GRADED ARTIN ALGEBRAS, COVERINGS AND FACTOR RINGS

by

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(ABSTRACT)

Let (Γ,ρ) be a directed graph with relations. Let $F:\Gamma'\longrightarrow\Gamma$ be a topological covering. It is proved in this thesis that there is a set of relations $\bar{\rho}$ on Γ such that the category of K-representations of Γ' whose images under the covering functor satisfy ρ is equivalent to the category of finite-dimensional, graded $K\Gamma/\langle\bar{\rho}\rangle$-modules. If Γ' is the universal cover of Γ, then this category is called the category of unwindable $K\Gamma/\langle\rho\rangle$-modules. For arrow unique graphs it is shown that the category of unwindable $K\Gamma/\langle\rho\rangle$-modules does not depend on $\langle\rho\rangle$. Also, it is shown that for arrow unique graphs the finite dimensional uniserial $K\Gamma/\langle\rho\rangle$-modules are unwindable.

Let Γ be an arrow unique graph with commutativity relations, ρ. In Section 2, the concept of unwindable modules is used to determine whether a certain factor ring of $K\Gamma/\langle\rho\rangle$ is of finite representation type.

In a different vein, the relationship between almost split sequences over Artin algebras and the almost split sequences over factor rings of such algebras is studied. Let Λ be an Artin algebra and let $\bar{\Lambda}$ be a factor ring of Λ. Two sets of necessary and sufficient conditions are obtained for determining when an almost split sequence of $\bar{\Lambda}$-modules remains an almost split sequence when viewed as a sequence of Λ-modules.

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Introduction

In the representation theory of Artin algebras the quiver of an algebra has been a useful tool. Specifically, if Λ is a finite dimensional algebra over a field K, then the quiver of Λ is a finite, directed graph whose vertex set depends on $\Lambda/\underline{r}$ and whose arrow set depends on $\underline{r}/\underline{r}^2$, where $\underline{r}$ is the Jacobson radical of Λ. Conversely, if Γ is a finite, directed graph, then given a field K we can form a K-algebra $K\Gamma$, called the path algebra of Γ, which will be defined in Section 1. If Λ is a finite-dimensional K-algebra satisfying certain conditions, then Λ is a homomorphic image of $K\Gamma$ where Γ is the quiver of Λ. The reason that the quiver of an algebra is so interesting is that there is a definition of representations of the quiver so that the category of representations of the quiver which satisfy certain relations is equivalent to the category of left Λ-modules which are finite dimensional over K.

Another approach to studying Artin algebras has been to consider graded Artin algebras and the category of graded modules over such algebras. Gordon and Green [4] have shown that there is a strong connection between the category of modules and the category of graded modules over graded Artin algebras.

The topological theory of covering spaces links the study of graded modules to the study of directed graphs. In particular, Green [5] has shown that if Λ is a finite dimensional algebra over a field K, and Γ is the quiver of Λ, then there exists, under certain conditions, a regular topological cover Γ' of Γ which induces a G-grading on Λ where G is the automorphism group of the covering. It should be mentioned that Bongartz and Gabriel [1,3] also make use of the covering theory of graphs in a way which overlaps the covering theory discussed in this paper.

Section 1 of this paper contains formal definitions and background information pertaining to the relationship between graded modules, directed graphs, and covering spaces. The main theorem of this section gives a relationship between any regular covering of the quiver of a finite-dimensional K-algebra Λ and the category of graded modules over a certain factor ring of Λ. In particular, when Γ' is the universal cover of Γ, we obtain a correspondence between certain representations of Γ' and a subcategory of Λ-modules which we call unwindable modules. The unwindable modules form a class of Λ-modules which is relatively easy to understand because the universal cover of a directed graph is a tree. We will make use of this connection in Section 2. Section 1 also contains an isomorphism result concerning unwindable modules and concludes with examples, both of modules which are unwindable, and those which are not.

In general, a catagory is said to be of <u>finite representation type</u> if it contains only finitely many nonisomorphic, indecomposable objects. If Λ is a finite-dimensional algebra over a field K, then Λ is said to be of finite representation type if the category of left Λ-modules which are finite-dimensional over K is of finite representation type. In Section 2 we use the concept of unwindable modules to determine which algebras from a certain class of finite dimensional K-algebras are of finite representation type. We begin by considering particular partially ordered sets related to a directed graph. The representation theory of partially ordered sets developed by Kleiner, Nazarova and Roiter [6,9], when applied to our specific case, plays a major role in determining whether each of these algebras is of finite representation type. Bongartz and Ringel [2] have used partially ordered sets to study directed trees with relations. We state their result and use this information about directed trees, as well as information about partially ordered sets, to achieve our classification.

Section 3 deals with a special type of short exact sequence called an almost split sequence. These sequences have been widely used in the representation theory of Artin algebras (see, for example [10]). In this section we look at the connection between almost split sequences of Λ-modules and almost split sequences of $\bar{\Lambda}$-modules where $\bar{\Lambda}$ is a factor ring of Λ. We obtain a set of criteria which determines when an almost split sequence of $\bar{\Lambda}$-modules remains an almost split sequence when viewed as a sequence of Λ-modules.

Section 1: Coverings and Graded Factor Rings

Throughout this paper K will denote a fixed, algebraically closed field. All modules mentioned will be left modules. If Λ is an algebra over K, then mod(Λ) will denote the category of all (left) Λ-modules which are finite-dimensional over K. In addition, all graphs in this paper will be directed graphs.

Let Γ be a directed graph. We denote the vertex set of Γ by $V(\Gamma)$ and the arrow set by $A(\Gamma)$. If $u,v \in V(\Gamma)$ and $a \in A(\Gamma)$ with origin u and terminus v, then a^{-1} denotes traveling along a in the opposite direction; that is, from v to u. A path p of length n is a sequence $p=a_n^{e_n} \ldots a_1^{e_1}$ where for each i, $a_i \in A(\Gamma)$ and $e_i = \pm 1$. Moreover, the origin of $a_1^{e_1}$ is u, the terminus of $a_n^{e_n}$ is v and for $i=1,2,\ldots,n-1$, the origin of $a_{i+1}^{e_{i+1}}$ is the terminus of $a_i^{e_i}$. We also do not allow p to contain a subsequence of the form aa^{-1} or $a^{-1}a$ for any $a \in A(\Gamma)$. If $e_1=e_2=\ldots=e_n=1$, then p is called a directed path of length n. Vertices are directed paths of length zero. For the remainder of this paper we will assume that all graphs are path connected; that is, given any two vertices in the graph there is a path of finite length between them. We will also assume that all graphs are locally finite; that is, for any vertex v, there are only finitely many arrows in the graph which have origin or terminus v.

A *set of relations* ρ on a directed graph Γ is a set of K-linear combinations of directed paths of length at least 2. The ordered pair (Γ,ρ) is called a *graph with relations*. Suppose $t=\sum_{\alpha\in A}\mu_\alpha p_\alpha$ where for every α, p_α is a directed path and $\mu_\alpha\in K$ is nonzero for only finitely many α. The *(u,v)-component of t* is defined by

$$c_{u,v}(t) = \sum_{\alpha\in A'}\mu_\alpha p_\alpha$$

where $A' = \{\alpha\in A: p_\alpha$ has origin u and terminus v$\}$.

Now given the graph Γ we define the *category of K-representations of* Γ, denoted by rep(Γ), as follows. The objects in rep(Γ) are tuples of the form

$$X = (X_v,\alpha(a))_{v\in V(\Gamma),a\in A(\Gamma)}$$

where for every $v\in V(\Gamma)$, X_v is a finite-dimensional K-vector space, and in the case that Γ is infinite, all but finitely many X_v are zero. Also, for every $a\in A(\Gamma)$ with origin u and terminus v, $\alpha(a):X_u \longrightarrow X_v$ is a K-linear map. A morphism in rep(Γ) from $X = (X_v,\alpha(a))$ to $Y = (Y_v,\beta(a))$ is a tuple $(f_v)_{v\in V(\Gamma)}$ making the following diagram

$$\begin{tikzcd} X_u \arrow[r, "f_u"] \arrow[d, "\alpha(a)"'] & Y_u \arrow[d, "\beta(a)"] \\ X_v \arrow[r, "f_v"] & Y_v \end{tikzcd}$$

commute for every $u,v \in V(\Gamma)$ and $a \in A(\Gamma)$. If p is a directed path of nonzero length with origin u and terminus v such that $p=a_n \ldots a_1$ with each $a_i \in A(\Gamma)$, then we define $p(X):X_u \longrightarrow X_v$ by

$$p(X) = \alpha(a_n)\circ \ldots \circ\alpha(a_1).$$

If t is a K-linear combination of directed paths each of length greater than zero such that $c_{u,v}(t) = \sum \mu_i p_i$, then by definition $c_{u,v}(X):X_u \longrightarrow X_v$ is given by

$$c_{u,v}(X) = \sum \mu_i p_i(X).$$

A representation X of Γ is said to <u>satisfy a set</u> ρ <u>of relations</u> on Γ if for every $t \in \rho$ and for every $u,v \in V(\Gamma)$, $c_{u,v}(t)(X)=0$. If (Γ,ρ) is a graph with relations, we denote by $\mathrm{rep}(\Gamma,\rho)$ the full subcategory of $\mathrm{rep}(\Gamma)$ whose objects are the representations of Γ which satisfy ρ.

For a finite graph Γ we can form the <u>path algebra</u> $K\Gamma$ as follows. As a K-vector space the set of directed paths forms a basis for $K\Gamma$. Multiplication of directed paths is composition when defined and 0 otherwise. The field K acts centrally and multiplication is extended using the distributive laws. If ρ is a set of relations on Γ and I is the ideal of $K\Gamma$ generated by ρ, then the following proposition gives the relationship between $K\Gamma/I$ and (Γ,ρ).

<u>Proposition 1.1</u>: There is an equivalence of categories

$$E:\mathrm{rep}(\Gamma) \longrightarrow \mathrm{mod}(K\Gamma)$$

which restricts to an equivalence

$$E: \mathrm{rep}(\Gamma,\rho) \longrightarrow \mathrm{mod}(K\Gamma/I).$$

For future reference we will describe the action of E on objects. If $X = (X_v, \alpha(a))$ is an object in $\mathrm{rep}(\Gamma)$, then $E(X)$ is the $K\Gamma$-module M which is equal to $\oplus\sum_v X_v$ as a K-vector space. If $m=\sum_v m_v \in M$ and $u \in V(\Gamma)$, then $u \cdot m = m_u$ and if p is a directed path of nonzero length from u_1 to u_2 then $p \cdot m = p(X)(m_{u_1})$. This multiplication makes M a $K\Gamma$-module which is finite-dimensional over K. Moreover, X satisfies ρ if and only if $IM=0$. The inverse functor $E^{-1}: \mathrm{mod}(K\Gamma) \longrightarrow \mathrm{rep}(\Gamma)$ takes a finite-dimensional $K\Gamma$-module M to a representation $X=(X_v,\alpha(a))$ of Γ where for every $v \in V(\Gamma)$, $X_v = vM$ and for each $a \in A(\Gamma)$ with origin u_1 and terminus u_2, $\alpha(a): X_{u_1} \longrightarrow X_{u_2}$ is given by $\alpha(a)(x) = a \cdot x$ where $x \in X_{u_1} = u_1 M$.

Suppose that Γ' and Γ are directed graphs and that $F: \Gamma' \longrightarrow \Gamma$ is a covering projection of graphs in the topological sense. We assume that all coverings are regular (see [8]). If p' is a directed path in Γ', then $F(p')$ is a directed path in Γ. If $t = \sum \mu_\alpha p'_\alpha$ is a K-linear combination of directed paths in Γ', we define

$F(t) = \sum\mu_\alpha F(p'_\alpha)$ which is a K-linear combination of directed paths in Γ. A <u>lifting</u> from Γ to Γ' is a function $L:V(\Gamma) \longrightarrow V(\Gamma')$ such that for every $v \in V(\Gamma)$, $L(v) \in F^{-1}(v)$. If p is a directed path in Γ from u_1 to u_2, then we let $L(p)$ denote the unique directed path in Γ' with origin $L(u_1)$ such that $F(L(p))=p$. Note that if u'_2 is the terminus of $L(p)$, then $u'_2 \in F^{-1}(u_2)$; however, u'_2 is not necessarily equal to $L(u_2)$. Finally, if $t=\sum\mu_\alpha p_\alpha$ is a K-linear combination of directed paths in Γ, then by definition $L(t) = \sum\mu_\alpha L(p_\alpha)$.

<u>Definition 1.2</u>: If (Γ',ρ') and (Γ,ρ) are graphs with relations, we say that $F:(\Gamma',\rho') \longrightarrow (\Gamma,\rho)$ is a <u>morphism of graphs with relations</u> if:

(A) $F:\Gamma' \longrightarrow \Gamma$ is a regular covering.

(B) $\rho' = \{L(t): t \in \rho$ and $L:V(\Gamma) \longrightarrow V(\Gamma')$ is a lifting$\}$.

(C) If $t \in \rho'$ and $u,v \in V(\Gamma)$, there exist $u',v' \in V(\Gamma')$ such that $F(c_{u',v'}(t)) = c_{u,v}(F(t))$.

<u>Remark</u>: Suppose p_1 and p_2 are directed paths and $\mu_1, \mu_2 \in K$ such that $\mu_1 p_1$ and $\mu_2 p_2$ are summands of t for some $t \in \rho$. Property (C) of Definition 1.2 implies that $\mu_1 p_1$ and $\mu_2 p_2$ are both summands of $c_{u,v}(t)$ for some $u,v \in V(\Gamma)$ if and only

if $\mu_1 L(p_1)$ and $\mu_2 L(p_2)$ are both summands of $c_{u',v'}(L(t))$ for some $u',v' \in V(\Gamma')$.

<u>Proposition</u> <u>1.3</u>: Let $F:(\Gamma',\rho') \longrightarrow (\Gamma,\rho)$ be a morphism of graphs with relations. Then F induces a functor $f:\mathrm{rep}(\Gamma',\rho') \longrightarrow \mathrm{rep}(\Gamma,\rho)$ which is exact and additive.

The functor f is called the <u>covering functor associated to F</u> and is defined on objects as follows. Let $X = (X_{w'},\beta(a'))_{w'\in V(\Gamma'),a'\in A(\Gamma')}$ be an object in $\mathrm{rep}(\Gamma',\rho')$. Then $f(X) = (Y_w,\alpha(a))_{w\in V(\Gamma),a\in A(\Gamma)}$ for which

$$Y_w = \coprod_{w'\in F^{-1}(w)} X_{w'} = \coprod X_{L(w)}$$

where the second sum is taken over all liftings $L:V(\Gamma) \longrightarrow V(\Gamma')$. If a is an arrow in Γ from u to v, then $\alpha(a):Y_u \longrightarrow Y_v$ is

$$\alpha(a) = \coprod\beta(L(a)):\coprod X_{L(u)} \longrightarrow \coprod X_{L(v)}$$

where each sum is taken over all liftings from $V(\Gamma)$ to $V(\Gamma')$ and $\coprod\beta(L(a))$ is defined in the obvious fashion. It follows that X satisfies ρ' if and only if f(X) satisfies ρ; however, given any set of relations ρ_1 on Γ, it is not always possible to find a set of relations ρ_1' on Γ' such that $F:(\Gamma',\rho_1') \longrightarrow (\Gamma,\rho_1)$ is a morphism of graphs with relations. In this case we would still like to know which representations X in $\mathrm{rep}(\Gamma')$ have the property that f(X) satisfies ρ_1. The next theorem answers this question.

Theorem 1.4: Let (Γ,ρ) be a finite graph with relations. Let $F:\Gamma' \longrightarrow \Gamma$ be a regular topological covering of Γ. Let $f:\mathrm{rep}(\Gamma') \longrightarrow \mathrm{rep}(\Gamma)$ be the covering functor associated to F. Then there exists a unique set of relations ρ' on Γ' and a unique set of relations $\bar{\rho}$ on Γ such that:

(1) $F:(\Gamma',\rho') \longrightarrow (\Gamma,\bar{\rho})$ is a morphism of graphs with relations.

(2) If C is the full subcategory of $\mathrm{rep}(\Gamma')$ with the property that X is an object in C if and only if $f(X)$ satisfies ρ, then $C=\mathrm{rep}(\Gamma',\rho')$.

Proof: (1) Let $L_0:V(\Gamma) \longrightarrow V(\Gamma')$ be a lifting. Let $t\in\rho$ and $u,v\in V(\Gamma)$. Write $c_{u,v}(t) = \sum_{i=1}^{n}\mu_i p_i$ where each $\mu_i \in K$ and each p_i is a directed path. Let $\{v_1',v_2',\ldots v_\ell'\}$ be the set of distinct vertices in $F^{-1}(v)$ such that $c_{L_0(u),v_j'}(L_0(c_{u,v}(t))\neq 0$; in other words, each v_j' is the terminus of $L_0(p_i)$ for at least one p_i. Define $c_{u,v}(t)_j = \sum_{i\in A_j} \mu_i p_i$ where $A_j = \{i:v_j'$ is the terminus of $L_0(p_i)\}$. Set

$$\bar{\rho}=\{s\in K\Gamma:\exists t\in\rho,\ u,v\in V(\Gamma) \text{ with } s=c_{u,v}(t)_j \text{ for some } j\}$$

and

$$\rho' = \{L(s) : s \in \bar{\rho} \text{ and } L:V(\Gamma) \longrightarrow V(\Gamma') \text{ is a lifting}\}.$$

To show that $F:(\Gamma',\rho') \longrightarrow (\Gamma,\bar{\rho})$ is a morphism of graphs with relations we need to verify (A), (B) and (C) of Definition 1.2. But (A) is true by hypothesis and (B) is true by the definitions of $\bar{\rho}$ and ρ', so it remains to prove (C). Let $r \in \rho'$ and $u,v \in V(\Gamma)$. By definition of ρ', $r=L(s)$ for some lifting $L:V(\Gamma) \longrightarrow V(\Gamma')$ and some $s \in \bar{\rho}$. By definition of $\bar{\rho}$ there exist $w,z \in V(\Gamma)$ such that $s=c_{w,z}(s)$. Furthermore, $s = \sum_{i=1}^{m} \mu_i p_i$ where each $\mu_i \in K$ and the p_i's are directed paths with the property that the terminus of $L_0(p_i)$ is the same for each i. Since $F:\Gamma' \longrightarrow \Gamma$ is a regular covering, the terminus of $L(p_i)$ must also be the same for each i. Call this terminus z'. Of course, the origin of $L(p_i)$ is also the same for each i, namely $L(w)$, which we will denote by w'. With this notation $r = c_{w',z'}(r)$. Now look at $c_{u,v}(s)$ which is nonzero if and only if $u=w$ and $v=z$. If $c_{u,v}(s)=0$, let u' be a vertex in $V(\Gamma')$ such that $u' \neq w'$ (the case where $V(\Gamma')$ contains only one element being trivial) and let $v' \in V(\Gamma)$ be any vertex. Then $c_{u',v'}(r)=0$, hence $F(c_{u',v'}(r))=F(0)=c_{u,v}(s)=c_{u,v}(F(r))$. If $c_{u,v}(s) \neq 0$, then $u=w$ and $v=z$. In this case let $u'=w'$ and $v'=z'$. Then

$$
\begin{aligned}
F(c_{u',v'}(r)) &= F(c_{w',z'}(r)) \\
&= F(r) \\
&= s \\
&= c_{w,z}(s) \\
&= c_{u,v}(s) \\
&= c_{u,v}(F(r)).
\end{aligned}
$$

So $F:(\Gamma',\rho') \longrightarrow (\Gamma,\bar{\rho})$ is a morphism of graphs with relations.

(2) Let $X=(X_{w'},\beta(a'))$ be an object in $\mathrm{rep}(\Gamma')$ such that $f(X)$ satisfies ρ. We need to show that X satisfies ρ'. Let $r\in\rho'$ and let $u',v'\in V(\Gamma')$ be the vertices for which $c_{u',v'}(r)=r$. By definition of ρ' there exists $s\in\bar{\rho}$ and a lifting $L_1:V(\Gamma) \longrightarrow V(\Gamma')$ such that $L_1(s)=r$. If $u = F(u')$ and $v=F(v')$ then $c_{u,v}(s)=s$. Write $s = \sum_{i=1}^{m}\mu_i p_i$ where for each i, $\mu_i\in K$ and p_i is a directed path in Γ. By definition of $\bar{\rho}$, there is a $t\in\rho$ with $t = \sum_{i=1}^{n}\mu_i p_i$ for some $n\geqslant m$. For each i, $i=1,\ldots,n$, let v_i' denote the terminus of $L_1(p_i)$. Then $v_i'=v'$ for $1\leqslant i\leqslant m$ and $v_i'\neq v'$ for $m<i\leqslant n$. Let $f(X)=(Y_w,\alpha(a))$. Let x be an element in $Y_u = \coprod X_{L(u)}$ such that when x is written as a tuple, $x=(x_{L(u)})$, then $x_{L(u)}=0$ for $L\neq L_1$. Since $f(X)$ satisfies ρ, $t(f(X))=0$. In

particular,

$$0=t(f(X))(x) = \sum_{i=1}^{n} \mu_i p_i(f(X))(x).$$

Consider $p_i(f(X))(x) \in Y_v = \coprod X_{L(v)}$. If $p_i(f(X))(x)$ is written as a tuple, $(z_{L(v)})$, then by definition of x,

$$z_{v_i'} = L_1(p_i)(X)(x_{L(u)})$$

and

$$z_{L(v)}=0, \text{ if } L(v) \neq v_i'.$$

Therefore, if $y=(y_{L(v)}) = \sum_{i=1}^{n} \mu_i p_i(f(X))(x)$, then

$$y_{v'} = \sum_{i=1}^{m} \mu_i L_1(p_i)(X)(x_{L(u)}) = r(X)(x_{L(u)})$$

But $y=0$, so $y_{v'}=0$ and since $x_{L(u)}$ was an arbitrary element in $X_{L(u)}$, $r(X)=0$, hence X satisfies ρ' as desired.

Conversely, if X is an object in $\operatorname{rep}(\Gamma',\rho')$, then since $F:(\Gamma',\rho') \longrightarrow (\Gamma,\bar{\rho})$ is a morphism of graphs with relations, $f(X)$ satisfies $\bar{\rho}$. But each relation in ρ is a sum of relations in $\bar{\rho}$. Thus $f(X)$ also satisfies ρ. Therefore $C=\operatorname{rep}(\Gamma',\rho')$.

The uniqueness of $\bar{\rho}$ follows from the remark after Definition 1.2. The uniqueness of ρ' is then clear by property (B) of Definition 1.2.

In order to apply Theorem 1.4 we need some results from [5]. But first we need more terminology.

Let (Γ,ρ) be a finite graph with relations. A weight function on Γ is a function $W{:}A(\Gamma) \longrightarrow G$ where G is a group. For a path $p=a_n^{e_n}\ldots a_1^{e_1}$ where each $a_i \in A(\Gamma)$ and $e_i=\pm 1$, we let $W(p)=W(an)^{e_n}\ldots W(a_1)^{e_1}$. A weight function $W{:}A(\Gamma) \longrightarrow G$ is called <u>connected</u> if for each pair of vertices $u,v \in V(\Gamma)$ there is a path p in Γ from u to v with $W(p)=g$.

Let G be a group and Λ a K-algebra. A <u>G-grading on Λ</u> is given by $\Lambda = \coprod_{g\in G} \Lambda_g$ where each Λ_g is a K-vector space and if $g,h \in G$, $\Lambda_g\Lambda_h \subseteq \Lambda_{gh}$. Now consider the path algebra $K\Gamma$. If $W{:}A(\Gamma) \longrightarrow G$ is a weight function, then W induces a G-grading on $K\Gamma$ given by letting $K\Gamma_g$ be the set of all K-linear combinations of directed paths of weight g. If I is a homogeneous ideal in $K\Gamma$, then $\Lambda=K\Gamma/I$ can be given a G-grading induced by the G-grading on $K\Gamma$. This grading on Λ is called the <u>G-grading induced by W</u>.

If $\Lambda = \coprod_{g\in G} \Lambda_g$ is a G-grading on a K-algebra Λ, then a Λ-module M is called a <u>graded</u> Λ-module if $M=\coprod_{g\in G} M_g$ where each M_g is a K-vector space and for $g,h \in G$, $\Lambda_g M_h \subseteq M_{gh}$. A Λ-module map $f{:}\coprod_{g\in G} M_g \longrightarrow \coprod_{g\in G} N_g$ between graded modules is called a <u>degree e map</u> if $f(M_g) \subseteq N_g$ for all $g \in G$. Let $gr_G(\Lambda)$ denote the category whose objects are the finite-dimensional graded Λ-modules and whose morphisms are the degree e maps. Let $\phi:gr_G(\Lambda) \longrightarrow mod(\Lambda)$ denote the forgetful functor. We have the following theorem from [5].

<u>Theorem</u> <u>1.5</u>: Let (Γ,ρ) be a finite, directed, path connected graph Γ with a set of relations ρ. Let $F:(\Gamma',\rho') \longrightarrow (\Gamma,\rho)$ be a morphism of graphs with relations and let $f:rep(\Gamma',\rho') \longrightarrow rep(\Gamma,\rho)$ be the covering functor associated to F. Let G be the automorphism group of covering $F:\Gamma' \longrightarrow \Gamma$. Then:

(1) There is a connected G-grading on $K\Gamma$ induced by a weight function W such that the ideal I generated by ρ is a homogeneous ideal.

(2) If $\Lambda=K\Gamma/I$ is given the G-grading induced by W, then there is an equivalence of categories $E':rep(\Gamma',\rho') \longrightarrow gr_G(\Lambda)$ such that the following diagram

$$\begin{tikzcd} rep(\Gamma',\rho') \arrow[r, "E'"] \arrow[d, "f"'] & gr_G(\Lambda) \arrow[d, "\phi"] \\ rep(\Gamma,\rho) \arrow[r, "E"] & mod(\Lambda) \end{tikzcd}$$

commutes, where E is the equivalence in Proposition 1.1.

<u>Remark</u>: The weight function, W, of Theorem 1.5 is given as follows. Fix a lifting $L:V(\Gamma) \longrightarrow V(\Gamma')$. The automorphism group G acts on $V(\Gamma')$ and we denote this action by v^g for

$v \in V(\Gamma')$ and $g \in G$. Let $a \in A(\Gamma)$ have origin u and terminus v. Then $W(a)=g$ if $L(a)$ has terminus $L(v)^g$. The function W induces a G-grading on Λ such that $gr_G(\Lambda)$ is equivalent to $rep(\Gamma',\rho')$ regardless of the lifting used to define W.

There is a further result from [5] we will need. Namely, if $\Lambda = \amalg \Lambda_g'$, is another connected G-grading of Λ induced by a weight function $W_1: A(\Gamma) \longrightarrow G$, then there is a lifting $L_1: V(\Gamma) \longrightarrow V(\Gamma')$ such that the G-grading induced by W_1 is the same as the G-grading induced by W provided that the lifting L_1 is used to define W.

Using Theorem 1.5 we obtain the following obvious corollary to Theorem 1.4.

<u>Corollary</u> <u>1.6</u>: Using the notation of Theorem 1.4, let $\bar{I}$ be the ideal of $K\Gamma$ generated by $\bar{\rho}$. Let $\bar{\Lambda}=K\Gamma/\bar{I}$. Note that if I is the ideal of $K\Gamma$ generated by ρ, then $I \subseteq \bar{I}$ so $K\Gamma/\bar{I}$ is a factor ring of $K\Gamma/I$. Let G be the automorphism group of the covering $F:\Gamma' \longrightarrow \Gamma$. Then there is a G-grading induced by a weight function such that the category $\mathcal{C}$ is equivalent to $gr_G(\bar{\Lambda})$.

<u>Proof</u>: By Theorem 1.4, $\mathcal{C}=rep(\Gamma',\rho')$ and $F:(\Gamma',\rho') \longrightarrow (\Gamma,\bar{\rho})$ is a morphism of graphs with relations. By Theorem 1.5

$rep(\Gamma',\rho')$ is equivalent to $gr_G(\bar{\Lambda})$.

For a graph with relations (Γ,ρ) we define:

$\rho_0=\{p\in K\Gamma: p$ is a directed path and there exists a K-linear combination of directed paths $t=\sum\mu_i p_i\in\rho$ with $p=p_i$ for some $i\}$.

The set ρ_0 is called the <u>set of zero relations associated to ρ</u>.

Now suppose Γ_0' is the universal cover of Γ in the topological sense (see [8]) with covering projection $F_0:\Gamma_0' \longrightarrow \Gamma$. Then Γ_0' is a locally finite, path connected, directed tree. Let $\mathcal{C}_0$ denote the full subcategory of $rep(\Gamma_0')$ whose objects are the representations X for which $f_0(X)$ satisfies ρ, where $f_0:rep(\Gamma_0') \longrightarrow rep(\Gamma)$ is the covering functor associated to F_0. Let I be the ideal in $K\Gamma$ generated by ρ and let $E:rep(\Gamma,\rho) \longrightarrow mod(K\Gamma/I)$ be the equivalence of Proposition 1.1. The subcategory

$$\mathcal{Q}=Ef_0(\mathcal{C}_0)$$

of mod $(K\Gamma/I)$ is called the <u>category of unwindable modules over $K\Gamma/I$</u>. This gives us another corollary to Theorem 1.4.

<u>Corollary 1.7</u>: Let (Γ,ρ) be a finite graph with relations and let $\Lambda=K\Gamma/\langle\rho\rangle$. Let $F_0:\Gamma_0' \longrightarrow \Gamma$ be the universal cover of

Γ and let G be the automorphism group of the covering F_0. Then there is a G-grading of Λ_0 such that the category of unwindable modules over Λ is equivalent to the category of finite-dimensional graded modules over Λ_0.

<u>Proof</u>: Apply Theorem 1.4 with $\Gamma' = \Gamma_0'$. In this case the set $\bar{\rho}$ is just ρ_0, so $F_0 : \mathrm{rep}(\Gamma_0', \rho') \to \mathrm{rep}(\Gamma, \rho_0)$ is a morphism of graphs with relations. By Theorem 1.5 there is a G-grading on Λ_0 and an equivalence $E' : \mathrm{rep}(\Gamma_0', \rho') \longrightarrow \mathrm{gr}_G(\Lambda_0)$ such that the following diagram

$$\begin{tikzcd} \mathrm{rep}(\Gamma_0', \rho') \arrow[r, "E'"] \arrow[d, "f_0"'] & \mathrm{gr}_G(\Lambda_0) \arrow[d, "\phi"] \\ \mathrm{rep}(\Gamma, \rho_0) \arrow[r, "E_0"] & \mathrm{mod}(\Lambda_0) \end{tikzcd}$$

commutes where ϕ is the forgetful functor and E_0 is the restriction of $E : \mathrm{rep}(\Gamma, \rho) \longrightarrow \mathrm{mod}(\Lambda)$. Consequently, if $\mathcal{a}$ is the category of unwindable Λ-modules, then

$$\begin{aligned} \mathcal{a} &= Ef_0(\mathrm{rep}(\Gamma_0', \rho')) \\ &= \phi E'(\mathrm{rep}(\Gamma_0', \rho')) \\ &= \phi(\mathrm{gr}_G(\Lambda_0)). \end{aligned}$$

Thus $\mathcal{a}$ is equivalent to $\mathrm{gr}_G(\Lambda_0)$.

We show now that if Γ is a finite graph and I is an ideal of $K\Gamma$ which is generated by a set of relations, then the category of unwindable modules over $K\Gamma/I$ does not depend on the choice of this generating set.

<u>Proposition</u> <u>1.8</u>: Let (Γ,ρ) be a finite graph with relations and let I be the ideal of $K\Gamma$ generated by ρ. Suppose that σ is a set of relations on Γ which also generates I. Let ρ_0 and σ_0 be the sets of zero relations associated to ρ and σ respectively. Then $\langle\rho_0\rangle=\langle\sigma_0\rangle$.

<u>Proof</u>: Suppose $p\in\rho_0$. Then there is an element $t\in\rho$ such that $t=\sum_{k=1}^{N}\mu_k p_k$ where for each k, $\mu_k\in K$, p_k is a directed path and $p=p_1$. Since σ generates I,

$$t=\sum_{i=1}^{n}x_i s_i y_i$$

where for each i, $s_i\in\sigma$ and $x_i,y_i\in K\Gamma$. Moreover, every s_i can be written as a k-linear combination of directed paths

$$s_i=\sum_{j=1}^{n_i}\eta_{ij}q_{ij}$$

where for each i,j, $\eta_{ij}\in K$ and $q_{ij}\in\sigma_0$. Also, for each i, we can write x_i and y_i as k-linear combinations of paths

$$x_i = \sum_{\ell=1}^{m_i} \beta_{i\ell}\tilde{p}_{i\ell} \text{ and } y_i=\sum_{r=1}^{m_i'}\gamma_{ir}\tilde{q}_{ir}.$$

Thus

$$t = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \sum_{\ell=1}^{m_i} \sum_{r=1}^{m_i'} \beta_{i\ell}\eta_{ij}\gamma_{ir}\tilde{p}_{i\ell}q_{ij}\tilde{q}_{ir}$$

and since the directed paths form a K-basis for $K\Gamma$, there must exist i_0, ℓ_0, j_0, r_0 such that $p=p_1=\tilde{p}_{i_0\ell_0}q_{i_0j_0}\tilde{q}_{i_0r_0}$. But $q_{i_0j_0}\epsilon\sigma_0$, so $p\epsilon\langle\sigma_0\rangle$, and hence $\langle\rho_0\rangle\subseteq\langle\sigma_0\rangle$. Switching the roles of ρ and σ, an analogous proof gives that $\langle\sigma_0\rangle\subseteq\langle\rho_0\rangle$ and therefore the equality $\langle\rho_0\rangle=\langle\sigma_0\rangle$.

A more difficult question is the following. Suppose Γ is a finite graph with two sets of relations, ρ and σ. Let I and J be the ideals of $K\Gamma$ generated by ρ and σ respectively. Assume $K\Gamma/I\cong K\Gamma/J$. Under what conditions is the category of unwindable modules over $K\Gamma/I$ equivalent to the category of unwindable modules over $K\Gamma/J$? We proceed to answer this question.

We now consider a directed graph Γ with the property that if $a\epsilon A(\Gamma)$ has origin u and terminus v, then there is no directed path $p\neq a$ in Γ with origin u and terminus v. Note that this property implies that Γ has no oriented cycles. We call such a graph an <u>arrow unique graph</u>.

Given an arrow unique graph Γ, let $K\Gamma^+$ denote the ideal of $K\Gamma$ generated by $A(\Gamma)$. Since Γ has no oriented cycles, $(K\Gamma^+)^n=0$ where n-1 is the length of the longest directed path in Γ. In this case $K\Gamma$ is a finite-dimensional K-algebra with Jacobson radical $K\Gamma^+$. Note that $t \in K\Gamma$ is in $K\Gamma^+$ if and only if $t^n=0$, because the vertices are idempotents in $K\Gamma$.

For the following three lemmas we will assume that Γ is a finite arrow unique graph, I and J are ideals of $K\Gamma$ which are contained in $(K\Gamma^+)^2$ and $\varphi: K\Gamma/I \longrightarrow K\Gamma/J$ is a K-algebra isomorphism.

<u>Lemma</u> <u>1.9</u>: If $v \in V(\Gamma)$, then there is a unique $\tilde{v} \in V(\Gamma)$ such that $\varphi(v+I)=\tilde{v}+x+J$ where $x \in K\Gamma^+$.

<u>Proof</u>: Since $v^n=v$ for all natural numbers n, $\varphi(v+I) \notin K\Gamma^+/J$. Thus

$$\varphi(v+I) = \sum_{i=1}^{m} \mu_i v_i + x + J$$

where $x \in K\Gamma^+$ and for $i=1,\ldots,m$, $0 \neq \mu_i \in K$ and the v_i are distinct vertices in Γ. Now

$$\begin{aligned}\sum_{i=1}^{m} \mu_i v_i + x + J &= \varphi(v+I) \\ &= (\varphi(v+I))^2 \\ &= (\sum_{i=1}^{m} \mu_i v_i + x)^2 + J \\ &= \sum_{i=1}^{m} \mu_i^2 v_i + y + J \quad \text{where } y \in K\Gamma^+.\end{aligned}$$

Consequently, $\sum_{i=1}^{m} (\mu_i - \mu_i^2) v_i + x - y \in J$. Since $x-y \in K\Gamma^+$ and

$v_i \notin K\Gamma^+$ for any i, $\mu_i - \mu_i^2 = 0$ and hence $\mu_i = 1$ for all i. Therefore $\varphi(v+I) = \sum_{i=1}^{m} v_i + x + J$. Let $\{w_1, \ldots, w_r\}$ be a complete set of distinct vertices in Γ. Then for each j, $j=1,\ldots,r$, $\varphi(w_j+I) = \sum_{i=1}^{m_j} w_{ji} + x_j + J$, where the $w_{ji} \in V(\Gamma)$ and $x_j \in K\Gamma^+$. Suppose $u \in V(\Gamma)$. Then, since

$$1+J=\varphi(1+I)=\varphi(w_1+\ldots+w_r+I) = \sum_{j=1}^{r}\left(\sum_{i=1}^{m_j} w_{ji}+x_j\right)+J,$$

$\sum_{j=1}^{r}\sum_{i=1}^{m_j} w_{ji}=1$ and hence $u=w_{ji}$ for some j and i. Also, since $w_j w_\ell = 0$ if $j \neq \ell$, u cannot equal $w_{\ell s}$ for any s, $1 \leq s \leq m_\ell$ and $\ell \neq j$. Thus each vertex in $V(\Gamma)$ appears once and only once as a w_{ji}. Therefore, given $v \in V(\Gamma)$ there is a unique $\tilde{v} \in V(\Gamma)$ such that $\varphi(v+I) = \tilde{v}+x+J$ with $x \in K\Gamma^+$.

<u>Lemma</u> <u>1.10</u>: If $a \in A(\Gamma)$, then there exists a unique $\tilde{a} \in A(\Gamma)$ such that $\varphi(a+I) = \eta\tilde{a}+x+J$ where $\eta \in K$ and $x \in (K\Gamma^+)^2$.

<u>Proof</u>: Write $\varphi(a+I) = \sum_{i=1}^{m} \eta_i p_i + J$ where $\eta_i \in K$ and the p_i are distinct directed paths in Γ. Since $a^2=0$, each p_i has length at least 1. Let u be the origin and v the terminus of a. By Lemma 1.9, there exist $\tilde{u}, \tilde{v} \in V(\Gamma)$ and $y, z \in K\Gamma^+$ such that $\varphi(u+I) = \tilde{u}+y+J$ and $\varphi(v+I) = \tilde{v}+z+J$. Therefore,

$$\sum_{i=1}^{m}\eta_i p_i+J = \varphi(a+I)$$

$$= \varphi(vau+I)$$

$$= (\tilde{v}+z)\ (\sum_{i=1}^{m}\eta_i p_i)(\tilde{u}+y)+J.$$

Hence,

$$t = \sum_{i=1}^{m}\eta_i p_i-\sum_{i=1}^{m}\eta_i\tilde{v}p_i\tilde{u}-\sum_{i=1}^{m}\eta_i zp_i\tilde{u}-\sum_{i=1}^{m}\eta_i\tilde{v}p_i y-\sum_{i=1}^{m}\eta_i zp_i y \in J.$$

Now $\sum_{i=1}^{m}\eta_i p_i \notin (K\Gamma^+)^2$ since $a\notin(K\Gamma^+)^2$, so at least one p_i is an arrow, say p_1. Since $J\subseteq(K\Gamma^+)^2$, $\eta_1 p_1\notin J$ and must therefore cancel with another term in t. Since $y,z\in K\Gamma^+$, it must be the case that $\eta_1 p_1=\eta_1\tilde{v}p_1\tilde{u}$; that is, p_1 is an arrow with origin $\tilde{u}$ and terminus $\tilde{v}$. For $i=2,\ldots,m$, p_i must either cancel with another term in t or $p_i\in J$. In either case, $p_i\in(K\Gamma^+)^2$. So letting $\tilde{a}=p_1$, and $\eta=\eta_1$ we have that $\varphi(a+I)=\eta\tilde{a}+x+J$ where $x\in(K\Gamma^+)^2$.

The uniqueness of $\tilde{a}$ comes from the fact that Γ is an arrow unique graph, for if $\varphi(a+I)=\lambda\tilde{b}+x_1+J$ where $\lambda\in K$, $x_1\in K\Gamma^+$ and $\tilde{b}\in A(\Gamma)$, then $\tilde{b}$ has to have origin $\tilde{u}$ and terminus $\tilde{v}$; that is, $\tilde{b}=\tilde{a}$, the unique arrow in Γ with origin $\tilde{u}$ and terminus $\tilde{v}$.

<u>Corollary</u> <u>1.11</u>: Assume the hypotheses of Lemma 1.10. Let p be a directed path in Γ of length $n>0$. Then there is directed path $\tilde{p}$ of length n in Γ such that $\varphi(p+I)=\alpha\tilde{p}+s+J$ where $\alpha \in K$ and $s\in(K\Gamma^{+})^{n+1}$.

<u>Proof</u>: Write $p=a_n\ldots a_2a_1$ where $a_i\in A(\Gamma)$. From Lemma 1.10 we have that for every i there exists a unique $\tilde{a}_i\in A(\Gamma)$ such that $\varphi(a_i+I)=\eta_i\tilde{a}_i+x_i+J$ where $\eta_i\in K$ and $x_i\in(K\Gamma^{+})^2$. Hence

$$\begin{aligned}\varphi(p+I)&=\varphi(a_n\ldots a_1+I)\\&=(\eta_n\tilde{a}_n+x_n)\ldots(\eta_1\tilde{a}_1+x_1)+J\\&=\alpha\tilde{p}+s+J\end{aligned}$$

where $\alpha=\eta_n\ldots\eta_1\in K$, $\tilde{p}=\tilde{a}_n\ldots\tilde{a}_1$ is a directed path Γ with length n and $s\in(K\Gamma^{+})^{n+1}$.

<u>Lemma</u> <u>1.12</u>: Suppose $a\in A(\Gamma)$ has origin u and terminus v. Using Lemma 1.9, write $\varphi(u+I)=\tilde{u}+\mathcal{X}+\mathcal{Y}+\mathcal{Z}+J$ where $\tilde{u}\in V(\Gamma)$ and $\mathcal{X},\mathcal{Y}$ and $\mathcal{Z}$ are linear combinations of distinct directed paths of length at least 1 with the following properties:

$$\begin{array}{ll}\mathcal{X}\tilde{u}=\mathcal{X} & \tilde{u}\mathcal{X}=0\\ \mathcal{Y}\tilde{u}=0 & \tilde{u}\mathcal{Y}=\mathcal{Y}\\ \mathcal{Z}\tilde{u}=0 & \tilde{u}\mathcal{Z}=0.\end{array}$$

Similarly, let $\varphi(v+I)=\tilde{v}+\mathcal{X}^{*}+\mathcal{Y}^{*}+\mathcal{Z}^{*}+J$ where $v\in V(\Gamma)$ and $\mathcal{X}^{*},\mathcal{Y}^{*}$ and $\mathcal{Z}^{*}$ are linear combinations of distinct directed paths of length at least 1 such that

$$\mathfrak{x}^*\tilde{v}=\mathfrak{x}^* \qquad \tilde{v}\mathfrak{x}^*=0$$

$$\mathfrak{y}^*\tilde{v}=0 \qquad \tilde{v}\mathfrak{y}^*=\mathfrak{y}^*$$

$$\mathfrak{z}^*\tilde{v}=0 \qquad \tilde{v}\mathfrak{z}^*=0.$$

Finally, using Lemma 1.10, let $\varphi(a+I)=\eta\tilde{a}+\mathcal{P}+\mathcal{Q}+\mathcal{S}+J$

where $\tilde{a}\in A(\Gamma)$, $\eta\in K$ and $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{S}$ are linear combinations of directed paths of length at least 2 such that

$$\mathcal{P}\tilde{u}=\mathcal{P} \qquad \tilde{v}\mathcal{P}=0$$

$$\mathcal{Q}\tilde{u}=0 \qquad \tilde{v}\mathcal{Q}=\mathcal{Q}$$

$$\mathcal{S}\tilde{u}=0 \qquad \tilde{v}\mathcal{S}=0.$$

Then the following elements of $K\Gamma$ are also elements of J:

(1) $\mathcal{Q}\mathfrak{x}$

(2) $\mathcal{S}\mathfrak{x}$

(3) $\mathcal{Q}-\eta\tilde{a}\mathfrak{y}$

(4) $\mathcal{P}-\eta\mathfrak{x}^*\tilde{a}$

(5) $\mathfrak{y}^*\mathcal{S}$.

<u>Proof</u>: $\tilde{u}+\mathfrak{x}+\mathfrak{y}+\mathfrak{z}+J = \varphi(u+I)$

$$= [\varphi(u+I)]^2$$

$$= (\tilde{u}+\mathfrak{x}+\mathfrak{y}+\mathfrak{z})(\tilde{u}+\mathfrak{x}+\mathfrak{y}+\mathfrak{z})+J$$

$$= \tilde{u}+\mathfrak{y}+\mathfrak{x}+\mathfrak{x}\mathfrak{y}+\mathfrak{y}\mathfrak{x}+\mathfrak{y}\mathfrak{z}+\mathfrak{z}\mathfrak{x}+\mathfrak{z}^2+J$$

hence, $\mathfrak{z}-\mathfrak{x}\mathfrak{y}-\mathfrak{y}\mathfrak{x}-\mathfrak{y}\mathfrak{z}-\mathfrak{z}\mathfrak{x}-\mathfrak{z}^2\in J$. Multiplying on the left by $\tilde{u}$ gives $\mathfrak{z}-\mathfrak{x}\mathfrak{y}-\mathfrak{z}\mathfrak{x}-\mathfrak{z}^2\in J$. Multiplying now on the right by $\tilde{u}$ gives $\mathfrak{z}-\mathfrak{x}\mathfrak{y}-\mathfrak{z}^2\in J$, and $\mathfrak{z}\mathfrak{x}\in J$. Therefore $\mathfrak{z}(\mathfrak{z}-\mathfrak{x}\mathfrak{y}-\mathfrak{z}^2)\in J$ and hence $\mathfrak{z}^2-\mathfrak{z}^3\in J$. Let ℓ be the smallest nonnegative integer such that $\mathfrak{z}^\ell\in J$. Suppose $\ell\geqslant 3$. Then

$$\mathfrak{z}^{\ell-3}(\mathfrak{z}^2-\mathfrak{z}^3)=\mathfrak{z}^{\ell-1}-\mathfrak{z}^\ell\in J$$

and thus $\mathfrak{z}^{\ell-1}\in J$, a contradiction. Therefore $\ell\leqslant 2$, $\mathfrak{z}^2\in J$ and hence $\mathfrak{z}-\mathfrak{x}\mathfrak{y}\in J$. A completely analogous computation gives us that $\mathfrak{z}^*-\mathfrak{x}^*\mathfrak{y}^*\in J$.

$$\begin{aligned}\text{Now } \eta\tilde{a}+\mathfrak{p}+\mathfrak{q}+\mathfrak{s} &= \varphi(a+I)\\ &= \varphi(a+I)\varphi(u+I)\\ &= (\eta\tilde{a}+\mathfrak{p}+\mathfrak{q}+\mathfrak{s})(\tilde{u}+\mathfrak{x}+\mathfrak{y}+\mathfrak{z})+J\\ &= \eta\tilde{a}+\eta\tilde{a}\mathfrak{y}+\mathfrak{p}+\mathfrak{p}\mathfrak{y}+\mathfrak{q}\mathfrak{x}+\mathfrak{q}\mathfrak{z}+\mathfrak{s}\mathfrak{x}+\mathfrak{s}\mathfrak{z}+J\end{aligned}$$

thus $\mathfrak{q}+\mathfrak{s}-\eta\tilde{a}\mathfrak{y}-\mathfrak{p}\mathfrak{y}-\mathfrak{q}\mathfrak{x}-\mathfrak{q}\mathfrak{z}-\mathfrak{s}\mathfrak{x}-\mathfrak{s}\mathfrak{z}\in J$. Multiplying on the left by $\tilde{v}$ gives $\mathfrak{q}-\eta\tilde{a}\mathfrak{y}-\mathfrak{q}\mathfrak{x}-\mathfrak{q}\mathfrak{z}\in J$ and $\mathfrak{s}-\mathfrak{p}\mathfrak{y}-\mathfrak{s}\mathfrak{x}-\mathfrak{s}\mathfrak{z}\in J$. Multiplying these on the right by $\tilde{u}$ gives $\mathfrak{q}\mathfrak{x}\in J$, $\mathfrak{q}-\eta\tilde{a}\mathfrak{y}-\mathfrak{q}\mathfrak{z}\in J$, $\mathfrak{s}\mathfrak{x}\in J$ and $\mathfrak{s}-\mathfrak{p}\mathfrak{y}-\mathfrak{s}\mathfrak{z}\in J$. In particular (1) and (2) are elements of J. Also, since $\mathfrak{z}-\mathfrak{x}\mathfrak{y}\in J$, $\mathfrak{q}-\eta\tilde{a}\mathfrak{y}-\mathfrak{q}\mathfrak{x}\mathfrak{y}\in J$, then $\mathfrak{q}-\eta\tilde{a}\mathfrak{y}\in J$. A similar calculation using $\tilde{v}$ gives that $\mathfrak{p}-\eta\mathfrak{x}^*\tilde{a}\in J$ and $\mathfrak{y}^*\mathfrak{s}\in J$.

<u>Theorem</u> <u>1.13</u>: Suppose Γ is a finite, arrow unique graph. Let ρ and σ be two sets of relations on Γ and let ρ_0 and σ_0 be the sets of zero relations associated to ρ and σ respectively. In $K\Gamma$ let $I=\langle\rho\rangle$, $J=\langle\sigma\rangle$, $I_0=\langle\rho_0\rangle$ and $J_0=\langle\sigma_0\rangle$. If $K\Gamma/I\cong K\Gamma/J$ as K-algebras, then $K\Gamma/I_0\cong K\Gamma/J_0$ as K-algebras.

<u>Proof</u>: Let $\varphi: K\Gamma/I \longrightarrow K\Gamma/J$ be a K-algebra isomorphism. From Lemma 1.9 and Lemma 1.10 we know that for every $v \in V(\Gamma)$ and for every $a \in A(\Gamma)$ there is a unique $\tilde{v} \in V(\Gamma)$ and a unique $\tilde{a} \in A(\Gamma)$ such that $\varphi(v+I)=\tilde{v}+x+J$ with $x \in K\Gamma^{+}$ and $\varphi(a+I)=\eta\tilde{a}+y+J$ where $\eta \in K$ and $y \in (K\Gamma^{+})^{2}$. Define $\theta: K\Gamma/I_0 \longrightarrow K\Gamma/J_0$ on vertices and arrows by

$$\theta(v+I_0)=\tilde{v}+J_0 \text{ for all } v \in V(\Gamma)$$

$$\theta(a+I_0)=\tilde{a}+J_0 \text{ for all } a \in A(\Gamma).$$

Extend θ to a K-algebra map from $K\Gamma/I_0$ to $K\Gamma/J_0$. We claim that θ is an isomorphism.

We need to show that θ is well-defined. Suppose $r \in \rho_0$. Then there is a linear combination $\sum_{j=1}^{m} \mu_j r_j \in \rho$ such that for each j, $0 \neq \mu_j \in K$, the r_j are distinct directed paths and $r=r_1$. For each j, let ℓ_j denote the length of r_j. Then by Corollary 1.11, $\varphi(r_j+I)=\alpha_j\tilde{r}_j+t_j+J$ where $\alpha_j \in K$, $\tilde{r}_j$ is a directed path of length ℓ_j and $t_j \in (K\Gamma^{+})^{\ell_j+1}$. Moreover, $\theta(r_j+I_0)=\tilde{r}_j+J_0$. We want to show that $\tilde{r}_1 \in J_0$. Since $\sum_{j=1}^{m} \mu_j r_j \in I$, $0=\varphi(\sum_{j=1}^{m} \mu_j r_j+I) = \sum_{j=1}^{m} \alpha_j\mu_j\tilde{r}_j + \sum_{j=1}^{m} \mu_j t_j+J$, thus $\sum_{j=1}^{m} \alpha_j\mu_j\tilde{r}_j + \sum_{j=1}^{m} \mu_j t_j \in J$. This would imply by Proposition 1.8

that $\tilde{r}_1 \in J_0$ and we would be done, unless a cancellation occurs because $\alpha_1\mu_1\tilde{r}_1$ is the negative of another term in the sum above. Assume that this is still the case when all other possible cancellations have been made. Because of the uniqueness assertion of Lemma 1.10, $\tilde{r}_1 \neq \tilde{r}_j$ for $j \neq 1$. Therefore, there must be a directed path $h=\tilde{r}_1$ such that ζh is a summand of some t_{j_0} for some $\zeta \in K$. Let $r_{j_0}=a_n \ldots a_1$ where each $a_i \in A(\Gamma)$. Let u_i denote the origin and v_i the terminus of a_i for $i=1,\ldots,n$. Thus $u_{i+1}=v_i$ for $i=1,\ldots,n-1$. For each i write

$$\varphi(u_i+I)=\tilde{u}_i+\mathscr{X}_i+\mathscr{Y}_i+\mathscr{Z}_i$$

$$\varphi(v_i+I)=\tilde{v}_i+\mathscr{X}_i^*+\mathscr{Y}_i^*+\mathscr{Z}_i^*$$

$$\varphi(a_i+I)=\eta_i\tilde{a}_i+\mathscr{P}_i+\mathscr{Q}_i+\mathscr{S}_i$$

where $\tilde{u}_i,\mathscr{X}_i,\mathscr{Y}_i,\mathscr{Z}_i,\tilde{v}_i,\mathscr{X}_i^*,\mathscr{Y}_i^*,\mathscr{Z}_i^*,\eta_i,\tilde{a}_i,\mathscr{P}_i,\mathscr{Q}_i$, and $\mathscr{S}_i$ have the same properties as $\tilde{u},\mathscr{X},\mathscr{Y},\mathscr{Z},\tilde{v},\mathscr{X}^*,\mathscr{Y}^*,\mathscr{Z}^*,\eta,\tilde{a},\mathscr{P},\mathscr{Q}$, and $\mathscr{S}$, respectively, in Lemma 1.12. Note that for $i=1,2,\ldots,n-1$, $\mathscr{X}_i^*=\mathscr{X}_{i+1}$, $\mathscr{Y}_i^*=\mathscr{Y}_{i+1}$ and $\mathscr{Z}_i^*=\mathscr{Z}_{i+1}$. Since

$$\varphi(r_{i_0}+I)=\varphi(a_n+I)\ldots\varphi(a_1+I)$$

$$= (\eta_n\tilde{a}_n+\mathscr{P}_n+\mathscr{Q}_n+\mathscr{S}_n)\ldots(\eta_1\tilde{a}_1+\mathscr{P}_1+\mathscr{Q}_1+\mathscr{S}_1)+J,$$

ζh is a product $\beta_n h_n \ldots \beta_1 h_1$ where for every i, $\beta_i \in K$ and h_i is a directed path such that one of the following is true:

(1) $\beta_i h_i = \eta_i \tilde{a}_i$

(2) $\beta_i h_i$ is a summand of $\mathcal{P}_i$

(3) $\beta_i h_i$ is a summand of $\mathcal{Q}_i$

(4) $\beta_i h_i$ is a summand of $\mathcal{S}_i$.

Moreover, since h is a summand of t_{i_0}, there is some ℓ, $1 \leqslant \ell \leqslant n$ such that $h_\ell \neq \tilde{a}_\ell$.

Case 1: Suppose $\beta_\ell h_\ell$ is a summand of $\mathcal{P}_\ell$, and assume that ℓ is maximal with this property. Now $\ell \neq n$ because $h = \tilde{r}_1$ implies that the terminus of h_n is $\tilde{v}_n$, but $\tilde{v}_n \mathcal{P}_n = 0$. Thus $\ell \leqslant n-1$. Therefore, by Lemma 1.12, $\mathcal{P}_\ell - \eta_\ell \mathcal{X}_\ell^* a_\ell = \mathcal{P}_\ell - \eta_\ell \mathcal{X}_{\ell+1} \tilde{a}_\ell \in J$. Consequently, either $h_\ell \in J_0$ and we are done, or there is a cancellation; that is, there is a directed path x and a $\gamma \in K$ such that γx is a summand of $\mathcal{X}_{\ell+1}$ and $\beta_\ell h_\ell = \eta_\ell \gamma x \tilde{a}_\ell$. If this happens, consider $\beta_{\ell+1} h_{\ell+1}$.

If $\beta_{\ell+1} h_{\ell+1}$ is a summand of $\mathcal{Q}_{\ell+1}$, then $\zeta h = \beta_n h_n \cdots \beta_{\ell+1} h_{\ell+1} \eta_\ell \gamma x \tilde{a}_\ell \beta_{\ell-1} h_{\ell-1} \cdots \beta_1 h_1$, and because $\beta_{\ell+1} h_{\ell+1} \gamma x$ is a summand of $\mathcal{Q}_{\ell+1} \mathcal{X}_{\ell+1}$ and $\mathcal{Q}_{\ell+1} \mathcal{X}_{\ell+1} \in J$, again we are done unless there is an element $z = (\sum_c \delta_c q_c)(\sum_d \epsilon_d y_d)$ such that for every c and d, $\delta_c, \epsilon_d \in K$, q_c and y_d are directed paths such that $q_c \neq h_{\ell+1}$, $y_d \neq x$, $\delta_c q_c$ is a summand of $\mathcal{Z}_{\ell+1}$, $\epsilon_d y_d$ is a summand of $\mathcal{X}_{\ell+1}$, and $z=-\beta_{\ell+1}h_{\ell+1}\gamma x$. But notice then that

$$\zeta h=\beta_n h_n\cdots\beta_{\ell+2}h_{\ell+2}(-z)\eta_\ell\tilde{a}_\ell\beta_{\ell-1}h_{\ell-1}\cdots\beta_1 h_1$$

$$=-\sum_{cd}\sum(\beta_n h_n\cdots\beta_{\ell+2}h_{\ell+2}\delta_c q_c\epsilon_d y_d\eta_\ell\tilde{a}_\ell\beta_{\ell-1}h_{\ell-1}\cdots\beta_1 h_1).$$

The fact that, for each d, $\epsilon_d y_d$ is a summand of $\mathcal{X}_{\ell+1}$ implies that $\eta_\ell\epsilon_d y_d\tilde{\alpha}_\ell$ is a summand of $\eta_\ell\mathcal{X}_{\ell+1}\tilde{a}_\ell$. Once more, since $\mathcal{P}_\ell-\eta_\ell\mathcal{X}_{\ell+1}\tilde{a}_\ell\epsilon J$, either $y_d\tilde{a}_\ell\epsilon J_0$ for some d and we are done, or there is for each d, a $\tau_d\epsilon K$ and a directed path p_d such that $\tau_d p_d$ is a summand of $\mathcal{P}_\ell$ and $\tau_d p_d=\eta_\ell\epsilon_d y_d\tilde{a}_\ell$. If this is true, then

$$\zeta h = -\sum_{cd}\sum(\beta_n h_n\cdots\beta_{\ell+2}h_{\ell+2}\delta_c q_c\tau_d p_d\beta_{\ell-1}h_{\ell-1}\cdots\beta_1 h_1)$$

and $p_d\neq h_\ell$ for any d since $p_d=y_d\tilde{a}_\ell$, $h_\ell=x\tilde{a}_\ell$ and $y_d\neq x$. Also, for every c and d,

$$\beta_n h_n\cdots\beta_{\ell+2}h_{\ell+2}\delta_c q_c\tau_d p_d\beta_{\ell-1}h_{\ell-1}\cdots\beta_1 h_1$$

is a summand of t_{j_0} so that in the sum t_{j_0}, ζh and $-\sum_{cd}\sum(\beta_n h_n\cdots\beta_{\ell+2}h_{\ell+2}\delta_c q_c\tau_d p_d\beta_{\ell-1}h_{\ell-1}\cdots\beta_1 h_1)$ would cancel out, a contradiction to the original assumption that all such cancellations had been made. Therefore, if $\beta_{\ell+1}h_{\ell+1}$ is a summand of $\mathcal{Z}_{\ell+1}$, then $h=\tilde{r}_1\epsilon J_0$.

If $\beta_{\ell+1}h_{\ell+1}$ is a summand of $\mathcal{S}_{\ell+1}$, then

$\beta_{\ell+1}h_{\ell+1}\beta_\ell h_\ell=\beta_{\ell+1}h_{\ell+1}\gamma x\eta_\ell\tilde{a}_\ell$, and $\beta_{\ell+1}h_{\ell+1}\gamma x$ is a summand of $\mathcal{S}_{\ell+1}\mathcal{X}_{\ell+1}$ which is an element of J by Lemma 1.12. Replacing $\mathcal{Z}_{\ell+1}$ by $\mathcal{S}_{\ell+1}$ and arguing as above gives $h=\tilde{r}_1\in J$.

If $\beta_{\ell+1}h_{\ell+1}=\eta_{\ell+1}\tilde{a}_{\ell+1}$, then $\beta_{\ell+1}h_{\ell+1}\beta_\ell h_\ell=\eta_{\ell+1}\tilde{a}_{\ell+1}\eta_\ell\gamma x\tilde{a}_\ell=\eta_{\ell+1}\eta_\ell\gamma\tilde{a}_{\ell+1}x\tilde{a}_\ell=0$ because the origin of $\tilde{a}_{\ell+1}$ is $\tilde{u}_{\ell+1}$ and $\tilde{u}_{\ell+1}\mathcal{X}_{\ell+1}=0$. This final contradiction shows that if $\beta_i h_i$ is a summand of $\mathcal{P}_i$ for some i, then $h=\tilde{r}_1\in J_0$.

Case 2: $\beta_i h_i$ is not a summand of $\mathcal{P}_i$ for any i, but $\beta_\ell h_\ell$ is a summand of $\mathcal{Z}_\ell$, where ℓ is minimal with this property. Now $\ell\neq 1$ because $h=\tilde{r}_1$ implies that the origin of h is $\tilde{u}_1$ and $\mathcal{Z}_1\tilde{u}_1=0$. Thus $\ell\geq 2$. By Lemma 1.12, $\mathcal{Z}_\ell-\eta_\ell\tilde{a}_\ell\mathcal{Y}_\ell\in J$ so either $h_\ell\in J_0$ and we are done, or $\beta_\ell h_\ell=\eta_\ell\tilde{a}_\ell\lambda y$ where $\lambda\in K$, y is a directed path and λy is a summand of $\mathcal{Y}_\ell$. If this occurs, consider $\beta_{\ell-1}h_{\ell-1}$.

If $\beta_{\ell-1}h_{\ell-1}$ is a summand of $\mathcal{S}_{\ell-1}$, then $\zeta h=\beta_n h_n\cdots\beta_{\ell+1}h_{\ell+1}\eta_\ell\tilde{a}_\ell\lambda y\beta_{\ell-1}h_{\ell-1}\cdots\beta_1 h_1$ where $\lambda y\beta_{\ell-1}h_{\ell-1}$ is a summand of $\mathcal{Y}_\ell\mathcal{S}_{\ell-1}=\mathcal{Y}^*_{\ell-1}\mathcal{S}_{\ell-1}$ which is an element of J by Lemma 1.12. Thus $yh_{\ell-1}\in J_0$ and we are done unless a cancellation occurs. But this would imply, as in case 1, that ζ cancels with another summand of t_{j_0}, a contradiction. Hence $h=\tilde{r}_1 \in J_0$.

If $\beta_{\ell-1}h_{\ell-1}=\eta_{\ell=-1}\tilde{a}_{\ell-1}$, then

$\beta_\ell h_\ell \beta_{\ell-1}h_{\ell-1}=\eta_\ell \tilde{a}_\ell \lambda \gamma \eta_{\ell-1}\tilde{a}_{\ell-1}=\eta_\ell \lambda \eta_{\ell-1}\tilde{a}_\ell \gamma \tilde{a}_{\ell-1}=0$ because the terminus of $\tilde{a}_{\ell-1}$ is $\tilde{v}_{\ell-1}=\tilde{u}_\ell$ and $\mathcal{Y}_\ell \tilde{u}_\ell=0$. Thus we have another contradiction, implying that if $\beta_i h_i$ is a summand of $\mathcal{Z}_i$ for some i, then $h=\tilde{r}_1 \in J_0$

<u>Case 3</u>: $\beta_i h_i$ is neither a summand of $\mathcal{Z}_i$ nor a summand of $\mathcal{P}_i$ for any i, but $\beta_\ell h_\ell$ is a summand of $\mathcal{S}_\ell$ for some ℓ. Because ζh contains no summand of $\mathcal{P}_i$ or $\mathcal{Z}_i$ the origin of h_ℓ must be the origin of $\tilde{a}_\ell$, namely $\tilde{u}_\ell$. Since $\mathcal{S}_\ell \tilde{u}_\ell=0$, this is impossible.

Thus we have shown that if $r\in\rho_0$ and $\Theta(r+I_0)=\tilde{r}+J_0$, then $\tilde{r}\in J_0$. Since ρ_0 generates I_0, Θ is well-defined. Replacing φ with φ^{-1} and defining $\Theta^{-1}:K\Gamma/J_0 \longrightarrow K\Gamma/I_0$ analogously we see that Θ^{-1} is well-defined and that $\Theta\Theta^{-1}=1_{K\Gamma/J_0}$ and $\Theta^{-1}\Theta=1_{K\Gamma/I_0}$. Consequently, Θ is an isomorphism.

<u>Remark</u>: Since any K-algebra map on $K\Gamma$ is completely determined by its action on $V(\Gamma)$ and $A(\Gamma)$, the function Θ defined in Theorem 1.13 induces an automorphism $\Theta:K\Gamma \longrightarrow K\Gamma$

where $\theta(v)=\tilde{v}$ and $\theta(a)=\tilde{a}$ for every $v\in V(\Gamma)$ and $a\in A(\Gamma)$. Of course this isomorphism also has the property that $\theta(I_0)=J_0$.

<u>Corollary 1.14</u>: Suppose Γ is a finite, arrow unique graph, I is an ideal of Λ which is contained in $(K\Gamma^{+})^{2}$ and $\Lambda=K\Gamma/I$. Then the category of unwindable modules over Λ does not depend on I.

<u>Proof</u>: Let ρ be a set of relations of Γ which generate I, ρ_0 the set of zero relations associated to ρ and $I_0=\langle\rho_0\rangle$. Suppose that J is another ideal of $K\Gamma$ which is contained in $(K\Gamma^{+})^{2}$ such that $K\Gamma/I\cong K\Gamma/J$. If σ is a set of generators for J, σ_0 the set of zero relations associated to σ, and $J_0=\langle\sigma_0\rangle$, then by Theorem 1.13, $K\Gamma/I_0\cong K\Gamma/J_0$; in fact, according to the remark above, there is an isomorphism $\theta:K\Gamma \longrightarrow K\Gamma$ such that $\theta(I_0)=J_0$.

Let $F_0:\Gamma_0' \longrightarrow \Gamma$ be the universal cover of Γ and let G be the automorphism group of the covering. Fix a lifting $L:V(\Gamma) \longrightarrow V(\Gamma_0')$. Let $W:A(\Gamma) \longrightarrow G$ be the weight function given in Theorem 1.5. Recall that if $a\in A(\Gamma)$ has origin u and terminus v, then $W(a)=g$ where $L(a)$ is an arrow from $L(u)$ to $L(v)^{g}$. The weight function W induces a G-grading on $K\Gamma$ and since both ρ_0 and σ_0 consist only of directed paths, both I_0 and J_0 are homogeneous with respect to this grading. Therefore W induces a G-grading on $K\Gamma/I_0$ and $K\Gamma/J_0$. Let $gr_G(K\Gamma/I_0)$ and $gr_G(K\Gamma/J_0)$ denote the categories of finite-dimensional graded $K\Gamma/I_0$ and $K\Gamma/J_0$-modules, respectively, under this grading.

We now define another weight function $W_1:A(\Gamma) \longrightarrow G$ which comes from the isomorphism θ. Namely, if $a \in A(\Gamma)$ we define $W_1(a)=W(\theta(a))$. The function W_1 induces a G-grading on $K\Gamma$ which is connected because the grading induced by W is connected. Again I_0 is homogeneous with respect to this grading and we let $gr_G(K\Gamma/I_0)_1$ denote the category of finite-dimensional graded $K\Gamma/I_0$-modules under the grading induced by W_1. Because $\theta:K\Gamma \longrightarrow K\Gamma$ takes I_0 to J_0, it is clear that the categories $gr_G(K\Gamma/J_0)$ and $gr_G(K\Gamma/I_0)_1$ are equivalent. But according to the remark after Theorem 1.5, there is a lifting $L_1:V(\Gamma) \longrightarrow V(\Gamma_0')$ such that when W is defined using L_1, the induced G-grading on $K\Gamma/I_0$ is the same as the G-grading on $K\Gamma/I_0$ induced by W_1. Since $gr_G(K\Gamma/I_0)$ does not depend on the lifting used to define W, $gr_G(K\Gamma/I_0)$ is equivalent to $gr_G(K\Gamma/I_0)_1$ which is equivalent to $gr_G(K\Gamma/J_0)$. Therefore the category of unwindable $K\Gamma/I$-modules is equivalent to the category of unwindable $K\Gamma/J$-modules.

The next example shows that if Γ is not an arrow unique graph, then there are cases when Theorem 1.13 does not hold.

<u>Example</u>. Let Γ be the following graph:

$$\begin{tikzcd}
v_1 \arrow[rr, "a_1"] \arrow[dr, "a_3"'] & & v_3 \arrow[rr, "a_2"] \arrow[dr, "a_5"'] & & v_5 \\
& v_2 \arrow[ur, "a_4"'] & & v_4 \arrow[ur, "a_6"'] &
\end{tikzcd}$$

and consider the map φ defined by

$\varphi(v_i)=v_i$, $i=1,\ldots,5$

$\varphi(a_1)=a_1+a_4a_3$

$\varphi(a_2)=a_2+a_6a_5$

$\varphi(a_i)=a_i$ $i=3,4,5,6$.

Extend φ to a K-algebra endomorphism of $K\Gamma$. Let $\rho=\langle a_2a_1-a_6a_5a_4a_3\rangle$ and $\sigma = \langle a_2a_1+a_2a_4a_3+a_6a_5a_1\rangle$. Let I and J be the ideals of $K\Gamma$ generated by ρ and σ, respectively. Define $\bar{\varphi}:K\Gamma \longrightarrow K\Gamma/J$ by $\bar{\varphi}(x)=\varphi(x)+J$. The kernel of $\bar{\varphi}$ is I since

$$\begin{aligned}
x\in I &\Leftrightarrow x=\mu(a_2a_1-a_6a_5a_4a_3) \text{ for some } \mu\in K \\
&\Leftrightarrow \varphi(x)=\mu\varphi(a_2a_1-a_6a_5a_4a_3) \\
&\Leftrightarrow \varphi(x)=\mu[(a_2+a_6a_5)(a_1+a_4a_3)-a_6a_5a_4a_3] \\
&\Leftrightarrow \varphi(x)=\mu[a_2a_1+a_2a_4a_3+a_6a_5a_1]
\end{aligned}$$

$\Leftrightarrow \varphi(x)\in J$.

Therefore $K\Gamma/I\cong K\Gamma/J$. However, the set of zero relations associated to ρ is $\rho_0=\{a_2a_1, a_6a_5a_4a_3\}$ and the set of zero relations associated to σ is $\sigma_0=\{a_2a_1, a_2a_4a_3, a_6a_5a_1\}$. Hence $\dim_K(K\Gamma/\langle\rho_0\rangle)$ is greater than $\dim_K(K\Gamma/\langle\sigma_0\rangle)$ so $K\Gamma/\langle\rho_0\rangle$ and $K\Gamma/\langle\sigma_0\rangle$ are not even isomorphic as K-vector spaces.

We now construct some unwindable modules when Γ is an arrow unique graph. If Λ is any ring, a Λ-module is called <u>uniserial</u> if it has a unique composition series. Suppose Λ is an Artin algebra with Jacobson radical $\underline{r}$. If n is the smallest integer such that $\underline{r}^n=0$, then by applying Nakayama's Lemma we see that a Λ-module M is uniserial if and only if $\underline{r}^iM/\underline{r}^{i+1}M$ is simple for $i=1,2,\ldots,n-1$. As the next result shows, if $\Lambda=K\Gamma/I$ where Γ is an arrow unique graph and $I \subseteq (K\Gamma^+)^2$, then every finite dimensional uniserial Λ-module is an unwindable Λ-module. Before proving this we need the following definition.

Let ρ be a set of relations which generates I. Suppose M is a Λ-module and $X = (X_v,\alpha(a))$ is the corresponding representation of (Γ,ρ) under the usual equivalence; that is, for every $v\in V(\Gamma)$, $X_v=vM=\bar{v}M$ (where $\bar{v}=v+I$) and for every $a\in A(\Gamma)$ from u to v, $\alpha(a):X_u \longrightarrow X_v$ is defined by $\alpha(a)(um)=aum=\bar{a}\bar{u}m$ for every $m\in M$ (again $\bar{a}=a+I$ and $\bar{u}=u+I$). Define a subgraph Γ_1 of Γ by $V(\Gamma_1)=\{u\in V(\Gamma):X_u\neq 0\}$ and $A(\Gamma_1)=\{a\in A(\Gamma)$: the origin and terminus of a is in $V(\Gamma_1)\}$. The graph Γ_1 is called the <u>support</u> of M.

<u>Lemma</u> <u>1.15</u>. Let $I \subseteq (K\Gamma^+)^2$ be an ideal of $K\Gamma$ where Γ is an arrow unique graph. Let $\Lambda=K\Gamma/I$ and let M be a finite dimensional uniserial Λ-module. Then the support of M has the following form:

$$w_1 \xrightarrow{a_1} w_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} w_n.$$

Moreover, if $(X_v,\alpha(a))$ is the representation of Γ corresponding to M, then $X_{w_i}\cong K$ for $i=1,2,\ldots,n$.

<u>Proof</u>: Let $\underline{r}$ denote the Jacobson radical of $K\Gamma/I$. If $x\in K\Gamma$, let $\bar{x}=x+I$. The proof is by induction on $n=|V(\Gamma_1)|$ where Γ_1 is the support of M.

If $n=1$, then $M=\bar{w}M$ where w denotes the single vertex in $V(\Gamma_1)$. Now $\underline{r}M=0$ since if $a\in A(\Gamma)$ and $m\in M$, $\bar{a}m=0$ unless the origin and terminus of a are both w, which is impossible since Γ contains no oriented cycles. Hence $\bar{w}M=M\cong M/\underline{r}M \cong K$ because M is uniserial. Consequently Γ_1 has the required form with $X_w\cong K$.

Suppose the conclusion to the lemma is true when the vertex set of the support of a module has less than n elements. Assume M is a uniserial Λ-module with support Γ_1 such that $|V(\Gamma_1)|=n$. Let $S=\{v_1,v_2,\ldots v_s\}$ be the subset of $V(\Gamma_1)$ consisting of the vertices with the property that no arrow $a\in A(\Gamma_1)$ has terminus v_j. Since Γ is finite and has no oriented cycles, $S\neq\phi$. Now $\bar{v}_1M+\ldots+\bar{v}_sM$ is a K-subspace of M and because $v_jv_\ell=0$ if $j\neq\ell$, the sum is direct. Moreover, for each $v_j\in S$, $\bar{v}_j\underline{r}M=0$ since there are no arrows in $A(\Gamma_1)$ with terminus v_j. Let $\varphi:\bar{v}_1M+\ldots+\bar{v}_sM \longrightarrow M/\underline{r}M$ be defined for every $m_1,m_2,\ldots m_s\in M$ by

$$\varphi(\bar{v}_1m_1+\ldots+\bar{v}_sm_s)=\bar{v}_1m_1+\ldots+\bar{v}_sm_s+\underline{r}M.$$

Then φ is 1-1, therefore

$$\begin{aligned} 1=\dim_K M/\underline{r}M &\geqslant \dim_K(\bar{v}_1M+\ldots+\bar{v}_sM) \\ &= \dim_K\bar{v}_1M+\ldots+\dim_K\bar{v}_sM. \end{aligned}$$

Each $\bar{v}_jM$ is nonzero by definition of Γ_1, and $S\neq\phi$, so S contains exactly one element. Call this vertex w_1. Then by the above we also have that $\bar{w}_1M\cong K$.

Let $w_2,\ldots,w_n$ denote the remaining vertices in $V(\Gamma_1)$. Let $N = \bar{w}_2M+\ldots+\bar{w}_nM$. Then $N\cong\underline{r}M$ and is therefore a uniserial Λ-submodule of M. Let Γ_2 be the support of N.

Then $|V(\Gamma_2)|=|\{w_2,\ldots,w_n\}|<n$, so by the induction hypothesis Γ_2 has the form

$$w_2 \xrightarrow{a_2} w_3 \xrightarrow{a_3} \cdots \xrightarrow{a_{n-1}} w_n$$

where for $i=2,\ldots,n$, $\bar{w}_iN=\bar{w}_iM\cong K$. Since there must be exactly one arrow in Γ_1 with terminus w_2, Γ_1 has the form

$$w_1 \xrightarrow{a_1} w_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} w_n$$

where $\bar{w}_iM\cong K$ for $i=1,\ldots,n$.

<u>Proposition</u> <u>1.16</u>. Let Γ be a finite, arrow unique graph and let ρ be a set of relations on Γ. Let $\Lambda=K\Gamma/I$ where I is the ideal of $K\Gamma$ generated by ρ and let M be a finite-dimensional uniserial Λ-module. Then M is an unwindable Λ-module.

<u>Proof</u>: Let $X=(X_v,\alpha(a))$ be the representation which corresponds to M under the usual equivalence $E:\mathrm{rep}(\Gamma,\rho) \longrightarrow \mathrm{mod}(\Lambda)$. Let Γ_1 be the support of M. Then by Lemma 1.15, Γ_1 has the form

$$w_1 \xrightarrow{a_1} w_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} w_n$$

where $X_{w_i}\cong K$ for $i=1,\ldots,n$. Let $F_0:\Gamma_0' \longrightarrow \Gamma$ be the universal cover of Γ. Let $L:V(\Gamma) \longrightarrow V(\Gamma_0')$ be a lifting. Let $u_1' = L(w_1)$ and let $a_1' = L(a)$. Define $u_2',\ldots,u_n'$ and

$a_2', \ldots, a_{n-1}'$ inductively by letting u_i' be the terminus of a_{i-1}' and by letting a_i' be the unique arrow in $F_0^{-1}(a_i)$ with origin u_i'. Define a representation $Y = (Y_{u'}, \beta(b'))$ of Γ_0' by

$$Y_{u_i'} = K \quad i=1,\ldots,n$$

$$Y_{u'} = 0 \text{ if } u' \neq u_i'$$

and $\beta(a_i') = \alpha(a_i)$ $i=1,\ldots,n-1$

$$\beta(b') = 0 \quad \text{if } b' \neq a_i'.$$

Then $f_0(Y)=X$ which satisfies ρ by definition of M. Hence $M=Ef_0(X)$ is an unwindable Λ-module, where f_0 is the covering functor associated to F_0.

When Γ is not an arrow unique graph, it is not the case that every uniserial Λ-module is an unwindable Λ-module as the following examples show.

<u>Example</u>: Let Γ be the following graph:

$$\begin{tikzcd} v_1 \arrow[r, bend left, "a_1"] \arrow[r, bend right, "a_2"'] & v_2 \arrow[r, "a_3"] & v_3 \end{tikzcd}$$

with the set of relations $\rho = \{a_3 a_1 - a_3 a_2\}$. Let $I = \langle \rho \rangle$ and let $\Lambda = K\Gamma / I$. Suppose M is the Λ-module whose corresponding representation of (Γ, ρ) is $X = (X_v, \alpha(a))$ where $X_{v_i} = K$, $i=1,2,3$

and $\alpha(a_i) = 1_K$, $i=1,2,3$. Then M is not an unwindable Λ-module because X does not satisfy $\rho_0=(a_3a_1,\ a_3a_2)$. M is, however, a uniserial Λ-module because $M/\underline{r}M \cong \underline{r}M/\underline{r}^2M \cong \underline{r}^2M \cong K$.

Example: Let Γ be the following graph.

$$\begin{tikzcd} & & v_4 \arrow[dl, "a_4"'] & & \\ v_1 \arrow[r, "a_1"] & v_2 \arrow[rr, "a_2"] & & v_3 \arrow[ul, "a_3"'] \arrow[r, "a_5"] & v_5 \end{tikzcd}$$

Let $\rho = (a_5a_2a_1 - a_5a_2a_4a_3a_2a_1)$, let I be the ideal of $K\Gamma$ generated by ρ and let $\Lambda=K\Gamma/I$. Let M be the Λ-module which corresponds to the representation $X=(X_v,\alpha(a))$ of (Γ,ρ) defined by $X_{v_1}=K$, $X_{v_2}=K\oplus K$, $X_{v_3}=K\oplus K$, $X_{v_4}=K$ and $X_{v_5}=K$ and $\alpha(a_1)(x)=(x,x)$, $\alpha(a_2)(x,y)=(x-y,x)$, $\alpha(a_3(x,y)=y$, $\alpha(a_4)(x)=(0,x)$, and $\alpha(a_5)(x,y)=-x+y$. The representation X does indeed satisfy ρ, since if $x\in X_{v_1} = K$, $\alpha(a_5)\alpha(a_2)\alpha(a_1)(x)=\alpha(a_5)\alpha(a_2)(x,x)=\alpha(a_5)(0,x)=x$ and

$$\begin{aligned} &\alpha(a_5)\alpha(a_2)\alpha(a_4)\alpha(a_3)\alpha(a_2)\alpha(a_1)(x) \\ &\qquad = \alpha(a_5)\alpha(a_2)\alpha(a_4)\alpha(a_3)\alpha(a_2)(x,x) \\ &\qquad = \alpha(a_5)\alpha(a_2)\alpha(a_4)\alpha(a_3)(0,x) \\ &\qquad = \alpha(a_5)\alpha(a_2)\alpha(a_4)(x) \\ &\qquad = \alpha(a_5)\alpha(a_2)(0,x) \\ &\qquad = \alpha(a_5)(-x,0) \end{aligned}$$

$= x.$

But X does not satisfy $\rho_0=\{a_5a_2a_1, a_5a_2a_4a_3a_2a_1\}$ since $a_5a_2a_1(X)\neq 0$, and therefore M is not an unwindable Λ-module. A straightforward calculation shows that $\underline{r}^iM/\underline{r}^{i+1}M \cong K$ for $i=0,\ldots,7$, so M is a uniseral Λ-module.

Section 2: The Classification Problem

Recall that a category is said to be of _finite representation type_ if it contains only finitely many nonisomorphic indecomposable objects. If A is a K-algebra, then A is said to be of _finite representation type_ if mod(A) is of finite representation type. In this section we use the concept of unwindable modules to determine for a specific class of K-algebras when a certain factor algebra of the K-algebra is of finite representation type, although the algebra itself may be of infinite representation type.

Let Γ be a directed graph with a set of relations ρ. Let ρ_0 be the set of zero relations associated to ρ. Suppose $p=a_n^{e_n}\ldots a_1^{e_1}$ is a path of nonzero length in Γ. A path $q=b_m^{f_m}\ldots b_1^{f_1}$ where $n\geq m$ is called a _subpath_ of p if there exists a natural number ℓ, $0\leq\ell\leq n-m$, such that for $j=1,\ldots,m$, $b_j=a_{\ell+j}$ and $f_j=e_{\ell+j}$. We use the notation $q|p$ to express the fact that q is a subpath of p. This notation is also used for paths of length 0, that is, vertices. The path p is called _walk of length n_ in (Γ,ρ) if p contains no subpath of the form r or r^{-1} where $r\in\rho_0$.

For each $v\in V(\Gamma)$ let S_v denote the set of all walks of nonzero length in (Γ,ρ) with terminus v. The set S_v can be partially ordered by the partial order defined by Bongartz and Ringel [2] which we give below.

<u>Definition</u> <u>2.1</u> For $w_1, w_2 \in S_v$, $w_1 \leqslant w_2$ if one of the following four conditions holds:

(1) $w_1 = w_2$

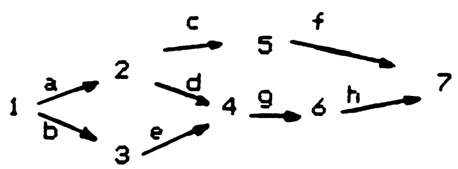
(2) $w_1 = w_2\tilde{w}_1$ and $w_2 = wa$ where $\tilde{w}_1$ and w are walks in (Γ, ρ) and $a \in A(\Gamma)$

(3) $w_2 = w_1\tilde{w}_2$ and $w_1 = wa^{-1}$ where $\tilde{w}_2$ and w are walks in (Γ, ρ) and $a \in A(\Gamma)$

(4) $w_1 = wr\tilde{w}_1$ and $w_2 = ws^{-1}\tilde{w}_2$ where $w, \tilde{w}_1, \tilde{w}_2$ are walks in (Γ, ρ) and $sr \in \rho_0$.

The following example illustrates Definition 2.1.

<u>Example</u> Let Γ be the directed graph drawn below



with the relation set $\rho = \{da - eb, hgd - fc\}$. Then

$\rho_0 = \{da, eb, hgd, fc\}$ and $S_3 = \{b, ba^{-1}, ba^{-1}c, e^{-1}, e^{-1}d,$

$e^{-1}dc^{-1}, e^{-1}g^{-1}, e^{-1}g^{-1}h^{-1}, e^{-1}g^{-1}h^{-1}f\}$

on which the partial order of Definition 2.1 has the form

$$\begin{tikzcd}[column sep=small]
 & e^{-1}g^{-1}h^{-1}f & \\
 & e^{-1}g^{-1}h^{-1} \arrow[u, dash] & \\
e^{-1}d \arrow[ur, dash] & & \\
 & & e^{-1}g^{-1} \arrow[uul, dash] \\
e^{-1}dc^{-1} \arrow[uu, dash] & & \\
 & e^{-1} \arrow[ul, dash] \arrow[uur, dash] & \\
 & b \arrow[u, dash] & \\
 & ba^{-1}c \arrow[u, dash] & \\
 & ba^{-1} \arrow[u, dash] &
\end{tikzcd}$$

where the notation $\begin{matrix} w_2 \\ | \\ w_1 \end{matrix}$ means that $w_1 < w_2$.

If S is any finite partially ordered set we can form the category rep(S) of *K-representations of S.* *A K-representation* $X=(U_X,(X_w)_{w\in S})$ *of the partially ordered set* S consists of a finite dimensional K-vector space, U_X, called the *total space* of X, and a tuple $(X_w)_{w\in S}$ where for each $w\in S$, X_w is a K-subspace of U_X such that if $w_1 \leqslant w_2$, then X_{w_1} is a subspace of X_{w_2}. If $Y=(U_Y,(Y_w)_{w\in S})$ is also a K-representation of S, then a morphism $f:X\longrightarrow Y$ in rep(S) is a K-linear map $f:U_X\longrightarrow U_Y$ such that $f(X_w)$ is a subspace of Y_w for every $w\in S$.

If S_1 and S_2 are partially ordered sets, we say that

S_1 is a *partially ordered subset* of S_2 if $S_1 \subseteq S_2$ and the partial order on S_1 is the restriction of the partial order on S_2. We say that S_1 and S_2 are *isomorphic as partially ordered sets* if there is a bijection $f: S_1 \longrightarrow S_2$ such that for $x, y \in S_1$, $x \leqslant y$ if and only if $f(x) \leqslant f(y)$.

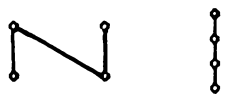
The following theorem of Kleiner [6] gives us information for determining when a partially ordered set is of finite representation type.

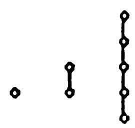
Theorem 2.2: Let S be a finite partially ordered set. Then rep(S) is of infinite representation type if and only if S contains a partially ordered subset which is isomorphic to one of the following:

PO_1

PO_2 

PO_3 

PO_4 

PO_5 

where again the notation $\overset{w_2}{\underset{w_1}{|}}$ means $w_1 < w_2$.

Now let T be a directed tree; that is, T is a directed graph with the property that if $u,v \in V(T)$, then there is exactly one path in T with origin u and terminus v. Let ρ be a set of relations on T and let ρ_0 be the set of zero relations associated to ρ. Then $\langle\rho\rangle=\langle\rho_0\rangle$. Because of this fact, and to simplify notation, we will assume throughout this section that any set of relations on a directed tree consists entirely of directed paths, that is, $\rho=\rho_0$.

Let $v \in V(T)$ and $a \in A(T)$ with origin u_1 and terminus u_2. We say that a is an <u>arrow leading to v</u> if u_2 and v belong to the same connected component of $T-\{a\}$. Otherwise, a is said to be an <u>arrow going away from v</u>. A K-representation $X=(X_v,\alpha(a))$ of T is said to have a <u>peak</u> at X_{v_0} if $\alpha(a)$ is an injection for every $a \in A(T)$ leading to v_0, and if $\alpha(a)$ is a surjection for every $a \in A(T)$ going away from v_0.

The following result of Bongartz and Ringel [2] relates the representations of a directed tree to the representations of partially ordered sets.

<u>Theorem 2.3</u>: Let (T,ρ) be a finite directed tree with relations. Let KT denote the path algebra and let I be the ideal in KT generated by ρ. For every $v \in V(T)$ let S_v denote the set of walks of nonzero length in (T,ρ) with terminus

v. Give S_v the partial order of Definition 2.1. Then KT/I is of finite representation type if and only if $\mathrm{rep}(S_v)$ is of finite representation type for every $v \in V(T)$. Moreover, in this case each indecomposable representation of (T,ρ) has a peak.

Remark: Bongartz and Ringel actually define an equivalence E_{v_0} between the category of representations of (T,ρ) with peak v_0 and $\mathrm{rep}(S_{v_0})$ such that if $X=(X_v,\alpha(a))$ is a representation of (T,ρ) with peak v_0, then X_{v_0} is the total space of $E_{v_0}(X)$.

The following theorem of Green [5] shows that we can use information about graded modules to determine whether or not certain K-algebras are of finite representation type.

Theorem 2.4: Let (Γ,ρ) be a finite directed graph with relations, let $F:\Gamma' \longrightarrow \Gamma$ be a regular covering and let ρ' be a set of relations on Γ' such that $F:(\Gamma',\rho') \longrightarrow (\Gamma,\rho)$ is a morphism of graphs with relations. Let G be the automorphism group of the covering and let $\mathrm{gr}_G(\Lambda)$ be the category of finite-dimensional graded modules defined in Section 1. Then Λ is of infinite representation type if and only if there are indecomposable objects in $\mathrm{gr}_G(\Lambda)$ of arbitrarily large dimension over K.

We will use this theorem in the following form:

<u>Corollary 2.5</u>: Let (Γ,ρ) be a finite directed graph with relations. Let ρ_0 be the set of zero relations associated to ρ. Let $\Lambda=K\Gamma/\langle\rho\rangle$ and $\Lambda_0=K\Gamma/\langle\rho_0\rangle$. Then Λ_0 is of infinite representation type if and only if there are indecomposable unwindable Λ-modules of arbitrarily large dimension over K.

<u>Proof</u>: Apply Theorem 2.4 where $F:\Gamma'\longrightarrow\Gamma$ is the universal cover of Γ and $\rho'=\{L(p): p\in\rho_0$ and $L:V(\Gamma)\longrightarrow V(\Gamma')$ is a lifting$\}$. Then $F:(\Gamma',\rho')\longrightarrow(\Gamma,\rho)$ is a morphism of graphs with relations. The result now follows from Corollary 1.7.

We now restrict our attention to a finite, arrow unique graph Γ. Let ρ be the set of relations on Γ defined by $\rho=\{p-q$: p has the same origin as q and p has the same terminus as $q\}$. The set ρ is called the <u>set of commutativity relations</u> on Γ and if $p-q\in\rho$ we say that p <u>commutes</u> with q. Let ρ_0 be the set of zero relations associated to ρ. Let $\Lambda=K\Gamma/\langle\rho\rangle$ and $\Lambda_0=K\Gamma/\langle\rho_0\rangle$. Loupias, [7], has classified the arrow unique graphs with commutativity relations for which Λ is of finite representation type. Our goal is to determine the arrow unique graphs with commutativity relations for which Λ_0 is of finite representation type, even though Λ may be of infinite representation type.

The following notation will be used for the remainder of this section. Let $F:\Gamma'\longrightarrow\Gamma$ be the universal cover of Γ. Let $\rho'=\{L(p): p\in\rho_0$ and $L:V(\Gamma)\longrightarrow V(\Gamma')$ is a lifting$\}$. For every $v'\in V(\Gamma')$ let $T_{v'}$ be the connected subtree of Γ' such that $A(T_{v'})=\{a'\in A(\Gamma'):a'|w'$ for some walk w' in (Γ',ρ') with terminus $v'\}$. Let $\rho_{v'}=\{p\in\rho': p$ is a path in $T_{v'}\}$. For every $v\in V(\Gamma)$ let S_v denote the partially ordered set of walks of nonzero length in (Γ,ρ) with terminus v. Similarly, for every $v'\in V(\Gamma')$ let $S_{v'}$ denote the partially ordered set of walks of nonzero length in (Γ',ρ') with terminus v'. Note that $S_{v'}$ is also the set of walks of nonzero length in $(T_{v'},\rho_{v'})$ with terminus v'. Also, $S_{v'}$ and $S_{F(v')}$ are isomorphic as partially ordered sets.

We now prove a theorem which is analogous to Theorem 2.3.

<u>Theorem</u> <u>2.6</u>: Let (Γ,ρ) be a finite, arrow unique graph with commutativity relations. Let ρ_0 be the set of zero relations associated to ρ. The following statements are equivalent:

(1) $\Lambda_0 = K\Gamma/\langle\rho_0\rangle$ is of finite representation type.

(2) $rep(T_{v'},\rho_{v'})$ is of finite representation type for each $v'\in V(\Gamma')$.

(3) S_v is a finite set and $rep(S_v)$ is of finite representation type for each $v\in V(\Gamma)$.

<u>Proof</u>: (1⇒2) Suppose Λ_0 is of finite representation type. Then by Corollary 2.5, there is a positive integer N such that the dimension over K of each indecomposable, unwindable Λ-module is less than N. Since the category of unwindable Λ-modules is equivalent to $f(rep(\Gamma',\rho'))$ where f is the covering functor associated to F, and since $rep(T_{v'},\rho_{v'})$ is a subcategory of $rep(\Gamma',\rho')$ for every $v'\in V(\Gamma')$, N is also a bound on the dimension of the indecomposable objects in $rep(T_{v'},\rho_{v'})$. Therefore $rep(T_{v'},\rho_{v'})$ is of finite representation type for every $v'\in V(\Gamma')$.

(2⇒3) If $rep(T_{v'},\rho_{v'})$ is of finire representation type, then $T_{v'}$ must be a finite tree. Consequently, $S_{v'}$ is a finite set. Theorem 2.3 applies and $rep(S_{v'})$ is of finite representation type. Now if $v\in V(\Gamma)$, then $v=F(v')$ for some $v'\in V(\Gamma')$ and S_v and $S_{v'}$ are isomorphic as partially ordered sets. Therefore S_v is of finite representation type.

(3⇒1) Suppose S_v is a finite set and $rep(S_v)$ is of finite representation type for every $v \in V(\Gamma)$. Then $S_{v'}$ is a finite set and $\mathrm{rep}(S_{v'})$ is of finite representation type for every $v' \in V(\Gamma')$. This implies that $T_{v'}$ is a finite tree for every $v' \in V(\Gamma')$. Let $L: V(\Gamma) \longrightarrow V(\Gamma')$ be a lifting. For each $v \in V(\Gamma)$, let $m_v = |V(T_{L(v)})|$ and let $M = \max_{v \in V(\Gamma)} \{m_v\}$. Also, for every $v \in V(\Gamma)$, let n_v be a bound on the dimension of the total spaces of the indecomposable representations of $S_{L(v)}$. Let $N = \max_{v \in V(\Gamma)} \{n_v\}$. Note that m_v and n_v do not depend on the choice of a lifting L and hence M and N likewise do not depend on L. If Λ_0 is of infinite representation type then by Corollary 2.5, there is an indecomposable representation $X = (X_{u'}, \alpha(a'))$ of (Γ', ρ') such that $\dim_K X = \sum_{u' \in V(\Gamma')} \dim_K X_{u'} > NM$. Let Γ'' be the support of X. Since X is indecomposable, Γ'' is a connected subtree of Γ'. Let $\rho'' = \{p \in \rho' : p \text{ is a path in } \Gamma''\}$. For every $u' \in V(\Gamma'')$ define $R_{u'}$ to be the partially ordered set of walks in (Γ'', ρ'') with terminus u'. The set $R_{u'}$ is a partially ordered subset of $S_{u'}$ and hence $\mathrm{rep}(R_{u'})$ is of finite representation type for every $u' \in V(\Gamma'')$. Since Γ'' is a finite tree, Theorem 2.3 gives that $K\Gamma''/\langle \rho'' \rangle$ is of finite representation type and that X has a peak at some $u_0' \in V(\Gamma'')$.

Let $E_{u_0'}$ be the equivalence between $\mathrm{rep}(\Gamma'', \rho'')$ and

$rep(R_{u_0}')$ discussed in the remark after Theorem 2.3. Then X_{u_0}' is the total space of $E_{u_0}'(X)$, hence $\dim_K(X_{u_0}') \leqslant N$. Moreover, since X has a peak at X_{u_0}', $\dim_K X_{u'} \leqslant \dim_K X_{u_0}'$ for all $u' \in V(\Gamma")$. Thus $\dim_K X \leqslant N|V(\Gamma")|$. We will show that $|V(\Gamma")| \leqslant M$ by showing $V(\Gamma") \subseteq V(T_{u_0}')$. Suppose $v' \in V(\Gamma")$ and $v' \notin V(T_{u_0}')$. Then the unique path p' in $\Gamma"$ with origin v' and terminus u_0' cannot be a walk in (Γ', ρ'). Thus there is a directed path $r' \in \rho'$ with origin u_1' and terminus u_2' such that $r' | p'$ or $(r')^{-1} | p'$. If $r' | p'$ then every arrow $a' \in A(\Gamma')$ such that $a' | r'$ is an arrow leading to u_0' and hence $r'(X)$ is an injection. But $r'(X)=0$, so X_{u_1} must be zero. This contradicts the fact that $\Gamma"$ is the support of X. Similarly, if $(r')^{-1} | p'$, then each arrow $a' \in A(\Gamma')$ such that $a' | r'$ is an arrow going away from u_0', thus $r'(X)$ is a surjection. But, again, $r'(X)=0$ so X_{u_2}' must be zero, another contradiction. Therefore $v' \in V(T_{u_0}')$. This means that $\dim_K X \leqslant N|V(\Gamma")| \leqslant N|V(T_{u_0}')| \leqslant NM$, a final contradiction. Therefore, if $rep(S_v)$ is of finite representation type for every $v \in V(\Gamma)$, then Λ_0 is of finite representation type.

Theorem 2.6 allows us to use partially ordered sets to determine whether Λ_0 is of finite representation type, therefore, Theorem 2.2 becomes useful. Theorem 2.6 also allows us to use directed trees to determine whether Λ_0 is of finite representation type. This, in turn, enables us to use further results of Bongartz and Ringel [2] which deal exclusively with trees.

We now apply Theorem 2.6 to some arrow unique graphs with special properties.

Lemma 2.7: Let (Γ,ρ) be a finite arrow unique graph with commutativity relations. Suppose there exists a walk w of nonzero length in (Γ,ρ) with origin and terminus both equal to $v\in V(\Gamma)$ such that w^2 is also a walk in (Γ,ρ). Then $\Lambda_0=K\Gamma/\langle\rho_0\rangle$ is of infinite representation type.

Proof: If w^2 is a walk in (Γ,ρ), then w^n is a walk in (Γ,ρ) for all $n\geq 1$. Therefore, the infinite set $\{w,w^2,w^3,\ldots\}$ is a subset of S_v. By Theorem 2.6, Λ_0 is of infinite representation type.

Lemma 2.8: Let (Γ,ρ) be a finite, arrow unique graph with commutativity relations. Suppose there is a walk of nonzero length in (Γ,ρ) which has the same origin and terminus as a directed path of nonzero length in Γ. Then

$\Lambda_0 = K\Gamma/\langle\rho_0\rangle$ is of infinite representation type.

Proof: Let w be a walk in (Γ,ρ) and let p be a directed path in Γ such that the origin of p is the origin of w and the terminus of p is the terminus of w. Assume w is of minimal length with this property. Let $p=a_n\ldots a_1$ where each $a_i \in A(\Gamma)$. Let u_i denote the terminus of a_i, $i=1,\ldots,n$, and let u_0 be the origin of a_1. Write $w=b_m^{e_m}\ldots b_1^{e_1}$ where $b_i \in A(\Gamma)$ and $e_i=\pm 1$. Because of the minimality condition on w, $e_1=1$ and $e_m=1$. Consider the path

$$w^{-1}p=b_1^{-1}b_2^{-e_2}\ldots b_m^{-1}a_n\ldots a_1.$$

If $w^{-1}p$ is not a walk, there are directed paths $q,r\in\rho_0$ such that $q-r\in\rho$ and $q|p$. The path r and the walk w have no common subpaths of nonzero length because of the minimality of the length of w. The existence of r implies that there is an arrow $c\in A(\Gamma)$ with origin u_j where $0\leqslant j\leqslant n-2$ such that $c\neq a_{j+1}$. Assume that j is minimal with this property. Similarly, there is an arrow $d\in A(\Gamma)$ with terminus u_ℓ, $2\leqslant \ell\leqslant n$ such that $d\neq a_\ell$. Assume that ℓ is maximal with this property. Then the set

$$\{a_\ell^{-1}\ldots a_n^{-1}w,\ d^{-1}\ a_{\ell+1}^{-1}\ldots a_n^{-1}w,\ a_{j+1}\ldots a_1,\ ca_j\ldots a_1\}$$

is a partially ordered subset of S_{u_0} which is isomorphic to PO_1. Therefore S_{u_0} is of infinite representation type and hence Λ_0 is of infinite representation type.

If $w^{-1}p$ is a walk in (Γ,ρ), then $(w^{-1}p)^2$ is a walk in (Γ,ρ) so by Lemma 2.7, Λ_0 is of infinite representation type.

A walk w of nonzero length in (Γ,ρ) is called a <u>circular walk</u> if the origin of w is the same as the terminus of w. With this definition we have the following:

<u>Proposition 2.9</u>: Let (Γ,ρ) be a finite, arrow unique graph with commutativity relations. Suppose there is a circular walk in (Γ,ρ). Then $\Lambda_0=K\Gamma/\langle\rho_0\rangle$ is of infinite representation type.

<u>Proof</u>: Let w be a circular walk in (Γ,ρ) with origin and terminus v. If w^2 is also a walk in (Γ,ρ) then Λ_0 is of infinite representation type by Lemma 2.7.

If w^2 is not a walk in (Γ,ρ), there must be a directed path $p\epsilon\rho_0$ such that $p|w^2$. However, since w is a walk, $v|p$. Thus p is a directed path in Γ which has the same origin and terminus as a walk in (Γ,ρ). By Lemma 2.8, Λ_0 is of infinite representation type.

<u>Proposition 2.10</u>: Let (Γ,ρ) be a finite, arrow unique

graph with commutativity relations. Suppose that $V(\Gamma)$ contains a vertex of degree 5. Then $\Lambda_0 = K\Gamma/\langle\rho_0\rangle$ is of infinite representation type.

<u>Proof</u>: Let $v \in V(\Gamma)$ have degree 5. For $i=1,\ldots,5$, let $b_i \in A(\Gamma)$ such that $b_i^{e_i}$ has terminus v for $e_i = \pm 1$. Assume that $b_i \neq b_j$ if $i \neq j$. Let u_i be the origin of $b_i^{e_i}$ for $i=1,\ldots,5$. The set $\{b_1^{e_1},\ldots,b_5^{e_5}\}$ is a subset of S_v. If B is a partially ordered subset of $\{b_1^{e_1},\ldots,b_5^{e_5}\}$ which is isomorphic to PO_1, then S_v is of infinite representation type and we are done. Therefore we will assume that if B is any subset of $\{b_1^{e_1},\ldots,b_5^{e_5}\}$ consisting of 4 elements then at least two of the elements in B are comparable under the partial order. In particular, this implies that $e_i = 1$ for at most 3 values and at least 2 values of i. Let us assume that $e_1 = e_2 = e_3 = 1$ and $e_4 = e_5 = -1$, the proposition being proved similarly if $e_i = 1$ for 2 values of i and $e_i = -1$ for 3 values of i. Thus the graph

$$\begin{tikzcd}[row sep=small]
b_1 \arrow[dr] & & b_4 \\
b_2 \arrow[r] & v \arrow[ur] \arrow[dr] & \\
b_3 \arrow[ur] & & b_5
\end{tikzcd}$$

is a subgraph of Γ. In addition, there must exist $j \neq \ell$, $1 \leqslant j \leqslant 3$, $1 \leqslant \ell \leqslant 3$, such that $b_4 b_j \in \rho_0$ and $b_5 b_\ell \in \rho_0$. Let us say that $b_4b_1 \in \rho_0$ and $b_5b_2 \in \rho_0$. Let q_{14} and q_{25} be directed paths in ρ_0 such that $q_{14}-b_4b_1 \in \rho$ and $q_{25}-b_5b_2 \in \rho$. We now consider the possibility of other relations in ρ_0 involving $b_1,\ldots,b_5$.

__Case 1__: Suppose $b_4b_3 \in \rho_0$. Let q_{34} be a directed path in Γ such that $q_{34}-b_4b_3 \in \rho$. Write $q_{14}=\tilde{q}_{14}a_{14}$, $q_{25}=\tilde{q}_{25}a_{25}$ and $q_{34}=\tilde{q}_{34}a_{34}$ where $a_{14},a_{25},a_{34} \in A(\Gamma)$. Then

$$\begin{array}{ccc} b_1 & b_2 & b_3 \\ | & | & | \\ b_1a_{14}^{-1} & b_2a_{25}^{-1} & b_3a_{34}^{-1} \end{array}$$

is a partially ordered subset of S_v which is isomorphic to PO_2; thus rep(S_v) is of infinite representation type. Similarly, if $b_5b_3 \in \rho_0$, then rep(S_v) is of infinite representation type.

__Case 2__: Suppose $b_4b_2 \in \rho_0$. Let q_{24} be a directed path in ρ_0 such that $q_{24}-b_4b_2 \in \rho$. Write $q_{24}=c_n\ldots c_1$ where each $c_i \in A(\Gamma)$. Since q_{24} and q_{25} have the same origin, there must exist an arrow $d \in A(\Gamma)$ such that $d \neq c_i$ for any i, but for some i, $1 \leqslant i \leqslant n-1$, d has the same origin as c_i. Pick i_0 to be the smallest value of i for which this occurs.
Then

$$\langle b_1,\ b_3,\ b_2c_1^{-1}\ldots c_{i_0}^{-1},\ b_2\ c_1^{-1}\ldots c_{i_0-1}^{-1}d^{-1}\rangle$$

is a partially ordered subset of S_v which is isomorphic to

PO_1. Thus S_v is of infinite representation type. Similarly, if $b_5b_1 \in \rho_0$, then S_v is of infinite representation type.

Case 3: Suppose $b_4b_3, b_5b_3, b_4b_2, b_5b_1 \notin \rho_0$, but $b_4b_1, b_5b_2 \in \rho_0$ as originally assumed. Then

$$\begin{array}{ccccc} b_4^{-1} & & b_5^{-1} & & \\ | & & | & & \\ b_1 & & b_2 & & b_3 \\ | & & | & & \\ b_1a_{14}^{-1} & & b_2a_{25}^{-1} & & \end{array}$$

is a partially ordered subset of S_v which is isomorphic to PO_3, and S_v is of infinite representation type.

Therefore, in all cases, S_v is of infinite representation type, so by Theorem 2.6, Λ_0 is of infinite representation type.

Again let Γ be a finite graph and let ρ be any set of relations on Γ. Let $\bar{\Gamma}$ be a directed, path connected subgraph of Γ. Let $t = \sum_{i=1}^{n} \mu_i p_i \in \rho$ where for each i, $\mu_i \in K$ and p_i is a directed path. Let Z_t be the subset of $\{p_1, \ldots p_n\}$ consisting of those p_i's which lie entirely in $\bar{\Gamma}$. If $Z_t \neq \phi$, let $\bar{t} = \sum_{p_i \in Z_t} \mu_i p_i$. Define $\bar{\rho} = \{\bar{t} : t \in \rho \text{ and } Z_t \neq \phi\}$. The graph with relations $(\bar{\Gamma}, \bar{\rho})$ is called a <u>branch graph with relations</u> of (Γ, ρ).

There is another type of graph with relations, $(\bar{\Gamma},\bar{\rho})$, which can be derived from (Γ,ρ). This involves the "shrinking" of an arrow $a \in A(\Gamma)$ from the graph Γ. To be more precise, let a be an arrow in $A(\Gamma)$ with origin u_1 and terminus u_2 such that for each $b \in A(\Gamma)$, $ab \notin \rho_0$ and $ba \notin \rho_0$ where ρ_0 is the set of zero relations associated to ρ. In other words, a is not part of a zero relation of length 2. We define $\bar{\Gamma}$ to be the directed graph obtained from Γ by shrinking $u_1 \xrightarrow{a} u_2$ to a point and identifying u_1 with u_2. Let u denote the vertex arising from this identification. Then $V(\bar{\Gamma})=(V(\Gamma)\cup\{u\})-\{u_1,u_2\}$ and $A(\bar{\Gamma})=A(\Gamma)-\{a\}$; however, if b is an arrow in $A(\Gamma)$ with origin (terminus) u_i, $i=1,2$, then in $\bar{\Gamma}$ b has origin (terminus) u. We now need to define $\bar{\rho}$. For each directed path p in Γ such that $a|p$ let $\bar{p}$ be the directed path in $\bar{\Gamma}$ which is created when a is shrunk to a vertex. For each directed path p in Γ such that a is not a subpath of p, let $\bar{p}$ be the path p thought of as a path in $\bar{\Gamma}$. If $t=\sum_{i=1}^{n} \mu_i p_i$ is a K-linear combination of directed paths in Γ, then $\bar{t}=\sum_{i=1}^{n} \mu_i \bar{p}_i$ is a K-linear combination of directed paths in $\bar{\Gamma}$. Define $\bar{\rho}=\{\bar{t}:t\in\rho\}$. The set $\bar{\rho}$ is a set of relations on $\bar{\Gamma}$ since if $t=\sum_{i=1}^{n} \mu_i p_i$ and $a|p_i$, then by assumption p_i has length at least 3 and therefore $\bar{p}_i$ has length at least 2. The graph with relations $(\bar{\Gamma},\bar{\rho})$ is called a <u>shrunk graph with relations</u> of (Γ,ρ).

Finally, we say that a graph with relations $(\bar{\Gamma},\bar{\rho})$ is <u>contained in the graph with relations</u> (Γ,ρ) if there is a sequence of graphs with relations,

$$(\bar{\Gamma},\bar{\rho})=(\Gamma_0,\rho_1),\ (\Gamma_2,\rho_2),\ldots,\ (\Gamma_n,\rho_n)=(\Gamma,\rho)$$

such that for $i=1,\ldots,n-1$, (Γ_i,ρ_i) is either a branch graph with relations or a shrunk graph with relations of $(\Gamma_{i+1},\rho_{i+1})$. If Γ is a tree, then this definition of containment corresponds to the definition given by Bongartz and Ringel [2]. We call the process of obtaining $(\bar{\Gamma},\bar{\rho})$ from (Γ,ρ) the <u>reduction</u> of (Γ,ρ) to $(\bar{\Gamma},\bar{\rho})$.

<u>Proposition 2.11</u>: Suppose $(\bar{\Gamma},\bar{\rho})$ is a finite graph with relations which is contained in a finite graph with relations (Γ,ρ). For each $\bar{v}\in V(\bar{\Gamma})$, let $\bar{S}_{\bar{v}}$ be the set of all walks of nonzero length in $(\bar{\Gamma},\bar{\rho})$ with terminus $\bar{v}$, and for every $v\in V(\Gamma)$, let S_v denote the set of all walks of nonzero length in (Γ,ρ) with terminus v. If there is a $\bar{v}\in V(\bar{\Gamma})$ such that $\mathrm{rep}(\bar{S}_{\bar{v}})$ is of infinite representation type, then there is a $v\in V(\Gamma)$ such that $\mathrm{rep}(S_v)$ is of infinite representation type.

Proof: Clearly it suffices to prove the proposition when $(\bar{\Gamma},\bar{\rho})$ is either a branch graph with relations or a shrunk graph with relations of (Γ,ρ).

If $(\bar{\Gamma},\bar{\rho})$ is a branch graph with relations of (Γ,ρ) and $\bar{v}\in V(\bar{\Gamma})$ such that $\mathrm{rep}(\bar{S}_{\bar{v}})$ is of infinite representation type, then $\bar{v}\in V(\Gamma)$, and $\bar{S}_{\bar{v}}$ is a partially ordered subset of $S_{\bar{v}}$, the set of all walks of nonzero length in (Γ,ρ) with terminus $\bar{v}$. Thus $\mathrm{rep}(S_{\bar{v}})$ is of infinite representation type.

Suppose that $(\bar{\Gamma},\bar{\rho})$ is a shrunk graph with relations of (Γ,ρ). Let $\bar{v}\in V(\bar{\Gamma})$ such that $\mathrm{rep}(\bar{S}_{\bar{v}})$ is of infinite representation type. Let $a\in A(\Gamma)$ be the arrow which is shrunk to a vertex in the reduction of (Γ,ρ) to $(\bar{\Gamma},\bar{\rho})$. Let $v\in V(\Gamma)$ be a vertex which becomes identified with $\bar{v}$ after shrinking a. It is possible that $v=\bar{v}$. Let $p=b_n^{e_n}\ldots b_1^{e_1}$ be a path in Γ of nonzero length where for each i, $b_i\in A(\Gamma)$ and $e_i=\pm 1$. We define a corresponding path $\bar{p}$ in $\bar{\Gamma}$ as follows. If a^e is not a subpath of p for $e=\pm 1$, then $\bar{p}=p$. If $a^e|p$, say $a^e=b_\ell^{e_\ell}$. In this case, let $\bar{p}=b_n^{e_n}\ldots b_{\ell+1}^{e_{\ell+1}}b_{\ell-1}^{e_{\ell-1}}\ldots b_1^{e_1}$, that is, the path in Γ to which p shrinks. Note that if q is a path in $\bar{\Gamma}$ of nonzero length, then there is a path p in Γ such that $q=\bar{p}$. Moreover, q is a walk in $(\bar{\Gamma},\bar{\rho})$ if and only if p is a walk in (Γ,ρ).

Since $\mathrm{rep}(\bar{S}_{\bar{v}})$ is of infinite representation type, there is a partially ordered subset $\bar{R}$ of $\bar{S}_{\bar{v}}$ which is isomorphic to PO_i for some PO_i in Theorem 2.2. Write $\bar{R}=\{\hat{w}_1,\ldots,\hat{w}_m\}$ where each $\hat{w}_i$ is a walk in $(\bar{\Gamma},\bar{\rho})$ with terminus $\bar{v}$. For each i, pick w_i in S_v such that $\bar{w}_i=\hat{w}_i$. Let $R=\{w_1,\ldots,w_m\}$. For notational convenience we will write $\bar{R}=\{\bar{w}_1,\ldots,\bar{w}_m\}$.

<u>Claim</u>: If $w_i \leqslant w_j$, then either $\bar{w}_i \leqslant \bar{w}_j$ or $\bar{w}_i \geqslant \bar{w}_j$.

<u>Case 1</u>: $w_i=w_jw_i^*$ and $w_j=w_j^*c$ where w_i^* and w_j^* are walks in (Γ,ρ) and $c\in A(\Gamma)$. If $c=a$, then $\bar{w}_i=\bar{w}_j\bar{w}_i^*$ and $\bar{w}_j=w_j^*$. This implies that either $\bar{w}_i \leqslant \bar{w}_j$ or $\bar{w}_j \leqslant \bar{w}_i$ depending on the structure of w_j^*. If $c\neq a$, then $\bar{w}_i=\bar{w}_j\bar{w}_i^*$ and $\bar{w}_j=\bar{w}_j^*c$, so $\bar{w}_i \leqslant \bar{w}_j$.

<u>Case 2</u>: Suppose $w_j=w_iw_j^*$ and $w_i=w_i^*c^{-1}$ where w_j^* and w_i^* are walks in (Γ,ρ) and $c\in A(\Gamma)$. Then arguing in a manner similar to Case 1, we see that either $\bar{w}_i \leqslant \bar{w}_j$ or $\bar{w}_j \leqslant \bar{w}_i$.

<u>Case 3</u>: Suppose $w_i=wrw_i^*$ and $w_j=ws^{-1}w_j^*$ where w, w_i^*, w_j^* are

walks in (Γ,ρ) and $sr\in\rho_0$, the set of zero relations associated to ρ. Then $\bar{w}_i=\overline{wrw}_i^*$ and $\bar{w}_j=\overline{ws}^{-1}\bar{w}_j^*$ where $\bar{w}$, $\bar{w}_i^*$, $\bar{w}_j^*$ are walks in $(\bar{\Gamma},\bar{\rho})$ and by defnintion of $\bar{\rho}$, $\bar{s}\bar{r}\in\bar{\rho}_0$ the set of zero relations associated to $\bar{\rho}$. So $\bar{w}_1\leqslant\bar{w}_2$.

With the claim established it is clear that either R and $\bar{R}$ are isomorphic as partially ordered sets or that R contains a partially ordered subset which is isomorphic to PO_1. In either case, rep(R), and hence rep(S_v), is of infinite representation type.

We know from Proposition 2.10 that if (Γ,ρ) is a finite, arrow unique graph with commutativity relations and Γ has a vertex of degree 5, then $\Lambda_0=K\Gamma/\langle\rho_0\rangle$ is of infinite representation type. This is also true if (Γ,ρ) contains a graph with relations, $(\bar{\Gamma},\bar{\rho})$ such that $\bar{\Gamma}$ has a vertex of degree 5, as the following proposition establishes.

<u>Proposition</u> <u>2.12</u>: Let (Γ,ρ) be a finite, arrow unique graph with commutativity relations which contains a graph with relations $(\bar{\Gamma},\bar{\rho})$ such that $\bar{\Gamma}$ has a vertex of degree 5. Suppose there are no circular walks in (Γ,ρ). Then $\Lambda_0=K\Gamma/\langle\rho_0\rangle$ is of infinite representation type.

<u>Proof</u>: Let $\bar{v}$ be a vertex of degree 5 in $V(\bar{\Gamma})$. Let

$b_1, b_2, \ldots b_5$ be distinct arrows in $A(\bar{\Gamma})$ such that for each i there exists $e_i = \pm 1$ such that $b_i^{e_i}$ has terminus $\bar{v}$. Let $v \in V(\Gamma)$ be a vertex which becomes identified with $\bar{v}$ in the reduction of (Γ, ρ) to $(\bar{\Gamma}, \bar{\rho})$. For each i, let u_i be the terminus of $b_i^{e_i}$ in Γ and let w_i be a walk in (Γ, ρ) with origin u_i and terminus v which shrinks to the vertex $\bar{v}$ in the reduction of (Γ, ρ) to $(\bar{\Gamma}, \bar{\rho})$.

Suppose that for some i and j there is a walk w_i^* of nonzero length which is a subpath of w_i and a walk w_j^* of nonzero length which is a subpath of w_j such that w_i^* and w_j^* have the same origin, x, and the same terminus, y, but w_i^* and w_j^* have no other common subpaths. Then $(w_j^*)^{-1} w_i^*$ is a path in Γ with origin and terminus x. In fact, $(w_j^*)^{-1} w_i^*$ must be a walk in (Γ, ρ) because it shrinks to the vertex $\bar{v}$ in the reduction of (Γ, ρ) to $(\bar{\Gamma}, \bar{\rho})$. This is a contradiction to the fact that there are no circular walks in (Γ, ρ).

Therefore, for each i and j we can write $w_i = w_{ij} \tilde{w}_{ij}$ and $w_j = w_{ji} \tilde{w}_{ji}$, where w_{ij}, $\tilde{w}_{ij}$, w_{ji}, $\tilde{w}_{ji}$ are walks in (Γ, ρ),

$w_{ij}=w_{ji}$ and $\tilde{w}_{ij}$ and $\tilde{w}_{ji}$ have no common subpaths except their common terminus.

Let $\bar{\rho}_0$ be the set of zero relations associated to $\bar{\rho}$. Suppose $b_i b_j \epsilon \bar{\rho}_0$. Then there exist a directed path p_{ij} in Γ such that $b_i p_{ij} b_j \epsilon \rho_0$, and p_{ij} shrinks to $\bar{v}$ in the reduction of (Γ,ρ) to $(\bar{\Gamma},\bar{\rho})$. Therefore p_{ij} is a walk in (Γ,ρ) and because there are no circular walks in (Γ,ρ), $p_{ij}=(\tilde{w}_{ij})^{-1}\tilde{w}_{ji}$. Hence $b_i(\tilde{w}_{ij})^{-1}\tilde{w}_{ji}b_j \epsilon \rho_0$ and no proper subpath of $b_i(\tilde{w}_{ij})^{-1}\tilde{w}_{ji}b_j$ is in ρ_0. For each i and j such that $b_i b_j \epsilon \bar{\rho}_0$, let q_{ij} be a directed path in ρ_0 which commutes with $b_i(\tilde{w}_{ij})^{-1}\tilde{w}_{ji}b_i$ in Γ.

We now define another graph with relations, $(\bar{\Gamma}_1,\bar{\rho}_1)$ which is contained in (Γ,ρ). First, let $\bar{\Gamma}_2$ be the connected subgraph of Γ consisting of $b_1,\ldots,b_5$, $w_1,\ldots,$ w_5 and the paths q_{ij} for which $b_i b_j \epsilon \bar{\rho}_0$. Let $\bar{\Gamma}_1$ be the graph obtained from $\bar{\Gamma}_2$ by shrinking each w_i to the same vertex $\bar{v}_1$. Define

$$\bar{\rho}_1=\{q_{ij}-b_i b_j:\ b_i b_j \epsilon \bar{\rho}_0\} \cup Z$$

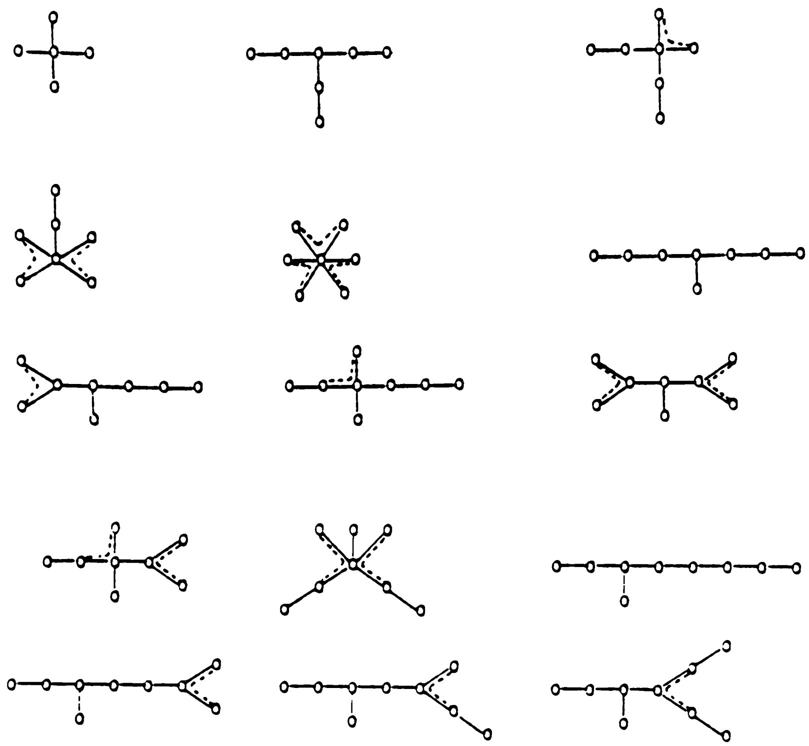
where $Z=\{p \epsilon \rho_0$ such that $p|q_{ij}$ for some i and j$\}$.

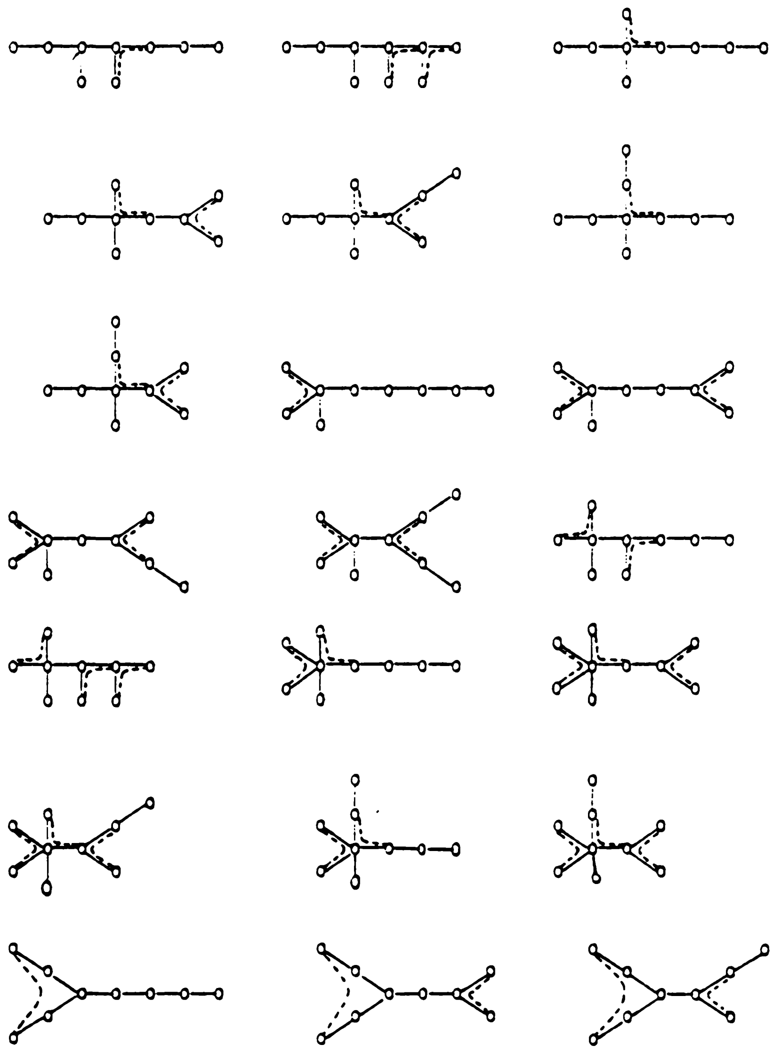
The graph with relations $(\bar{\Gamma}_1,\bar{\rho}_1)$ is contained in (Γ,ρ). If $Z\neq\phi$, then $\bar{S}_{\bar{v}_1}$, the set of all walks of nonzero length in $(\bar{\Gamma}_1,\bar{\rho}_1)$ with terminus $\bar{v}_1$, contains a partially ordered subset which is isomorphic to PO_1. Therefore $rep(\bar{S}_{\bar{v}_1})$ is of infinite representation type and by Proposition 2.11, there is a $v\in V(\Gamma)$ for which $rep(S_v)$ is of infinite representation type. Therefore, Λ_0 is of infinite representation type.

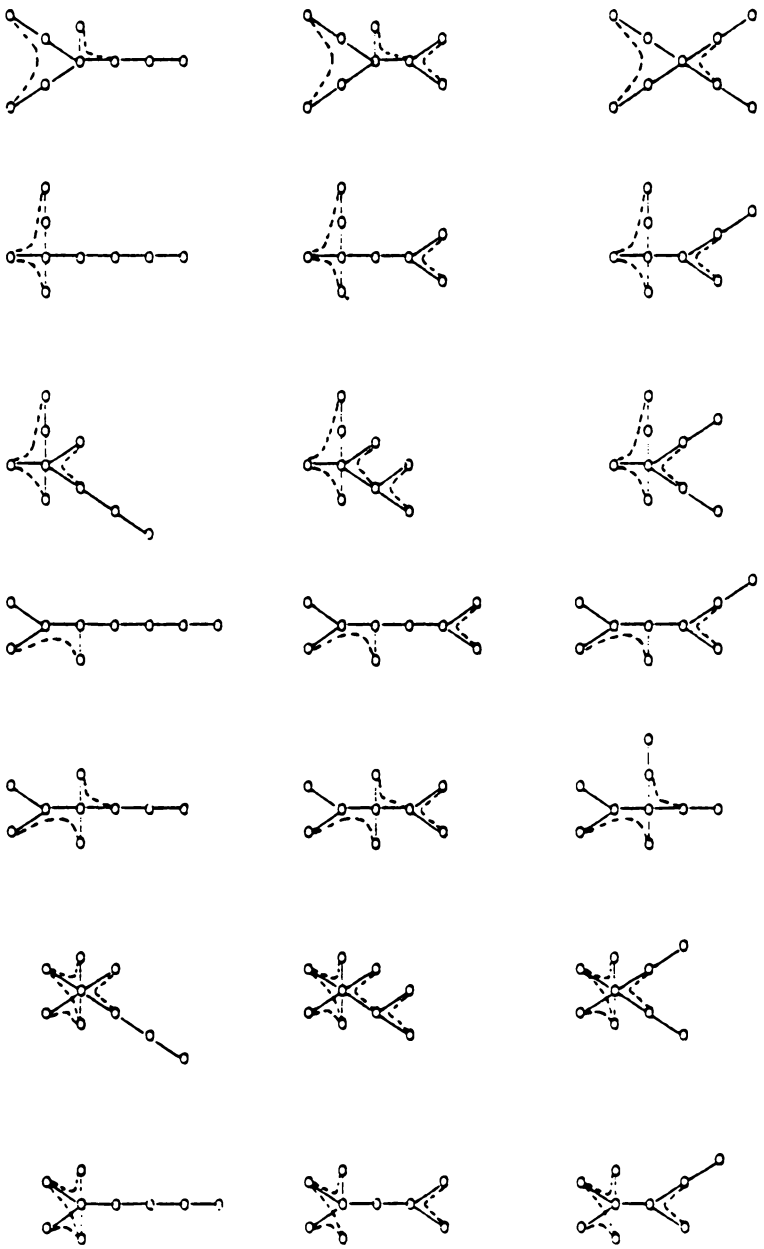
If $Z=\phi$, then $(\bar{\Gamma}_1,\bar{\rho}_1)$ is a finite, arrow unique graph with commutativity relations so by Proposition 2.10, $rep(\bar{S}_{\bar{v}_1})$ is of infinite representation type. Again this gives us that Λ_0 is of infinite representation type.

We now know that if (Γ,ρ) is a finite, arrow unique graph with commutativity relations, then $\Lambda_0=K\Gamma/\langle\rho_0\rangle$ is of infinite representation type if there is a circular walk in (Γ,ρ) or if (Γ,ρ) contains a graph with relations, $(\bar{\Gamma},\bar{\rho})$, such that $\bar{\Gamma}$ has a vertex of degree 5. In order to consider other cases we need the following result of Bongartz and Ringel [2].

<u>Theorem</u> <u>2.12</u>: Let (T,ρ) be a finite tree with relations. Then (T,ρ) is of infinite representation type if and only if (T,ρ) contains one of the following trees with relations:







The dotted lines represent relations in ρ. The direction of the arrows does not matter except that all of the arrows in a relation must, of course, point in the same direction.

<u>Lemma</u> <u>2.13</u>: Let (Γ,ρ) be a finite, arrow unique graph with commutativity relations. Suppose there are no circular walks in (Γ,ρ). Let $v \in V(\Gamma)$ and $v' \in F^{-1}(v)$. Consider $T_{v'}$. Suppose $u_1', u_2' \in V(T_{v'})$ such that $u_1' \neq u_2'$, but $F(u_1')=F(u_2')$. Let p' be the unique path in $T_{v'}$ with origin u_1' and terminus u_2'. Then there is a directed path $q' \in \rho_{v'}$ such that $q' \mathrm{I} p'$ or $q' \mathrm{I} (p')^{-1}$.

<u>Proof</u>: Let $u=F(u_1')=F(u_2')$. Then $F(p')$ is a path in Γ with origin u and terminus u. Since there are no circular walks in (Γ,ρ), there is a directed path $q \in \rho_0$ such that $q \mathrm{I} F(p')$ or $q \mathrm{I} (F(p'))^{-1}$. Thus there is a directed path q' in $T_{v'}$ such that $F(q')=q$ and $q' \mathrm{I} p'$ if $q \mathrm{I} F(p)$ or $q' \mathrm{I} (p')^{-1}$ if $q \mathrm{I} (F(p'))^{-1}$. Moreover, $q' \in \rho_{v'}$ since $q \in \rho_0$.

We are now ready to state the main and final result of this section.

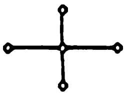
<u>Theorem</u> <u>2.14</u>: Let (Γ,ρ) be a finite, arrow unique graph

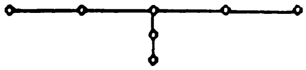
with commutativity relations. Let ρ_0 be the set of zero relations associated to ρ. Then $\Lambda_0 = K\Gamma/\langle\rho_0\rangle$ is of infinite representation type if and only if one of the following holds:

(1) There is a circular walk in (Γ,ρ).

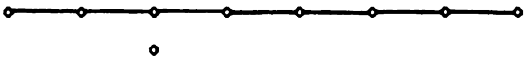
(2) There is a graph with relations $(\bar{\Gamma},\bar{\rho})$ contained in (Γ,ρ) such that $\bar{\Gamma}$ has a vertex of degree 5.

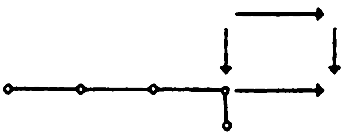
(3) (Γ,ρ) contains one of the following arrow unique graphs with commutativity relations:

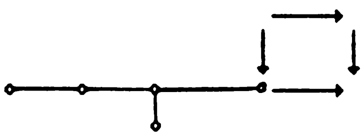
(a) 

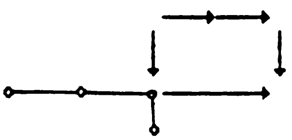
(b) 

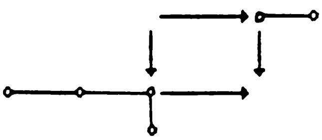
(c) 

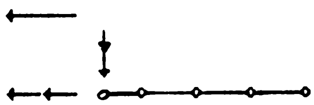
(d) 

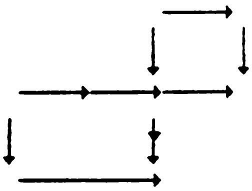
(e) 

(f) 

(g) 

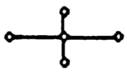
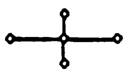
(h) 

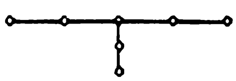
(i) 

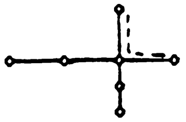
(j) 

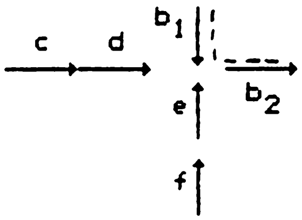
The direction of the unoriented edges does not matter.

<u>Proof</u>: Suppose A_0 is of infinite representation type and that (1) and (2) do not occur. Then there is a vertex $v \in V(\Gamma)$ such that $\operatorname{rep}(S_v)$ is of infinite representation type. Let $v' \in F^{-1}(v)$. Then $(T_{v'}, \rho_{v'})$ is of infinite representation type. Since (1) does not occur, $T_{v'}$ is a finite tree, therefore $(T_{v'}, \rho_{v'})$ contains some (T, σ) from the list in Theorem 2.12. Since (2) does not occur, T does not have a vertex of degree 5, nor does (T, σ) contain a tree with relations which has a vertex of degree 5. We begin a case by case study of possible (T, σ).

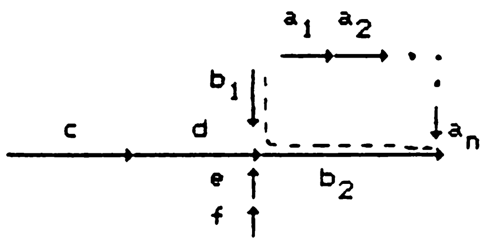
<u>Case 1</u>: Suppose (T, σ) is . Since $\sigma = \phi$, each arrow in $A(T)$ corresponds to a unique arrow in $A(\Gamma)$ by Lemma 2.13. Therefore (Γ, ρ) contains , which is (a).

<u>Case 2</u>: Suppose (T, σ) is . Then arguing as in Case 1, (Γ, ρ) contains  and this is (b).

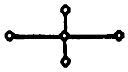
<u>Case 3</u>: (T,σ) is . For the sake of clarity we give directions and names to the arrows in T as follows:

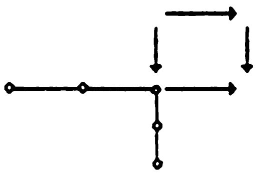


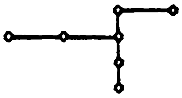
Again, by Lemma 2.13, each arrow in $A(T)$ corresponds to a unique arrow in $A(\Gamma)$. Let p' be the directed path in $\rho_{v'}$ that shrinks to b_2b_1 in the reduction of $(T_{v'},\rho_{v'})$ to (T,σ). Let $p=F(p')$ and let q be a directed path in ρ_0 such that $p-q\in\rho$. Note that p' can be chosen so that p and q have no common subpaths of nonzero length, and we assume that we have done so. Write $q=a_n\ldots a_1$ where each $a_i\in A(\Gamma)$. For $i=1,\ldots,n$, let u_i denote the terminus of a_i and let u_0 be the origin of a_1. Then (Γ,ρ) contains



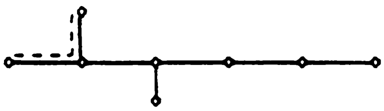
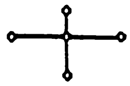
with the relation $b_2b_1-a_n\ldots a_1$ and with the possibility of some subpaths of q being zero relations. Suppose there is an arrow $h\in A(\Gamma)$ such that h has origin u_i for some i,

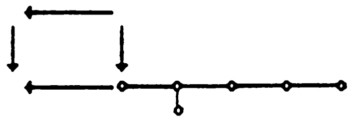
$0 \leqslant i \leqslant n-2$, or h has terminus u_i for some i, $2 \leqslant i \leqslant n$. Then (Γ,ρ) contains . If no such arrow h exists, then q can be shrunk to a path of length 2. In this case (Γ,ρ) contains the graph with commutativity relations

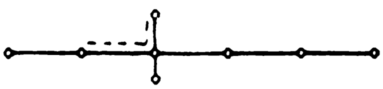
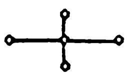
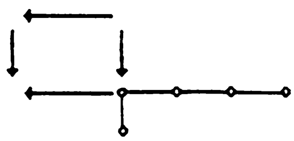


which contains , which is (b).

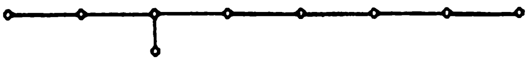
<u>Case</u> <u>4</u>: (T,σ) is . Then (Γ,ρ) contains , which is (c).

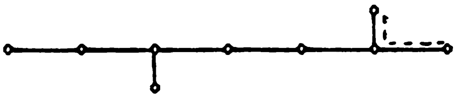
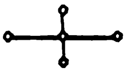
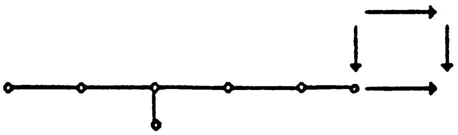
<u>Case</u> <u>5</u>: (T,σ) is . Then arguing as in Case 3, either (Γ,ρ) contains , or (Γ,ρ) contains

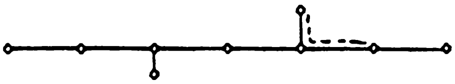
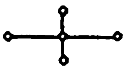
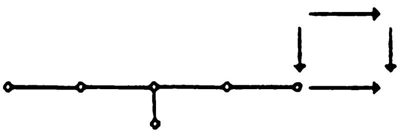
 with commutativity relations which contains (f).

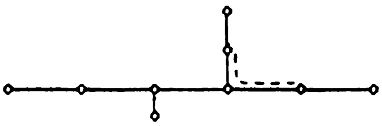
<u>Case</u> <u>6</u>: (T,σ) is . Then either (Γ,ρ) contains , or (Γ,ρ) contains  with

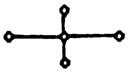
commutativity relations, which is (e).

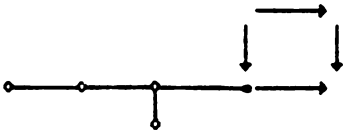
<u>Case 7</u>: (T,σ) is . Then (Γ,ρ) contains , which is (d).

<u>Case 8</u>: (T,σ) is . Then either (Γ,ρ) contains , or (Γ,ρ) contains  with commutativity relations which contains (f).

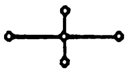
<u>Case 9</u>: (T,σ) is . Then either (Γ,ρ) contains , or (Γ,ρ) contains  with commutativity relations, which contains (f).

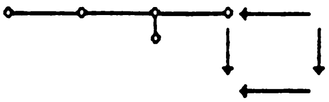
<u>Case 10</u>: (T,σ) is . Then either

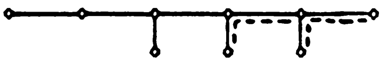
(Γ,ρ) contains , or (Γ,ρ) contains

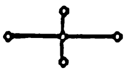
 with commutativity relations, which is (f).

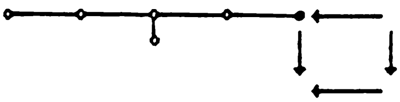
Case 11: (T,σ) is . Then either

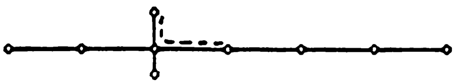
(Γ,ρ) contains , or (Γ,ρ) contains

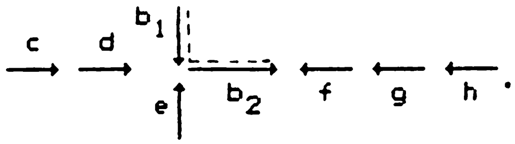
 with commutativity relations, which contains (f).

Case 12: (T,σ) is . Then either

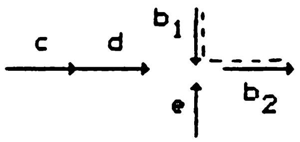
(Γ,ρ) contains , or (Γ,ρ) contains

 with commutativity relations, which contains (f).

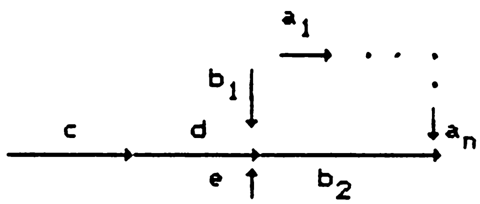
Case 13: (T,σ) is . In this case we need to examine (Γ,ρ) more carefully. We give names and directions to the arrows in (T,σ) as follows:

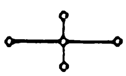


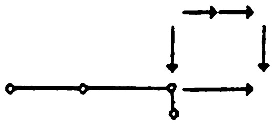
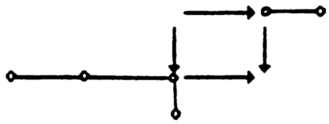
Let $p'\epsilon\rho_{v'}$ be a directed path in $T_{v'}$ which shrinks to b_2b_1 in the reduction of $(T_{v'},\rho_{v'})$ to (T,σ). Let $p=F(p')$ and let q be a directed path in ρ_0 such that $p-q\epsilon\rho$. We assume that we have chosen p' such that p and q have no common subpaths of nonzero length. Write $q=a_n\ldots a_1$ where each $a_i\epsilon A(\Gamma)$. For $i=1,\ldots,n$, let u_i be the terminus of a_i and let u_0 be the origin of u_1. By Lemma 2.14,

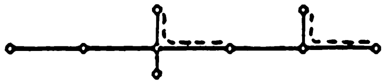
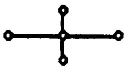
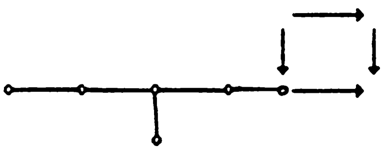


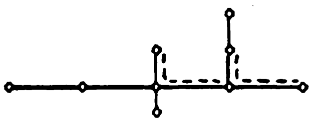
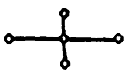
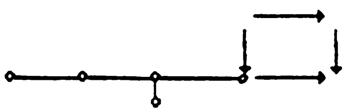
is contained in (Γ,ρ), hence (Γ,ρ) contains

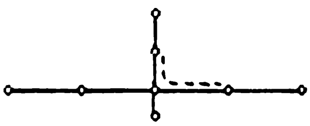


with the relation $b_2b_1-a_n\ldots a_1$, and with the possibility that some subpaths of q are zero relations. If there is an arrow $h\epsilon A(\Gamma)$ with origin u_i for some i, $0\leqslant i\leqslant n-2$, or with terminus u_i for some i, $2\leqslant i\leqslant n$, then (Γ,ρ) contains .

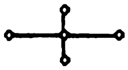
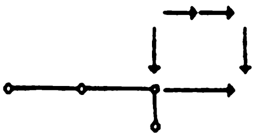
Suppose this is not the case. If the length of q is at least 3, then (Γ,ρ) contains  with commutativity relations, which is (g). If the length of q is 2, then because f,g and h correspond to arrows in $A(\Gamma)$, then (Γ,ρ) must contain  with commutativity relations, which is (h).

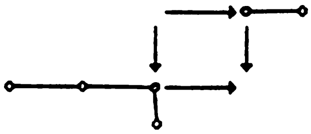
<u>Case</u> <u>14</u>: (T,σ) is . Then either (Γ,ρ) contains , or (Γ,ρ) contains  with commutativity relations, which contains (f).

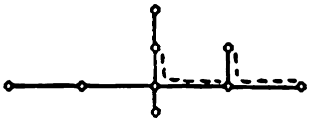
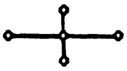
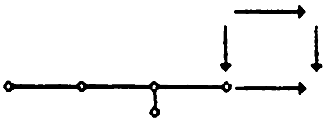
<u>Case</u> <u>15</u>: (T,σ) is . Then either (Γ,ρ) contains , or (Γ,ρ) contains  with commutativity relations, which is (f).

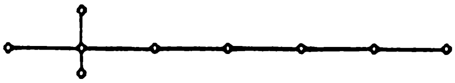
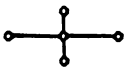
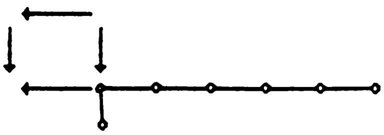
<u>Case</u> <u>16</u>: (T,σ) is . This is similar to

Case 13. (Γ,ρ) must contain at least one of the following graphs with commutativity relations:

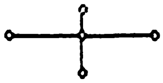
(a)  (g) 

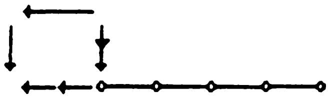
(h) 

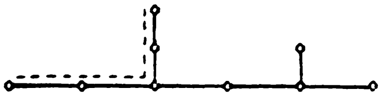
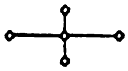
Case 17: (T,σ) is . Then either (Γ,ρ) contains , or (Γ,ρ) contains  with commutativity relations, which is (f).

Case 18: (T,σ) is . Then either (Γ,ρ) contains , or (Γ,ρ) contains  with commutativity relations, which contains  which is (d).

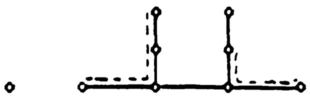
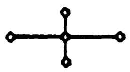
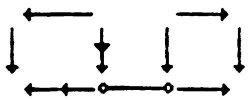
Case 19: (T,σ) is . Then either

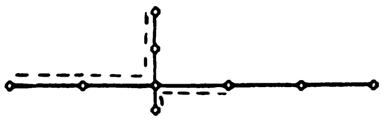
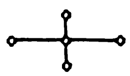
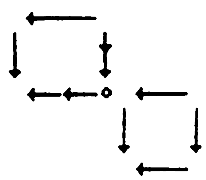
(Γ,ρ) contains , or (Γ,ρ) contains

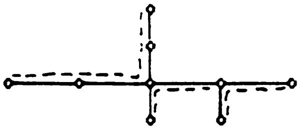
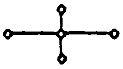
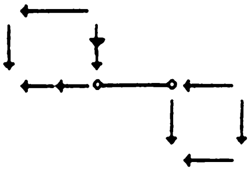
 with commutativity relations, which is (i).

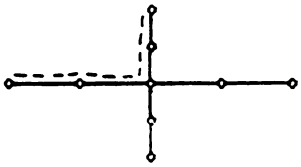
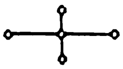
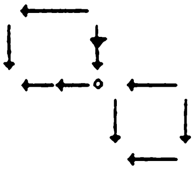
<u>Case</u> <u>20</u>: (T,σ) is . Then either (Γ,ρ) contains , or (Γ,ρ) contains

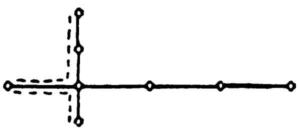
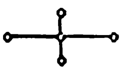
 with commutativity relations, which contains (j).

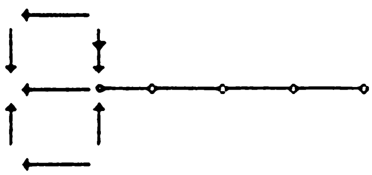
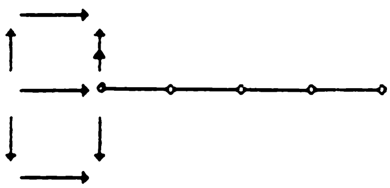
<u>Case</u> <u>21</u>: (T,σ) is . Then either (Γ,ρ) contains , or (Γ,ρ) contains  with commutativity relations, which contains (j).

<u>Case</u> <u>22</u>: (T,σ) is . Then either (Γ,ρ) contains , or (Γ,ρ) contains  with commutativity relations, which is (j).

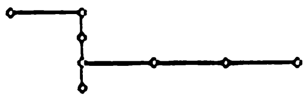
<u>Case</u> <u>23</u>: (T,σ) is . Then either (Γ,ρ) contains , or (Γ,ρ) contains  with commutativity relations, which contains (j).

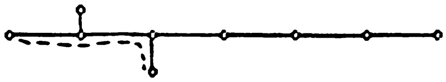
<u>Case</u> <u>24</u>: (T,σ) is . Then either (Γ,ρ) contains , or (Γ,ρ) contains  with commutativity relations, which is (j).

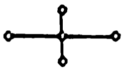
<u>Case</u> <u>25</u>: (T,σ) is . Then either (Γ,ρ) contains , or (Γ,ρ) contains

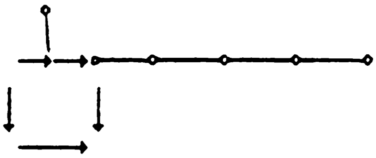
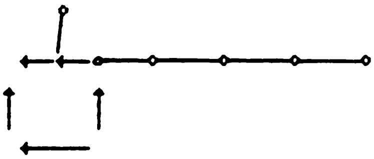
 or  with

commutativity relations, both of which contain

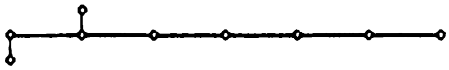
, which is (c).

<u>Case 26</u>: (T,σ) is . Then either

(Γ,ρ) contains , or (Γ,ρ) contains

 or  with

commutativity relations, both of which contain

, which is (d).

Conversely, suppose (Γ,ρ) satisfies (1), (2), or (3). If (1) holds, then Λ_0 is of infinite representation type by Proposition 2.9. If (2) holds, then Λ_0 is of infinite representation type by Proposition 2.12. If (3) holds, then Λ_0 is of infinite representation type by Proposition 2.11.

Section 3: Almost Split Sequences over Factor Rings

Throughout this section Λ will denote an Artin algebra. We will look at the relationship between almost split sequences over Artin algebras and their factor rings. An almost split sequence of Λ-modules is a short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ with the following 4 properties:

(1) A and C are indecomposable Λ-modules.

(2) $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is not a split exact sequence.

(3) Given any Λ-module X and a Λ-map $h:X \longrightarrow C$ which is not a splittable epimorphism, there exists a Λ-map $t:X \longrightarrow B$ such that $gt=h$.

(4) Given any Λ-module Y and a Λ-map $\ell:A \longrightarrow Y$ which is not a splittable monomorphism, there exists a Λ-map $s:B \longrightarrow Y$ such that $sf=\ell$.

Moreover, if C is any nonprojective, indecomposable Λ-module, there exist Λ-modules A and B, and Λ-maps f and g, such that $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is an almost split sequence. For a further discussion of almost split sequences see [10].

We will keep the following notation for the remainder of this section. Let I be an ideal of Λ and let $\bar{\Lambda}$ be the factor ring Λ/I. Let M denote an indecomposable, nonprojective $\bar{\Lambda}$-module. The short exact sequence

$$(*)\quad 0 \longrightarrow \bar{A} \xrightarrow{\bar{f}} \bar{B} \xrightarrow{\bar{g}} M \longrightarrow 0$$

will be an almost split sequence of $\bar{\Lambda}$-modules. Our aim is to determine when (*) remains an almost split sequence when viewed as a sequence of Λ-modules.

<u>Proposition</u> <u>3.1</u>: Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} M \longrightarrow 0$ be an almost split sequence of Λ-modules. Then there exist monomorphisms $\alpha:\bar{A}\longrightarrow A$ and $\beta:\bar{B}\longrightarrow B$ such that the following diagram commutes.

$$(3.2)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \bar{A} & \xrightarrow{\bar{f}} & \bar{B} & \xrightarrow{\bar{g}} & M & \longrightarrow & 0 \\ & & \alpha\downarrow & & \beta\downarrow & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & M & \longrightarrow & 0 \end{array}$$

<u>Proof</u>: Since $\bar{g}$ is not a splittable epimorphism when viewed as a $\bar{\Lambda}$-map, it is not a splittable epimorphism when viewed as a Λ-map. Therefore, there exists a map $\beta:\bar{B}\longrightarrow B$ such that $g\beta=\bar{g}$. It follows that there exists a map $\alpha:\bar{A}\longrightarrow A$ such that the diagram (3.2) commutes. Suppose β is not a monomorphism. Since $\ker(\beta) \subseteq \ker(\bar{g})=\mathrm{im}(\bar{f})$, there is a nonzero submodule L of $\bar{A}$ such that $\bar{f}(L)=\ker(\beta)$. Let $\pi_{\bar{A}}:\bar{A}\longrightarrow\bar{A}/L$ and $\pi_{\bar{B}}:\bar{B}\longrightarrow\bar{B}/L$ be the canonical surjections.

Since $L\neq 0$, $\pi_{\bar{A}}$ is not a splittable monomorphism. Therefore, there is a map $\gamma:\bar{B}\to\bar{A}/L$ such that $\gamma\bar{f}=\pi_{\bar{A}}$. Now $\gamma(\mathrm{Ker}(\beta))=\gamma\bar{f}(L)=\pi_{\bar{A}}(L)=0$, therefore, γ factors through $\bar{B}/\mathrm{Ker}(\beta)$; that is, there is a map $\gamma':\bar{B}/\mathrm{Ker}(\beta)\to\bar{A}/L$ such that $\gamma'\pi_{\bar{B}}=\gamma$. We have the following diagram:

$$\begin{tikzcd}
\bar{A} \arrow[r, "\bar{f}"] \arrow[d, "\pi_{\bar{A}}"'] & \bar{B} \arrow[d, "\pi_{\bar{B}}"] \arrow[dl, "\gamma"'] \\
\bar{A}/L \arrow[r, shift left, "f'"] & \bar{B}/\mathrm{Ker}(\beta) \arrow[l, shift left, "\gamma'"]
\end{tikzcd}$$

where f' is the map induced by f. Now $\gamma'f'\pi_{\bar{A}}=\gamma'\pi_{\bar{B}}\bar{f}=\gamma\bar{f}=\pi_{\bar{A}}$, thus, since $\pi_{\bar{A}}$ is an epimorphism, $\gamma'f'=1_{\bar{A}/L}$. Let $g':\bar{B}/\mathrm{Ker}(\beta)\to M$ and $\beta':\bar{B}/\mathrm{Ker}(\beta)\to B$ be the maps induced by $\bar{g}$ and β respectively. Then the short exact sequence $0\to\bar{A}/L\xrightarrow{f'}\bar{B}/\mathrm{Ker}(\beta)\xrightarrow{g'}M\to 0$ is split exact because $\gamma'f'=1_{\bar{A}/L}$. It follows that there is a map $h:M\to\bar{B}/\mathrm{Ker}(\beta)$ such that $g'h=1_M$. But this gives a splitting of $g:B\to M$ since $\bar{g}=g\beta \Rightarrow g'\pi_{\bar{B}}=g\beta'\pi_{\bar{B}} \Rightarrow g'=g\beta' \Rightarrow g'h=g\beta'h \Rightarrow 1_M=g\beta'h$, a contradiction to the fact that $0\to A\xrightarrow{f}B\xrightarrow{g}M\to 0$ is an almost split sequence. Therefore, β is a monomorphism and, according to the snake lemma, α is a monomorphism also.

Let D (respectively, $\bar{D}$) denote the duality between left and right Λ (respectively, $\bar{\Lambda}$) modules and let tr_{Λ} (respectively, $tr_{\bar{\Lambda}}$) denote the transpose. It is well known that if Λ is an Artin algebra and $0 \to A \to B \to C \to 0$ is an almost split sequence of Λ-modules, then A is isomorphic to $Dtr_{\Lambda}(M)$. We therefore have the following:

<u>Corollary</u> <u>3.3</u>: Let M be an indecomposable, nonprojective $\bar{\Lambda}$-module. Then $\bar{D}tr_{\bar{\Lambda}}(M)$ is isomorphic to a Λ-submodule of $Dtr_{\Lambda}(M)$.

Noting that if N is a $\bar{\Lambda}$-module, then the $\bar{\Lambda}$-length and the Λ-length of N are equal, we present our first result for determining when (*) is an almost split sequence of Λ-modules.

<u>Theorem</u> <u>3.4</u>: Let Λ be an Artin algebra and let $\bar{\Lambda}=\Lambda/I$ where I is an ideal in Λ. Suppose that (*) $0 \to \bar{A} \xrightarrow{\bar{f}} \bar{B} \xrightarrow{\bar{g}} M \to 0$ is an almost split sequence of $\bar{\Lambda}$-modules. The following statements are equivalent:

(1) The sequence (*) is an almost split sequence when

viewed as a sequence of Λ-modules.

(2) $Dtr_\Lambda(M)$ is isomorphic to $\bar{D}tr_{\bar{\Lambda}}(M)$.

(3) The $\bar{\Lambda}$-length of $\bar{D}tr_{\bar{\Lambda}}(M)$ is greater than or equal to the Λ-length of $Dtr_\Lambda(M)$.

<u>Proof</u>: (1⇒2) If $0 \longrightarrow \bar{A} \xrightarrow{\bar{f}} \bar{B} \xrightarrow{\bar{g}} M \longrightarrow 0$ is an almost split sequence when viewed as a sequence of Λ-modules, then $Dtr_{\bar{\Lambda}}(M) \cong \bar{A} \cong Dtr_\Lambda(M)$.

(2⇒3) If $Dtr_\Lambda(M) \cong \bar{D}tr_{\bar{\Lambda}}(M)$, then the Λ-length of $Dtr_\Lambda(M)$ equals the Λ-length of $\bar{D}tr_{\bar{\Lambda}}(M)$. But, as noted above, the latter is equal to the $\bar{\Lambda}$-length of $\bar{D}tr_{\bar{\Lambda}}(M)$.

(3⇒1) Suppose the $\bar{\Lambda}$-length of $\bar{D}tr_{\bar{\Lambda}}(M)$ is greater than or equal to the Λ-length of $Dtr_\Lambda(M)$. Then, as above, the Λ-length of $\bar{D}tr_{\bar{\Lambda}}(M)$ is greater than or equal to the Λ-length of $Dtr_\Lambda(M)$. Corollary 3.3 now implies that $\bar{D}tr_{\bar{\Lambda}}(M) \cong Dtr_\Lambda(M)$. From this it follows that the maps α and β of Proposition 3.1 are isomorphisms. Hence (*) is an almost split sequence when viewed as a sequence of Λ-modules.

We also have the following interesting consequence.

<u>Corollary 3.5</u>: Keeping the above notations, (*) is an almost split sequence of Λ-modules if and only if $Dtr_{\Lambda}(M)$ is a $\bar{\Lambda}$-module.

<u>Proof</u>: If $Dtr_{\Lambda}(M)$ is not a $\bar{\Lambda}$-module, then $IDtr_{\Lambda}(M)\neq 0$. But $I\bar{D}tr_{\bar{\Lambda}}(M)=0$, so $Dtr_{\Lambda}(M)$ and $\bar{D}tr_{\bar{\Lambda}}(M)$ are not isomorphic as Λ-modules. Hence (*) is not an almost split sequence of Λ-modules.

Conversely, suppose $Dtr_{\Lambda}(M)$ is a $\bar{\Lambda}$-module. Consider the monomorphism α of Proposition 3.1. Since $Dtr_{\Lambda}(M)$ is indecomposable, either α is an isomorphism and we are done, or α is not a splittable monomorphism. In the second case there exists a map $\delta:\bar{B}\longrightarrow Dtr_{\Lambda}(M)$ such that $\delta\bar{f}=\alpha$. We define a map ζ: $M\longrightarrow B$ as follows. Let $m\in M$. Then there is a $\bar{b}\in\bar{B}$ such that $\bar{g}(\bar{b})=m$. Define $\zeta(m)=(\beta-f\delta)(\bar{b})$. Note that ζ is well-defined, for if $\bar{g}(\bar{b}_1)$ is also equal to m, then $\bar{b}-\bar{b}_1\in Ker(\bar{g})=im(\bar{f})$. Thus $\bar{b}-\bar{b}_1=f(\bar{a})$ for some $\bar{a}\in\bar{A}$. Therefore,

$$
\begin{aligned}
(\beta-f\delta)(\bar{b}-\bar{b}_1)&=(\beta-f\delta)\bar{f}(\bar{a})\\
&=\beta\bar{f}(\bar{a})-f\alpha(\bar{a})\\
&=0.
\end{aligned}
$$

Also, $g\zeta(m)=g(\beta-f\delta)(\bar{b})=g\beta(\bar{b})=\bar{g}(\bar{b})=m$. Thus ζ gives a splitting of $g:B\longrightarrow M$, which is a contradiction.

Consequently, α must be an isomorphism and therefore, (*) is an almost split sequence of Λ-modules.

<u>Remark</u>: Corollary 3.5 implies that if (*) is not an almost split sequence of Λ-modules and $0 \longrightarrow A \longrightarrow B \longrightarrow M \longrightarrow 0$ is an almost split sequence of Λ-modules, then $IA \neq 0$ and $IB \neq 0$.

Although the above characterizations are interesting, it is somewhat hard in general to determine the dual of the transpose either over Λ or over $\bar{\Lambda}$. We now develop other criteria for determining when (*) remains an almost split sequence when viewed as a sequence of Λ-modules. The following result is of a very different nature than those above because it deals with the existence of certain modules and maps into M. We begin by remarking that if A and B are Λ-modules, then there is a natural inclusion of $\mathrm{Ext}^1_{\bar{\Lambda}}(A,B)$ into $\mathrm{Ext}^1_{\Lambda}(A,B)$ which we will denote by $\psi_{A,B}$. Let $f:A\longrightarrow X$ and $g:A\longrightarrow Y$ be Λ-module homomorphisms. We denote the pushout of f and g, that is, the cokernel of $\begin{pmatrix} f \\ g \end{pmatrix}:A\longrightarrow X\oplus Y$, by $\mathcal{P}(f,g)$. As before, let M denote an indecomposable, nonprojective $\bar{\Lambda}$-module and let (*) $0 \longrightarrow \bar{A} \xrightarrow{\bar{f}} \bar{B} \xrightarrow{\bar{g}} M \longrightarrow 0$ be an almost split sequence of $\bar{\Lambda}$-modules. Recall that a module N is called <u>local</u> if $N/\underline{r}N$ is a simple module where $\underline{r}$ is the Jacobson radical of the ring.

Theorem 3.6: The exact sequence (*) is not the almost split sequence for M when viewed as a sequence of Λ-modules if and only if one of the following conditions holds:

(1) There exists a simple A-module T such that the monomorphism $\psi_{M,T}:\mathrm{Ext}^1_{\bar{\Lambda}}(M,T) \longrightarrow \mathrm{Ext}^1_{\Lambda}(M,T)$ is not an epimorphism.

(2) The following statements hold:

(i) For all simple A-modules T, $\psi_{M,T}$ is an isomorphism.

(ii) There exists a simple A-module T such that:

(a) There is a nonsplit short exact sequence $0 \longrightarrow T \xrightarrow{\sigma} M^* \longrightarrow M \longrightarrow 0$.

(b) There is a local Λ-module L with $IL \neq 0$ and a monomorphism τ: $T \longrightarrow L$ satisfying:

(b_1) The pushout $P(\sigma,\tau)$ is an indecomposable Λ-module.

(b_2) For each simple summand T′ of the socle of L, if π':$L \longrightarrow L/T'$ is the canonical surjection, then there exists a map θ: $M^* \longrightarrow L/T'$ such that $\theta\sigma = \pi'\tau$.

Proof: We begin by showing that if (1) holds, then (*) is not an almost split sequence of Λ-modules. Let T be a simple A-module such that $\psi_{M,T}$ is not an isomorphism. Then there exists a nonsplit short exact sequence $0 \longrightarrow T \xrightarrow{\alpha} N \xrightarrow{p} M \longrightarrow 0$ which is not in the image of $\psi_{M,T}$. This implies that $IN \neq 0$. However, since $IM=0$, $IN \subseteq \mathrm{Ker}(p)$, thus $IN=\alpha(T)$. If (*) were an almost split sequence of Λ-modules, then there would exist a map $q:N \longrightarrow \bar{B}$ such that $\bar{g}q=p$. Now $I\bar{B}=0$ since $\bar{B}$ is $\bar{\Lambda}$-module, hence $IN \subseteq \mathrm{Ker}(q)$. Thus q would factor through N/IN. But $N/IN \cong M$, therefore this factorization would give a splitting of $\bar{g}:\bar{B} \longrightarrow M$, a contradiction. Consequently, (*) is not an almost split sequence of Λ-modules.

We show next that (2) implies that (*) is not an almost split sequence of Λ-modules. Let X denote the pushout $\mathcal{P}(\sigma,\tau)$. Since τ and σ are monomorphisms, we may view L as a submodule of X. From this we get a short exact sequence $0 \longrightarrow L \longrightarrow X \xrightarrow{\nu} M \longrightarrow 0$. The module X is indecomposable by hypothesis, so ν is not a splittable epimorphism. Therefore, if (*) were an almost split sequence of Λ-modules, there would exist a map $\mu:X \longrightarrow \bar{B}$ such that $\bar{g}\mu=\nu$. We are given that $IL \neq 0$; however, $I\bar{B}=0$. Hence μ restricted to L is not a monomorphism.

Let T' be a simple submodule of $\mathrm{Ker}(\mu) \cap L$. Let $\pi':X \longrightarrow X/T'$ be the canonical surjection and let $\mu':X/T' \longrightarrow B$ and $\nu':X/T' \longrightarrow M$ be the induced maps. We have the following diagram:

$$\begin{tikzcd} & X \arrow[d, "\pi'"] \\ & X/T' \arrow[dl, "\mu'"'] \arrow[d, "\nu'"] \\ \bar{B} \arrow[r, "\bar{g}"] & M \end{tikzcd}$$

where $\bar{g}\mu'=\nu'$ since $\bar{g}\mu'\pi'=\nu'\pi'$ and π' is a surjection. By 2(b)(b_2), there is a map $\theta:M^*\to L/T'$ such that $\theta\sigma=\pi'\tau$. This induces a map $\delta:X/T'\to L/T'$ such that $\delta\circ\mathrm{incl}=1_{L/T'}$, where incl denotes the inclusion of L/T' into X/T'. Consequently, the short exact sequence $0 \to L/T' \xrightarrow{\mathrm{incl}} X/T' \xrightarrow{\nu'} M \to 0$ splits. Thus, there is a map $\gamma:M\to X/T'$ such that $\nu'\gamma=1_M$. It follows that $\bar{g}\mu'\gamma=\nu'\gamma=1_M$, contradicting the fact that $\bar{g}:\bar{B}\to M$ is not splittable. Therefore, if (2) holds, then (*) is not an almost split sequence of Λ-modules.

For the remainder of the proof, assume (*) is not an almost split sequence of Λ-modules. Then there exists a Λ-module X and a Λ-morphism $\alpha:X\to M$ which is not a splittable epimorphism such that there does not exist a map $\beta:X\to\bar{B}$ so that $\bar{g}\beta=\alpha$. Assume that X is a Λ-module of minimal length with this property. Since (*) is an almost split sequence over $\bar{\Lambda}$, it follows that $IX\neq 0$. Moreover, α is a surjection. For if not, then $\alpha(X)$ is a proper submodule of M, so the inclusion of $\alpha(X)$ into M is not a splittable epimorphism. However, since $\alpha(X) \subseteq M$, $I\alpha(X)=0$, hence $\alpha(X)$ is a $\bar{\Lambda}$-module. Therefore, there is a map $\delta:\alpha(X)\longrightarrow\bar{B}$ such that $\bar{g}\delta=\mathrm{incl}:\alpha(X)\longrightarrow M$. This contradicts the fact that there is no map β with $\bar{g}\beta=\alpha$. Thus α is a surjection. Furtherfore, the minimality of X assures the indecomposability of X.

<u>Case 1</u>: Suppose the length of M is 1 less that the length of X. Then, letting $T=\ker(\alpha)$, we have that T is a simple Λ-module and the sequence $0 \longrightarrow T \longrightarrow X \xrightarrow{\alpha} M \longrightarrow 0$ is a nonzero element of $\mathrm{Ext}^1_\Lambda(M,T)$ which is not in the image of $\psi_{M,T}$ since $IX\neq 0$. Therefore, $\psi_{M,T}$ is not an epimorphism, (1) holds, and we are done.

<u>Case 2</u>: The length of M is at least 2 less than the length of X. Again, if (1) holds, we are done. If not, then for all simple Λ-modules T, $\psi_{M,T}$ is an isomorphism. We proceed to show that the rest of (2) holds.

Consider $\mathrm{soc}(X)\cap IX$ where $\mathrm{soc}(X)$ denotes the socle of X. Let T be a simple summand of $\mathrm{soc}(X)\cap IX$. Then $T \subseteq IX \subseteq \ker(\alpha)$. Thus we get a factorization of α: $X\longrightarrow M$ into $X \xrightarrow{\pi} X/T \xrightarrow{\bar{\alpha}} M$ where $\pi:X\longrightarrow X/T$ is the canonical surjection. If $X/T \xrightarrow{\bar{\alpha}} M$ is not split, then, by the minimality of the length of X, $\bar{\alpha}$ factors through $\bar{B}$. This again contradicts the assumption that there is no map $\beta: X \longrightarrow \bar{B}$ such that $\bar{g}\beta = \alpha$. Therefore, X/T decomposes into $\bar{X} \oplus \bar{M}$ where $\bar{X}$ is the kernel of $\bar{\alpha}$ and $\bar{M}$ is the submodule of X/T such that $\bar{\alpha}: \bar{M} \longrightarrow M$ is an isomorphism. Let M^* be the inverse image of $\bar{M}$ under π. We get a short exact sequence,

$$(* \ *) \quad 0 \longrightarrow T \xrightarrow{\text{incl}} M^* \xrightarrow{\alpha} M \longrightarrow 0.$$

This sequence is not split since $\alpha(M^*) = M$ and α is not split. Now, the length of M^* is 1 more than the length of M. Thus, by assumption, the length of M^* is less than the length of X. Therefore $\alpha: M^* \longrightarrow M$ factors through $\bar{B}$; that is, there exists a map $\epsilon: M^* \longrightarrow \bar{B}$ such that $\bar{g}\epsilon = \alpha$. This implies that $\ker(\epsilon) \subseteq \ker(\alpha) = T$. If $\ker(\epsilon) = T$, then ϵ factors through M^*/T. But M^*/T is isomorphic to M, so this factorization would give a splitting of $\bar{g}: \bar{B} \longrightarrow M$ which is a contradiction. Therefore, ϵ is a monomorphism. In particular, $IM^* = 0$. So $(* \ *)$ is a nonzero element of $\mathrm{Ext}^1_{\bar{\Lambda}}(M, t)$.

Next we consider the summand $\bar{X}$ of X/T. Let $\bar{B}_1, \ldots, \bar{B}_r$ be local submodules of $\bar{X}$ such that $\sum_{i=1}^{r} \bar{B}_i = \bar{X}$. Let $L_1, \ldots, L_r$ be local submodules of X such that $\pi(L_i) = \bar{B}_i$ for each i. Note that each $L_i \subseteq \ker(\alpha)$ since $\bar{X}=\ker(\bar{\alpha})$. Furthermore, $(\sum_{i=1}^{r} L_i)+M^*=X$. Since $IX^*\neq 0$, there exists an i_0 such that $IL_{i_0}\neq 0$. We now replace T by a simple summand of $IL_{i_0} \cap \mathrm{soc}(L_{i_0})$. We redefine M^* as in the preceeding paragraph using this new choice of T. Let $L=L_{i_0}$. Since $L\subset\ker(\alpha)$ and $M^* \cap \ker(\alpha) = T$, we see that $L \cap M^* = T$.

Let $\tau:T\longrightarrow L$ and $\sigma:T\longrightarrow M^*$ denote the inclusion maps. Consider the pushout $\mathcal{P}(\sigma,\tau)$. Since $M^*\cap L = T$, $\mathcal{P}(\sigma,\tau)=L+M^* \subseteq X$. We claim that $L+M^*=X$.

Suppose $L+M^*$ is a proper submodule of X. Then, by the minimality of the length of X, either the composition $L+M^* \xrightarrow{\text{incl}} X \xrightarrow{\alpha} M$ is a splittable epimorphism or there exists a map $\zeta:L+M^* \longrightarrow \bar{B}$ such that $\bar{g}\zeta = \alpha\circ(\text{incl})$. However, since $\alpha:X\longrightarrow M$ is not split, $\alpha\circ(\text{incl}):L+M^*\longrightarrow M$ is not split. Therefore, ζ must indeed exist. Since $I\bar{B}=0$, $I(L+M^*) \subseteq \ker(\zeta)$. Hence, since $T \subseteq IL$, it follows that $T \subseteq \ker(\zeta)$. Therefore we obtain a factorization of ζ through $(L+M^*)/T$; that is, $\zeta=\bar{\zeta}\pi^*$ where $\pi^*:L+M^* \longrightarrow (L+M^*)/T$ is the canonical surjection and $\bar{\zeta}:(L+M^*)\ T \longrightarrow \bar{B}$ is the induced map. But $(L+M^*)/T \cong (L/T) \oplus M$. This yields a splitting of $\bar{g}:\bar{B}\longrightarrow M$ which is a contradiction. Thus the claim has been verified and $L+M^*=X$. Hence, the pushout $\mathcal{P}(\sigma,\tau)$ is indecomposable since $\mathcal{P}(\sigma,\tau)=X$ and X is indecomposable.

The proof will be complete once we show that (b_2) holds. Let T' be a simple submodule of L. Let $\pi':X\longrightarrow X/T'$ be the canonical surjection. If $T'=T$, then $\pi'\tau=0$, so we may take $\theta=0$. Thus we may assume that $T'\neq T$. Since L is contained in the kernel of $\alpha:X\longrightarrow M$, we get an induced map $\alpha':X/T'\longrightarrow M$. By the minimality of the length of X, we have that either $\alpha':X/T'\longrightarrow M$ splits, or there exists $h':X/T'\longrightarrow\bar{B}$ such that $\bar{g}h'=\alpha'$. But the existence of h' gives a contradiction to the fact that there is no $\beta:X\longrightarrow\bar{B}$ with $\bar{g}\beta=\alpha$. Therefore, it must be the case that $\alpha':X/T'\longrightarrow M$ splits. This implies that the short exact sequence $0\longrightarrow L/t'\xrightarrow{\text{incl}} X/T'\xrightarrow{\alpha'} M$ is split exact. Let $\varphi:X/T'\longrightarrow L/T'$ be a map satisfying $\varphi\circ(\text{incl})$. Recalling that $X=L+M^*$, we define $\theta:M^*\longrightarrow L/T'$ by $\theta(m^*)=\varphi(m^*+T')$. Then, if $t\in T$, $\theta\sigma(t)=\theta(t)=\varphi(t+T')=\varphi\circ(\text{incl})(t+T')=t+T'=\pi'\tau(t)$. Thus $\theta\sigma(t)=\pi'\tau(t)$. This completes the proof.

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