

Damping optimization of parameter dependent mechanical systems by rational interpolation

Zoran Tomljanović* Christopher Beattie† Serkan Gugercin†

Abstract

We consider an optimization problem related to semi-active damping of vibrating systems. The main problem is to determine the best damping matrix able to minimize influence of the input on the output of the system. We use a minimization criteria based on the \mathcal{H}_2 system norm. The objective function is non-convex and the associated optimization problem typically requires a large number of objective function evaluations. We propose an optimization approach that calculates ‘interpolatory’ reduced order models, allowing for significant acceleration of the optimization process.

In our approach, we use parametric model reduction (PMOR) based on the Iterative Rational Krylov Algorithm, which ensures good approximations relative to the \mathcal{H}_2 system norm, aligning well with the underlying damping design objectives. For the parameter sampling that occurs within each PMOR cycle, we consider approaches with predetermined sampling and approaches using adaptive sampling, and each of these approaches may be combined with three possible strategies for internal reduction. In order to preserve important system properties, we maintain second-order structure, which through the use of modal coordinates, allows for very efficient implementation.

The methodology proposed here provides a significant acceleration of the optimization process; the gain in efficiency is illustrated in numerical experiments.

Keywords: Model reduction, Interpolation, Second-order systems, semi-active damping
msc2010: 49J15, 74P10, 70Q05, 41A05

1 Introduction

We consider the following vibrational system described by

$$\begin{aligned} M\ddot{q}(t) + C\dot{q}(t) + Kq(t) &= Ew(t), \\ z(t) &= Hq(t) \end{aligned} \tag{1}$$

*Department of Mathematics, University J.J. Strossmayer in Osijek, Osijek, Croatia, ztomljan@mathos.hr

†Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, USA, {beattie, gugercin}@vt.edu

where the mass matrix, M , and stiffness matrix, K , are real, symmetric positive-definite matrices of order n . The state variables comprise displacement and rotational degrees of freedom and are collected in the coordinate vector $q \in \mathbb{R}^n$. The vector $z \in \mathbb{R}^{m_{out}}$ denotes the observed output comprising the system displacements in which we are interested, which in turn determines the (constant) output port matrix $H \in \mathbb{R}^{m_{out} \times n}$. The vector $w(t) \in \mathbb{R}^{m_{in}}$ represents a dynamic disturbance, the primary excitation. The associated locations of the primary excitation determine the primary excitation matrix, $E \in \mathbb{R}^{n \times m_{in}}$.

The damping matrix is given by

$$C = C_{int} + C_{ext},$$

where C_{int} and C_{ext} represent contributions from internal and external damping, respectively. We are principally concerned with optimizing the placement and geometry of external dampers whose dynamic effects are described through C_{ext} . The internal damping contribution as encoded in C_{int} is typically both small in magnitude and difficult to model in detail. Often it is modeled by taking C_{int} to be a small multiple of critical damping (see, e.g., [18, 19, 47]):

$$C_{int} = 2\alpha_c M^{1/2} (M^{-1/2} K M^{-1/2})^{1/2} M^{1/2}, \quad \alpha_c \ll 1. \quad (2)$$

Other models of internal damping may be considered as well, but whatever model may be chosen (i.e., however C_{int} is determined), it is part of the description of the underlying structure and not accessible to modification in the course of optimization.

We assume that the external damping, the component of damping that is accessible to modification and optimization, creates dynamic effects that may be modeled as $C_{ext} = B G B^T$ where $G = \text{diag}(g_1, g_2, \dots, g_p) \in \mathbb{R}^{p \times p}$ is a diagonal matrix and $B \in \mathbb{R}^{n \times p}$ determines the placement and geometry of the external dampers. The entries $\{g_i\}_{i=1}^p$ are usually called gains or viscosities and represent friction coefficients of the corresponding dampers. These coefficients are non-negative and may be either constant or vary over time. Here, we consider the g_i to have constant values that will be chosen within fixed feasible margins, $g_i \in [g_i^-, g_i^+]$ for $i = 1, \dots, p$. Usually the number of dampers p is much smaller than the full dimension: $p \ll n$. More details regarding system stability and model description can be found in [16, 20].

Damping optimization has a long history in the service of structural engineers and these problems continue to be widely investigated from the perspectives of both engineering and mathematics. A frequent context for damping optimization comes from system stabilization goals, wherein one strategically adds damping to a structure with light internal damping so as to mute resonances or move them away from the frequencies of ambient oscillatory loads or to suppressing damaging effects of external impacts on the structure. There is a vast literature in this field of research, see, e.g., [34, 35, 39, 40, 44]. Depending on the particular application, different damping criteria may be appropriate. For example, for stationary systems criteria that involve spectral abscissa are useful (see [27]), while criteria that involve total average energy was used in, e.g., [45, 47]. Structured dimension reduction methods using total average energy criterion were considered in [18, 19]. For non-stationary systems one may consider in addition particular

external forces that potentially play an important role in system behaviour. In this case, criteria that involve average energy amplitude and average displacement amplitude can be considered (see, e.g., [35,36,46]). Overviews on different possible damping criteria can be found in [29,49].

Since optimization of gains g_1, g_2, \dots, g_p will be our main interest, we collect these parameters into a vector $g = (g_1, g_2, \dots, g_p)$ and write $G(g)$ to express the parametric dependence on gains. The transfer function matrix for system (1) is given by

$$F(s; g) = H(s^2 M + s(C_{int} + BG(g)B^T) + K)^{-1} E, \quad s \in \mathbb{C}. \quad (3)$$

Note that for any $s \in \mathbb{C}$, $F(s; g)$ is an $m_{out} \times m_{in}$ complex matrix with rational functions of s as elements. We may also rewrite the second-order system representing our vibrating system as a first-order system of differential equations:

$$\begin{aligned} \dot{x}(t) &= \hat{A}x(t) + \hat{E}w(t), \\ z(t) &= \hat{H}x(t), \end{aligned} \quad (4)$$

where

$$\begin{aligned} x &= \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \quad \hat{A}(g) = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}(C_{int} + BG(g)B^T) \end{bmatrix}, \\ \hat{E} &= \begin{bmatrix} 0 \\ M^{-1}E \end{bmatrix}, \quad \text{and} \quad \hat{H} = [H \quad 0], \end{aligned} \quad (5)$$

leading to an alternate realization of $F(s; g)$: $F(s; g) = \hat{H} (sI - \hat{A}(g))^{-1} \hat{E}$. The main task in our setting will be to determine a damping configuration, as encoded in G and B , which minimizes the influence of the input disturbance, w , on observed output states, z .

One may consider different criteria to achieve this goal, some are based on system-theoretic norms (see, e.g., [16, 20, 29]). We focus on the \mathcal{H}_2 norm and define a cost function in the frequency domain using the transfer function matrix defined above in (3):

$$\|F(\cdot; g)\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{trace}(F(j\omega; g)^* F(j\omega; g)) d\omega \right)^{\frac{1}{2}}. \quad (6)$$

For single-input/single-output (SISO) systems, this criterion can be identified with the response energy resulting from an impulsive input:

$$\|F(\cdot; g)\|_{\mathcal{H}_2}^2 = \int_0^{+\infty} \|z_D(t)\|_2^2 dt. \quad (7)$$

Here, z_D is the (scalar) output response of the SISO system (4) resulting from a Dirac function input, $w(t)$. Moreover, in the multi-input/multi-output (MIMO) case, the \mathcal{H}_2 -norm provides a uniform bound on the time response magnitude assuming a disturbance with unit L_2 -energy: $\max_{t \geq 0} \|z(t)\|_\infty \leq \|F(\cdot; g)\|_{\mathcal{H}_2}$ when $\|w\|_{L_2} \leq 1$.

In order to minimize uniformly the output z under the influence of the disturbance, w , we will proceed by determining the “best” damping such that $\|F\|_{\mathcal{H}_2}$ is minimal. Using standard theory (see, e.g., [20, 24, 51]), it can be shown that the \mathcal{H}_2 -norm of the system, $\|F\|_{\mathcal{H}_2}$, can be expressed via the solution of a Lyapunov equation:

$$\|F(\cdot; g)\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi} \text{trace } \hat{E}^T \hat{\mathbb{X}} \hat{E} \right)^{\frac{1}{2}}, \quad (8)$$

where $\hat{\mathbb{X}}$ solves

$$\hat{A}^T(g) \hat{\mathbb{X}} + \hat{\mathbb{X}} \hat{A}(g) = -\hat{H}^T \hat{H}, \quad (9)$$

and the matrices $\hat{A}(g)$, \hat{E} , and \hat{H} are given in (5).

Although this makes evaluation of $\|F(\cdot; g)\|_{\mathcal{H}_2}$ amenable to numerical computation, the computational resources required to approach realistic problems is still substantial. Moreover, one observes that the objective function, $\|F(\cdot; g)\|_{\mathcal{H}_2}$ will have many local minima with respect to damping positions as encoded in B , and for each B there may be many local minimizers with respect to the damping gains, $g = (g_1, g_2, \dots, g_p)$. Not surprisingly, many function evaluations are necessary to carry out optimization with respect to both g and B and this frequently creates an unmanageable computational burden.

We introduce an approach here which calculates reduced second-order systems in such a way that allows efficient approximation of the \mathcal{H}_2 norm, which in turn brings us a significant acceleration of the optimization process. We focus first on efficient optimization with respect to g , and then optimization with respect to damping positions; both can be well approximated and cheaply obtained using a reduced order model. Since we are dealing with structured second-order systems, we use structure preserving methods which are derived particularly for a parametric setting.

There are several different methods for calculating a reduced system for second-order settings; see, e.g., [6, 9, 10, 21, 22, 25, 38, 41, 43]. A review of different methods of dimension reduction, both parametric and nonparametric, can be found in [2, 3, 5, 8, 14, 15, 17, 33]. The approximation of optimal damping using dimension reduction for stationary second-order systems was studied in [18, 19] where the authors considered optimization of passive damping. Another approach based on dominant poles, presented first in [42], was studied in [16]. The approach that we present here uses interpolatory projections to produce high fidelity reduced-dimension second-order systems that are then optimized with respect to damping as measured with the \mathcal{H}_2 system norm. We employ a variant of the Iterative Rational Krylov Algorithm (IRKA), which is a popular approach for producing high quality reduced models. IRKA produces locally \mathcal{H}_2 -optimal reduced models, dovetailing perfectly with the optimization task at hand. In Section 2, we describe the original IRKA approach as well as our variant, and organize its use in damping optimization. Implementation issues are discussed in Section 3. In Section 4, we describe a variety of numerical experiments that show the advantages of our approach.

2 IRKA and Damping Optimization

The use of reduced models in parameter optimization involves iterating a two step process wherein one performs parameter optimization on a high-fidelity, parameterized surrogate model in the first, comparatively inexpensive step, which is expected to bring the parameters into the vicinity of their optimal values, followed by a (generally more expensive) update step that corrects deviations that the surrogate model may have with the true parameterized system in the vicinity of the current parameter values. This approach is aligned with classical trust region strategies, see for example, [1, 31]. Since we are seeking to minimize the \mathcal{H}_2 system norm with respect to a variety of damping configurations, an effective choice of surrogate models will be \mathcal{H}_2 optimal reduced order models. Efficient interpolatory projection methods have been developed to derive locally \mathcal{H}_2 -optimal reduced order models in related settings (e.g. see [7, 12, 26, 32]).

Projection-based methods for model reduction can be described quite simply. We approximate the full state, $q(t)$, using $q(t) \approx X_r q_r(t)$ where $X_r \in \mathbb{R}^{n \times r}$ has linearly independent columns spanning a *right modeling space* that is still to be determined. A complementary *left modeling space* spanned by the columns of a second matrix, $W_r \in \mathbb{R}^{n \times r}$, allows us to enforce Petrov-Galerkin conditions that define reduced model dynamics:

$$\begin{aligned} M_r \ddot{q}_r(t) + C_r \dot{q}_r(t) + K_r q_r(t) &= E_r w(t), \quad \text{where} \\ M_r = W_r^* M X_r, \quad C_r(g) &= W_r^* C(g) X_r, \quad K_r = W_r^* K X_r, \quad \text{and} \quad E_r = W_r^* E. \end{aligned} \quad (10)$$

The reduced model output then appears as $z_r(t) = H_r q_r(t)$ with $H_r = H X_r$. Evidently, one should choose X_r and W_r , or equivalently, right and left modeling subspaces, so as to ensure $z_r(t) \approx z(t)$ over a wide range of inputs, $w(t)$. In [16], the authors have chosen the columns of X_r and W_r to contain eigenvectors of the polynomial pencil, $\lambda^2 M + \lambda C(g) + K$, that correspond to dominant poles of the transfer function, $F(s; g)$, defined in (3). With this approach one maintains the dominant terms from $F(s; g)$ in the reduced transfer function,

$$F_{2r}(s; g) = H_r (s^2 M_r + s C_r(g) + K_r)^{-1} E_r. \quad (11)$$

Note that $F_{2r}(s; g)$ is an $m_{out} \times m_{in}$ complex matrix (same size as $F(s; g)$) but now with rational functions of lower order ($2r - 1, 2r$) as elements.

We will make a different choice for W_r and X_r , choosing them instead so as to enforce *tangential interpolation* conditions: for selected interpolation points $\sigma_1, \sigma_2, \dots, \sigma_r \in \mathbb{C}$ and directions b_1, \dots, b_r and c_1, \dots, c_r , we will choose W_r and X_r so that

$$c_i^T F(\sigma_i) = c_i^T F_{2r}(\sigma_i), \quad F(\sigma_i) b_i = F_{2r}(\sigma_i) b_i, \quad \text{and} \quad c_i^T F'(\sigma_i) b_i = c_i^T F'_r(\sigma_i) b_i,$$

for $i = 1, \dots, r$. Calculation of W_r and X_r that enforce these interpolation properties is straightforward, and depending on the context, a variety of choices for interpolation points and tangent directions could be used, see, e.g., [4, 11, 13, 23, 28, 32]. This is the thrust of *interpolatory projection methods* for model reduction. Since we are dealing with structured second-order mechanical

systems, we will constrain the choice of W_r and X_r so as to preserve second-order structure as well as important system properties such as system stability, passivity, and reciprocity which is encoded in system structure through the symmetry and positive-definiteness of M , C , and K . This can be accomplished by setting $W_r = X_r$, and then (10) becomes:

$$\begin{aligned} M_r \ddot{q}_r(t) + C_r \dot{q}_r(t) + K_r q_r(t) &= E_r w(t), \quad \text{where} \\ M_r &= X_r^* M X_r, \quad C_r(g) = X_r^* C(g) X_r, \quad K_r = X_r^* K X_r, \quad \text{and } E_r = X_r^* E. \end{aligned} \quad (12)$$

Using either (10) or (12), one may express the reduced transfer function $F_{2r}(s; g)$ in (11) in terms of its $2r$ poles and residues:

$$F_{2r}(s) = \sum_{k=1}^{2r} \frac{c_k b_k^T}{s - \lambda_k} \quad (13)$$

First-order necessary conditions for an order $2r$ rational function, $\hat{F}_{2r}(s)$, having the form (13) (and having $2r$ distinct poles, $\{\lambda_k\}$) to be an *optimal* \mathcal{H}_2 reduced order approximation to $F(s)$ are due to Meier and Luenberger [37]. They require that $\hat{F}_{2r}(s)$ be a Hermite interpolant to the full-order system at points in the complex plane that reflect the reduced system poles across the imaginary axis and for MIMO systems this need only happen in particular directions in the input/output spaces see, e.g., [23, 32, 48]. Specifically, if $\hat{F}_{2r}(s) = \sum_{k=1}^{2r} \frac{\hat{c}_k \hat{b}_k^T}{s - \hat{\lambda}_k}$ and \hat{F}_{2r} is a local minimizer of $\|\hat{F}_{2r} - F\|_{\mathcal{H}_2}$, then \hat{F}_{2r} is also a tangential Hermite interpolant of $F(s)$ at $-\hat{\lambda}_k$, $k = 1, \dots, 2r$ in the sense that

$$\begin{aligned} \hat{F}_{2r}(-\hat{\lambda}_k) \hat{b}_k &= F(-\hat{\lambda}_k) \hat{b}_k, \quad \hat{c}_k^T \hat{F}_{2r}(-\hat{\lambda}_k) = \hat{c}_k^T F(-\hat{\lambda}_k), \\ \text{and } \hat{c}_k^T \hat{F}'_{2r}(-\hat{\lambda}_k) \hat{b}_k &= \hat{c}_k^T F'(-\hat{\lambda}_k) \hat{b}_k, \quad \text{for } k = 1, \dots, 2r. \end{aligned} \quad (14)$$

IRKA [32] is an algorithm that can produce such an \hat{F}_{2r} efficiently and will provide directly a standard realization for it:

$$\begin{aligned} \dot{x}_{2r}(t) &= \hat{A}_{2r} x_{2r}(t) + \hat{E}_{2r} w(t), \\ z_{2r}(t) &= \hat{H}_{2r} x_{2r}(t) \end{aligned} \quad (15)$$

so that $\hat{F}_{2r} = \hat{H}_{2r} (s I - \hat{A}_{2r})^{-1} \hat{E}_{2r}$, but significantly for our application, \hat{F}_{2r} will *not* generally have the form of a second-order system transfer function such as F_{2r} in (11), and this is what motivates the modification we propose for IRKA.

Assume that we have a damping configuration given by B and $G = \text{diag}(g_1, g_2, \dots, g_p)$, and we would like to obtain a reduced model, $F_{2r}(s; g)$, of the form (11) that will accurately represent $F(s; g)$ at least for small changes that may be made to the gains, g_1, g_2, \dots, g_p . Using

$\|F_{2r}(s; g)\|_{\mathcal{H}_2}$ as a surrogate for $\|F(s; g)\|_{\mathcal{H}_2}$, we choose g_1, g_2, \dots, g_p so as to minimize the value of $\|F_{2r}(s; g)\|_{\mathcal{H}_2}$ (and hopefully, by proxy, $\|F(s; g)\|_{\mathcal{H}_2}$).

Algorithm 1, which we will also refer to as sym2IRKA, calculates a modeling basis $X_r \in \mathbb{C}^{n \times r}$ using a one-sided projection approach inspired by IRKA. In each iteration, we form a $2r$ order reduced system transfer function given by (11) and (12) that interpolates the true transfer function at a set of r interpolation points in r directions that had been determined in the previous step. In order to proceed, in Step 4, the $2r$ order reduced system transfer function is further reduced to an order r transfer function represented in pole-residue form as $\tilde{F}_r(s) = \sum_{k=1}^r \frac{\tilde{c}_k \tilde{b}_k^T}{s - \mu_k}$. By reflection across the imaginary axis, this gives r interpolation points and tangent directions with which to continue to the next step. There are many strategies with which to carry out this ‘‘internal reduction’’ step, and in what follows we consider three different strategies:

- (a) **internal reduction based on balanced truncation:** we use the *balanced truncation method* applied to a linearized order $2r$ realization of F_{2r} , to produce an order r (standard system) realization. The r poles of this realization are reflected to produce the next set of interpolation points. Details regarding the balanced truncation method can be found, e.g., in [2, 3, 15].
- (b) **internal reduction based on IRKA:** We follow a process similar to (a) except we use a variant of IRKA to produce an order r (standard system) realization. A natural way of doing this would be to use the original formulation of IRKA, which, upon convergence, will produce a reduced system satisfying necessary conditions for \mathcal{H}_2 optimality. Interestingly, we found this approach did not perform as well as a (\mathcal{H}_2 suboptimal) one-sided modification of IRKA similar to Algorithm 1 (i.e., using $W_r = X_r$) but applied instead to a linearized order $2r$ realization of F_{2r} in order to reduce it to a standard order r realization. This modified approach appears to reduce the potential for a loss of stability at non-interpolating damping configurations and we generally observe faster convergence of the iteration.
- (c) **internal reduction based on dominant poles:** we choose the r most dominant poles that are closed under conjugation, see e.g., [16, 42]); the transfer function (3) can be represented

as $F(s) = \sum_{i=1}^{2n} \frac{R_i}{s - \lambda_i}$ with residues $R_i = (H x_i)(y_i^* E) \lambda_i \in \mathbb{C}^{s \times m}$, where $\lambda_i \in \mathbb{C}$ and $x_i, y_i \in \mathbb{C}^n \setminus \{0\}$ are, respectively, eigenvalues, right eigenvectors, and left eigenvectors of the quadratic eigenvalue problem

$$(\lambda_i^2 M + \lambda_i C + K)x_i = 0, \quad y_i^*(\lambda_i^2 M + \lambda_i C + K) = 0, \quad i = 1, \dots, 2n. \quad (16)$$

Although there are a variety of definitions for ‘‘dominant’’ poles; we take as dominant those poles producing the largest values of $\frac{\|R_i\|}{|\operatorname{Re}(\lambda_i)|}$. This choice has been shown to have good performance within the optimization setting that we consider here. If the r dominant poles fail to be closed under conjugation then we consider taking $r + 1$ dominant poles, so as to be closed under conjugation.

Algorithm 1: sym2IRKA. Symmetrized IRKA for second-order systems

Input: System matrices defining (1) with (fixed) gains $\hat{g} = (g_1, g_2, \dots, g_p)^T$.

itMax - maximum number of iterations allowed for system reduction

Initial shift selection: $\{\sigma_1, \dots, \sigma_r\}$ (closed under conjugation).

Initial tangent directions: $\{b_1, \dots, b_r\}$ (also closed under conjugation).

Output: Modeling basis $X_r(\hat{g})$ producing an interpolatory ROM at $g = \hat{g}$.

1: **for** $j = 1, \dots, \text{itMax}$ **do**

2: $X_r = [(\sigma_1^2 M + \sigma_1 C + K)^{-1} E b_1, \dots, (\sigma_r^2 M + \sigma_r C + K)^{-1} E b_r];$

3: Form reduced system determined by

$$M_r = X_r^* M X_r, C_r = X_r^* C X_r, K_r = X_r^* K X_r, E_r = X_r^* E \text{ and } H_r = H X_r$$

4: Internal reduction step using strategies (a), (b), or (c):

Reduce the order $2r$ transfer function, F_{2r} , defined in (13) to an order r transfer function,

$$\tilde{F}_r(s) = \sum_{k=1}^r \frac{\tilde{c}_k \tilde{b}_k^T}{s - \mu_k}, \text{ such that } \mu_1, \dots, \mu_r, \text{ are closed under conjugation}$$

5: Assign $\sigma_k = -\mu_k$ and $b_k = \tilde{b}_k$ for $k = 1, \dots, r$

6: **if** $\{\sigma_k\}$ converged **then**

7: **break**;

8: **end if**

9: **end for**

10: $X_r(\hat{g}) \leftarrow X_r;$

Other potential strategies for internal reduction may be found in [50]. Note that the **for** loop in sym2IRKA goes up to a specified maximum number of iterations, itMax, but is stopped earlier if two consecutive sets of interpolation points, $\{\sigma_i\}$ have changed only slightly within a given tolerance. In terms of computational effort, note that Step 4 deals only with reduced systems, requiring effort that scales with $r \ll n$ rather than n .

Although we are solving large linear systems repeatedly within Step 1 and Step 6, we find that these tasks can be implemented efficiently through the use of the modal coordinates. This is discussed in the next subsection.

Through the use of sym2IRKA, we obtain a reduced model that has the same structure as the true system and should replicate the response of the true system (and hence replicate its \mathcal{H}_2 norm) to relatively high accuracy at the given sampling gain (g_1, g_2, \dots, g_p) . In order to approximate the \mathcal{H}_2 norm of $F(s, g)$ with the (easily computed) \mathcal{H}_2 norm of $F_{2r}(s, g)$ over a sufficiently broad range of parameter values in g , we will employ a parametric model order reduction approach described in [7]. In this approach we calculate \mathcal{H}_2 -based interpolatory reduced systems for a small number of selected sampling parameters and then form an aggregate interpolatory reduced model. The original and reduced model then have virtually the same \mathcal{H}_2 norms on the given set of sampling gains and generally nearly so at points in between. This makes the optimization process more robust and damping optimization can be performed more efficiently. A similar approach was also used in [16] where the authors considered local approximations based on dominant poles of the system.

In our setting, parameterized model order reduction (PMOR) is organized so that we calculate reduced systems given by (12) for several, say m , damping gain configurations, $g^{(i)}$, $i = 1, \dots, m$ and for each of these sampled gain configurations (represented as a p -tuple of individual damper gains), we calculate a modeling basis, $X_r^{(i)}$, that corresponds to a reduced system given by (12). We then merge all $X_r^{(i)}$ $i = 1, \dots, m$ into an aggregate modeling basis, $X = [X_r^{(1)}, X_r^{(2)}, \dots, X_r^{(m)}]$ and form an aggregate reduced system model,

$$\begin{aligned} X^* M X \ddot{q}_k(t) + X^* C(g) X \dot{q}_k(t) + X^* K X q_k(t) &= X^* E w(t), \\ z(t) &= H X q_k(t). \end{aligned} \tag{17}$$

By using this approach, we are assured of a good approximation of system response (and \mathcal{H}_2 norm approximation) for any of the particular gain configurations, $g^{(i)}$, $i = 1, \dots, m$, and we will generally retain good approximation for nearby gain configurations as well. Since we are interested in optimizing these gains, we may organize the procedure so that sampled gain configurations are determined adaptively during the optimization process, and used to augment the aggregate modeling basis. Gain configurations that were useful earlier in the process but that no longer contribute significant information can be dropped from the aggregate modeling basis. How best to produce a balanced parameter sampling strategy is an important and generally unresolved issue, but for particular problems there have been some advances (e.g., see the recent survey [15]). For our damping optimization problem, we consider two main strategies to sample damping gain configurations:

- (i) **predetermined sampling of damping gain configurations:** We use a predetermined set of m damping gain configurations: $g^{(1)}, \dots, g^{(m)}$ that has been chosen from the set of feasible damping gain configurations, $g^{(k)} \in \times_{i=1}^p [g_i^-, g_i^+]$. This choice can be done via uniform sampling across a fixed mesh in the feasible region, or it can include *ad hoc* choices of damping gain configurations as well. For example, we include among our predetermined damping gain configurations, the trivial configuration, $g = 0$, that is, we include a system having only internal damping and no external damping. This approach is described in more detail as Algorithm 2.
- (ii) **adaptive sampling during optimization.** Starting with an initial damping gain configuration (say, $g^{(1)} = 0$), construct a reduced order model, $F_{2r}(s; g^{(1)})$ using sym2IRKA. Then, allowing g to vary, determine the next damping gain configuration to be sampled by finding $g^{(2)}$ that solves (approximately)

$$g^{(2)} = \arg \min_g \|F_{2r}(\cdot; g)\|_{\mathcal{H}_2}.$$

Repeat this process, each time augmenting the modeling subspace used to construct reduced models with information from the newest reduced order model. This is discussed below and described in more detail as Algorithm 3.

Algorithm 2 Approximation of optimal gains using predetermined gain configuration samples

Input: System matrices defining (1); A set of m (different) gain configurations $g^{(1)}, \dots, g^{(m)}$.

The number of retained poles $nRetPoles$ for each gain configuration;

Initial choices for shift selection $\sigma_1 \dots, \sigma_r$ and directions b_1, \dots, b_r ;

Output: Approximate optimal gains.

- 1: **for** $j = 1, \dots, m$ **do**
 - 2: Using the gain configuration $g^{(j)}$, calculate a reduced order modeling subspace, V^j , with `sym2IRKA` (Algorithm 1).
 - 3: **end for**
 - 4: Aggregate modeling spaces, into $X = \text{orth}([V^1, \dots, V^m])$.
 - 5: Form a global reduced system using X as in (17).
 - 6: Find (approximate) optimal gains by optimizing the gains of the global reduced system, (17) (using an appropriate optimization tool).
-

Algorithm 3 Computation of optimal gains with adaptive sampling

Input: System matrices defining (1); An initial gain configuration $\hat{g}^{(0)} = (g_1^{(0)}, g_2^{(0)}, \dots, g_p^{(0)})^T$.

Initial choices for shift selection $\sigma_1 \dots, \sigma_r$ and directions b_1, \dots, b_r ;

$nRetPoles$ - number of retained poles for each gain configuration;

$itMax$ - maximum number of iterations for `sym2IRKA` (Algorithm 1).

$tolDiff$ - termination criterion for gain optimization

Output: Approximate optimal gains.

- 1: $j = 0$;
 - 2: **repeat**
 - 3: Using the gain configuration $g^{(j)}$, calculate a reduced order modeling subspace, V^j , with `sym2IRKA`.
 - 4: Form a reduced system using $X = \text{orth}([V^0, V^1, \dots, V^j])$ as in (17).
 - 5: $j = j + 1$
 - 6: Find an (approximately) optimal gain configuration by optimizing the gains of the global reduced system, (17) (using an appropriate optimization tool), and denote it by $g^{(j)}$
 - 7: **until** $|g^{(j)} - g^{(j-1)}| < tolDiff$
 - 8: **return** $g^{(j)}$
-

Strategies (i) and (ii) (and the associated Algorithms 2 and 3) could be viewed as representing two extremes and one can easily construct sensible sampling strategies that combine elements of both in various ways. The common element in both strategies is the use of a reduced order model that inherits from the original model, (3), the parameterization with respect to damping. The optimal damping configuration then is sought by optimizing the reduced order model, using it as an inexpensive surrogate for the original model. This is effective when the \mathcal{H}_2 -norm of the reduced order model closely tracks the \mathcal{H}_2 -norm of the original model as the damping parameters in g change. In order to assure that, one may include in the optimization goals a *greedy search*

for the damping parameter \hat{g} , such that the deviation of \mathcal{H}_2 -norm values between full-order and reduced-order models is largest:

$$\hat{g} = \arg \max_g | \|F(\cdot; g)\|_{\mathcal{H}_2} - \|F_{2r}(\cdot; g)\|_{\mathcal{H}_2} |, \quad (18)$$

where, as before, F is transfer function of full system while F_{2r} is transfer function of reduced system given by (17). Augmenting the current reduced model so as to interpolate the full order model at \hat{g} is likely to improve fidelity over a wide range of intervening g . One may repeat the procedure until the deviation given by (18) is acceptable. Such an approach ensures that the reduced system is an accurate surrogate for the full system viewed as a function of damping parameters, g .

There are two principal disadvantages to this approach in our setting. The first is that the determination of \hat{g} from (18) requires evaluation of the \mathcal{H}_2 -norm of the full system, which either is computationally very demanding, or if the \mathcal{H}_2 -norm is computed only approximately, this introduces an additional layer of approximation to the approach. The second disadvantage is that it may happen that the greedy selection of gains produced by (18) may be far away from the gains that minimize the \mathcal{H}_2 -norm of the full system, so we expend significant effort in producing a reduced model that has high fidelity at damping configurations that are not interesting for us.

Therefore, we propose an adaptive sampling strategy that samples gain configurations that have approximately minimized the \mathcal{H}_2 -norm of an intermediate reduced order system. Assume that we have obtained at some stage in the process a reduced system given by (17) with $X = V^1$. We then determine a gain configuration, \hat{g} , that minimizes the surrogate objective function:

$$\hat{g} = \arg \min_g \|F_{2r}(\cdot; g)\|_{\mathcal{H}_2},$$

where F_{2r} is the transfer function of the reduced system given by (17). We generate another modeling subspace, V^2 , that would produce a high fidelity reduced model at the system with a damping configuration of \hat{g} , (i.e., we calculate V^2 using Algorithm 1 for the gain configuration \hat{g}), but we update the reduced system by augmenting the modeling subspace with V^2 , instead of replacing it with V^2 : $X = \text{orth}([V^1, V^2])$. These steps are repeated (and the modeling subspace grows), until the difference between consecutively sampled gains drops below the prescribed tolerance `tolDiff`. until the difference between consecutive sampling gains is equal or smaller than prescribed tolerance `tolDiff`. In this way, the modeling subspace spanned by the columns of X continues to grow and the reduced order model generally provides progressively higher fidelity approximations to the full order system in the vicinity of the sampled damping configurations. However, this added fidelity might not provide improved information about the optimum damping configuration. Note also that if too low an order is used for the reduced model then it may have poor fidelity notwithstanding its local \mathcal{H}_2 -optimality and the reduced models could fail to recover even coarse \mathcal{H}_2 -norm information for the full system resulting (potentially) in a false optimum. As a practical matter, we have found that quite small reduced system orders will still produce reasonable \mathcal{H}_2 -norm estimates and we have not observed such failures. It is significant that we are able to avoid entirely the evaluation of the \mathcal{H}_2 norm of the full system, which is a forbidding computational challenge for large scale systems.

3 Some Implementation Details

Our approach relies on the effectiveness of sym2IRKA (Algorithm 1), in generating structured reduced order models with good fidelity. The efficiency of this process is improved if we introduce *modal coordinates*: Since M and K are symmetric positive definite matrices there exists a nonsingular matrix Φ (the *modal matrix*) which simultaneously diagonalizes both M and K :

$$\Phi^T K \Phi = \Omega^2 = \text{diag}(\omega_1^2, \dots, \omega_n^2) \quad \text{and} \quad \Phi^T M \Phi = I, \quad (19)$$

where $0 < \omega_1 \leq \omega_2 \leq \dots \leq \omega_n$ are the undamped natural frequencies of the system while the columns of the matrix Φ are eigenvectors (*modes*) of the undamped system. We have adopted the usual assumption on internal damping, namely that internal damping is *modal damping*, meaning that Φ diagonalizes the internal damping matrix, C_{int} , as well. We have taken internal damping to be modeled as a small multiple of critical damping, so $\Phi^T C_{int} \Phi = 2\alpha_c \Omega$ with small α_c . Evidently, this approach can be adapted immediately to other models of internal damping as well, e.g., Rayleigh damping or more general proportional damping (see, e.g. [36]).

Now, rewrite the system (1) in modal coordinates. By using (19) and substituting $q(t) = \Phi \hat{q}(t)$, we obtain:

$$\ddot{\hat{q}}(t) + (\alpha\Omega + \Phi^T B G B^T \Phi) \dot{\hat{q}}(t) + \Omega^2 \hat{q}(t) = \Phi^T E w(t), \quad (20)$$

$$z(t) = H \Phi \hat{q}(t). \quad (21)$$

We are focused in optimizing parameters in matrix $G = \text{diag}(g_1, g_2, \dots, g_p)$ where number of dampers is usually much smaller than the number of states. This means that within the damping matrix during the optimization process we have small rank update which one should use in order to use the structure of system matrices efficiently. In particular, in steps 1 and 6 of Algorithm 1, one drawback from the computational point of view, is that we need to solve many linear systems especially within the main loop of the algorithm. Here, since we are in modal coordinates we use the following approach.

In applying sym2IRKA, it is necessary to solve repeatedly systems of linear equations having the form $(\sigma^2 M + \sigma C + K)^{-1} B b$, for varying shifts σ and directions b . In modal coordinates,

$$\begin{aligned} (\sigma^2 M + \sigma C + K)^{-1} B b &= \Phi (\sigma^2 I + \sigma \alpha_c \Omega + \sigma \Phi^T B G B^T \Phi + \Omega^2)^{-1} \Phi^T B b \\ &= \Phi \mathbb{D}(\sigma)^{-1} \Phi^T B b - \sigma \Phi \mathbb{D}(\sigma)^{-1} \Phi^T B \mathbb{M}(\sigma, g) B^T \Phi \mathbb{D}(\sigma)^{-1} \Phi^T B b \end{aligned}$$

where

$$\begin{aligned} \mathbb{D}(\sigma) &= \sigma^2 I + \sigma \alpha_c \Omega + \Omega^2, \quad \mathbb{G}(g) = \text{diag}(\sqrt{g_1}, \dots, \sqrt{g_p}), \\ \mathbb{M}(\sigma, g) &= \mathbb{G}(g) (I_p + \sigma \mathbb{G}(g) B^T \mathbb{D}(\sigma)^{-1} B \mathbb{G}(g))^{-1} \mathbb{G}(g), \end{aligned}$$

and we have used the Sherman-Morrison-Woodbury formula [30] in passing from the first to the second line.

Note that $\mathbb{D}(\sigma)$ is an $n \times n$ diagonal matrix and $\mathbb{M}(\sigma, g)$ is a $p \times p$ matrix. Recalling that $p \ll n$ is the number of dampers, the evaluation of products of the form $\mathbb{M}(\sigma, g)z$ or even outright evaluation of the full matrix $\mathbb{M}(\sigma, g)$ can be implemented very efficiently.

The choice of initial gain in Algorithm 3 follows [16], where it was found effective to take initial gains uniformly equal to zero ($g^{(0)} = 0$), so that the optimization process starts with a modally damped system having each mode with a presumed fixed initial fraction, α_c , of critical damping. In [16], modal truncation was used to form reduced order models to be used as surrogates in the damping optimization process (see also, [19, 29, 49]); here we improve on the standard modal approximation by using IRKA approach on system without external damping. It is shown that one can take undamped eigenvectors that corresponds to the smallest r undamped eigenfrequencies, in order to obtain modal approximation of the system. This improves robustness of optimization process and it can also be calculated off-line since it does not include information on external damping. Here we will improve standard modal approximation (see, e.g. [16, 19, 29, 49]) by using sym2IRKA (Algorithm 1) without external damping ($g=0$).

The application of Krylov-based model reduction of second-order systems with proportional damping was studied in [10], but since in this particular case (with $g = 0$) in Step 1 and Step 6 of sym2IRKA, we need only solve diagonal systems and running sym2IRKA can be done off-line without significant cost. This first approximation determines an initial shift selection $\sigma_1 \dots, \sigma_r$. The corresponding tangent directions b_1, \dots, b_r are calculated as right singular vectors of reduced system transfer function evaluations on the corresponding initial shifts. In this way, the first cycle of either Algorithm 2 or 3 involving the development of a reduced order model for the case $g = 0$ can be counted as part of the off-line phase. We have observed in the numerical examples described below, that this initial model reduction approach provides ultimately a better approximation of optimal gains than if we were to use modal approximation for the initial surrogate model.

Note that in Step 3 of Algorithm 3, we recycle information obtained from earlier sym2IRKA runs: optimal shifts and directions obtained from sym2IRKA can be used as starting shifts and directions for the next run of sym2IRKA. This further improves the convergence behaviour of sym2IRKA, especially when consecutive gain configurations are close to one another.

4 Numerical experiments

We illustrate here the advantages that accrue with the use of sym2IRKA, which is our variant of IRKA introduced as Algorithm 1. We present results obtained using different sampling strategies for intermediate damping configurations and different approaches taken for the internal reduction step (Step 4) in sym2IRKA. These results will be compared with the approach presented in [16], which describes another approach to damping optimization.

Example 4.1. We consider a mechanical system of n masses connected in sequence by simple springs (see [45]), so that each mass is connected via a spring to two neighboring masses. The

We consider different damping configurations by varying the indices k and j . In particular, we consider the following possibilities:

$$\{(j, k) \mid j \in \{50, 150, 250, 350\} \text{ and } k \in \{850, 950, 1050, \dots, 1850\}\}.$$

This gives 44 different damping configurations over which we optimize.

During the optimization procedure, we use the following parameters in Algorithms 2 and 3:

$$r = 60, \quad \text{tolDiff} = 0.001, \quad \text{itMax} = 40.$$

A convergence tolerance of 0.001 was used for termination in step 6 of sym2IRKA.

Viscosities were optimized by MATLAB's built-in `fminsearch` using a termination tolerance of 10^{-4} , ending the optimization if either function values or variable values change by less than 10^{-4} . The starting point for the optimization procedure was $g^{(0)} = (1000, 1000)$.

We perform several comparisons with this example. The first comparison is between our approach using sym2IRKA and the dominant pole approach using [16, Algorithm 1 and 2].

For clearer comparisons, we use preset gain configurations with four parameter samples: $g^{(1)} = (0, 0)$ (internal damping only), $g^{(2)} = (1000, 1000)$, $g^{(3)} = (100, 1000)$ and $g^{(4)} = (1000, 100)$. When we used adaptive sampling, then the starting initial gain was taken as $g^{(1)} = (0, 0)$ allowing only for internal damping.

We note that the magnitude of the first and second gains varies between 500 and 4000, thus for the sake of better comparisons with the preset gain configurations, we have chosen values in this range. Usually such information is not known in advance and an adaptive sampling strategy could be advantageous.

As one can see in the upper plot of Figure 1, approaches using sym2IRKA have smaller relative errors in optimal gain. Relative error of the optimal gain was calculated by $\|g^* - g\|/\|g\|$, where g and g^* denote the optimal gain calculated with and without dimension reduction, respectively. The influence of our \mathcal{H}_2 -based reduction approach is emphasized even further in the lower plot of Figure 1 where we depict the relative errors in the \mathcal{H}_2 cost function. sym2IRKA consistently yields smaller error.

Next, we consider different damping configuration sampling strategies coupled with different internal reduction strategies ((a), (b) and (c) as described above) used in Step 4 of sym2IRKA. For Algorithm 2, a predetermined damping configuration sampling was used. For Algorithm 3 the first initial gain was set to an approximation of optimal gains obtained by using sym2IRKA applied to a system that has only internal damping. In Figure 2 we show relative errors for Algorithm 2 and 3 with internal reduction strategies based on balanced truncation, sym2IRKA, and the dominant pole algorithm related to strategies (a), (b), and (c) described above.

Besides accuracy, an important comparison criterion is the speed-up of the underlying optimization algorithm. Table 1 shows the average speed-up for the optimization process obtained by sym2IRKA for different strategies of internal reduction. We show here the average ratio between the time required for the gain optimization with and without the approximation technique

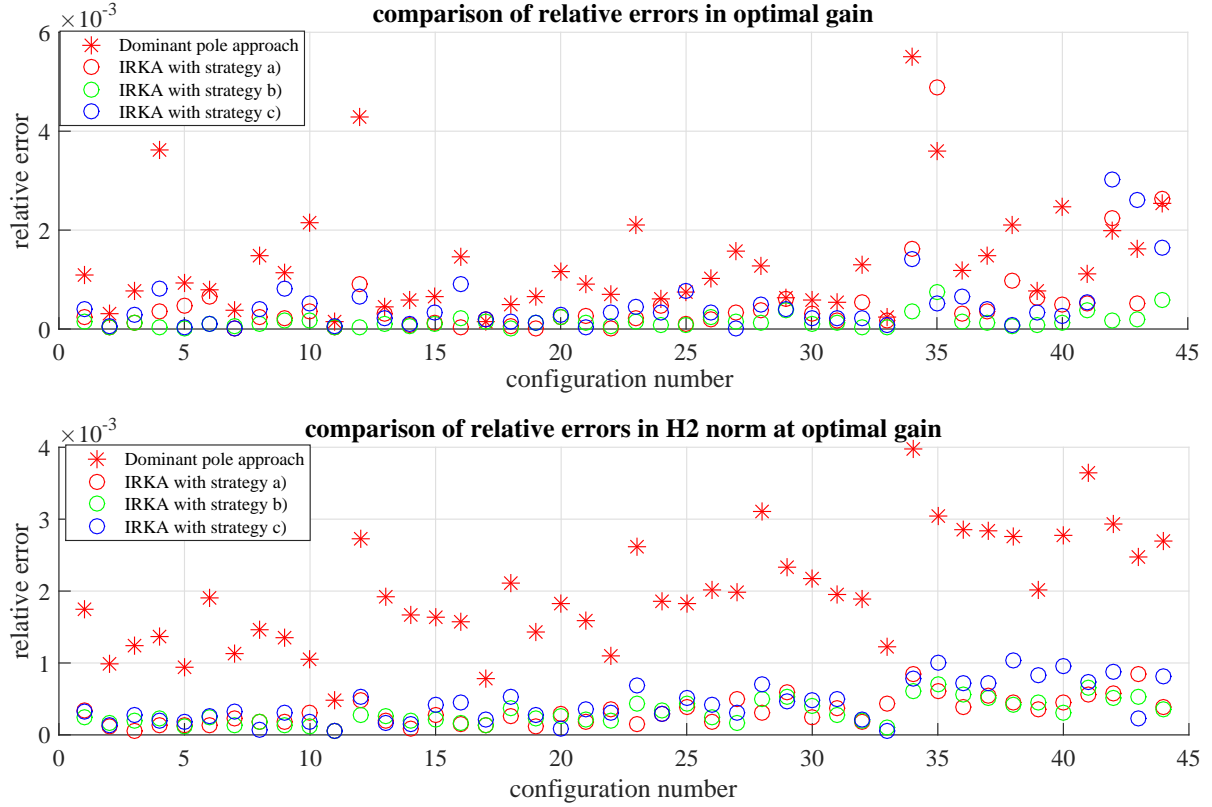


Figure 1: Relative errors for the gain and the objective function.

using sym2IRKA. Overall, Algorithm 3 yields bigger speed-ups. For example, Algorithm 3 with Strategy (c) has yielded an average speed up of 346.20. By way of contrast, note that the dominant pole algorithm based on [16, Algorithm 1 and 2] accelerated the optimization process only by a factor of 43.99 in average cases.

We conclude that we have obtained satisfactory relative errors in these trials and the approach using sym2IRKA significantly accelerated the optimization process. Moreover, we see that an approach that includes adaptive sampling additionally improves the efficiency of the sym2IRKA approach, especially if feasible intervals of optimal gains are not known in advance.

For the predetermined sampling strategy, we have obtained similar reduced dimension for approaches that use sym2IRKA, however an approach that uses the dominant pole approach had smaller reduced dimension. In particular, the average reduced dimension for predetermined sampling with strategy (a), strategy (b), and strategy (c) was 150, 178 and 141, respectively, while for the approach based on dominant poles, we obtained an average reduced dimension of 98. This means that an approach that uses sym2IRKA provides a modeling subspace that spans a “richer” subspace, if we compare with the subspace that is obtained by using the dominant pole approach. The reduced dimension has a direct impact not only on the time ratio, but also on the

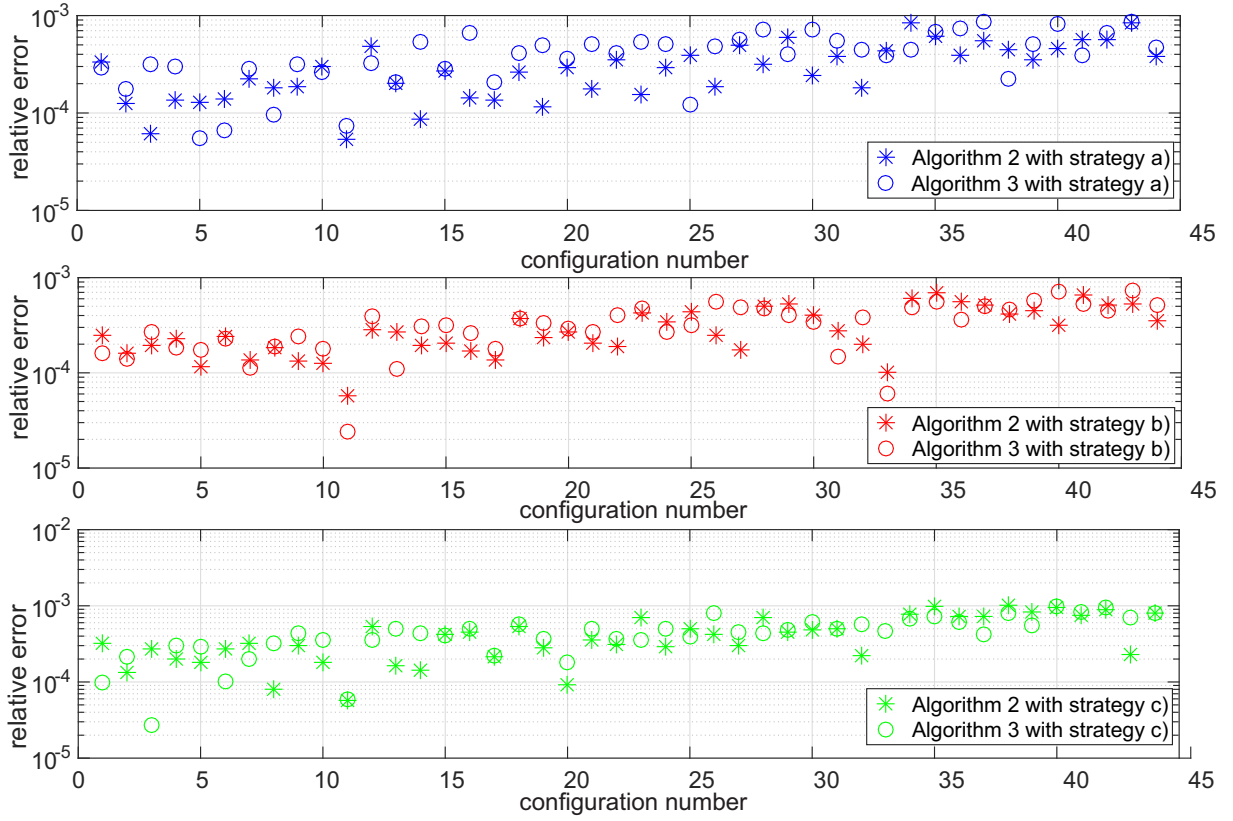


Figure 2: Example 1, relative errors for \mathcal{H}_2 norm at optimal gain for Algorithm 2 and Algorithm 3 with strategies a), b) and c)

rate of convergence of sym2IRKA.

Example 4.2. We now consider a different structure that allows for more variables within the optimization process. Consider a mass oscillator with $2d + 1$ masses and $2d + 3$ springs (see [16, 19]). There are two rows of d masses connected with springs; the first row of masses have stiffnesses k_1 and the second row have stiffnesses k_2 . The first masses on the left edge (m_1 and m_{d+1}) are connected to a fixed boundary while on the other side of rows the masses (m_d and m_{2d}) are connected to mass m_{2d+1} with a stiffness k_3 connected to a fixed boundary.

As in the previous example, the mathematical model is given by (1), with diagonal mass

Acceleration factor for:	Algorithm 2	Algorithn 3
Strategy (a)	187.37	288.62
Strategy (b)	146.12	338.29
Strategy (c)	228.63	346.20

Table 1: Time ratios for Example 1, i.e., acceleration factor of each approach using model reduction accelerated optimization

matrix $M = \text{diag}(m_1, m_2, \dots, m_n)$, but now the mass and stiffness matrices are given by

$$K = \begin{bmatrix} K_{11} & & -\kappa_1 \\ & K_{22} & -\kappa_2 \\ -\kappa_1^T & -\kappa_2^T & k_1 + k_2 + k_3 \end{bmatrix}, \quad \text{with} \quad K_{ii} = k_i \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad \kappa_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k_i \end{bmatrix},$$

for $i = 1, 2$. We will consider the n -mass oscillator with $d = 1000$, which means that we consider 2001 masses with the following configuration for the masses:

$$m_{2001} = 100, \quad m_i = \begin{cases} 100 - \frac{i}{10}, & i = 1, \dots, 499, \\ \frac{i}{30} + 33, & i = 501, \dots, 1000, \\ 100 - (i - 999)\frac{5}{20} + \frac{(i-999)^2}{5000}, & i = 1001, \dots, 2000, \end{cases}$$

The stiffness values are given by

$$k_1 = 400, k_2 = 100, k_3 = 300.$$

The parameter α_c associated with internal damping (2) is set to 0.003 (i.e., 0.3% critical damping).

The primary excitation are 21 disturbances applied to 21 masses that are closest to ground. The 10 masses closest to the left-hand side are such that masses closest to the ground are exposed to higher magnitude disturbances. Additionally, a disturbance is applied to the mass on the right-hand side. That is $E \in \mathbb{R}^{n \times 21}$ with

$$\begin{aligned} E(1 : 10, 1 : 10) &= \text{diag}(1000 : -100 : 100), \\ E(1001 : 1010, 11 : 20) &= \text{diag}(1000 : -100 : 100), \\ E(2001, 21) &= 2000; \end{aligned}$$

all other entries are equal to zero.

Regarding the output, we are interested in the 21 displacements of the first row of masses and 21 displacements of the second row of masses. In particular, we have that

$$z(t) = [q_{490}(t) \ q_{491}(t) \ \dots \ q_{510}(t) \ q_{1490}(t) \ q_{1491}(t) \ \dots \ q_{1510}(t)]^T,$$

so that $H \in \mathbb{R}^{42 \times 2001}$.

In this example we illustrate how our approach based on sym2IRKA performs within a setting having more optimization variables. We still consider four dampers ($p = 4$) but now with different viscosities. The geometry of external damping is determined by a matrix B given by

$$B = [e_j - e_{j+5} \quad e_{j+20} - e_{j+25} \quad e_k - e_{k+5} \quad e_{k+20} - e_{k+25}], \quad (24)$$

where e_j is the j th canonical vector and indices j and k determine damping positions. Since the dampers now have (potentially) all different viscosities, we have that

$$G = \text{diag}(g_1, g_2, g_3, g_4) \in \mathbb{R}^{4 \times 4}.$$

The positions are chosen such that the first two dampers are applied on the first row of masses, while the third and the fourth damper are applied on the second row of masses. Similarly as in the previous example, in order to consider different damping configurations, we vary the indices k and j . The following configurations are considered:

$$\{(j, k) \mid j \in \{250, 450, 650, 850\} \text{ and } k \in \{1150, 1250, 1350, 1450, 1550, 1650, 1750\}\}.$$

This gives 28 different damping configurations over which we optimize.

During optimization, we use the following parameters for Algorithms 2 and 3:

$$r = 120, \quad \text{tolDiff} = 0.05, \quad \text{itMax} = 40.$$

A tolerance of 0.001 was used for checking convergence in Step 6 of sym2IRKA. The starting point for optimization (as performed by `fminsearch`) was set to (1000, 1000, 1000, 1000). The termination tolerance for `fminsearch` was set to $5 \cdot 10^{-4}$.

We observed that the magnitudes of all optimal gains vary from 350 to 7000, thus, as we have done in the previous example, for the predetermined gain sampling we choose values that belong to this range. In particular, five sampling parameters were used as predetermined samples of damping gain configurations: $g^{(1)} = (0, 0, 0, 0)$ – representing a system that has only internal damping, $g^{(2)} = (1000, 1000, 1000, 1000)$, $g^{(3)} = (1000, 1000, 4000, 4000)$, $g^{(4)} = (4000, 4000, 1000, 1000)$, and $g^{(5)} = (4000, 500, 4000, 500)$.

For the case of adaptive sampling, the initial gain was (0, 0, 0, 0), as before, but since we have more variables and in order to improve robustness, in the generation of the initial subspace we have added a starting point informed by the optimization procedure (that is, at an initial gain of (1000, 1000, 1000, 1000)).

Note that in these examples the starting point for optimization also included predetermined sampling of damping gain configurations.

Similarly as in the first example, in Figure 3 we show relative errors for the \mathcal{H}_2 norm at optimal gain. This figure shows results obtained by Algorithm 2 and Algorithm 3 with strategies of internal reduction based strategies from cases (a), (b) and (c). We can see that within the

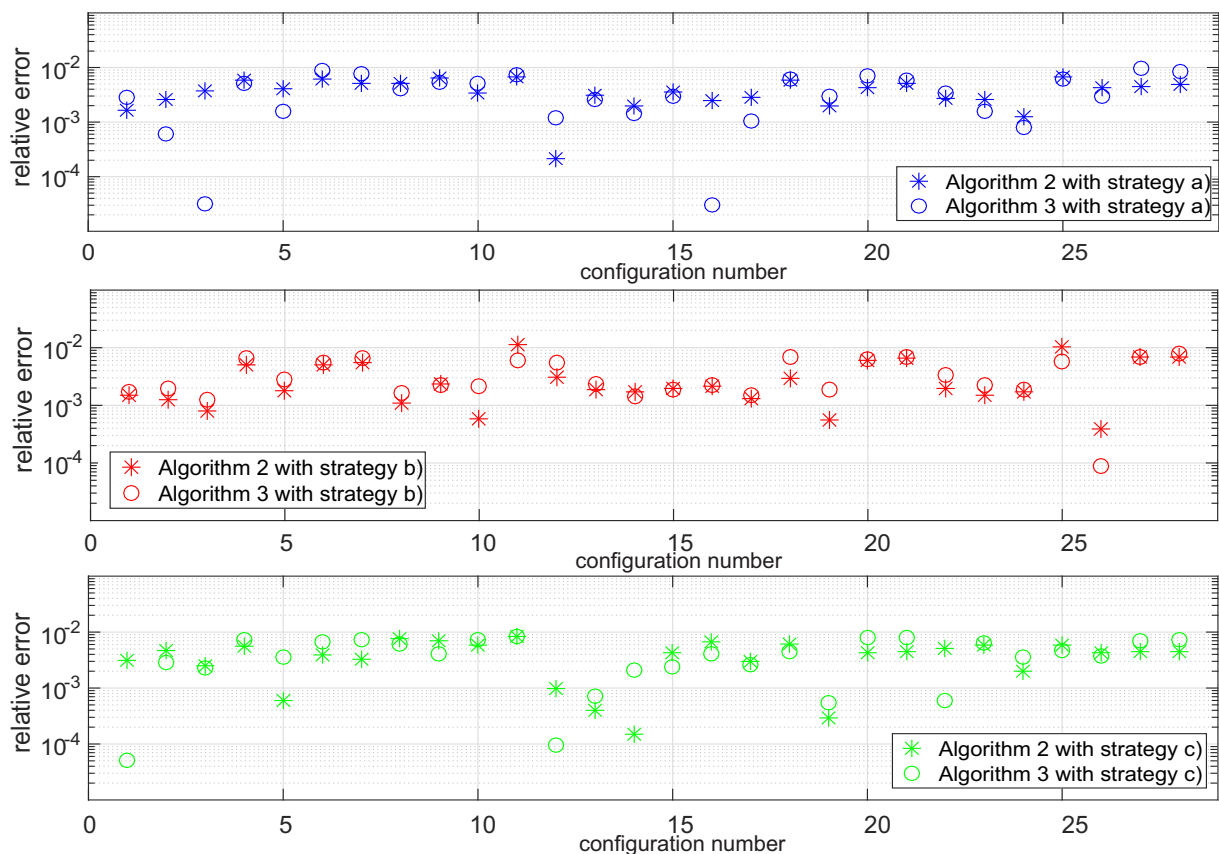


Figure 3: Example 2, relative errors for \mathcal{H}_2 norm at optimal gain for Algorithm 2 and Algorithm 3 with strategies a), b) and c)

given tolerances the proposed methodology for damping optimization once again yields satisfactory approximation together with significant acceleration of optimization. As in the previous example, our approach based on sym2IRKA returned approximation of optimal gains with better relative errors compared to the one obtained by dominant pole approach based on [16, Algorithm 1 and 2] in this example as well.

Similar to the first example, in Table 2 we illustrate average time speed-up for the optimization process obtained using sym2IRKA for different strategies of internal reduction. In this example dominant pole algorithm based on [16, Algorithm 1 and 2] accelerated optimization process 83.87 times. This is also significant acceleration factor, but new approach based on sym2IRKA was even more efficient, yielding an accelerating factor as high as 208. From this table we can see that adaptive strategy is more efficient for strategies a) and b), but provides slightly smaller time ratio for strategy b). Here adaptive sampling needed more updates of initial gains which have had impact in the final time ratio, but this strategy do not need to have additional information on the area where are the optimal gains.

Acceleration factor for	Algorithm 2	Algoritm 3
Strategy (a)	124.72	157.98
Strategy (b)	133.53	126.16
Strategy (c)	171.93	208.80

Table 2: Time ratios for Example 2, i.e., acceleration factor with model reduction accelerated optimization

Although the time speed ups depend on the tuning tolerances, for both examples we can conclude, based on performed numerical experiments, that the approach based on sym2IRKA is significantly faster than the dominant pole approach while also providing better relative errors. For the predetermined sampling strategy, we have used gain samples that are somehow in the vicinity of the optimal gains and this is, in general, hard to know. Thus, in practice, the adaptive sampling strategy is expected to be much more efficient. The adaptive sampling strategy yields similar relative errors as the predetermined sampling strategy and usually provides better or at least similar acceleration speed up.

Acknowledgements

The work of Zoran Tomljanović was supported in part by the Croatian Science Foundation under the project Optimization of parameter dependent mechanical systems, Grant No. IP-2014-09-9540 and project Control of Dynamical Systems, Grant No. IP-2016-06-2468. The work of Christopher Beattie was supported in part by the Einstein Foundation Berlin. The work of Serkan Gugercin was supported in part by the Alexander von Humboldt Foundation and by the NSF through Grant DMS-1522616.

References

- [1] N.M. Alexandrov, J.E. Dennis, R.M. Lewis, and V. Torczon, *A trust-region framework for managing the use of approximation models in optimization*, Structural and Multidisciplinary Optimization **15** (1998), no. 1, 16–23.
- [2] A. C. Antoulas, D. C. Sorensen, and S. Gugercin, *A survey of model reduction methods for large-scale systems*, Contemporary Mathematics **280** (2001), 193–219.
- [3] A.C. Antoulas, *Approximation of large-scale dynamical systems*, SIAM Publications, Philadelphia, PA, 2005.

- [4] A.C. Antoulas, C.A. Beattie, and S. Gugercin, *Interpolatory model reduction of large-scale dynamical systems*, Efficient Modeling and Control of Large-Scale Systems, Springer, 2010, pp. 3–58.
- [5] Z. Bai, J. Demmel, J. Dongarra, A. Ruhe, and H. van der Vorst, *Templates for the solution of algebraic eigenvalue problems: A practical guide*, SIAM, Philadelphia, 2000.
- [6] Z. Bai and Y. Su, *Dimension reduction of second order dynamical systems via a second-order Arnoldi method*, SIAM Journal on Scientific Computing **5** (2005), 1692–1709.
- [7] U. Baur, C. A. Beattie, P. Benner, and S. Gugercin, *Interpolatory projection methods for parameterized model reduction*, SIAM Journal on Scientific Computing **33** (2011), no. 5, 2489–2518.
- [8] U. Baur, P. Benner, and L. Feng, *Model order reduction for linear and nonlinear systems: A system-theoretic perspective*, Arch. Comput. Methods Eng. **21** (2014), no. 4, 331–358.
- [9] C. Beattie and S. Gugercin, *Interpolatory projection methods for structure-preserving model reduction*, Systems and Control Letters **58** (2009), 225–232.
- [10] C.A. Beattie and S. Gugercin, *Krylov-based model reduction of second-order systems with proportional damping*, Proceedings of the 44th IEEE Conference on Decision and Control, December 2005, pp. 2278–2283.
- [11] _____, *Model reduction by rational interpolation*, To appear in Model Reduction and Approximation: Theory and Algorithms. Available as <http://arxiv.org/abs/1409.2140> (P. Benner, A. Cohen, M. Ohlberger, and K. Willcox, eds.), SIAM, Philadelphia, PA, USA, 2017.
- [12] P. Benner and T. Breiten, *Interpolation-based \mathcal{H}_2 -model reduction of bilinear control systems*, SIAM Journal on Matrix Analysis and Applications **33** (2012), no. 3, 859–885.
- [13] P. Benner, T. Breiten, and T. Damm, *Generalized tangential interpolation for model reduction of discrete-time mimo bilinear systems*, International Journal of Control **84** (2011), no. 8, 1398–1407.
- [14] P. Benner, A. Cohen, M. Ohlberger, and K. Willcox, *Model reduction and approximation: Theory and algorithms*, SIAM, 2005.
- [15] P. Benner, S. Gugercin, and K. Willcox, *A survey of projection-based model reduction methods for parametric dynamical systems.*, SIAM Review **57** (2015), no. 4, 483–531.
- [16] P. Benner, P. Kürschner, Z. Tomljanović, and N. Truhar, *Semi-active damping optimization of vibrational systems using the parametric dominant pole algorithm*, Journal of Applied Mathematics and Mechanics (2015), 1–16, DOI:10.1002/zamm201400158.

- [17] P. Benner, V. Mehrmann, and D.C. Sorensen, *Dimension reduction of large-scale systems*, Lecture Notes in Computational Science and Engineering, Springer-Verlag, Berlin/Heidelberg, Germany, 2005.
- [18] P. Benner, Z. Tomljanović, and N. Truhar, *Dimension reduction for damping optimization in linear vibrating systems*, *Z. Angew. Math. Mech.* **91** (2011), no. 3, 179 – 191, DOI: 10.1002/zamm.201000077.
- [19] ———, *Optimal Damping of Selected Eigenfrequencies Using Dimension Reduction*, *Numer. Linear Algebr.* **20** (2013), no. 1, 1–17, DOI: 10.1002/nla.833.
- [20] F. Blanchini, D. Casagrande, P. Gardonio, and S. Miani, *Constant and switching gains in semi-active damping of vibrating structures*, *Int. J. Control* **85** (2012), no. 12, 1886–1897.
- [21] T. Bonin, H. Faßbender, A. Soppa, and M. Zaeh, *A fully adaptive rational global arnoldi method for the model-order reduction of second-order mimo systems with proportional damping*, *Mathematics and Computers in Simulation* **122** (2016), 1–19.
- [22] T. Breiten, *Structure-preserving model reduction for integro-differential equations*, *SIAM Journal on Control and Optimization* **54** (2016), no. 6, 2992–3015.
- [23] A. Bunse-Gerstner, D. Kubalinska, G. Vossen, and D. Wilczek, *\mathcal{H}_2 -norm optimal model reduction for large scale discrete dynamical {MIMO} systems*, *Journal of Computational and Applied Mathematics* **233** (2010), no. 5, 1202 – 1216, Special Issue Dedicated to William B. Gragg on the Occasion of His 70th Birthday.
- [24] J. B. Burl, *Linear optimal control: \mathcal{H}_2 and \mathcal{H}_∞ methods*, 1st ed., Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA, 1998.
- [25] J. Fehr, M. Fischer, B. Haasdonk, and P. Eberhard, *Greedy-based approximation of frequency-weighted gramian matrices for model reduction in multibody dynamics*, *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik* **93** (2013), no. 8, 501–519.
- [26] G. Flagg and S. Gugercin, *Multipoint Volterra series interpolation and \mathcal{H}_2 optimal model reduction of bilinear systems*, *SIAM Journal on Matrix Analysis and Applications* **36** (2015), no. 2, 549–579.
- [27] P. Freitas and P. Lancaster, *On the optimal value of the spectral abscissa for a system of linear oscillators*, *SIAM Journal on Matrix Analysis and Applications* **21** (1999), no. 1, 195–208.
- [28] K. Gallivan, A. Vandendorpe, and P. Van Dooren, *Model reduction of MIMO systems via tangential interpolation*, *SIAM Journal on Matrix Analysis and Applications* **26** (2005), no. 2, 328–349.

- [29] W.K. Gawronski, *Advanced structural dynamics and active control of structures*, Springer, New York, USA, 2004.
- [30] G.H. Golub and Ch.F. van Loan, *Matrix computations*, 3rd ed., J. Hopkins University Press, Baltimore, 1996.
- [31] S. Gratton and L. N. Vicente, *A surrogate management framework using rigorous trust-region steps*, Optimization Methods Software **29** (2014), no. 1, 10–23.
- [32] S. Gugercin, A. C. Antoulas, and C. Beattie, *\mathcal{H}_2 model reduction for large-scale linear dynamical systems*, SIAM Journal on Matrix Analysis and Applications **30** (2008), no. 2, 609–638.
- [33] J. S Hesthaven, G. Rozza, and B. Stamm, *Certified reduced basis methods for parametrized partial differential equations*, Springer, 2016.
- [34] D. J. Inman and A. N. Jr. Andry, *Some results on the nature of eigenvalues of discrete damped linear systems*, ASME J. Appl. Mech. **47** (1980), 927–930.
- [35] Y. Kanno, *Damper placement optimization in a shear building model with discrete design variables: a mixed-integer second-order cone programming approach*, Earthquake Engineering and Structural Dynamics **42** (2013), 1657–1676.
- [36] I. Kuzmanović, Z. Tomljanović, and N. Truhar, *Optimization of material with modal damping*, Applied Mathematics and Computation **218** (2012), 7326–7338.
- [37] L. Meier III and D.G. Luenberger, *Approximation of linear constant systems*, Automatic Control, IEEE Transactions on **12** (1967), no. 5, 585–588.
- [38] D.G. Meyer and S. Srinivasan, *Balancing and model reduction for second-order form linear systems*, IEEE Transactions on Automatic Control **41** (1996), no. 11, 1632–1644.
- [39] P.C. Müller and W.O. Schiehlen, *Linear vibrations*, Martinus Nijhoff Publishers, 1985.
- [40] B. Nour-Omid and M. E. Regelbrugge, *Lanczos method for dynamic analysis of damped structural systems*, Earthquake Engrg. and Structural Dynamic **18** (1989), 1091–1104.
- [41] T. Reis and T. Stykel, *Balanced truncation model reduction of second-order systems*, Mathematical and Computer Modelling of Dynamical Systems **14** (2008), no. 5, 391–406.
- [42] M. Saadvandi, K. Meerbergen, and W. Desmet, *Parametric dominant pole algorithm for parametric model order reduction*, Tech. report, KU Leuven, Department of Computer Science, March 2013.
- [43] T.-J. Su and R.R. Craig Jr, *Model reduction and control of flexible structures using krylov vectors*, Journal of guidance, control, and dynamics **14** (1991), no. 2, 260–267.

- [44] I. Takewaki, *Optimal damper placement for minimum transfer functions*, Earthquake Engineering and Structural Dynamics **26** (1997), 1113–1124.
- [45] N. Truhar, Z. Tomljanović, and M. Puvača, *An efficient approximation for optimal damping in mechanical systems*, International journal of numerical analysis and modeling **14** (2017), no. 2, 201–217.
- [46] N. Truhar, Z. Tomljanović, and K. Veselić, *Damping optimization in mechanical systems with external force*, Applied Mathematics and Computation **250** (2015), 270–279.
- [47] N. Truhar and K. Veselić, *An efficient method for estimating the optimal dampers' viscosity for linear vibrating systems using Lyapunov equation*, SIAM J. Matrix Anal. Appl. **31** (2009), no. 1, 18–39.
- [48] P. Van Dooren, KA Gallivan, and P.A. Absil, *\mathcal{H}_2 -optimal model reduction of MIMO systems*, Applied Mathematics Letters **21** (2008), no. 12, 1267–1273.
- [49] K. Veselić, *Damped Oscillations of Linear Systems*, Springer Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2011.
- [50] S. Wyatt, *Issues in interpolatory model reduction: Inexact solves, second-order systems and daes*, Dissertation, Virginia Polytechnic Institute and State University, Blacksburg, 2012.
- [51] K. Zhou, J.C. Doyle, and K. Glover, *Robust and optimal control*, Upper Saddle River, New Jersey: Prentice Hall, 1996.