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## Modified Dark Matter: Relating Dark Energy, Dark Matter and Baryonic Matter

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Modified dark matter (MDM) is a phenomenological model of dark matter, inspired by gravitational thermodynamics. For an accelerating Universe with positive cosmological constant ( $\Lambda$ ), such phenomenological considerations lead to the emergence of a critical acceleration parameter related to  $\Lambda$ . Such a critical acceleration is an effective phenomenological manifestation of MDM, and it is found in correlations between dark matter and baryonic matter in galaxy rotation curves. The resulting MDM mass profiles, which are sensitive to  $\Lambda$ , are consistent with observational data at both the galactic and cluster scales. In particular, the same critical acceleration appears both in the galactic and cluster data fits based on MDM. Furthermore, using some robust qualitative arguments, MDM appears to work well on cosmological scales, even though quantitative studies are still lacking. Finally, we comment on certain non-local aspects of the quanta of modified dark matter, which may lead to novel non-particle phenomenology and which may explain why, so far, dark matter detection experiments have failed to detect dark matter particles.

*Keywords:* Dark Matter; Dark Energy; Baryonic Matter.

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### 1. Introduction and Overview

The ‘missing mass’ problem is one of the fundamental puzzles in contemporary physics and astronomy.<sup>1</sup> Since the pioneering work of Oort,<sup>2</sup> Zwicky,<sup>3</sup> and Rubin and Ford,<sup>4–6</sup> observational evidence for substantial mass discrepancies between dynamical studies and observations of visible (baryonic) matter have become overwhelming on all scales spanning from the galactic to the cosmological.

From the perspective of Einstein’s equations

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}, \quad (1)$$

where  $G_{ab}$  is the Einstein tensor,  $T_{ab}$  is the energy-momentum tensor, and  $\Lambda$  is the cosmological constant, the problem amounts to a mismatch between the left-

and right-hand sides. The curvature of spacetime (on the left-hand side) which determines the dynamics is larger than what is expected from contributions of baryonic matter to the energy-momentum tensor on the right-hand side. Note that we will be treating the dark energy as the cosmological constant or vacuum energy<sup>7</sup> in this review, which meshes well with our view of the modified dark matter, that aims to relate the dark matter, dark energy and baryonic matter sectors of Einstein's equations.

This mismatch can be alleviated by the modification of, or addition of extra terms to, either the left- or the right-hand sides of the equation. The problem with modifications to the geometric/gravitational (left-hand) side<sup>8-11</sup> is that they are difficult to motivate from general physical principles as elegantly as the original formulation of general relativity (GR) by Einstein. Moreover, such modifications are necessarily classical, and thus their quantum nature is obscure and almost certainly more opaque than the quantum nature of Einstein's gravity, which itself remains controversial.

Attention has thus focused mainly on modifications to the source (right-hand) side of Eq. (1), the simplest of which is to add contributions to the energy-momentum tensor from heretofore unknown and unobserved degrees of freedom, *i.e.* Dark Matter (DM). An obvious candidate for DM is baryons that are not easily detectable by observations of photons, either in absorption or emission. This led to Massive Compact Halo Object (MACHO) models, in which the DM consists of brown dwarfs, neutron stars, black holes and/or other collapsed objects. However, exhaustive searches for microlensing events that would signify the presence of such objects in our Milky Way's halo turned up far too few events to make MACHOs a significant source of DM.<sup>12-16</sup>

Instead, observational evidence favors that DM is largely non-baryonic in nature. In particular, comparisons of the observed deuterium to hydrogen ratio to that expected from Big Bang nucleosynthesis shows that the bulk of matter cannot be baryonic.<sup>17</sup> Observations of the power spectrum of anisotropies in the cosmic microwave background (CMB) are consistent with at least most DM being non-relativistic (*i.e.* cold) and diffuse.<sup>18-22</sup> The evolution of large-scale structures (LSS) of galaxies across the history of the Universe are also consistent with this idea.<sup>23-27</sup> Other evidence includes the observations of colliding galaxy clusters, which are straightforward to explain with non-baryonic cold DM<sup>28</sup> but convoluted to explain in competing models.<sup>29,30</sup>

This paradigm of a cold, diffuse, non-relativistic DM is known as the Cold Dark Matter (CDM) paradigm.<sup>31-34</sup> It is however notable that CDM is not particularly restrictive with models that fit this framework; all that is required is a Weakly Interacting Massive Particle (WIMP) in which extra and independent (from the baryonic matter) degrees of freedom are described by new weakly-interacting quantum fields,<sup>35</sup> but in which (at least from the astrophysical perspective) the mass and

interaction cross section of individual WIMPs are barely constrained<sup>a</sup>. Note that the standard cosmological model,  $\Lambda$ CDM, assumes that the dark energy sector is modeled with the cosmological constant (i.e. vacuum energy) and this will remain the case in our discussion of the MDM proposal.

The popularity of the WIMP models is in part due to the possibility of directly detecting the WIMPs using laboratory-based experiments. Starting in the 1980s, increasingly sensitive searches have been made for direct signatures of DM particles.<sup>39</sup> However, no direct detections have been made, and even the latest results of experiments searching for recoil events<sup>40</sup> only set limits on the mass of the assumed DM particle and its cross section with baryonic matter.

In the absence of any direct detection of DM, it is important to look for other constraints that we should place on the nature and properties of DM in order to narrow down the list of possibilities. For this, we note that hints may be discernible in the tensions between observations and CDM models. First, there is a set of ‘problems’ between  $N$ -body simulations of galaxy/cluster evolution and observations:

- **Missing Satellite Problem:**

Simulations predict a much larger number of satellite galaxies in CDM haloes than is typically observed.<sup>41–45</sup> Also, there are tensions between the dispersion in mass predicted by simulations, and the dispersion that is observed.<sup>46,47</sup> These discrepancies can be at least partly resolved by suppression of dwarf galaxy formation via the UVB (ultraviolet background) heating of the IGM (intergalactic medium) gas (e.g. Refs. 48, 49) or supernova feedback.<sup>50,51</sup> Moreover, recent simulations at high spatial resolution do reproduce the observed Milky Way satellite number.<sup>52,53</sup>

- **Core/Cusp Problem:**

Simulations predict that the central CDM distribution in galaxy clusters should be sharply peaked, but observations instead favor a much flatter central density profile.<sup>54–58</sup>

- **Too-Big-to-Fail Problem:**

Observations infer that luminous dwarf galaxies inhabit lower mass CDM halos than those predicted by simulations.<sup>59–63</sup>

- **Satellite Planes Problem:**

Observations find that satellite galaxies are distributed much more anisotropically than is predicted by simulations.<sup>64–67</sup>

Summaries of these observational tensions can be found in e.g. Refs. 68–75.

These problems, however, may not pose insurmountable challenges for CDM. For example, some studies suggest that all of these problems are due to the limitations of our current simulation capabilities, not to our limited understanding of the nature of CDM, and that they can be (at least partly) overcome by includ-

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<sup>a</sup>In this review we are not going to concentrate on alternative dark matter scenarios that include warm dark matter models,<sup>36</sup> axions,<sup>37</sup> and hidden sectors.<sup>38</sup>

ing a sufficiently complete suite of baryonic physical processes in simulations.<sup>76–80</sup> Moreover, modifications to cold dark matter which introduce relativistic degrees of freedom, such as warm dark matter (sterile neutrinos), or other popular relativistic degrees of freedom, such as axions, do solve some of these problems.<sup>81</sup>

The second, and in our view more serious, tension is the set of observations at the galactic scale which strongly suggests that DM may not simply be extra, independent degrees of freedom. These observations include:

(1) **Presence of a Universal Acceleration Scale:**

A recent work by McGaugh and collaborators claims a precise correlation between the mass profiles of dark and baryonic matter in disk galaxies spanning an extremely wide ranges in scale, mass, and age.<sup>82–84</sup> This correlation is expressed as a relation between the observed acceleration  $a_{\text{obs}}$  and the expected acceleration  $a_{\text{bar}}$  from baryonic matter only as

$$a_{\text{obs}} = \frac{a_{\text{bar}}}{1 - e^{-\sqrt{a_{\text{bar}}/\bar{a}_0}}}, \quad (2)$$

where  $\bar{a}_0$  is a universal constant which has been fit to  $\bar{a}_0 = (1.20 \pm 0.02) \times 10^{-10} \text{m/s}^2$ . (Here we have written  $\bar{a}_0$  rather than  $a_0 \approx cH_0$ , which will be introduced later. Note that  $\bar{a}_0 = a_0/2\pi$ , in parallel with  $\hbar = h/2\pi$  and  $\lambda = \lambda/2\pi$ .) Given that the Hubble parameter is  $H_0 = (67.74 \pm 0.46) \text{ km/s/Mpc}$  (see Table 4 of Ref. 85)<sup>b</sup> it has been noted that<sup>86</sup>

$$\bar{a}_0 \approx \frac{cH_0}{2\pi}, \quad (3)$$

which suggests that the constant  $a_0$  may be cosmological in origin.

Notice that the above correlation (Eq.2) is difficult to motivate within purely collision-less CDM, since some dispersion might be expected between the collision-less CDM particles, and the free baryons that act as fuel for the star formation. Nevertheless, it may be possible to reconcile this result with CDM models<sup>58,87</sup> by relying on dissipative baryonic dynamics. On the other hand, this correlation could be pointing to a yet unknown property of DM which connects its distribution to that of baryonic matter. Perhaps the more mysterious part of this relation is the appearance of the universal acceleration constant  $a_0$ . If this acceleration  $a_0$  is cosmological in origin and related to  $H_0$ , how can the DM mass profile be sensitive to it?

(2) **Baryonic Tully-Fisher Relation:**

The Tully-Fisher Relation<sup>88</sup> is a universal relation between the total observed baryonic mass (stars + gas) of a galaxy  $M_{\text{bar,total}}$  and the asymptote of the

<sup>b</sup>That is,  $H_0 = [(6.581 \pm 0.045) \times 10^{-8} \text{cm/s}^2] / c$ .

galactic rotation curve  $v_\infty$ :

$$M_{\text{bar,total}} = Av_\infty^4, \quad A = (47 \pm 6) M_\odot \text{s}^4/\text{km}^4. \quad (4)$$

This relation holds regardless of the value of  $M_{\text{bar,total}}$ , or how it is distributed. Even when the shapes of the rotation curves are different, the asymptote is always the same for galaxies with the same baryonic mass. This is remarkable when one recognizes that the rotation velocity  $v(r)$  at a distance  $r$  from the center of the galaxy is determined by the distribution of the sum of baryonic and dark matter, and not by baryonic matter alone. Nevertheless, the asymptotic velocity depends only on the total baryonic mass, again suggesting a correlation between the baryonic and DM mass distributions.

While it is not clear that current DM models naturally explain these relations, they are natural consequences of Milgrom's MODified Newtonian Dynamics (MOND).<sup>89–91</sup> (Note that one of the aims of our work is to provide a DM model that explains the baryonic Tully-Fisher relation and accounts for the acceleration scale  $a_0$ .)

In MOND, it is postulated that Newton's equation of motion  $F = ma$  should be modified to

$$F = \begin{cases} ma & (a \gg \bar{a}_0) \\ ma^2/\bar{a}_0 & (a \ll \bar{a}_0) \end{cases}. \quad (5)$$

More specifically,

$$F = ma \mu(a/\bar{a}_0), \quad (6)$$

where  $\mu(x) = 1$  for  $x \gg 1$  and  $\mu(x) = x$  for  $x \ll 1$ . The choice of interpolating functions  $\mu(x)$  is arbitrary. This implies

$$a_{\text{obs}} = \begin{cases} a_{\text{bar}} & (a_{\text{bar}} \gg \bar{a}_0) \\ \sqrt{\bar{a}_0 a_{\text{bar}}} & (a_{\text{bar}} \ll \bar{a}_0) \end{cases}, \quad (7)$$

*i.e.* the same relation implied by Eq. (2). Far away from the galactic center, we can expect the following baryonic acceleration

$$a_{\text{bar}}(r) = \frac{GM_{\text{bar,total}}}{r^2}, \quad (8)$$

and thus

$$v^2(r) = r a_{\text{obs}}(r) \xrightarrow{r \rightarrow \infty} r \sqrt{\bar{a}_0 a_{\text{bar}}(r)} = \sqrt{\bar{a}_0 GM_{\text{bar,total}}} \equiv v_\infty^2, \quad (9)$$

which gives us flat rotation curves<sup>c</sup> and

$$M_{\text{bar,total}} = \frac{v_\infty^4}{\bar{a}_0 G} = (63 M_\odot \text{s}^4/\text{km}^4) v_\infty^4, \quad (10)$$

<sup>c</sup>In reality, rotation curves are not all flat; they display a variety of properties. See, e.g. Ref. 92.

cf. Eq. (4). Other studies of MOND<sup>d</sup> in the context of rotation curves include Ref. 94,95.

We note, however, that MOND can also be interpreted as the introduction of a very specific type of DM. Consider a spherically symmetric distribution of baryonic matter where  $M_{\text{bar}}(r)$  is the total baryonic mass enclosed in a sphere of radius  $r$ . Then, the gravitational force on a test mass  $m$  placed at  $r$  due to this distribution will be given by

$$F(r) = \frac{GM_{\text{bar}}(r)m}{r^2}. \quad (11)$$

Eq. (6) in this case can be rewritten as:

$$a(r) = \frac{1}{\mu(a(r)/\bar{a}_0)} \frac{GM_{\text{bar}}(r)}{r^2} \equiv \frac{G[M_{\text{bar}}(r) + M_{\text{DM}}(r)]}{r^2}, \quad (12)$$

where we identify

$$M_{\text{DM}}(r) = \left[ \frac{1}{\mu(a(r)/\bar{a}_0)} - 1 \right] M_{\text{bar}}(r), \quad (13)$$

as the total DM mass within a radius of  $r$  from the center. Thus, to reproduce the success of MOND at galactic scales, we need a DM model which predicts such a mass distribution. (Note that such a dark matter model is *not* going to be an inversion of Milgrom's MOND, *i.e.* it is *not* going to be a "phantom" dark matter, because it will have to work on all scales: galactic, cluster and cosmological.) We still have to account for the impressive successes of the canonical  $\Lambda$ CDM model on cluster and galactic scales.

However, there are problems with MOND at the cluster and cosmological scales. <sup>e</sup> In general MOND fails to address the dynamics of galactic clusters (more later in Section 3.2) and other cosmological measurements, in particular, it cannot explain the third and higher CMB peaks, and the shape of matter power spectrum. Nevertheless, the fundamental acceleration parameter, also known as Milgrom's scaling, appears to be both real and a potentially important signpost towards a deeper understanding of the dark sector. CDM, for example, can only reproduce Milgrom's scaling through a somewhat convoluted argument of Kaplinghat and Turner.<sup>96</sup> The fundamental meaning (if there is any) of Milgrom's scaling is obscure in this argument.<sup>f</sup>

Thus the question regarding the relation between this fundamental acceleration parameter  $a_0$  and dark matter is still outstanding. The nature of this question motivated us to examine a new model for non-baryonic dark matter, which we

<sup>d</sup>There are also the relativistic versions AQUAL, RAQUEL and TeVeS; but they tend to be more limited in their predictive power. See Ref. 93 and references therein.

<sup>e</sup>The reason may be due to the lack of a fundamental relativistic quantum theory of MOND.

<sup>f</sup>We should note that MOND has been formulated in a relativistic context in Ref. 8 and even argued to be a consequence of quantum gravity in Ref. 97.

term modified (or “Mondian”) dark matter, or simply, MDM. The idea here is that by taking into account the existence of the fundamental acceleration as well as of the baryonic Tully-Fisher relation, without modifying the Einstein equations, and thus Newtonian dynamics in the non-relativistic regimes, and by combining it with the non-baryonic dark matter paradigm, we should be able to sharpen the CDM proposal, and point towards a more focused origin of dark matter quanta. (At the moment, the nature of dark matter quanta is not constrained at all, and these can span enormous energy scales.)

The defining feature of the MDM proposal is that the modified dark matter profile should be sensitive to the fundamental acceleration  $a_0$ , or alternatively, to the cosmological constant, at all scales (galactic, cluster and cosmological) and that on galactic scales the modified dark matter mass profile should be correlated to the baryonic mass profile. The question really is, whether we might be able to modify the energy momentum tensor in such a way so that this modification depends both on the original baryonic source, and on the inertial properties, such as the acceleration, associated with the geometric side of Einstein’s equation.

The canonical formulation of General Relativity is ignorant of any modified inertial properties like the ones suggested by MOND. Moreover, the effective field theory (which is used to model dark matter particles, as in WIMP models) does not know about inertial properties at all. So, whatever one does to implement the dependence of the dark matter profile on some fundamental acceleration, it has to go beyond the classical Einstein theory and the usual local effective quantum field theory without violating these two pillars of modern physics in their domains of validity.

One way to do so is by appealing to quantum gravity, which should reproduce Einstein’s gravity and the energy momentum tensor of the sources described by effective field theory. We will comment on the role of quantum gravity in the conclusion, when we talk about the recent new formulation of string theory and quantum gravity in terms of metastring theory.

Regardless of the lofty origins of MDM, our working proposal is much simpler and more practical: we propose to look at the thermodynamic reformulation of Einstein’s theory of gravity and search for a consistent modification of the energy momentum tensor in that thermodynamics approach, so that the fundamental acceleration is included *ab initio*. The reason for this is that gravitational thermodynamics (the prototype of which is black hole thermodynamics) is the only sure place where quantum theory and physics in accelerating frames are precisely related. Thus, whatever quantum gravity is, it should be consistent with gravitational thermodynamics, and the proposal for modified dark matter, MDM, should be based in robust features of gravitational thermodynamics which include the sensitivity of the dark matter profile on the fundamental acceleration.

The main result of our investigation can be summarized in the following formula for the mass profile of non-baryonic dark matter, which relates the mass of the dark

matter ( $M'$ ) with the mass of the baryonic matter ( $M$ ) via an acceleration parameter  $a_0$ ;

$$\frac{M'}{M} = \frac{\alpha}{[1 + (r/r_{MDM})]} \left[ \frac{a_0^2}{(a_{\text{obs}} + a_0)^2 - a_0^2} \right], \quad (14)$$

where  $a_{\text{obs}}$  is the observed acceleration,  $r$  the radial distance,  $r_{MDM}$  is a dark matter distance scale, and  $\alpha$  is constant factor that is of order 1 for galaxies and 100 for galaxy clusters.<sup>8</sup> Note that for the case of galaxies  $r/r_{MDM} \rightarrow 0$ , and then (see Eq. 35 in §2.2) the mass profile reduces simply to

$$\frac{M'}{M} = \frac{1}{2} \left[ \frac{a_0^2}{(a_{\text{obs}} + a_0)^2 - a_0^2} \right]. \quad (15)$$

Thus this profile works both on galactic and cluster scales, and how well it works can be pictorially represented in Fig. 1 and Fig. 2.

Fig. 1 shows a very tight correlation between baryonic matter and dark matter in galaxies. This figure is similar to the one presented in Ref. 83, but for a different data set. We use galactic rotation curve data from the sample of Ursa Major galaxies represented in Fig. 3. The data is fit with a modified version of Eq. 2:

$$a_{\text{obs}} = \frac{a_{\text{bar}}}{1 - e^{-\sqrt{a_{\text{bar}}/z\bar{a}_0}}}, \quad (16)$$

where

$$z = \frac{\alpha}{[1 + (r/r_{MDM})]} \quad (17)$$

is the prefactor in Eq. 14 appropriate for galaxies. Of course, numerically, this is the same formula used in Ref. 83. However, inclusion of  $z$  allows for consistency when we go to the galaxy cluster scale, where Eq. 2 does not fit the data well.

Fig. 2 shows a correlation between baryonic matter and dark matter in the sample of thirteen galaxy clusters presented in §3.2. The black squares represent the data fitting functions developed in Ref. 121. Our fitting function for galaxy clusters has the same form as the function used for galaxies, Eq. 16, with  $z$  appropriate for the galaxy cluster scale. We plot the function for two values of the acceleration scale: For the dashed red line, we use  $\bar{a}_0$ , and for the solid red line, we use  $a_0$ . Note that we use the same scale distance  $r_{MDM}$  for all galaxy clusters in this plot, while in the fits presented later, this scale is allowed to vary for different clusters. Using a single value increases the scatter in the data.

The paper is organized as follows: First we give a heuristic argument for the modified dark matter (MDM) profile based on gravitational thermodynamics. In particular, we review and then generalize the entropic gravity/gravitational thermodynamics arguments of Jacobson<sup>98</sup> and Verlinde<sup>99,100</sup> to de-Sitter space with positive cosmological constant to construct a dark matter model which is sensitive

<sup>8</sup>The value of  $\alpha$  for galaxy clusters is currently not well-constrained. Values between  $\sim 50 - 100$  fit the data in our sample well. In this paper, we use  $\alpha = 50$  for our galaxy cluster fits.



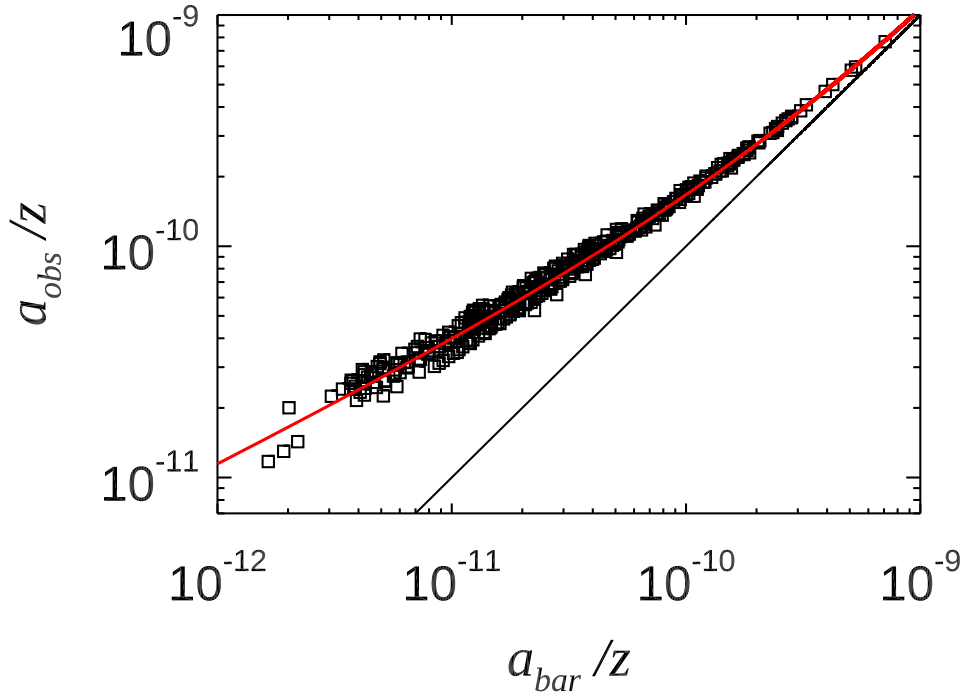


Fig. 1. Comparison of observed accelerations and accelerations expected from baryons in galaxies. The black squares are 386 data points from a sample of 30 galaxies presented in this paper. The black line is what we expect from Newtonian physics and no dark matter. The red line is the prediction of Eq. 16.

to  $\Lambda$  and thus to the fundamental acceleration.<sup>101–103</sup> In this context we explain the relation between MDM and the entropic gravity proposal of Verlinde. Then we discuss how MDM captures the observed data on galactic, cluster, and even cosmological scales. We emphasize that modified dark matter captures the successful features of CDM at cluster and cosmological scales, but it effectively behaves like MOND at the galactic scales. (The may explain the apparent failure of MOND at large scales.) We subject MDM to observational tests with galactic rotation curves for 30 galaxies and observed mass profiles for 13 galactic clusters.<sup>104,105</sup> We show that MDM is in some sense more economical than CDM in fitting data at the galactic scale, and it is superior to MOND at the cluster scale. Also, based on very general arguments, MDM also works on cosmological scales where it is consistent with the  $\Lambda$ CDM paradigm. We conclude the review with a few comments about some non-local non-particle properties of MDM as well as about its possible fundamental origin in quantum gravity.

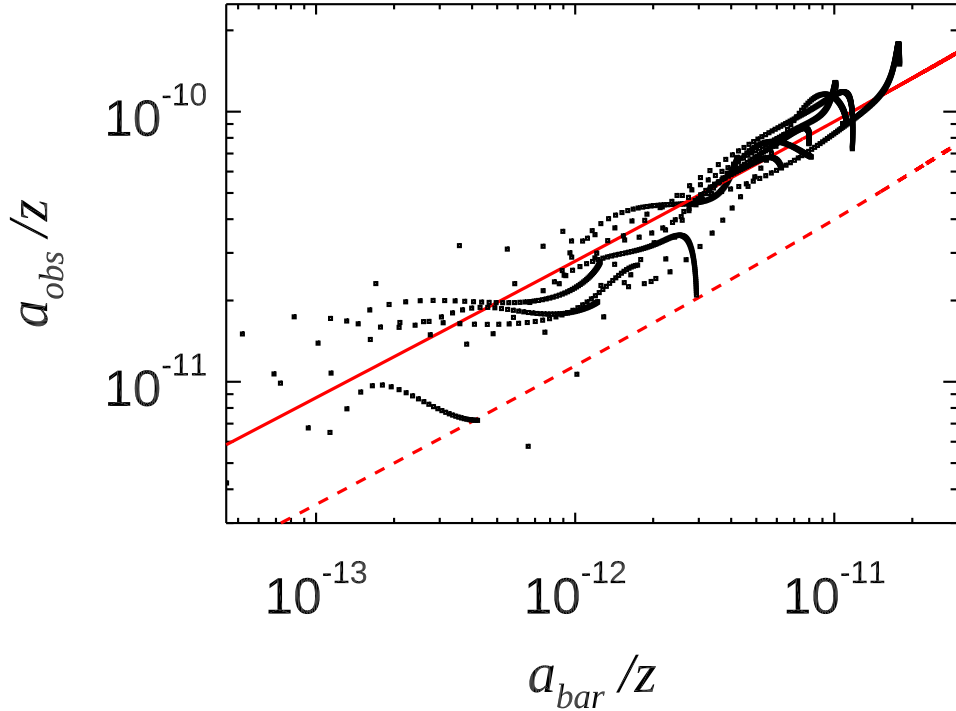


Fig. 2. Comparison of observed accelerations and accelerations expected from baryons in galaxy clusters. The black squares represent fits to data from a sample of 13 galaxy clusters.<sup>121</sup> The solid and dashed red lines are the predictions of Eq. 16 using  $a_0$  and  $\bar{a}_0$ , respectively.

## 2. Constructing Modified Dark Matter (MDM)

In this section we aim to outline the basic logic of our proposal. We want to construct a mass profile for non-baryonic dark matter, that is sensitive to the fundamental acceleration whose value is set by the observed cosmological constant. We also want to tie this dark matter mass profile to the baryonic mass profile via the fundamental acceleration parameter on galactic scales, as indicated by data. However, we also aim to have enough flexibility not to correlate the dark matter mass profile to the baryonic mass profile on cluster and cosmological scales. So the question is: how can this be achieved? One might think that this is impossible, given the fact that the canonical mass profiles for CDM are arrived at after laborious numerical simulations of structure formation.

The idea here is that the acceleration can be re-interpreted in terms of temperature of the Unruh-Hawking kind,<sup>106,107</sup> and that in turn, such temperature can also be corrected by the presence of the cosmological constant, due to the fact that maximally symmetric spaces with positive cosmological constant, that is, asymptotically de Sitter spaces, also have a characteristic temperature associated with their cosmological horizons. This temperature can be rephrased as the fundamental

acceleration. Furthermore, any excess temperature can be interpreted as excess energy, and thus as extra matter source. Thus, the fundamental origin of dark matter is tied to the thermal properties associated with gravity in the context of effective quantum field theory in curved spacetime. Note that according to this proposal any excess source and the usual visible matter sources could be related via the corresponding temperature. Therefore, in principle, the dark matter and visible, baryonic matter, could be related, and this relation can, in principle, involve the fundamental acceleration parameter. Finally, given the fact that temperature can be red-shifted by using the well-known Tolman-Ehrenfest formula, we can in principle argue for different mass profiles on different scales, while still maintaining the explicit dependence of the dark matter profile on the fundamental acceleration. As we will, see this flexibility allows us to account for the observed data both on galactic and cluster scales.

Technically, our approach can be traced to the work of Jacobson<sup>98</sup> regarding the thermodynamics of Einstein's gravitational equations. We are interested in a slight modification of his argument that utilizes the thermodynamics of de Sitter space. Thus we consider a local observer with acceleration  $a$  in a spatially flat de Sitter space. Jacobson's idea was to start with the thermodynamic relation

$$dE = TdS. \quad (18)$$

in Rindler spacetime. By examining the ways in which temperature  $T$ , entropy  $S$  and energy  $E$  are defined, and by utilizing the Raychaudhuri focusing equation, Jacobson deduced the integral form of the Einstein equation. For  $T$ , it is natural to use the Unruh temperature associated with the local accelerating (Rindler) observer<sup>106,107</sup>

$$T = \frac{\hbar a}{2\pi c k_B}. \quad (19)$$

For  $S$ , the holographic principle<sup>108,109</sup> can be invoked to give

$$S = \frac{c^3 A}{4G\hbar}, \quad (20)$$

where  $A$  is the area of the Rindler horizon. Here  $E$  denotes the integral of the energy momentum tensor of matter

$$E = \int T_{\alpha\beta} k^\alpha k^\beta, \quad (21)$$

where  $k^\alpha$  are appropriate unit vectors.

The variation of the area  $A$  in the holographic principle expression is given in terms of the expansion of the horizon generators the focusing of which is associated with the energy flux flowing across the horizon. For an instantaneously stationary local Rindler horizon (required for equilibrium thermodynamics), the shear and vorticity terms can be neglected in the integration of the Raychaudhuri equation for the expansion of the horizon generators to yield

$$\frac{\delta A}{\delta \lambda} = R_{\alpha\beta} k^\alpha k^\beta + \dots, \quad (22)$$

where  $\lambda$  is the appropriate affine parameter. Using the above relations, Jacobson obtained

$$8\pi G \int T_{\alpha\beta} k^\alpha k^\beta = \int R_{\alpha\beta} k^\alpha k^\beta . \quad (23)$$

The cosmological constant appears as an integration constant in this approach, and application of local conservation of energy and momentum finally yields the Einstein equation.<sup>h</sup> In the following two sub-sections, we construct MDM by two related arguments both inspired by Jacobson's insight.

### 2.1. MDM and Gravitational Thermodynamics

We can argue for the MDM profile by using a heuristic argument as follows. Consider modifying Jacobson's argument<sup>98</sup> by introducing a fundamental acceleration that is related to the cosmological constant. We assume that Einstein's theory of gravity is valid and we want a standard energy-momentum tensor. The first condition requires that we preserve the holographic scaling of the area. Consequently the second condition, in conjunction with the form of the thermodynamic relation, demands that we change the temperature while preserving the entropy. Our model is therefore given by the thermodynamic relation

$$d\tilde{E} = \tilde{T}dS . \quad (24)$$

We note that, since the Unruh temperature knows the inertial properties and is fixed by the background, the additional part of the energy-momentum tensor (coming from a modified temperature) will also know the inertial properties and the background.

Here we want to point out that in these arguments one always has to absorb the acceleration factor in the Einstein tensor, in order to end up with the correct normalization of Einstein's equations. This acceleration-dependent factor will be crucial in our argument for the emergence of the dark matter mass profile that depends on the observed acceleration as well as the fundamental acceleration parameter, and that is still subject to the canonical force law, as implied by the canonical Einstein equations.

Now consider a local observer with local acceleration  $a$  in de Sitter space, where  $a_0 = c^2\sqrt{\Lambda/3} = cH_0$ . The Unruh temperature experienced by this observer is<sup>i110, 111</sup>

$$T_{a_0+a} = \frac{\hbar}{2\pi ck_B} \sqrt{a^2 + a_0^2} . \quad (25)$$

<sup>h</sup> Note that, in the derivation, the Ricci tensor  $R_{\alpha\beta}$  enters the discussion via the expression for  $dS$  in Eq. (20). Thus, to remain consistent with general relativity, we should preserve the holographic scaling of the area. This fact will be useful in the construction of MDM mass profile.

<sup>i</sup>If the four-dimensional de Sitter space is envisioned as a hyperboloid embedded in a flat, five-dimensional space, the effective five-dimensional acceleration ( $a_5$ ) is related to the four-dimensional acceleration ( $a_4$ ) as  $a_5 = \sqrt{a_4^2 + a_0^2}$ .

However, since de Sitter space has a cosmological horizon, it has a horizon temperature associated with  $a_0$ . We thus define the following *effective* temperature (so that for zero acceleration we get zero temperature)

$$\tilde{T} \equiv T_{a_0+a} - T_{a_0} = \frac{\hbar}{2\pi c k_B} \left( \sqrt{a^2 + a_0^2} - a_0 \right) \equiv \frac{\hbar \tilde{a}}{2\pi c k_B}. \quad (26)$$

In analogy with the normalized temperature  $\tilde{T}$ , the normalized energy is given by  $\tilde{E} = E_{a_0+a} - E_{a_0}$ . Writing the temperature as  $T_{a_0+a} = T + T'$  where the  $T$  part corresponds to the Unruh temperature of the observer moving with the Newtonian acceleration  $a_N$  (in the correspondence limit  $a_0 \ll a = a_N$ ), and using  $dE_{a_0+a} = T_{a_0+a} dS$ , we can also write  $dE + dE' = T dS + T' dS$ . If we interpret the original  $dE = T dS$  as corresponding to baryonic (unprime) matter, then  $dE' = T' dS = \frac{T'}{T} dE$  corresponds to dark (prime) matter. By expanding the formula for the de Sitter temperature Eq. (25) (relating  $T'$  to  $T$ ), using Eq. (21) (relating  $E$  to  $T_{\alpha\beta}$ ) and the fact that the energy density is the 00 component of the energy-momentum tensor, we have a relation between the dark and visible matter

$$M' = \frac{a_0^2}{2a^2} M. \quad (27)$$

This dark matter profile is qualitatively the same as the one (Eq. (35)) in our original papers on MDM.<sup>101–103</sup>

## 2.2. MDM and Entropic Gravity

The argument given in Ref. 101 is a simple generalization of Verlinde's recent proposal of entropic gravity,<sup>99</sup> inspired by Jacobson's work,<sup>98</sup> for  $\Lambda = 0$  to the case of de-Sitter space with a positive  $\Lambda$ . Let us first review Verlinde's view of Newton's second law  $\vec{F} = m\vec{a}$ . Consider a particle with mass  $m$  approaching a holographic screen at temperature  $T$ . Using the first law of thermodynamics to introduce the concept of entropic force

$$F = T \frac{\Delta S}{\Delta x}, \quad (28)$$

and invoking Bekenstein's original arguments<sup>112</sup> concerning the entropy  $S$  of black holes,

$$\Delta S = 2\pi k_B \frac{mc}{\hbar} \Delta x, \quad (29)$$

Verlinde found

$$F = 2\pi k_B \frac{mc}{\hbar} T. \quad (30)$$

With the aid of the formula for the Unruh temperature associated with a uniformly accelerating (Rindler) observer,

$$k_B T = \frac{\hbar a}{2\pi c}, \quad (31)$$

Verlinde then obtained  $\vec{F} = m\vec{a}$ .

The already cited formula for the effective temperature,<sup>106,107,113</sup> as measured by a non-inertial observer with acceleration  $a$  relative to an inertial observer, is

$$\tilde{T} = \frac{\hbar\tilde{a}}{2\pi k_B c}, \quad (32)$$

with  $\tilde{a} = \sqrt{a^2 + a_0^2} - a_0$ .<sup>110</sup> This reduces to the usual temperature for Rindler observers by neglecting  $a_0$ . Note that

$$\sqrt{a^2 + a_0^2} - a_0 \approx \begin{cases} a & (a \gg a_0), \\ \frac{a^2}{2a_0} & (a \ll a_0). \end{cases} \quad (33)$$

The entropic force (in de-Sitter space) is hence given by the replacement of  $T$  and  $a$  by  $\tilde{T}$  and  $\tilde{a}$  respectively, leading to

$$F = m \left[ \sqrt{a^2 + a_0^2} - a_0 \right] \equiv m a_{\text{obs}}, \quad (34)$$

where we have changed the notation from  $\tilde{a}$  defined in Eq. (26) to  $a_{\text{obs}}$  to emphasize its meaning as the observed acceleration. This means that the above mass profile needs to be rewritten as

$$M' = \frac{1}{2} \left( \frac{a_0}{a} \right)^2 M \equiv \frac{1}{2} \left( \frac{a_0^2}{(a_{\text{obs}} + a_0)^2 - a_0^2} \right) M, \quad (35)$$

Here we emphasize one very important point. Given the fact that we have not changed Einstein's equations, we cannot change the Newton law of motion. However, the Newtonian, observed acceleration  $a_{\text{obs}}$ , does depend on the fundamental acceleration parameter  $a_0$ , as well as the auxiliary acceleration  $a$  that appears in the Unruh formula for the de Sitter space. This hidden dependence, originating in gravitational thermodynamics, finds its explicit realization in the dark matter profile, which now depends on  $a_0$  and  $a_{\text{obs}}$ . Thus, in some sense, we are managing to bootstrap the observed acceleration, the dark matter mass  $M'$ , and the visible matter mass  $M$ . Thus the relevant Newtonian dynamics is described via

$$a_{\text{obs}} = \frac{G(M + M')}{r^2} \equiv \frac{GM}{r^2} \left[ 1 + \frac{1}{2} \left( \frac{a_0^2}{(a_{\text{obs}} + a_0)^2 - a_0^2} \right) \right]. \quad (36)$$

To summarize: dark matter exists in our scheme, but its mass profile is tied to the baryonic mass profile (at least on galactic scales) via the observed acceleration and the fundamental acceleration parameter, set by the cosmological constant. Such dark matter is most probably of a non-local, non-particle kind, because of the fact that its mass profile is tied to the baryonic mass profiles as well as to the inertial properties, encoded in the observed acceleration. Furthermore, as we will show next, the MOND-like scaling, found to work beautifully at the galactic scale, is simply a manifestation of such non-local dark matter, answering the question raised in Ref. 96. Dark matter of this kind can behave as if there is no dark matter but

MOND. (That was why we used to call it “MONDian dark matter”. However, we find the Modified Dark Matter more appropriate, given the new, modified, form of the dark matter profile.) Intriguingly the dark matter profile we have obtained relates, at the galactic scale, dark matter ( $M'$ ), dark energy ( $\Lambda$ ) and ordinary matter ( $M$ ) to one another. In the concluding section, we will speculate on a microscopic basis for the dark matter’s dependence on  $\Lambda$ .

Finally, let us comment on the origin of an effective MOND in our scheme. In order to compare to the usual MOND reasoning, we revert to the notation that involves the auxiliary acceleration  $a$ . First we note that for  $a \gg a_0$ , we have  $F/m \approx a$  which gives  $a = a_N \equiv GM/r^2$ , the familiar Newtonian value for the acceleration due to the source  $M$ . But for  $a \ll a_0$ ,  $F \approx m \frac{a^2}{2a_0}$ , so the terminal velocity  $v$  of the test mass  $m$  in a circular motion with radius  $r$  should be determined from  $ma^2/(2a_0) = mv^2/r$ . In this small acceleration regime, the observed flat galactic rotation curves ( $v$  being independent of  $r$ ) now require  $a \approx (2a_N a_0^3 / \pi)^{\frac{1}{4}}$ . But that means  $F \approx m \sqrt{a_N \bar{a}_0}$ . This is the modified Newtonian dynamics (MOND) scaling,<sup>89–91</sup> discovered by Milgrom who introduced the critical acceleration parameter  $\bar{a}_0 = a_0/(2\pi) = cH_0/(2\pi)$  (with  $H_0$  being the Hubble parameter) by hand to phenomenologically explain the flat galactic rotation curves.<sup>j</sup> Thus, as advertised, we have recovered MOND with the correct magnitude for the critical galactic acceleration parameter  $\bar{a}_0 \sim 10^{-8} \text{ cm/s}^2$ .<sup>k</sup> From our perspective, MOND is a *classical* phenomenological consequence of the MDM mass profile, obtained from gravitational thermodynamics in de Sitter space (with the  $\hbar$  dependence in  $T \propto \hbar$  and  $S \propto 1/\hbar$  canceled out).<sup>101</sup> As a bonus, we have also recovered the observed Tully-Fisher relation ( $v^4 \propto M$ ).

### 3. Observational Tests of MDM

As a preparation for our discussion on testing MDM (and CDM and MOND), let us first collect all the relevant formulas for the various (effective) mass profiles. For MDM, the equation of motion (Eq. (36)) reads

$$a_N \left[ 1 + f_{\text{MDM}}(a/a_0) \right] = \sqrt{a^2 + a_0^2} - a_0 \equiv a_{\text{obs}} , \quad (37)$$

where

$$\frac{M'}{M} = f_{\text{MDM}}(a/a_0) \equiv f_{\text{MDM}} \left( \frac{\sqrt{(a_{\text{obs}} + a_0)^2 - a_0^2}}{a_0} \right) . \quad (38)$$

<sup>j</sup>As far as the knowledge of  $\bar{a}_0 \sim cH_0$  is concerned, Kaplinghat and Turner<sup>96</sup> have argued that the acceleration scale  $cH_0$  may arise naturally within CDM models. The CDM obtains information on  $cH_0$  from the simple fact that it is evolving in a universe expanding at the rate of  $H_0$ , whereas the coincidence  $\bar{a}_0 \sim O(1)cH_0$  is more of a numerical accident. But even with this knowledge, CDM has problems at the galactic scale as mentioned above.

<sup>k</sup>The emergence of this scale in non-standard dark matter models has been also addressed in Ref.<sup>114–116</sup>

For comparison, let us include the dynamical masses predicted by CDM and MOND. For CDM, we use the<sup>117,118</sup> (NFW) density profile,

$$\rho_{\text{NFW}}(r) = \frac{\rho_0}{\frac{r}{r_{\text{CDM}}} \left(1 + \frac{r}{r_{\text{CDM}}}\right)^2}, \quad (39)$$

to determine the mass predicted by CDM, where  $r_{\text{CDM}} = r_{200}/c$ , and  $r_{200}$  designates the edge of the halo, within which objects are assumed to be virialized, usually taken to be the boundary at which the halo density exceeds 200 times that of the background. The parameter  $c$  (not to be confused with the speed of light) is a dimensionless number that indicates how centrally concentrated the halo is.

For an effective MOND, one can write the left-hand-side of Newton's equation as

$$\frac{1}{\mu(a/\bar{a}_0)} \frac{GM}{r^2} = \frac{G(M + M')}{r^2}, \quad (40)$$

and interpret  $M$  and

$$M' = M \left[ \frac{1}{\mu(a/\bar{a}_0)} - 1 \right] \equiv M f_{\text{MOND}}(a/\bar{a}_0), \quad (41)$$

as the mass of baryonic matter and the *effective* mass of non-baryonic dark matter respectively enclosed within the same sphere. Then we have that

$$a_N \left[ 1 + f_{\text{MOND}}(a/\bar{a}_0) \right] = a. \quad (42)$$

Solving this equation for the acceleration  $a$  will also determine  $M' = M f_{\text{MOND}}(a/\bar{a}_0)$ . The dark matter distribution determined in this fashion would precisely reproduce the results of MOND without modifying inertia or the law of gravity. Therefore for MOND, we have, from Eq. (41),

$$\frac{M'(r)}{M(r)} = f_{\text{MOND}}(a(r)/\bar{a}_0) = \frac{1}{\mu(a(r)/\bar{a}_0)} - 1. \quad (43)$$

Assuming a spherically symmetric distribution, the effective dark matter density profile for MOND is given by

$$\rho_{\text{MOND}}(r) = \frac{1}{4\pi r^2} \frac{d}{dr} M'(r). \quad (44)$$

### 3.1. Galactic Rotation Curves

In order to test MDM with galactic rotation curves, we fit computed rotation curves to a selected sample of Ursa Major galaxies given in Ref. 95. The sample contains both high surface brightness (HSB) and low surface brightness (LSB) galaxies. The rotation curves, predicted by MDM as given above by

$$F = m \left[ \sqrt{a^2 + a_0^2} - a_0 \right] = m a_N \left[ 1 + \frac{1}{2} \left( \frac{a_0}{a} \right)^2 \right], \quad (45)$$



or equivalently, in terms of the observed acceleration  $a_{\text{obs}}$ ,

$$F = ma_{\text{obs}} = ma_N \left[ 1 + \frac{1}{2} \left\{ \frac{a_0^2}{(a_{\text{obs}} + a_0)^2 - a_0^2} \right\} \right], \quad (46)$$

along with  $F = mv^2/r$  for circular orbits, can be solved for  $a_{\text{obs}}(r)$  (or  $a(r)$ ) and  $v(r)$ . In Ref. 104, we fit these to the observed rotation curves as determined in Ref. 95, using a least-squares fitting routine. As in Ref. 95, the mass-to-light ratio  $M/L$ , which is our *only* fitting parameter for MDM, is assumed constant for a given galaxy but allowed to vary between galaxies. Once we have, for example, the auxiliary acceleration  $a(r)$ , we can find the MDM density profile by using  $M' \approx \frac{1}{2} \left( \frac{a_0}{a} \right)^2 M$ , to give

$$\rho'(r) = \left( \frac{\bar{a}_0}{r} \right)^2 \frac{d}{dr} \left( \frac{M}{a^2} \right). \quad (47)$$

Rotation curves predicted by MDM are shown in Fig. 3. Both the MDM and CDM models fit the data well; but note that while the MDM fits use only 1 free parameter, for the CDM fits one needs to use 3 free parameters. Thus the MDM model is a more economical model than CDM in fitting data at the galactic scale.

Shown in Fig. 4 are the dark matter density profiles predicted by MDM (solid lines) and CDM (dashed lines).

Finally we should point out that the rotation curves predicted by MDM and MOND have been found<sup>104</sup> to be virtually indistinguishable over the range of observed radii and both employ only 1 free parameter.

### 3.2. Mass Profiles of Galaxy Clusters

In principle, the above mass profile, Eq. (35) or Eq. (27), fixed by the ratio of the corresponding Unruh-Hawking temperatures (in the limit of small  $a_0/a$ ) can be modified due to some well-known physical effects associated with a change of scale. For example, in the presence of gravity, the temperature is not constant in space at equilibrium. As a result, it can be changed due to the so-called Tolman-Ehrenfest effect<sup>119,120</sup>

$$T\sqrt{g_{00}} = 2\tilde{\alpha}, \quad \text{where } g_{00} = 1 + 2\Phi/c^2, \quad (48)$$

with  $\Phi$  being the gravitational potential, and the factor  $\tilde{\alpha}$  is determined by the boundary conditions of the problem.

But what gravitational potential  $\Phi$  should be used in our case and what sets the value of  $\tilde{\alpha}$ ? In general, we should expend the gravitational potential in powers of  $r$ . The leading  $r$  dependence corresponds to a constant background gravitational field, i.e. the linear potential (this is indeed the first term in the Taylor expansion for the potential, up to a physically irrelevant constant piece). Thus we end up with the following modification of the mass profile :

$$f_{\text{MDM}}(a/a_0) = \frac{M'}{M} = \frac{\alpha}{[1 + (r/r_{\text{MDM}})]} \left[ \frac{a_0^2}{(a_{\text{obs}} + a_0)^2 - a_0^2} \right], \quad (49)$$

18 *Edmonds, Farrah, Minic, Ng, Takeuchi*

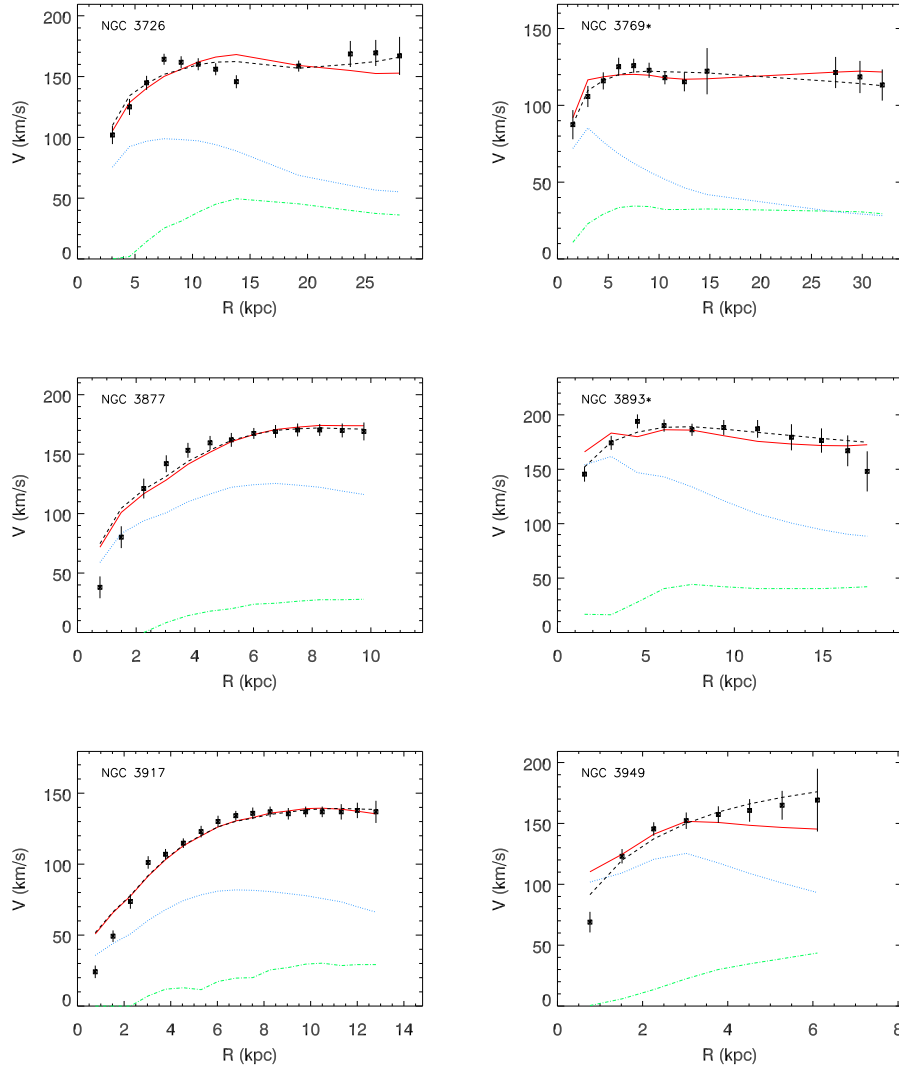


Fig. 3. Galactic rotation curves. The observed rotation curve is depicted by points with error bars. The solid red and dashed black lines are the MDM and CDM rotation curves, respectively. Newtonian curves for the stellar and gas components of the baryonic matter are depicted by dotted blue and dot-dashed green lines, respectively. The mass of the stellar component is derived from the  $M/L$  ratio determined from MDM fits to the rotation curve. These figures are based on the generalized mass profile

where the dimensionless factor  $\alpha$  is determined by the ratio of  $\tilde{\alpha}$  at different scales (in our case, the cluster and galactic scales).<sup>1</sup> Note that the prefactor  $\alpha/[1 + (r/r_{\text{MDM}})]$

<sup>1</sup>Interestingly the same prefactor appears in the context of conformal gravity when one rewrites

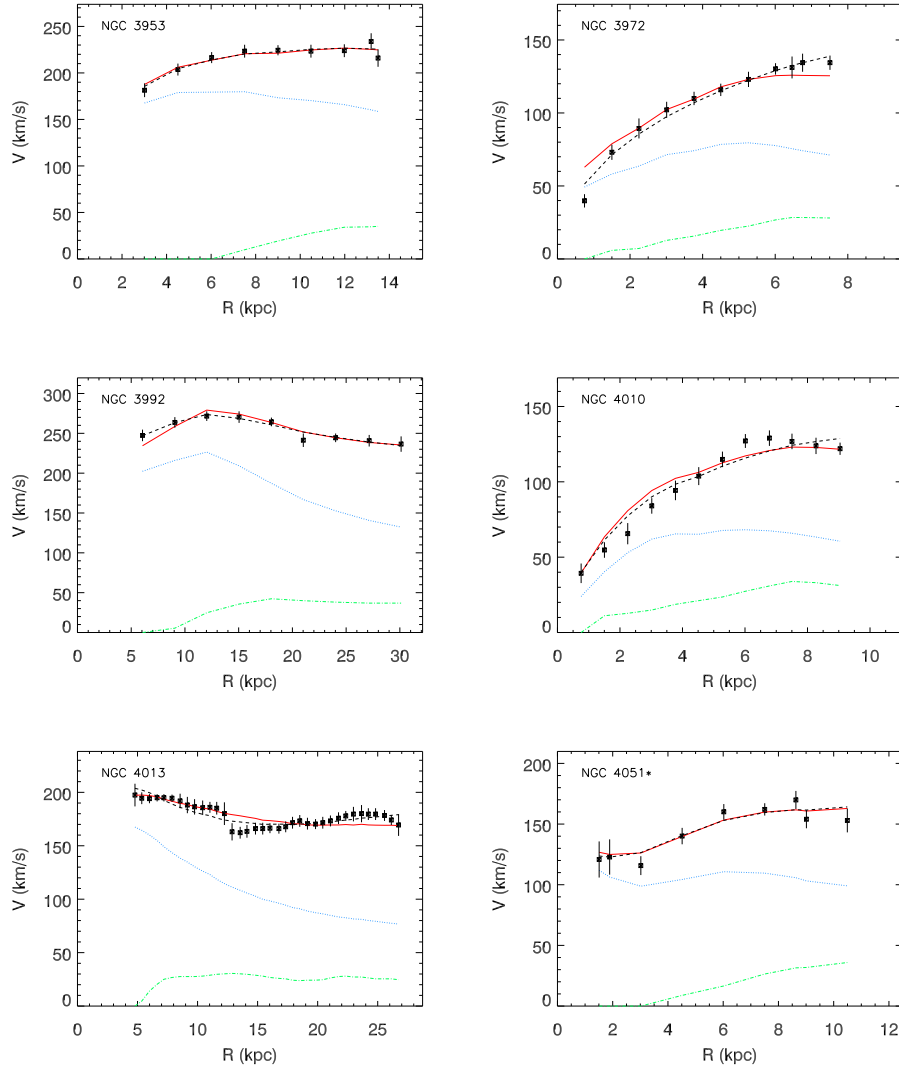


Fig. 3 (Cont.).

is only the leading term in a more general expression that involves higher order terms in  $r$ .

In Ref. 105 we compared MDM mass profiles with the observed (virial) mass profiles in a sample of 13 relaxed galaxy clusters given in Ref. 121. These authors of Ref. 121 analyzed all available *Chandra* data which were of sufficient quality to

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the FRW cosmological line element in the Schwarzschild coordinate system (the linear potential also being the direct analogue of the Newtonian potential for conformal gravity; see Ref. 9).

20 *Edmonds, Farrah, Minic, Ng, Takeuchi*

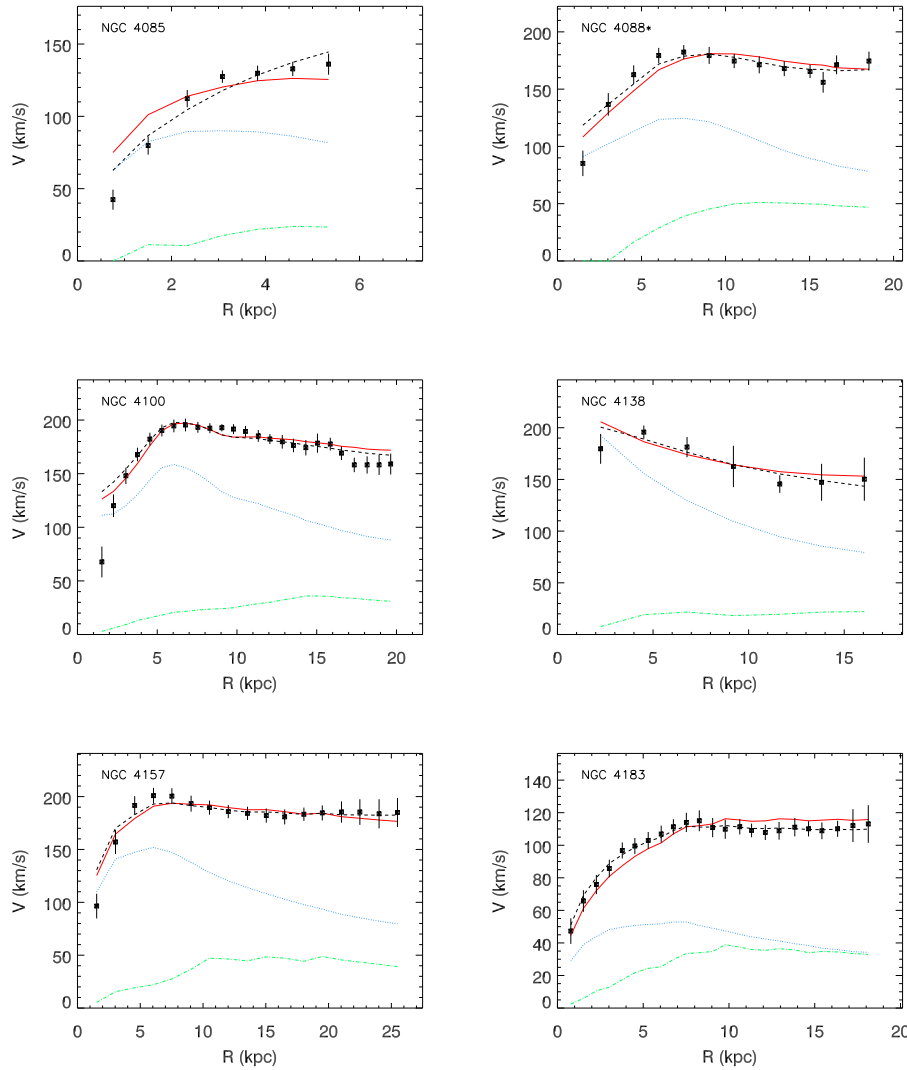


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determine mass profiles robustly out to large radii ( $\sim 0.75 r_{500}$  and extended past  $r_{500}$  in five clusters). With the dark matter mass profile predicted by MDM given by Eq. (49), the total mass, the sum of dark matter and baryonic matter, is compared with the virial mass for the sample of galaxy clusters in Fig. 5. For comparison, we include dynamical masses predicted by CDM (Eq. (39)) and MOND (Eq. (43)).

We see in Fig. 5 that, with  $\alpha \sim 100$  and  $r_{\text{MDM}} \sim 10$  kpc, the MDM mass profiles fit the virial mass data well. The fits for MDM mass profiles are as good as those for NFW. On the other hand, the MOND (effective) mass profiles fail to reproduce

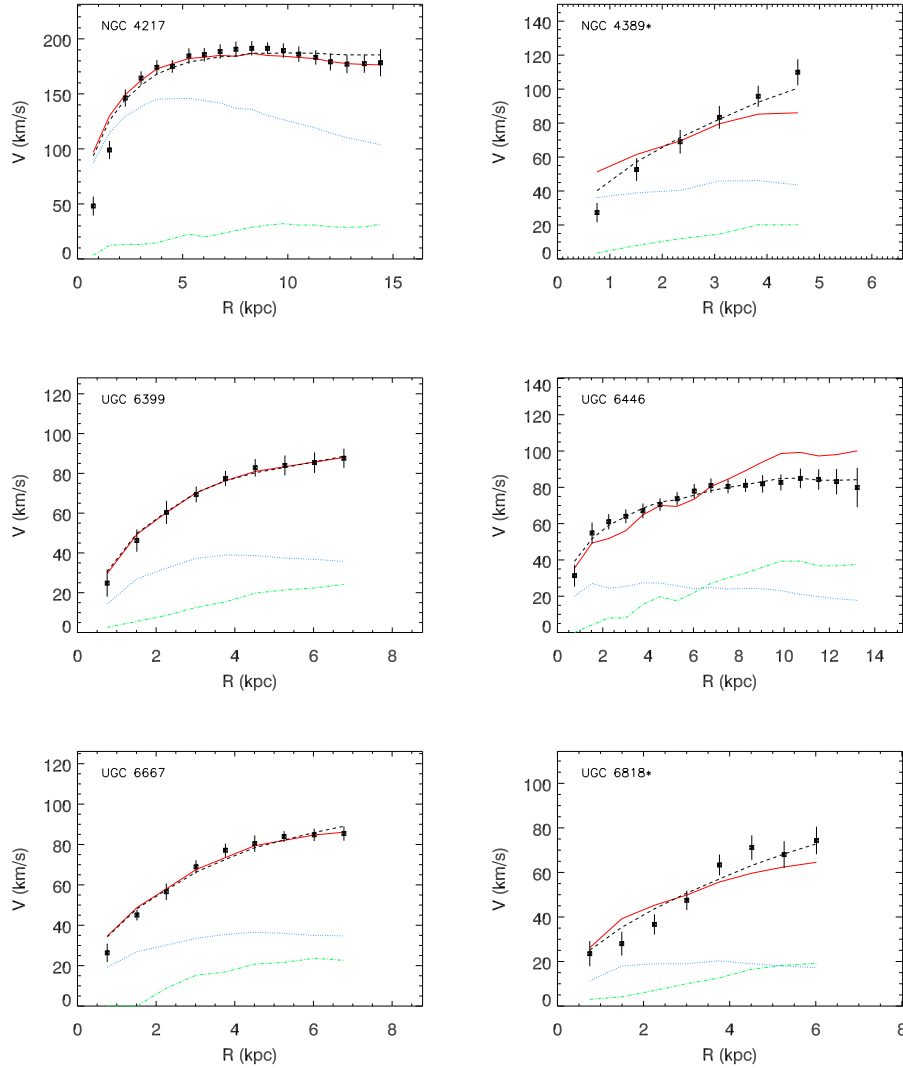


Fig. 3 (Cont.).

the virial mass profiles in magnitudes and in shape.

Finally we should recall that, in Ref. 104, we fitted a sample of galactic rotation curves using the MDM mass profile Eq. (35) (see previous sub-section). We would like to see if the new mass profile can work at galactic scales as well as it does at cluster scales. For the galaxy clusters in our sample, we found  $\alpha \sim 50 - 100$  for the mass profile given in Eq. (49) fits the data well, while values  $\lesssim 50$  do not. For the galaxy cluster figures produced in this paper, we used  $\alpha = 50$ . For comparison of data fits using  $\alpha = 100$ , please see Ref. 105. Since the fits to galactic rotation

22 *Edmonds, Farrah, Minic, Ng, Takeuchi*

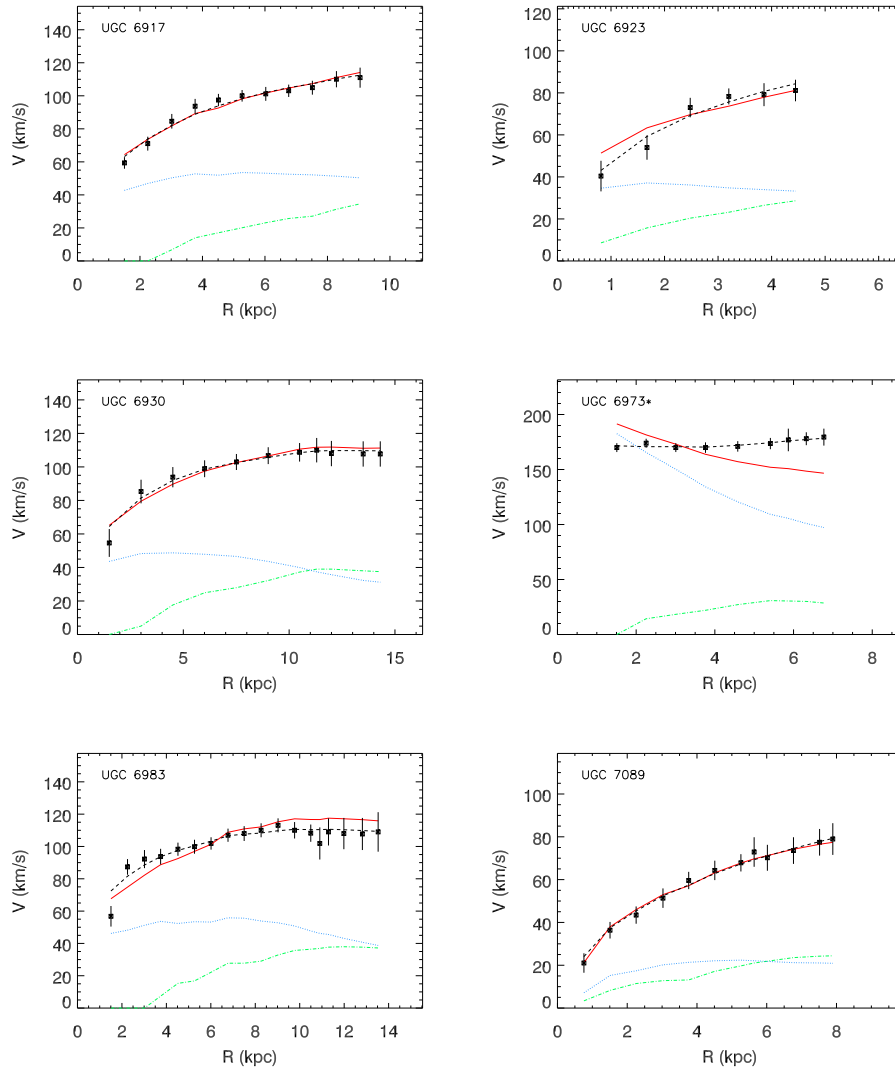


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curves presented in Ref. 104 were fit so well with Eq. (35), we expect  $\alpha \sim 1$  with  $r_{\text{MDM}} \gg r$  in the generalized mass profile.

One might be disturbed by the appearance of such two radically different values for  $\alpha$ :  $\alpha \sim 1$  for galactic scales and  $\alpha \sim 100$  for the scale of clusters. Here we note that  $\alpha$  is essentially a boundary condition for the effective Unruh-Hawking-like temperature, and thus it can be expected to have radically different values on radically different scales. We note that there is a numerical coincidence involving

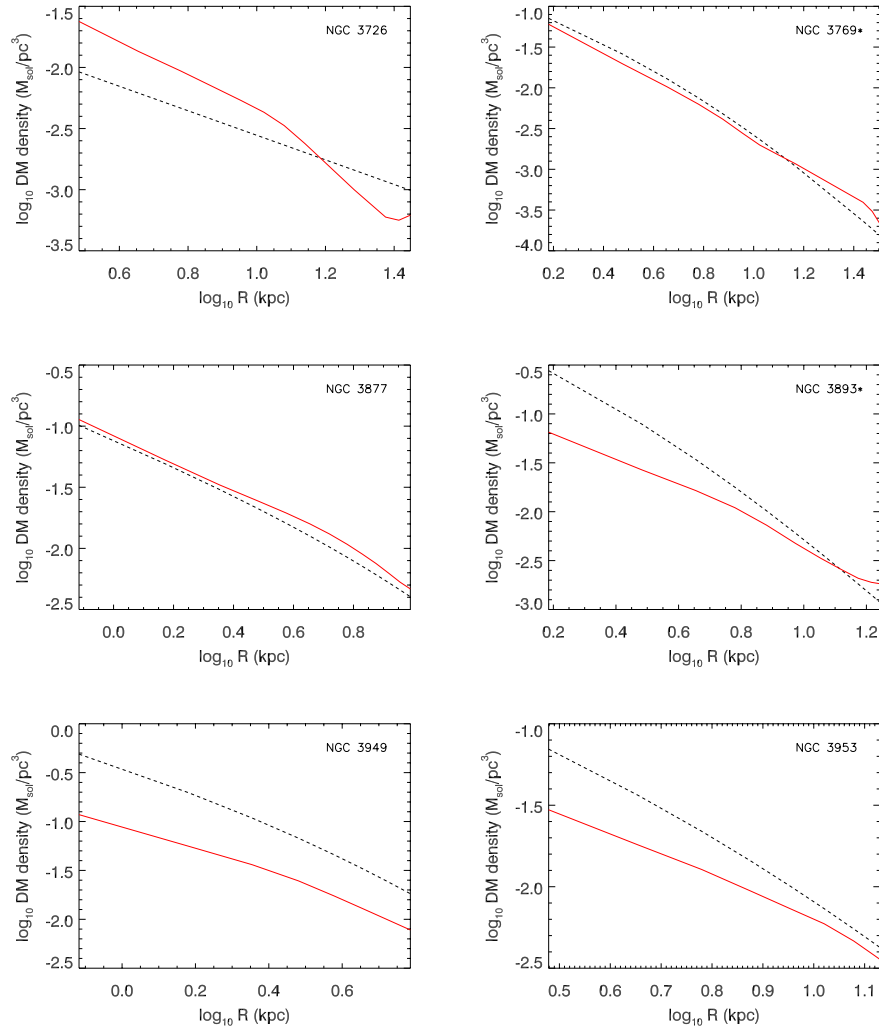


Fig. 4. Dark matter density profiles for our sample of galaxies. Density profiles for MDM and CDM are depicted as solid and dashed lines, respectively.

the ratio of temperatures of the intracluster medium as compared to the interstellar medium in galaxies that can be around 100. Of course, this could be a coincidence and one would have to understand the origin of  $\alpha$  better in order to say anything that is more definitive. Finally, note that the difference in the values of  $\alpha$  can be understood as an effective “renormalization” of  $a_0$ , needed to fit the cluster data given the successful fits on galactic scales. Note however that the rescaling of  $a_0$  is not enough to make MOND work on cluster scales. MOND has a hard time reproducing the necessary profiles which are captured by Tolman-like  $r$  dependent

24 *Edmonds, Farrah, Minic, Ng, Takeuchi*

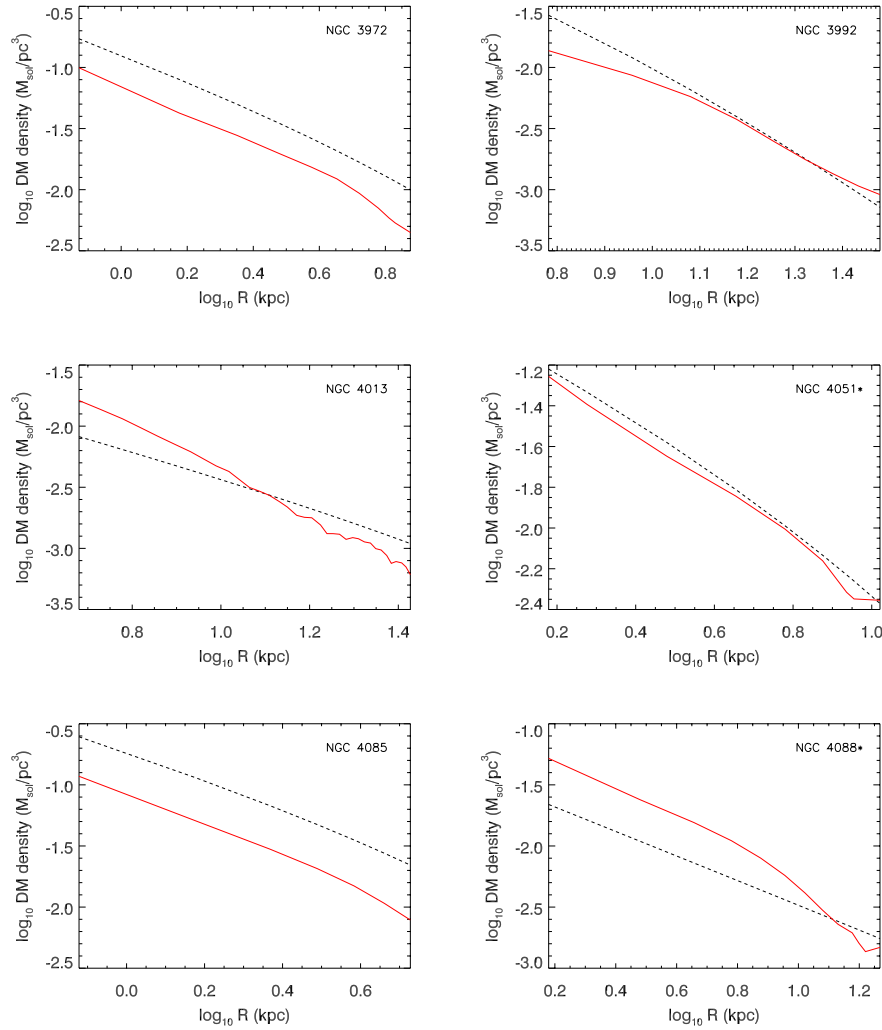


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factors in the context of our MDM proposal.

### 3.3. MDM and Cosmology

To recapitulate: By generalizing the canonical gravitational thermodynamics arguments to de-Sitter space, we are led to a new model of dark matter, which takes into account the observed correlation between dark matter and baryonic matter on galactic scales. The resulting dark matter mass profile is, by construction, sensitive to the fundamental acceleration parameter found in the observed galactic data.



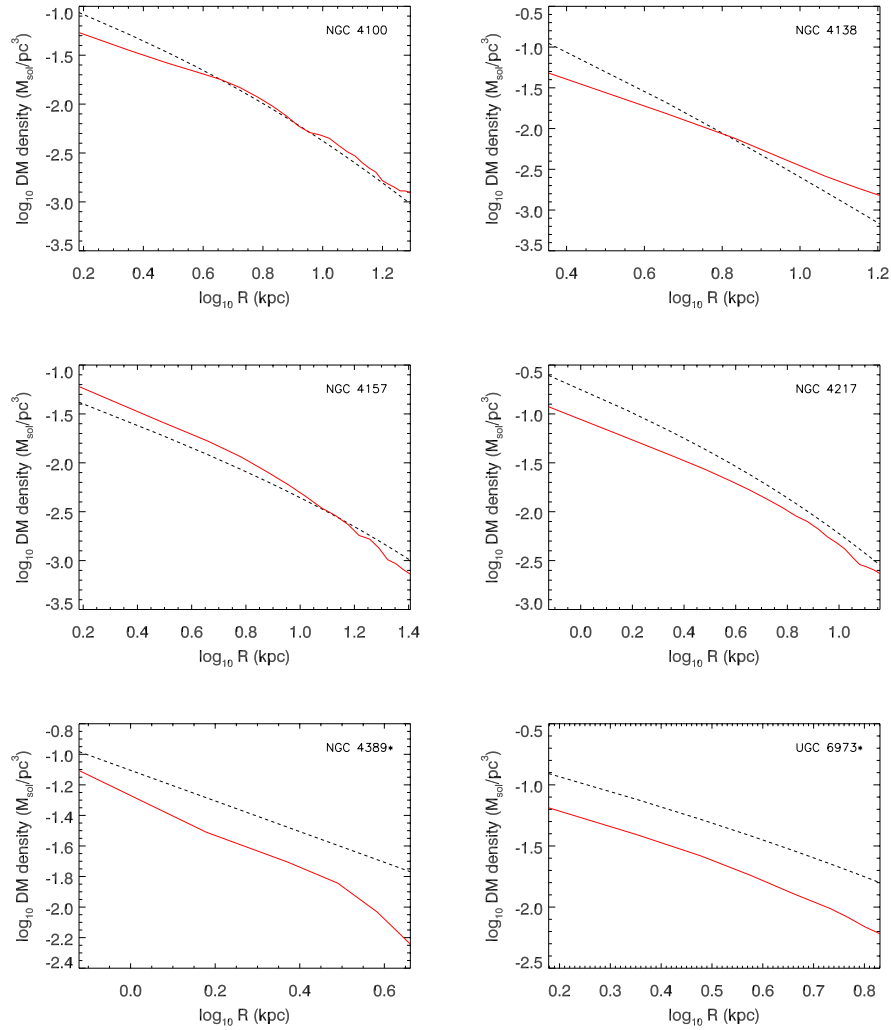


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Moreover, by taking into account the temperature variation at difference scales, and by translating it into the equivalent acceleration variation with scale, we obtained a successful dark matter profile at cluster scales. As already emphasized, our MDM behaves more like MOND at galactic scales, but more like CDM at cluster scales.

However, so far we have not discussed the cosmological implications of MDM. Cosmology provides perhaps the most persuasive evidence for the “missing mass” in the form of the canonical  $\Lambda$ CDM model, so we have to answer the following question: How does MDM fare at cosmic scales? While we do not have concrete

26 *Edmonds, Farrah, Minic, Ng, Takeuchi*

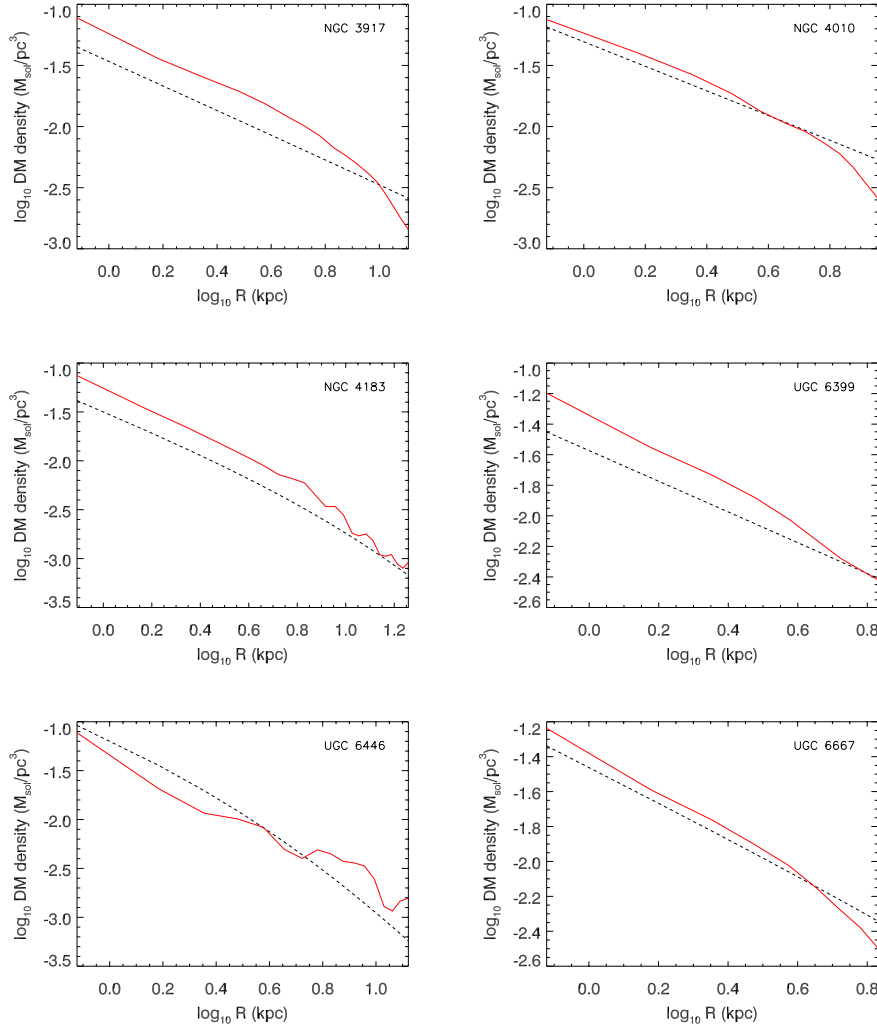


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results yet, qualitative and heuristic arguments seem to yield an optimistic picture. Let us briefly touch on the issues of cosmology and gravitational lensing. As shown in Ref. 101, at cosmological scales we need to take into account of the fully relativistic sources. First, we concentrate on the isotropic and homogeneous situation described with the FRW metric

$$ds^2 = -dt^2 + R(t) (dr^2 + r^2 d\Omega^2) , \quad (50)$$

and we assume that the matter sources form a perfect fluid with the energy-

*MDM: Relating Dark Energy, Dark Matter and Baryonic Matter* 27

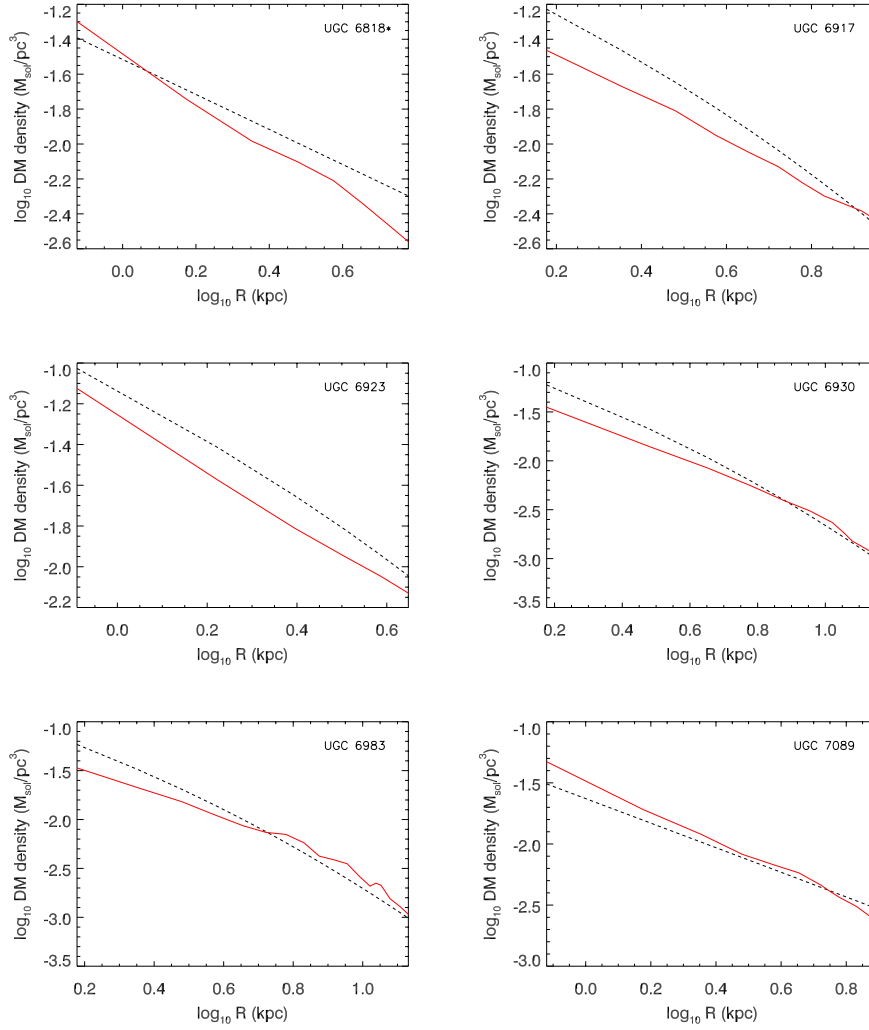


Fig. 4 (Cont.).

momentum tensor

$$T_{ab} = (\rho + p) u_a u_b + p g_{ab} . \quad (51)$$

Next, for a fully relativistic description, we need to replace  $M + M'$  in

$$a_{\text{obs}} = \frac{G(M + M')}{\tilde{r}^2} + 4\pi G p \tilde{r} - \frac{\Lambda}{3} \tilde{r} , \quad (52)$$

by the active gravitational mass (Tolman-Komar (TK) mass)

$$M_{\text{TK}} = \frac{1}{4\pi G} \int dV R_{ab} u^a u^b , \quad (53)$$

28 *Edmonds, Farrah, Minic, Ng, Takeuchi*

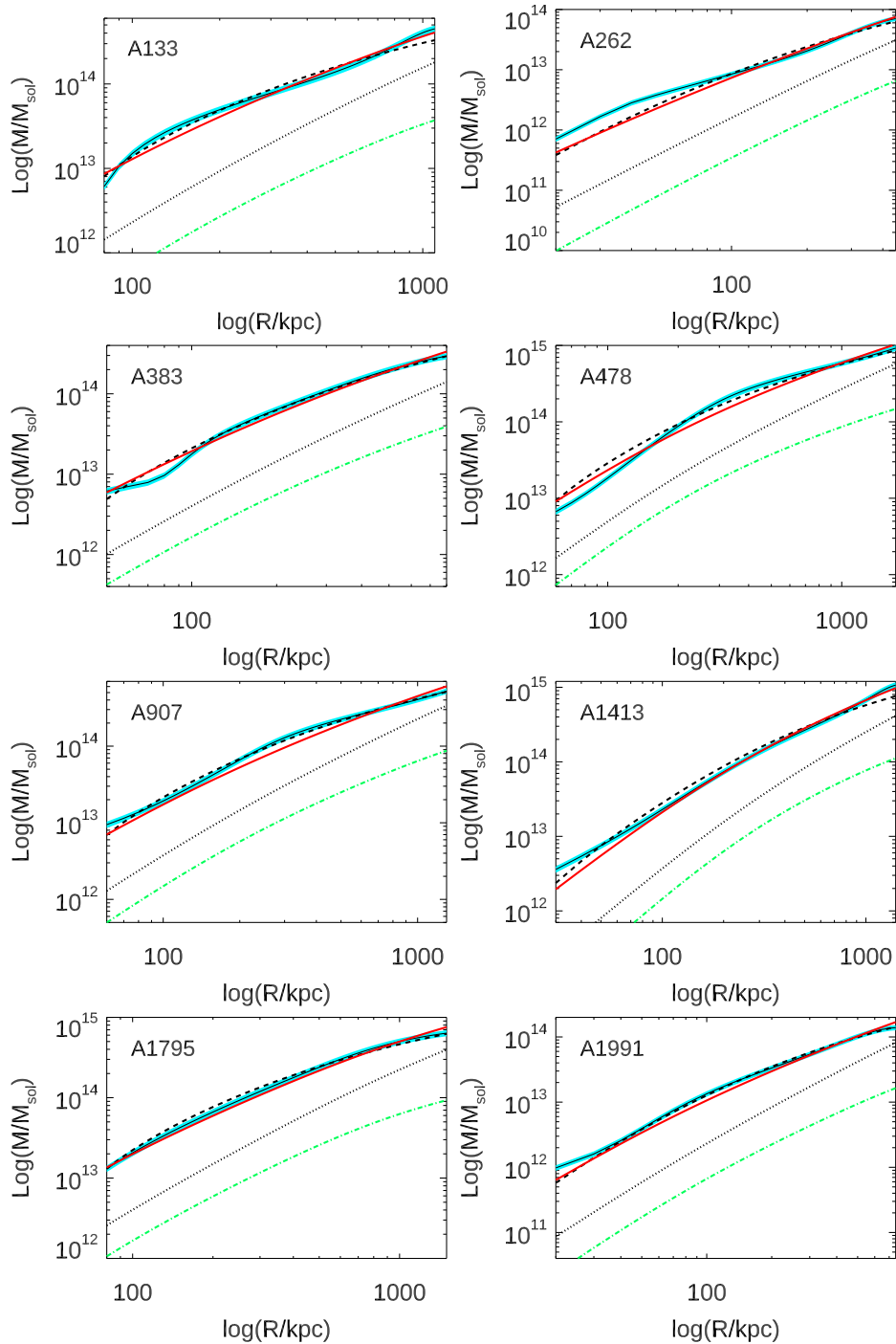


Fig. 5. Plots of total mass of galactic clusters within radius  $R$  (assuming spherical symmetry). The solid black line is the virial mass and surrounding the black line is a blue shaded region depicting  $1 - \sigma$  errors; The dot-dashed green line is gas mass; The dotted black line is MOND (effective mass); The dashed black line is CDM; The solid red line is MDM with  $\alpha = 100$ .

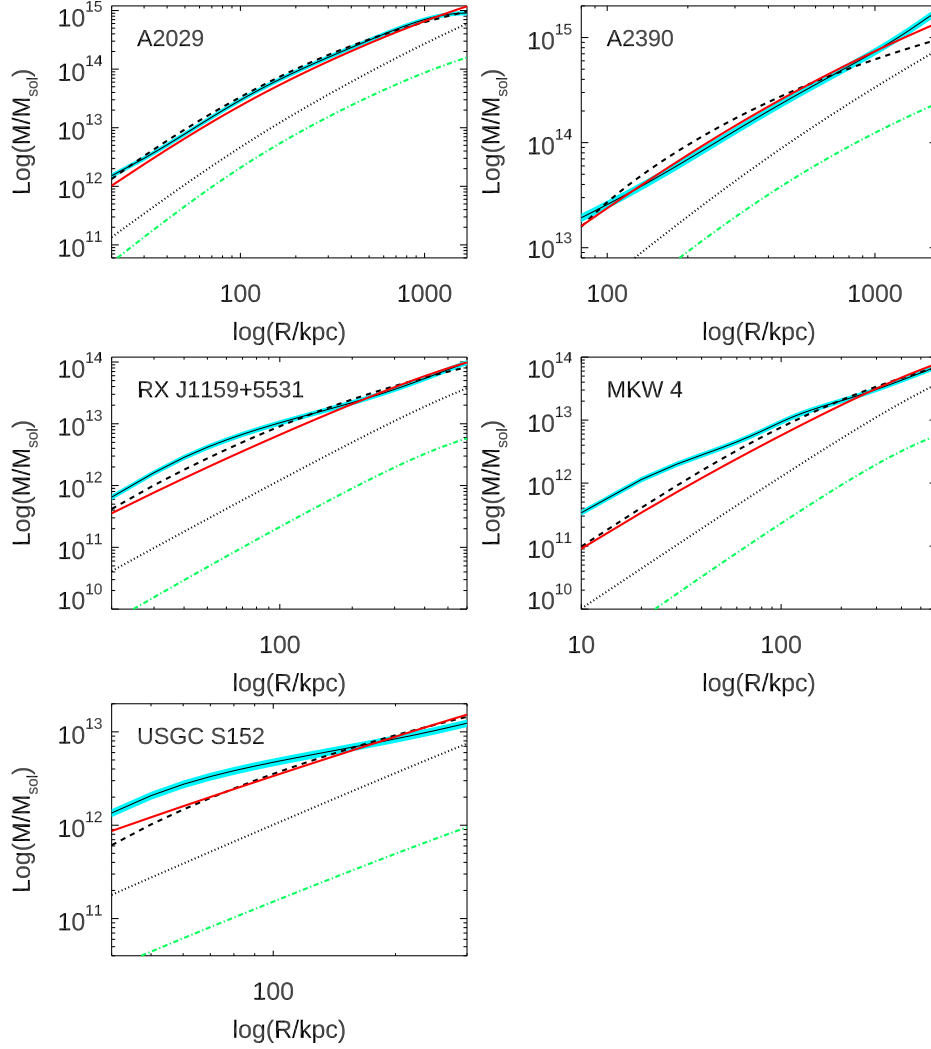


Fig. 5 (Cont.).

which by the Einstein equations of general relativity becomes

$$M_{\text{TK}} = 2 \int dV \left( T_{ab} - \frac{1}{2} g_{ab} T + \frac{\Lambda}{8\pi G} g_{ab} \right) u^a u^b, \quad (54)$$

or alternatively

$$M_{\text{TK}} = \left( \frac{4}{3} \pi r^3 R^3 \right) \left[ (\rho + 3p) - \frac{\Lambda}{4\pi G} \right]. \quad (55)$$

30 *Edmonds, Farrah, Minic, Ng, Takeuchi*

Then it can be shown<sup>101</sup> that

$$\frac{1}{R} \frac{d^2 R}{dt^2} = -\frac{4\pi G}{3} (\rho + 3p) + \frac{\Lambda}{3}, \quad (56)$$

with  $p$  being the pressure. Here  $\tilde{r} = r R(t)$  denotes the physical radius, where  $r$  and  $R(t)$  are the comoving radius and scale factor respectively. At cluster and cosmological scales, either  $4\pi G p \tilde{r}$  or  $\Lambda \tilde{r}/3$  or both could be significant and this may explain why MOND does not work well at the cluster and cosmological scales. The main point is that, continuing the argument of Ref. 101, using the above equation and the continuity equation  $\dot{\rho} + 3H(\rho + p) = 0$  one obtains, for the cosmology of MDM, the other canonical Friedmann equation

$$H^2 = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3}. \quad (57)$$

We anticipate that this fact will allow MDM to predict the correct cosmic microwave background (CMB) spectrum shapes as well as its characteristic alternating peaks.

Next, let us comment on strong gravitational lensing in the context of MDM and MOND. (Recall that strong lensing refers to the formation of multiple images of background sources by the central regions of some clusters.) It is known that the critical surface density required for strong lensing is

$$\Sigma_c = \frac{1}{4\pi} \frac{cH_0}{G} F(z_l, z_s), \quad (58)$$

with  $F \approx 10$  for typical clusters and background sources at cosmological distances. Sanders<sup>122</sup> argued that, in the deep MOND limit,  $\Sigma_{\text{MOND}} \approx \bar{a}_0/G$ . Recalling that numerically  $\bar{a}_0 \approx cH_0/6$ , Sanders concluded that MOND cannot produce strong lensing on its own:  $\Sigma_c \approx 5\Sigma_{\text{MOND}}$ . On the other hand, MDM mass distribution appears to be *sufficient* for strong lensing since the natural scale for the critical acceleration for MDM is  $a_0 = cH_0 = 2\pi\bar{a}_0 \approx 6\bar{a}_0$ , five to six times that for MOND.

#### 4. Non-local aspects of MDM

In this section we comment on the crucial feature of MDM contained in Eq. (3) and Eq. (35) which indicate that modified dark matter profile is sensitive to the cosmological constant, or equivalently, the Hubble parameter. But what is the microscopic basis for this correlation? We will now argue that it may have something to do with the quanta of MDM being non-local (and non-particle-like), or more specifically, the MDM quanta obeying an exotic statistics known as infinite statistics or quantum Boltzmann statistics. The evidence for such non-local nature of MDM quanta is most apparent in the effective dark matter mass profile derived from gravitational thermodynamics. It is very hard to see why the mass profiles coming from particle-like quanta of some local effective field theory would be sensitive to a fundamental acceleration parameter, set by the cosmological scale, even if one assumes some unusual dissipative baryonic dynamics. Thus, it is not completely outlandish to expect that the MDM dark matter quanta should be non-local and non-particle like.

In order to get some intuition about such putative non-local features, without assuming anything about the microscopic dynamics of MDM quanta, let us<sup>103</sup> first reformulate MOND via an effective gravitational dielectric medium, motivated by the analogy<sup>123</sup> between Coulomb's law in a dielectric medium and Milgrom's law for MOND. We start with the nonlinear electrostatics embodied in the Born-Infeld theory,<sup>124</sup> and write the corresponding gravitational Hamiltonian density as

$$H_g = \frac{b^2}{4\pi} \left( \sqrt{1 + \frac{D_g^2}{b^2}} - 1 \right), \quad (59)$$

where  $D$  stands for the electric displacement vector and  $b$  is the maximum field strength in the Born-Infeld theory. With  $\mathcal{A}_0 \equiv b^2$  and  $\vec{\mathcal{A}} \equiv b\vec{D}_g$ , the Hamiltonian density becomes

$$H_g = \frac{1}{4\pi} \left( \sqrt{\mathcal{A}^2 + \mathcal{A}_0^2} - \mathcal{A}_0 \right). \quad (60)$$

If we invoke energy equipartition ( $H_g = \frac{1}{2}k_B T_{\text{eff}}$ ) and the Unruh temperature formula ( $T_{\text{eff}} = \frac{\hbar}{2\pi k_B c} a_{\text{eff}}$ ), and apply the equivalence principle (in identifying, at least locally, the local accelerations  $\vec{a}$  and  $\vec{a}_0$  with the local gravitational fields  $\vec{\mathcal{A}}$  and  $\vec{\mathcal{A}}_0$  respectively), then the effective acceleration  $a_{\text{eff}}$  is identified as  $a_{\text{eff}} \equiv \sqrt{a^2 + a_0^2} - a_0$ . But this, in turn, implies that the Born-Infeld inspired force law takes the form  $F_{\text{BI}} = m \left( \sqrt{a^2 + a_0^2} - a_0 \right)$ , for a given test mass  $m$ , which is precisely the MONDian force law.

To be a viable cold dark matter candidate, the quanta of the MDM must be much heavier than  $k_B T_{\text{eff}}$  since  $T_{\text{eff}}$ , with its quantum origin (being proportional to  $\hbar$ ), is a very low temperature. Now recall that the equipartition theorem in general states that the average of the Hamiltonian is given by  $\langle H \rangle = -\frac{\partial \log Z(\beta)}{\partial \beta}$ ,

where  $\beta^{-1} = k_B T$ . To obtain  $\langle H \rangle = \frac{1}{2}k_B T$  per degree of freedom, even for very low temperature, we require the partition function  $Z$  to be of the Boltzmann form  $Z = \exp(-\beta H)$ . But this is precisely the case of infinite statistics.<sup>103, 125</sup>

What is infinite statistics? Succinctly, a Fock space realization of infinite statistics is provided by (a  $q$  deformation of) the commutation relations of the oscillators:  $a_k a_\ell^\dagger = \delta_{k\ell}$  as described by the average of the bosonic and fermionic algebras. It is known that a theory of particles obeying infinite statistics cannot be local.<sup>125</sup> For example, the expression for the number operator,

$$n_i = a_i^\dagger a_i + \sum_k a_k^\dagger a_i^\dagger a_i a_k + \sum_\ell \sum_k a_\ell^\dagger a_k^\dagger a_i^\dagger a_i a_k a_\ell + \dots, \quad (61)$$

is both nonlocal and nonpolynomial in the field operators, and so is the Hamiltonian. The lack of locality may make it difficult to formulate a relativistic version of the theory; but it appears that a non-relativistic theory can be developed. Lacking locality also means that the familiar spin-statistics relation is no longer valid for

particles obeying infinite statistics; hence they can have any spin. Remarkably, the CPT theorem and cluster decomposition have been shown to hold despite the lack of locality.<sup>125</sup> A good example of infinite statistics is the large  $N$  (planar) limit of  $SU(N)$  Yang-Mills theory.

This type of statistics suggests that the MDM quanta might be of quasi-one-dimensional nature, as is natural for string-like excitations found in the context of the planar Yang-Mills theory. Such non-local excitations are believed to be an important ingredient in the formulation of quantum gravity, and thus the lack of locality for theories of infinite statistics may not be a defect; it can actually be a virtue. Perhaps it is this lack of locality that makes it easier to incorporate gravitational interactions in the theory. Quantum gravity and infinite statistics appear to fit together nicely, and non-locality seems to be a common feature of both of them.<sup>101,103</sup> Conceivably it is the extended nature of the MDM quanta that connects them to the cosmological constant/dark energy and the Hubble parameter, two (related) global aspects of spacetime. As we will argue in the concluding section, such extended, non-particle excitations may be the generic features of quantum theory of gravity, pointing towards a more fundamental origin of the MDM quanta.

## 5. Summary and Discussion

In the conclusion of this review we summarize our main points and address some fundamental underpinnings of our proposal.

At the moment the MDM proposal is not rooted in any fundamental physics, and there is no specific candidate for the MDM quantum. Here we want to make some comments about the possible fundamental rationale for MDM and the nature of the MDM quantum. At this point one might ask: why do we need such unusual quanta? The answer lies, as already mentioned in the introduction, in the comments made by Kaplinghat and Turner<sup>96</sup> in the context of CDM regarding the observed Milgrom scaling in the galactic rotations curves. It is very hard for CDM to reproduce the apparent *universality* of this scaling, even though the work on this topic is continuing. Note that Milgrom's scaling is usually associated with modified Newtonian dynamics (MOND) which denies dark matter. As emphasized in this review, this is NOT our point. What we have been asking is the following: Given the unknown nature of dark matter, what constraint does the Milgrom scaling impose on its quanta (at all scales)? Can this scaling be accounted for by modifying CDM-like mass profiles? Is this scaling compatible with a non-particle nature of dark matter, thus explaining why we have, so far, not seen the assumed particle quanta of dark matter? If so, what is the "smoking gun" signal for such non-local, non-particle quanta of dark matter?

Our central observation is that Milgrom's scaling implies a non-local mixing of physics in the dark matter and dark energy sector. Simply put, the observed value for the cosmological constant can be associated with the acceleration parameter found in Milgrom's scaling. Then the question is, whether a dark matter profile can



be found which accounts for the galactic rotation curves and is sensitive to this acceleration parameter. As summarized in this review, we have proposed precisely such a dark matter profile based on a heuristic viewpoint rooted in gravitational thermodynamics. At cluster as well as cosmological scales, modified dark matter behaves as CDM, but at the galactic scale, modified dark matter implies the scaling behavior usually associated with MOND. However, we emphasize once again that MDM is *not* MOND – MDM is an unconventional, and most probably non-local form of dark matter. Note that the modified dark matter profile can be related to the classic CDM mass profiles, such as the NFW mass profile,<sup>117,118</sup> in certain limits, which might be viewed as another justification for the MDM proposal. Also, the MDM proposal can be viewed as a unification between the dark energy, dark matter and baryonic matter sectors.

We have already made a comment regarding the extended nature of the MDM quanta. This might seem very surprising given the fact that the successful existing theoretical tools of fundamental physics are all rooted in the concept of locality in spacetime. Locality is indeed one of the cornerstones of modern physics. It is one of the key properties underlying effective field theory, which is widely considered as a universal language for describing the fundamental physics and which captures the main features of known particle physics at low energy scales, and thus it is a prominent tool in the standard approaches towards the particle-like CDM quanta. However, it is becoming increasingly clear that non-locality may play a central role in solving some of the most outstanding puzzles in fundamental, such as the vacuum energy problem, the black hole information paradox, as well as the deep and central non-local features of quantum theory and quantum field theory, including the naturalness and hierarchy problems. If this is the case, then the tools of effective field theory are inadequate, and we must develop new ideas and techniques. Similarly, the fundamental concepts of differential geometry constitute the basic mathematical language of general relativity, our deepest theory of space, time and gravity. However, these concepts seem to be just a limiting case of the mathematical language of generalized geometry required to talk about various new non-local phenomena encountered in string theory, such as T-duality (the intrinsic relation between short and long distances). The concept of non-locality is brought to the forefront in the context of the so-called metastring theory, proposed by Freidel, Leigh and Minic.<sup>126–132</sup> The metastring formulation of quantum gravity introduces a new concept of quantum spacetime called modular spacetime which, surprisingly, sheds light on the foundational issues in quantum theory and quantum field theory. Also, by pointing out that the effective field theoretic description of strings is generically non-commutative, yet covariant, at long distance, metastring theory sheds light on why effective (Wilsonian) local quantum field theory is so successful in so many domains of physics, and why it is bound to be transcended in more general situations.

Given the successful phenomenology of MDM presented in this review, we expect that the fundamental non-locality and non-commutativity advocated by metastring

theory show up in the context of the dark sector, involving dark energy and dark matter. In particular, such fundamental non-commutativity and non-locality of the metastring imply that the spectrum of low energy excitations can be non-particle-like. This happens in non-commutative field theories<sup>133</sup> which can be viewed as toy models of the metastring.<sup>126–132</sup> Such excitations can also have unusual statistics. We expect that such non-local excitations (as part of a covariant non-commutative formulation, as implied by the metastring<sup>131,132</sup>) can serve as unusual dark matter quanta. Furthermore, the zero mode sector of the metastring, does not look like the classic relativistic particle, but as a non-local extension of the usual quanta of free local fields<sup>m</sup>. Roughly, the zero mode sector looks like two entangled particles, and it could be viewed as a model for modified dark matter quanta discussed in this review, implying, perhaps naturally, the observed relation between dark matter and baryonic mass profiles at certain scales.

We can speculate about the extended nature of MDM as follows: for such extended excitations the change in momentum is proportional to the change in distance (i.e. such excitations expand with an influx of energy)  $\delta p \sim \alpha \delta x$ . Assuming a non-relativistic situation for which  $p \sim \dot{x}$  then we get that change acceleration is proportional to change velocity, or equivalently, momentum, and after using the Heisenberg relation  $\delta p \delta x \sim \hbar$ , we get that the acceleration is proportional to the inverse of distance, which is exactly what the Milgrom scaling demands. Thus, extended, one-dimensional, excitations could model non-local MDM quanta.

We conclude with our to-do list. First we aim to understand other static clusters in which the CDM apparently encounters certain issues (such as Abell 1689). Then we plan to study concrete constraints from gravitational lensing and the dynamical clusters, such as the famous bullet cluster, on MDM. Specifically we would like to answer these questions: Can we distinguish MDM from CDM in these physical situations? How strongly coupled is MDM to baryonic matter? How does MDM self-interact? We would also like to test MDM at cosmic scales by studying the acoustic peaks in the CMB as well as by doing simulations of structure formation. If the quanta of MDM indeed obey infinite statistics, as suggested in Ref. 103, it is possible that there are (dark) stars made of such quanta. If so, what are some of their observational signatures?

Finally, it is imperative to develop a deeper understanding of the fundamental nature of the non-local MDM quanta. Recent reformulations of quantum gravity and string theory<sup>126–130</sup> may be helpful in this effort. A more concrete theory for MDM quanta will allow us to test the whole scheme of MDM at colliders, dark matter direct detection experiments and indirect detection experiments. Possibly unusual non-particle phenomenology awaits to be discovered. And from our perspective, this may be regarded as quantum gravity phenomenology in disguise.

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<sup>m</sup>The theory of such “metaparticles” is currently being developed by Laurent Freidel, Jerzy Kowalski-Glikman, Rob Leigh and Djordje Minic.

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42 *Edmonds, Farrah, Minic, Ng, Takeuchi*

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