon N}$ is a sequence in ∂D such that $f|_{\partial D}$ is not open at each a_n. Because f is nonconstant, we may assume that $f(a_n) \neq f(a_m)$ if $n \neq m$, and, for all n, $f'(a_n) \neq 0$. For each n, there is a $\delta_n > 0$ and a sequence $\{b_{n,k}\}_{k\varepsilon N}$ so that

$$f|_{B(a_n,\delta_n)} \text{ is one-to-one,}$$

$$f(b_{n,k}) \to f(a_n) \quad \text{as} \quad k \to \infty,$$

$$f(b_{n,k}) \varepsilon f(B(a_n,\delta_n)) - f(B(a_n,\delta_n) \cap \partial D),$$

and there exists b_n such that

$$b_{n,k} \to b_n \quad \text{as} \quad k \to \infty.$$

Again, by considering subsequences, we may assume that $a_n \to a$ and $b_n \to b$. Let arg: $\partial D \to [-\pi,\pi)$ denote the principal branch of the argument function. By composing f with a rotation, we may assume that arg is continuous on $\{a_n\}^- \cup \{b_n\}^-$, and, by passing to subsequences if necessary, that $\{\arg a_n\}$ and $\{\arg b_n\}$ are monotonic. The other cases being similar, for definiteness suppose that $\{\arg a_n\}$ and $\{\arg b_n\}$ are increasing. Assume further that $f(a) = 0$.

Let T_a and T_b be Möbius transformations such that $T_a^{-1}(\partial D) = T_b^{-1}(\partial D) = R$, $T_a^{-1}(a) = T_b^{-1}(b) = 0$, and that $T_a^{-1}(a_n)$ and $T_b^{-1}(b_n)$ are positive. Let p and q denote the orders of the zeroes of f at a and b respectively and define

$$g_a = f o T_a o z^q \quad ,$$

and

$$g_b = f o T_b o z^p \quad .$$

Then both g_a and g_b have zeroes at 0 of order pq. Let α_k, β_k, and $\beta_{k,j}$ be positive such that

$$\alpha_k^q = T_a^{-1}(a_k),$$

$$\beta_k^p = T_b^{-1}(b_k), \text{ and}$$

$$\beta_{k,j}^p = T_b^{-1}(b_{k,j}) \quad .$$

Then $g_a(\alpha_k) = f(a_k)$

$= f(b_k)$

$= g(\beta_k)$,

and for each $n \in N$ and $\varepsilon > 0$, there is an M so that if $k \geq M$ then

(1) $\emptyset \neq g_a^{-1}(g_b(\beta_{n,k})) \cap B(\alpha_n, \varepsilon)$.

Let $r > 0$ be such that, on $B(0,r)$, $g_a(z) = z^{pq}\, h_a(z)$ and $g_b(z) = z^{pq}\, h_b(z)$ where h_a and h_b are analytic, $|h_a(0)|$ and $|h_b(0)|$ are positive. By multiplying f by a constant of modulus 1 if necessary, we may assume that $h_a(0)$ and $h_b(0)$ are not elements of $(-\infty,0]$. Thus there exists $\varepsilon > 0$ so that the principal branch of Log z is analytic on $h_a(B(0,\varepsilon)) \cup h_b(B(0,\varepsilon))$.

Let $\xi_a = z \exp(\frac{1}{pq} \operatorname{Log} h_a)$ and

$\xi_b = z \exp(\frac{1}{pq} \operatorname{Log} h_b)$ on $B(0,\varepsilon)$. Then $\xi_a'(0)$ and $\xi_b'(0)$ are nonzero and so, by choosing ε sufficiently small, we may assume ξ_a and ξ_b are one-to-one on $B(0,\varepsilon)$. Moreover, $\xi_a^{pq} = g_a$ and $\xi_b^{pq} = g_b$ on $B(0,\varepsilon)$.

As $z \to 0$, $z > 0$:

$$\arg(g_a(z)) = \arg(h_a(z)) \to \arg(h_a(0)) ,$$

$$\arg(g_b(z)) = \arg(h_b(z)) \to \arg(h_b(0)) ,$$

$$\arg(\xi_a(z)) = \frac{1}{pq} \arg(h_a(z)) \to \frac{1}{pq} \arg(h_a(0)) ,$$

and $$\arg(\xi_b(z)) = \frac{1}{pq} \arg(h_b(z)) \to \frac{1}{pq} \arg(h_b(0)) .$$

But $\arg(g_a(\alpha_k)) = \arg(g_b(\beta_k))$ and both α_k and β_k converge to 0 as $k \to \infty$; thus $\mathrm{Arg}(h_a(0)) = \mathrm{Arg}(h_b(0))$.

Let
$$\Theta_0 = \frac{1}{pq} \arg(h_a(0)) = \lim_{\substack{z\to 0\\ z>0}} \arg(\xi_a(z)) = \lim_{\substack{z\to 0\\ z>0}} \arg(\xi_b(z)) ,$$

and define

$$\Delta = \{z \in \xi_a(B(0,\varepsilon)) \cap \xi_b(B(0,\varepsilon)) \mid z = 0 \text{ or } |\arg z - \Theta_0| < \tfrac{\pi}{2pq} \}.$$

Let $U = B(0,\varepsilon) \cap \xi_a^{-1}(\Delta) \cap \xi_b^{-1}(\Delta)$. Then U is open, and, since $\lim_{\substack{z\to 0\\ z>0}} \xi_a(z) = \lim_{\substack{z\to 0\\ z>0}} \xi_b(z) = \Theta_0$, there is a $\delta > 0$ such that $(0,\delta) \subset U$.

Since z^{pq} is one-to-one on Δ, and ξ_a and ξ_b are one-to-one on $B(0,\varepsilon)$, the functions g_a and g_b are one-to-one on U. By Lemma 3,

(2) $((\xi_a|_{B(0,\varepsilon)})^{-1}\xi_b)\ (0,\sigma) \subset (0,\varepsilon)$.

Choose N such that for $n \geq N$, α_n and β_n are contained in $(0,\delta)$. Choose $\eta > 0$ such that $B(\alpha_N,\eta) \subset U$ and $g_a(B(\alpha_N,\eta)) \subset f(B(a_N,\delta_N))$. By

(1) there is an M such that if $k \geq M$ then $\beta_{N,k} \varepsilon (0,\delta)$ and $g_b(\beta_{N,k}) = f(b_{N,k})$ is contained in $g_a(B(\alpha_N,\eta) - R)$. Thus $\xi_b(\beta_{N,k}) \varepsilon \xi_a(B(\alpha_N,\eta) - R)$, contradicting (2) ∎

Let $S = \{a \varepsilon \partial D \mid f'(a) = 0 \text{ or } f|_{\partial D} \text{ is not open at } a\}$.

Lemma 5: Let U be a bounded region with $\partial U \subset K$. Then

$$(f|_{\partial D})^{-1}(\partial U) - S \subset \mathrm{int}_{\partial D}((f|_{\partial D})^{-1}(\partial U)) ;$$

that is, $$\partial_{\partial D}((f|_{\partial D})^{-1}(\partial U)) \subset S .$$

Proof: Let $a \varepsilon (f|_{\partial D})^{-1}(\partial U) - S$. We wish to show there exists an $\varepsilon > 0$ such that $f(B(a,\varepsilon) \cap \partial D) \subset \partial U$. Choose $\delta > 0$ such that

$$f|_{B(a,\delta)} \text{ is one to one,}$$

and $$f|_{B(a,\delta) \cap \partial D} : B(a,\delta) \cap \partial D \to K$$

is an open map.

Then there is an open set V in the plane such that $V \subset f(B(a,\delta))$ and $V \cap K = f(B(a,\delta) \cap \partial D)$. Choose $\varepsilon > 0$ such that $B(a,\varepsilon) \subset (f|_{B(a,\delta)})^{-1}(V)$.

Now $B(a,\varepsilon)$ is homeomorphic to $f(B(a,\varepsilon))$ and is disconnected by $B(a,\varepsilon) \cap \partial D$; moreover, if L is a proper subset of $B(a,\varepsilon) \cap \partial D$, then $B(a,\varepsilon) - L$ is connected. Thus $f(B(a,\varepsilon))$ is disconnected by $f(B(a,\varepsilon) \cap \partial D)$ and by no proper subset. Since

$$f(B(a,\varepsilon)) - \partial U = (f(B(a,\varepsilon)) \cap U) \cup (f(B(a,\varepsilon)) - U)$$

is disconnected and

$$\begin{aligned} \partial U \cap f(B(a,\varepsilon)) &\subset K \cap f(B(a,\varepsilon)) \\ &= f(B(a,\varepsilon) \cap \partial D); \end{aligned}$$

$$\partial U \cap f(B(a,\varepsilon)) = f(B(a,\varepsilon) \cap \partial D) \blacksquare$$

<u>Theorem 6</u>: C-K has only finitely many components.

<u>Proof</u>: Let $\mathcal{T}$ denote the set of compact, connected proper subsets of ∂D with end points contained in $\mathcal{S}$. $\mathcal{T}$ is a finite set, and if U is a component of C-K then, by Lemma 5, there is a subset F_U of $\mathcal{T}$ such that $(f|_{\partial D})^{-1}(\partial U) = \bigcup_{L \varepsilon F_U} L$.

The following Lemma shows that the map $U \to F_U$ from the components of C-K into a finite set is one-to-one.

<u>Lemma 7</u>: Let U and V be bounded regions whose boundaries are contained in K. If $\partial(\overline{U})^{\wedge} \subset \partial(\overline{V})^{\wedge}$ then $U \cap V \neq \emptyset$ and $(\overline{U})^{\wedge} = (\overline{V})^{\wedge}$.

Proof: By the Maximum Modulus Theorem,

$$(\overline{U})^{\wedge} = \{z \mid |p(z)| \leq \|p\|_{\partial(\overline{U})^{\wedge}} \text{ for all polynomials } p\}$$

$$\subset \{z \mid |p(z)| \leq \|p\|_{\partial(\overline{V})^{\wedge}} \text{ for all polynomials } p\}$$

$$= (\overline{V})^{\wedge} .$$

By Lemma 5, there exist $a \in (f|_{\partial D})^{-1}(\partial(\overline{U})^{\wedge}) - S$ and $\delta > 0$ such that

$$B(a,\delta) \cap \partial D \subset (f|_{\partial D})^{-1}(\partial(\overline{U})^{\wedge}) - S ,$$

$$f|_{B(a,\delta)} \text{ is one-to-one } ,$$

$$f(B(a,\delta)) \cap K = f(B(a,\delta) \cap \partial D) ,$$

and
$$f_{B(a,\delta) \cap \partial D} : B(a,\delta) \cap \partial D \to K$$

is an open map. Since $f(a) \in \partial(\overline{U})^{\wedge} \subset \partial(\overline{V})^{\wedge}$,

$$f(B(a,\delta)) \cap U \neq \emptyset ,$$

$$f(B(a,\delta)) \cap V \neq \emptyset ,$$

and $$f(B(a,\delta)) - (\overline{V})^{\wedge} \neq \emptyset .$$

Let $U_1 = f(B(a,\delta) \cap D)$ and

$U_2 = f(B(a,\delta) - D^{-})$.

Either $U_1 \cap U \neq \emptyset$ or $U_2 \cap U \neq \emptyset$. Suppose that $U_1 \cap U \neq \emptyset$; the proof in the other case is exactly the same.

As in the proof of Lemma 5,

$$\begin{aligned}\partial(\overline{U})^{\wedge} \cap f(B(a,\delta)) &\subset \partial U \cap f(B(a,\delta)) \\ &\subset K \cap f(B(a,\delta)) \\ &= f(B(a,\delta) \cap \partial D)) \\ &= f(B(a,\delta)) \cap \partial(\overline{U})^{\wedge} .\end{aligned}$$

Therefore,

$$\begin{aligned}f(B(a,\delta)) &= U_1 \cup U_2 \cup f(B(a,\delta) \cap \partial D) \\ &= (f(B(a,\delta) \cap U) \cup (f(B(a,\delta)) - \overline{U}) \\ &\quad \cup (f(B(a,\delta) \cap \partial D) \\ &= (f(B(a,\delta) \cap \operatorname{int}(\overline{U})^{\wedge}) \cup (f(B(a,\delta))-(\overline{U})^{\wedge}) \\ &\quad \cup (f(B(a,\delta) \cap \partial D) ,\end{aligned}$$

Since U_1 is connected,

$$\begin{aligned}U_1 &= f(B(a,\delta)) \cap U \\ &= f(B(a,\delta)) \cap \operatorname{int}(\overline{U})^{\wedge},\end{aligned}$$

and similarly

$$U_2 = f(B(a,\delta)) - \overline{U} ,$$

$$U_2 = f(B(a,\delta))-(\overline{U})^{\wedge}$$
$$\supset f(B(a,\delta))-(\overline{V})^{\wedge} \quad .$$

Now $U_2 = (\mathrm{int}(\overline{V})^{\wedge} \cap U_2) \cup (U_2-(\overline{V})^{\wedge})$

since $U_2 \cap (\overline{V})^{\wedge} = \emptyset$. If $V \cap U = \emptyset$ then $\mathrm{int}(\overline{V})^{\wedge} \cap U_2 \neq \emptyset$ since $\emptyset \neq f(B(a,\delta)) \cap V$, and

$$\begin{aligned} f(B(a,\delta)) \cap V &= (f(B(a,\delta))-\overline{U}) \cap V \\ &= U_2 \cap V \\ &\subset U_2 \cap \mathrm{int}(\overline{V})^{\wedge} \quad . \end{aligned}$$

Also, $U_2-(\overline{V})^{\wedge} = f(B(a,\delta))-(\overline{V})^{\wedge}$ is nonempty since $f(a) \in \partial(\overline{V})^{\wedge}$. This contradicts the fact that $U_2 = f(B(a,\delta)-\overline{D})$ is connected. Thus $U \cap V \neq \emptyset$ and therefore

$$\begin{aligned} V &= (V \cap (\overline{U})^{\wedge}) \cup (V-(\overline{U})^{\wedge}) \\ &= (V \cap \mathrm{int}(\overline{U})^{\wedge}) \cup (V-(\overline{U})^{\wedge}) \end{aligned}$$

Since $\partial(\overline{U})^\wedge \subset \partial(\overline{V})^\wedge \subset \partial V$. Thus $V \subset (\overline{U})^\wedge$ and so $(\overline{V})^\wedge \subset (\overline{U})^\wedge$. ∎

The results above depend heavily of course on K's being the analytic image of ∂D. Consider the following example.

If $z \in C-[-2i,2i]$ then define $q(z) = \sqrt{\rho_1\rho_2}\ \exp(i(\frac{\Theta_1+\Theta_2}{2}))$ where $z = \rho_1 e^{i\Theta_1} + 2i = \rho_2 e^{i\Theta_2} - 2i$, the arguments Θ_1 and Θ_2 being chosen between $\frac{\pi}{2}$ and $\frac{5\pi}{2}$. Then q is analytic on $C-[-2i,2i]$. Define ξ on $S^2-[-2i,2i]$ by $\xi(z) = \frac{i}{2}(-z + q(z))$ if $z \in C-[-2i,2i]$, and $\xi(\infty) = 0$. Then ξ is a conformal map from $S^2-[-2i,2i]$ onto D, and $\xi^{-1}(z) = i(z + \frac{1}{z})$. Define

$$G_1 = \{x + iy \mid x \neq 0 \text{ and } y > \sin\tfrac{1}{x}, \text{ or } x = 0 \text{ and } y > 2\}$$

and

$$G_2 = \{x + iy \mid x \neq 0 \text{ and } y < \sin\tfrac{1}{x}, \text{ or } x = 0 \text{ and } y < -2\}$$

(see Figure 2). Then G_1 and G_2 are simply connected and

$$S^2-[-2i,2i] = G_1 \cup G_2 \cup \{x + 2i \sin\tfrac{1}{x} \mid x \in R \cup \{\infty\},\ x \neq 0\}.$$

Here we set $x + 2i\sin\frac{1}{x} = \infty$ if $x = \infty$. If T is the Möbius transformation such that $T(1) = 0$, $T(-1) = i$, and $T(-i) = \infty$ then the image of ∂D under T is $R^{\#}$, the extended Real Line in S^2. Define $\psi = \xi \circ \rho \circ T$ where $\rho(z) = z + 2i\sin\frac{1}{z}$. Then ψ is analytic on $\partial D - \{1\}$; observe ψ has a removable singularity at $-i$. Let $\Lambda = \psi(\partial D - \{1\})$.

A routine but tedious calculation establishes that

$$\{\operatorname{Arg}\xi(x + 2i\sin\tfrac{1}{x}) \mid x < 0\} = (-\pi, 0)$$

and

$$\{\operatorname{Arg}\xi(x + 2i\sin\tfrac{1}{x}) \mid x > 0\} = (0, \pi)\ .$$

Moreover, as x approaches 0 from the left, $\operatorname{Arg}\xi(x + 2i)$ increases to 0 and $\operatorname{Arg}\xi(x - 2i)$ decreases to $-\pi$. As x approaches 0 from the right, $\operatorname{Arg}\xi(x + 2i)$ and $\operatorname{Arg}\xi(x - 2i)$ monotonically approach 0 and π, respectively. Also, notice that if $x > 0$ then $\xi(x) = \frac{i}{2}(-x + \sqrt{x^2 + 4})$ and if $x < 0$ then $\xi(x) = \frac{-i}{2}(x + \sqrt{x^2 + 4})$. Finally, since $\xi^{-1}(e^{i\Theta}) = i(e^{i\Theta} + e^{-i\Theta}) = 2i\cos\Theta$, we have that

$$|\xi(x + 2i\sin\tfrac{1}{x})| \to 1 \quad \text{as } x \to 0\ .$$

These facts enable us to sketch $\psi(\partial D - \{1\})$ in Figure 2. Also they show that if $U_1 = \xi(G_1)$ and $U_2 = \xi(G_2)$ then

$$D = U_1 \cup U_2 \cup \Lambda\ ,$$

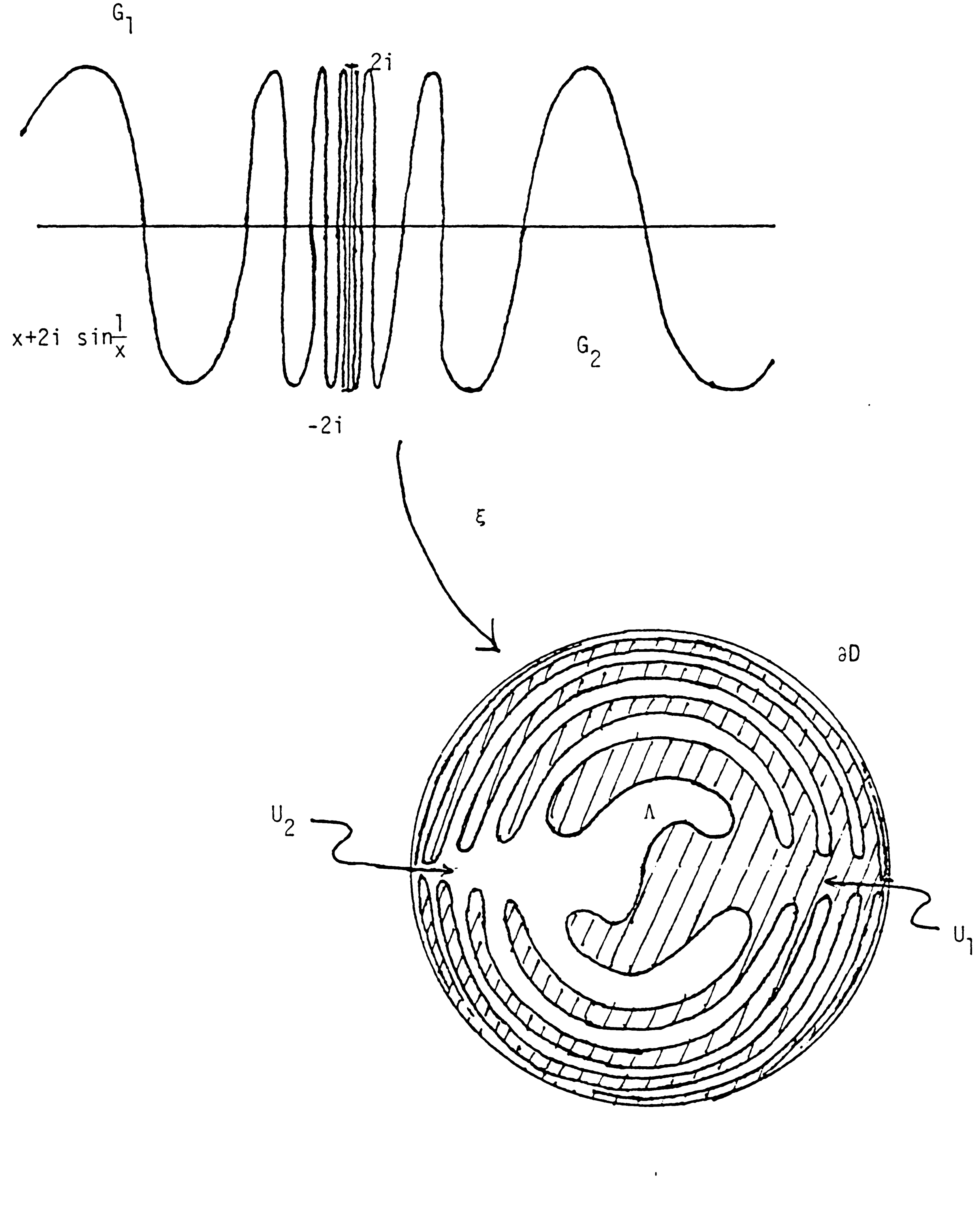

Figure 2.

the sets U_1 and U_2 are simply connected, and

$$\begin{aligned} \partial U_1 &= \partial U_2 \\ &= \Lambda^- \\ &= \Lambda \cup \partial D \, . \end{aligned}$$

Thus $\partial U_1 = \partial U_2$ is almost the analytic image of ∂D, but the conclusion of Lemma 7 fails.

Consider also the Jordan curve defined by

$$\gamma(\Theta) = \begin{cases} e^{i\Theta} \text{ if } \frac{\pi}{2} \leq \Theta \leq 2\pi \\ e^{i\Theta}(1 + e^{i(\Theta(\Theta - \frac{\pi}{2}))^{-1}} \sin(\Theta(\Theta - \frac{\pi}{2}))^{-1}) \text{ if } 0 < \Theta < \frac{\pi}{2} \, . \end{cases}$$

See Figure 3 for a sketch of γ .

Then γ is a C^∞ curve and if $f(z) = z^2$ then $C - f(\gamma)$ has infinitely many components. If $\phi : D \to \operatorname{int}\gamma$ is a Riemann map then (see [14], pg. 59) ϕ extends to a homeomorphism of D^- onto $(\operatorname{int}\gamma)^-$. Here and throughout this paper, if Γ is a Jordan curve then $\operatorname{int}\Gamma$ and $\operatorname{ext}\Gamma$ denote the bounded and unbounded components of $C - \Gamma$ respectively. Thus ϕ^2 is a disc algebra function so that $C - \phi^2(\partial D)$ has infinitely many components.

Let U be a bounded, simply connected region with ∂U contained in K. This section ends with the decomposition of ∂U into a finite

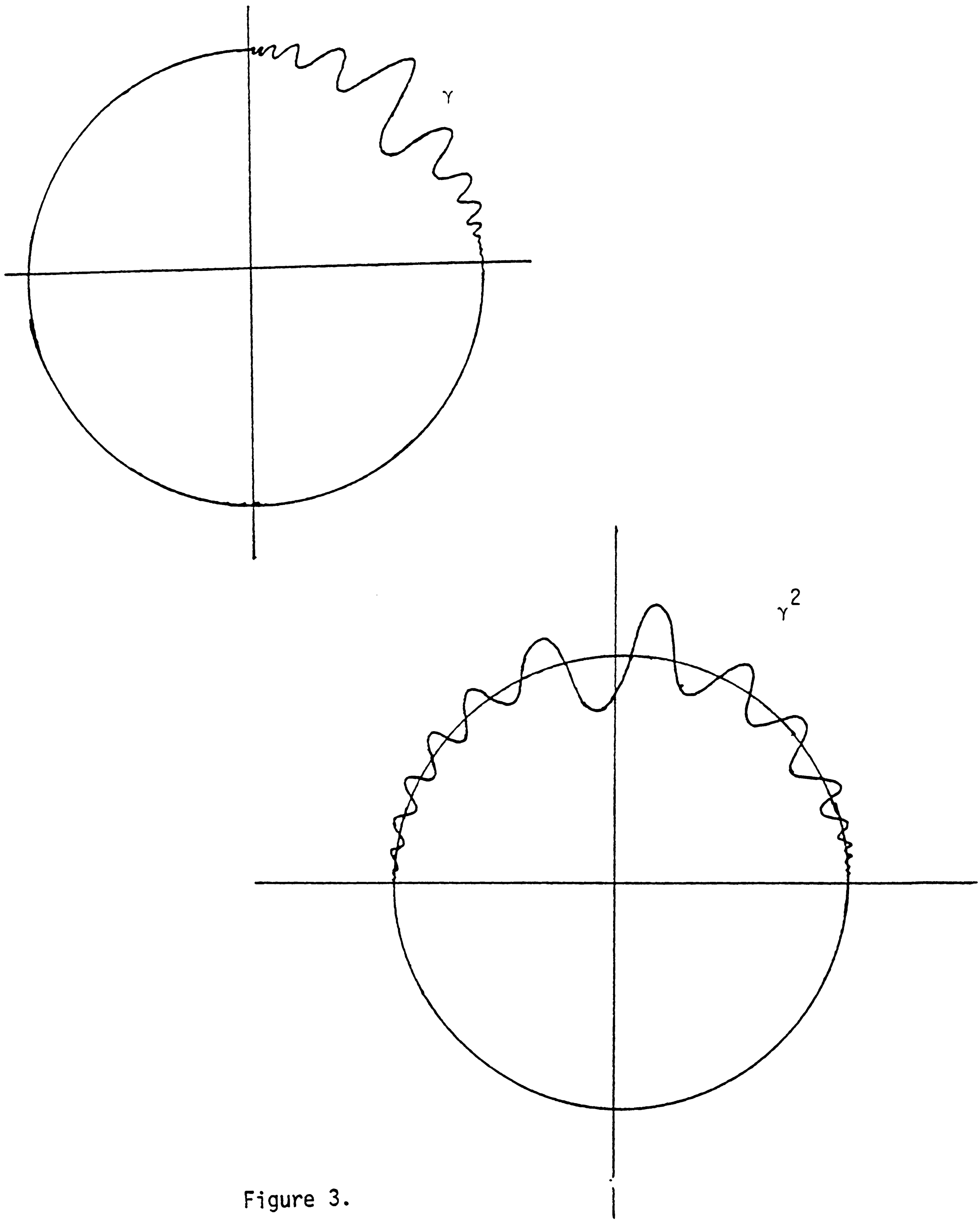

Figure 3.

union of rectifiable Jordan curves.

<u>Theorem 8</u>: If U is a bounded, simply connected region with $\partial U \subset K$ then

(a) $C-\overline{U}$ has only finitely many bounded components, say $U_1, U_2, \ldots, U_m$;

(b) $U_n = \text{int}(\overline{U}_n)\hat{}$ for each n,

and

(c) $\partial(\overline{U})\hat{}$ and $\partial U_n = \partial(\overline{U}_n)\hat{}$ are rectifiable Jordan curves.

<u>Proof</u>: (a) Let $\{U_1, U_2, \ldots\}$ denote the possibly infinite set of bounded components of $C-\overline{U}$, and fix $V = U_n$ for some n. Since V is open, $V-K \neq \emptyset$ and there exists a component G of $C-K$ that meets V. Then $G = (G \cap V) \cup (G-\overline{V})$ since $\partial V \subset \partial U \subset K$, and so $G \subset V$. Thus the number of components of $C-\overline{U}$ is less than the number of components of $C-K$.

(b) By Lemma 7, $\partial(\overline{U}_j)\hat{} \not\subset \partial(\overline{U})\hat{}$ and $\partial(\overline{U}_j)\hat{} \not\subset \partial(\overline{U}_k)\hat{}$ for all j and for all $k \neq j$. Fix j; then

$$U = (\text{int}(\overline{U}_j)\hat{} \cap U) \cup (U-(\overline{U}_j)\hat{}\ )$$

since $\partial(\overline{U}_j)\hat{} \subset \partial U_j \subset \partial U$. If $U-(\overline{U}_j)\hat{} = \emptyset$ then $(\overline{U})\hat{} = (\overline{U}_j)\hat{}$, a contradiction. Thus $U \cap \text{int}(\overline{U}_j)\hat{} = \emptyset$ and consequently $\overline{U} \cap \text{int}(\overline{U}_j)\hat{} = \emptyset$. Similarly, if $j \neq k$ then $\overline{U}_k \cap (\overline{U}_j)\hat{} = \emptyset$. Therefore

$$U_j = \text{int}(\overline{U}_j)\hat{}$$

and $$\partial U_j = \partial(\overline{U}_j)\hat{} \; .$$

(c) First notice that $\text{int}(\overline{U})\hat{}$ is connected. Otherwise, by reordering the components of $C-\overline{U}$, we may assume that U and U_1 are contained in distinct components of $\text{int}(\overline{U})\hat{}$. Since $U \cup U_1$ is connected in $(\overline{U})\hat{}$, we must then have $\partial U_1 \subset \partial\,\text{int}(\overline{U})\hat{}$, a contradiction via Lemma 7.

Thus it suffices to show that if V is a bounded simply connected region with $\partial V \subset K$ and such that $V = \text{int}(\overline{V})\hat{}$ then $\partial V = \partial(\overline{V})\hat{}$ is a rectifiable Jordan curve.

For $\Theta_0 \leq \Theta_1 < \Theta_0 + 2\pi$, let $[e^{i\Theta_0}, e^{i\Theta_1}] = \{e^{i\Theta} | \Theta_0 \leq \Theta \leq \Theta_1\}$,

and if $a = e^{i\Theta_0}$ and $0 < \varepsilon < \pi$, let

$$(a-\varepsilon, a+\varepsilon) = \{e^{i\Theta} |\; \Theta_0 - \varepsilon < \Theta < \Theta_0 + \varepsilon\}.$$

By Lemma 5, we may write $f^{-1}(\partial V) \cap \partial D$ as the finite union of nonoverlapping intervals $k_j = [a_j, b_j]$, where $1 \leq j \leq n$ and $k_j \cap S = \{a_j, b_j\}$.

<u>Claim 3</u>: ∂V is a rectifiable Jordan curve or $f|_{k_j}$ is one-to-one for each j.

Proof of 3: As in the proof of Lemma 4, there is an $\varepsilon > 0$ such that $f|_{[a_j,a_j+\varepsilon)}$ and $f|_{(b_j-\varepsilon,b_j]}$ are one-to-one. Let

$\delta_j = \sup\{\delta > 0 |\ a_j < a_j + \delta \le b_j$ and $f|_{[a_j,a_j+\delta)}$ is one-to-one.

If $a_j + \delta_j = b_j$ and $f|_{k_j}$ is not one-to-one then there exists a point $c_j \varepsilon [a_j,b_j)$ such that $f(c_j) = f(b_j)$. Clearly $f|_{[c_j,b_j)}$ is one-to-one. On the otherhand, if $d_j = a_j + \delta_j < b_j$ then there is an $r > 0$ such that $B(d_j,r) \cap \partial D \subset (a_j,b_j)$, and $f|_{B(d_j,r)}$ is one-to-one since $d_j \notin S$. Again, by the definition of δ_j, there exists $c_j \varepsilon [a_j,d_j)$ so that $f(c_j) = f(d_j)$ and $f|_{[c_j,d_j)}$ is one-to-one. In either case, if $f|_{k_j}$ is not one-to-one then there exists a Jordan curve, Γ, contained in $f(k_j) \subset \partial V$. Then $\operatorname{int}\Gamma = V$ since $\operatorname{int}\Gamma \subset (\overline{V})\hat{}$ and V is connected.

Therefore we may without loss of generality assume that for each j the function $f|_{k_j}$ is one-to-one.

By considering several cases and using Lemma 4, we see that if

$$f(\operatorname{int} k_i) \cap f(k_j) \neq \emptyset$$

then $f(k_i) \subset f(k_j)$ or $f(k_j) \subset f(k_i)$. Order the set $\{f(k_j)\}_{j=1}^{n}$

by inclusion, and let $\gamma_1,\ldots,\gamma_q$ denote its maximal elements. Then each γ_j is an arc, and $\partial V = \bigcup_{j=1}^{q} \gamma_j$. Moreover, if $\phi_j : [0,1] \to C$ is a parameterization of γ_j then $\gamma_i \cap \gamma_j \subset \{\phi_i(0), \phi_i(1)\}$ whenever $i \neq j$.

Let $\mathcal{T}$ be the set of subsets F of $\{1,2,\ldots,q\}$ with the property that $C - \bigcup_{j\in F} \gamma_j$ is not connected. Order $\mathcal{T}$ by inclusion, and let F_0 be a minimal member of $\mathcal{T}$.

Claim 4: $\Gamma = \bigcup_{j\in F_0} \gamma_j$ is a Jordan curve.

Proof of 4: Without loss of generality, we may assume $F_0 = \{1,2,\ldots,P\}$ for some p less than or equal to q. Since F_0 is minimal, clearly Γ is connected, and therefore

$$\gamma_1 \cap \left(\bigcup_{j=2}^{p} \gamma_j\right) \neq \emptyset \quad .$$

By reordering $\{\gamma_j\}_{j=2}^{p}$ if necessary, we may assume $\gamma_1 \cap \gamma_2 \neq \emptyset$, and by reparameterizing γ_2 if necessary, that $\phi_1(1) = \phi_2(0)$.

Now $\phi_2(1) \in \bigcup_{j\in F} \gamma_j$; otherwise an easy argument shows that $F_0 - \{2\} \in \mathcal{T}$. If $\phi_2(1) = \phi_1(0)$ then $F_0 = \{1,2\}$ and $\Gamma = \gamma_1 \cup \gamma_2$ is a Jordan curve; we are done. So we assume that there is a j with

$3 \leq j \leq p$ and such that $\phi_2(1) \varepsilon \gamma_j$. We may assume that $\phi_2(1) = \phi_3(0)$. Proceeding by induction, reordering and reparameterizing the arcs γ_j as necessary, we obtain an arc $\bigcup_{j=1}^{P-1} \gamma_j$ such that for $i < j$

$$\gamma_i \cap \gamma_j = \begin{cases} \emptyset \text{ if } j \neq i+1 \\ \{\phi_i(1)\} = \{\phi_j(0)\} \text{ if } j = i+1 \end{cases}$$

If $\phi_{P-1}(1) \notin \gamma_P$ then $F_0 - \{P-1\} \varepsilon \mathcal{T}$, and so again we may assume $\phi_{P-1}(1) = \phi_P(0)$. Since Γ disconnects the plane, it is not an arc and so there is a j less than p such that $\phi_j(0) = \phi_P(1)$. It follows that $\bigcup_{i=j}^{P} \gamma_i$ is a Jordan curve and so $j = 1$ by the minimality of F_0. Since V is connected and

$$V = (V \cap \operatorname{int}\Gamma) \cup (V - (\operatorname{int}\Gamma)^-) ,$$

$V = \operatorname{int}\Gamma$ and $(\overline{V})\hat{} = (\operatorname{int}\Gamma)^-$. Thus $F_0 = \{1,2,\ldots,q\}$ and $\partial V = \bigcup_{j=1}^{q} \gamma_j$ is a rectifiable Jordan curve since each γ_j is an analytic arc ∎

If V is a bounded, simply connected region then, by the Riemann Mapping Theorem, there is a conformal map ϕ from D onto V. Moreover, V is a Jordan domain if, and only if ϕ extends continuously in a one-to-one fashion to ∂D. In addition, if V is a Jordan domain then ∂V is rectifiable if, and only if $\phi|_{\partial D}$ is absolutely continuous. (see [14], pgs. 59-73). While the preceding proof is elementary, it is perhaps more natural to show directly that ϕ extends continuously to D^-.

In order to proceed, some preliminaries are necessary. If G is a bounded, simply connected region and $\lambda \in \partial G$ then λ is a simple boundary point of G provided that for every sequence $\{\lambda_n\}$ in G that converges to λ there is an arc $\gamma: [0,1] \to G^-$ and a sequence

$$0 < t_1 < t_2 < \ldots < t_n < \ldots < 1$$

so that $\gamma([0,1)) \subset G$ and $\gamma(t_n) = \lambda_n$. If λ is a simple boundary point and ψ a conformal mapping of G onto D then ψ extends continuously to $G \cup \{\lambda\}$ and $|\psi(\lambda)| = 1$. Moreover, if λ_1 and λ_2 are simple boundary points of G then $\psi(\lambda_1) \neq \psi(\lambda_2)$ (see [19], pgs. 308-9). G is a Jordan domain if, and only if each boundary point of G is simple.

Let h be a complex-valued function on G and $\lambda_0 \in G^-$. The Cluster Set $C(h,\lambda_0)$ of h at λ_0 is defined to be the set of all values α in S^2 so that there is a sequence $\{\lambda_n\}$ in $G-\{\lambda\}$ converging to λ_0 and such that $\{h(\lambda_n)\}$ converges to α. Equivalently,

$$C_G(h,\lambda_0) = \bigcap_{r>0} (h((B(\lambda_0,r)\cap G)-\{\lambda_0\}))^{-}.$$

For h defined on D, the Boundary Cluster Set of h at $e^{i\Theta_0}$ is defined to be the set

$$C_B(h,e^{i\Theta_0}) = \bigcap_{\eta>0} \Big(\bigcup_{0<|\Theta-\Theta_0|<\eta} C(h,e^{i\Theta})\Big)^{-}.$$

In the case that h is continuous on D, $C(h,e^{i\Theta_0})$ is connected (see [4], pg. 3). If h is analytic on D, Iversen's Theorem ([4], pg. 91) states that

$$\partial(C(h,e^{i\Theta_0}))\subset C_B(h,e^{i\Theta_0}) \quad .$$

Consider V as in the proof of Theorem 8(c); that is, V is a simply connected, bounded region such that $V = \operatorname{int}(\overline{V})^{\wedge}$ and $\partial V\subset K$. Let ϕ be a conformal map from D onto V. Then each point of $\partial V-f(\mathcal{S})$ is simple. Indeed, if $a \in f^{-1}(\partial V)\cap(\partial D-\mathcal{S})$then, for each positive ε sufficiently small, $f(B(a,\varepsilon))\cap V$ is a Jordan domain. Thus f(a) is a simple boundary point of $f(B(a,\varepsilon))\cap V$, and, since the collection of open sets $f(B(a,\varepsilon))$ is a neighborhood base at f(a), f(a) is a simple boundary point of V.

The function ϕ^{-1} therefore can be extended in a continuous and one-to-one fashion to $\overline{V}-(f(\mathcal{S})\cap \partial V)$, and ϕ is continuous and one-to-one on $D \cup \phi^{-1}(\partial V-f(\mathcal{S}))$. Notice that $\phi^{-1}(\partial V-f(\mathcal{S}))$ can be written as the union of connected open sets in ∂D:

$$\phi^{-1}(\partial V - f(\mathcal{S})) = \bigcup_{j=1}^{n} (a_j, b_j)$$

where $a_j < b_j \leq a_{j+1}$. If a point λ in (b_j, a_{j+1}) is such that $C(\phi,\lambda) = \{\omega\}$ then ϕ can be extended continuously to the set

$$D \cup \phi^{-1}(\partial V - f(\mathcal{S})) \cup \{\lambda\}$$

by defining $\phi(\lambda) = \omega$. Since

$$\begin{aligned} C(\phi^{-1},\phi(\lambda)) &= \phi^{-1}(C(\phi,\lambda)) \\ &= \{\lambda\} \quad , \end{aligned}$$

$\phi(\lambda)$ is a simple boundary point of V; therefore the extension of ϕ is one-to-one, and $\phi(\lambda) \in f(\mathcal{S}) \cap \partial V$.

Thus if $b_j < a_{j+1}$ then there is a point λ_0 in (b_j, a_{j+1}) such that $C(\phi,\lambda_0)$ contains more than one point. In fact, the argument above shows that $C(\phi,\lambda)$ contains more than one point for all but finitely many $\lambda \in (b_j, a_{j+1})$. Since $C(\phi,\lambda_0)$ is compact and connected,

$$C(\phi,\lambda_0) \cap (V \cup (\partial V - f(\mathcal{S}))) \neq \emptyset \quad .$$

If λ_1 is an element of $D \cup \phi^{-1}(\partial V - f(\mathcal{S}))$ such that $\phi(\lambda_1) \in C(\phi,\lambda_0)$ then

$$\begin{aligned} C(\phi^{-1},\phi(\lambda_1)) &= \phi^{-1}(C(\phi,\lambda_1)) \\ &= \{\lambda_1\} \ . \end{aligned}$$

But, at the same time, $C(\phi^{-1},\phi(\lambda_1)) = C(\phi^{-1},\phi(\lambda_0))$

$$= \phi^{-1}(C(\phi,\lambda_0)) \ ,$$

not a singleton set. This contradiction followed from the assumption that

$$\partial D - \phi^{-1}(\partial V - f(S))$$

was not finite.

Thus we may write

$$\phi^{-1}(\partial V - f(S)) = \partial D - \{c_0, c_1, \ldots, c_{n-1}\}$$

where $c_j = e^{i\Theta_j}$ and $\Theta_0 < \Theta_1 < \ldots < \Theta_{n-1} < \Theta_0 + 2\pi$. Since ϕ is one-to-one and continuous on (c_0,c_1) and on (c_1,c_2) , both $\lim\limits_{\Theta\to\Theta_1^-} \phi(e^{i\Theta})$ and $\lim\limits_{\Theta\to\Theta_1^+} \phi(e^{i\Theta})$ exist. (Recall that we are working on the curve K.) Moreover, $C(\phi,e^{i\Theta}) = \{\phi(e^{i\Theta})\}$ for each $e^{i\Theta} \in (c_0,c_1) \cup (c_1,c_2)$, and so $C_B(\phi,c_1)$ contains at most two points. Since $C(\phi,c_1)$ is compact and connected, it follows from Iversen's Theorem that $C(\phi,c_1)$ contains only one point. Thus ϕ extends continuously to $D^- - \{c_0,c_2,\ldots c_{n-1}\}$ and this extension is one-to-one ([19], pg. 309). Continuing the process above, we see that

ϕ extends to a homeomorphism of D^- onto $\overline{V}$ and ∂V is a Jordan curve. That ∂V is rectifiable follows from Lindelöf's Theorem ([9], pg. 44]) and the assumption that $\partial V \subset f(\partial D)$ where f is analytic.

Let U be a bounded, simply connected region with $\partial U \subset K$. Then, by Theorem 8, we may write

$$\partial U = \bigcup_{j=0}^{n} \Gamma_j$$

where $\Gamma_0 = \partial(\overline{U})^\wedge$ and, for $1 \leq j \leq n$, the arc Γ_j is the boundary of a bounded component U_j of $C-\overline{U}$.

Lemma 9: $\Gamma_i \cap \Gamma_j$ is finite whenever $i \neq j$.

Proof: Fix i and j positive and suppose $a \in (f|_{\partial D})^{-1}(\Gamma_i \cap \Gamma_j) - S$. Then there exists $\varepsilon > 0$ such that

$$f|_{B(a,\varepsilon)} \text{ is one-to-one,}$$

$$B(a,\varepsilon) \cap \partial D \subset f^{-1}(\partial U) \cap \partial D\ ,$$

$$\text{and} \quad f(B(a,\varepsilon) \cap \partial D) = f(B(a,\varepsilon)) \cap K\ .$$

Let $V_+ = f(B(a,\varepsilon) \cap D)$ and

$V_- = f(B(a,\varepsilon) - D^-)$.

Then, as in Lemma 5,

$$f(B(a,\varepsilon) \cap \partial D) = f(B(a,\varepsilon)) \cap \partial U \quad ,$$

and therefore

$$f(B(a,\varepsilon)) = V_+ \cup V_- \cup f(B(a,\varepsilon) \cap \partial D)$$

$$= (f(B(a,\varepsilon)) \cap U) \cup (f(B(a,\varepsilon) \cap U_i)$$

$$\cup (f(B(a,\varepsilon)) \cap U_j) \cup (f(B(a,\varepsilon)) \cap K).$$

Since each of the sets $f(B(a,\varepsilon)) \cap U$, $f(B(a,\varepsilon)) \cap U_i$ and $f(B(a,\varepsilon)) \cap U_j$ are nonempty, we obtain a contradiction to the fact that both V_+ and V_- are connected. Thus $\Gamma_1 \cap \Gamma_2 \subset f(S)$. Similarly, if $j > 0$ then $\Gamma_j \cap \Gamma_0 \subset f(S)$ ∎

Nonreductive Normal Operators Arising as Functions of Unitary Operators

Henceforth let μ be a positive regular Borel measure with support contained in ∂D, and define $\nu = \mu \circ f^{-1}$ on $f(\partial D) = K$. In order to determine $P^{\infty}(\nu)$ via Sarason's process, we must concern ourselves with compact sets L such that R(L) is a Dirichlet algebra and $\partial L \subset K$.

<u>Notation:</u> In the sequel, L will always denote such a set.

If h is a complex-valued function on an interval [a , b] contained in ∂D, then h is said to be absolutely continuous if for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that

$$\sum_{j=1}^{n} | f(b_j) - f(a_j)| < \varepsilon$$

whenvever $\{(a_j,b_j) \mid 1 \leq j \leq n\}$ is a finite set of disjoint subintervals of [a,b] for which $\sum_{j=1}^{n} m([a_j,b_j]) < \delta$. A function h is absolutely continuous on $[a,b] \subset \partial D$ if, and only if there exists $g \in L^1[a,b]$ such that

$$h(\lambda) = \int_{[a,\lambda]} g \, dm + h(a)$$

for all $\lambda \in [a,b]$ (see for example [11], pg. 286). If h is absolutely continuous on $[a,b] \subset \partial D$ and with range $[c,d] \subset \partial D$, then

$m(h(E)) = 0$ whenever E is a subset of $[a,b]$ such that $m(E) = 0$. (See [11], pg. 288).

Let U be a component of $\operatorname{int}L$, and write

$$\partial U = \bigcup_{j=0}^{n} \Gamma_j$$

where $\Gamma_0 = \partial(\overline{U})\hat{}$, and $U_j = \operatorname{int}\Gamma_j$ is a bounded component of $C-\overline{U}$ for each $j \geq 1$. Choose $a \in U$ and $a_j \in U_j$ for each $j \geq 1$, and define $g_j(\lambda) = \frac{1}{\lambda - a_j}$. Let $\Lambda_j = g_j(\Gamma_j)$. Then Λ_j is a rectifiable Jordan curve; $0 \in \operatorname{int}\Lambda_j$, and there exists a conformal map ψ_j from D onto $\operatorname{int}\Lambda_j = g_j(\operatorname{ext}\Gamma_j)$ such that $\psi_j(0) = g_j(a)$. Then $\psi_j|_{\partial D}$ is absolutely continuous; in fact, $(\psi_j|_{\partial D})' \in H^1(\partial D)$. (See [9], pg. 44). If $\phi_j = g_j^{-1} \circ \psi_j$ then ϕ_j is a homeomorphism from D^- onto $\operatorname{ext}\Gamma_j$ viewed as a subset of S^2, $\phi_j(0) = a$ and $(\phi_j|_{\partial D})' = -\frac{1}{\psi_j^2}\psi_j'$. Thus $(\phi_j|_{\partial D})' \neq 0$ almost everywhere m. Since $\phi_j(D) = \operatorname{ext}\Gamma_j$, $\operatorname{ind}(\phi_j, a_j) = -1$. Here $\operatorname{ind}(\gamma,\lambda)$ denotes the winding number of the closed curve γ about a point $\lambda \notin \gamma$.

In the case that $j = 0$ let ϕ_0 be a conformal map of D onto $\operatorname{int}\Gamma_0$ such that $\phi_0(0) = a$. Then, as above, $\phi_0|_{\partial D}$ is absolutely continuous, $\phi_0|_{\partial D}$ is a homeomorphism from ∂D onto Γ_0, and $(\phi_0|_{\partial D})' \in H^1(\partial D)$. Define the measure σ_j on Γ_j by $\sigma_j = m \circ \phi_j^{-1}$, and define $\sigma_a = \sum_{j=0}^{n} \sigma_j$.

<u>Lemma 10</u>: If ω_a is harmonic measure for $\overline{U}$ evaluated at a then σ_a and ω_a are mutually absolutely continuous.

<u>Proof</u>: If $u \in C_r(\partial U)$ then, by Tietze's Extension Theorem, u extends to a real-valued continuous function on ∂L, also denoted u. There is a sequence $\{g_n\}$ in R(L) so that $\|Re(g_n) - u\|_{\partial L} \to 0$, and so $\{g_n|_{\overline{U}}\}$ is a sequence in $R(\overline{U})$ such that $\|g_n|_{\overline{U}} - u\|_{\partial U} \le \|g_n - u\|_{\partial L}$.

Thus $R(\overline{U})$ is a Dirichlet algebra. Let q be a rational function with poles off $\overline{U}$. Then

$$q(a) = \frac{1}{2\pi i} \int_{\Gamma_0} \frac{q(\omega)}{\omega - a}\, d\omega - \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\Gamma_j} \frac{q(\omega)}{\omega - a}\, d\omega$$

$$= \frac{1}{2\pi i} \int_{\partial D} \frac{q(\phi_0(\xi))}{\phi_0(\xi) - a}\, \phi_0'(\xi)\, d\xi$$

$$+ \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\partial D} \frac{q(\phi_j(\xi))}{\phi_j(\xi)-a} \phi'(\xi)\, d\xi$$

$$= \int_{\partial D} (q \circ \phi) \left(\left(\frac{(z\phi_0') \circ \phi_0^{-1}}{z-a}\right) \circ \phi_0\right) dm$$

$$+ \sum_{j=1}^{n} \int_{\partial D} (q \circ \phi_j) \left(\left(\frac{(z\phi_j') \circ \phi_j^{-1}}{z-a}\right) \circ \phi_j\right) dm$$

$$= \int_{\Gamma_0} q\, k_0\, d\sigma_0 + \sum_{j=1}^{n} \int_{\Gamma_j} q\, k_j\, d\sigma_j$$

where $k_0(\lambda) = \dfrac{\phi_0^{-1}(\lambda)\phi_0'(\phi^{-1}(\lambda))}{\lambda - a}$, and $k_j(\lambda) = \dfrac{\phi_j^{-1}(\lambda)\phi_j'(\phi_j^{-1}(\lambda))}{\lambda - a}$.

Thus

$$q(a) = \int_{\partial U} q\, k\, d\sigma_a$$

where $k\big|_{\Gamma_j} = k_j$. The function k is defined almost everywhere - σ_a since $\sigma_a(f(S)) = 0$.

The measure $kd\sigma_a$ is therefore a complex representing measure for a on $R(\overline{U})$, and consequently, $\omega_a << |k\sigma_a|$ (see [10],pg. 33). Thus $\omega_a << \sigma_a$.

If $\lambda = k\sigma_a$ has Lebesgue decomposition $\lambda = h\omega_a + \lambda_s$ with respect to ω_a then $\lambda - \omega_a = (h-1)\omega_a + \lambda_s$ is orthogonal to $R(\overline{U})$. It follows from Ahern's Theorem ([10], pg. 44) that $(h-1)\omega_a$ and λ_s are orthogonal to $R(\overline{U})$.

If $\lambda_s \perp \lambda_b$ for each representing measure λ_b where $b \in \overline{U}$, then it follows from Wilkin's Theorem ([10], pg. 47) that $\lambda_s = 0$. To this end, notice that for each j, the measure $\sigma_j = m \circ \phi_j^{-1}$ is continuous (that is, it has no atoms) since ϕ_j is one-to-one. Thus λ_s is continuous, and, in particular, $\lambda_s \perp \lambda_b$ for each $b \in \partial U$. If $b \in U$ then $\lambda_s \perp \lambda_b$ since λ_b and ω_a are mutually absolutely continuous.

Thus $k\sigma_a = \lambda$ is absolutely continuous with respect to ω_a, and, since $k \neq 0$ almost everyhere - σ_a, σ_a is absolutely continuous with respect to ω_a ■

Lemma 11: Let γ be a Jordan curve contained in K and $I = (a,b)$ a component of $f^{-1}(\gamma) \cap (\partial D - \mathcal{S})$. Then there is a partition of I into a finite number of subintervals $J_k = (c_{k-1}, c_k)$ such that:

$$a = c_0 < c_1 < \ldots < c_p = b$$

and

(i) For each k there is a connected set V_k, open in C, such that $V_k \cap \partial D = J_k$;

(ii) $f|_{V_k}$ is one-to-one;

(iii) $f(V_k) \cap K = f(J_k)$;

(iv) $f(V_k) \cap \operatorname{int}\gamma$ and $f(V_k) \cap \operatorname{ext}\gamma$ are connected, and

(v) $V_k \cap V_j = \emptyset$ whenever $j \neq k$.

The idea of the proof is to find a finite collection of open, connected sets V_k such that $f|_{V_k}$ is one-to-one, $V_k \cap \partial D = (a_k, b_k)$, $V_k \cap D$ and $V_k - D^-$ are connected. Intersecting the sets V_k with sections of the form $\{re^{i\Theta} \mid 0 < r$ and $\Theta_1 < \Theta < \Theta_2\}$ makes them disjoint.

Proof: There exists an $\varepsilon > 0$ and natural numbers n and m such that $f|_{B(a,\varepsilon)} = f(a) + (z-a)^n h_a$ and $f|_{B(b,\varepsilon)} = f(b) + (z-b)^m h_b$ where h_a and h_b are analytic and nonzero on $B(a,\varepsilon)$ and $B(b,\varepsilon)$ respectively. Let $a = e^{i\Theta_0}$ and $b = e^{i\Theta_1}$ where $\Theta_0 < \Theta_1 \leq \Theta_0 + 2\pi$ and define

$$U_a = \{ a + re^{i\Theta} \mid 0 < r < \varepsilon, |\Theta - \Theta_0 - \tfrac{\pi}{2}| < \tfrac{\pi}{n+1} \}.$$

Then $f|_{U_a}$ is one-to-one, $U_a \cap \partial D = (a, a+\delta)$ for some $\delta > 0$, and so

$f|_{U_a \cap \partial D}$ is open from $(a,a+\delta)$ into K. Then there is a connected open set $V_a \subset f(U_a)$ such that $f(a,a+\delta) = V_a \cap K = V_a \cap \gamma$. Let $\Delta_a = U_a \cap f^{-1}(V_a)$, and define Δ_+ and Δ_- by

$$\Delta_+ = \Delta_a \cap D$$

$$\Delta_- = \Delta_a - D^- .$$

Then Δ_+ and Δ_- are connected, and $V_a = f(\Delta_a) = f(\Delta_+) \cup f(\Delta_-) \cup f(a,a+\delta)$

$$= (V_a \cap \text{int}\gamma) \cup (V_a \cap \text{ext}\gamma) \cup (V_a \cap \gamma) \quad .$$

Thus $f(\Delta_+) = V_a \cap \text{int}\gamma$ and $f(\Delta_-) = V_a \cap \text{ext}\gamma$, or $f(\Delta_-) = V_a \cap \text{int}\gamma$ and $f(\Delta_+) = V_a \cap \text{ext}\gamma$. In either case, $f(\Delta_a) \cap \text{int}\gamma$ and $f(\Delta_a) \cap \text{ext}\gamma$ are connected.

Define U_b analogously to U_a:

$$U_b = \{b + re^{i\Theta} \mid 0 < r < \varepsilon/2, |\Theta - \Theta_1 + \tfrac{\pi}{2}| < \tfrac{\pi}{m+1} \} .$$

Let $V_b \subset f(U_b)$ be an open connected set such that $V_b \cap K = f(U_b \cap (a,b))$, and let $\Delta_b = U_b \cap f^{-1}(V_b)$. Then

$$\Delta_b \cap \partial D = U_b \cap \partial D$$

$$= (b-\delta', b)$$

for some $\delta' > 0$; Δ_b is connected, and $f(\Delta_b) \cap \operatorname{int}\gamma$ and $f(\Delta_b) \cap \operatorname{ext}\gamma$ are connected.

For each $\lambda \in (a,b)$, fix ε_λ such that $0 < \varepsilon_\lambda < \frac{|b-(b-\delta')|}{4}$, the restriction of f to $B(\lambda, \varepsilon_\lambda)$ is one-to-one,

$$B(\lambda, \varepsilon_\lambda) \cap \partial D \subset (a,b) \quad ,$$

and

$$f(B(\lambda, \varepsilon_\lambda)) \cap K = f(B(\lambda, \varepsilon_\lambda) \cap \partial D) \quad .$$

If $\Delta_\lambda = B(\lambda, \varepsilon_\lambda)$ then

$$Q = \{\Delta_a \cup \{a\}, \Delta_b \cup \{b\}\} \cup \{\Delta_\lambda \cap [a,b] \mid \lambda \in (a,b)\}$$

is an open cover of $[a,b]$. Let U be the collection of all finite subcovers in Q and order U by inclusion. Since each chain in U is finite, it has a lower bound, and so U has a minimal element: $C = \{\Delta_a \cup \{a\}\} \cup \{\Delta_{\lambda_i} \mid 2 \leq i \leq m\}$. We may assume $a < \lambda_2 < \ldots < \lambda_m = b$. Notice that since C is a cover, $\Delta_1 = \Delta_a \cup \{a\}$ must be in C, and since C is minimal, if $\lambda_j \neq b$ then $|\lambda_j - b| > |\frac{b-(b-\delta')}{2}| > 2\varepsilon_j$.

Since C is minimal,

$$(\Delta_2 - \overline{\Delta}_1) \cap \partial D = (e^{it_1}, e^{it_2})$$

for some $\Theta_0 < t_1 < t_2$. Let $c_0 = a$ and $c_j = e^{it_j}$ for $j = 1,2$.

Define the sets

$$V_1 = \Delta_1 \cap \{re^{i} \mid 0 < r \text{ and } \Theta_0 < \theta < t_1\} ,$$

and

$$V_2 = \Delta_2 \cap \{re^{i\Theta} \mid 0 < r,\ t_1 < \Theta < t_2\} .$$

If $J_i = (c_{i-1}, c_i)$ for $i = 1,2$, then the sets V_i and J_i satisfy (i) - (v)

Suppose that for $1 \leq k \leq m-1$ the points $c_j = e^{it_j}$ are defined for $0 \leq j \leq k$ such that $\Theta_0 = t_0 < t_1 < \ldots < t_k$, and the sets V_j for $1 \leq j \leq k$ are defined such that for $i \geq 2$,

$$\partial D \cap (\Delta_i - \overline{V}_{i-1}) = (c_{i-1}, c_i) = J_i$$

and V_i, J_i satisfy (i) - (iv).

Since C is minimal,

$$\partial D \cap (\Delta_{k+1} - \bigcup_{j=1}^{K} \overline{V}_j) = \partial D \cap (\Delta_{k+1} - \overline{V}_k)$$

$$= (c_k, c_{k+1})$$

for some $c_{k+1} = e^{it_{k+1}}$ where $t_k < t_{k+1}$. Then $V_{k+1} = \Delta_{k+1} \cap \{re^{i\theta} | 0 < r$ and $t_k < \theta < t_{k+1}\}$ and $J_{k+1} = (c_k, c_{k+1})$ satisfy (i) - (iv) and $V_{k+1} \cap V_j = \emptyset$ for each $j \leq k$. Proceed inductively to construct the intervals

$$J_i = (\Delta_i - \overline{V}_{i-1}) \cap \partial D$$

$$= (c_{i-1}, c_i)$$

$$= (e^{it_{i-1}}, e^{it_i}) \quad ,$$

and

$$V_i = \Delta_i \cap \{re^{i\theta} | 0 < r, t_{i-1} < \theta < t_i\}$$

for each $i \leq m-1$.

It remains to construct V_m. Let $\varepsilon' = |b - c_{m-1}|$. If $\varepsilon' = 0$ then

$$\partial D \cap B(\lambda_{m-1}, \varepsilon_{m-1}) \subset (b-\delta', b)$$

since $\varepsilon_{m-1} = |c_{m-1} - \lambda_{m-1}| < \frac{|(b-\delta')-b|}{4}$. Thus $0 < \varepsilon' < |b - \lambda_{m-1}| < \frac{3\varepsilon}{4}$.

Take $V_m = \Delta_b \cap \{re^{i\theta} | 0 < r, t_{m-1} < \theta < \theta_1\}$, and $J_m = (c_{m-1}, b)$.

Then V_m and J_m satisfy (i) - (iv), and $V_m \cap V_j = \emptyset$ for each $j \leq m-1$ ∎

Recall the notation of Lemma 10:

<u>Theorem 12</u>: For each j, the measures $\sigma_j = m \circ \phi_j^{-1}|_{\Gamma_j}$ and $m \circ f^{-1}|_{\Gamma_j}$ are mutually absolutely continuous.

<u>Proof</u>: Let $\{I_q\}_{q=1}^{p}$ denote the components of $(f^{-1}(\Gamma_j) \cap \partial D) - S$, and partition each component I_q into subintervals $\{J_{q,k}\}_{k=1}^{m_q}$ according to Lemma 11:

$$J_{q,k} = (c_{q,k-1}, c_{q,k})$$

$$= \partial D \cap V_{q,k} \ .$$

Let $W_{q,k} = V_{q,k} \cap f^{-1}(\phi_j(D^-))$. Then either $W_{q,k} \subset D^-$ or $W_{q,k} \cap D = \emptyset$ since $\phi_0(D^-) = (\mathrm{int}\Gamma_0)^-$ and $\phi_j(\overline{D}) = (\mathrm{ext}\,\Gamma_j)^-$ if $j \geq 1$. Also notice that $W_{q,k} \cap \partial D = J_{q,k}$ and $W_{q,k} - \partial D$ is open in the complex plane. Define $G_{q,k}$ to be the open set

$$G_{q,k} = W_{q,k} \cup \{\lambda \mid \frac{1}{\lambda} \in W_{q,k}\},$$

and define $\xi_{q,k}$ on $G_{q,k}$ by

$$\xi_{q,k}(\lambda) = \begin{cases} \phi_j^{-1}(f(\lambda)) & \text{if } \lambda \in W_{q,k} \\ \overline{(\phi^{-1}(f((\overline{\lambda})^{-1})))^{-1}} & \text{otherwise.} \end{cases}$$

Then $\xi_{q,k}$ is analytic on $G_{q,k}$, and since $\phi_j^{-1} \circ f|_{W_{q,k}}$ is one-to-one, $\xi_{q,k}$ is one-to-one on $G_{q,k}$. Moreover, $\xi_{q,k}(J_{q,k}) \subset \partial D$. Since $\xi_{q,k}|_{J_{q,k}}$ and $\xi^{-1}_{q,k}|_{\xi_{q,k}(J_{q,k})}$ are the restrictions of analytic functions, they are absolutely continuous.

Define ξ_j on $f^{-1}(\Gamma_j) \cap \partial D$ by $\xi_j|_{J_{q,k}} = \xi_{q,k}$. If E is a Borel set contained in $f^{-1}(\Gamma_j) \cap \partial D$ such that $m(E) = 0$ then

$$\begin{aligned} m(\xi_j(E)) &= m\Big(\bigcup_{q=1}^{p} \bigcup_{k=1}^{m_q} \xi_j(E \cap J_{q,k})\Big) \\ &\leq \sum_q \sum_k m(\xi_{q,k}(E \cap J_{q,k})) \\ &= 0 \, . \end{aligned}$$

If F is a Borel set on ∂D such that $m(F) = 0$ then

$$m(\xi_j^{-1}(F)) = m(\xi_j^{-1}(F \cap (\bigcup_q \bigcup_k \xi_{q,k}(J_{q,k}))))$$

$$\leq \sum_q \sum_k m(\xi_{q,k}^{-1}(F \cap \xi_{q,k}(J_{q,k})))$$

$$= 0 \ .$$

Thus $m \circ \xi_j$ and $m \circ \xi_j^{-1}$ are absolutely continuous with respect to m.

Let P be a Borel set in Γ_j such that $m(f^{-1}(P)) = 0$. Then

$$0 = m(\xi_j(f^{-1}(P))),$$

and so

$$0 = m(\phi_j^{-1}(P)) \ .$$

On the otherhand, if Q is a Borel set in Γ_j such that $m(\phi_j^{-1}(Q)) = 0$

then

$$0 = m(\phi_j^{-1}(f(f^{-1}(Q))))$$

$$= m(\xi_j(f^{-1}(Q)))$$

and so

$$0 = m(\xi_j^{-1}(\xi_j(f^{-1}(Q))))$$

$$= m(f^{-1}(Q)) \blacksquare$$

Theorem 12 establishes that ω_a, harmonic measure for L evaluated at $a \in U$, and $m \circ f^{-1}|_{\partial U}$ are mutually absolutely continuous whenever U is a component of intL; therefore harmonic measure for L and $m \circ f^{-1}|_{\partial(\text{intL})}$ are mutually absolutely continuous.

Corollary 13: If U_1 and U_2 are distinct components of intL, then

$$\partial U_1 \cap \partial U_2 \subset f(\mathcal{S}) .$$

Therefore $\partial U_1 \cap \partial U_2$ is a finite set.

Proof: For i = 1,2, let $a_i \in U_i$. If ω_i is harmonic measure for L evaluated at a_i then $\omega_1 \perp \omega_2$. If

$$a \in f^{-1}(\partial U_1 \cap \partial U_2) \cap (\partial D - \mathcal{S})$$

then, by Lemma 4 there is an $\varepsilon > 0$ such that $B = B(a,\varepsilon) \cap \partial D$ is contained in $f^{-1}(\partial U_1 \cap \partial U_2) \cap \partial D$. Thus

$$0 < m(B) \leq m(f^{-1}(f(B)))$$

Thus $\omega_1(f(B))$ and $\omega_2(f(B))$ are positive, a contradiction ∎

Corollary 13 is not true for general Dirichlet Algebras. The "Lakes of Wada" (see [12], pg. 143) is an example of a compact set

J such that R(J) is a Dirichlet algebra. The interior of J has two components, U_1 and U_2, but $\partial U_1 = \partial U_2 = \partial J$.

Recall that μ is a positive regular Borel measure supported on ∂D and that $\nu = \mu \circ f^{-1}$ on K. Let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of μ with respect to m.

<u>Lemma 14</u>: The Lebesgue decomposition of $\nu|_{\partial U}$ with respect to $m \circ f^{-1}|_{\partial U}$ is

$$\nu|_{\partial U} = \nu_a + \nu_s$$

where

$$\nu_a = \mu_a \circ f^{-1}|_{\partial U}$$

and

$$\nu_s = \mu_s \circ f^{-1}|_{\partial U} \quad .$$

<u>Proof</u>: Let $\partial U = \bigcup_{j=0}^{n} \Gamma_j$. It suffices to show that

$$\mu_s \circ f^{-1}|_{\Gamma_j} \perp m \circ f^{-1}|_{\Gamma_j}$$

because clearly

$$\mu_a \circ f^{-1}|_{\Gamma_j} << m \circ f^{-1}|_{\Gamma_j} \quad .$$

Let $I_1, I_2, \ldots, I_p$ be the components of $f^{-1}(\Gamma_j) \cap (\partial D - S)$, and for each k, let $\{J_{k,i} \mid 1 \leq i \leq p_k\}$ be a partition of I_k where, as in Lemma 11, $J_{k,i} = V_{k,i} \cap \partial D$ and $V_{k,i}$ satisfy (i) - (iv).

For simplicity, label the set $\{J_{k,i}\} | 1 \leq k \leq p, 1 \leq i \leq p_k\}$ as $\{J_\ell\}_{\ell=1}^m$ and let $J_\ell = V_\ell \cap \partial D$.

An easy argument shows that if $f(J_i) \cap f(J_k) \neq \emptyset$ then

$$\ell_{i,k} = \partial D \cap (V_i \cap f^{-1}(f(V_k)))$$

is an open interval in J_i; notice in particular that $\ell_{i,i} = J_i$. Define $g_{i,k}$ on the region $V_i \cap f^{-1}(f(V_k))$ by

$$g_{i,k} = (f|_{V_k})^{-1} \; f|_{V_i \cap f^{-1}(f(V_k))} \quad .$$

Again notice that $g_{i,i} = z$ on V_i. Then $g_{i,k}$ is one-to-one and analytic on $V_i \cap f^{-1}(f(V_k))$ and so its restriction to $\ell_{i,k}$ is absolutely continuous. Moreover, since $g_{i,k}(V_i \cap f^{-1}(f(V_k))) = V_k \cap f^{-1}(f(V_i))$ and since $g_{i,k}(\ell_{i,k}) = \ell_{k,i}$, we have $(g_{i,k})^{-1} = g_{k,i}$.

Let F be a Borel set contained in $f^{-1}(\Gamma_j) \cap \partial D$ such that $m(F) = 0$.

Then, since $\ell_{i,i} = J_i$,

$$(5) \qquad m(f^{-1}(f(F))) = m(f^{-1}(f(F \cap (\bigcup_{i,k} \ell_{i,k}))))$$

$$= \sum_k m(J_k \cap f^{-1}(\bigcup_i f(F \cap \ell_{i,k})))$$

$$\leq \sum_k \sum_i m(g_{i,k}(F \cap \ell_{i,k}))$$

$$= 0 \ .$$

Therefore, if $E \subset \partial D$ is such that

$$m(E) = \mu_s(\partial D - E)$$

$$= 0 \ ,$$

then

$$\mu_s(f^{-1}(\Gamma_j - f(E)))$$

$$= \mu_s(f^{-1}(\Gamma_j) - f^{-1}(f(E)))$$

$$\leq \mu_s(\partial D - f^{-1}(f(E)))$$

$$\leq \mu_s(\partial D - E)$$

$$= 0 \ .$$

But, by (5), $m(f^{-1}(f(E))) = 0$. Thus $\mu_s f^{-1}|_{\Gamma_j} \perp m \circ f^{-1}|_{\Gamma_j}$ and the Lemma is proved. ∎

<u>Theorem 15</u>: Suppose μ is a positive, regular Borel measure supported on D and has Lebesgue decomposition with respect to m given by

$$\mu = \mu_a + \mu_s$$

and suppose $\nu = \mu \circ f^{-1}$ where f is analytic in a neighborhood of ∂D. Then $P^\infty(\nu) \neq L^\infty(\nu)$ if and only if there is a Jordan curve Γ contained in $K = f(\partial D)$ such that $m \circ f^{-1}|_\Gamma$ is absolutely continuous with respect to $\mu_a \circ f^{-1}$.

<u>Proof</u>: If $\tilde{K}$ is the Sarason hull of ν then $P^\infty(\nu) \neq L^\infty(\nu)$ if, and only if $\mathrm{int}\tilde{K} \neq \emptyset$. For each countable ordinal α, define the compact set K_α as in Sarason's process, that is, $K_0 = \hat{K}$, and, if K_α is defined then

$$K_{\alpha+1} = (\text{support } \nu) \cup \{\lambda \in \mathrm{int}K_\alpha \mid \text{for each } g \in H^\infty(\mathrm{int}K_\alpha), |q(\lambda)| \leq \|q\|_{\nu_\alpha}\}$$

where ν_α is the K -admissable component of ν.

Since $K_\alpha - K_{\alpha+1}$ is the union of some components of $(\mathrm{int}K_\alpha)-$ (support ν), and since C-K has only finitely many components, there is a natural number N such that $\tilde{K} = K_N$; that is, Sarason's process stops after a finite number of steps.

We wish to show that $\mathrm{int}\tilde{K} \neq \emptyset$ if and only if there is a Jordan curve Γ contained in K such that $m\circ f^{-1}|_\Gamma << \mu_a\circ f^{-1}|_\Gamma$. Suppose then that Γ is such a curve and let $V = \mathrm{int}\Gamma$; clearly $V \subset \mathrm{int}K_0$. Suppose that $V \subset \mathrm{int}K_n$ for some n. Let $\lambda \in V$ and $g \in H^\infty(\mathrm{int}K_n)$. If ω_λ is harmonic measure for $\overline{V}$ evaluated at λ then, by Theorem 12 and Lemma 14, ω_λ and $m\circ f^{-1}|_\Gamma$ are mutually absolutely continuous. Thus ω_λ and $\mu_a\circ f^{-1}|_\Gamma$ are mutually absolutely continuous and

$$|g(\lambda)| \leq ||g||_{\omega_\lambda} = ||g||_{\mu_a\circ f^{-1}|_\Gamma} \leq ||g||_{\nu_n}$$

since, by Lemma 14,

$$\begin{aligned}\nu_n &= \mu_a\circ f^{-1}|_{\partial(\mathrm{int}K_n)} + \nu|_{\mathrm{int}K_n} \\ &= \mu_a\circ f^{-1}|_{(\mathrm{int}K_n)^-} + \mu_s\circ f^{-1}|_{\mathrm{int}K_n} .\end{aligned}$$

Therefore $\lambda \in \mathrm{int}K_{n+1}$ and so, by induction, $V \subset \bigcap_n \mathrm{int}K_n$

$$\subset \mathrm{int}\,\tilde{K} .$$

Conversely, suppose that $\mathrm{in}\tilde{K} \neq \emptyset$. Let $\mathrm{int}\tilde{K} = \bigcup_{j=1}^{n} U_j$, where $\{U_j\}_{j=1}^{n}$ is the set of components of $\mathrm{int}\tilde{K}$. Let ω_i denote harmonic measure for $\overline{U}_i$, and ν_i the $\overline{U}_i$-admissable component of ν. Then $\omega = \sum_{i=1}^{n} \omega_i$ is harmonic measure for $\bar{K}$ and the $\tilde{K}$-admissable component of ν, $\tilde{\nu}$, is given by

$$\tilde{\nu} = (\nu|_{\partial(\mathrm{int}\tilde{K})})_a + \nu|_{\mathrm{int}\tilde{K}}$$

$$= \sum_{j=1}^{n} ((\nu|_{\partial U_i})_a + \nu|_{U_i})$$

$$= \sum_{j=1}^{n} \nu_i$$

Thus, by Theorems 1 and 2 in the Introduction, $P^\infty(\tilde{\nu})$ is isometrically isomorphic to $\bigoplus_{i=1}^{n} H^\infty(U_i)$, and $P^\infty(\nu_i)$ is isometrically isomorphic to $H^\infty(U_i)$.

Fix $U = U_i$ and write ∂U as $\bigcup_{j=0}^{n} \Gamma_j$ where Γ_j is a rectifiable Jordan curve and $\Gamma_0 = \partial(\overline{U})^{\wedge}$. Let $\nu = \nu_i$. It follows that the Sarason hull of ν is $\overline{U}$, but for $q \in H^\infty(\mathrm{int}\Gamma_0)$, by the Maximum Modulus Theorem,

$$\|q\|_{H^\infty(\mathrm{int}\Gamma_0)} = \|q\|_\nu .$$

Thus $(\overline{U})^\wedge = (\mathrm{int}\Gamma_0)^-$ is the Sarason hull of ν and so $\overline{U} = (\overline{U})^\wedge$; $U = \mathrm{int}\Gamma_0$.

It remains to show that $m\circ f^{-1}|_{\Gamma_0} \ll \mu_a \circ f^{-1}|_{\Gamma_0}$. If $f(a) \in \Gamma_0 \cap (\text{support } \nu|_U)$ then each open set about $f(a)$ meets $K \cap U$ and consequently, $a \in S$. Thus $\Gamma_0 \cap (\text{support } \nu|_U)$ is finite, and so $P^\infty(\nu) = P^\infty(\nu|_\Gamma)$. Therefore $P^\infty(\nu|_\Gamma) \neq L^\infty(\nu|_\Gamma)$ and consequently there exists a nonzero element q of $L^1(\nu|_{\Gamma_0})$ so that for all polynomials p,

$$\int_{\Gamma_0} pq\,d\nu = 0.$$

Since $(\mathrm{int}\Gamma_0)^-$ is the Sarason hull of ν, if ϕ is the conformal map from D onto $\mathrm{int}\Gamma$ then ϕ is a weak-star sequential generator of $H^\infty(D)$ (see [16]). Thus if $\lambda = \nu\circ\phi$ then for all polynomials p

$$\int (p\circ\phi)(q\circ\phi)d\lambda = \int pq\,d\nu = 0.$$

Since ϕ is a weak-star generator of $H^\infty(D)$,

$$\int z^n(q\circ\phi)d\lambda = 0$$

for each $n \geq 0$. Thus, by the F. and M. Riesz Theorem, $(q\circ\phi)\lambda \ll m$ and $(q\circ\phi)\frac{d\lambda}{dm} \in H^1(m)$. In particular, $(q\circ\phi)\frac{d\lambda}{dm} \neq 0$ almost everywhere-m, and so $m \ll \lambda$. Thus $m\circ\phi^{-1}|_{\Gamma_0} \ll \nu|_{\Gamma_0}$, and since $m\circ\phi^{-1}$ and harmonic measure are mutually absolutely continuous (Theorem 12),

$m\circ f^{-1}|_{\Gamma_0} \ll \mu_a\circ f^{-1}|_{\Gamma_0}$. ∎

<u>Corollary 16</u>: If $E = \{\lambda | \frac{d\lambda}{dm}(\lambda) > 0\}$ then $P^\infty(\nu) \neq L^\infty(\nu)$ if, and only if there is a Jordan curve Γ contained in K such that

$$m(f^{-1}(\Gamma - f(E))) = 0.$$

<u>Proof</u>: If $P^\infty(\nu) \neq L^\infty(\nu)$ then there is a Jordan curve Γ contained in K such that $m\circ f^{-1}|_\Gamma \ll \mu_a\circ f^{-1}|_\Gamma$. Then $m(f^{-1}(\Gamma - f(E))) = 0$ since $\mu_a(f^{-1}(\Gamma - f(E))) = \mu_a(f^{-1}(\Gamma) - f^{-1}(f(E)))$

$$\leq \mu_a(\partial D - f^{-1}(f(E))),$$

$$\mu_a(f^{-1}(\Gamma - f(E))) \leq \mu_a(\partial D - E)$$
$$= 0 .$$

To see the converse, suppose that Γ is a Jordan curve contained in K such that $m(f^{-1}(\Gamma - f(E))) = 0$. As in the proof of Lemma 14, we can show that if F is a Borel set in $f^{-1}(\Gamma) \cap \partial D$ such that $m(F) = 0$, then $m(f^{-1}(f(F)) = 0$. If A is a Borel set in Γ such that $\mu_a(f^{-1}(A)) = 0$ then

$$\begin{aligned} m(f^{-1}(A)) &= m(f^{-1}(A \cap f(E))) \\ &\quad + m(f^{-1}(A - f(E))) \\ &= m(f^{-1}(f(f^{-1}(A) \cap E))) \\ &= 0 \end{aligned}$$

provided that $m(f^{-1}(A) \cap E) = 0$. Since $m|_E \ll \mu_a$ and $\mu_a(f^{-1}(A)) = 0$, $m(f^{-1}(A) \cap E) = 0$, and the corollary is proved via Theorem 15 ∎.

An operator T on the Hilbert space $\mathcal{H}$ is said to be reductive if every subspace of $\mathcal{H}$ invariant under T is also invariant under T^*. As mentioned in the Introduction, a normal operator with scalar-valued spectral measure λ is reductive if, and only if $P^\infty(\lambda) = L^\infty(\lambda)$. If U is a unitary operator, Theorem 15 and its corollary give criteria for determining whether f(U) is reductive. Intuitively, Corollary 16 states that f(U) is non-reductive provided that f wraps the carrier of $\frac{d\lambda}{dm}$ around some hole in the plane.

The Continuity of Subnormal Representations of Bounded Analytic Functions

Throughout this Section let S be a subnormal operator on $\mathcal{H}$ with minimal normal extension N on $\mathcal{K}$. Let E be a spectral measure for N and G a bounded region of the plane. For i = 1,2, let

$$\pi_i : H^\infty(G) \to \mathcal{B}(\mathcal{H})$$

denote a norm-continuous homomorphism such that $\pi_i(1) = 1$ and $\pi_i(z) = S$.

Lemma 17: If $\lambda \in G$ and $g \in H^\infty(G)$ then

$$\operatorname*{Lim\,sup}_{r \to 0^{\pm}} \frac{\|E(B(\lambda,r))(\pi_1(g) - \pi_2(g))\|}{r^2} = 0 \quad .$$

Proof: Choose $r > 0$ such that $B(\lambda,r) \subset G$ and define $h_\lambda(z)$ to be to be the first 3 terms of the Taylor series expansion of g:

$$h_\lambda(z) = g(\lambda) + g'(\lambda)(z-\lambda) + g''\frac{(\lambda)}{2}(z-\lambda)^2.$$

Then $g = h_\lambda + (z-\lambda)^3 q_\lambda$ where

$$q_\lambda(z) = \sum_{k=3}^{\infty} \frac{g^{(k)}(\lambda)}{k!} (z-\lambda)^{k-3}$$

is analytic in G. In fact, q_λ is bounded and $\|q\|_{H^\infty(G)}$ is independent of r. Then

$$\|E(B(\lambda,r))(\pi_1(g) - \pi_2(g))\|$$

$$= \|E(B(\lambda,r))(\pi_1(h_\lambda + (z-\lambda)^3 q_\lambda) - \pi_2(h_\lambda + (z-\lambda)^3 q_\lambda))\|$$

$$= \|E(B(\lambda,r))(S-\lambda)^3(\pi_1(q_\lambda) - \pi_2(q_\lambda))\|$$

$$\leq \|E(B(\lambda,r))(S-\lambda)^3\|(\|\pi_1(q_\lambda)\| + \|\pi_2(q_\lambda)\|)$$

$$= \|(N-\lambda)^3 E(B(\lambda,r))\| \; (\|\pi_1\| + \|\pi_2\|)\|q_\lambda\|_\infty$$

$$\leq \; r^3(\|\pi_1\| + \|\pi_2\|) \; \|q_\lambda\| \quad \blacksquare .$$

<u>Lemma 18</u>: $\|E(G)(\pi_1(g) - \pi_2(g))\| = 0$ for all $g \in H^\infty(G)$.

<u>Proof</u>: If $\|E(G)(\pi_1(g) - \pi_2(g))\| > 0$ then, since E is regular, there exists a compact set $K \subset G$ such that

$$\eta \equiv \|E(K)(\pi_1(g) - \pi_2(g))\| > 0.$$

Let d be the diameter of K; that is $d = \sup\{|\omega-\lambda| \mid \omega,\lambda \in K\}$. Construct a square with sides of length d containing K and partition it into four congruent squares with sides of length d/2 and centers $\{\lambda_{1,k}\}_{k=1}^{4}$. One of these centers, λ_1, has the property that

$$\|E(B(\lambda_1, \tfrac{d}{2}) \cap K)(\pi_1(g) - \pi_2(g))\| \geq \eta/4, \text{ and}$$

thus

$$\frac{\|E(B(\lambda_1, \frac{d}{2}) \cap K)\ (\pi_1(g) - \pi_2(g))\|}{(\frac{d}{2})^2} \geq \eta/d^2 \quad .$$

Continuing by induction, we construct a sequence of points $\{\lambda_n\}$ such that

$$|\lambda_n - \lambda_{n+1}| \leq \frac{\sqrt{2}d}{2^{n+2}} \quad ,$$

$$B(\lambda_{n+1}, \frac{d}{2^{n+1}}) \subset B(\lambda_n, \frac{d}{2^n}) \ ,$$

and

$$\frac{\|E(B(\lambda_n, \frac{d}{2^n}) \cap K)(\pi_1(g) - \pi_2(g))\|}{(\frac{d}{2^n})^2} \geq \eta/d^2 \quad .$$

Let $\lambda = \lim\limits_{n\to\infty} \lambda_n$. The point λ belongs to K; otherwise, there is an $\varepsilon > 0$ such that $E(B(\lambda,\varepsilon) \cap K) = 0$. Thus $E(B(\lambda_n,\frac{d}{2^n}) \cap K) = 0$ for all n sufficiently large.

Fix N, and let $r = |\lambda - \lambda_N| + \frac{d}{2^N}$. If N is sufficiently large then

$$B(\lambda_{N+1}, \frac{d}{2^{N+1}}) \subset B(\lambda, r) \subset G .$$

Thus $\dfrac{\|E(B(\lambda,r))(\pi_1(g) - \pi_2(g))\|}{r^2}$

$$\geq \|E(B(\lambda_{N+1}, \frac{d}{2^{N+1}}) \cap K)(\pi_1(g) - \pi_2(g))\|$$

$$\geq \eta/d^2 \quad ;$$

a contradiction of Lemma 17 ∎

<u>Corollary 19</u>: Let S be a subnormal on $\mathcal{H}$ with minimal normal extension N on $\mathcal{K}$, and π a norm continuous unital representation from $H^\infty(G)$ into $\mathcal{B}(\mathcal{H})$ such that $\pi(z) = S$. Suppose that E is the spectral measure for N. If $E(\partial G) = 0$ then π is unique.

<u>Proof</u>: Notice that $\sigma(N) \subset \sigma(s) \subset G^-$. Since $I = E(\partial G) + E(G)$, the result follows ∎

Let S be a pure subnormal operator; that is, there is no nonzero reducing subspace $\mathcal{M}$ of S so that $S|_{\mathcal{M}}$ is normal. If $\pi: H^\infty(G) \to \mathcal{B}(\mathcal{H})$ is a norm continuous unital representation. Define $G^* = \{\lambda \in C \mid \bar{\lambda} \in G\}$, and if $f \in H^\infty(G^*)$ then $\tilde{f}$, defined by $\tilde{f}(z) = (f(\bar{z}))^-$, is a bounded analytic function on G. The map $f \to \tilde{f}$ is an isometry from $H^\infty(G^*)$ onto

$H^\infty(G)$ that is also a weak-star homeomorphism and conjugate linear. Define a representation π^* on $H^\infty(G^*)$ by $\pi^*(f) = (\pi(f))^*$.

The proof of the following Lemma may be found in [1].

<u>Lemma 20</u>: Let $\{f_n\}_n$ be a sequence in $H^\infty(G)$ such that $f_n \to 0$ weak-star, and suppose that ϕ is a rational function with poles off ∂G. Then there is a sequence of polynomials, $\{P_n\}$, such that

(i) $\|P_n\|_\infty \to 0$ as $n \to \infty$, and

(ii) $\phi(f_n - P_n) \in H^\infty$ and

$\phi(f_n - P_n) \to 0$ weak-star.

<u>Theorem 21</u>: If Π is a unital, norm continuous representation of $H^\infty(G)$ into $\mathcal{B}(\mathcal{H})$ so that $\pi(z) = S$ is pure subnormal, then π^* is weak-star, sot sequentially continuous.

<u>Proof</u>: If π^* is not weak*-sot sequentially continuous then there is a sequence $\{f_n\}_n$ in $H^\infty(G)$ such that $\|f_n\|_\infty \leq 1$ for each n and $f_n \to 0$ pointwise on G, and a vector $x \in \mathcal{H}$ so that

$$\|\pi(f_n)^* x\| \to \alpha > 0$$

as $n \to \infty$. Let $x_n = \dfrac{\pi(f_n)^* x}{\|\pi(f_n)^* x\|}$. Then, by considering a subsequence if necessary, we may assume that $\{\pi(f_n)x_n\}_n$ converges weakly to a vector y in $\mathcal{H}$. Notice that $y \neq 0$ since

$$< y,x > = \operatorname*{Lim}_n < \pi(f_n)x_n, x >$$

$$= \operatorname*{Lim}_n < x_n, \pi(f_n)^* x >$$

$$= \alpha > 0.$$

Claim 6: There is a continuous linear map, Γ, from $C(\partial G)$ into $\mathcal{H}$ such that for all $f \in C(\partial G)$,

$$\Gamma(zf) = S\Gamma(f) .$$

Proof: Let $\phi = p/q$ be a rational function with poles off ∂G, and construct via Lemma 20 a sequence of polynomials $\{P_n\}$ so that

$$\|P_n\|_\infty \to 0 \quad , \qquad \text{and}$$

$$\phi(f_n - P_n) \to 0$$

weak-star in $H^\infty(G)$. Then the sequence $\pi(\phi(f_n - P_n))x_n$ is bounded and thus clusters weakly. Suppose

$u = \text{weak-limit } \pi(\phi(f_{n_k} - P_{n_k}))x_{n_k}$. Then

$$(7)\qquad q(S)u = \underset{k\to\infty}{\text{weak-limit}}\ \pi(q)\pi(\phi(f_{n_k}-P_{n_k}))x_{n_k}$$

$$= \underset{k\to\infty}{\text{weak-limit}}\ \pi(q\phi(f_{n_k}-P_{n_k}))x_{n_k}$$

$$= \underset{k\to\infty}{\text{weak-limit}}\ \pi(P(f_{n_k}-P_{n_k}))x_{n_k}$$

$$= p(S)\ \underset{k\to\infty}{\text{weak-limit}}\ \pi(f_{n_k})x_{n_k} - P_{n_k}(S)x_{n_k}$$

$$= P(S)y\ .$$

Since S is pure, it has no eigenvalues, and, consequently, neither does $q(S)$. In fact, $q(S)$ is also pure [15] . Thus u is unique and the sequence

$$\{\pi(\phi(f_n-P_n))x_n\}_n$$

is weakly convergent.

Define $\Gamma(\phi) = \underset{n\to\infty}{\text{weak-limit}}\ \pi(\phi(f_n-P_n))x_n$. The computation (7) then also shows that $\Gamma(\phi)$ is independent of the sequence $\{P_n\}$. That Γ is linear is immediate. It remains to show that Γ is norm-continuous on the rational functions. Since $R(\partial G) = C(\partial G)$, [17] , Γ then extends to $C(\partial G)$. By the Maximum Modulus Theorem,

$$\| \phi(f_n-P_n)\|_\infty = \sup\{\|\phi(f_n-P_n)\|_{G-K} \mid K \text{ is a compact set, } K\subset G\}$$

$$\leq \|\phi\|_{\partial G}\| f_n-P_n\|_\infty .$$

Thus $\|\Gamma(\phi)\| \leq \|\pi\| \operatorname{Lim\,sup} \|\phi(f_n-P_n)\|_\infty$

$$\leq \|\pi\| \, \|\phi\|_{\partial G} \operatorname{Lim\,sup} \|f_n-P_n\|_\infty$$

$$= \|\pi\| \, \|\phi\|_{\partial G} \operatorname{Lim\,sup} \|f_n\|_\infty .$$

Remark: This construction of Γ, on which the theorem depends heavily, is due to C. Apostol and B. Chevreau [1].

If p/q is a rational function with poles off ∂G, then the argument above has shown

$$q(S)\,\Gamma\left(\frac{p}{q}\right) = p(S)y \quad .$$

Moreover, if $\lambda \in C$ then

$$\begin{aligned} q(S)\,\Gamma\left(\frac{(z-\lambda)p}{q}\right) &= (S-\lambda)p(S)y \\ &= (S-\lambda)q(S)\,\Gamma\left(\frac{p}{q}\right) \\ &= q(S)((S-\lambda)\Gamma\left(\frac{p}{q}\right) \quad . \end{aligned}$$

Thus $\Gamma\left(\frac{(z-\lambda)p}{q}\right) = (S-\lambda)\,\Gamma\left(\frac{p}{q}\right)$, and by continuity,

$$\Gamma((z-\lambda)f) = (S-\lambda)\Gamma(f)$$

for all $f \in C(\partial G)$. Thus, for $f \in C(\partial G)$ and $\lambda \in G$,

$$(N-\lambda)\Gamma\left(\frac{f}{z-\lambda}\right) = \Gamma(f) \quad .$$

If Δ is a Borel subset of the plane and if

$$x \in \bigcap_{\lambda \in \Delta} \mathcal{R}\mathrm{an}(N-\lambda)$$

then $E(\Delta)x = 0$ (see [3], pg. 19). Thus

$$\mathcal{R}\mathrm{an}\Gamma \subset E(\partial G)\mathcal{K} \cap \mathcal{H}.$$

In particular, $y = \Gamma(1)$ is contained in $E(\partial G)\mathcal{K} \cap \mathcal{H}$.

Let q be a polynomial without roots on ∂G. Then

$q(N)|_{E(\partial G)\mathcal{K}} = q(E(\partial G)N)$ is invertible on $E(\partial G)\mathcal{K}$, and, for any polynomial p,

$$\begin{aligned} \frac{p}{q}(NE(\partial G))y &= q^{-1}(NE(\partial G))(q(N)\,\Gamma(\tfrac{p}{q})) \\ &= \Gamma(\tfrac{p}{q}) \quad . \end{aligned}$$

Thus $\Gamma(f) = f(N)y$ for all $f \in C(\partial G)$ by the continuity of the maps Γ and $C(\partial G) \to \mathcal{B}(K)$ defined by $f \to f(N)|_{E(\partial G)K}$. Therefore the closure of the manifold $\{\Gamma(f) | f \in C(\partial G)$ is a nonzero reducing subspace of S on which it is normal, a contradiction to the assumption that S is pure ∎

<u>Corollary 22</u>: If S is a pure subnormal contradiction then

$$S^{*n} \to 0 \text{ sot} \quad .$$

<u>Proof</u>: The map $f \to f(S) \equiv f(N)|_H$ is a norm continuous unital representation of $H^\infty(D)$ into $\mathcal{B}(\mathcal{H})$ with $\pi(z) = S$, [7]. Since $D = D^*$ and $z^n \to 0$ weak-star, the result follows from Theorem 21 ∎ .

<u>Corollary 23</u>: π is weak-star, weak-star continuous.

<u>Proof</u>: Since $H^\infty(G)$ is the dual of a separable Banach space, it suffices to show that π is weak-star, weak-star sequentially continuous [6] . In fact, it suffices to show that π is weak-star, wot sequentially continuous since π is norm continuous and the weak-star and wot topologies agree on balls in $\mathcal{B}(\mathcal{H})$.

Thus let $\{f_n\}$ be a sequence in $H^\infty(G)$ converging weak-star to zero. Then $\{\tilde{f}_n\}$ converges to zero weak-star in $H^\infty(G^*)$ and consequently $\pi^*(\tilde{f}_n) = (\pi(f_n))^*$ converges to zero sot. The result now follows, noting that the map $A \to A^*$ is wot continuous on $\mathcal{B}(\mathcal{H})$ ∎ .

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FUNCTIONS OF SUBNORMAL OPERATORS

by

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(ABSTRACT)

If f is analytic in a neighborhood of $\partial D = \{z \mid |z| = 1\}$ and if $\mathcal{K} = f(\partial D)$, then $C - \mathcal{K}$ has only finitely many components; moreover, if U is a bounded simply connected region of the plane, then

$$\partial U = \bigcup_{j=0}^{n} \Gamma_j$$

where each Γ_j is a rectifiable Jordan curve and $\Gamma_i \cap \Gamma_j$ is a finite set whenever $i \neq j$.

Let μ be a positive regular Borel measure supported on ∂D and let m denote normalized Lebesgue measure on ∂D. If L is a compact set such that $\partial L \subset \mathcal{K}$ and $\mathcal{R}(L)$ is a Dirichlet algebra and if $\nu = \mu o f^{-1}$, then the Lebesgue decomposition of $\nu|_{\partial V}$ with respect to harmonic measure for L is

$$\nu|_{\partial V} = \mu_a o f^{-1}|_{\partial V} + \mu_s o f^{-1}|_{\partial V}$$

where V = intL and $\mu = \mu_a + \mu_s$ is the Lebesgue decomposition of μ with respect to m.

Applying Sarason's process, we obtain $P^{\infty}(\nu) \neq L^{\infty}(\nu)$ if, and only if there is a Jordan curve Γ contained in $\mathcal{K}$ such that $m \circ f^{-1}|_{\Gamma} << \mu_a \circ f^{-1}|_{\Gamma}$. If U is a unitary operator with scalar-valued spectral measure μ then f(U) is non-reductive if and only if there is a Jordan curve $\Gamma \subset \mathcal{K}$ such that $m \circ f^{-1}|_{\Gamma} << \mu_a \circ f^{-1}|_{\Gamma}$.

Let G be a bounded region of the plane and $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators on the separable Hilbert space $\mathcal{H}$. If π: $H^{\infty}(G) \to \mathcal{B}(\mathcal{H})$ is a norm-continuous homomorphism such that $\pi(1) = 1$ and $\pi(z)$ is pure subnormal then π is weak-star, weak-star continuous. Moreover, if S is a pure subnormal contraction, then $S^{*n} \to 0$ sot.