Variable Sampling Interval Control Charts

by

Raid Widad Amin

Dissertation submitted to the Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

in

Statistics

APPROVED:

Marion R. Reynolds, Jr. (Co-chairman)

Jesse C. Arnold (Co-chairman)

Klaus H. Hinkelmann

Joel A. Nachlas

Robert V. Foutz

July 10, 1987

Blacksburg, Virginia

Variable Sampling Interval Control Charts

by

Raid Widad Amin

Marion R.Reynolds, Jr. , Co-chairman

Jesse C. Arnold , Co-chairman

Statistics

(ABSTRACT)

Process control charts are widely used to display sample data from a process for purposes of determining whether a process is in control, for bringing an out-of-control process into control, and for monitoring a process to make sure that it stays in control. The usual practice in maintaining a control chart is to take samples from the process at fixed length sampling intervals. This research investigates the modification of the standard practice where the sampling interval or time between samples is not fixed but can vary depending on what is observed from the data. Variable sampling interval process control procedures are considered for monitoring the outcome of a production process. The time until the next sample depends on what is being observed in the current sample. Sampling is less frequent when the process is at a high level of quality and vise versa. Properties such as the average number of samples until signal, average time to signal and the variance of the time to signal are developed for the variable sampling interval Shewhart and cusum charts. A Markov chain is utilized to approximate the average time to signal and the corresponding variance for the cusum charts. Properties of the variable sampling interval Shewhart chart are investigated through Renewal Theory and Markov chain approaches for the cases of a sudden and gradual shift in the process mean respectively. Also considered is the case of a shift occuring in the time between two samples without the simplifying assumption that the process mean remains the same from time zero onward. For such a case, the adjusted time to signal is developed for both the Shewhart and cusum charts in addition to the variance of the adjusted time to signal.

Results show that the variable sampling interval control charts are considerably more efficient than the corresponding fixed sampling interval control charts. It is preferable to use only two sampling intervals which keeps the complexity of the chart to a reasonable level and has practical implications. This feature should make such charts very appealing for use in industry and other fields of application where control charts are used.

Dedication :

This dissertation is dedicated with love and gratitude to my parents, brother and sister, who continuously encouraged and supported me in every way possible throughout the past years.

Acknowledgements

I am greatly indebted to my co-advisors, Drs. Marion R.Reynolds, Jr. and Jesse C. Arnold for their inspiring guidance, patience, and concern. Without their assistance and their willingness to meet with me and to do work at all hours, this dissertation would not have been written.

I would like to thank Dr. Klaus H. Hinkelmann for his concern and understanding throughout the years of my graduate study and for serving on my advisory committee.

Also, I would like to thank Drs. Joel A. Nachlas and Robert V. Foutz for serving on my committee.

My endless struggles with the computer were made a lot easier with the help of Ms. Sharon Myer to whom I express my gratitude.

Finally I wish to thank the Faculty, Staff and Students of the Virginia Tech Statistics Department for a productive and enjoyable experience.

Table of Contents

List of Tables

Table Page

List of Figures

Figure Page

Chapter I

Introduction

In many process control procedures it is possible to express the quality of the output of a process by some numerical quality characteristic such as the process mean, the process standard deviation, or the process proportion of defective products. Process control procedures are designed for the purpose of continuously monitoring the production process in order to quickly detect any deterioration in quality. A control chart is maintained by taking samples from a process and plotting in time order on the chart some statistic computed from the samples. Control limits on the chart represent the limits within which the plotted points would fall with high probability if the process is operating in control. A point outside the control limits is taken as an indication that something, sometimes called an *"assignable cause"* of variation, has happened to change the process. When the chart signals that an assignable cause is present then rectifying action is taken to remove the assignable cause and bring the process back into control. The most widely used process control procedures are the Shewhart chart which was proposed by Shewhart (1931) and the cusum chart which was introduced by Page (1954). After each sample is taken the decision is either to signal that there is a potential problem with the process or to continue to the next sampling point and take another sample. The random time at which the procedure signals is called the *Run Length* (N) and the expected value of N is called the *Average Run Length* (ARL). When evaluating the properties of a control procedure or when comparing more than one procedure, the ARL is the most widely used criterion. When the process is out of control the ARL should be small so that the number of samples required for the procedure to signal is small. When the process is in control the ARL should be large so that the frequency of false alarms is low. The usual practice in maintaining a control chart is to take samples from the process at fixed length sampling intervals. This dissertation investigates the modification of the standard practice where the sampling interval between samples is not fixed but instead can vary depending on what is observed from the data. The idea of using a variable sampling interval should seem to be intuitively appealing to users of control charts. If a sampling point falls close to a control limit (but not actually outside where a signal is produced) then one would naturally wonder whether subsequent points may actually be outside the control limits. Since it is important to quickly detect changes in the process, the natural inclination in this situation would be to take another sample quickly rather than wait the usual delay for the next sample. In this way the suspicion about the point close to the control limit could be more quickly confirmed or denied. On the other hand, if the current point is close to the target so that there is no indication of trouble, it might be reasonable to wait longer than the usual delay for the next sample. Thus the proposed control procedure is to let the time until the next sample depend on what is observed in the current sample. The time will be short if there is some indication that there may be a problem, and longer if there is no indication of a problem. If the indication of a problem is strong enough then a signal will be produced at the current sample in the same way as in the usual chart using a fixed sampling interval. This dissertation investigates the situation where the observations from the process are normally distributed and the objective is to control the process mean. The main objective of this dissertation is to investigate the properties of the Shewhart chart and the cusum chart when the variable sampling interval feature is used. The results of this study indicate that these two process control procedures are significantly more efficient when using variable sampling intervals between samples. This is accomplished without increasing the expected number of samples to signal. Since the traditional meaning of the ARL relates to both number of samples and time, we define two additional measures for use with variable sampling interval process control charts. These additional measures are the *Average number of Samples to Signal (ANSS)* and the *Average Time to Signal (ATS)* . Both the *ANSS* and the *ATS* are function of the process mean μ.

This dissertation is organized in seven chapters as follows. Chapter II gives a review of the development of process control procedures with fixed sampling intervals. The construction of the Shewhart chart and the cusum chart is briefly explained and the advantages and disadvantages are discussed. Continuous sampling plans which alternate between more than one level of inspection level are reviewed in addition to lot-by-lot sampling plans in which provision is made for inspecting only a fraction of the produced lots. These plans motivated several authors to investigate the problem of determining a sampling plan with variable sampling intervals between samples. A complete review of publications related to this topic is given. Chapter III presents the properties of the Variable Sampling Interval (VSI) Shewhart chart. The expression for the *ATS* is derived using the Markov chain approach used by all previous publications on variable sampling interval plans in addition to a Renewal Theory approach. The properties of the *VSI* Shewhart chart are investigated for the case of a sudden shift in the process mean and the case of a drift in the mean. In Chapter IV, we propose a model which takes into account that the process may shift at some random time between samples. Such an assumption is more realistic in most practical applications. The adjusted properties of the *VSI* Shewhart chart are developed and extensive comparisons are given. Limited simulation studies using several prior distributions for the time of the shift are given in support of the simplifying assumption that the shift occurs uniformly over a sampling interval. The optimal sampling intervals are obtained numerically by minimizing the adjusted *ATS* function for a VSI Shewhart chart with two sampling intervals. Also considered is the *VSI* Shewhart chart with runs rules. In Chapter V, we develop the properties of the *VSI* cusum chart by using a Markov chain approach as proposed by Brook and Evans (1972). An expression for the variance of the time to signal is developed and comparisons are made with the Fixed Sampling Interval (FSI) Cusum chart. Chapter VI develops the properties of the *VSI* cusum chart when the adjustment factor is taken into account. Extensive tables are given to assist in the choice of the parameters which yield the most efficient sampling plans.

Chapter II

Literature Review

2.1 Fixed Sampling Interval Control Charts

The purpose of this section is to give an overview of the development of process control charts which use fixed sampling intervals. One of the simplest and most widely used control charts is the $\overline{X}$- chart developed by Shewhart (1931) for detecting assignable causes which lead to a change in the process mean. The basic structure of this chart consists of a central line corresponding to a target value μ_0 and two action (control) lines which usually are set at $\mu_0 \pm 3$ standard errors. Sample means are plotted independently over time and if a mean value exceeds any of the two action lines it is taken as an indication that the process is out of statistical control. Shewhart charts are simple to use and fast in detecting large shifts from the target value. However, they are relatively insensitive to moderate and small shifts in the process mean. In order to correct this disadvantage, additional criteria are sometimes added to signal that the process is out of control. Page (1955) investigated some Shewhart-type control charts modified by adding warning lines which are drawn within the action lines in order to use the information from the last few samples. An additional rule is that if r out of the last N points fall between the warning and action lines, then an investigation is demanded. Page (1955) showed numerically that the standard Shewhart chart is less efficient in detecting small shifts in the process mean than a $\overline{X}$- chart with warning lines. This is so because the standard Shewhart chart uses only the information of one sample at a time to decide whether the mean has shifted. Weiler (1953) and Moore (1958) suggested rules using runs of points on the control chart. Action is taken if r consecutive points fall outside an action line even though no additional points fall outside an action line. Page (1962) showed that the control charts with warning lines are generally better than Moore's (1958) runs rules. The performance of these two rules is similar for small shifts in the mean, but the ARL out-of-control is smaller for the chart with warning lines in the case of large shifts. Toda (1958) described the use of a test in which a control chart is partitioned into zones by the central line in addition to 1σ, 2σ and 3σ limit lines. Each of the zones is labeled with a non-zero score and scores are cumulated. Corrective action is taken when the magnitude of the cumulative score reaches a selected value. These types of schemes allow us to narrow the range between the control limits and as a result it improve the sensitivity for small shifts in the mean. Reynolds and Ghosh (1986) discuss several procedures for detecting simultaneous changes in the mean and variance of a production process when Shewhart-type charts are used. Extensions to the multivariate case are also discussed. All of the control charts discussed so far are Shewhart-type control charts.

A significant development in process control procedures was the introduction of the cumulative sum (cusum) control chart by Page (1954). The cusum chart directly incorporates all of the information in the sequence of sample values by plotting the cumulative sums of the deviations of the sample values from a target value. The properties of a cusum chart are difficult to evaluate because because there is no exact expression available for the run length distribution. Page (1954) and Ewan and Kemp (1960) showed that the run length distribution of a one-sided cusum is approximately geometric when the ARL is large. Ewan and Kemp (1960) show that the run length distribution is approximated for large i by

$$P(N \leq i) \cong 1 - e^{-i/ARL(\mu_0)}$$

Van Dobben de Bruyn (1968) described how to compute the exact ARL by solving an integral equation. A contour nomogram which is useful in the design of cusum charts for controlling the normal mean was provided by Goel and Wu (1971). Johnson and Bagshaw (1974) and Reynolds (1975) studied the use of a Brownian Motion Process to approximate the cusum process. Khan (1978) gives Wald's approximations to the ARL in cusum procedures for an exponential family of densities. Using these approximations it is shown that Page's (1954) cusum is in a sense identical with a cusum procedure defined in terms of likelihood ratios. A Markov chain approach to compute the ARL for a one-sided cusum was introduced by Brook and Evans (1972). This method may be used with any discrete distribution and yields accurate approximations with continuous distributions. The Markov chain approach provides a practical method for investigating modifications of the standard cusum chart. Lucas and Crosier (1982c) and Woodall (1984) use a similar Markov chain to determine the ARL of the two-sided cusum chart. A combined Shewhart-cusum chart is developed by Lucas (1982a) to improve the ability of the cusum to detect large shifts in the process mean. The cusum chart can easily be modified by adding the Shewhart control limits and the properties are found readily by using the Markov chain approach proposed by Brook and Evans (1972). The cusum chart is also used in continuous sampling plans. Beattie (1962) discussed the use of a cusum chart for the number of defectives in a continuous acceptance sampling procedure. Many other authors have studied the design and properties of the cumulative sum control chart, such as Barnard (1959), Goldsmith and Whitfield (1961), Johnson (1961), Johnson and Leone (1962a, 1962b, 1962c), Woodall (1986a) and Lucas (1976).

2.2 Variable Sampling Interval Plans

A lot of work has been done in the literature on sampling plans with more than one level of sampling. Such plans can be categorized as follows:

1. Continuous sampling plans which alternate between more than one level of inspection.

2. Lot-by-lot sampling plans in which a provision is made for inspecting only a fraction of the produced lots.

3. Variable sampling interval plans in which the sampling interval used reflects the perceived quality of the quality characteristic observed.

A review of the work done in the these categories is helpful in motivating the use of variable sampling interval process control charts as proposed in this dissertation.

A continuous sampling plan was first proposed by Dodge (1943) and was later labeled CSP-1. This plan alternates between 100% inspection, where every unit is inspected, and partial inspection, where only a fraction f of the units is inspected. These sample units are selected over time at random from the production process and they are categorized either as defectives or non-defectives. Several variations in the original CSP-1 plan followed. Wald and Wolfowitz (1945) proposed a similar sampling inspection plan which was more efficient in detecting long sequences of low quality quickly. Dodge and Torrey (1951) developed CSP-2 and CSP-3 which don't require reversion to 100% inspection whenever a defective item is found during inspection. Multilevel sampling inspection plans were designed by Lieberman and Solomon (1955) to overcome the disadvantages of abrupt switching between the two levels of the previous CSP's. Inspection intensity is increased when the manufacturing process isn't operating well, and decreased during periods of good production. Their sampling plan allows for $t \geq 1$ levels of partial inspection (f, f^1, f^2, ... , f^t) subject to the restriction that transitions can occur only between adjacent levels. For $t = 1$ this plan reduces directly to Dodge's original CSP-1 plan. The multi-level plan instructs a reduced amount of inspection when the quality is relatively good and requires 100% inspection when quality is poor. Derman, Littauer and Solomon (1957) gave three generalizations to the Lieberman-Solomon multi-level plans. Their plans make provisions for quick shifting to tightened inspection levels. Accordingly, their plans are known as "tightened" multi-level plans.

The principles of continuous sampling can be applied to individual lots received from a production line. Just as units are "skipped" in a continuous sampling plan, lots may be skipped under skip-lot sampling plans. Lot-by-lot sampling plans are employed when the product is formed into lots and the purpose of the sampling plan is to make some decision about the individual lots. Skip-lot sampling plans make provisions for skipping the inspection of some fraction of the submitted lots when the quality of the inspected lots is good. Dodge (1956) first introduced the skip-lot sampling plan labeled SkSP-1 as an extension of CSP-1 sampling plans for individual units of production. This scheme is very similar to the CSP-1, but instead of units, lots are skipped whenever r consecutive lots are found non-defecive. If a defective lot is detected then the plan switches to 100% inspection of the lots. Skip-lot sampling plans in which each lot inspected is sampled according to some attribute lot inspection plan were investigated by Perry (1973) and labeled SkSP-2. Similar plans with modifications followed in later years.

The ideas of switching inspection levels in acceptance sampling carry over to statistical control charts for monitoring the output of a production process. Arnold (1970) developed a Markov process as a mathematical model to study the water quality monitoring of streams. He proposed sampling procedures which use variable sampling intervals, sampling less frequently when the process is operating well. Several alternative sampling plans were evaluated on the basis of the expected sample size for which an expression was given. A dissertation by Crigler (1973) followed in which an economically optimal sampling policy was formulated using Arnold's (1970) Markov process. Crigler and Arnold (1979,1986) extended their previous results in the following years. Smeach and Jernigan (1977) developed an expression for the variance of the sample size as well as simplifying approximations to both the expected sample size and its variance. Hui (1980) and Hui and Jensen (1980) gave extensions for the variable sampling interval plans in the multivariate case. They derived expressions for the expected sample size and its variance when one or more rejection regions are used. Previous work on the Markovian sampling policy did not consider the idea of including a rejection region into their schemes. The next logical step will be to develop process control procedures with variable sampling intervals and to evaluate their performance relative to fixed sampling interval control charts.

Chapter III

Shewhart $\overline{X}$ - Charts

3.1 Fixed Sampling Interval (FSI) Shewhart Charts

Shewhart control charts are widely used to display sample data from a process for purposes of determining whether a process is in control, for bringing an out-of-control into control, and for monitoring a process to make sure that it stays in control. Consider the situation where the observations from the process are normally distributed with mean μ and known variance σ^2. We denote the the target value for the mean by μ_0. Assume that random samples of size n are taken at each sampling point and let $\underline{X}_i = (X_{i1}, X_{i2}, ..., X_{in})$ represent the sample taken at the i^{th} point. When the i^{th} sample is taken the sample mean $\overline{X}_i$ is computed and plotted on a control chart with center line μ_0 and control limits $\mu_0 \pm \gamma\sigma/\sqrt{n}$ where γ is frequently taken to be 3. If $\overline{X}_i$ falls outside of the control limits then a signal is given. In practice it will frequently be necessary to estimate μ_0 and σ^2 from past data but for simplicity it will be assumed here that μ_0 and σ^2 are known. Shewhart charts with appropriate control statistics can be used for several control purposes. For example, if the control statistic is the sample mean, the sample range or the fraction defective, the Shewhart chart can be used to control the process mean, the process standard deviation or the proportion of defective items, respectively. If the objective is to detect a shift in the process mean, then the Shewhart $\overline{X}$- chart signals at time i for which $\overline{X}_i \geq \mu + \gamma\sigma_{\bar{x}}$ or $\overline{X}_i \leq \mu - \gamma\sigma_{\bar{x}}$. The chart can be easily carried out using estimated values of μ_0 and σ^2, although the properties of the resulting chart will be different from the case where μ_0 and σ^2 are known.

3.1.1 Properties of the FSI Shewhart Charts

The properties of a control chart are determined by the length of time it takes the chart to produce a signal. If the process is in control then this time should be long so that the rate of false alarms is low, but if the process mean shifts then the time from the shift to the signal should be short so that detection is quick. The number of samples until a signal is usually called the run length in the quality control literature and the expected number of samples until a signal is called the average run length (ARL). With a fixed sampling interval between samples, the ARL can easily be converted to the expected time to signal by multiplying the ARL by the fixed interval length so that the ARL can be thought of as the expected time to signal where the time unit is the interval between samples. In addition the sampling rate will be constant regardless of the value of the process mean μ. However with a variable sampling interval chart, as proposed in this dissertation , the time to signal is not a constant multiple of the number of samples to signal and the sampling rate depends on the value of μ. Thus for the variable sampling interval chart it is necessary to keep track of both the number of samples to signal and the time to signal. Since the traditional meaning of the ARL relates to both the number of samples and time, it seems to be preferable to define new quantities so that comparisons with the proposed variable sampling interval chart are feasible.

Define the *number of samples* to be the number of samples taken from the start of the process to the time the chart signals, and let the *average number of samples to signal (ANSS)* be the expected value of the number of samples to signal. Define the *time to signal* to be the time from the start of the process to the time where the chart signals, and similarly let the *average time to signal (ATS)* be the expected value of the time to signal. Both the *ANSS* and the *ATS* would of course be functions of the process mean μ. When $\mu = \mu_0$ the *ATS* should be large and when μ shifts from μ_0 the *ATS* should be small. To avoid large sampling costs the *ANSS* should not be excessively large compared to the *ATS*, particularly when $\mu = \mu_0$ since the process will presumably operate with $\mu = \mu_0$ most of the time.

As long as the control limits remain fixed the variable sampling interval feature will have no effect on the probability that $\overline{X}$ falls outside the control limits. Thus the probability that $\overline{X}$ falls outside the control limits will be

$$q = P(\overline{X} \leq \mu_0 - \gamma\sigma/\sqrt{n} \text{ or } \overline{X} \geq \mu_0 + \gamma\sigma/\sqrt{n})$$

regardless of the interval between samples. If

$$N = \text{number of samples to signal}$$

then N has a geometric distribution with parameter q when the process does not change. Thus the *ANSS* is

$$E(N) = \frac{1}{q} \tag{3.1}$$

and the variance of N is

$$var(N) = \frac{1-q}{q^2}. \tag{3.2}$$

If

$$R = \text{time to signal}$$

and

$$R_i = \text{sampling interval used before the } i^{th} \text{ sample is taken}$$

then

$$R = \sum_{l=1}^{N} R_l. \tag{3.3}$$

For simplicity it is assumed here that the chart is started at time zero and that the fixed sampling interval is d. Hence the ATS for the FSI Shewhart chart is just

$$\begin{aligned} E(R) &= d\,E(N) \\ &= \frac{d}{q} \end{aligned} \tag{3.4}$$

Since the expected time to signal R is just the product of the $ANSS$ and the fixed sampling interval d, the variance of R can be expressed as

$$\begin{aligned} var(R) &= d^2\,var(N) \\ &= \frac{d^2\,(1-q)}{q^2} \end{aligned} \tag{3.5}$$

The usual practice in maintaining a control chart is to take samples from the process at fixed length sampling intervals. The following section investigates the modification of the standard practice where the sampling interval or time between samples is not fixed but instead can vary depending on what is observed from the data.

3.2 *Variable Sampling Interval (VSI) Shewhart Charts*

In the standard Shewhart $\overline{X}$ -chart the length of the time interval between samples is fixed but in the VSI Shewhart chart the interval between samples $\overline{X}_i$ and $\overline{X}_{i+1}$ will depend on the value of $\overline{X}_i$. It will be assumed that the variable interval chart uses a finite number of interval lengths

$d_1, d_2, \ldots, d_t$ where $d_1 < d_2 < \ldots < d_t$ and these possible interval lengths must be chosen to satisfy $m_1 \le d_i \le m_2$. The minimum possible interval length $m_1 > 0$ might be determined by physical considerations such as the time required to take a sample or to assure independence between samples. The maximum interval length m_2 might be determined by the maximum amount of time that process engineers are willing to allow the process to run without sampling. The choice of a sampling interval as a function of $\overline{X}_i$ can be represented by a function $d(x)$ which specifies the sampling interval to be used when $\overline{X}_i = x$ is observed. Let the interval $(\mu_0 - \gamma\sigma/\sqrt{n},\ \mu_0 + \gamma\sigma/\sqrt{n})$ be partitioned into t regions $I_1, I_2, \ldots, I_t$ such that

$$d(x) = d_j \text{ when } x \in I_j.$$

Thus the sampling interval used between $\overline{X}_i$ and $\overline{X}_{i+1}$ is $d(\overline{X}_i)$. In practice it might be preferable to use the shortest interval d_1 at first when the process is just starting up to give additional protection against problems that arise during start-up. This kind of modification could be easily handled with a slight increase in the complexity of the resulting expressions but this refinement will not be pursued in this dissertation. This section investigates the situation where the observations from the process are normally distributed and the objective is to control the process mean by using the sample means in an $\overline{X}$ -chart.

An example of a variable interval chart is shown in Figure 3.1. This chart uses two interval lengths d_1 and d_2, with

$$I_1 = (\mu_0 - \gamma\frac{\sigma}{\sqrt{n}},\ \mu_0 - \gamma'\frac{\sigma}{\sqrt{n}}] \ \cup\ [\mu_0 + \gamma'\frac{\sigma}{\sqrt{n}},\ \mu_0 + \gamma\frac{\sigma}{\sqrt{n}})$$

and

$$I_2 = (\mu_0 - \gamma'\frac{\sigma}{\sqrt{n}},\ \mu_0 + \gamma'\frac{\sigma}{\sqrt{n}}),$$

where $0 < \gamma' < \gamma$. The way that the variable interval chart works to improve the detection ability of the $\overline{X}$-chart can be explained with reference to Figure 3.1. Suppose that in Figure 3.1, $\gamma = 3$

and $\gamma' = 1$. Then when $\mu = \mu_0$, $P(\overline{X} \in I_1) = .3146$ and $P(\overline{X} \in I_2) = .6827$ so that the longer interval will be used approximately twice as often as the shorter interval. Now suppose that μ shifts to $\mu_1 = \mu_0 + 2\sigma/\sqrt{n}$. In this case $P(\overline{X} \in I_1) = .6840$ and $P(\overline{X} \in I_2) = .1573$ and the shorter interval will be used much more often than the longer interval. By using the shorter interval more often when μ shifts, the frequency of sampling is increased and the time required to obtain a sample mean outside the control limits (to produce a signal) is substantially reduced.

The sample means in the chart in Figure 3.1 are plotted against the sample number. In practical applications of the chart it would be necessary to record on the chart the times that the samples are taken since the constant interval between the points on the chart disguises the fact that the actual time intervals between samples are not the same. For example as the points are plotted the interval between samples 1 and 2 is d_2 while the interval between samples 2 and 3 is d_1. An alternate way to construct a chart would involve plotting the sample means against time on the horizontal axis. Then the intervals between points on the chart would vary corresponding to the actual time intervals between samples. This alternate construction would be relatively easy to implement by hand if $d_2 = cd_1$ where c is a small positive integer. If d_2 is very much larger than d_1 then large blank spaces will appear on the chart whenever d_2 is used.

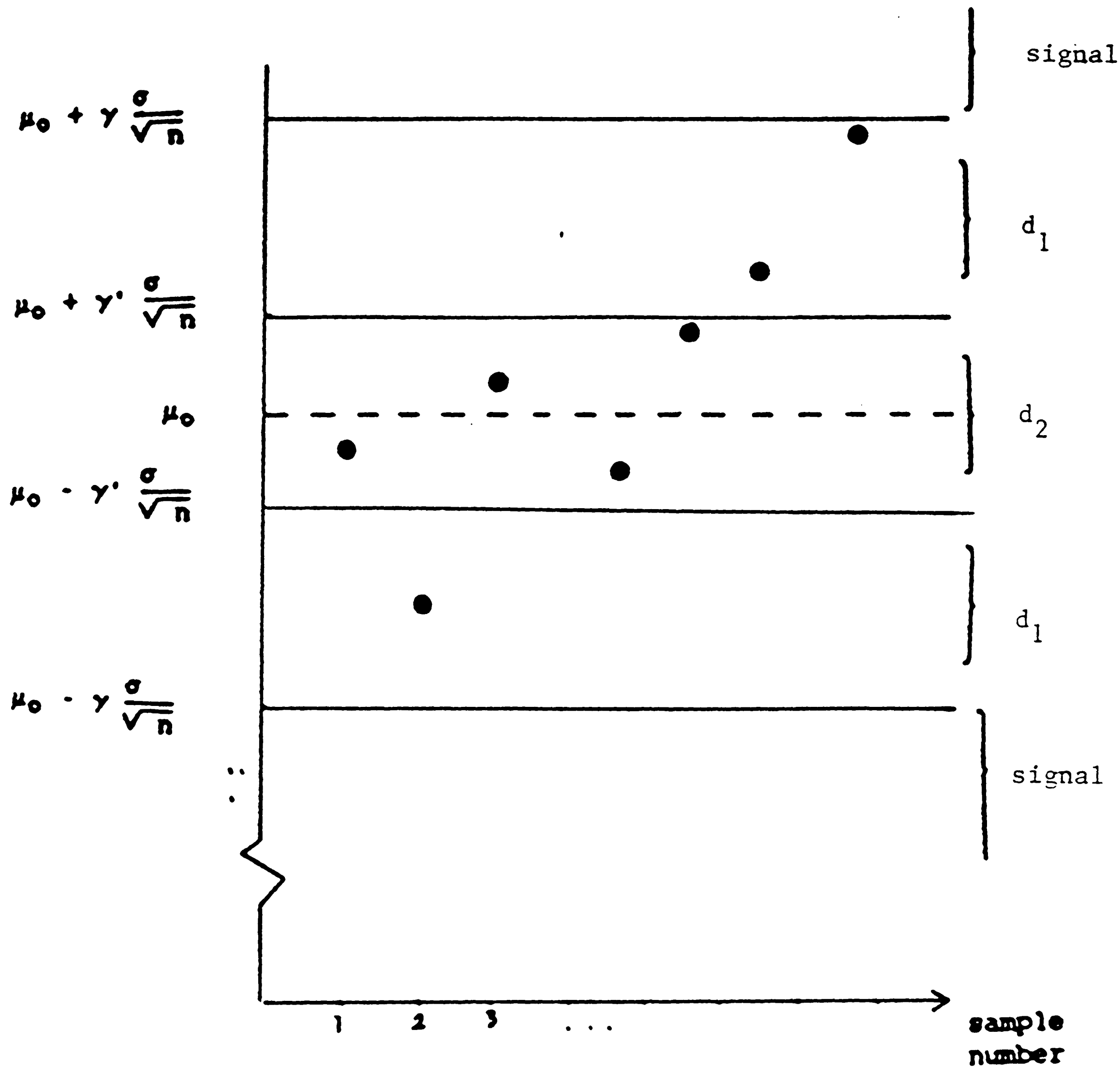


Figure 3.1: $\bar{X}$-chart with variable sampling intervals.

3.2.1 Properties of the VSI Shewhart Chart

In evaluating the performance of a variable sampling interval Shewhart $\bar{X}$-chart it would be helpful to have a standard for comparison and the Shewhart $\bar{X}$-chart using a fixed sampling interval seems to be the natural standard to use. If both fixed and variable interval charts have control limits at $\mu_0 \pm \gamma\sigma/\sqrt{n}$ then both will have the same value of q and the same $ANSS$ function. This means that the interval used in the variable interval chart will have no effect on the number of samples required to signal. It will be convenient to take the interval used by the fixed interval chart as the unit of time. For example if the fixed interval chart takes samples every 8 hours then a time unit will be taken as an 8 hour period. In this case the numerical value of the ATS function will be the same as the numerical value of the $ANSS$ function. If the variable interval chart is set up so that $E(R_i)$, the expected interval length, is equal to 1 time unit when $\mu = \mu_0$ then the two charts will have the same ATS functions when $\mu = \mu_0$. Then the variable interval chart will be "matched" to the fixed interval chart in the sense that they both require the same number of samples to signal and when $\mu = \mu_0$ they both have the same average sampling rate and false alarm rate over time. Then the values of the ATS functions for the two charts can be compared for various values of μ_1 to determine which chart will do a better job of detecting a change in μ.

If two intervals are used the requirement that $E(R_i) = 1$ when $\mu = \mu_0$ means that

$$d_1 p_{01} + d_2 p_{02} = 1 - q_0 \tag{3.6}$$

where q_0 is the probability of a signal when $\mu = \mu_0$ and p_{0i} is the probability of using sampling interval d_i when $\mu = \mu_0$. When it is necessary to distinguish between μ_0 and some other value $\mu_1(\neq \mu_0)$ that is of interest, the probabilities under μ_0 will be denoted by q_0 and p_{oj} and the probabilities under μ_1 by q_1 and p_{1j}. If γ is fixed then q_0 is fixed and (3.6) can be satisfied by specifying d_1 and d_2 and allowing these sampling intervals to determine p_{01} and p_{02} and then I_1 and I_2. Alternately I_1 and I_2 can be specified and then these regions will determine p_{01} and p_{02} and then d_1 and d_2. The approach taken here for investigating the behavior of the variable interval chart will be to specify d_1 and d_2 and let these sampling intervals determine the probabilities and the regions required to match the fixed interval chart. Then for given values of γ and $0 < d_1 < 1 < d_2$, p_{01} and p_{02} must satisfy

$$p_{01} = \frac{d_2 - 1}{d_2 - d_1}(1 - q_0) \quad and \quad p_{02} = \frac{1 - d_1}{d_2 - d_1}(1 - q_0) \tag{3.7}$$

in order to satisfy (3.6), and (3.7) then determines γ' which determines I_1 and I_2. If the process mean is constant then using Wald's identity the ATS can be written as

$$E(R) = E(N)E(R_i). \tag{3.8}$$

If

$$p_j = P(d(\overline{X}) = d_j) = P(\overline{X} \in I_j)$$

then

$$E(R_i) = \sum_{j=1}^{t} d_j \frac{p_j}{1 - q} \tag{3.9}$$

where the probabilities are conditional on no signal since $\sum_{i=1}^{t} p_i = 1 - q < 1$. The R_i 's are i.i.d., and thus ATS is

$$E(R) = \frac{\sum_{j=1}^{t} d_j p_j}{q(1 - q)}. \tag{3.10}$$

Since the distribution of each R_i is the conditional distribution of $d(\overline{X})$ given that $\overline{X}$ is within the control limits, it follows that R_1, R_2, ... are conditionally independent of N and the variance of R can be expressed as

$$\text{Var}(R) = E(N)\text{Var}(R_i) + \text{Var}(N)E(R_i)^2$$

$$= \frac{\sum_{j=1}^{t} d_j^2 p_j}{q(1-q)} + \frac{(1-2q)(\sum_{j=1}^{t} d_j p_j)^2}{q^2(1-q)^2}. \tag{3.11}$$

The probabilities q and p_j used above depend on the value of μ. Equation (3.11) above reduces to equation (3.5) for the FSI Shewhart chart when a fixed sampling interval d is used.

3.3.2 Properties of the VSI Shewhart Chart when the Process Mean Drifts

The properties of the variable sampling interval control chart have been evaluated for the case where the process mean is constant. This means that the occurrence of an assignable cause results in a shift in the process mean that remains constant until the shift is detected and the assignable cause is eliminated. Another type of change in the process mean that can occur in applications is a gradual drift of the mean away from the target value. This drift can continue until it is detected and eliminated. In some applications the drift may be predictable to some degree, such as with tool wear in machining operations. In the case of tool wear the problem may be to monitor the process to decide on the optimal time to replace the tool. In other applications the drift may be unpredictable in onset, direction, and magnitude and it is primarily for this situation that the variable sampling interval chart is being considered.

In some situations it may be useful to know the actual distribution of the time to signal R. The distribution of R can be derived for certain special cases. Consider the case where each possible sampling interval is an integer multiple of a constant, i.e.

$$d_j = m_j h, \tag{3.12}$$

for $j = 1, 2, ..., t$, where $h > 0$ and m_j is an integer. Then the only times where samples can be taken will be integer multiples of h. If $\pi(i)$ is defined by

$$\pi(i) = P(\text{a sample is taken at time } hi), \tag{3.13}$$

for $i = 1, 2, \ldots$, then $\pi(i)$ satisfies

$$\pi(i) = \sum_{j=1}^{t} \pi(i - m_j)p_j, \tag{3.14}$$

for $i = 1, 2, \ldots$, where in addition we define $\pi(i) = 0$ for $i < 0$ and $\pi(0) = \frac{1}{1-q}$. When a sample is taken at a point the probability of a signal at that point is q and thus

$$P(R = hi) = \pi(i)q. \tag{3.15}$$

Then the ATS can be expressed as

$$E(R) = \sum_{i=0}^{\infty} hi\pi(i)q. \tag{3.15}$$

When there is a drift in the process mean the probabilities $p_1, p_2, \ldots, p_t, q$ change over time. The properties of the variable interval chart can be evaluated for the special case (3.13) where each sampling interval is an integer multiple of a constant $h > 0$. In this case denote the sampling interval and signal probabilities at time hi by $p_j(i)$ and $q(i)$, respectively. Then (3.14) becomes

$$\pi(i) = \sum_{j=1}^{t} \pi(i - m_j)p_j(i - m_j) \tag{3.17}$$

and the ATS is

$$E(R) = \sum_{i=0}^{\infty} hi\pi(i)q(i). \tag{3.18}$$

In addition the $ANSS$ will be

$$E(N) = \sum_{i=0}^{\infty} \pi(i). \tag{3.19}$$

A SAS program for the case of a drifting mean is given in Appendix 3.

3.3.3 The VSI Shewhart Chart with Runs Rules

The variable sampling interval technique can also be used with other types of charts such as Shewhart $\overline{X}$ -charts with run rules and cusum charts. When run rules are used with Shewhart $\overline{X}$ -charts the decision to signal depends on past samples in addition to the current sample. With the cusum chart the decision to signal depends on a cumulative sum of past sample means. When the variable sampling interval feature is added to such charts the sampling interval function could be allowed to depend on past samples. Although the principle is relatively straightforward, the actual calculation of properties and determination of optimal procedures is much more complicated. A simple extension of the variable sampling interval technique is investigated in this section to show that the variable sampling interval technique can significantly improve the performance of Shewhart charts other than the standard $\overline{X}$ -chart.

Consider a control chart where the decision to signal after the i^{th} sample is taken can depend on previous samples in addition to current samples. For example, suppose that warning limits are added to the Shewhart $\overline{X}$ -chart at $\mu_0 \pm \xi\sigma/\sqrt{n}$ where the control limits (or action limits) are at $\mu_0 \pm \gamma\sigma/\sqrt{n}$ and $\xi < \gamma$. In many cases γ will be 3 an ξ will be between 1 and 2. As an example of a chart with run rules using the warning limits consider the chart which signal if

– r' of r sample means fall between the warning and action limits on one side of μ_0, or

- r^{*} consecutive sample means are on one side of μ_0, or

- the current mean falls outside the action limits.

Suppose that the variable interval modification is added to a chart of this type where the sampling interval function depends only on the current sample. Allowing the sampling interval function to depend only on past samples would presumably be even better but this extension will not be considered here.

When the variable sampling interval feature with two sampling intervals d_1 and d_2 is used with the runs rules in application it would be convenient to choose $\gamma' = \xi$ so that the warning limits are the same as the limits that determine the sampling intervals d_1 and d_2. This would keep the chart simple and enable the user to quickly determine whether to signal and what sampling interval to use if the decision is not to signal. In this case it will be preferable to choose $\xi\ (= \gamma')$ to be less than 2, for example $\xi = 1$, so that the probability of using d_1 is not excessively small.

When runs rules are added to the Shewhart $\overline{X}$ -chart the $ANSS$ is no longer given by (3.1) since the distribution of N will not be geometric. Thus in this case a different method for evaluating $E(N)$ must be used. Methods for evaluating the ARL have been developed by Page (1962), Weindling, Littauer and Oliveira (1970) and Champ and Woodall (1987). Since the ARL of a fixed interval chart is numerically equal to the $ANSS$, these methods of computing the ARL can be applied to the case of the variable interval chart. The Markov chain approach used by Champ and Woodall (1987) was used to compute the $ANSS$. The ATS was computed by using the Markov chain to determine the expected number of times that each sampling interval is used before a signal is given.

Consider a two-sided Shewhart chart which uses lengths d_1 and d_2 with corresponding probabilities P_1 and P_2 respectively, and

$$\begin{aligned} I_1 &= I_{1u} \cup I_{1l} \\ &= (\mu_0 - \gamma\frac{\sigma}{\sqrt{n}},\ \mu_0 - \gamma'\frac{\sigma}{\sqrt{n}}] \cup [\mu_0 + \gamma'\frac{\sigma}{\sqrt{n}},\ \mu_0 + \gamma\frac{\sigma}{\sqrt{n}}) \end{aligned}$$

and

$$I_2 = (\mu_0 - \gamma'\frac{\sigma}{\sqrt{n}},\ \mu_0 + \gamma'\frac{\sigma}{\sqrt{n}}),$$

and signal region

$$A = (-\infty,\ \mu_0 - \gamma\frac{\sigma}{\sqrt{n}}] \cup [\mu_0 + \gamma\frac{\sigma}{\sqrt{n}},\ \infty)$$

where $0 < \gamma' < \gamma$.

The *ATS* for a *VSI* chart where a random mechanism is used to determine the first sampling interval used at the start of the process is

$$ATS = \frac{d_1 p_1 + d_2 p_2}{1 - q} + \sum_{j=2}^{t} M_{1j}\, d(j) + (M_{1,1} - 1)\, d(1)$$
$$= \frac{d_1 p_1 + d_2 p_2}{1 - q} + \sum_{j=1}^{t} M_{1j}\, d(j) - d_2$$

where d(j) is the sampling interval used when the process is in state E_j and M_{1j} is the expected number of times the process is in state j given that the process started with state 1.

A SAS program for the computations of *ATS* values for this case is given in Appendix 4.

3.3.4 A Markov Chain Approach to Evaluate the Properties of the VSI Shewhart Chart

An alternative method to evaluate the properties of the VSI Shewhart chart is to use the Markov chain approach as introduced by Arnold (1970). Denote the transition matrix by $P = \{p_{ij}\}$ where each row of P sums to unity. Let E_i = the sampling state in which the most recent sample was taken i time units previously ($i = 1,2,3,\ldots, d_t$) and let A be the absorbing state. Define π_i to be the probability of delaying exactly i time units before taking the next sample, and define p_{ij} to be the probability of a state transition from i to j in one unit of time. Then the general element of the one-step transition matrix P can be expressed as

$$
p_{ij} = \begin{cases} \dfrac{(1 - P_A)\,\pi_i}{\sum_{v=i}^{t} \pi_v}, & j = 1, \quad i = 1,2,\ldots,d_t \\ \dfrac{P_A\,\pi_i}{\sum_{v=i}^{t} \pi_v}, & j = A, \quad i = 1,2,\ldots,d_t \\ 1 - \dfrac{\pi_i}{\sum_{v=i}^{t} \pi_v}, & j = i + 1, \quad i = 1,2,\ldots,d_t - 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.20}
$$

The Markovian sampling scheme is a finite absorbing Markov chain in whch d_t states are transient. We can obtain the transition matrix P as

$$
P = \begin{bmatrix} p_{1,1} & p_{1,2} & \cdot & \cdot & \cdot & p_{1,d_t} & p_{1,A} \\ p_{2,1} & p_{2,2} & \cdot & \cdot & \cdot & p_{2,d_t} & p_{2,A} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{d_t,1} & p_{d_t,2} & \cdot & \cdot & \cdot & p_{d_t,d_t} & p_{d_t,A} \\ 0 & 0 & . & . & . & 0 & 1 \end{bmatrix}.
$$

The matrix P can be written in the form

$$
P = \begin{bmatrix} Q & (I - Q)\underline{1} \\ \underline{0}' & 1 \end{bmatrix} \tag{3.21}
$$

where $p_{i,j}$ = P(going to $E_j \mid in\ E_i$) and where Q is the submatrix of P obtained by deleting the last row and column, i.e. removing the rows and columns associated with the absorbing state. The properties of the VSI scheme can be obtained by working with the submatrix Q which simplifies the task. The fundamental matrix M is

$$M = (I - Q)^{-1} = \{M_{ij}\} \tag{3.22}$$

where M_{ij} = expected number of times the process is in state E_j given that the process starts in E_i. The average number of samples until a signal is expressed as the $(1,1)^{th.}$ element of the matrix M as

$$ANSS = M_{1,1} \tag{3.23}$$

where $M_{1,1}$ is the expected number of times the process is in state E_1 , given that the process starts in in state E_1 It is assumed that the process always starts in state E_1, i.e. at time 1 the last sample was taken 1 time unit earlier when the process was started. Another expression for the $ANSS$ is

$$ANSS = \sum_{j=1}^{d_t} (M_{1j} - M_{1j+1}) + M_{1,d_t} \tag{3.24}$$

where the difference $(M_{1_j} - M_{1_{j+1}})$ is the number of times the process uses a sampling interval of exactly j time units between two consecutive samples. As for the average number of time units until signal, it can expressed as

$$ATS = \sum_{j=1}^{d_t} M_{1j} \tag{3.25}$$

where M_{1_j} is the expected number of times the process will use sampling intervals of length at least j time units. Summing over all possible values of j to the maximum sampling interval d_t , we obtain the average time to signal. The ATS can also be expressed as

$$ATS = \sum_{j=1}^{d_t - 1} (M_{1j} - M_{1j+1})\, d_j + M_{d_t} d_t \tag{3.26}$$

where d_j represents the sampling interval of length d_j time units. The difference $(M_{1_j} - M_{1_{j+1}})$ is the expected number of times the process uses a sampling interval of exactly j time units. A third expression for the ATS is

$$ATS = \frac{\sum_{j=1}^{d_t} d_j \pi_j}{q(1-q)}. \tag{3.27}$$

Equation (3.27) can be obtained from (3.25) as follows:

Consider the transition matrix P and the matrix M where

$$P = \begin{bmatrix} Q & (I-Q)\underline{1} \\ \underline{0}' & 1 \end{bmatrix}$$

and

$$ATS = \sum_{j=1}^{d_t} M_{1j}.$$

Recall that by definition of the inverse of a matrix

$$\begin{aligned} M &= (I - Q)^{-1} \\ &= \frac{(I-Q)_c}{|(I-Q)|} \end{aligned}$$

where

$$(I - Q)_c = \text{matrix of cofactors } (\alpha).$$

It can be shown by induction that

$$\begin{aligned} \alpha_{1,1} &= 1 \\ \alpha_{1,2} &= 1\, p_{1,2} \\ \alpha_{1,3} &= 1\, p_{1,2}\, p_{2,3} \\ . &= \ldots \\ \alpha_{1,d_t} &= 1\, p_{1,2}\, p_{2,3} \cdots p_{d_t-1,\, d_t} \end{aligned}$$

Parzen (1962) showed that

$$|(I - Q)| = p_A.$$

The ATS is

$$\begin{aligned} ATS &= \sum_{j=1}^{d_t} M_{1j} \\ &= \sum_{j=1}^{d_t} \frac{\alpha_{1j}}{|I - Q|} \\ &= \sum_{j=1}^{d_t} \frac{\alpha_{1j}}{p_A} \\ &= \frac{(1 + p_{1,2} + p_{1,2}p_{2,3} + \dots \prod_{i=1}^{d_t - 1} p_{i,i+1})}{p_A} \end{aligned}$$

Let

$$\pi(i) = \sum_{v=i}^{d_t} \pi_v.$$

It follows that

$$p_{1,2} = 1 - \frac{\pi_1}{\pi(1)} = \frac{(\pi(1) - \pi_1)}{\pi(1)} = \frac{\pi(2)}{\pi(1)}$$

$$p_{2,3} = 1 - \frac{\pi_2}{\pi(2)} = \frac{(\pi(2) - \pi_2)}{\pi(2)} = \frac{\pi(3)}{\pi(2)}$$

$$p_{3,4} = 1 - \frac{\pi_3}{\pi(3)} = \frac{(\pi(3) - \pi_3)}{\pi(3)} = \frac{\pi(4)}{\pi(3)}$$

$$. = . . = . = .$$

$$p_{d_{t-1},d_t} = 1 - \frac{\pi_{d_{t-1}}}{\pi(d_{t-1})} = \frac{(\pi(d_{t-1}) - \pi_{d_{t-1}})}{\pi(d_{t-1})} = \frac{\pi(d_t)}{\pi(d_{t-1})}.$$

Hence the average time to signal is

$$
\begin{aligned}
ATS &= \left(1 + \frac{\pi(2)}{\pi(1)} + \frac{\pi(2)\,\pi(3)}{\pi_{(1)}\,\pi(2)} + \dots + \frac{\prod_{i=2}^{d_t} \pi(i)}{\prod_{i=1}^{d_t - 1} \pi(i)}\right)/p_A \\
&= (1 + \pi(2) + \pi(3) + \dots + \pi\, d_t)/p_A\, \pi(1) \\
&= \left(\sum_{i=1}^{d_t} \pi_i + \sum_{i=2}^{d_t} \pi_i + \sum_{i=3}^{d_t} \pi_i + \dots + \pi_{d_t}\right)/p_A\, \pi(1) \\
&= (\pi(1) + 2\pi(2) + 3\pi(3) \dots + d_t \pi_{d_t})/p_A\, \pi(1) \\
&= \frac{\sum_{i=1}^{d_t} d_i\, \pi_{d_i}}{p_A\,(1 - p_A)}
\end{aligned}
$$

where $\pi(1) = (1 - p_A)$. The Markov chain approach can be expanded to the multivariate case where several characteristics are monitored at the same time through a multivariate statistic. This approach may be a valuable tool when complicated cases are considered for which other approaches could be tedious to use.

3.3.5 The Relative Efficiency of the VSI Shewhart Chart

A measure of relative efficiency in comparing two process control procedures such as the FSI Shewhart chart and VSI Shewhart chart is the ratio of the ATS when the process it out out of control while both procedures have the same in-control ATS and $ANSS$. The ratio of the average time to signal depends on the value of μ_1 since the ATS changes with μ. Let

$$
RE = \frac{ATS \text{ of } (FSI)}{ATS \text{ of } (VSI)}
$$

denote the relative efficiency of the VSI Shewhart chart to the FSI Shewhart chart. Then the relative efficiency is

$$
\begin{aligned}
RE &= \frac{d / p_A}{\sum_{i=1}^{d_t} d_i \pi_{d_i} / p_A (1 - p_A)} \\
&= \frac{d}{\sum_{i=1}^{d_t} d_i \pi_{d_i} / (1 - p_A)}
\end{aligned}
\tag{3.28}
$$

where

d_i = sampling interval of length exactly d_i time units
d = fixed sampling interval of length d

The relative efficiency is just the ratio of the expected sampling intervals. When the process is in control the fixed sampling interval of the FSI Shewhart chart and the expected sampling interval of the VSI Shewhart chart are chosen such that they are equal. When the process goes out of control the procedure with the shorter expected sampling interval is more efficient with respect to the ATS. Consider the limiting relative efficiency of the VSI Shewhart chart as the process mean $\mu \rightarrow \infty$

$$
\lim_{\mu \to \infty} \frac{ATS(FSI)}{ATS(VSI)} = \frac{d}{d_1} \tag{3.29}
$$

where d_1 = minimum sampling interval used. This implies that the RE will increase as d_1 decreases and all other d_i $(i > 1)$ increase. In the simple case of using two sampling intervals, this result implies that d_1 and d_2 should be spaced apart.

3.4 *Computational Results*

The usefulness of the variable sampling interval feature is compared numerically to the FSI Shewhart chart for both one-sided and two-sided charts. In order to be able to make valid comparisons between the FSI Shewhart chart and VSI Shewhart chart it is necessary to choose the parameters of the charts such that both the ANSS and the ATS are the same for both kinds of charts when the process is in control. As long as the control limits are chosen to be the same for both charts, the ANSS will also be the same since the ANSS depends only on q, the probability of a signal. Three sigma control limits are used such that the in-control ANSS is 740.8 and 370.4 samples for one-sided and two-sided charts respectively. For convenience it is assumed that the FSI Shewhart chart uses a sampling interval $d = 1$ time unit, i.e. the ANSS will equal the ATS when the process is in control. The sampling intervals of the VSI Shewhart chart are chosen such that the expected sampling interval is unity when the process is in control, i.e. $E(R_i) = 1$. This will assure that the FSI Shewhart chart and the VSI Shewhart chart have the same ATS and same ANSS when the process is in control. Then the values of the ATS functions for the two charts can be compared numerically fo several values of μ_1 to determine which chart will do a better job of detecting a change in μ for the same ANSS.

3.4.1 The ATS for Shewhart Charts with Runs Rules

The *ATS* was computed by using the Markov chain to determine the expected number of times that each sampling interval is used before a signal is given. A comparison of the $\overline{X}$- chart using run rules both with and without the variable sampling interval feature can be done in the same way as the comparison for the charts without run rules. If both charts use the same rule to decide when to signal then the *ANSS* function will be equal for all μ . If the fixed interval is chosen 1 and the sampling interval function of the variable interval chart is chosen so that the ATS is equal to the $ANSS$ when $\mu = \mu_0$ then the two charts will be matched.

Values of the unadjusted ATS were computed for the fixed and variable interval $\overline{X}$- chart using run rules of the form explained above with $\gamma = 3$ for two cases. In the first case, called rule I, $d_1 = 0.1$, $d_2 = 1.078$, $\xi = \gamma' = 2, r' = 2$, and $r = 3$, and in the second case, called rule II, $d_1 = 0.1$, $d_2 = 1.415$, $\xi = \gamma' = 1, r' = 5$, and $r = 5$. The value of d_2 for rules I and II was selected so that the ATS values for the fixed and variable interval charts matched when $\mu = \mu_0$. In all practical applications it would be preferable to pick d_2 to be a convenient value since it would not be necessary to match a fixed interval chart. The results in Table (3.1) show that the variable sampling interval feature significantly improves the two $\overline{X}$- charts with run rules. It can be noted that the (variable) rule I is less efficient than the (variable) rule II when compared to their corresponding FSI Shewhart chart with runs rules because the region corresponding to d_2 has a significantly higher probability than the region corresponding to d_1 when $\mu = \mu_0$. A more extensive investigation of this topic will be required before recommendations can be given on the best choice of runs rules for various situations.

Table 3.1: Values of the ATS for two-sided symmetric Shewhart charts with runs rules

Shift = $\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	RULE I FIXED	RULE I VARIABLE	RULE II FIXED	RULE II VARIABLE
0.00	225.868	225.868	349.388	349.388
0.10	211.745	211.431	329.381	328.384
0.25	158.058	156.607	249.820	245.143
0.50	77.735	74.965	121.801	113.022
1.00	20.005	17.475	27.737	20.817
1.50	7.301	5.544	9.411	5.107
2.00	3.646	2.327	4.675	1.680
2.50	2.292	1.181	2.874	0.665
3.00	1.676	0.675	1.948	0.313
4.00	1.669	0.280	1.188	0.132

The following sections of this chapter give the numerical results for the FSI and VSI Shewhart charts when the mean shifts suddenly. A SAS program (PROC MATRIX) for computing the following results is given Appendix 2.

3.4.2 The Number of Sampling Intervals

One of the main reasons why the Shewhart chart is such a popular process control procedure is its ease of use. It is important to keep the process control chart as simple as possible and still obtain satisfactory performance in detecting an out-of-control situation as early as possible. The FSI Shewhart chart is simpler to use than the VSI Shewhart chart because only one sampling interval is necessary and so it is of importance to us to know how many intervals are really necessary in the VSI Shewhart chart to obtain best results. Shewhart charts with different number of sampling intervals ($t = 2,3,5,9$) are evaluated numerically as shown in Tables (3.2,3.3,3.4,3.5,3.6,3.7). The numerical results show clearly that the VSI Shewhart chart with only two sampling intervals has uniformly the shortest ATS when compared to the other VSI Shewhart charts. All variable charts have a better performance than the FSI Shewhart chart with respect to the ATS. The more sampling intervals are used the more the properties of the VSI Shewhart chart approaches the properties of the FSI Shewhart chart. As for the standard deviation of the time to signal the FSI Shewhart chart has a smaller SD for very small shifts only ($\mu < 0.1$). Table 3.4 reveals that among one-sided VSIS charts using two sampling intervals yields the smallest SD values for all shifts considered in the range (0.1,4.0). As the number of sampling intervals are increased the SD of the VSI Shewhart chart approaches the SD of the FSI Shewhart chart. Among two-sided VSI Shewhart charts it seems that using two intervals only results in smaller values of standard deviation for small and moderately large shifts. However the properties of the VSI Shewhart charts are very similar for the two-sided case and no real preference can be detected in Table (3.5). The coefficient of variation is lower for the FSI Shewhart chart and for all shifts considered for both the one-sided and two-sided charts. The coefficient of variation appears to increase with an increase of the size of the shift in he process mean and then to decrease for very large shifts. Among one-sided VSI Shewhart charts the chart which uses only two sampling intervals has the lowest values for the coefficient of variation whereas among two-sided VSI Shewhart charts the chart which uses nine intervals has the lowest coefficient of variation however the differences are practically insignificant in the two-sided case.

Table 3.2: Values of the ATS for one-sided symmetric Shewhart charts with representative number of sampling intervals

Shift = $\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	A $t=1$	B $t=2$	C $t=3$	D $t=5$	E $t=9$
0.00	740.800	740.800	740.800	740.800	740.800
0.10	535.959	497.722	501.100	501.570	505.322
0.25	335.597	276.302	281.466	282.787	287.322
0.50	161.039	105.926	110.466	111.831	116.163
1.00	43.956	17.208	18.958	19.567	21.344
1.50	14.968	3.419	3.930	4.135	4.726
2.00	6.303	0.936	1.072	1.136	1.325
2.50	3.241	0.376	0.412	0.432	0.494
3.00	2.000	0.210	0.219	0.226	0.248
4.00	1.189	0.119	0.120	0.121	0.124

A = (1) B = (.1/1.9) C = (.1/1/1.9) D = (.1/.5/1/1.5/1.9)

E = (.1/.3/.5/.7/1/1.3/1.5/1.7/1.9)

Table 3.3: Values of the ATS for two-sided symmetric Shewhart charts with representative number of sampling intervals

Shift = $\sqrt{n}(\mu_1 - \mu_0)/\sigma$	A t = 1	B t = 2	C t = 3	D t = 5	E t = 9
0.00	370.400	370.400	370.400	370.400	370.400
0.10	352.931	351.491	351.596	351.638	351.740
0.25	281.153	274.572	275.044	275.221	275.680
0.50	155.224	141.428	142.385	142.743	143.684
1.00	43.895	30.604	31.406	31.718	32.548
1.50	14.968	6.951	7.332	7.492	7.919
2.00	6.303	1.821	1.972	2.042	2.231
2.50	3.241	0.603	0.659	0.688	0.769
3.00	2.000	0.271	0.292	0.304	0.340
4.00	1.189	0.125	0.128	0.131	0.139

A = (1) B = (.1/1.9) C = (.1/1/1.9) D = (.1/.5/1/1.5/1.9)

E = (.1/.3/.5/.7/1/1.3/1.5/1.7/1.9)

Table 3.4: Values of the standard deviation of the time to signal for one-sided symmetric Shewhart charts with representative number of sampling intervals

Shift =	A	B	C	D	E
$\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	t = 1	t = 2	t = 3	t = 5	t = 9
0.00	740.297	740.708	740.836	740.872	741.090
0.10	535.459	497.691	501.207	502.374	505.513
0.25	335.097	276.363	281.670	283.446	288.130
0.50	160.538	106.124	110.816	112.415	116.592
1.00	43.453	17.570	19.498	20.199	21.980
1.50	14.460	3.758	4.460	4.737	5.396
2.00	5.781	1.128	1.427	1.560	1.851
2.50	2.695	0.437	0.582	0.659	0.822
3.00	1.414	0.198	0.272	0.319	0.422
4.00	0.473	0.055	0.074	0.093	0.141

A = (1) B = (.1/1.9) C = (.1/1/1.9) D = (.1/.5/1/1.5/1.9)

E = (.1/.3/.5/.7/1/1.3/1.5/1.7/1.9)

Table 3.5: Values of the standard deviation of the time to signal for two-sided symmetric Shewhart charts with representative number of sampling intervals

Shift = $\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	A t = 1	B t = 2	C t = 3	D t = 5	E t = 9
0.00	369.898	370.175	370.050	370.023	369.980
0.10	352.430	351.400	351.369	351.376	351.425
0.25	280.652	274.498	274.833	274.974	275.378
0.50	154.723	141.413	142.226	142.545	143.426
1.00	43.392	30.770	31.411	31.677	32.429
1.50	14.459	7.275	7.493	7.599	7.941
2.00	5.781	2.170	2.193	2.214	2.327
2.50	2.695	0.854	0.837	0.832	0.865
3.00	1.414	0.402	0.390	0.383	0.396
4.00	0.473	0.114	0.114	0.111	0.119

A = (1) B = (.1/1.9) C = (.1/1/1.9) D = (.1/.5/1/1.5/1.9)

E = (.1/.3/.5/.7/1/1.3/1.5/1.7/1.9)

Table 3.6: Values of the coefficient of variation of the time to signal for one-sided symmetric Shewhart charts with representative number of sampling intervals

Shift = $\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	A t = 1	B t = 2	C t = 3	D t = 5	E t = 9
0.00	0.9993	0.9999	1.0000	1.0001	1.0002
0.10	0.9991	0.9999	1.0002	1.0003	1.0004
0.25	0.9985	1.0002	1.0007	1.0008	1.0010
0.50	0.9969	1.0019	1.0032	1.0035	1.0037
1.00	0.9886	1.0210	1.0285	1.0299	1.0298
1.50	0.9660	1.0991	1.1349	1.1427	1.1418
2.00	0.9173	1.2052	1.3314	1.3698	1.3970
2.50	0.8315	1.1613	1.4139	1.5214	1.6631
3.00	0.7071	0.9459	1.2401	1.4090	1.7047
4.00	0.3983	0.4608	0.6197	0.7684	1.1316

Table 3.7: Values of the coefficient of variation of the time to signal for two-sided symmetric Shewhart charts with representative number of sampling intervals

Shift = $\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	A t = 1	B t = 2	C t = 3	D t = 5	E t = 9
0.00	0.9986	0.9997	0.9994	0.9993	0.9991
0.10	0.9986	0.9997	0.9994	0.9993	0.9991
0.25	0.9982	0.9997	0.9992	0.9991	0.9989
0.50	0.9968	0.9999	0.9989	0.9986	0.9982
1.00	0.9885	1.0054	1.0002	0.9987	0.9964
1.50	0.9660	1.0465	1.0219	1.0144	1.0028
2.00	0.9173	1.1915	1.1118	1.0843	1.0434
2.50	0.8315	1.4164	1.2693	1.2095	1.1247
3.00	0.7071	1.4846	1.3336	1.2595	1.1647
4.00	0.3983	0.9148	0.8883	0.8508	0.8543

3.4.3 Choice of Interval Width

Numerical results in the previous section suggest the use of two sampling intervals in order to obtain the lowest out-of-control ATS. When a VSI Shewhart chart with two intervals is used the question of how to choose these intervals arises. Consider VSI Shewhart charts with two sampling intervals d_1 and d_2 :(0.1,1.9), (0.3,1.7) , (0.6,1.4) and (0.9,1.1) as shown in Tables (3.8,3.9,3.10, 3.11,3.12,3.13). The ATS is lowest for both one-sided and two-sided charts when the two sampling intervals are chosen far apart, i.e. $d_1 = 0.1$ and $d_2 = 1.9$. As d_1 increases and d_2 decreases the properties of the VSI Shewhart chart approach the properties of the FSI Shewhart chart. All VSI Shewhart charts have a lower ATS than the corresponding FSI Shewhart chart. The standard deviation of the time to signal is lowest when the sampling intervals are chosen far apart and for all shifts considered in this study. The standard deviation increases as the sampling intervals approach the fixed interval d. The coefficient of variation is lowest for the FSI Shewhart chart for both the one-sided and two-sided charts, however the difference in the coefficient of variation is very small when compared to the VSIS charts. It appears that choosing the sampling intervals far apart works very well with respect to the ATS and the SD.

Table 3.8: Values of the ATS for one-sided symmetric Shewhart charts with representative interval widths

	A	B	C	D	E
Shift =	d1 = 1	d1 = .1	d1 = .3	d1 = .6	d1 = .9
$\sqrt{n}\ (\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.9	d2 = 1.7	d2 = 1.4	d2 = 1.1
0.00	740.800	740.800	740.800	740.800	740.800
0.10	535.959	497.722	506.217	518.964	531.710
0.25	335.597	276.302	289.478	309.243	329.009
0.50	161.039	105.926	118.173	136.544	154.916
1.00	43.956	17.208	23.152	32.068	40.984
1.50	14.968	3.419	5.986	9.836	13.685
2.00	6.303	0.936	2.129	3.918	5.707
2.50	3.241	0.376	1.013	1.968	2.923
3.00	2.000	0.210	0.608	1.204	1.801
4.00	1.189	0.119	0.357	0.713	1.070

Table 3.9: Values of the ATS for two-sided symmetric Shewhart charts with representative interval widths

	A	B	C	D	E
Shift =	d1 = 1	d1 = .1	d1 = .3	d1 = .6	d1 = .9
$\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.9	d2 = 1.7	d2 = 1.4	d2 = 1.1
0.00	370.400	370.400	370.400	370.400	370.400
0.10	352.931	351.491	351.811	352.291	352.771
0.25	281.153	274.572	276.034	278.228	280.421
0.50	155.224	141.428	144.494	149.093	153.691
1.00	43.895	30.604	33.557	37.988	42.418
1.50	14.968	6.951	8.733	11.405	14.077
2.00	6.303	1.821	2.617	4.311	5.805
2.50	3.241	0.603	1.189	2.068	2.948
3.00	2.000	0.271	0.655	1.231	1.808
4.00	1.189	0.125	0.361	0.716	1.070

Table 3.10: Values of the standard deviation of the time to signal for one-sided symmetric Shewhart charts with representative interval widths

	A	B	C	D	E
Shift =	d1 = 1	d1 = .1	d1 = .3	d1 = .6	d1 = .9
$\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.9	d2 = 1.7	d2 = 1.4	d2 = 1.1
0.00	740.297	740.704	740.544	740.378	740.302
0.10	535.459	497.691	506.002	518.561	531.219
0.25	335.097	276.363	289.319	308.866	328.523
0.50	160.538	106.124	118.092	136.201	154.438
1.00	43.453	17.570	23.141	31.761	40.518
1.50	14.460	3.758	5.948	9.535	13.222
2.00	5.781	1.128	2.034	3.608	5.235
2.50	2.695	0.437	0.875	1.642	2.431
3.00	1.414	0.198	0.442	0.854	1.274
4.00	0.473	0.055	0.144	0.284	0.426

Table 3.11: Values of the standard deviation of the time to signal for two-sided symmetric Shewhart charts with representative interval widths

	A	B	C	D	E
Shift =	d1 = 1	d1 = .1	d1 = .3	d1 = .6	d1 = .9
$\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.9	d2 = 1.7	d2 = 1.4	d2 = 1.1
0.00	369.898	370.175	370.043	369.921	369.889
0.10	352.430	351.400	351.558	351.872	352.276
0.25	280.652	274.498	275.792	277.813	279.927
0.50	154.723	141.413	144.289	148.694	153.200
1.00	43.392	30.770	33.459	37.635	41.937
1.50	14.459	7.275	8.712	11.087	13.602
2.00	5.781	2.170	2.800	4.002	5.327
2.50	2.695	0.854	1.125	1.747	2.453
3.00	1.414	0.402	0.539	0.885	1.279
4.00	0.473	0.114	0.165	0.289	0.426

Table 3.12: Values of the coefficient of variation of the time to signal for one-sided symmetric Shewhart charts with representative interval widths

	A	B	C	D	E
Shift =	d1 = 1	d1 = .1	d1 = .3	d1 = .6	d1 = .9
$\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.9	d2 = 1.7	d2 = 1.4	d2 = 1.1
0.00	0.9993	0.9999	0.9997	0.9994	0.9993
0.10	0.9991	0.9999	0.9996	0.9992	0.9991
0.25	0.9985	1.0002	0.9995	0.9988	0.9985
0.50	0.9969	1.0019	0.9993	0.9975	0.9969
1.00	0.9886	1.0210	0.9995	0.9904	0.9886
1.50	0.9660	1.0991	0.9937	0.9694	0.9661
2.00	0.9173	1.2052	0.9554	0.9209	0.9174
2.50	0.8315	1.1613	0.8639	0.8344	0.8316
3.00	0.7071	0.9459	0.7269	0.7088	0.7072
4.00	0.3983	0.4608	0.4028	0.3987	0.3983

Table 3.13: Values of the coefficient of variation of the time to signal for two-sided symmetric Shewhart charts with representative interval widths

	A	B	C	D	E
Shift =	d1 = 1	d1 = .1	d1 = .3	d1 = .6	d1 = .9
$\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.9	d2 = 1.7	d2 = 1.4	d2 = 1.1
0.00	0.9986	0.9997	0.9931	0.9989	0.9987
0.10	0.9986	0.9997	0.9993	0.9988	0.9986
0.25	0.9982	0.9997	0.9991	0.9985	0.9982
0.50	0.9968	0.9999	0.9986	0.9973	0.9968
1.00	0.9885	1.0054	0.9971	0.9907	0.9887
1.50	0.9660	1.0465	0.9977	0.9722	0.9663
2.00	0.9173	1.1915	0.9938	0.9283	0.9176
2.50	0.8315	1.4164	0.9465	0.8447	0.8320
3.00	0.7071	1.4846	0.8223	0.7185	0.7074
4.00	0.3983	0.9148	0.4556	0.4034	0.3985

3.4.4 Asymmetry of Sampling Intervals

Define a *symmetric* Shewhart chart to have sampling intervals d_i $i=1,2,\ldots,$ d_t such that $P(d_i) = P(d_{t+1-i})$ when the process is in control. When only two sampling intervals are used this translates to choosing $P(d_1) = P(d_2)$ when $\mu = \mu_0$. All numerical results in the previous sections of this chapter are symmetric charts. Asymmetric charts are such that $P(d_i) \neq P(d_{t+1-i})$. In a chart with only two sampling intervals d_1 and d_2 the choice of the corresponding probabilities $P(d_1)$ and $P(d_2)$ affects the ATS as shown in Tables (3.14,3.15). Five different VSI Shewhart charts with sampling intervals $d_1 = 0.1$ and $d_2 =$ (1.1,1.5,1.9,4.0,10.0) are compared to the corresponding FSI Shewhart chart with the same ANSS and ATS when the process is in control. This means that whenever we increase d_2 we are also decreasing $P(d_2)$ in order to obtain the same ATS in control as the FSI Shewhart chart. Hence a chart with $d_2 > 1.9$ implies that the probability of the larger sampling interval (when in control) will be less than the probability of the smaller sampling interval when in control. The results in Tables(3.14,3.15) suggest that when only two sampling intervals are used and where $d_1 = 0.1$ it is best to choose the maximum interval as large as possible. This result holds for all shifts considered and for both one-sided and two-sided charts. The standard deviations are lowest when $d_2 = 10$ in the one-sided charts for shifts in the range (0.1,10) whereas among two-sided charts the symmetric chart has the lowest values of the standard deviation for a wide range of shifts. The properties of the VSI Shewhart chart approach the properties of FSI Shewhart chart as d_2 decreases. Two-sided VSI Shewhart charts with symmetric sampling intervals have a small standard deviation for a wide range of shifts, however the difference among the VSI Shewhart charts is very small. The coefficient of variation is lowest for the FSI Shewhart chart, however the VSI Shewhart charts have similar values for the coefficient of variation.

Table 3.14: Values of the ATS for one-sided asymmetric Shewhart charts with representative interval widths

	A	B	C	D	E	F
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.1	d2 = 1.5	d2 = 1.9	d2 = 4.0	d2 = 10
0.00	740.800	740.800	740.800	740.800	740.800	740.800
0.10	535.959	526.105	507.544	497.722	475.090	454.851
0.25	335.597	318.754	290.443	276.302	245.962	221.351
0.50	161.039	142.626	117.206	105.926	84.375	69.315
1.00	43.956	31.755	20.905	17.208	11.620	8.734
1.50	14.968	8.088	4.368	3.419	2.288	1.859
2.00	6.303	2.385	1.164	0.936	0.721	0.662
2.50	3.241	0.841	0.431	0.376	0.335	0.327
3.00	2.000	0.369	0.223	0.210	0.201	0.200
4.00	1.189	0.143	0.120	0.119	0.119	0.119

Table 3.15: Values of the ATS for two-sided asymmetric Shewhart charts with representative interval widths

	A	B	C	D	E	F
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.1	d2 = 1.5	d2 = 1.9	d2 = 4.0	d2 = 10
0.00	370.400	370.400	370.400	370.400	370.400	370.400
0.10	352.931	352.392	351.758	351.491	352.931	350.672
0.25	281.153	278.370	275.483	274.572	273.438	273.050
0.50	155.224	149.110	143.174	141.428	139.529	139.056
1.00	43.895	37.296	32.027	30.604	29.152	28.839
1.50	14.968	10.355	7.610	6.951	6.313	6.182
2.00	6.303	3.304	1.972	1.821	1.591	1.546
2.50	3.241	1.229	0.659	0.603	0.525	0.511
3.00	2.000	0.544	0.292	0.271	0.245	0.241
4.00	1.189	0.186	0.128	0.125	0.122	0.121

Table 3.16: Values of the standard deviation of the time to signal for one-sided asymmetric Shewhart charts with representative interval widths

	A	B	C	D	E	F
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}\ (\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.1	d2 = 1.5	d2 = 1.9	d2 = 4.0	d2 = 10
0.00	740.297	740.364	740.353	740.798	740.871	744.089
0.10	535.459	525.667	507.317	497.691	476.027	458.449
0.25	335.097	318.346	290.291	276.363	247.003	224.954
0.50	160.538	142.278	117.175	106.124	85.512	72.691
1.00	43.453	31.556	21.071	17.570	12.608	10.819
1.50	14.460	8.046	4.612	3.758	2.815	2.609
2.00	5.781	2.460	1.357	1.128	0.888	0.825
2.50	2.695	0.961	0.524	0.437	0.347	0.320
3.00	1.414	0.472	0.240	0.198	0.161	0.151
4.00	0.473	0.164	0.066	0.055	0.049	0.048

Table 3.17: Values of the standard deviation of the time to signal for two-sided asymmetric Shewhart charts with representative interval widths

	A	B	C	D	E	F
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}\ (\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.1	d2 = 1.5	d2 = 1.9	d2 = 4.0	d2 = 10
0.00	369.898	369.971	370.095	370.175	370.765	373.170
0.10	352.430	351.938	351.486	351.400	351.829	354.208
0.25	280.652	277.924	275.226	274.498	274.309	276.591
0.50	154.723	148.691	142.971	141.413	140.454	142.607
1.00	43.392	36.980	31.995	30.770	30.218	32.268
1.50	14.459	10.182	7.753	7.275	7.377	8.977
2.00	5.781	3.275	2.305	2.170	2.394	3.308
2.50	2.695	1.303	0.900	0.854	0.981	1.421
3.00	1.414	0.657	0.432	0.402	0.453	0.656
4.00	0.473	0.263	0.133	0.114	0.115	0.159

Table 3.18: Values of the coefficient of variation of the time to signal for one-sided asymmetric Shewhart charts with representative interval widths

	A	B	C	D	E	F
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}\ (\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.1	d2 = 1.5	d2 = 1.9	d2 = 4.0	d2 = 10
0.00	0.9993	0.9994	0.9996	0.9999	1.0012	1.0048
0.10	0.9991	0.9992	0.9996	0.9999	1.0020	1.0079
0.25	0.9985	0.9976	0.9995	1.0002	1.0042	1.0163
0.50	0.9969	0.9937	0.9997	1.0019	1.0135	1.0487
1.00	0.9886	0.9948	1.0079	1.0210	1.0850	1.2388
1.50	0.9660	1.0315	1.0559	1.0991	1.2307	1.4031
2.00	0.9173	1.1427	1.1659	1.2052	1.2323	1.2462
2.50	0.8315	1.2792	1.2160	1.1613	1.0365	0.9798
3.00	0.7071	1.1459	1.0733	0.9459	0.7997	0.7556
4.00	0.3983	0.9995	0.5465	0.4608	0.4091	0.4014

Table 3.19: Values of the coefficient of variation of the time to signal for two-sided asymmetric Shewhart charts with representative interval widths

	A	B	C	D	E	F
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}\ (\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.1	d2 = 1.5	d2 = 1.9	d2 = 4.0	d2 = 10
0.00	0.9986	0.9988	0.9993	0.9997	1.0023	1.0096
0.10	0.9986	0.9987	0.9992	0.9997	1.0024	1.0101
0.25	0.9982	0.9984	0.9991	0.9997	1.0032	1.0130
0.50	0.9968	0.9972	0.9986	0.9999	1.0066	1.0255
1.00	0.9885	0.9915	0.9990	1.0054	1.0365	1.1189
1.50	0.9660	0.9833	1.0188	1.0465	1.1686	1.4521
2.00	0.9173	0.9912	1.1110	1.1915	1.5046	2.1398
2.50	0.8315	1.0598	1.2957	1.4164	1.8681	2.7814
3.00	0.7071	1.2096	1.4204	1.4846	1.8465	2.7250
4.00	0.3983	1.4109	1.0243	0.9148	0.9459	1.3146

3.4.5 Gradual Shift in the Process Mean

The ATS was evaluated numerically for the case where there are multiple sampling intervals and the process mean increases linearly over time. The amount of the drift at time hi is $\sqrt{n}\,\theta hi/\sigma$, where θ is the drift per unit time in units of the standard deviation of $\overline{X}$. To evaluate (3.17) and (3.18) numerically it is necessary at each i to store only $\pi(i-m), \pi(i-m+1), \ldots, \pi(i-1)$ in order to compute $\pi(i)$. The sums in (3.18) and (3.19) can be truncated when $q(i)$ is close to one. As in the case of a shift in the mean to a constant value , t = 2 intervals was best with respect to the ATS and the variable interval charts detects the shift before the fixed interval chart, although the detection is not dramatically faster. In contrast to the constant shift case, the VSI Shewhart chart takes more samples on the average to detect the drifting shift. An apparent explanation is that for a slow drift it takes the process mean a relatively long time to get to a level where detection is fast. For both the FSI Shewhart chart and the VSI Shewhart chart there is a relatively long period when detection is unlikely. When the process mean finally rises to a reasonably high level the variable interval chart will signal quickly but the long initial period when the process mean is low tends to mask the difference between the detection time for the two charts. Tables (3.20,3.21,3.22,3.23,3.24) show that the VSI Shewhart chart with t = 2 detects the drifting mean faster than VSI cusum charts with more than two sampling intervals and the FSI Shewhart chart, but it takes more samples to detect the shift. As the number of sampling intervals increase the properties of the VSI Shewhart chart approach the properties of the FSI Shewhart chart, i.e. the ATS increases and the $ANSS$ decreases. Although the performance of the VSI Shewhart chart in the following tables is not as dramatic as in the case of a sudden shift, when the drift does get to the point where the process mean is far from the target the VSI Shewhart chart will detect this situation much faster than the FSI Shewhart chart. More samples will be required for detection but faster detection will usually be worth the extra sampling cost. The process will presumably operate with the mean at the target most of the time and in this case the FSI Shewhart chart and the VSI Shewhart chart have the same average sampling rate. The ATS values computed below for the case of a drift in the process assume that the drift starts at time 0. Table (3.25) shows that the standard deviation of the time to signal is lower for the VSI Shewhart chart compared to the FSI Shewhart chart. As the number of sampling intervals increase the standard deviation tends to increase for small shifts and decrease for large shifts. Table (3.26) compares the ATS and the $ANSS$ for two-sided charts with $d_1 = 0.1$ and $d_2 = 1.1$, 1.9 and 2.5. As d_2 increases the properties of the VSI Shewhart chart approach the properties of the FSI Shewhart chart.

Table 3.20: Values of the $ANSS$ and the ATS for two-sided symmetric Shewhart charts with two sampling intervals in the case of a gradual shift

Shift/hr	$ANSS$ (F)	$ANSS$(V)	ATS(F)	ATS(V)
0.000	370.4000	370.4000	370.4000	370.4000
0.005	134.1050	139.1720	134.1050	127.3910
0.010	89.5601	95.3417	89.5601	83.4387
0.025	49.3706	55.1429	49.3706	44.6616
0.050	30.4519	35.6804	30.4519	26.8959
0.100	18.4285	22.8556	18.4285	15.8986
0.250	9.3122	12.5316	9.3122	7.8373
0.500	5.5186	7.8125	5.5186	4.6108
1.000	3.2772	4.7844	3.2772	2.6833

d_i = (0.1,1.9)

ANSS(F) = ANSS for the FSI chart

ANSS(V) = ANSS for the VSI chart

ATS(F) = ATS for the FSI chart

ATS(V) = ATS for the VSI chart

Table 3.21: Values of the *ANSS* and the *ATS* for two-sided symmetric Shewhart charts with 3 sampling intervals in the case of a gradual shift

Shift/hr	*ANSS* (F)	*ANSS*(V)	*ATS*(F)	*ATS*(V)
0.000	370.4000	370.4000	370.4000	370.4000
0.005	134.1050	138.8010	134.1050	127.7980
0.010	89.5601	94.9150	89.5601	83.7801
0.025	49.3706	54.7150	49.3706	44.8860
0.050	30.4519	35.2941	30.4519	27.0358
0.100	18.4285	22.5343	18.4285	15.9689
0.250	9.3122	12.3151	9.3122	7.8410
0.500	5.5186	7.6986	5.5186	4.5863
1.000	3.2772	4.6998	3.2772	2.7230

$d_i = (0.1, 1.0, 1.9)$

ANSS(F) = ANSS for the FSI chart

ANSS(V) = ANSS for the VSI chart

ATS(F) = ATS for the FSI chart

ATS(V) = ATS for the VSI chart

Table 3.22: Values of the *ANSS* and the *ATS* for two-sided symmetric Shewhart charts with 5 sampling intervals in the case of a gradual shift

Shift/hr	*ANSS* (F)	*ANSS*(V)	*ATS*(F)	*ATS*(V)
0.000	370.4000	370.4000	370.4000	370.4000
0.005	134.1050	138.5290	134.1050	128.1230
0.010	89.5601	94.6021	89.5601	84.0612
0.025	49.3706	54.3987	49.3706	45.0832
0.050	30.4519	35.0051	30.4519	27.1704
0.100	18.4285	22.2886	18.4285	16.0508
0.250	9.3122	12.1392	9.3122	7.8713
0.500	5.5186	7.5786	5.5186	4.5892
1.000	3.2772	4.6362	3.2772	2.7126

d_i = (0.1,0.5,1.0,1.5,1.9)

ANSS(F) = ANSS for the FSI chart

ANSS(V) = ANSS for the VSI chart

ATS(F) = ATS for the FSI chart

ATS(V) = ATS for the VSI chart

Table 3.23: Values of the *ANSS* and the *ATS* for two-sided symmetric Shewhart charts with 9 sampling intervals in the case of a gradual shift

Shift/hr	*ANSS* (F)	*ANSS*(V)	*ATS*(F)	*ATS*(V)
0.000	370.4000	370.4000	370.4000	370.4000
0.005	134.1050	138.3120	134.1050	128.3880
0.010	89.5601	94.3521	89.5601	84.2962
0.025	49.3706	54.1447	49.3706	45.2550
0.050	30.4519	34.7711	30.4519	27.2934
0.100	18.4285	22.0868	18.4285	16.1316
0.250	9.3122	11.9896	9.3122	7.9099
0.500	5.5186	7.4709	5.5186	4.6050
1.000	3.2772	4.5678	3.2772	2.7136

d_i = (0.1,0.3,0.5,0.7,1.0,1.3,1.5,1.7,1.9)

ANSS(F) = ANSS for the FSI chart

ANSS(V) = ANSS for the VSI chart

ATS(F) = ATS for the FSI chart

ATS(V) = ATS for the VSI chart

Table 3.24: Values of the $ANSS$ and the ATS for two-sided symmetric Shewhart charts with 19 sampling intervals in the case of a gradual shift

Shift/hr	$ANSS$ (F)	$ANSS$(V)	ATS(F)	ATS(V)
0.000	370.4000	370.4000	370.4000	370.4000
0.005	134.1050	138.0260	134.1050	128.7440
0.010	89.5601	94.0234	89.5601	84.6130
0.025	49.3706	53.8119	49.3706	45.4889
0.050	30.4519	34.4655	30.4519	27.4630
0.100	18.4285	21.8237	18.4285	16.2458
0.250	9.3122	11.7940	9.3122	7.9643
0.500	5.5186	7.3284	5.5186	4.6343
1.000	3.2772	4.4742	3.2772	2.7236

$d_i = 0.1i$ where i = 0.1,0.2,0.3,...,1.9.

ANSS(F) = ANSS for the FSI chart

ANSS(V) = ANSS for the VSI chart

ATS(F) = ATS for the FSI chart

ATS(V) = ATS for the VSI chart

Table 3.25: Values of the standard deviation of the time to signal for two-sided symmetric Shewhart charts with representative number of sampling intervals for a gradual shift

	A	B	C	D	E	F
Shift/hr	t = 1	t = 2	t = 3	t = 5	t = 9	t = 19
0.000	370.4000	370.4000	370.4000	370.4000	37.400	370.400
0.005	68.0787	62.2443	62.5241	62.7708	62.9874	62.9874
0.010	39.3305	34.7404	34.9316	35,.1107	35.2735	35.2735
0.025	18.3321	15.3497	15.4376	15.5357	15.6322	15.6322
0.050	10.1758	8.1650	8.1968	8.2491	8.3071	8.3071
0.100	5.6530	4.3675	4.3615	4.3817	4.4118	4.4118
0.250	2.6326	1.9849	1.9504	1.9444	1.9508	1.9508
0.500	1.5046	1.1487	1.1156	1.0982	1.0933	1.0933
1.000	0.8852	0.7892	0.7037	0.6788	0.6650	0.6650

A : $d_i = (1)$

B : $d_i = (.1,1.9)$

C : $d_i = (.1,1,1.9)$

D : $d_i = (.1,.5,1,1.5,1.9)$

E : $d_i = (.1,.3,.5,.7,1,1.3,1.5,1.7,1.9)$

E : $d_i = (.1,.2,.3,...,1.8,1.9)$

Table 3.26: Values of the $ANSS$ and the ATS for two-sided asymmetric Shewhart charts with two sampling intervals in the case of a gradual shift

Shift/hr	$ANSS$ (V1)	$ANSS$(V2)	$ANSS$(V3)	ATS(V1)	ATS(V2)	ATS(V3)
0.000	370.4000	370.4000	370.4000	370.4000	370.4000	370.4000
0.005	136.6000	139.1720	139.4180	130.7510	127.3910	127.0720
0.010	92.4502	95.3417	95.6090	86.4046	89.5601	83.1759
0.025	52.3454	55.1429	55.3686	46.8015	44.6616	44.5088
0.050	33.2367	35.6804	35.8328	28.4074	26.8959	26.8235
0.100	20.8953	22.8556	22.9115	16.8832	15.8986	15.8924
0.250	11.2759	12.5316	12.4435	8.3184	7.8373	7.8984
0.500	7.0900	7.8125	7.7421	4.8444	4.6108	4.7585
1.000	4.4900	4.7844	3.8508	2.8408	2.6833	2.8379

$V1$: $d_1 = 0.1 \quad d_2 = 1.1$

$V2$: $d_1 = 0.1 \quad d_2 = 1.9$

$V3$: $d_1 = 0.1 \quad d_2 = 2.5$

Chapter IV

Adjusted Properties of a Shewhart Chart

The numerical comparisons given in the previous chapter assume a shift in the mean at the beginning of the process. Though there are situations where such an assumption is realistic, in most practical application a process would shift at some random point between samples. A proposed model for this situation, along with comparisons of its effect on the variable interval chart follows.

When $\mu = \mu_1 \neq \mu_0$ the ATS is a measure of how long the chart will take to detect this deviation from target. The expressions for the ATS developed in the previous section gives the ATS for any μ_1 under the simplifying assumption that this value of μ is the process mean from time zero onward. But in practice the process may start out with $\mu = \mu_0$ and then shift form μ_0 to μ_1 at some random time in the future. In this case the detection time that is of interest is the time from the process shift to the point where the chart signals. For the type of chart considered here, the only difference between these two ways of determining the time to signal is that a shift which occurs at a random time in the future may occur in the time interval between two samples.

4.1 Development of the Adjusted Properties of the Shewhart Chart

To develop a model which adjusts the time to signal to allow for the fact that a shift can occur between samples, let

R^* = adjusted time to signal

= time from the process shift until a signal

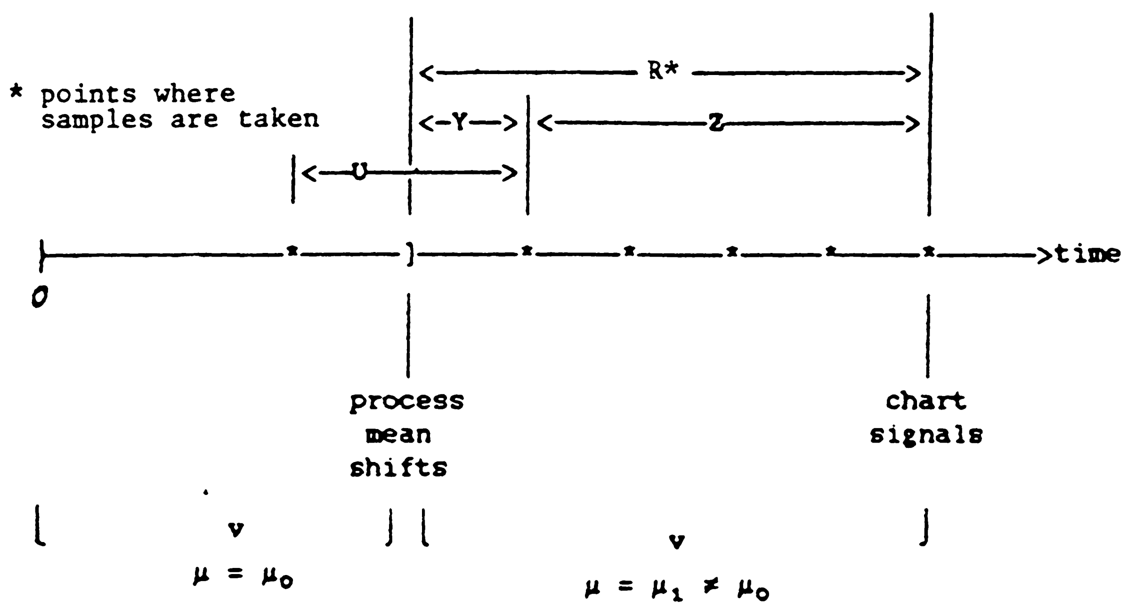
U = length of the interval in which the shift occurs

Y = time from the process shift until the next sample

Z = time from the next sample after the process shift until a signal

N = number of samples after the shift until a signal.

The relationship between R^*, U, Y, *and* Z is shown in the following sketch:



From above it is clear that

$$R^* = Y + Z \tag{4.1}$$

and that Z has the same distribution as $\sum_{i=1}^{N-1} R_i(\mu_1)$ where each $R_i(\mu_1)$ has the conditional distribution of $d(\overline{X})$ given no signal, when the process mean has shifted. The ATS which has been adjusted for the shift occurring between samples is

$$\begin{aligned} E(R^*) &= E(Y) + E(Z) \\ &= E(Y) + E(N-1)E(R_i(\mu_1)) \end{aligned} \tag{4.2}$$

The distribution of N is geometric with parameter q as before and an expression for $E(R_i)$ is given by (3.8). Before proceeding with the development of a model for determining the distribution of Y it may be useful to discuss the differences between the time to signal defined by R and the adjusted time to signal defined by R^*. The time R^* includes the time Y from the shift to the next sample where $0 \leq Y \leq U$ and the distribution of U is determined when the process mean is at μ_0 since the sample immediately before the shift was taken when $\mu = \mu_0$. In contrast, the distribution of R is the same as the distribution of $V + Z$ where V has the same distribution as $d(\overline{X})$ conditional on no signal when $\mu = \mu_1$. This follows from the fact that when the process starts at time zero with $\mu = \mu_1$ the distribution of the first interval is assumed to be the same as the distribution of future intervals determined when $\mu = \mu_1$. Thus the difference between R and R^* is the difference between V and Y.

The ATS determined by $E(R^*)$ is particularly relevant when a variable interval chart has a large value of d_t and the magnitude of shift $(\mu_1 - \mu_0)$ is large. For example suppose that a variable interval chart using intervals of 1 hour and 20 hours is being compared with a fixed interval chart using a interval of 8 hours. The intuitive objection to the variable interval chart would be that a large shift in μ could occur early in a 20 hour sampling interval and then the process would operate for many hours at the wrong level before the next sample is taken. With the fixed interval chart, on the other hand, the maximum possible time until the next sample is only 8 hours. The use of $E(R^*)$ accounts for the possibility of the shift occurring in a long sampling interval when a variable interval chart is being used. The difference between $E(R)$ and $E(R^*)$ will be important only when the shift in μ is large since small or moderate shifts will require a relatively large number of samples for detection and in this case Y is a relatively small component of R^*. When two fixed interval charts each using the same interval are being compared the use of $E(R^*)$ may not be necessary since both charts have the same possibility of a shift between samples.

To actually evaluate $E(R^*)$ a model must be developed so that $E(Y)$ can be determined. The distribution of Y depends on the time that the shift in μ occurs and one approach to developing a model would involve using a "prior" distribution for the time of this shift and then deriving the distribution of Y. Since this approach appears to be quite difficult, an alternate approach is to assume that when the shift falls in a particular sampling interval the position within the interval is uniformly distributed over the interval. The probability of the shift falling in an interval of a particular length is proportional to the product of the length of the interval and the probability of the interval when the process is in control. This means that

$$f_{Y|U}(y|u) = \frac{1}{u}, \qquad 0 \leq y \leq u, \tag{4.3}$$

and

$$P(U = d_j) = d_j p_{0j} / \sum_{j=1}^{t} d_j p_{0j}, \qquad j = 1, 2, \ldots, t. \tag{4.4}$$

Then

$$\begin{aligned} f_Y(y) &= \sum_{j=1}^{t} f_{Y|U}(y|d_j)\, P(U = d_j) \\ &= \sum_{\{j: d_j \geq y\}} p_{0j} / \sum_{j=1}^{t} d_j p_{0j}, \qquad 0 \leq y \leq d_t, \end{aligned} \tag{4.5}$$

and

$$E(Y) = \int_0^{d_t} y f_Y(y)dy$$
$$= \sum_{j=1}^{t} d_j^2 p_{0j} \Big/ \left(2 \sum_{i=1}^{t} d_j p_{0j} \right). \tag{4.6}$$

which reduces to $d/2$ for the FSI Shewhart Chart where d is the fixed sampling interval used. The variance of Y can be expressed as follows;

$$var(Y) = \frac{\sum_{j=1}^{t} d_i^3 p_{oi}}{3 \sum_{i=1}^{t} d_i p_{oi}} - \frac{(\sum_{i=1}^{t} d_i^2 p_{oi})^2}{(2 \sum_{i=1}^{t} d_i p_{oj})^2} \tag{4.7}$$

which reduces in the FSI Shewhart Chart to $d^2/12$.

The adjusted ATS when the shift is to $\mu = \mu_1$ is then

$$E(R^*) = \frac{\sum_{j=1}^{t} d_j^2 p_{0j}}{2 \sum_{j=1}^{t} d_j p_{0j}} + \frac{1}{q_1} \sum_{j=1}^{t} d_j p_{1j} \tag{4.8}$$

which reduces in the fixed sampling interval case to $d(\frac{1}{2} + \frac{1 - q_1}{q_1})$. The variance of the adjusted time to signal is

$$\text{Var}(R^*) = \text{Var}(Y) + E(N - 1)\text{Var}(R_i) + \text{Var}(N - 1)E(R_i)^2$$

$$= \frac{\sum_{j=1}^{t} d_j^3 p_{0j}}{3 \sum_{j=1}^{t} d_j p_{0j}} - \frac{\left(\sum_{j=1}^{t} d_j^2 p_{0j} \right)^2}{4 \left(\sum_{j=1}^{t} d_j p_{0j} \right)^2} + \frac{\sum_{j=1}^{t} d_j^2 p_{1j}}{q_1} + \frac{\left(\sum_{j=1}^{t} d_j p_{1j} \right)^2}{q_1^2} \tag{4.9}$$

For the FSI Shewhart Chart the equation above can be expressed as follows;

$$\text{Var}(R^*) = \text{Var}(Y) + \text{Var}(N - 1)E(R_i)^2$$
$$= \frac{d^2}{12} + \frac{(1 - q_1)\, d^2}{q_1^2} \tag{4.10}$$

Several assumptions about the distribution of Y were made above and the following section will investigate whether such assumptions are reasonable.

4.2 Simulation Study on the Behavior of the Time of the Shift

Limited simulation studies using several prior distributions for the time of the shift indicated that the assumptions that Y is uniform and that U is proportional to the product of length and probability of an interval are quite reasonable. In this simulation study a standard normal random variable is monitored for a *VSI* Shewhart chart with three sampling intervals with equal probability when the process is under control and the time of the shift T^* is assumed to have a prior distribution with mean μ^*. The results were obtained from 10000 runs using a random number generator (IMSL). Any out-of-control signal which occurs before the simulated shift is ignored. In each run it is registered in which sampling interval the shift occurs in addition to the position of the shift within the interval. The average shift location and the standard deviation of the shift location are calculated from the runs and Tables (4.1,4.2,4.3) give the simulation results for the cases

- $T^* \sim \exp(\mu^*)$
- $T^* \sim$ truncated $N(\mu^*, 1)$
- $T^* \sim$ truncated $N(\mu^*, 9)$

Under the assumption that $Y \sim U(0, d_j)$ the expected value of Y is μ^* is (0.50,1.00,1.50) for sampling intervals of length (1.0, 2.0, 3.0) respectively. The frequency of a shift falling in an interval of a particular length is assumed to be proportional to the product of the length of the interval and the probability of the interval when the process is in control. This implies that the frequency of a shift should be (1666.6, 3333.3,5000.0) for the sampling intervals (1.0,2.0,3.0) respectively since the sampling intervals have equal probability when the process is in control. The simulation results indicate that it is reasonable to assume that Y $\sim$ U(0, d_j) and that the probability of a shift falling in an interval of a particular length is proportional to the product of the length of the interval and the probability of the interval when the process is in control. The only case where these assumptions may not be reasonable seems to occur when the mean of the prior distribution is very small and the shift occurs very soon after the chart is started.

4.3 Computational Results

4.3.1 Simulation Study on the Behaviour of Y

Tables 4.1, 4.2 and 4.3 give the simulation results on the behaviour using several prior distributions for the time of the shift. The notation used in the tables is as follows:

μ^* = mean of the distribution of the time of the shift

E(Y) = expected time from the shift to the next sample taken

d_j = sampling interval in which the shift occurred

FREQ = frequency of shifts in interval of length d_j

SD = estimated standard deviation of Y

The *ATS* values computed in Chapter 3 for the case of a drift in the process mean assume that the drift starts at time 0. An adjusted *ATS* will not be developed for this case because of the difficulty in computing an adjusted *ATS*. If the drift starts between samples then the position of the process mean at the next sample depends on where the drift started in the interval. This means that the distribution of Z depends on the value of Y. As long as the drift is not too fast the *ATS* will be relatively large and there will be little difference between adjusted and unadjusted *ATS* values.

Table 4.1: Simulation results on the properties of the variable Y when the time of the shift T* follows the Exponential (μ^*) distribution

μ^*	E(Y)	d_j	FREQ	SD
3.0	0.5235	1.0	2071	0.2867
3.0	1.1213	2.0	3543	0.5721
3.0	1.7283	3.0	4386	0.8538
5.0	0.5203	1.0	1894	0.2899
5.0	1.0582	2.0	3422	0.5744
5.0	1.6664	3.0	4684	0.8581
10.0	0.5092	1.0	1806	0.2862
10.0	1.0496	2.0	3402	0.5875
10.0	1.5527	3.0	4792	0.8707
20.0	0.4999	1.0	1697	0.2886
20.0	1.0098	2.0	3416	0.5742
20.0	1.5440	3.0	4889	0.8711
50.0	0.5026	1.0	1705	0.2875
50.0	0.9976	2.0	3306	0.5746
50.0	1.4983	3.0	4989	0.8760
100.0	0.4954	1.0	1677	0.2897
100.0	1.0048	2.0	3376	0.5765
100.0	1.5104	3.0	4947	0.8665

Table 4.2: Simulation results on the properties of the variable Y when the time of the shift T* follows the truncated N(μ*,1) distribution

μ^*	E(Y)	d_j	FREQ	SD
3.0	0.4986	1.0	1766	0.2891
3.0	0.9814	2.0	3500	0.5799
3.0	1.5708	3.0	4734	0.8228
5.0	0.5117	1.0	1738	0.2835
5.0	1.0106	2.0	3323	0.5668
5.0	1.4972	3.0	4939	0.8553
10.0	0.4899	1.0	1648	0.2885
10.0	1.0011	2.0	3342	0.5784
10.0	1.4954	3.0	5010	0.8622
20.0	0.4752	1.0	1717	0.2952
20.0	1.0077	2.0	3318	0.5803
20.0	1.4849	3.0	4965	0.8636
50.0	0.4878	1.0	1691	0.2919
50.0	0.9699	2.0	3401	0.5834
50.0	1.4992	3.0	4908	0.8669
100.0	0.4943	1.0	1657	0.2849
100.0	1.0024	2.0	3382	0.5843
100.0	1.5138	3.0	4961	0.8644

Table 4.3: Simulation results on the properties of the variable Y when the time of the shift T^* follows the truncated $N(\mu^*,9)$ distribution

μ^*	E(Y)	d_j	FREQ	SD
3.0	0.5105	1.0	1811	0.2896
3.0	1.0350	2.0	3444	0.5808
3.0	1.6152	3.0	4745	0.8480
5.0	0.5018	1.0	1750	0.2944
5.0	1.0101	2.0	3421	0.5772
5.0	1.5520	3.0	4829	0.8565
10.0	0.4944	1.0	1630	0.2955
10.0	0.9937	2.0	3395	0.5774
10.0	1.5244	3.0	4975	0.8724
20.0	0.4979	1.0	1678	0.2862
20.0	1.0034	2.0	3270	0.5775
20.0	1.4953	3.0	5052	0.8649
50.0	0.4995	1.0	1713	0.2913
50.0	0.9856	2.0	3356	0.5777
50.0	1.4698	3.0	4931	0.8633
100.0	0.5014	1.0	1713	0.2906
100.0	0.9896	2.0	3338	0.5792
100.0	1.4949	3.0	4949	0.8717

The results given in Tables 4.1, 4.2 and 4.3 indicate that our assumption about the variable Y are reasonable. We know want to compute the properties of the FSI and VSI Shewhart charts when the adjusted time to signal is considered. The usefulness of the variable sampling interval feature is compared numerically to the FSI Shewhart chart for both one-sided and two-sided charts. In order to be able to make valid comparisons among the FSI and VSI Shewhart chart it is necessary to choose the parameters of the charts such that both the ANSS and the unadjusted ATS are the same for both kinds of charts when the process is in control. As long as the control limits are chosen to be the same for both charts, the ANSS will also be the same since the ANSS depends only on q, the probability of a signal. Three-sigma control limits are used such that the in-control ANSS is 740.8 and 370.4 samples for one-sided and two-sided charts respectively. For convenience it is assumed that the FSI Shewhart chart uses a sampling interval $d = 1$ time unit, i.e. the ANSS will equal the unadjusted ATS when the process is in control. The sampling intervals of the VSI Shewhart chart are chosen such that the expected sampling interval is unity when the process is in control, i.e. $E(R_i) = 1$. This will assure that the FSI Shewhart chart and the VSI Shewhart chart have the same unadjusted ATS and same ANSS when the process is in control. Then the values of the ATS functions for the two charts can be compared numerically fo several values of μ_1 to determine which chart will do a better job of detecting a change in μ for the same ANSS. Numerical comparisons are also made of the standard deviation of the adjusted time to signal SD(R^*) and the coefficient of variation of the adjusted time to signal CV(R^*) which is just the ratio of the SD(R^*) to the adjusted ATS. In the computations of the adjusted ATS and SD(R^*) equations (4.7) and the square root of (4.9) are used respectively. SAS programs (PROC MATRIX) for computing the results of this chapter are provided in Appendix 2.

4.3.2 The Number of Sampling Intervals

The Shewhart chart is a popular process control procedure because of its ease of use. It is important to keep the process control chart as simple as possible and still obtain satisfactory performance in detecting an out-of-control situation as early as possible. The FSI Shewhart chart is simpler to use than the VSI Shewhart chart because only one sampling interval is necessary and so we want to investigate how many intervals are really necessary in the *VSI* Shewhart chart to obtain best results. Shewhart charts with selected numbers of sampling intervals ($t = 2,3,5,9$) are evaluated numerically as shown in Tables (4.4,4.5,4.6,4.7,4.8,4.9). The numerical results show clearly that the one-sided *VSI* Shewhart chart with only two sampling intervals has the shortest *ATS* when compared to the other one-sided *VSIS* charts. All variable one-sided charts have a better performance than the one-sided *FSI* Shewhart chart with respect to the adjusted *ATS* for shifts smaller than 4.0.. The more sampling intervals used the more the properties of the one-sided *VSI* Shewhart chart approaches the properties of the *FSI* Shewhart chart. The two-sided *VSI* charts show similar behavior when compared to each other however the difference in their corresponding adjusted *ATS* values is much smaller than in the one-sided cases. As for the standard deviation of the adjusted time to signal the one-sided *VSI* Shewhart charts have a smaller SD than the one-sided *FSI* Shewhart chart for all shifts considered. Table (4.7) reveals that among one-sided *VSIS* charts using two sampling intervals yields the smallest SD values for shifts (0.1,3.0). As the number of sampling intervals are increased the SD of the *VSI* Shewhart chart approaches the SD of the *FSI* Shewhart chart. Among two-sided *VSI* Shewhart charts it seems that using two intervals only results in smaller SD values for small and moderately large shifts. However the properties of the *VSI* Shewhart charts are very similar for the two-sided case and no real preference can be detected in Table (4.7). The coefficient of variation is lower for the *VSI* Shewhart charts and for all shifts considered for both the one-sided and two-sided charts. The CV appears to increase with an increase of the size of the shift in he process mean and then to decrease for very large shifts. Among one-sided *VSI* Shewhart charts the chart which uses only two sampling intervals has the lowest

CV for shifts in the range (0.25,2.00) whereas among two-sided *VSI* Shewhart charts it is better to use more than two intervals, however the differences are insignificant in the two-sided case.

Table 4.4: Values of the adjusted ATS for one-sided symmetric Shewhart charts with representative number of sampling intervals

Shift = $\sqrt{n}(\mu_1 - \mu_0)/\sigma$	A $t=1$	B $t=2$	C $t=3$	D $t=5$	E $t=9$
0.00	740.800	740.800	740.800	740.800	740.800
0.10	535.459	497.695	501.205	502.370	505.506
0.25	335.097	276.382	281.667	283.437	288.115
0.50	160.539	106.173	110.820	112.406	116.569
1.00	43.456	17.721	19.567	20.240	21.986
1.50	14.468	4.096	4.707	4.943	5.538
2.00	5.803	1.692	1.942	2.033	2.242
2.50	2.741	1.165	1.325	1.374	1.469
3.00	1.500	1.010	1.150	1.187	1.251
4.00	0.689	0.924	1.059	1.094	1.147

A = (1) B = (.1/1.9) C = (.1/1/1.9) D = (.1/.5/1/1.5/1.9)

E = (.1/.3/.5/.7/1/1.3/1.5/1.7/1.9)

Table 4.5: Values of the adjusted ATS for two-sided symmetric Shewhart charts with representative number of sampling intervals

Shift =	A	B	C	D	E
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	t = 1	t = 2	t = 3	t = 5	t = 9
0.00	370.400	370.400	370.400	370.400	740.800
0.10	352.431	351.400	351.386	351.396	351.449
0.25	280.653	274.500	274.873	275.021	275.435
0.50	154.724	141.422	142.301	142.633	143.532
1.00	43.395	30.812	31.555	31.844	32.628
1.50	14.468	7.392	7.722	7.857	8.234
2.00	5.803	2.437	2.545	2.591	2.730
2.50	2.741	1.322	1.343	1.351	1.389
3.00	1.500	1.040	1.034	1.028	1.028
4.00	0.689	0.925	0.908	0.897	0.880

A = (1) B = (.1/1.9) C = (.1/1/1.9) D = (.1/.5/1/1.5/1.9)

E = (.1/.3/.5/.7/1/1.3/1.5/1.7/1.9)

Table 4.6: **Values of the standard deviation of the adjusted time to signal for one-sided symmetric Shewhart charts with representative number of sampling intervals**

Shift =	A	B	C	D	E
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	t = 1	t = 2	t = 3	t = 5	t = 9
0.00	740.297	740.704	740.836	740.871	741.089
0.10	535.459	497.688	501.206	502.374	505.513
0.25	335.097	276.361	281.669	283.444	288.129
0.50	160.539	106.121	110.813	112.412	116.589
1.00	43.454	17.566	19.490	20.189	21.968
1.50	14.463	3.773	4.456	4.727	5.374
2.00	5.789	1.229	1.483	1.597	1.853
2.50	2.711	0.698	0.776	0.817	0.915
3.00	1.443	0.596	0.617	0.626	0.649
4.00	0.554	0.572	0.578	0.575	0.568

A = (1) B = (.1/1.9) C = (.1/1/1.9) D = (.1/.5/1/1.5/1.9)

E = (.1/.3/.5/.7/1/1.3/1.5/1.7/1.9)

Table 4.7: Values of the standard deviation of the adjusted time to signal for two-sided symmetric Shewhart charts with representative number of sampling intervals

Shift = $\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	A t=1	B t=2	C t=3	D t=5	E t=9
0.00	369.898	370.174	370.050	370.022	369.979
0.10	352.431	351.399	351.369	351.376	351.425
0.25	280.652	274.497	274.833	274.974	275.378
0.50	154.724	141.411	142.225	142.544	143.425
1.00	43.393	30.763	31.407	31.674	32.428
1.50	14.462	7.261	7.486	7.595	7.940
2.00	5.789	2.175	2.206	2.231	2.348
2.50	2.711	0.952	0.937	0.937	0.968
3.00	1.443	0.651	0.631	0.628	0.633
4.00	0.554	0.573	0.553	0.550	0.546

A = (1) B = (.1/1.9) C = (.1/1/1.9) D = (.1/.5/1/1.5/1.9)

E = (.1/.3/.5/.7/1/1.3/1.5/1.7/1.9)

Table 4.8: Values the of coefficient of variation of the adjusted time to signal for one-sided symmetric Shewhart charts with representative number of sampling intervals

Shift =	A	B	C	D	E
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	t = 1	t = 2	t = 3	t = 5	t = 9
0.00	1.0000	1.0000	1.0000	1.0000	1.0000
0.10	1.0000	1.0000	1.0000	1.0001	1.0000
0.25	1.0000	0.9999	1.0001	1.0003	1.0001
0.50	1.0000	0.9995	0.9999	1.0005	1.0002
1.00	1.0000	0.9913	0.9961	0.9975	0.9992
1.50	0.9996	0.9211	0.9467	0.9562	0.9705
2.00	0.9975	0.7264	0.7636	0.7855	0.8265
2.50	0.9889	0.5986	0.5859	0.5949	0.6225
3.00	0.9623	0.5900	0.5369	0.5268	0.5189
4.00	0.8053	0.6186	0.5462	0.5262	0.4949

Table 4.9: Values the of coefficient of variation of the adjusted time to signal for two-sided symmetric Shewhart charts with representative number of sampling intervals

Shift = $\sqrt{n}(\mu_1 - \mu_0)/\sigma$	A $t = 1$	B $t = 2$	C $t = 3$	D $t = 5$	E $t = 9$
0.00	1.0000	1.0000	1.0000	1.0000	1.0000
0.10	1.0000	1.0000	1.0000	0.9999	0.9999
0.25	1.0000	1.0000	0.9999	0.9998	0.9998
0.50	1.0000	0.9999	0.9947	0.9994	0.9993
1.00	1.0000	0.9984	0.9953	0.9947	0.9939
1.50	0.9996	0.9823	0.9694	0.9666	0.9642
2.00	0.9975	0.8923	0.8667	0.8611	0.8600
2.50	0.9889	0.7200	0.6975	0.6934	0.6970
3.00	0.9623	0.6255	0.6105	0.6111	0.6160
4.00	0.8053	0.6197	0.6083	0.6132	0.6206

4.3.3 Choice of Interval Width

Numerical results in the previous section suggest the use of two sampling intervals in order to obtain the lowest adjusted *ATS* out-of-control for a wide range of shifts in the process mean except extremely large shifts. How should we choose these two sampling intervals? Additional information about good choices for d_1 and d_2 can be obtained by finding the values of d_1 and d_2 which minimize the adjusted *ATS* at a specific shift, subject to the constraint that the chart be matched to a specified fixed interval chart when $\mu = \mu_0$. Analytically minimizing the adjusted *ATS* is difficult, so a numerical procedure was used for the cases where the fixed interval chart had $\gamma = 3$ and a fixed sampling interval of 1.0 and d_1 and d_2 are constrained to fall in the intervals $0.1 \leq d_1 \leq 0.9$ and $1.1 \leq d_2 \leq 10.0$. The lower bound of $m_1 = 0.1$ for d_1 seems to be realistic since it would usually be impractical to take a sample sooner than 1/10 of the fixed sampling interval.

Consider a one-sided VSI Shewhart chart using two interval lengths with intervals $I_1 = (\gamma^*, 3)$ and $I_2 = (-\infty, \gamma^*)$. In order to match the fixed procedure (for given d_1 and d_2) γ^* must be chosen such that

$$p_{01} d_1 + p_{02} d_2 = 1 - q_0$$

which implies that

$$p_{01} = \frac{(d_2 - 1)(1 - q_0)}{d_2 - d_1}$$

and

$$p_{02} = \frac{(1 - d_1)(1 - q_0)}{d_2 - d_1}.$$

Suppose that $\mu_0 = 0$ and $\sigma/\sqrt{(n)} = 1$. It follws that

$$p_{02} = P(\overline{X} \leq \gamma^* \mid \mu_0) = \Phi(\gamma^*)$$

and

$$p_{12} = P(\overline{X} \leq \gamma^* \mid \mu_1) = \Phi(\gamma^* - \mu_1).$$

Hence we can rewrite γ^* as

$$\begin{aligned} \gamma^* &= \Phi^{-1}\{p_{02}\} \\ &= \Phi^{-1}\{\frac{(1 - d_1)(1 - q_0)}{d_2 - d_1}\} \end{aligned} \tag{4.11}$$

where $p_{11} + p_{12} + q_1 = 1$.

The adjusted ATS for a one-sided VSI Shewhart chart is

$$E(R^*) = E(Y) + E(Z)$$

where

$$\begin{aligned} E(Y) &= \frac{d_1^2 p_{01} + d_2^2 p_{02}}{2(d_1 p_{01} + d_2 p_{02})} \\ &= \frac{d_1^2 (d_2 - 1) + d_2^2 (1 - d_1)}{2(d_2 - d_2)} \end{aligned}$$

and

$$\begin{aligned} E(Z) &= d_1 p_{11} + d_2 p_{12}/q_1 \\ &= d_1 (1 - q_1) + (d_2 - d_{(1)}) \Phi(\gamma^* - \mu_1)/q_1 \\ &= d_1 (1 - q_1) + (d_2 - d_{(1)}) \Phi (\Phi^{-1} \frac{(1 - d_1)(1 - q_0)}{d_2 - d_1} - \mu_1)/q_1 \end{aligned}$$

For a given value of μ_1 we can now numerically find the values of d_1 and d_2 which minimize $E(R^*)$. For all shifts considered the optimal solution always has $d_1 = 0.1$. The optimal values of d_2 for several values of $\sqrt{n}(\mu_1 - \mu_0)/\sigma$ are as follows:

$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	0.1	0.5	1.0	1.5	2.0	3.0	3.5	4.0
d_2	9.95	9.97	6.46	2.79	1.69	1.27	1.13	1.10 .

These results suggest that d_1 should be taken as small as is practical while d_2 should be large if interest is in small shifts and small if interest is in large shifts. Although for some processes it may be possible to specify the magnitude of the shift that is likely to occur, it will usually be necessary to guard against a range of possible shifts. For this reason it may not be wise to use extremely large values of d_2 since large values of d_2 do not give as much protection against large shifts. Thus for most applications it seems appropriate to recommend $d_1 = t_1 = 0.1$ and d_2 in the range $1 < d_2 < 2$.

Consider a two-sided VSI Shewhart chart using two interval lengths with $I_1 = (-3, -\gamma^*)\,U\,(\gamma^*, 3)$ and $I_2 = (-\gamma^*, \gamma^*)$. In this case it is implied that

$$\begin{aligned} p_{02} &= P(\overline{X} \le \gamma^* \mid \mu_0) - P(\overline{X} \le -\gamma^* \mid \mu_0) \\ &= 0.5 + 0.5p_{02} \end{aligned}.$$

Hence we can rewrite γ^* as

$$\gamma^* = \Phi^{-1}\{0.5 + \frac{(1 - d_1)(1 - q_0)}{d_2 - d_1}\} \tag{4.12}$$

and

$$\begin{aligned} E(Z) &= d_1 p_{11} + d_2 p_{12}/q_1 \\ &= d_1(1 - q_1) + (d_2 - d_{(1)})[\Phi(\gamma^* - \mu_1) - \Phi(-\gamma^* - \mu_1)]/q_1. \end{aligned}$$

where γ^* is given by equation (4.12). For a given value of μ_1 we can now numerically find the values of d_1 and d_2 which minimize $E(R^*)$. For all shifts considered the optimal solution always has $d_1 = 0.1$. The optimal values of d_2 for several values of $\sqrt{n}(\mu_1 - \mu_0)/\sigma$ are as follows:

$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	0.1	0.5	1.0	1.5	2.0	3.0	3.5	4.0
d_2	1.54	3.21	2.82	2.23	1.61	1.38	1.18	1.10 .

These results suggest that d_1 should be taken as small as is practical while d_2 should be large if interest is in small shifts and small if interest is in large shifts unless the shift is extremely small in which case a small sampling interval is preferred. The results above show that the optimal sampling interval ranges between (1.1 and 3.2) for the two-sided *VSI* Shewhart chart compared to (1.1 and 9.9) for the one-sided *VSI* Shewhart chart It will usually be necessary to guard against a range of possible shifts and for this reason it may not be wise to use extremely large values of d_2 since large values of d_2 do not give as much protection against large shifts as the results above indicate for the two-sided charts. Thus for most applications it seems appropriate to recommend $d_1 = t_1 = 0.1$ and d_2 in the range $1 < d_2 < 2$.

The following tables show numerical results on several *VSI* Shewhart charts with two sampling intervals. Consider *VSI* Shewhart charts with sampling intervals d_1 and d_2 :(0.1,1.9), (0.3,1.7) , (0.6,1.4) and (0.9,1.1) as shown in Tables (4.10,4.11,4.12,4.13, 4.14,4.15). The adjusted *ATS* is lowest over a wide range of shifts for both one-sided and two-sided charts when the two sampling intervals are chosen far apart, i.e. $d_1 = 0.1$ and $d_2 = 1.9$. As d_1 increases and d_2 decreases the properties of the *VSI* Shewhart chart approach the properties of the *FSI* Shewhart chart. All *VSI* Shewhart charts have a lower *ATS* than the corresponding *FSI* Shewhart chart except for a shift of 4 where only plan (E) still beats the *FSI* Shewhart chart. The standard deviation of the adjusted time to signal is lowest when the sampling intervals are chosen far apart and for all shifts considered in this study except $\mu_1 = 4$. The *SD* increases as the sampling intervals approach the fixed interval d. The coefficient of variation is lowest for the *VSI* Shewhart chart for both the one-sided and two-sided charts. For larger shifts the *CV* values decrease as the two sampling intervals are chosen far apart. The following tables show that choosing the sampling intervals far apart works very well with respect to the ATS , the SD and the CV. A FORTRAN program for minimizing the adjusted ATS function is given in Appendix 1.

Table 4.10: Values of the adjusted ATS for one-sided symmetric Shewhart charts with representative interval widths.

	A	B	C	D	E
Shift =	d1 = 1	d1 = .1	d1 = .3	d1 = .6	d1 = .9
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.9	d2 = 1.7	d2 = 1.4	d2 = 1.1
0.00	740.800	740.800	740.800	740.800	740.800
0.10	535.459	497.695	506.017	518.575	531.223
0.25	335.097	276.382	289.360	308.902	328.533
0.50	160.539	106.173	118.184	136.276	154.459
1.00	43.456	17.721	23.370	31.918	40.556
1.50	14.468	4.096	6.331	9.758	13.276
2.00	5.803	1.692	2.536	3.876	5.306
2.50	2.741	1.165	1.445	1.941	2.526
3.00	1.500	1.010	1.049	1.182	1.406
4.00	0.689	0.924	0.802	0.693	0.675

Table 4.11: **Values of the adjusted ATS for two-sided symmetric Shewhart charts with representative interval widths**

	A	B	C	D	E
Shift =	d1 = 1	d1 = .1	d1 = .3	d1 = .6	d1 = .9
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.9	d2 = 1.7	d2 = 1.4	d2 = 1.1
0.00	370.400	370.400	370.400	370.400	370.400
0.10	352.431	351.400	351.559	351.873	352.276
0.25	280.653	274.500	275.797	277.818	279.929
0.50	154.724	141.422	144.308	148.712	153.206
1.00	43.395	30.812	33.538	37.702	41.957
1.50	14.468	7.392	8.894	11.223	13.641
2.00	5.803	2.437	3.115	4.207	5.389
2.50	2.741	1.322	1.567	2.010	2.543
3.00	1.500	1.040	1.073	1.196	1.409
4.00	0.689	0.925	0.802	0.694	0.675

Table 4.12: Values of the standard deviation of the adjusted time to signal for one-sided symmetric Shewhart charts with representative interval widths

	A	B	C	D	E
Shift =	d1 = 1	d1 = .1	d1 = .3	d1 = .6	d1 = .9
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.9	d2 = 1.7	d2 = 1.4	d2 = 1.1
0.00	740.297	740.704	740.836	740.872	740.302
0.10	535.459	497.688	506.002	518.561	531.219
0.25	335.097	276.361	289.319	308.866	328.523
0.50	160.539	106.121	118.091	136.201	154.439
1.00	43.454	17.566	23.141	31.762	40.519
1.50	14.463	3.773	5.960	9.541	13.225
2.00	5.789	1.229	2.086	3.627	5.243
2.50	2.711	0.698	1.008	1.687	2.449
3.00	1.443	0.596	0.677	0.940	1.308
4.00	0.554	0.572	0.537	0.488	0.519

Table 4.13: Values of the standard deviation of the adjusted time to signal for two-sided symmetric Shewhart charts with representative interval widths

	A	B	C	D	E
Shift =	d1 = 1	d1 = .1	d1 = .3	d1 = .6	d1 = .9
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.9	d2 = 1.7	d2 = 1.4	d2 = 1.1
0.00	369.898	370.174	370.043	369.921	369.889
0.10	352.431	351.399	351.558	351.872	352.276
0.25	280.652	274.497	277.792	277.813	279.927
0.50	154.724	141.411	144.288	148.694	153.201
1.00	43.393	30.763	33.457	37.635	41.938
1.50	14.462	7.261	8.709	11.090	13.605
2.00	5.789	2.175	2.815	4.014	5.335
2.50	2.711	0.952	1.202	1.783	2.470
3.00	1.443	0.651	0.722	0.963	1.313
4.00	0.554	0.573	0.539	0.488	0.520

Table 4.14: Values the of coefficient of variation of the adjusted time to signal for one-sided symmetric Shewhart charts with representative interval widths

	A	B	C	D	E
Shift =	d1 = 1	d1 = .1	d1 = .3	d1 = .6	d1 = .9
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.9	d2 = 1.7	d2 = 1.4	d2 = 1.1
0.00	1.0000	1.0000	1.0000	1.0000	1.0000
0.10	1.0000	1.0000	1.0000	1.0000	1.0000
0.25	1.0000	0.9999	0.9999	0.9999	1.0000
0.50	1.0000	0.9995	0.9992	0.9994	0.9999
1.00	1.0000	0.9913	0.9902	0.9951	0.9991
1.50	0.9996	0.9211	0.9414	0.9777	0.9962
2.00	0.9975	0.7264	0.8227	0.9359	0.9882
2.50	0.9889	0.5986	0.6976	0.8695	0.9694
3.00	0.9623	0.5900	0.6454	0.7953	0.9305
4.00	0.8053	0.6186	0.6702	0.7033	0.7699

Table 4.15: Values the of coefficient of variation of the adjusted time to signal for two-sided symmetric Shewhart charts with representative interval widths

	A	B	C	D	E
Shift =	d1 = 1	d1 = .1	d1 = .3	d1 = .6	d1 = .9
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.9	d2 = 1.7	d2 = 1.4	d2 = 1.1
0.00	1.0000	1.0000	1.0000	1.0000	1.0000
0.10	1.0000	1.0000	1.0000	1.0000	1.0000
0.25	1.0000	1.0000	1.0000	1.0000	1.0000
0.50	1.0000	0.9999	0.9999	0.9999	1.0000
1.00	1.0000	0.9984	0.9976	0.9982	0.9996
1.50	0.9996	0.9823	0.9792	0.9881	0.9973
2.00	0.9975	0.8923	0.9035	0.9541	0.9899
2.50	0.9889	0.7200	0.7672	0.8872	0.9712
3.00	0.9623	0.6255	0.6728	0.8054	0.9317
4.00	0.8053	0.6197	0.6713	0.7042	0.7701

4.3.4 Asymmetry of Sampling Intervals

Define a *symmetric* Shewhart chart to have sampling intervals d_i i = 1,2,..., d_t such that $P(d_i) = P(d_{t+1-i})$ when the process is in control. All numerical results in Tables 4.4 to 4.15 are symmetric charts. Asymmetric charts are such that $P(d_i) \neq P(d_{t+1-i})$. In a chart with only two sampling intervals d_1 and d_2 the choice of the corresponding probabilities $P(d_1)$ and $P(d_2)$ affects the adjusted ATS as shown in Tables (4.16,4.17). Five different VSI Shewhart charts with sampling intervals $d_1 = 0.1$ and $d_2 =$ (1.1,1.5,1.9,4.0,10.0) are compared to the corresponding FSI Shewhart chart with the same ANSS and ATS when the process is in control. This means that whenever we increase d_2 we are also decreasing $P(d_2)$ in order to obtain the same ATS in control as the FSI Shewhart chart. Hence a chart with $d_2 > 1.9$ implies that the probability of the larger sampling interval (when in control) will be less than the probability of the smaller sampling interval when in control. The results in Tables(4.16,4.17) suggest that when only two sampling intervals are used and where $d_1 = 0.1$ choose d_2 small when a large shift is expected and vice versa. Similar results are obtained in Tables (4.18,4.19) for SD of the adjusted time to signal and in Tables (4.20,4.21) for the CV. The results in the following tables indicate that there is no single VSI Shewhart chart with the best properties for all shifts.

Table 4.16: Values of the adjusted ATS for one-sided asymmetric Shewhart charts with representative interval widths

	A	B	C	D	E	F
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.1	d2 = 1.5	d2 = 1.9	d2 = 4.0	d2 = 10
0.00	740.800	740.800	740.800	740.800	740.800	740.800
0.10	535.459	525.669	507.322	497.698	476.054	458.552
0.25	335.097	318.349	290.303	276.384	247.079	225.242
0.50	160.539	142.285	117.203	106.174	85.701	73.435
1.00	43.456	31.578	21.154	17.721	13.206	13.085
1.50	14.468	8.093	4.801	4.096	3.985	6.285
2.00	5.803	2.551	1.704	1.692	2.456	5.107
2.50	2.741	1.126	1.023	1.165	2.082	4.776
3.00	1.500	0.729	0.837	1.010	1.951	4.650
4.00	0.689	0.568	0.744	0.924	1.869	4.569

Table 4.16: Values of the adjusted ATS for one-sided asymmetric Shewhart charts with representative interval widths

	A	B	C	D	E	F
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.1	d2 = 1.5	d2 = 1.9	d2 = 4.0	d2 = 10
0.00	370.400	370.400	370.400	370.400	370.400	370.400
0.10	352.431	351.939	351.486	351.400	351.831	354.229
0.25	280.653	277.925	275.228	274.500	274.316	276.628
0.50	154.724	148.694	142.977	141.422	140.480	142.711
1.00	43.395	36.991	32.022	30.812	30.338	32.732
1.50	14.468	10.208	7.827	7.392	7.741	10.319
2.00	5.803	3.325	2.471	2.437	3.189	5.851
2.50	2.741	1.395	1.205	1.322	2.213	4.903
3.00	1.500	0.817	0.877	1.040	1.973	4.670
4.00	0.689	0.575	0.746	0.925	1.869	4.569

Table 4.18: Values of the standard deviation of the adjusted time to signal for two-sided asymmetric Shewhart charts with representative interval widths

	A	B	C	D	E	F
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.1	d2 = 1.5	d2 = 1.9	d2 = 4.0	d2 = 10
0.00	740.297	740.364	740.353	740.708	741.870	744.090
0.10	535.459	525.667	507.317	497.690	476.026	458.452
0.25	335.097	318.346	290.290	276.363	247.003	224.964
0.50	160.539	142.277	117.174	106.122	85.512	72.735
1.00	43.454	31.554	21.067	17.557	12.643	11.211
1.50	14.463	8.037	4.608	3.773	3.037	4.021
2.00	5.789	2.440	1.388	1.229	1.495	3.199
2.50	2.711	0.994	0.654	0.698	1.268	3.114
3.00	1.443	0.498	0.489	0.596	1.234	3.102
4.00	0.554	0.331	0.448	0.572	1.226	3.099

Table 4.19: Values of the standard deviation of the adjusted time to signal for one-sided asymmetric Shewhart charts with representative interval widths

	A	B	C	D	E	F
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.1	d2 = 1.5	d2 = 1.9	d2 = 4.0	d2 = 10
0.00	369.898	369.971	370.095	370.174	370.764	373.172
0.10	352.431	351.938	351.485	351.399	351.827	354.210
0.25	280.652	277.924	275.226	274.497	274.307	276.594
0.50	154.724	148.691	142.970	141.411	140.451	142.615
1.00	43.393	36.979	31.990	30.763	30.211	32.336
1.50	14.462	10.175	7.740	7.261	7.401	9.338
2.00	5.789	3.253	2.290	2.175	2.581	4.374
2.50	2.711	1.264	0.928	0.952	1.492	3.325
3.00	1.443	0.626	0.561	0.651	1.272	3.136
4.00	0.554	0.343	0.450	0.573	1.227	3.100

Table 4.20: Values the of coefficient of variation of the adjusted time to signal for one-sided asymmetric Shewhart charts with representative interval widths

	A	B	C	D	E	F
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.1	d2 = 1.5	d2 = 1.9	d2 = 4.0	d2 = 10
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.10	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998
0.25	1.0000	1.0000	1.0000	0.9999	0.9997	0.9988
0.50	1.0000	1.0000	0.9998	0.9995	0.9978	0.9905
1.00	1.0000	0.9992	0.9959	0.9913	0.9573	0.8568
1.50	0.9996	0.9931	0.9599	0.9211	0.7621	0.6398
2.00	0.9975	0.9562	0.8147	0.7264	0.6085	0.6264
2.50	0.9889	0.8383	0.6390	0.5986	0.6092	0.6519
3.00	0.9623	0.6823	0.5842	0.5900	0.6328	0.6671
4.00	0.8053	0.5824	0.6015	0.6186	0.6560	0.6783

Table 4.21: Values the of coefficient of variation of the adjusted time to signal for two-sided asymmetric Shewhart charts with representative interval widths

	A	B	C	D	E	F
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d1 = 1	d2 = 1.1	d2 = 1.5	d2 = 1.9	d2 = 4.0	d2 = 10
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.25	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
0.50	1.0000	1.0000	1.0000	0.9999	0.9999	0.9993
1.00	1.0000	0.9997	0.9990	0.9984	0.9958	0.9879
1.50	0.9996	0.9968	0.9889	0.9823	0.9561	0.9049
2.00	0.9975	0.9784	0.9268	0.8923	0.8094	0.7476
2.50	0.9889	0.9063	0.7697	0.7200	0.6741	0.6781
3.00	0.9623	0.7669	0.6391	0.6255	0.6450	0.6714
4.00	0.8053	0.5976	0.6039	0.6197	0.6563	0.6784

Chapter V

Cusum Charts

5.1 Fixed Sampling Interval (FSI) Cusum Charts

The cumulative sum (cusum) control chart for monitoring the process mean introduced by Page (1954) has been shown to be much more efficient than the simpler Shewhart $\overline{X}$- chart in detecting small and moderate shifts in the process mean. The cusum chart is usually maintained by taking samples at fixed time intervals and plotting a cumulative sum of differences between the sample means and the target value in time order on the chart. The process is considered to be in statistical control as long as the cusum statistic computed from the samples does not fall into the signal region(s) of the cusum chart. A value of the cusum statistic in the signal region would be taken as an indication that the process mean has changed and that the possible causes of the shift should be investigated.

Suppose that the process of interest has a quality characteristic X whose distribution is normal with mean μ and standard deviation σ. The objective is to detect any shift in the process mean from a target value μ_0 by taking a sequence of samples each of size n from the process. For the j^{th} sample

(j = 1,2,...) consider the standardized statistic $C_j = \sqrt{n}\,(\overline{X}_j - \mu_0)/\sigma$ where $\overline{X}_j$ is the sample mean. In practice it may be necessary in many cases to estimate both μ_0 and σ from past data, but for simplicity we assume here that μ_0 and σ are known. A cusum scheme accumulates deviations of the sample means which are more than k standardized units from μ_0. The parameter k is usually called the reference value of the cusum chart. For detecting positive changes in μ the cusum control statistic that is usually used at the j^{th} sample is

$$\hat{S}_j = \max\left\{\hat{S}_{j-1} + (C_j - k), 0\right\} \tag{5.1}$$

where $\hat{S}_0$ is a specified constant. For purposes of adding the *VSI* feature to the cusum chart we will use the modified cusum statistic

$$S_j = \max\{S_{j-1}, 0\} + (C_j - k) \tag{5.2}$$

where $S_0 = $ c and c is a constant (frequently 0). The difference between $\hat{S}_j$ and S_j is that $\hat{S}_j$ is never negative while S_j records negative values that may be needed to specify the sampling intervals. The cusum chart signals that the process is out-of-control whenever $S_j \geq h$ (which is equivilant to $\hat{S}_j \geq h$) where h is usually called the decision interval.

The parameters h and k are determined to achieve certain properties. k is usually determined by the shift in the mean which the cusum is designed to detect. Bissel (1969) concluded that for a control scheme designed to detect a specific shift from μ_0 to some specified value μ_1 , the recommended value of k is $\delta/2$ where $\delta = \sqrt{n}\,(\mu_1 - \mu_0)/\sigma$ is the size of the shift expressed in units of standard error of $\overline{X}$. The parameter h is usually chosen to give a specified expected number of samples to signal when $\mu = \mu_0$. For detecting negative shifts in μ another cusum statistic similar to (5.1) or (5.2) would be used. For detecting both negative and positive shifts in μ both one-sided cusums can be used simultaneously. The resulting two-sided cusum chart signals when either one-sided cusum signals. Adopting the *VSI* feature to the two-sided cusum chart is not a straightforward problem and in this dissertation we consider only the one-sided case.

5.2 *Variable Sampling Interval (VSI) Cusum Charts*

A *VSI* cusum chart which is designed to detect a positive shift in the mean uses a long sampling interval whenever the cumulative sum S_j takes on small values which indicate that the process mean is close to target. Shorter sampling intervals are used when S_j takes on large values which tend to indicate that the process mean has shifted. We assume that the variable sampling interval cusum uses a finite number t of interval lengths $d_1, d_2, ..., d_t$ where $d_1 < d_2 < ... < d_t$. These possible sampling interval lengths are chosen to satisfy $l_1 \le d_i \le l_2$ where the minimum sampling interval length $l_1 > 0$ may be determined by considerations such as the minimum amount of time required to take a sample, and the maximum sampling interval length l_2 might be determined by the maximum amount of time that it is reasonable to allow the process to run without sampling. The choice of a sampling interval as a function of the cusum statistic can be represented by a function d(.) which specifies the sampling interval to be used. Let the interval $(-\infty, h)$ be partitioned into t disjoint regions $L_1, L_2, \ldots, L_t$ such that

$$d(s) = d_i \text{ if } s \in L_i.$$

Thus the sampling interval between samples j and j + 1 is $d(S_j)$.

5.2.1 Properties of the VSI Cusum Chart

Several methods to calculate the *ANSS* (or *ARL*) for a cusum scheme have been presented in the literature. A Markov chain approach for a one-sided cusum chart was given by Brook and Evans (1972). Lucas and Crosier (1982) and Woodall (1984) expanded the Markov chain to evaluate a two-sided cusum. The Markov chain is relatively easy to use and it can be utilized to calculate the exact *ANSS* for a cusum with a discrete state space and yields very good approximations when the state space is continuous. In the latter case the continuous state space of the cusum statistic is partitioned into a finite number of discrete class intervals and the probability distribution of the cusum is discretized. Brook and Evans (1972) established that for the normal distribution about ten class intervals are sufficient to give estimates of the *ARL* correct to about three significant figures.

Suppose that the interval $(-\infty, h)$ for the cusum statistic S_j is partitioned into r intervals $E_1, E_2, ..., E_r$ where each interval corresponds to a state of the Markov chain. In addition there is an absorbing state A corresponding to S_j falling in the signal region $[h, \infty)$. Except for E_r and A, the intervals are of equal length and S_j is discretized into a statistic such that $S_j \in E_i$ corresponds to the discrete version of S_j equal to the midpoint of E_i. In what follows the notation will not explicitly distinguish between S_j and its discrete version. We assume that the region L_i where the sampling interval d_i is used corresponds to one or more of the states of the Markov chain so that the set of r intervals $E_1, E_2, ..., E_r$ is partitioned into t disjoint regions for purposes of determining the sampling interval to be used. Let b_i represent the sampling interval used when $S_j \in E_i$, i.e. $b_i = d_\ell$ if $E_i \subset L_\ell$, and let $\underset{\sim}{b} = \{b_1, b_2, ..., b_r\}'$. As an illustration, Figure 5.1 shows a *VSI* cusum chart with r=11 states and t=5 possible sampling intervals. The 5 regions L_i, the 5 sampling intervals d_i, and the b_i corresponding to the states of the Markov chain are shown in the figure. In the Brook and Evans model the interval $(-\infty, 0]$ forms one state since the standard cusum statistic can not take on negative values. In the current model it will be convenient to subdivide $(-\infty, 0]$ into more than one state to allow for a better selection of sampling intervals. All numerical results given later in this dissertation are based on the division of the intervals $(0, h)$ and $(-\infty, 0]$ into 15 states each which corresponds to r=30.

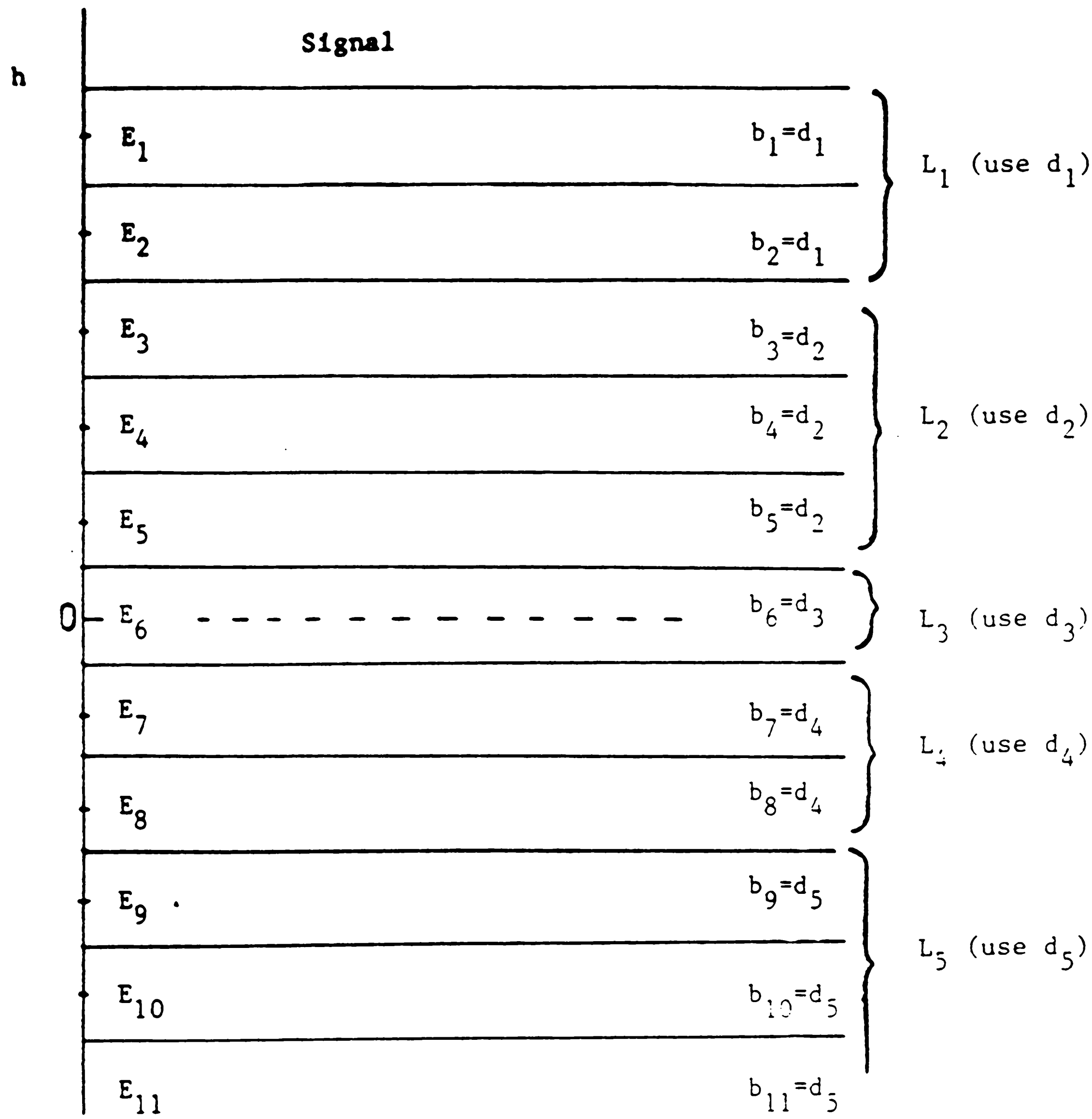


Figure 5.1 : Cusum chart with variable sampling intervals.

The transition matrix P for the Markov chain is

$$P = \begin{bmatrix} p_{1,1} & p_{1,2} & \cdot & \cdot & \cdot & p_{1,r} & p_{1,A} \\ p_{2,1} & p_{2,2} & \cdot & \cdot & \cdot & p_{2,r} & p_{2,A} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{r,1} & p_{r,2} & \cdot & \cdot & \cdot & p_{r,r} & p_{r,A} \\ 0 & 0 & . & . & . & 0 & 1 \end{bmatrix}$$

and it can be written as

$$P = \begin{bmatrix} Q & (I - Q)\underline{1} \\ \underline{0}' & 1 \end{bmatrix} \tag{5.3}$$

where Q is the submatrix of P corresponding to the r transient states, $\underline{0}$ is a vector of 0's of dimension rx1, and $\underline{1}$ is a vector of 1's of dimension rx1. Properties of the cusum chart can be obtained by working with the submatrix Q. Define the fundamental matrix M as

$$\begin{aligned} M &= (I - Q)^{-1} \\ &= \{M_{ij}\}. \end{aligned} \tag{5.4}$$

It can be shown that M_{ij} is the expected number of times the process is in the transient state E_j before absorption into state A, given that the process starts in state E_i. Let N_i be the number of samples to signal when the cusum starts in state i and let $\underline{N}' = (N_1, N_2, ..., N_r)$. Since a transition in the Markov chain occurs every time a sample is taken, the $ANSS$ when the Markov chain starts in state i is

$$E(N_i) = \sum_{j=1}^{r} M_{i,j}. \tag{5.5}$$

The vector of $ANSS$ values $E(\underline{N}) = M\underline{1}$ gives the expected number of samples to signal corresponding to the various initial cusum values. In many cases the cusum chart will start with $S_0 = 0$ and in this case the relevant $ANSS$ value is the one corresponding to the state which contains zero. Other starting states are sometimes needed, for example, when a fast initial response feature (Lucas and Crosier 1982b) is used or when a steady state ATS (see Chapter 6) is considered. The value of the $ANSS$ of a cusum depends on the two parameters k and h and a VSI cusum chart and a fixed sampling interval (FSI) cusum chart which have the same k and h values will have the same $ANSS$ values.

When the cusum starts in state i the expected number of time units that the VSI cusum chart spends in a transient state E_j before being absorbed in state A is , $M_{i,j}\, b_j$, the product of the expected number of visits to state i and the corresponding sampling interval. The ATS is then just the sum of the expected times in each transient state. Let T_i be the time to signal when the cusum starts in state i and let $\underline{T}' = (T_1, T_2, ..., T_r)$. Then the vector of ATS values is

$$E(\underline{T}) = M\,\underline{b}. \tag{5.6}$$

In addition to the expected values of N and T, other moments, such as the variance, may be of interest. The variance of N, as given by Kemeny and Snell (1960) is

$$var(\underline{N}) = (2M - I)\,E(\underline{N}) - (E(\underline{N}))^2 \tag{5.7}$$

where we use the notation $\underline{v}^2$ for $(v_1^2, v_2^2, ..., v_r^2)'$ when $\underline{v} = (v_1, v_2, ..., v_r)'$. Thus $(E(\underline{N}))^2$ should be interpreted as $((E(N_1))^2,(E(N_2))^2, ...,(E(N_r))^2)'$. To develop an expression for the variance of the time to signal note that

when the process starts in state i, it will either go to A in which case the time to absorption will be b_i, or else it will go to some transient state j in which case the time to absorption will be $b_i + T_j$. Using this basic argument gives the following equation

$$
\begin{aligned}
E(T_i^2) &= p_{i,A}\, b_i^2 + \sum_{j=1}^{r} p_{i,j}\, E((T_j + b_i)^2) \\
&= (p_{i,A} + \sum_{j=1}^{r} p_{i,j})\, b_i^2 + \sum_{j=1}^{r} p_{i,j}\, E(T_j^2) + 2\sum_{j=1}^{r} p_{i,j} b_i\, E(T_i) \qquad (5.8) \\
&= b_i^2 + \sum_{j=1}^{r} p_{i,j}\, E(T_j^2) + 2\sum_{j=1}^{r} p_{i,j}\, b_i\, E(T_j)
\end{aligned}
$$

Let B be a diagonal matrix with $b_1, b_2, \ldots, b_r$ on the diagonal. Then the expected value of $\underline{T}^2$ can be written as

$$
E(\underline{T}^2) = \underline{b}^2 + QE(\underline{T}^2) + 2BQ.
$$

Solving this equation for $E(Tbarunder^2)$ gives

$$
\begin{aligned}
E(\underline{T}^2) &= (I - Q)^{-1} (\underline{b}^2 + 2BQE(\underline{T})) \\
&= M\underline{b}^2 + 2MBQE(\underline{T}) \\
&= MB\underline{b} + 2MBQE(\underline{T}) \\
&= MB(\underline{b} + 2QM\underline{b}) \qquad (5.9) \\
&= MB(I + 2(M - I))\underline{b} \\
&= MB(2M - I)\underline{b}
\end{aligned}
$$

where we have used the fact that $B\underline{b} = \underline{b}^2$ and Q M = (M - I). The variance vector of the time to signal is then

$$
\begin{aligned}
var(\underline{T}) &= E(\underline{T}^2) - (E(\underline{T}))^2 \qquad (5.10) \\
&= MB(2M - I)\underline{b} - (M\underline{b})^2.
\end{aligned}
$$

This expression reduces to $var(\underline{N})$ when all sampling intervals are equal to one and to the product of the squared fixed sampling interval and $var(\underline{N})$ in the *FSI* cusum.

5.3 Computational Results

In evaluating the properties of the *VSI* cusum chart it seems reasonable to compare it to the corresponding *FSI* cusum chart. As long as the *VSI* cusum chart and the *FSI* cusum chart have the same values for h and k then both procedures will have the same *ANSS*. This means that adding the variable sampling interval feature to the cusum procedure will not affect the number of samples to signal. If the regions $L_1, L_2, ..., L_t$ and the corresponding sampling intervals $d_1, d_2, ..., d_t$ are selected so that the two cusum charts have the same *ATS* values when $\mu = \mu_0$ then both charts will have the same average sampling rate when $\mu = \mu_0$. When the *VSI* and *FSI* cusum charts are matched in this way the performance of the two charts can be evaluated by computing the *ATS* values at various values of μ_1 to determine which chart is more efficient in detecting changes in the process mean. A measure of relative efficiency is the ratio of the *ATS* for the *FSI* cusum relative to the *ATS* for the *VSI* cusum. If d is the sampling interval used by the *FSI* cusum chart then the relative efficiency for a specified value of μ_1 and when the Markov chain starts in state i is defined as

$$\frac{ATS\text{ of }(FSI\text{ cusum})}{ATS\text{ of }(VSI\text{ cusum})} = \frac{d\sum_{j=1}^{r} M_{i,j}}{\sum_{j=1}^{r} M_{i,j}\, b_j} = \frac{d}{\sum_{j=1}^{r} M_{i,j}\, b_j \,/ \sum_{j=1}^{r} M_{i,j}} \tag{5.11}$$

The last expression is simply the ratio of the fixed sampling interval to the average variable sampling interval. When the process goes out of control the cusum scheme with the shorter expected sampling interval (higher relative efficiency) is considered the better cusum procedure.

In addition to evaluating the gain in efficiency from the *VSI* feature, it is also necessary to determine how to specify the parameters such as h, k, $\{d_i\}$, and $\{L_i\}$ that define the *VSI* cusum chart. A *VSI* cusum chart will be said to be *symmetric* with respect to the regions L_i if for each i the expected number of visits to the states corresponding to region L_i is the same as the expected number of visits to the states corresponding to region L_{r-i+1} when the process is in control. A chart will be said to be symmetric with respect to the sampling intervals if there is a value d such that for each i $d - d_i = d_{r+1-i} - d$. In the special case of a symmetric *VSI* cusum chart using with two sampling intervals d_1 and d_2 , symmetric regions imply that the chart uses d_1 and d_2 the same expected number of times when the process is in control. Symmetric charts are easy to design for and evaluate computationally and yield good results over a wide range of shifts in the process mean.

In the numerical results presented in this dissertation the unit of time was chosen as the sampling interval of the *FSI* chart so that d = 1 and the *ATS* and the *ANSS* functions of the *FSI* cusum chart have the same numerical value. The parameters h and k were chosen so that the *ANSS* function at $\mu = \mu_0$ is 740.8, the same as that of a one-sided Shewhart $\overline{X}$ chart with a 3-sigma limit. Matching the cusum chart with the Shewhart chart in this way will enable comparisons to be made with previous results on the *VSI* and *FSI* Shewhart $\overline{X}$-charts. In this chapter the computed *ATS* values are based on the assumption that $S_0 = 0$. A SAS program (PROC MATRIX) for computing the results of this chapter is provided in Appendix 5.

5.3.1 The Number of Sampling Intervals

In a process control procedure it is important to keep the scheme as simple as possible and still be able to obtain satisfactory performance in detecting an out-of-control situation as early as possible. Cusum charts with different number of sampling intervals (t = 2,3,5,9) are evaluated. Previous results and results in Reynolds and Arnold (1987) indicate that two sampling intervals (t = 2) are preferred in the Shewhart $\overline{X}$-chart. This result appears to carry over to the cusum chart as shown in Tables (5.1,5.2,5.3,5.4.5.5,5.6). The numerical results indicate that for $k = 0.25$ the *ATS* values of the *VSI* cusum chart are considerably smaller with t = 2 for most shifts and gradually increasing as t increases. When larger shifts are considered ($\mu \geq 2.5$) the ATS appears to be shorter for cusum plans with multiple sampling intervals. This is however is misleading because the maximum sampling interval used in these plans is not the same due to computational restrictions so that particular ANSS and ATS values are obtained when the process is in control. The larger the number of sampling intervals used the larger the maximum sampling interval had to be chosen. In section (5.3.2) it will be shown that the ATS is lowest when d_1 and d_t are taken far apart. The chart using only two sampling intervals has a smaller standard deviation of the time to signal for small and large shifts and the plan with five sampling intervals is superior for moderate shifts. In general there is only a small difference between the performance of variable cusums when you change the number of sampling intervals. The variable cusums have a much smaller ATS and corresponding standard deviation than the corresponding fixed cusum for $k = 0.25$ and for all shifts in the range (0.1,4.0). The relative efficiency of the variable cusum is highest for moderate shifts in the mean for that k for ($\mu_1 = 0.5, 1.0, 1.5$). Recall that the fixed cusum with k is optimal for shifts of size 2 k , and there the variable cusum performs very well relative to the fixed cusum. As the size of the shift increases more than two standard deviations, the ATS of the variable cusum decreases only slightly since it is very small. The coefficient of variation (CV) decreases as the size of the shift gets large (Table 5.5). When only two sampling intervals are used the CV is smaller for large shifts and larger for smaller shifts compared to cusum schemes with multiple sampling intervals. In general the VSI cusum chart has a larger CV for shifts smaller than 2.5 for $k = 0.25$ when compared to a FSI cusum chart. For a variable cusum with $k = 1$ the ATS and the corresponding standard deviation are lowest for all shifts considered when only two sampling intervals are used. Computational restrictions made the maximum sampling interval largest for the VSI cusum chart with two sampling intervals. The relative efficiency of all VSI cusum schemes increase with an increase in the size of the shift in the process mean. There are only slight differences in the ATS values for the different VSI cusum schemes. The CV's are larger for $k = 1$ compared to $k = 0.25$. Table (5.6) shows that the FSI cusum chart has smaller CV's for shifts in the range (0.1,2.0) compared to the VSI cusum schemes. The VSI cusum chart with 9 sampling intervals has the lowest CV values for all shifts considered when compared to the other VSI cusum schemes.

Table 5.1: Values of the ATS for matched fixed and variable sampling interval cusum charts with k = 0.25 for representative number of sampling intervals

Shift = $\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	A t = 1	B t = 2	C t = 3	D t = 5	E t = 9
0.00	740.800	740.800	740.800	740.800	740.800
0.10	267.236	217.572	219.709	224.164	228.023
0.25	85.498	50.485	51.706	54.468	56.514
0.50	29.333	13.075	13.525	14.424	15.052
1.00	11.603	4.930	5.129	5.303	5.452
1.50	7.235	3.244	3.390	3.418	3.446
2.00	5.298	2.568	2.680	2.656	2.611
2.50	4.214	2.244	2.321	2.269	2.166
3.00	3.527	2.082	2.128	2.042	1.896
4.00	2.710	1.955	1.968	1.774	1.596

A = (1) B = (.1/1.78) C = (.1/1/1.79) D = (.1/.5/1/1.5/1.85)

E = (.1/.3/.5/.7/1/1.3/1.5/1.7/1.96)

Table 5.2: Values of the ATS for matched fixed and variable sampling interval symmetric cusum charts with k = 1.00 for representative number of sampling intervals

Shift =	A	B	C	D	E
$\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	t = 1	t = 2	t = 3	t = 5	t = 9
0.00	740.800	740.800	740.800	740.800	740.800
0.10	442.669	402.050	408.378	409.633	411.428
0.25	211.485	163.777	170.017	171.538	173.786
0.50	69.549	39.251	42.422	43.346	44.862
1.00	13.580	3.730	4.315	4.532	5.081
1.50	5.464	0.880	1.025	1.089	1.403
2.00	3.268	0.396	0.443	0.467	0.715
2.50	2.353	0.251	0.266	0.275	0.497
3.00	1.862	0.189	0.194	0.197	0.406
4.00	1.322	0.132	0.132	0.133	0.334

A = (1) B = (.1/1.98) C = (.1/1/1.85) D = (.1/.5/1/1.5/1.9)

E = (.1/.3/.5/.7/1/1.3/1.5/1.7/1.95)

Table 5.3: Values of the standard deviation of the time to signal for matched fixed and variable sampling interval symmetric cusum charts with $k = 0.25$ for representative number of sampling intervals

Shift =	A	B	C	D	E
$\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	t = 1	t = 2	t = 3	t = 5	t = 9
0.00	725.087	737.557	737.119	736.303	736.019
0.10	250.922	214.093	215.780	219.349	222.952
0.25	69.878	46.869	47.646	49.540	51.380
0.50	17.165	9.972	10.011	10.275	10.744
1.00	4.330	2.759	2.633	2.541	2.612
1.50	2.110	1.413	1.307	1.215	1.253
2.00	1.309	0.848	0.784	0.702	0.752
2.50	0.924	0.521	0.502	0.446	0.503
3.00	0.705	0.306	0.318	0.315	0.358
4.00	0.531	0.092	0.110	0.206	0.201

A = (1) B = (.1/1.78) C = (.1/1/1.79) D = (.1/.5/1/1.5/1.85)

E = (.1/.3/.5/.7/1/1.3/1.5/1.7/1.96)

Table 5.4: Values of the standard deviation of the time to signal for matched fixed and variable sampling interval symmetric cusum charts with k = 1.00 for representative number of sampling intervals

Shift = $\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	A t=1	B t=2	C t=3	D t=5	E t=9
0.00	738.849	739.612	741.302	741.262	741.017
0.10	440.585	402.708	408.850	410.060	411.607
0.25	209.250	164.392	170.438	171.910	173.905
0.50	67.106	39.780	42.751	43.622	44.878
1.00	11.129	4.046	4.475	4.642	4.935
1.50	3.403	1.027	1.094	1.124	1.199
2.00	1.597	0.427	0.451	0.457	0.487
2.50	0.970	0.205	0.223	0.226	0.244
3.00	0.703	0.107	0.119	0.122	0.134
4.00	0.482	0.050	0.051	0.052	0.055

A = (1) B = (.1/1.98) C = (.1/1/1.85) D = (.1/.5/1/1.5/1.9)

E = (.1/.3/.5/.7/1/1.3/1.5/1.7/1.95)

Table 5.5: Values of the coefficient of variation of the time to signal for matched fixed and variable sampling interval symmetric cusum charts with $k = 0.25$ for representative number of sampling intervals

Shift = $\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	A t = 1	B t = 2	C t = 3	D t = 5	E t = 9
0.10	0.9390	0.9840	0.9821	0.9785	0.9778
0.25	0.8173	0.9284	0.9215	0.9095	0.9092
0.50	0.5852	0.7627	0.7402	0.7123	0.7138
1.00	0.3732	0.5596	0.5134	0.4792	0.4791
1.50	0.2917	0.4356	0.3857	0.3554	0.3635
2.00	0.2472	0.3303	0.2927	0.2646	0.2878
2.50	0.2194	0.2322	0.2162	0.1964	0.2325
3.00	0.1998	0.1471	0.1495	0.1544	0.1887
4.00	0.1959	0.0471	0.0561	0.1164	0.1258

A = (1) B = (.1/1.78) C = (.1/1/1.79) D = (.1/.5/1/1.5/1.85)
E = (.1/.3/.5/.7/1/1.3/1.5/1.7/1.96)

Table 5.6: Values of the coefficient of variation of the time to signal for matched fixed and variable sampling interval symmetric cusum charts with $k = 1.00$ for representative number of sampling intervals

Shift =	A	B	C	D	E
$\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	t = 1	t = 2	t = 3	t = 5	t = 9
0.10	0.9953	1.0016	1.0012	1.0010	1.0004
0.25	0.9894	1.0038	1.0025	1.0022	1.0007
0.50	0.9649	1.0135	1.0078	1.0064	1.0004
1.00	0.8195	1.0848	1.0372	1.0244	0.9711
1.50	0.6227	1.1667	1.0672	1.0322	0.8547
2.00	0.4888	1.0774	1.0178	0.9801	0.6813
2.50	0.4124	0.8188	0.8380	0.8220	0.4913
3.00	0.3775	0.5641	0.6139	0.6174	0.3298
4.00	0.3643	0.3757	0.3875	0.3919	0.1647

A = (1) B = (.1/1.98) C = (.1/1/1.85) D = (.1/.5/1/1.5/1.9)

E = (.1/.3/.5/.7/1/1.3/1.5/1.7/1.95)

5.3.2 Choice of Interval Width

Consider a variable cusum with two sampling intervals d_1 and d_2. How far apart should the two intervals be taken assuming that the chart is matched to a *FSI* cusum chart whose fixed interval is 1 ? Nearly symmetric VSI cusum schemes are considered with sampling intervals (d_1, d_2) = (0.1,1.8) , (0.5,1.4) and (0.9,1.1) for $k = 0.25$ and with sampling intervals (0.1,1.98) , (0.5,1.5) and (0.9,1.1) for $k = 1.00$. Numerical results in Tables (5.7,5.8,5.9,5.10) show that both the ATS and the standard deviation are lowest when the two sampling intervals are as far apart as possible. This results holds for both $k = 0.25$ and $k = 1.00$. The properties of the VSI cusum chart approach the properties of the FSI cusum chart as d_1 and d_2 approach the fixed sampling interval d = 1. As for the coefficient of variation it can be seen that the FSI cusum chart has a slightly less variability than the scheme with $d_1 = 0.5$ and $d_2 = 1.5$.

Table 5.7: Values of the ATS for matched fixed and variable sampling interval symmetric cusum charts with k = 0.25 for representative interval widths

Shift = $\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	A d1 = 1 d2 = 1	B d1 = .1 d2 = 1.8	C d1 = .5 d2 = 1.4	D d1 = 0.9 d2 = 1.1
0.00	740.800	740.800	740.800	740.800
0.10	267.236	217.572	239.642	261.717
0.25	85.498	50.485	66.045	81.607
0.50	29.333	13.075	20.301	27.527
1.00	11.603	4.930	7.896	10.862
1.50	7.235	3.244	5.018	6.792
2.00	5.298	2.568	3.781	4.994
2.50	4.214	2.244	3.119	3.995
3.00	3.527	2.082	2.724	3.366
4.00	2.710	1.955	2.291	2.626

Table 5.8: Values of the ATS for matched fixed and variable sampling interval symmetric cusum charts with k = 1.00 for representative interval widths

Shift = $\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	A d1 = 1 d2 = 1	B d1 = .1 d2 = 1.9	C d1 = .5 d2 = 1.5	D d1 = .9 d2 = 1.1
0.00	740.800	740.800	740.800	740.800
0.10	442.669	402.050	420.656	438.255
0.25	211.485	163.777	185.203	206.227
0.50	69.549	39.251	52.767	66.192
1.00	13.580	3.730	8.111	12.486
1.50	5.464	0.880	2.918	4.955
2.00	3.268	0.396	1.673	2.949
2.50	2.353	0.251	1.185	2.119
3.00	1.862	0.189	0.933	1.676
4.00	1.322	0.132	0.661	1.190

Table 5.9: Values of the standard deviation of the time to signal for matched fixed and variable sampling interval symmetric cusum charts with $k = 0.25$ for representative interval widths

	A	B	C	D
Shift =	d1 = 1	d1 = .1	d1 = .5	d1 = 0.9
$\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.8	d2 = 1.4	d2 = 1.1
0.00	725.087	737.557	731.367	726.210
0.10	250.922	214.093	229.529	246.481
0.25	69.878	46.869	55.906	66.921
0.50	17.165	9.972	12.229	16.072
1.00	4.330	2.759	3.0054	4.027
1.50	2.110	1.413	1.497	1.960
2.00	1.309	0.848	0.904	1.211
2.50	0.924	0.521	0.597	0.848
3.00	0.705	0.306	0.415	0.641
4.00	0.531	0.092	0.271	0.478

Table 5.10: Values of the standard deviation of the time to signal for matched fixed and variable sampling interval symmetric cusum charts with k = 1.00 for representative interval widths

	A	B	C	D
Shift =	d1 = 1	d1 = .1	d1 = .5	d1 = .9
$\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.9	d2 = 1.5	d2 = 1.1
0.00	738.849	739.612	740.179	739.080
0.10	440.585	402.708	419.949	436.425
0.25	209.250	164.392	184.366	204.241
0.50	67.106	39.780	51.734	64.001
1.00	11.129	4.046	6.940	10.272
1.50	3.403	1.0027	1.919	3.097
2.00	1.597	0.427	0.860	1.445
2.50	0.970	0.205	0.504	0.876
3.00	0.703	0.107	0.357	0.633
4.00	0.482	0.050	0.241	0.433

Table 5.11: Values of the coefficient of variation of the time to signal for matched fixed and variable sampling interval symmetric cusum charts with k = 0.25 for representative interval widths

	A	B	C	D
Shift =	d1 = 1	d1 = .1	d1 = .5	d1 = 0.9
$\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.8	d2 = 1.4	d2 = 1.1
0.10	0.9390	0.9840	0.9578	0.9418
0.25	0.8173	0.9284	0.8465	0.8200
0.50	0.5852	0.7627	0.6024	0.5839
1.00	0.3732	0.5596	0.3868	0.3708
1.50	0.2917	0.4356	0.2983	0.2886
2.00	0.2472	0.3303	0.2392	0.2424
2.50	0.2194	0.2322	0.1914	0.2124
3.00	0.1998	0.1471	0.1523	0.1905
4.00	0.1959	0.0471	0.1184	0.1822

Table 5.12: Values of the coefficient of variation of the time to signal for matched fixed and variable sampling interval symmetric cusum charts with $k = 1.00$ for representative interval widths

	A	B	C	D
Shift =	d1 = 1	d1 = .1	d1 = .5	d1 = .9
$\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.9	d2 = 1.5	d2 = 1.1
0.10	0.9953	1.0016	0.9983	0.9958
0.25	0.9894	1.0038	0.9955	0.9904
0.50	0.9649	1.0135	0.9804	0.9669
1.00	0.8195	1.0848	0.8556	0.8227
1.50	0.6227	1.1667	0.6576	0.6249
2.00	0.4888	1.0774	0.5142	0.4902
2.50	0.4124	0.8188	0.4257	0.4131
3.00	0.3775	0.5641	0.3823	0.3777
4.00	0.3643	0.3757	0.3645	0.3643

5.3.3 Asymmetry of Sampling Intervals

All results presented in the previous sections of this chapter are for (nearly) symmetric charts. The question of how many states of the Markov chain should be assigned to interval d_j arises. Three VSI cusum schemes for $k = 0.25$ and $k = 1.00$ are considered with minimum sampling interval $d_1 = 0.1$ and maximum sampling interval (1.3,1.8,2.4) and (1.2,1.98,5.0) respectively. For a VSI cusum chart with $k = 0.25$ the ATS is lowest when the size of d_2 is chosen small for large shifts and large for small shifts. For shifts in the mean less than 1.0 using $d_2 = 2.4$ yields the lowest ATS whereas the VTSC plans with sampling intervals $d_2 = 1.8$ and $d_2 = 1.3$ are more efficient for shifts in the range (1.0,2.0) and (2.5,4.0) respectively. All VSI cusum schemes have smaller ATS values than t he corresponding FSI cusum chart. The standard deviations and the coefficients of variation are lowest when $d_2 = 2.4$ for all shifts considered. All VSI cusum schemes have smaller standard deviations than the FSI cusum chart. The coefficient of variation of the FSI cusum chart is smaller for shifts in the range (0.1,1.50). For the case $k = 1.0$ the ATS and the standard deviation are shortest when $d_2 = 4.9$ for all shifts considered. The FSI cusum chart has a larger ATS and standard deviation when compared to the VSI cusum schemes, however the CV is smaller for small and moderate shifts.

Table 5.13: Values of the ATS for matched fixed and variable sampling interval asymmetric cusum charts with $k = 0.25$ for representative interval widths

	A	B	C	D
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.3	d2 = 1.8	d2 = 2.4
0.00	740.800	740.800	740.800	740.800
0.10	267.236	231.521	217.572	210.133
0.25	85.498	57.848	50.484	47.118
0.50	29.333	15.410	13.075	12.304
1.00	11.603	5.618	4.930	4.974
1.50	7.235	3.540	3.244	3.526
2.00	5.298	2.644	2.568	2.991
2.50	4.214	2.142	2.244	2.757
3.00	3.527	1.827	2.082	2.645
4.00	2.710	1.522	1.955	2.549

Table 5.14: Values of the ATS for matched fixed and variable sampling interval asymmetric cusum charts with $k = 0.25$ for representative interval widths

	A	B	C	D
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}\ (\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.1	d2 = 1.9	d2 = 5.0
0.00	740.800	740.800	740.800	740.800
0.10	442.660	426.101	402.050	379.739
0.25	211.485	189.792	163.777	141.622
0.50	69.549	53.579	39.251	29.689
1.00	13.580	7.427	3.730	2.504
1.50	5.464	2.660	0.880	0.659
2.00	3.268	1.727	0.396	0.343
2.50	2.353	1.428	0.251	0.238
3.00	1.862	1.308	0.189	0.187
4.00	1.322	1.223	0.132	0.132

Table 5.15: Values of the standard deviation of the time to signal for matched fixed and variable sampling interval asymmetric cusum charts with $k = 0.25$ for representative interval widths

	A	B	C	D
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.3	d2 = 1.8	d2 = 2.4
0.00	725.087	736.394	737.557	737.713
0.10	250.922	226.915	214.093	206.777
0.25	69.878	53.222	46.869	43.571
0.50	17.165	11.472	9.972	9.205
1.00	4.330	2.954	2.759	2.603
1.50	2.110	1.507	1.413	1.290
2.00	1.309	0.988	0.848	0.715
2.50	0.924	0.732	0.521	0.396
3.00	0.705	0.551	0.306	0.210
4.00	0.531	0.245	0.092	0.066

Table 5.16: Values of the standard deviation of the time to signal for matched fixed and variable sampling interval asymmetric cusum charts with k = 1.00 for representative interval widths

	A	B	C	D
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.1	d2 = 1.9	d2 = 5.0
0.00	738.849	740.047	739.612	742.829
0.10	440.585	425.329	402.710	381.709
0.25	209.250	188.994	164.393	143.479
0.50	67.106	52.735	39.780	31.267
1.00	11.129	6.509	4.046	3.248
1.50	3.403	1.716	1.027	0.862
2.00	1.597	0.751	0.427	0.332
2.50	0.970	0.398	0.205	0.149
3.00	0.703	0.220	0.107	0.082
4.00	0.482	0.070	0.050	0.048

Table 5.17: Values of the coefficient of variation of the time to signal for matched fixed and variable sampling interval asymmetric cusum charts with k = 0.25 for representative interval widths

	A	B	C	D
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}\ (\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.3	d2 = 1.8	d2 = 2.4
0.10	0.9390	0.9801	0.9840	0.9840
0.25	0.8173	0.9200	0.9284	0.9247
0.50	0.5852	0.7444	0.7627	0.7481
1.00	0.3732	0.5258	0.5596	0.5233
1.50	0.2917	0.4256	0.4356	0.3659
2.00	0.2472	0.3738	0.3303	0.2392
2.50	0.2194	0.3419	0.2322	0.1436
3.00	0.1998	0.3013	0.1471	0.0792
4.00	0.1959	0.1609	0.0471	0.0257

Table 5.18: Values of the coefficient of variation of the time to signal for matched fixed and variable sampling interval asymmetric cusum charts with k = 1.00 for representative interval widths

	A	B	C	D
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.1	d2 = 1.9	d2 = 5.0
0.10	0.9953	0.9982	1.0016	1.0052
0.25	0.9894	0.9958	1.0038	1.0131
0.50	0.9649	0.9843	1.0135	1.0532
1.00	0.8195	0.8764	1.0848	1.2970
1.50	0.6227	0.6449	1.1666	1.3080
2.00	0.4888	0.4349	1.0774	0.9682
2.50	0.4124	0.2788	0.8188	0.6250
3.00	0.3775	0.1682	0.5641	0.4371
4.00	0.3643	0.0569	0.3757	0.3657

Chapter VI

Adjusted Properties of a Cusum

The expression for the ATS developed in Chapter 5 gives the average time to signal for any μ_1 under the simplifying assumption that this value of μ is the process mean from time zero onward. A more realistic assumption is that the process starts out with $\mu = \mu_0$ and then shifts at some random time to $\mu = \mu_1$ where the time to shift may fall in an interval between samples. It is of interest to calculate the time from the shift to the time when the process signals. The numerical results in the previous chapter were obtained under the additional assumption that the cusum starts with a zero value. If the shift in μ occurs after the chart has been running for some time then the cusum schemes may not be at zero when the process mean shifts. Corrective adjustments to the ATS and the corresponding standard deviation are given which allow for the case where the cusum statistic can fall anywhere in $(-\infty, h)$ at the time of the shift which occurs after the process has been operating for some time. In the evaluation of *VSI* control charts it is important to allow the shift to occur between samples since a large shift that occurs during a a long sampling interval may take a relatively long time to detect.

6.1 The Adjusted *ATS* for the Cusum

One approach to modeling the shift between samples uses the steady state distribution of the cusum scheme. Roberts (1966), Taylor (1968), Yachin (1985) and Crosier (1986) are among the authors who discussed various approches to the steady state distribution of the cusum scheme. The steady state *ATS* is defined as the *ATS* computed after the process has been running for a time and it is assumed that the shift occurs immediately after a sample is taken. The steady state *ATS* is computed as the weighted average of the *ATS's* conditioned on the initial state the scheme was in when the shift occurred, where the weights are the steady state probabilities. Roberts (1966) noticed that the steady state *ARL*'s of one-sided cusum charts are smaller than the conventional *ARL* values when the cusum is at zero at the time of the shift. However, Crosier (1986) pointed out that this may not always be true in the case of two-sided cusum charts. Let J be the number of observations taken (under μ_0) from time zero on until the time of the shift and let N^* be the number of observations taken (under μ_1) from the time of the shift on until the time of the signal. Then the total number of observations is J + N^*. Taylor (1968) used a simulation study with 200 runs to determine the effect of the first J observations taken before the process mean shifts on the run length distribution. The numerical results he obtained indicated that there were no practical differences in the run lengths as J varied between 0 and 50. Taylor (1968) used two different ways to simulate the steady state *ARL* depending on whether any false alarms before the time of the shift are ignored or not. In the first of Taylor's (1968) simulation studies, any signal occuring before the actual time of the shift is ignored and a new run is started after the count of observations is reset to zero. This will ensure that there are always J observations taken before the time of the shift. Crosier (1986) called such an *ARL conditional steady state ARL*, because the sequences of observations are conditioned on not producing any false alarms during the first J observations. The other approach used by Taylor (1968) resets the cusum to its initial value without resetting the count of observations if a there is a signal before the shift occurs. Since the cusum is recycled back to its initial value, Crosier (1986) called *ARL*'s computed this way *cyclical steady state ARL's* . Taylor (1968) recognized the cyclical steady state ARL as an average of conditional steady state ARL's. Numerical results on the two types of steady state ARL's obtained by Crosier (1986) show that for practical purposes there are no real differences between cyclical and conditional ARL's. Yachin (1985) used results of Darroch and Seneta (1965) to come up with a different approach to the steady state ARL of the cusum chart. He used the fact that the conditional steady state probabilities of the states of the Markov chain can be expressed by the eigenvectors of the principal minor of the transition matrix of the Markov chain.

The steady state ATS used here is a conditional steady state ATS where any false alarm before the shift occurs is ignored and the cusum is started again. To develop a model which adjusts the time to signal allowing for the fact that a shift may occur between samples, let

T^* = the adjusted time to signal

N^* = number of samples taken from the time of the shift until signal

$\underline{p}$ = vector of steady state probabilities for the states of the cusum

U = length of the interval in which the shift occurs

Y = the time from the shift until the next sample

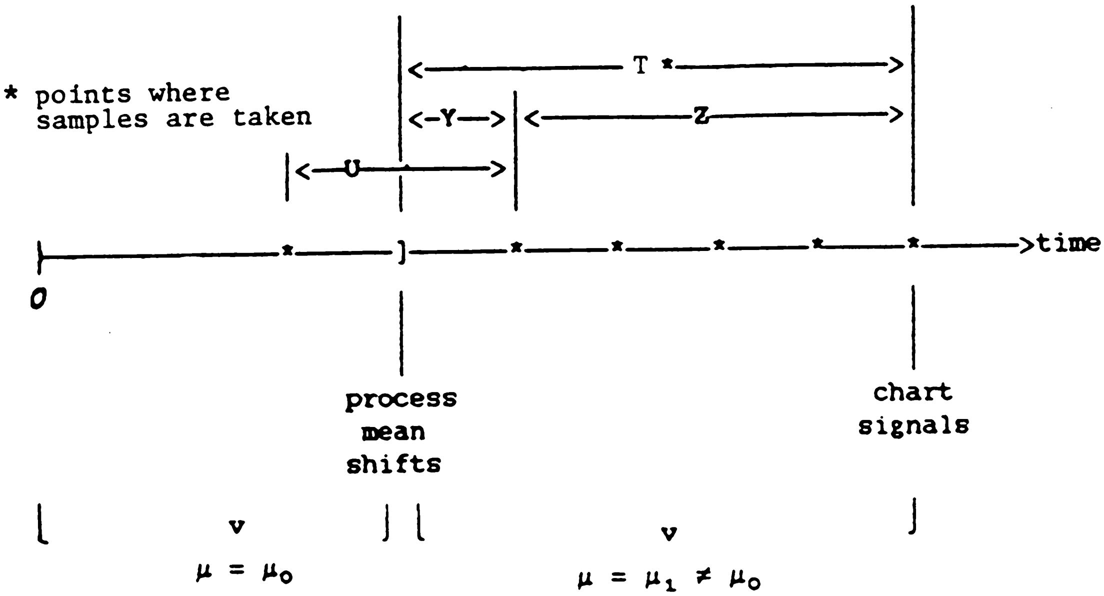
Z = the time from the first sample after the shift to signal

F = the event that a shift occurs between samples j and j + 1.

The relationship between U,Y,Z and T^* is along with hypothetical positions for the cusum statistic are shown in the following sketch. It is clear from the sketch that T^* = Y + Z and the ATS adjusted for the shift occurring between samples is

$$E(T^* \mid F) = E(Y \mid F) + E(Z \mid F) \tag{6.1}$$

where the expected values of the variables R^*, Y and Z are conditional on the event F that a shift occurred between two samples. The variables Y and Z are not independent since the value of the cusum statistic at sample j + 1 depends on the value at sample j.



Crosier (1986) expanded the Markov chain approach given by Brook and Evans (1972) to evaluate the steady state *ARL* (or *ANSS*). The expected number of samples to signal is then $ANSS = \underline{p}'M\underline{1}$ where $\underline{p}$ is the conditional steady state probability vector and $M\underline{1}$ is the vector of the expected number of samples to signal. The vector $\underline{p}$ is found by forming a matrix $\hat{Q}$ by conditioning the elements of Q such that each row sums up to unity. Then $\underline{p}$ is the solution of the equation

$$\underline{p} = \hat{Q}'\underline{p}. \tag{6.2}$$

The adjusted *ATS* differs from the steady state *ATS* in that it is not assumed that the shift occurs immediately after a sample is taken, but instead the shift is allowed to occur at some random point between samples.

To compute the adjusted *ATS* we need to know the value of the cusum at the time of the shift. One approach would be to assume a distribution for the time of the shift and then try to derive the distribution of the cusum at this random time. Instead we will assume that the shift occurs in some sampling interval and the probability of falling in an interval of a particular length is proportional to the product of the length of this interval and the frequency with which this interval is used when the Markov chain is in steady state. This implies that $P(F \mid S_j = v) \propto b_v$ where, in this chapter, S_j denotes the discrete version of the cusum statistic. Using Bayes Theorem we can obtain the conditional distribution of S_n given F as

$$
\begin{aligned}
P(S_j = v \mid F) &= \frac{P(F \mid S_j = v)\, P_0(S_j = v)}{\sum_{v<h} P(F \mid S_j = v)\, P_0(S_j = v)} \\
&= \frac{d(v)\, P_0(S_j = v)}{\sum_{i=1}^{t} \sum_{v \in D_i} d(v)\, P_0(S_j = v)} \\
&= \frac{d(v)\, P_0(S_j = v)}{\sum_{i=1}^{t} d_i\, P_0(d_i)}
\end{aligned}
\tag{6.3}
$$

where $D_i = \{v{:}d(v) = d_i\}$, and $P_0(d_i)$ is the steady state probability of using the sampling interval d_i. In addition we assume that when the shift falls in a particular sampling interval the position within the interval is uniformly distributed over the interval so that

$$
\begin{aligned}
E(Y \mid F) &= \sum_{v<h} E(Y \mid S_j = v, F)\, P_0(S_j = v, \mid F) \\
&= \sum_{i=1}^{t} \sum_{v \in D_i} E(Y \mid S_j = v, F)\, P_0(S_j = v \mid F) \\
&= \sum_{i=1}^{t} \frac{d_i}{2} \frac{d_i\, P_0(d_i)}{\sum_{l=1}^{t} d_l\, P_0(d_l)} \\
&= \frac{1}{2} \sum_{i=1}^{t} \frac{d_i^2\, P_0(d_i)}{\sum_{l=1}^{t} d_l\, P_0(d_l)}
\end{aligned}
\tag{6.4}
$$

The expected value of Z conditional on F is

$$
\begin{aligned}
E(Z \mid F) &= \sum_{s<h} E(Z \mid S_{j+1} = s, F)\, P(S_{j+1} = s \mid F) \\
&+ \sum_{s \ge h} E(Z \mid S_{j+1} = s, F)\, P(S_{j+1} = s \mid F) \\
&= \sum_{s<h} E(Z \mid S_{j+1} = s, F) \sum_{v<h} P_0(S_j = v \mid F)\, P(C^*_{j+1} = s - v \mid F) \\
&= \sum_{s<h} E(T_s) \sum_{v<h} P_0(S_j = v \mid F)\, P(C^*_{j+1} = s - v \mid F)
\end{aligned}
\tag{6.5}
$$

where C^*_{j+1} is the discrete version of $C_{j+1} - k$. Combining (6.4) and (6.5) gives the adjusted ATS

$$
\begin{aligned}
E(T^* \mid F) &= \sum_{s<h} E(T_s) \sum_{v<h} P_0(S_j = v \mid F)\, P(C^*_{j+1} = s - v \mid F) \\
&+ \sum_{i=1}^{t} \frac{d_i}{2} \frac{d_i P_0(d_i)}{\sum_{l=1}^{t} d_l P(d_l)} \\
&= \sum_{s<h} E(T_s) \sum_{v<h} \frac{d(v)\, P_0(S_j = v)}{\sum_{i=1}^{t} d_i P_0(d_i)} P(C^*_{j+1} = s - v \mid F) \\
&+ \sum_{i=1}^{t} \frac{d_i}{2} \frac{d_i P_0(d_i)}{\sum_{l=1}^{t} d_l P(d_l)}
\end{aligned}
\tag{6.6}
$$

The conditional variance of $T^* = Y + Z$ is

$$
var(T^* \mid F) = var(Y \mid F) + var(Z \mid F) + 2\, cov(Y, Z \mid F). \tag{6.7}
$$

By an argument similar to (6.4) it follows that

$$
\begin{aligned}
E(Y^2 \mid F) &= \sum_{i=1}^{t} \sum_{v \in D_i} E(Y^2 \mid S_j = v, F)\, P_0(S_j = v \mid F) \\
&= \frac{1}{3} \sum_{i=1}^{t} \frac{d_i^3 P_0(d_i)}{\sum_{l=1}^{t} d_l P_0(d_l)}.
\end{aligned}
\tag{6.8}
$$

Using (6.5) and (6.8) gives the conditional variance of Y

$$var(Y \mid F) = E(Y^2 \mid F) - (E(Y \mid F))^2$$

$$= \frac{1}{3} \sum_{i=1}^{t} \frac{d_i^3 \, P_0(d_i)}{\sum_{l=1}^{t} d_l P_0(d_l)} - \left\{ \frac{1}{2} \sum_{i=1}^{t} \frac{d_i^2 \, P_0(d_i)}{\sum_{l=1}^{t} d_l P_0(d_l)} \right\}^2 \tag{6.9}$$

which reduces to $d^2/12$ for the fixed sampling interval cusum with sampling interval d. By an argument similar to (6.5) it follows that

$$E(Z^2 \mid F) = \sum_{s<h} E(T_s^2) \sum_{v<h} P_0(S_j = v \mid F) \, P(C_{j+1}^* = s - v \mid F). \tag{6.10}$$

Using (6.5) and (6.10), the conditional variance of Z is

$$\begin{aligned} var(Z \mid F) = & \sum_{s<h} E(T_s^2) \sum_{v<h} P_0(S_j = v \mid F) \, P(C_{j+1}^* = s - v \mid F) \\ & - \left[\sum_{s<h} E(T_s) \sum_{v<h} P_0(S_j = v \mid F) \, P(C_{j+1}^* = s - v \mid F) \right]^2. \end{aligned} \tag{6.11}$$

The conditional covariance of Y and Z is

$$cov(Y,Z \mid F) = E(YZ \mid F) - E(Y \mid F) \, E(Z \mid F) \tag{6.12}$$

The last two factors of (6.11) are given by (6.4) and (6.5). The conditional expectation of YZ is determined as follows. Y and C_{j+1}^* are independent and, conditional on $S_j = v$ and $S_{j+1} = v + x$, Y and Z are conditionally independent. Thus

$$\begin{aligned} E(YZ \mid S_j = v, F) &= \sum_{x} E(Y \mid S_j = v, C_{j+1}^* = x, F) \, E(Z \mid S_j = v, C_{j+1}^* = x, F) \, P(C_{j+1}^* = x \mid F) \\ &= \sum_{x<h-v} E(Y \mid S_j = v, F) E(Z \mid S_{j+1} = v + x, F) P(C_{j+1}^* = x \mid F) \\ &= \sum_{x<h-v} \frac{d(v)}{2} E(T_{v+x}) \, P(C_{j+1}^* = x \mid F) \end{aligned} \tag{6.13}$$

It follows that

$$E(YZ \mid F) = \sum_{v<h} E(YZ \mid S_j = v, F)\, P(S_j = v \mid F) \tag{6.14}$$

$$= \sum_{v<h} \sum_{x<h-v} \frac{b_v}{2} E(T_{v+x})\, P(C^*_{j+1} = x, F) \frac{d(v)\, P_0(S_j = v)}{\sum_{i=1}^{t} b_i P_0(b_i)}$$

and hence $cov(YZ \mid F)$ can be obtained by substituting (6.4), (6.5) and (6.14) into (6.12). Finally, $var(T^* \mid F)$ can be obtained by substituting (6.9), (6,11) and (6.12) into (6.7). The covariance between Y and Z is zero for the fixed cusum since the sampling interval is fixed and doesn't depend on the actual value of the control statistic at sample j.

6.2 Computational Results

A Markov chain similar to the one used in the previous chapter is being utilized to evaluate the adjusted properties of the cusum. All computations in this chapter are based on the division of the interval (0 , h) and ($-\infty$, 0] into 30 states which corresponds to r = 30 as shown in the SAS program in Appendix 5. In each case the value of h was chosen to give an in-control *ANSS* of 740.8 when $\mu = \mu_0$. The numerical results indicate that the variable sampling interval cusum charts are more efficient than their corresponding FSI cusum chart for all shifts in the range (0.1,4.0) with respect to the adjusted ATS which takes into account the time between the shift and the first sample taken out of control.

6.2.1 The Number of Sampling Intervals

In the previous chapter it was concluded that in most cases it is more efficient (with respect to the unadjusted ATS) to use only two sampling intervals. In this section the problem of how many sampling intervals to choose is investigated numerically when the adjusted properties are considered. Cusum charts with different number of sampling intervals ($t=2,3,5,9$) are evaluated as shown in Tables (6.1,6.2,6.3,6.4.6.5,6.6). The numerical results indicate that for $k = 0.25$ a choice of only two sampling intervals is most efficient with respect to the ATS for a wide range of shifts (0.1,1.5). When larger shifts are considered (≥ 1.5) the adjusted ATS appears to be shorter for cusum plans with multiple sampling intervals. A similar result was obtained in Chapers 3 and 4 for the Shewhart $\overline{X}$-chart. The maximum sampling interval used in these plans is not the same due to computational restrictions so that particular ANSS and ATS values are achieved when the process is in control. The more sampling intervals are used the larger the maximum sampling interval had to be chosen. There is a VSI cusum chart which is more efficient than the FSI cusum chart for all shifts considered in the range (0.1,4.0). As for the adjusted standard deviation of the time to signal it is more efficient to use only two sampling intervals for large shifts (3.0,4.0) and the plans with 3 and 9 sampling intervals are superior for shifts (1.5,2.5) and (.1,1.0) respectively. The adjusted ATS values for the different VSI cusum charts are very similar and this suggests that it is not necessary to use more than two sampling intervals. The variable cusums have a much smaller adjusted ATS and corresponding standard deviation than the corresponding fixed cusum for $k = 0.25$ and for all shifts in the range (0.1,4.0) and (0.5,4.0) respectively. The relative efficiency of the variable cusum is highest for moderate shifts in the mean for that k for ($\mu_1 = 0.5,1.0,1.5$). Recall that the fixed cusum with k is optimal for shifts of size 2 k , and there the variable cusum performs very well relative to the fixed cusum. As the size of the shift increases more than two standard deviations, the adjusted ATS of the variable cusum decreases only slightly since it is very small. The coefficient of variation (CV) decreases as the size of the shift gets large (Table 6.5). When nine sampling intervals are used the CV of the FSI cusum chart is smaller for all shifts than the VSI cusum schemes.

The VSI cusum chart has a larger CV than the corresponding FSI cusum chart for all shifts in the range (0.1,4.0). For a variable cusum with $k = 1$ the adjusted ATS is lowest for shifts in the range (0.1,1.0) when only two sampling intervals are used. However as the size of the shift increases it is more efficient with respect to the adjusted ATS to use multiple sampling intervals as shown in Table (6.2). There is only a slight difference in the performance of the VSI plans and for practical considerations 2 or 3 sampling intervals are sufficient. Computational restrictions made the maximum sampling interval largest for the VSI cusum chart with two sampling intervals. The relative efficiency of all VSI cusum schemes increase with an increase for moderate shifts in the process mean and start to decrease as the size of the shift gets large. Table (6.4) reveals that the the VSI cusum schemes have much lower adjusted standard deviations than the FSI cusum chart except for a very large shift (4.0). Among the VSI cusum schemes using five or nine sampling intervals yields the lowest adjusted standard deviations however the difference among the VSI cusum schemes is only small. The CV's are larger for $k = 1$ compared to $k = 0.25$ and Multiple Sampling Intervals. Table (6.6) shows that the FSI cusum chart has smaller CV's for shifts in the range (0.1,0.5) compared to the VSI cusum schemes. There is no VSI cusum chart which uniformly has the lowest CV values for all shifts considered when compared to the other VSI cusum schemes.

Table 6.1: Values of the adjusted ATS for matched fixed and variable sampling interval symmetric cusum charts with $k = 0.25$ for representative number of sampling intervals

Shift = $\sqrt{n}(\mu_1 - \mu_0)/\sigma$	A t=1	B t=2	C t=3	D t=5	E t=9
0.00	740.800	740.800	740.800	740.800	740.800
0.10	255.800	214.949	216.700	220.477	224.111
0.25	78.784	48.517	49.416	51.796	53.688
0.50	25.481	11.529	11.734	12.522	13.073
1.00	9.459	3.695	3.727	3.995	4.145
1.50	5.676	2.149	2.156	2.343	2.410
2.00	4.027	1.550	1.536	1.692	1.711
2.50	3.111	1.268	1.230	1.360	1.343
3.00	2.533	1.129	1.067	1.164	1.122
4.00	1.841	1.015	0.928	0.935	0.879

A = (1) B = (.1/1.78) C = (.1/1/1.79) D = (.1/.5/1/1.5/1.85)

E = (.1/.3/.5/.7/1/1.3/1.5/1.7/1.96)

Table 6.2: Values of the adjusted ATS for matched fixed and variable sampling interval symmetric cusum charts with k = 1.00 for representative number of sampling intervals

Shift =	A	B	C	D	E
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	t = 1	t = 2	t = 3	t = 5	t = 9
0.00	740.800	740.800	740.800	740.800	740.800
0.10	440.965	401.882	408.020	409.232	410.968
0.25	210.107	163.760	169.794	171.268	173.447
0.50	68.493	39.447	42.390	43.261	44.690
1.00	12.811	4.242	4.583	4.737	5.156
1.50	4.807	1.581	1.493	1.493	1.636
2.00	2.663	1.187	1.023	0.990	1.036
2.50	1.777	1.076	0.898	0.857	0.860
3.00	1.304	1.025	0.845	0.803	0.785
4.00	0.792	0.971	0.790	0.748	0.719

A = (1) B = (.1/1.98) C = (.1/1/1.85) D = (.1/.5/1/1.5/1.9)

E = (.1/.3/.5/.7/1/1.3/1.5/1.7/1.95)

Table 6.3: Values of the standard deviation of the adjusted time to signal for matched fixed and variable sampling interval symmetric cusum charts with k = 0.25 for representative number of sampling intervals

Shift = $\sqrt{n}(\mu_1 - \mu_0)/\sigma$	A t = 1	B t = 2	C t = 3	D t = 5	E t = 9
0.00	725.087	737.557	737.119	736.303	736.019
0.10	249.421	213.879	215.564	219.098	222.693
0.25	65.869	46.628	47.400	49.252	51.083
0.50	16.496	9.580	9.631	9.881	10.344
1.00	3.946	2.346	2.280	2.249	2.303
1.50	1.888	1.242	1.200	1.200	1.178
2.00	1.206	0.994	0.950	0.953	0.899
2.50	0.916	0.941	0.886	0.881	0.815
3.00	0.776	0.930	0.866	0.853	0.783
4.00	0.669	0.926	0.856	0.829	0.757

A = (1) B = (.1/1.78) C = (.1/1/1.79) D = (.1/.5/1/1.5/1.85)

E = (.1/.3/.5/.7/1/1.3/1.5/1.7/1.96)

Table 6.4: Values of the standard deviation of the adjusted time to signal for matched fixed and variable sampling interval symmetric cusum charts with k = 1.00 for representative number of sampling intervals

Shift = $\sqrt{n}(\mu_1 - \mu_0)/\sigma$	A $t=1$	B $t=2$	C $t=3$	D $t=5$	E $t=9$
0.00	738.849	739.612	741.302	741.262	741.017
0.10	440.151	402.591	408.731	409.941	411.487
0.25	208.927	164.300	170.343	171.814	173.807
0.50	66.871	39.704	42.669	43.538	44.792
1.00	10.918	4.040	4.421	4.578	4.861
1.50	3.192	1.357	1.271	1.266	1.306
2.00	1.431	1.100	0.933	0.897	0.881
2.50	0.874	1.072	0.895	0.855	0.829
3.00	0.669	1.069	0.889	0.849	0.821
4.00	0.571	1.068	0.888	0.847	0.820

A = (1) B = (.1/1.98) C = (.1/1/1.8) D = (.1/.5/1/1.5/1.9)

E = (.1/.3/.5/.7/1/1.3/1.5/1.7/1.95)

Table 6.5: Values of the coefficient of variation of the adjusted time to signal for matched fixed and variable sampling interval symmetric cusum charts with k = 0.25 for representative number of sampling intervals

Shift = $\sqrt{n}(\mu_1 - \mu_0)/\sigma$	A t = 1	B t = 2	C t = 3	D t = 5	E t = 9
0.00	0.9788	0.9956	0.9950	0.9939	0.9935
0.10	0.9751	0.9950	0.9948	0.9937	0.9937
0.25	0.8745	0.9611	0.9592	0.9509	0.9515
0.50	0.6474	0.8310	0.8207	0.7891	0.7923
1.00	0.4172	0.6349	0.6117	0.5629	0.5557
1.50	0.3326	0.5781	0.5569	0.5119	0.4889
2.00	0.2994	0.6411	0.6184	0.5634	0.5256
2.50	0.2944	0.7423	0.7202	0.6477	0.6067
3.00	0.3065	0.8245	0.8119	0.7331	0.6977
4.00	0.3635	0.9131	0.9220	0.8861	0.8616

A = (1) B = (.1/1.78) C = (.1/1/1.79) D = (.1/.5/1/1.5/1.85)

E = (.1/.3/.5/.7/1/1.3/1.5/1.7/1.96)

Table 6.6: Values of the coefficient of variation of the adjusted time to signal for matched fixed and variable sampling interval symmetric cusum charts with k = 1.00 for representative number of sampling intervals

Shift = $\sqrt{n}(\mu_1 - \mu_0)/\sigma$	A $t=1$	B $t=2$	C $t=3$	D $t=5$	E $t=9$
0.00	0.9974	0.9984	1.0007	1.0006	1.0003
0.10	0.9982	1.0018	1.0017	1.0017	1.0013
0.25	0.9944	1.0033	1.0032	1.0032	1.0021
0.50	0.9763	1.0065	1.0066	1.0064	1.0023
1.00	0.8523	0.9525	0.9647	0.9666	0.9428
1.50	0.6642	0.8584	0.8512	0.8479	0.7982
2.00	0.5374	0.9271	0.9125	0.9061	0.8496
2.50	0.4922	0.9965	0.9966	0.9967	0.9641
3.00	0.5129	1.0423	1.0530	1.0572	1.0466
4.00	0.7214	1.0991	1.1236	1.1322	1.1395

A = (1) B = (.1/1.98) C = (.1/1/1.8) D = (.1/.5/1/1.5/1.9)

E = (.1/.3/.5/.7/1/1.3/1.5/1.7/1.95)

6.2.2 Choice of Interval Width

Consider a variable cusum with two sampling intervals d_1 and d_2. How far apart should the two intervals be taken assuming that the fixed interval is 1 ? Nearly symmetric VSI cusum schemes are considered with sampling intervals (d_1, d_2) = (0.1,1.8) , (0.5,1.4) and (0.9,1.1) for $k = 0.25$ and d_1 = 0.9 d_2 = 1.1) and with sampling intervals (0.1,1.98) , (0.5,1.5) and (0.9,1.1) for $k = 1.00$. Numerical results in Table (6.7) show that adjusted ATS is lowest when the two sampling intervals are as far apart as possible for $k = 0.25$. The properties of the VSI cusum chart approach the properties of the FSI cusum chart as d_1 and d_2 approach the fixed sampling interval d = 1. The adjusted standard deviation of the VSI cusum schemes is lower for a wide range of shifts and using d_1 =0.5 and d_2 = 1.4 yields the lowest adjusted standard deviation for shifts in the range (0.5,2.5). Table (6.11) shows that the FSI cusum chart has the lowest coefficient of variation for small and moderately large shifts. Among the VSI cusum schemes it seems that choosing the sampling intervals close together reduces the coefficient of variation for a wide range of shifts. The results for k = 1 differ slightly from above as shown in Tables (6.8,6.10,6.12). The VSI cusum chart with sampling intervals d_1 = 0.1 and d_2 = 1.9 has the lowest adjusted ATS and adjusted standard deviation for small and moderately large shifts. The numerical results indicate that the two sampling intervals should be spaced well apart when small shifts are anticipated. The FSI cusum chart has larger values for both the adjusted ATS and adjusted standard deviations compared to the VSI cusum schemes except for a shift of 0.1 in which case it has the lowest standard deviation. Table (6.12) which gives the adjusted coefficient of variation for several cusum plans shows that for k = 1.0 the VSI cusum schemes have a lower CV and that in general the CV decreases as the size of the shift increases.

Table 6.7: **Values of the adjusted ATS for matched fixed and variable sampling interval symmetric cusum charts with $k = 0.25$ for representative interval widths**

	A	B	C	D
Shift =	d1 = 1	d1 = .1	d1 = .5	d1 = .9
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.8	d2 = 1.4	d2 = 1.1
0.00	740.800	740.800	740.800	740.800
0.10	255.800	214.949	232.789	251.135
0.25	78.784	48.517	61.895	75.391
0.50	25.481	11.529	17.743	23.936
1.00	9.459	3.695	6.287	8.830
1.50	5.676	2.149	3.740	5.293
2.00	4.027	1.550	2.664	3.757
2.50	3.111	1.268	2.090	2.908
3.00	2.533	1.129	1.745	2.374
4.00	1.841	1.015	1.355	1.739

Table 6.8: Values of the adjusted ATS for matched fixed and variable sampling interval symmetric cusum charts with $k = 1.00$ for representative interval widths

	A	B	C	D
Shift =	d1 = 1	d1 = .1	d1 = .5	d1 = .9
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.98	d2 = 1.5	d2 = 1.1
0.00	740.800	740.800	740.800	740.800
0.10	440.965	401.882	419.716	436.692
0.25	210.107	163.760	184.507	204.970
0.50	68.493	39.447	52.345	65.251
1.00	12.811	4.242	7.986	11.832
1.50	4.807	1.581	2.936	4.416
2.00	2.663	1.187	1.756	2.464
2.50	1.777	1.076	1.295	1.662
3.00	1.304	1.025	1.053	1.234
4.00	0.792	0.971	0.790	0.771

Table 6.9: Values of the standard deviation of the adjusted time to signal for matched fixed and variable sampling interval symmetric cusum charts with k = 0.25 for representative interval widths

	A	B	C	D
Shift =	d1 = 1	d1 = .1	d1 = .5	d1 = .9
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.8	d2 = 1.4	d2 = 1.1
0.00	725.087	737.557	731.367	726.210
0.10	249.421	213.879	228.704	245.111
0.25	65.869	46.628	55.314	66.018
0.50	16.496	9.580	11.729	15.445
1.00	3.946	2.346	2.725	3.670
1.50	1.888	1.242	1.376	1.770
2.00	1.206	0.994	0.991	1.151
2.50	0.916	0.941	0.856	0.894
3.00	0.776	0.930	0.799	0.771
4.00	0.669	0.926	0.757	0.676

Table 6.10: Values of the standard deviation of the adjusted time to signal for matched fixed and variable sampling interval symmetric cusum charts with k = 1.00 for representative interval widths

	A	B	C	D
Shift =	d1 = 1	d1 = .1	d1 = .5	d1 = .9
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.98	d2 = 1.5	d2 = 1.1
0.00	738.849	739.612	740.179	739.080
0.10	440.151	402.591	419.698	436.029
0.25	208.927	164.300	184.182	203.948
0.50	66.871	39.704	51.600	63.789
1.00	10.918	4.040	6.835	10.083
1.50	3.192	1.357	1.912	2.918
2.00	1.431	1.100	1.036	1.325
2.50	0.874	1.072	0.841	0.837
3.00	0.669	1.069	0.787	0.663
4.00	0.571	1.068	0.763	0.581

Table 6.11: Values of the coefficient of variation of the adjusted time to signal for matched fixed and variable sampling interval symmetric cusum charts with k = 0.25 for representative interval widths

	A	B	C	D
Shift =	d1 = 1	d1 = .1	d1 = .5	d1 = .9
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.8	d2 = 1.4	d2 = 1.1
0.00	0.9788	0.9956	0.9873	0.9803
0.10	0.9751	0.9950	0.9825	0.9760
0.25	0.8745	0.9611	0.8937	0.8757
0.50	0.6474	0.8310	0.6611	0.6453
1.00	0.4172	0.6349	0.4335	0.4156
1.50	0.3326	0.5781	0.3681	0.3345
2.00	0.2994	0.6411	0.6372	0.3065
2.50	0.2944	0.7423	0.4097	0.3074
3.00	0.3065	0.8245	0.4582	0.3247
4.00	0.3635	0.9131	0.5590	0.3889

Table 6.12: Values of the coefficient of variation of the adjusted time to signal for matched fixed and variable sampling interval symmetric cusum charts with k = 1.00 for representative interval widths

	A	B	C	D
Shift =	d1 = 1	d1 = .1	d1 = .5	d1 = .9
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.98	d2 = 1.5	d2 = 1.1
0.00	0.9974	0.9984	0.9992	0.9990
0.10	0.9982	1.0018	0.9999	0.9985
0.25	0.9944	1.0033	0.9982	0.9950
0.50	0.9763	1.0065	0.9858	0.9776
1.00	0.8523	0.9525	0.8559	0.8522
1.50	0.6642	0.8584	0.6511	0.6607
2.00	0.5374	0.9271	0.5902	0.5377
2.50	0.4922	0.9965	0.6493	0.5036
3.00	0.5129	1.0423	0.7470	0.5370
4.00	0.7214	1.0991	0.9655	0.7541

6.2.3 Asymmetry of Sampling Intervals

Three VSI cusum schemes for $k = 0.25$ and $k = 1.00$ are considered with minimum sampling interval $d_1 = 0.1$ and maximum sampling interval (1.3,1.8,2.4) and (1.2,1.98,5.0) respectively. For a VSI cusum chart witk $k = 0.25$ the ATS is lowest when d_2 is chosen small for large shifts and large for small shifts. For shifts in the mean less than 1.0 using $d_2 = 2.4$ yields the lowest ATS whereas the VTSC plans with sampling intervals $d_2 = 1.8$ and $d_2 = 1.3$ are more efficient for shifts in the range (1.0,2.0) and (2.5,4.0) respectively. These results are similar to the ones obtained in chapter 5 for the unadjusted ATS. All VSI cusum schemes have smaller ATS values than the corresponding FSI cusum chart. The adjusted standard deviation of the time to signal is lower over a wide range of shifts for the VSI cusum chart as shown in Table (6.15). As the maximum sampling interval increases the standard deviation also increases for the variable sampling interval charts. The coefficient of variation of the FSI cusum chart is smaller for shifts in the range (0.1,4.0) and the VSI cusum chart with the lowest CV uses a $d_2 = 1.3$. Tables (6.14,6.16,6.18) show the numerical results when $k^* = 1.0$. The adjusted ATS is always lower when using a VSI cusum chart even when there is a very large shift in the process mean. Among the VSI cusum schemes the maximum sampling interval should be chosen large if interest is in detecting small shifts and vice versa. The (nearly) symmetric case is best with respect to the adjusted ATS for shifts in the range (1.0,3.0), as shown in Table (6.14). As for the adjusted standard deviation the VSI cusum chart is more efficient than the FSI cusum chart for all shifts considered. The (nearly) symmetric case yields the smallest adjusted standard deviations for shifts in the range (0.1,0.5). When large shifts are expected a smaller d_2 is more efficient with respect to the adjusted standard deviation. Table (6.18) reveals that the VSI cusum schemes have a smaller adjusted coefficient of variation for all shifts considered. When the maximum sampling interval is increased for VSI cusum schemes the CV increases for large shifts and decreases for small shifts. In general we can conclude that the VSI cusum schemes are significantly more efficient than the corresponding FSI cusum chart with respect to both the adjusted ATS and adjusted standard deviation and for a wide range of shifts.

Table 6.13: Values of the adjusted ATS for matched fixed and variable sampling interval asymmetric cusum charts with k = 0.25 for representative interval widths

	A	B	C	D
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.3	d2 = 1.8	d2 = 2.4
0.00	740.800	740.800	740.800	740.800
0.10	255.80	227.888	214.949	207.662
0.25	78.784	55.100	48.517	45.250
0.50	25.481	13.369	11.529	10.769
1.00	9.459	4.185	3.695	3.619
1.50	5.676	2.384	2.149	2.225
2.00	4.027	1.651	1.550	1.711
2.50	3.111	1.262	1.268	1.486
3.00	2.533	1.032	1.129	1.379
4.00	1.841	0.816	1.015	1.284

Table 6.14: Values of the adjusted ATS for matched fixed and variable sampling interval asymmetric cusum charts with k = 1.00 for representative interval widths

	A	B	C	D
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.2	d2 = 1.98	d2 = 5.0
0.00	740.800	740.800	740.800	740.800
0.10	440.965	425.375	401.882	380.957
0.25	210.107	189.102	163.760	143.061
0.50	68.493	52.925	39.447	31.401
1.00	12.811	6.803	4.242	4.524
1.50	4.807	2.048	1.581	2.800
2.00	2.663	1.119	1.187	2.522
2.50	1.777	0.822	1.076	2.426
3.00	1.304	0.703	1.025	2.377
4.00	0.792	0.619	0.971	2.324

Table 6.15: Values of the standard deviation of the adjusted time to signal for matched fixed and variable sampling interval asymmetric cusum charts with $k = 0.25$ for representative interval widths

	A	B	C	D
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.3	d2 = 1.8	d2 = 2.4
0.00	725.087	736.394	737.557	737.713
0.10	249.421	226.677	213.879	206.572
0.25	65.869	52.940	46.628	43.342
0.50	16.496	11.044	9.580	8.846
1.00	3.946	2.511	2.346	2.340
1.50	1.888	1.203	1.242	1.419
2.00	1.206	0.846	0.994	1.242
2.50	0.916	0.742	0.941	1.210
3.00	0.776	0.715	0.930	1.203
4.00	0.669	0.705	0.926	1.200

Table 6.16: Values of the standard deviation of the adjusted time to signal for matched fixed and variable sampling interval asymmetric cusum charts with k = 1.00 for representative interval widths

	A	B	C	D	
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1	
$\sqrt{n}\,(\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.2	d2 = 1.98	d2 = 5.0	
0.00	738.849	740.047	739.612	740.179	742.829
0.10	440.151	425.206	402.591	419.698	381.603
0.25	208.927	188.889	164.300	184.182	143.411
0.50	66.871	52.634	39.704	51.600	31.280
1.00	10.918	6.364	4.040	6.835	3.994
1.50	3.192	1.602	1.357	1.912	2.630
2.00	1.431	0.824	1.100	1.036	2.547
2.50	0.874	0.689	1.072	0.841	2.540
3.00	0.669	0.668	1.069	0.787	2.539
4.00	0.571	0.664	1.068	0.763	2.538

Table 6.17: Values of the coefficient of variation of the adjusted time to signal for matched fixed and variable sampling interval asymmetric cusum charts with k = 0.25 for representative interval widths

	A	B	C	D
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.3	d2 = 1.8	d2 = 2.4
0.00	0.9788	0.9941	0.9956	0.9958
0.10	0.9751	0.9947	0.9950	0.9948
0.25	0.8745	0.9608	0.9611	0.9578
0.50	0.6474	0.8261	0.8310	0.8214
1.00	0.4172	0.5999	0.6349	0.6467
1.50	0.3326	0.5045	0.5781	0.6377
2.00	0.2994	0.5121	0.6411	0.7259
2.50	0.2944	0.5881	0.7423	0.8140
3.00	0.3065	0.6926	0.8245	0.8727
4.00	0.3635	0.8644	0.9131	0.9347

Table 6.18: Values of the coefficient of variation of the adjusted time to signal for matched fixed and variable sampling interval asymmetric cusum charts with k = 1.00 for representative interval widths

	A	B	C	D
Shift =	d1 = 1	d1 = .1	d1 = .1	d1 = .1
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	d2 = 1	d2 = 1.2	d2 = 1.98	d2 = 5.0
0.00	0.9974	0.9990	0.9984	1.0027
0.10	0.9982	0.9996	1.0018	1.0017
0.25	0.9944	0.9783	1.0033	1.0025
0.50	0.9763	0.9945	1.0065	0.9961
1.00	0.8523	0.9355	0.9525	0.8830
1.50	0.6642	0.7822	0.8584	0.9393
2.00	0.5374	0.7362	0.9271	1.0100
2.50	0.4922	0.8377	0.9965	1.0467
3.00	0.5129	0.9500	1.0423	1.0679
4.00	0.7214	1.0729	1.0991	1.0924

Table 6.19: Values of the adjusted ATS for matched fixed sampling interval symmetric cusum charts with d = 1 for representative values of *k*

	A	B	C	D	E	F
Shift =	h = 8.14	h = 4.79	h = 3.35	h = 2.52	h = 1.99	h = 1.6
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	k = .25	k = .5	k = .75	k = 1	k = 1.25	k = 1.5
0.00	740.800	740.800	740.800	740.800	740.800	740.800
0.10	255.800	338.570	369.951	400.965	476.535	502.817
0.25	78.784	121.341	166.931	210.107	250.753	285.017
0.50	25.481	33.465	48.728	68.493	91.573	115.121
1.00	9.459	8.747	9.989	12.811	17.328	23.396
1.50	5.676	4.591	4.437	4.807	5.684	7.096
2.00	4.027	3.052	2.719	2.663	2.815	3.160
2.50	3.111	2.268	1.929	1.777	1.748	1.815
3.00	2.533	1.800	1.486	1.304	1.222	1.208
4.00	2.015	1.301	0.974	0.792	0.716	0.686

Table 6.20: Values of the adjusted ATS for matched variable sampling interval symmetric cusum charts with two sampling intervals for representative values of k

	A	B	C	D	E	F
	d1 = .1	d1 = .1	d1 = .1	d1 = .1	d1 = 0.1	d1 = 0.1
	d2 = 1.8	d2 = 1.97	d2 = 1.93	d2 = 1.98	d2 = 1.86	d2 = 1.93
Shift =	h = 8.14	h = 4.79	h = 3.35	h = 2.52	h = 1.99	h = 1.6
$\sqrt{n}(\mu_1 - \mu_0)/\sigma$	k = .25	k = .5	k = .75	k = 1	k = 1.25	k = 1.5
0.00	740.800	740.800	740.800	740.800	740.800	740.800
0.10	214.949	298.343	359.361	401.880	440.246	464.588
0.25	48.517	85.288	126.307	163.759	202.431	230.766
0.50	11.529	15.236	26.010	39.447	56.958	72.535
1.00	3.695	2.669	3.167	4.242	6.208	8.561
1.50	2.149	1.509	1.479	1.581	1.755	2.088
2.00	1.550	1.236	1.173	1.187	1.160	1.239
2.50	1.268	1.136	1.074	1.076	1.018	1.058
3.00	1.129	1.083	1.026	1.025	0.961	0.992
4.00	1.015	1.029	0.971	0.971	0.909	0.939

The numerical results in this chapter reveal that variable sampling interval cusum charts are significantly more efficient than the corresponding fixed sampling interval cusum chart with respect to a shorter adjusted ATS which in fact penalizes the VSI cusum chart charts in the case of large shifts in the process mean. In most cases it is sufficient to use two interval widths in a symmetric way such that the expected number of times the process is in states corresponding to the minimum sampling interval is approximately equal to the expected number of times the process is in states corresponding to the maximum sampling interval when the process is in control.

Tables 6.19 and 6.20 give values of the adjusted *ATS* for *FSI* and *VSI* charts respectively and for selected values of k. We considered six schemes with reference values ranging between 0.25 and 1.50. The numerical results indicate that when the interval widths d_1 and d_2 are chosen perfectly symmetric, then the optimal value for k is δ /2 as in the fixed case. In Table 6.20 there are some discrepancies from this rule, because the cusum schemes are not all symmetric. Table 6.20 shows that the adjusted *ATS* values for large k are much larger for smaller shifts and only slightly smaller for large shifts when compared to *ATS* values for small k. In practical situations it may not be possible to specify one particular shift of interest and thus it seems reasonable to recommend a relatively small k such as 0.25 for most applications.

6.3 GUIDELINES FOR THE DESIGN OF VSI CUSUM CHARTS

A number of papers have been written on methods for designing cusum charts, for example, a general discussion and many references are given in Woodall (1986). The design of a *VSI* cusum chart requires the specification of the usual parameters h, k and n and in addition to the specification of the sampling intervals d_1, d_2, ..., d_t and the corresponding regions L_1, L_2, ..., L_t. One relatively simple approach to the design of a *VSI* cusum chart for a particular application is to choose h, k, and n as would be done for a *FSI* cusum chart and then choose the sampling intervals and regions of the *VSI* cusum chart to match the sampling interval of the *FSI* chart. This would give a *VSI* chart with the same average sampling rate as the *FSI* chart, but the *VSI* chart would have faster detection of shifts in μ.

When the sample size n is specified the parameters h an k of the *FSI* cusum chart are usually chosen to give a specified *ANSS* (*ARL*) when $\mu = \mu_0$ and to give fast detection of a specified shift. The results in Chapter 6 indicate that a relatively small value of k such as k = 0.25 may be appropriate for the *VSI* cusum chart. Methods for choosing h to give a specified *ANSS* when $\mu = \mu_0$ are given, for example, in Lucas (1976).

The numerical results in Chapter 6 suggest using a *VSI* with two sampling intervals, d_1 and d_2. Symmetric charts perform well over a wide range of shifts and are relatively easy to design. The value of d_1 should be the smallest value that is practical while a vlaue of d_2 around 2 should work well for most applications. For example, choose $d_1 = 0.1$ and $d_2 = 1.9$. The numerical results in this dissertation assume that the unit of time is the sampling interval of the *FSI* cusum chart. For example, if the *FSI* cusum chart uses 2 hour sampling intervals then $d_1 = 0.1$ and $d_2 = 1.9$ would correspond to 12 minutes and 228 minutes respectively. In practice it might be preferable to modify these intervals to more convenient intervals such as 10 minutes and 4 hours.

When the sampling intervals d_1 and d_2 have been chosen the corresponding regions, say $L_1 = (-\infty, g)$ and $L_2 = (g, h)$, must be specified. If the two regions are symmetric then, when $\mu = \mu_0$, the cusum statistic will be on the average half the time in L_1 and half the time in L_2. To avoid having to use the Markov chain to determine g each time a cusum chart is designed, a regression equation was developed to predict g as a function of h and k. The equation

$$\hat{g} = -1.34 + 0.71 \log(\frac{h}{k}) \tag{6.15}$$

was based on 24 combinations of (h,k) values which were selected to give *ATS* values of 100, 740.8 and 1500 when the process is in control. The fit is quite good, with coefficient of determination R-squared of 0.95. The resulting equation does not depend on the values of d_1 and d_2 but is assumes that the corresponding regions they correspond to are symmetric. For asymmetric choices of d_2 and for $d_1 = 0.1$ we give the following regression equation

$$\hat{g} = -1.40 + 0.78 \log(\frac{h}{k}) + 0.6(1.9 - d_2) \tag{6.16}$$

which is based on 7 combinations of (h,k) which were selected to give an ATS value of 740.8. The fit for (6.16) is good, with a coefficient of variation of 0.98. Note that the estimated slope and intercept of equations 6.15 and 6.16 are very similar and that equation 6.16 becomes to equation 6.15 in the symmetric case.

Example: Consider designing a VSI cusum chart with two sampling interval $d_1 = 0.1$ and $d_2 = 1.9$ such that the in-control ATS is around 1000. Using tables by Van Dobben de Bruyn (1968) we choose k = 0.4 and h = 6 to obtain an in-control ATS of 940. Substituting in (6.16) we obtain $g \cong -0.5$. This implies that d_1 should be used when the cusum statistic $S_j \leq -0.5$ and d_2 when $-0.5 < S_j \leq 6$.

An additional question is the sampling interval to use before the first sample when the chart is first put into operation. Cusum charts have traditionally started with $S_0 = 0$ and if $\delta = \sqrt{n}\,(\mu_1 - \mu_0)/\sigma$ is used then the first sampling interval would be d(0). Lucas and Crosier (1982b) proposed the fast initial response feature where the starting state $S_0 = c$ with $c > 0$. This feature gives extra protection against problems that might arise during startup of the process since starting the cusum closer to h gives the cusum a headstart to the signal region. By the same reasoning it may be useful to start the VSI cusum with $c > \{0,g\}$ so that the shortest sampling interval d_1 is used first. This gives extra protection against start-up problems in two ways: (1) by giving the cusum a headstart and (2) by using short sampling intervals at first. Using d_1 as the first interval should correspond to the intuition of most control chart users.

Chapter VII

Summary and Conclusions

7.1 Summary of Results

The usual practice in maintaining a control chart is to take samples from the process at fixed time intervals. This dissertation proposes a modification of the standard practice where the time between samples can vary depending on what is observed in the current sample. The properties of the VSI Shewhart chart and the VSI cusum chart are investigated through Renewal theory and Markov chain approaches. The average number of samples to signal ($ANSS$) and the average time to signal (ATS) are defined and used to evaluate the performance of these proposed charts. These two new quantities are more appropriate criteria than the conventional average run length (ARL) which relates to both, number of samples *and* and time to signal. Sampling schemes with different number of sampling intervals are considered in addition to varying the probabilities of the regions corresponding to the sampling intervals used. The properties of the Shewhart chart and the cusum chart are investigated under different assumptions about the time of the shift. A standard assumption in process control is that the value of μ is the process mean from time zero on. Though there are situations where this is realistic, in most practical applications a process shifts at some random point in time between samples. A model is developed which adjusts the time to signal to allow for the fact that a sudden shift can occur between samples. Expressions for the average adjusted time to signal ATS^* and the corresponding standard deviation are developed for the Shewhart chart and the cusum chart under the assumption that the shift occurs uniformly over a sampling interval. This simplifying assumption was supported by a simulation study using several prior distributions for the time of the shift. The only case where these assumptions may not be reasonable seems to occur when mean of the prior distribution is very small and the shift occurs very soon after the chart is started. Also considered are the properties of the Shewhart chart when there is a drift in the process mean and the case when runs rule are used. The properties of the cusum chart are considered when it is assumed that the process has been running for a time before the shift occurs. The steady state distribution of the cusum is considered and the average adjusted time to signal is actually the steady state ATS with the adjustment factor incorporated. Conclusions about the variable sampling interval control charts should be based on the adjusted time to signal which in fact penalizes theses charts when a large shifts occurs at the start.

7.2 Conclusions

The following conclusions are based on the average adjusted time to signal ATS^* which can be considered a realistic measure of how well a process control chart performs.

Variable sampling interval control charts are significantly more efficient than the corresponding fixed sampling interval control charts. The ATS^* is significantly smaller for the VSI Shewhart chart and the VSI cusum charts when compared to the corresponding FSI charts. It is known that the *FSI* $\overline{X}$-chart is more efficient than the *FSI* cusum chart in detecting very large shifts in the process mean. We are interested in comparing these two types of control charts when the *VSI* feature is added to them. Figures 7.1, 7.2, 7.3 and 7.4 show the values of the adjusted ATS for $\overline{X}$ charts and cusum chart with two sampling intervals ($d_1 = 0.1, d_2 \cong 1.9$) and k = 0.25, 0.50, 1.0 and 1.5. Note that the adjusted *ATS* of the *VSI* cusum chart is significantly lower for most shifts and that a small reference value yields more improvements over a wide range of shifts than a large reference value. It is clear that the *FSI* Shewhart chart can not compete with the *FSI* cusum chart in detecting small and moderately large shifts in the mean. However, adding the variable sampling interval feature to the $\overline{X}$- chart makes the *VSI* Shewhart chart more sensitive to smaller shift as shown in the plots. The *VSI* cuusm chart is substantially more efficient than the *VSI* Shewhart chart for small and moderate shifts in the mean. We conclude that adding the variable sampling interval feature improves both the *FSI* Shewhart chart and the *FSI* cusum chart, but the *VSI* Shewhart chart can't compete with the proposed variable sampling interval cusum chart.

It is sufficient to use only two sampling intervals. In most cases considered this is actually the most efficient plan to use in Shewhart charts and cusum charts. This result keeps the complexity of variable sampling interval charts to a reasonable level and can be thought of as a control chart with two warning lines where the sampling interval changes when a point falls between a warning line and an action line. Results in previous chapters suggest that increasing the number of sampling intervals will not change the ATS^* significantly. The minimum sampling interval d_1 should be taken as small as possible while d_2, the maximum sampling interval should be large if interest is in small shifts, and small if interest is in large shifts. A "symmetric" choice of the sampling intervals yields a chart which performs well over a wide range of shifts. The two sampling intervals d_1 and d_2 should be taken reasonably far apart in order to improve significantly upon the fixed sampling interval control charts.

Some of the possible extensions of our results are as follows: The *VSI* technique can also be used with two-sided cusum charts for detecting both negative and positive shifts in μ by using two one-sided charts simultaneously. In this case it may happen that the two one-sided charts will not specify the same sampling interval to be used before the next sample. One possible rule to solve this problem is to use the minimum of the two sampling intervals corresponding to the one-sided cusum charts. Although this is straightforward in principle, the calculation of the properties of such a rule requires a more complicated model. Other rules based on the positions of the two cusums could also be developed. A thorough investigation of this problem will require considerable effort and was not attempted in this dissertation.

The approach to the design of *VSI* cusum charts that was discussed in the previous chapter involves designing a *FSI* chart and then simply adding the *VSI* feature to obtain a matched *VSI* chart. Another possibility is to develop an economic model and then choose the chart parameters to minimize the expected cost. In some cases a *VSI* chart may have a higher administrative cost than a *FSI* chart since the sampling rate is not constant in the *VSI* chart. An economic model would be useful in this case for weighing the benefits from faster detection of process changes against the higher administrative costs.

The *VSI* cusum chart as proposed here uses sampling intervals that may not be convenient. For example if the chart uses 10 minute and 100 minute intervals the sampling throughout a work shift will be irregular. As discussed in Reynolds et al (1987), another way to use the variable sampling idea is the establish a fixed interval of, say , one hour when samples will be taken on the hour and then take additional samples within an when necessary. Samples could be taken at, say 10 minute intervals within an hour if there is an indication of a problem, but when there is no indication of a problem the chart would revert to taking samples on the hour. This modification of the *VSI* idea might be administratively simpler and also be easier to explain to process operators since the standard hourly spacing is retained.

In most cases where a control chart is being used to control the process mean, some other chart will be used to control the process variance. In addition there may be more than one process variable being monitored. When monitoring multiple parameters for one variable or multiple parameters resulting from multiple variables, some of the information to decide when to take the next sample. One possibility is to use separate charts for each variable and then use, say, the minimum sampling interval from all the charts. Another possibility is to combine all of the information into a multivariate-type control statistic and then use this statistic in one *VSI* control chart.

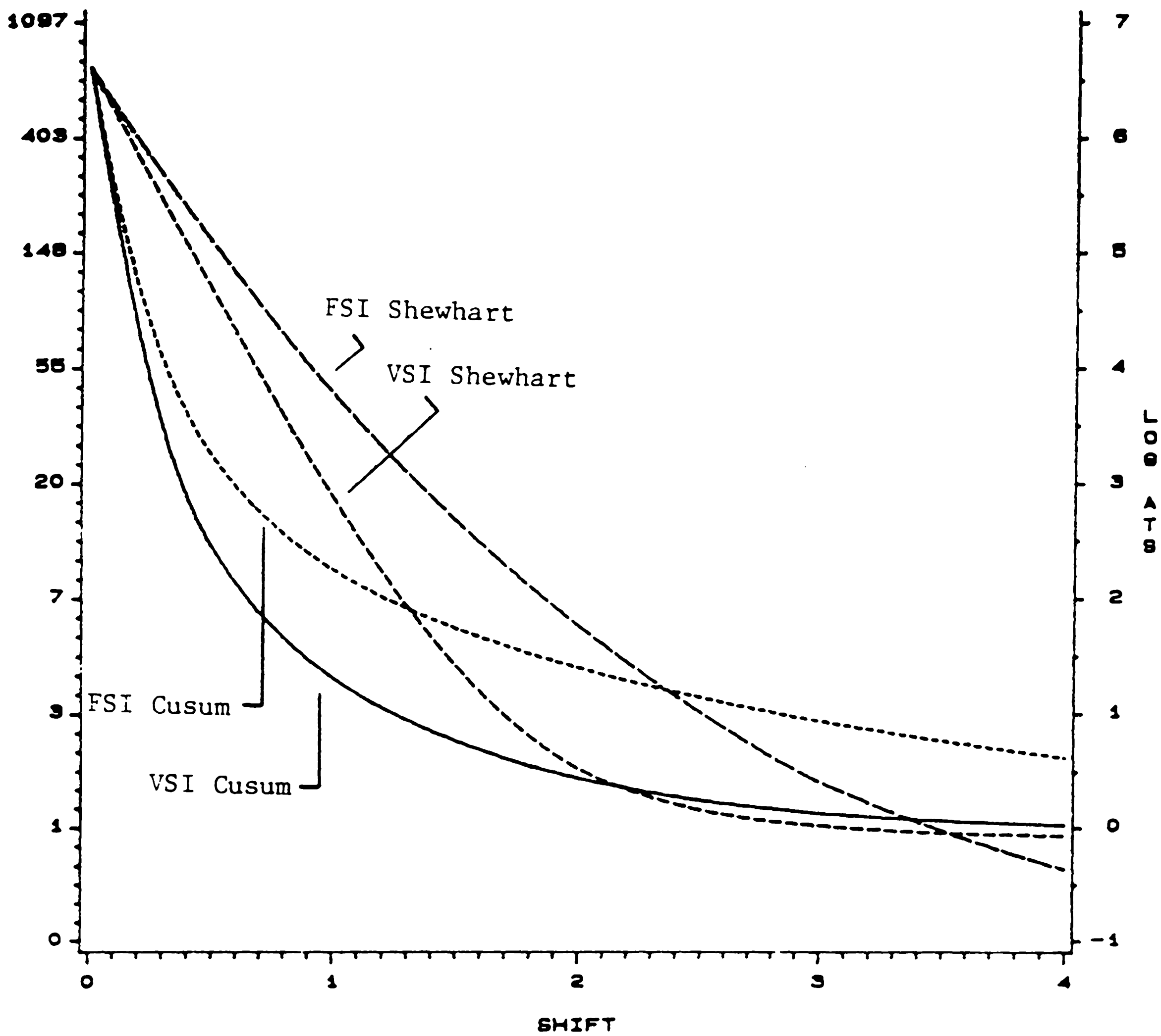


FIGURE 7.1 PLOT OF THE LOG (ATS) FOR K=0.25

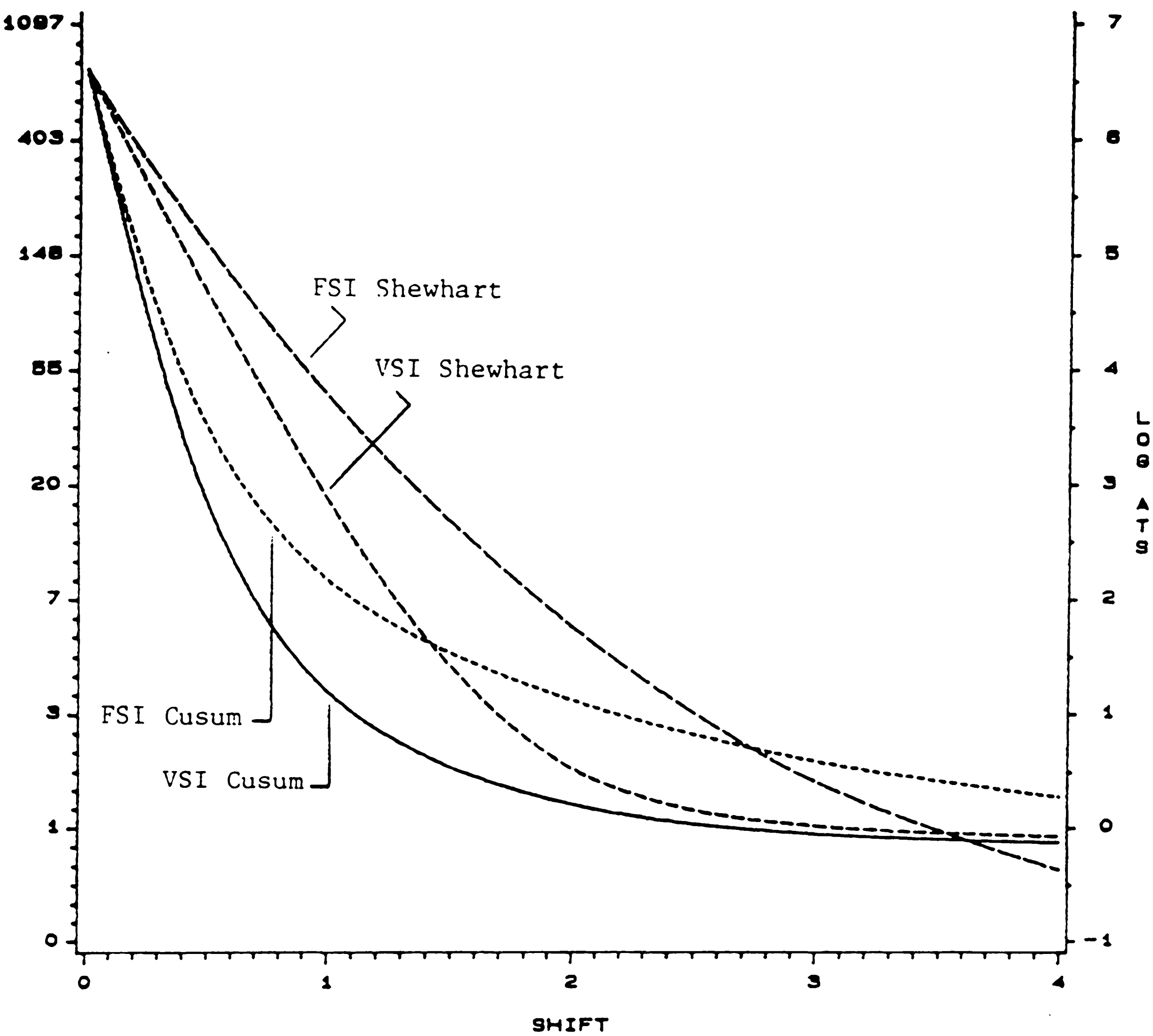


FIGURE 7.2 PLOT OF THE LOG (ATS) FOR K=0.50

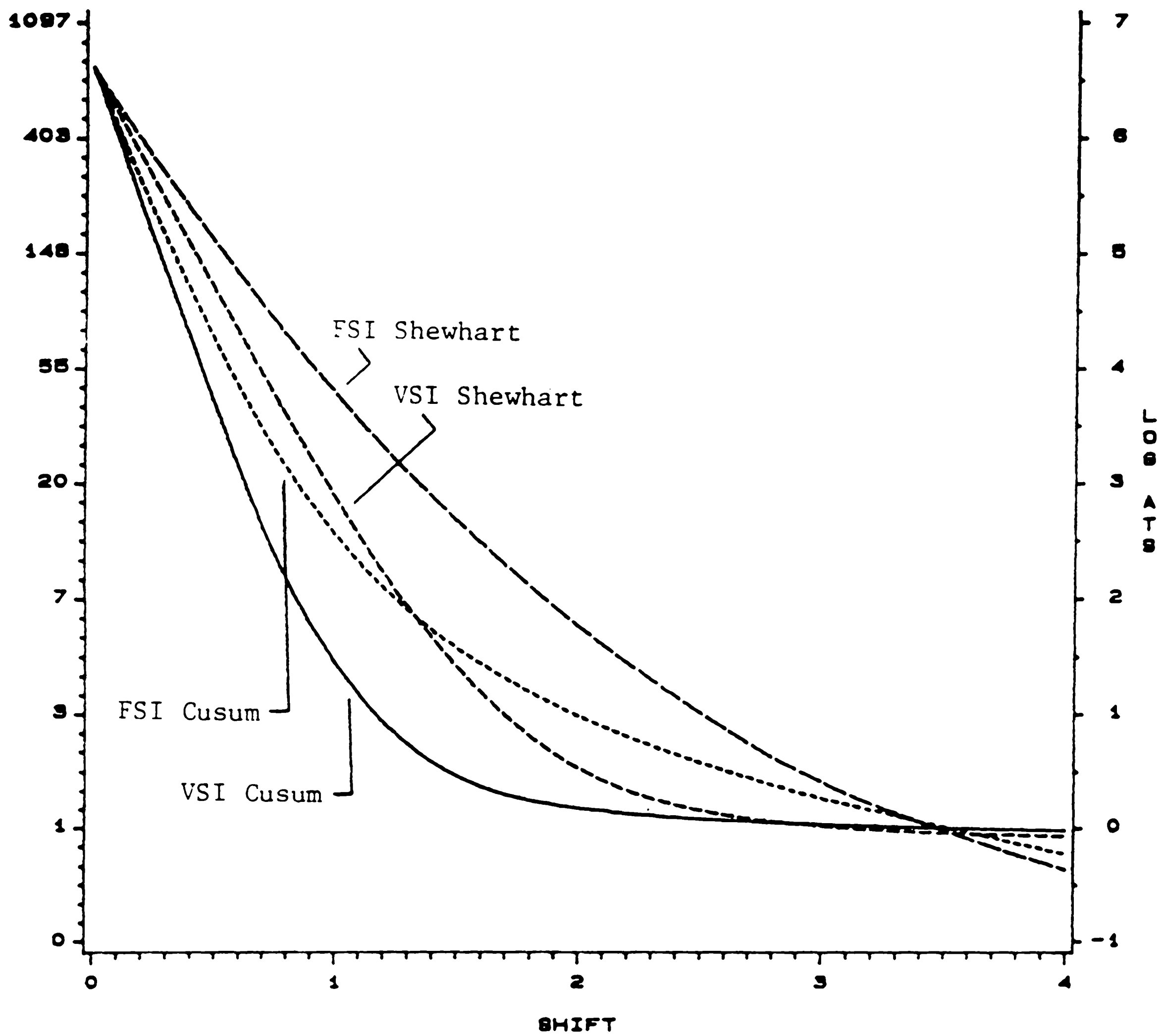


FIGURE 7.3 PLOT OF THE LOG (ATS) FOR K=1.00

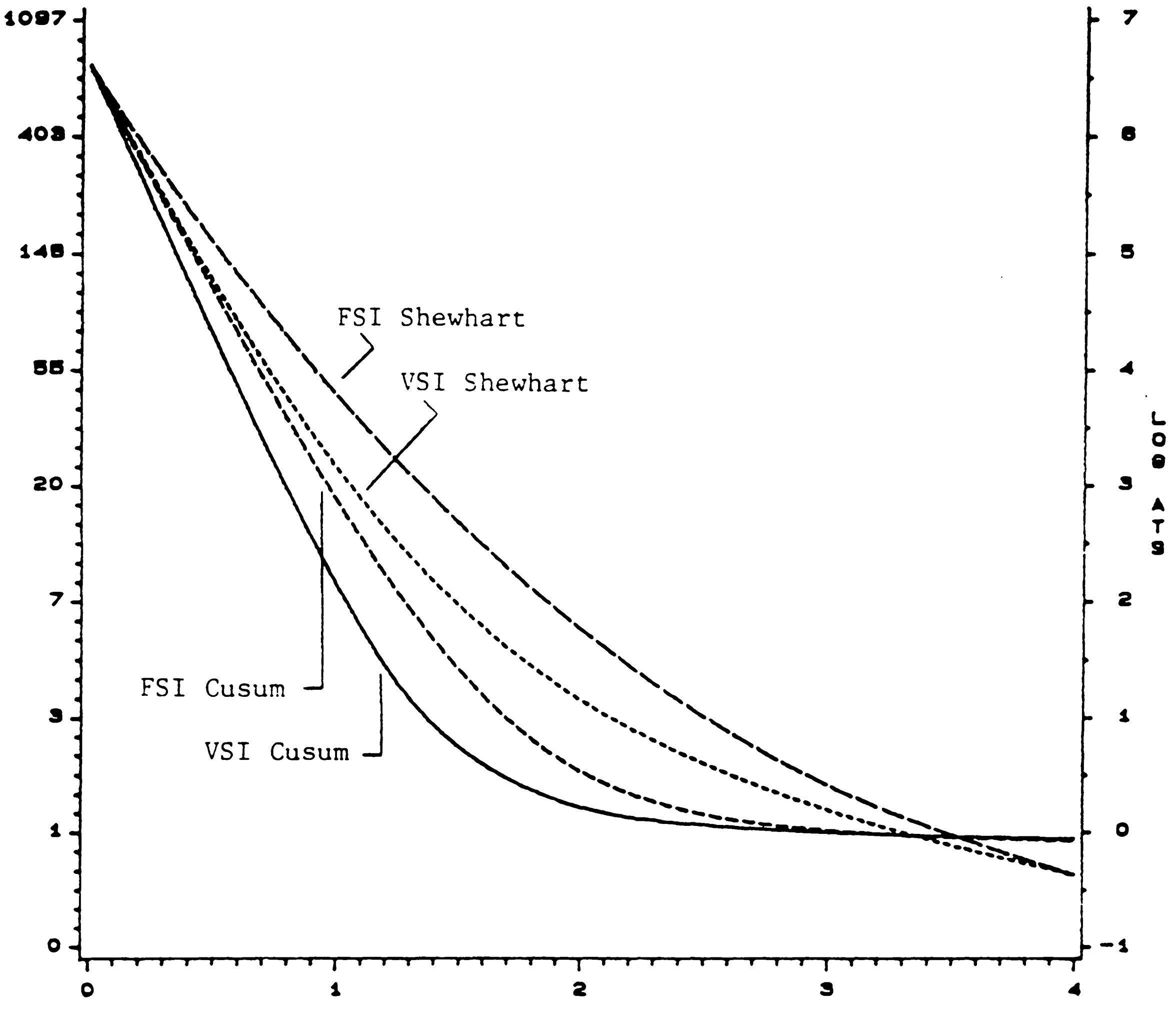


FIGURE 7.4 PLOT OF THE LOG (ATS) FOR K=1.50

Bibliography

Arnold, J.C. (1970). "A Markovian Sampling Policy Applied to Quality Monitoring of Streams." *Biometrics*,26, 739-747 .

Barnard, G.A. (1959). "Control Charts and Stochastic Processes." *Journal of the Royal Statistical Society* , Ser. B, Vol. 21, 239-271.

Beattie, D.W. (1962). "A Continuous Acceptance Sampling Procedure Based Upon a Cumulative Sum Chart for the Number of Defectives." *Applied Statistics* , Vol. 11, No. 3, 137-147.

Bissel, A.F. (1969). "CUSUM Techniques for Quality Control." *Applied Statistics*,Vol. 18, 1-30 .

Brook, D. and Evans, D.A. (1972). "An approach to the probability distribution of cusum run length ." *Biometrika* , Vol. 59, 539-549.

Champ, C.W. and Woodall, W.H. (1987). "Exact Results for Shewhart Control Charts with Supplementary Run Rules ", Submitted for Publication.

Crigler, J.R. (1973). "An Economically Optimal Markovian Sampling Policy for Monitoring Continuous Processes ". Ph.D. Dissertation , Virginia Polytechnic Institute and State University , Blacksburg, Virginia .

_________ and Arnold, J.C. (1979). "On Economically Optimal Markovian Sampling Policies with Applicatons to Water Pollution Monitoring ". *Proceedings of the International Statistical Institute* , 42nd Session, Manila, Philippines, p. 501-525.

_________ and Arnold, J.C. (1986). "Economically Optimum Markovian Sampling Policies for Process Monitoring ". *Communications in Statistics* , Vol. 15, No. 6, 1772-1802.

Crosier, R.B. (1986). "A New Two-Sided Cumulative Sum Quality Control Scheme ". *Technometrics* , Vol. 28, No. 3, 187-194.

Darroch, J.N. and Seneta, E. (1965). "On Quasi-Stationary Distributions in Absorbing Discrete-Time Markov Chains". *Journal of Applied Probability*, 2, 88-100.

Derman, C., Littauer.S. and Solomon.H. (1957). "Tightened Multi-Level Continuous Sampling Plans " . *Annals of Mathematical Statistics* , Vol. 28, 395-404.

Dodge, H.F. (1943). "A Sampling Inspection Plan for Continuous Production." *Annals of Mathematical Statistics*, Vol. 28, 264-279.

_________ (1955). "Skip-Lot Sampling Plans " . *Industrial Quality Control* , Vol. 11, No. 6, 3-5.

_________ and Torrey, M.N. (1951). "Additional Continuous Sampling Inspection Plans " . *Industrial Quality Control* ,

_________ and Perry, R.L. (1971). "A System of Skip-Lot Plans for Lot by Lot Inspection " . *Annual Technical Conference Transactions* , American Society for Quality Control, 469-477.

Duncan, A.J. (1974). "Quality Control and Industrial Statistics. 4th Ed. " . Richard D. Irwin. Inc., Honewood, Illinois.

Ewan, W.D. and Kemp, K.W. (1960). "Sampling Inspection of Continuous Processes with No Autocorrelation between Successive Results ." *Biometrika* , Vol. 47, 363-380.

Goel, A.L. and Wu, S.M. (1971). "Determination of A.R.L. and a Contour Nomogram for Cusum Charts to Control Normal Mean.". *Technometrics*, Vol. 13, No. 2, 221-230.

Goldsmith, P.L. and Whitfield, H.W. (1961). "Average Run Lengths in Cumulative Chart Quality Control Schemes ." *Technometrics*,3 , 11-20.

Hui, Y.V. (1980). "Topics in Statistical Process Control." Ph.D. Dissertation , Virginia Polytechnic Institute and State University , Blacksburg, Virginia .

_________ and Jensen, D.R. (1980). "Markovian Time-delay Sampling Policies " U.S. Army Research Office, Grant No. DAAG-29-78-G-0172.

Johnson, N.L. (1961). "A Simple Theoretical Approach to Cumulative Sum Control Charts." *Journal of the American Statistical Association* , Vol. 56 , 835-840 .

_________ and Leone, F.C. (1962-a). "Cumulative Sum Control Charts - Mathematical Principles Applied to their Construction and Use." *Industrial Quality Control*, June 1962 15-21.

_________ and Leone, F.C. (1962-b). "Cumulative Sum Control Charts - Mathematical Principles Applied to their Construction and Use." *Industrial Quality Control*, July 1962, 29-36.

_________ and Leone, F.C. (1962-c). "Cumulative Sum Control Charts - Mathematical Principles Applied to their Construction and Use." *Industrial Quality Control*, Aug. 1962 22-28 .

Kemeny, J.G. and Snell, J.L. (1960). "Finite Markov Chains". Princeton, D. Van Nostrand Company, Inc.

Kemp, K.W. (1961). "The Average Run Length of a Cumulative Sum Chart when a V-mask is used." *Journal of the Royal Statistical Society* , B, Vol. 23, 149-153.

_________ (1971). "Formal Expressions which can be applied to Cusum Charts". *Journal of the Royal Statistical Society*, Ser. B, Vol. 33, 331-360.

Khan, R.A. (1978). "Wald's Approximations to the average run length in cusum procedures." *Journal of Statistical Planning and Inference*,2 , 63-77.

_________ (1984). "On cumulative sum procedures and the SPRT application". *Journal of the Royal Statistical Society* , B 46, No. 1, 79-85.

Lieberman, G.J. and Solomon, H. (1955). "Multi-Level Continuous Sampling Plans ". *Annals of Mathematical Statistics* , 26, 685-704.

Lucas, J.M. (1976). "The Design and Use of Cumulative Sum Quality Control Schemes." *Journal of Quality Technology* , Vol. 8, No. 1, 1-12 .

_________ (1982a). "Combined Shewhart-CUSUM Quality Control Schemes". *Journal of Quality Technology*, Vol. 14, No.2, 51-59.

_________ and Crosier, R.B. (1982b). "Fast Initial Response for CUSUM Quality Control Schemes". *Technometrics*, Vol.24, 199-205.

_________ and Crosier, R.B. (1982c). "Robust Cusum ." *Communications in Statistics* ,Theor. Meth. Vol. 11, No. 23, 2667-2687 .

Moore, P.G. (1958). "Some Properties of Runs to Quality Control Charts ." *Biometrika* , Vol. 45, 89-95.

Page, E.S. (1954). "Continuous Inspection Schemes ". *Biometrika* , 41, 100-114 .

_________ (1955). "Control Charts with Warning Lines " . *Biometrika*, 42, 243-259.

_________ (1962). "A Modified Control Chart with Warning Lines." *Biometrika*, Vol. 49, 171-176.

Parzen, E. (1962). "Stochastic Processes". San Fransisco, Holden-Day, Inc..

Perry, R.L. (1973). "Skip-Lot Sampling Plans ". *Journal of Quality Technology* , Vol. 5, 123-136.

Reynolds, M.R..Jr. (1975). "Approximations to the Average Run Length in Cumulative Sum Control Charts". *Technometrics* , Vol. 17, 65-71 .

_________ and Ghosh, B.K. (1981). "Designing control charts for means and variances". ASQC Quality Congress Transactions, San Francisco, 400-407.

_________ , Amin, R.W.. Arnold, J.C. and Nachlas, J.A. (1986). "$\overline{X}$- Charts with Variable Sampling Intervals ". Submitted for publication.

Roberts, S.W. (1959)."Control Charts Based on Geometric Moving Averages ." *Technometrics* ,1, 239-250 .

_________ (1966). "A Comparison of Some Control Chart Procedures", *Technometrics*, Vol. 8, No. 3, 411-430.

Smeach, S.C. and Jernigan, R.W. (1977). "Further Aspects of a Markovian Sampling Policy for Water Quality Monitoring ." *Biometrics*,33. 41-46 .

Shewhart, W.A. (1931). "The Economic Control of Quality Manufactured Products ." Van Nostrand , New York .

Taylor, H.M. (1968). "The Economic Design of Cumulative Sum Control Charts ", *Technometrics*, Vol. 10, No. 3, 479-488.

Toda, H. (1958). "Band Score Control Charts ." *Reports of Statistical Applications Research* . Union of Japanese Scientists and Engineers ,5, 20-24.

Van Dobben de Bruyn, C.S. (1968). "Cumulative Sum Tests." Hafner, New York.

Wald, A. (1947) . "Sequential Analysis ." Wiley, New York .

_________ and Wolfowitz, J. (1945). "Sampling Inspection Plans for Continuous Production which Ensure a Prescribed Limit on the Outgoing Quality." *Annals of Mathematical Statistics*, 16, 30-49 .

Weiler, H. (1953). "The use of runs to control the mean in quality control ." *Journal of the American Statistical Association*, Vol. 48, 816-825 .

Weindling, J.I.. Littauer, S.B. and De Oliveira, J.T. (1970). "Mean action Time of the $\overline{X}$ − Control Chart with Warning Limits ". *Journal of Quality Technology* , 2, 79-85.

Woodall, W.H. (1984). "On the Markov Chain Approach to the Two-Sided CUSUM Procedure". *Technometrics*, Vol. 26, No. 1.

________ (1986). "The Design of Cusum Quality Control Charts". *Journal of Quality Technology*, 18, 99-102.

________ and Ncube, M.M. (1985). "Multivariate CUSUM Quality Control Procedures" *Technometrics*, Vol. 27, No. 3.

Yachin, E. (1985). "On the Analysis and Design of CUSUM-Shewhart Control Schemes". *IBM Journal of Research and Development*, 29, 377-391.

Appendix A

Appendix

Appendix 1 : Minimizing the ATS of the VSI Shewhart Chart

```
*************************************************************************************
*   THIS PROGRAM FINDS THE VALUES FOR D1 AND D2 SUCH THAT
*   THE ATS FUNCTION OF THE SHEWHART CHART IS MINIMIZED.
*************************************************************************************

//STEP1 EXEC FORTVCG,LIB='VPI.IMSL77.DP'
//FORT.SYSIN  DD *
      DOUBLE PRECISION A(2),B(2),X(2),WORK(25),F
      INTEGER N,NSIG,NSRCH,IWORK(2),IER
      EXTERNAL FCN
```

```
      N = 2
      NSIG = 3
      NSRCH = 12
*******************************************************************************
*     MINIMUM DELAY D1 IS DENOTED BY X(1) AND HAS LOWER BOUND A1
*     AND UPPER BOUND B1.
*     MAXIMUM DELAY D2 IS DENOTED BY X(2) AND HAS LOWER BOUND A2
*     AND UPPER BOUND B2.
*******************************************************************************
      A(1) = 0.10D0
      A(2) = 1.10D0
      B(1) = 0.90D0
      B(2) = 10.00D0
      CALL ZXMWD (FCN,N,NSIG,A,B,NSRCH,X,F,WORK,IWORK,IER)
      WRITE(6,20) X(1),X(2)
   20 FORMAT(2X,'X(1) X(2)',2F8.4)
      STOP
      END
      SUBROUTINE FCN (N,X,F)
      DOUBLE PRECISION  R,Z,F,X(N),Q0,Q1,U1
      INTEGER N
      REAL Y,P
*******************************************************************************
*     Q0 IS THE PROBABILITY OF A SIGNAL, GIVEN MU = 0
*     Q1 IS THE PROBABILITY OF A SIGNAL, GIVEN MU = U1
*     U1 IS THE SHIFT IN THE MEAN
*     F if the ATS function
*******************************************************************************
```

```
      Q0 = 0.00270D0
      Q1 = 0.002833D0
      U1 = 0.1D0
      Y = (1.0-X(1))/(X(2)-X(1))*(1.0-Q0)
      CALL MDNRIS (Y,P)
      Z = P-U1
      CALL MDNORD (Z,R)
      F = (X(1)*X(1)*(X(2)-1.0D0) + X(2)*X(2)*(1.0D0-X(1)))
     1 /(2.0D0*(X(2)-X(1)))
     1  + ((X(1)*(1.0D0-Q1) + ((X(2)-X(1))*R)))/Q1
      WRITE(6,11) F
   11 FORMAT(2X,'F=',F14.4)
      RETURN
      END
/*
//
```

Appendix 2 : Properties of the Shewhart Chart

```
**************************************************************************************
* THIS PROGRAM FINDS THE PROPERTIES OF A TWO-SIDED VSI SHEWHART CHART
**************************************************************************************
```

```
// EXEC SAS
OPTIONS LS=80;
DATA ONE;
INPUT T1-T8 @@;
D1=.1;  D2=.1;  D3=.1;  D4=1;  D5=1;
D6=1;  D7=1.9;  D8=1.9;  D9=1.9;  D=1;
DD1=D1*D1; DD2=D2*D2; DD3=D3*D3; DD4=D4*D4; DD5=D5*D5;
DD6=D6*D6; DD7=D7*D7; DD8=D8*D8; DD9=D9*D9;
DT1=D1*DD1; DT2=D2*DD2; DT3=D3*DD3; DT4=D4*DD4; DT5=D5*DD5;
DT6=D6*DD6; DT7=D7*DD7; DT8=D8*DD8; DT9=D9*DD9;
DO Z=0 , .1, .25, .5, 1, 1.5, 2, 2.5, 3, 4;
X1=T1-Z; X2=T2-Z; X3=T3-Z; X4=T4-Z; X5=T5-Z;
X6=T6-Z; X7=T7-Z; X8=T8-Z;
A=-3-Z;
B=3-Z;
Y1=-T1; Y2=-T2; Y3=-T3; Y4=-T4; Y5=-T5;
Y6=-T6; Y7=-T7; Y8=-T8;
V1=Y1-Z; V2=Y2-Z; V3=Y3-Z; V4=Y4-Z; V5=Y5-Z;
V6=Y6-Z; V7=Y7-Z; V8=Y8-Z;
********************************************************************************
* FOLLOWING VARIABLES ARE CALCULATED UNDER MU-1
********************************************************************************
P0=PROBNORM(B)-PROBNORM(A);
P1=PROBNORM(X1)-PROBNORM(V1);
P2=PROBNORM(X2)-PROBNORM(V2);
P3=PROBNORM(X3)-PROBNORM(V3);
P4=PROBNORM(X4)-PROBNORM(V4);
P5=PROBNORM(X5)-PROBNORM(V5);
```

```
P6 = PROBNORM(X6)-PROBNORM(V6);
P7 = PROBNORM(X7)-PROBNORM(V7);
P8 = PROBNORM(X8)-PROBNORM(V8);
PR = 1-P0;
RP = 1-PR;

Q9 = P8;
Q8 = P7-P8;
Q7 = P6-P7;
Q6 = P5-P6;
Q5 = P4-P5;
Q4 = P3-P4;
Q3 = P2-P3;
Q2 = P1-P2;
Q1 = P0-P1;
*******************************************************************************
* FOLLOWING VARIABLES ARE CALCULATED UNDER MU-0
*******************************************************************************
PP0 = PROBNORM(3)-PROBNORM(-3);
PP1 = PROBNORM(T1)-PROBNORM(V1);
PP2 = PROBNORM(T2)-PROBNORM(V2);
PP3 = PROBNORM(T3)-PROBNORM(V3);
PP4 = PROBNORM(T4)-PROBNORM(V4);
PP5 = PROBNORM(T5)-PROBNORM(V5);
PP6 = PROBNORM(T6)-PROBNORM(V6);
PP7 = PROBNORM(T7)-PROBNORM(V7);
PP8 = PROBNORM(T8)-PROBNORM(V8);
PPR = 1-PP0;
```

```
QQ9 = PP8;
QQ8 = PP7-PP8;
QQ7 = PP6-PP7;
QQ6 = PP5-PP6;
QQ5 = PP4-PP5;
QQ4 = PP3-PP4;
QQ3 = PP2-PP3;
QQ2 = PP1-PP2;
QQ1 = PP0-PP1;

C1 = D1*Q1 + D2*Q2 + D3*Q3 + D4*Q4 + D5*Q5 + D6*Q6 + D7*Q7 + D8*Q8 + D9*Q9;
C2 = DD1*Q1 + DD2*Q2 + DD3*Q3 + DD4*Q4 + DD5*Q5 + DD6*Q6 + DD7*Q7 +
   DD8*Q8 + DD9*Q9;
C3 = DT1*Q1 + DT2*Q2 + DT3*Q3 + DT4*Q4 + DT5*Q5 + DT6*Q6 + DT7*Q7 +
   DT8*Q8 + DT9*Q9;

CC1 = D1*QQ1 + D2*QQ2 + D3*QQ3 + D4*QQ4 + D5*QQ5
 + D6*QQ6 + D7*QQ7 + D8*QQ8 + D9*QQ9;
CC2 = DD1*QQ1 + DD2*QQ2 + DD3*QQ3 + DD4*QQ4 + DD5*QQ5
 + DD6*QQ6 + DD7*QQ7 + DD8*QQ8 + DD9*QQ9;
CC3 = DT1*QQ1 + DT2*QQ2 + DT3*QQ3 + DT4*QQ4 + DT5*QQ5
 + DT6*QQ6 + DT7*QQ7 + DT8*QQ8 + DT9*QQ9;

G1 = ((1-PR)/PR)*(C2/(1-PR) - (C1/(1-PR))*(C1/(1-PR)));
G2 = (1-PR)/(PR*PR)*(C1/(1-PR))*(C1/(1-PR));
G3 = (1/3)*(CC3/CC1)-0.25*(CC2/CC1)*(CC2/CC1);
*******************************************************************************
```

```
* THE FOLLOWING STATISTICS ARE CALCULATED UNDER MU-0
* ARLF, VARF, SDF AND CVF ARE FOR THE FSI SHEWHART CHART
* ARLV, VARV, SDV AND CVV ARE FOR THE VSI SHEWHART CHART
**********************************************************************************
ARLF=1/PR;
VARF=(1-PR)/(PR*PR);
SDF=SQRT(VARF);
CVF=SDF/ARLF;

ARLV=C1/((1-PR)*PR);
VARV=(1/PR)*(C2/RP-(C1/RP)*(C1/RP))+(RP/(PR*PR))*(C1/RP)*(C1/RP);
SDV=SQRT(VARV);
CVV=SDV/ARLV;

IF Z=0 THEN ARLV=ARLF;
DIFF=ARLF-ARLV;
**********************************************************************************
* THE FOLLOWING STATISTICS ARE CALCULATED UNDER MU-1
* AND THEY ARE THE ADJUSTED PROPERTIES OF THE FSI AND VSI SHEWHART CHARTS
**********************************************************************************
ADJF=0.5+(1-PR)/PR;
VADJF=(1/12 +(1-PR)/(PR*PR));
SADJF=SQRT(VADJF);
ACVF=SADJF/ADJF;

EY=0.5*CC2/CC1;
ADJV=EY+C1/PR;
VADJV=G1+G2+G3;
```

```
SADJV=SQRT(VADJV);
ACVV=SADJV/ADJV;

IF Z=0 THEN ADJF=ARLF;
IF Z=0 THEN ADJV=ARLF;
ADIFF=ADJF-ADJV;

LIST;

OUTPUT;
END;
CARDS;
1.58294 1.21525 0.96376 0.76196 0.58731 0.42905 0.28101 0.13903
PROC PRINT;VAR Z D1-D9 ADJF ADJV  ADIFF SADJF SADJV  ACVF ACVV
                ARLF ARLV  DIFF  SDF   SDV    CVF  CVV;
//
```

Appendix 3 : The ATS of the Shewhart Chart in the Case of a Gradual Shift

```
****************************************************************************************
* THIS PROGRAM COMPUTES THE ATS AND THE STANDARD DEVIATION OF
```

```
* THE TIME TO SIGNAL WHEN THERE IS A GRADUAL SHIFT IN THE MEAN
********************************************************************************

// EXEC SAS
PROC MATRIX;
N=120;
S=0.05;
Q0=1-(PROBNORM(3)-PROBNORM(-3));
P1=J(N,1,0);
P2=J(N,1,0);
Q=J(N,1,0);
PX=J(N,1,0);
X1=0.0; X2=0.0;
********************************************************************************
* THIS DO LOOP GENERATES THE INTERVAL PROBABILITIES FOR
* A GRADUAL LINEAR SHIFT IN THE PROCESS MEAN
********************************************************************************
        DO I=1 TO N;
        Q(I,1)=1-(PROBNORM(3-I*S)-PROBNORM(-3-I*S));
        P2(I,1)=PROBNORM(0.113610-I*S)-PROBNORM(-0.113610-I*S);
        P1(I,1)=1-Q(I,1)-P2(I,1);
        END;
PX(1,1)=0.90664;
X1=X1+PX(1,1)*Q(1,1);
X2=X2+PX(1,1)*Q(1,1);
********************************************************************************
* THIS DO LOOP GENERATES THE PROBABILITY OF SAMPLE/SIGNAL
```

```
* FOR THE FIRST D2-1 STAGES.
*****************************************************************************

          DO I=2 TO 99;
          C=I-1;
          PX(I,1)=PX(C,1)*P1(C,1);
          X1=X1+I*PX(I,1)*Q(I,1);
          X2=X2+I*I*PX(I,1)*Q(I,1);
          END;
PX(100,1)=PX(99,1)*P1(99,1)+0.090664;
X1=X1+100*PX(100,1)*Q(100,1);
X2=X2+100*100*PX(100,1)*Q(100,1);
*****************************************************************************
* THIS DO LOOP GENERATES THE PROBABILTY OF SAMPLE/SIGNAL FOR
* THE GENERAL CASE WHEN BOTH SAMPLING INTERVALS ARE POSSIBLE
*****************************************************************************
          DO I=101 TO N;
          C=I-1;
          R=I-100;
          PX(I,1)=PX(C,1)*P1(C,1)+PX(R,1)*P2(R,1);
          X1=X1+I*PX(I,1)*Q(I,1);
          X2=X2+I*I*PX(I,1)*Q(I,1);
          END;
ANSS=PX(+,1);
ATS= X1#/10;
VTS=(X2#/100-ATS*ATS);
SDTS=SQRT(VTS);
PRINT N S ANSS ATS SDTS ;
```

```
*/
//
```

Appendix 4 : The Properties of the Shewhart Chart with Runs Rules

```
*******************************************************************************
* THE FOLLOWING PROGRAM COMPUTES THE UNADJUSTED
* ATS FOR A TWO-SIDED SHEWHART CHART WITH TWO SAMPLING INTERVAL
* WHEN THE FOLLOWING RUNS RULE IS USED;
* SIGNAL IF 2 OUT OF 3 MEANS FALL BETWEEN THE 2-SIGMA LINES
* AND THE ACTION LINES.
*******************************************************************************

PROC MATRIX;
A=J(7,7,0);
D=J(7,7,1);
V=J(81,10,0);
NM=0;
DO Z=0 to 4 by 0.05;
NM =NM+1;
*******************************************************************************
```

```
* THE FOLLOWING STEPS CALCULATE THE TRANSITION PROBABILITIES.
* PR = PROBABILITY OF A SIGNAL (X EXCEEDING 3-SIGMA LIMITS)
* P1 = PROBABILITY OF INTERVAL CORRESPONDING TO D1
*    = P1U + P1L
* P2 = PROBABILITY OF INTERVAL CORRESPONDING TO D2
*******************************************************************************
PR = 1- ( PROBNORM(3-Z) - PROBNORM(-3-Z) );
P2 = PROBNORM(2-Z) - PROBNORM(-2-Z) ;
P1U= PROBNORM(3-Z) - PROBNORM( 2-Z) ;
P1L= PROBNORM(-2-Z) - PROBNORM(-3-Z) ;
P1 = P1U + P1L;

A(1,1)=P2; A(1,2)=P1U; A(1,5)=P1L;
A(2,3)=P2; A(2,4)=P1L;
A(3,1)=P2; A(3,5)=P1L;
A(4,1)=P2;
A(5,6)=P2; A(5,7)=P1U;
A(6,1)=P2; A(6,2)=P1U;
A(7,3)=P2;

D= 1.07832/
  0.1/
  0.1/
  0.1/
  0.1/
  0.1/
  0.1;
ID= I(7);
```

```
M= INV(ID-A);
M1=M(1,+);
MD=M*D;
MD1=MD(1,+);
D2=D(1,1);
ED0=(D1*P1 + D2*P2) #/ (1-PR);
ATSV=MD1 + ED0 - D2;
ATSF=M1;
E=ATSF #/ ATSV;
V(NM,1)= Z;
V(NM,2)= P1U;
V(NM,3)= P1L;
V(NM,4)= P1;
V(NM,5)= P2;
V(NM,6)= D1;
V(NM,7)= D2;
V(NM,8)= ATSF;
V(NM,9)= ATSV;
```

Appendix 5 : The Properties of the Cusum Chart

```
*******************************************************************************
* THE FOLLOWING PROGRAM COMPUTES THE ADJUSTED AND UNADJUSTED
```

```
* ATS AND THE CORRESPONDING STANDARD DEVIATION AND COEFFICIENT
* OF VARIATION OF THE TIME TO SIGNAL FOR A CUSUM CHART.
************************************************************************************

// EXEC SAS
PROC MATRIX;
V=J(19,19,0);
A=J(30,30,0) ;
N10=J(30,10,0);
N30=J(30,20,0);
B=J(30,30,0);
F=J(30,29,0);
D=J(30,1,1);
D2=J(30,1,0);
D3=J(30,1,0);
NN=J(30,30,0);
DE=J(30,1,0);
L1=J(30,1,0);
MND=J(30,30,0);
XPS=J(30,1,0);
G=J(30,30,0);
XPU=J(30,1,0);
X1=J(43,1,0);
ESV=J(30,1,0);
DS=J(30,30,0);
M=J(30,1,1);
UU=J(30,30,0);
FA=J(30,1,0);
```

```
VA = J(30,1,0);
INTERVAL = 2;
  D = 1.9622052/
    1.9622052/
    1.9622052/
    1.9622052/
    1.9622052/
    1.9622052/
    1.9622052/
    1.9622052/
    1.9622052/
    1.9622052/
    1.9622052/
    1.9622052/
    1.9622052/
    1.9622052/
    0.1/
    0.1/
    0.1/
    0.1/
    0.1/
    .1/
    .1/
    0.1/
    0.1/
    0.1/
    0.1/
    0.1/
```

```
   0.1/
   0.1/
   0.1/
   0.1;
D2 = D#D;
D3 = D2#D;
*******************************************************************************
* THE FOLLOWING STEPS CALCULATE THE AVERAGE DELAY WHICH
* MATCHES THE ATS FOR THE FIXED AND THE VARIABLE CUSUM
*******************************************************************************
*K = 0.00;
*K = 0.10;
*K = 0.25;
*K = 0.35;
*K = 0.50
*K = 0.75;
 K = 1.00;
*K = 1.25;
*K = 1.50;
NM = 0;
*******************************************************************************
* T, K AND K ARE THE PARAMETERS OF THE CUSUM CHARTS
* W IS THE WIDTH OF THE DISCRETIZED INTERVALS AS SUGGESTED BY
* BROOK AND EVANS. THE FOLLOWING DO LOOPS GENERATE THE ENTRIES
* OF THE TRANSITION MATRIX B WHEN THE PROCESS MEAN IS ZERO.
*******************************************************************************
T = 15;
*H = 29.0384;
```

```
*H = 14.1074;
*H = 8.1365;
*H = 6.376465;
*H = 4.798952;
*H = 3.346737;
 H = 2.519035;
*H = 1.987302;
*H = 1.604639;
*H = 2.519035;
W = 2*H#/(2*T-1);
DO L = 16 TO 30;
DO WW = 16 TO 30;
B(L,WW) = PROBNORM((WW-L)*W + 0.5*W + K)-PROBNORM((WW-L)*W-.5*W + K);
END;
END;
DO L = 16 TO 30;
DO WW = 2 TO 15;
SA = WW-L;
B(L,WW) = PROBNORM((SA + 0.5)*W + K)-PROBNORM((SA-0.5)*W + K);
END;
B(L,1) = PROBNORM((1-L + 0.5)*W + K);
END;
DO L = 1 TO 15;
DO WW = 1 TO 30;
B(L,WW) = B(16,WW);
END;
END;
I30 = I(30);
```

```
NB = INV(I30-B);
************************************************************************
* THE FOLLOWING STEPS ARE TO SOLVE EQUATIONS IN ORDER TO OBTAIN
* STEADY STATE PROBABILITIES WHICH ARE THEN USED TO ADJUST THE
* ATS FOR THE CUSUM. A NEW MATRIX A IS FORMED SUCH THAT EACH
* ROW SUMS UP TO UNITY.
************************************************************************
S = J(29,1,0)//J(1,1,1);
DO AA = 1 TO 30;
DO LL = 1 TO 29;
F(AA,LL) = B(AA,LL)#/B(AA,+);
END;
END;
G = F||J(30,1,1);
G = G-I(30);
G = G';
X = SOLVE(G,S);
XSS = X(+,1);
DO L = 1 TO 30;
X(L,1) = X(L,1)#/XSS;
END;
XS = X(+,1);
ID = I(30);
ASNB = NB(16,+);
NBD = NB*D;
ATSVB = NBD(16,+);
************************************************************************
*THE FOLLOWING LOOPS GENERATE THE TRANSITION PROBABILITIES WHEN
```

```
* THE MEAN SHIFTS BY THE AMOUNT Z.
*******************************************************************************
Z=0.0;
DO Z=.5 TO 5.0 BY 0.5;
IF Z=4.5   THEN Z=0.1;
IF Z=.6 THEN Z=0.25;
NM=NM+1;
DO L=16 TO 30;
DO WW=16 TO 30;
A(L,WW)=PROBNORM((WW-L)*W+0.5*W+K-Z)-PROBNORM((WW-L)*W-.5*W+K-Z);
END;
END;
DO L=16 TO 30;
DO WW=2 TO 15;
SA=WW-L;
A(L,WW)=PROBNORM((SA+0.5)*W+K-Z)-PROBNORM((SA-0.5)*W+K-Z);
END;
A(L,1)=PROBNORM((1-L+0.5)*W+K-Z);
END;
DO L=1 TO 15;
DO WW=1 TO 30;
A(L,WW)=A(16,WW);
END;
END;
*******************************************************************************
* X1 IS THE PROBABILITY THAT AN OBSERVATION FALLS IN A CERTAIN
* INTERVAL WHEN MU=MU1
*******************************************************************************
```

```
DO L=1 TO 15;
X1(L,1)=PROBNORM(((1.5-L)*W)+K-Z);
END;
DO LL=16 TO 43;
X1(LL,1)=PROBNORM(((LL-28.5)*W)+K-Z)-PROBNORM(((LL-29.5)*W)+K-Z);
END;
ID=I(30);
N=INV(ID-A);
ASN=N(16,+);
N3=N(16,3:30); N3S=N3(1,+); ASN2=0.1*N3S;
N1=N(16,1:2); N1S=N1(1,+);
D2X=(740.8-ASN2)#/N1S;
ASN1=D2X*N1S;
N1=N(1, );
XD=D'//D'//D'//D'//D'//D'//D'//D'//D'//D'//
  D'//D'//D'//D'//D'//D'//D'//D'//D'//D'//
  D'//D'//D'//D'//D'//D'//D'//D'//D'//D';
XD2=D2'//D2'//D2'//D2'//D2'//D2'//D2'//D2'//D2'//D2'//
   D2'//D2'//D2'//D2'//D2'//D2'//D2'//D2'//D2'//D2'//
   D2'//D2'//D2'//D2'//D2'//D2'//D2'//D2'//D2'//D2';
DO KL=1 TO 16;
DO L=16 TO 30;
UU(KL,L)=X1(L-15,1);
UU(KL,L-15)=UU(KL,16);
END;
END;
DO KL=17 TO 30;
DO L=16 TO 30;
```

```
AS = 29 + (KL-L);
UU(KL,L) = X1(AS,1);
UU(KL,L-15) = UU(KL,16);
END;
END;
UCSUM = UU( + , );
C = D'*X;
PD = X#D;
PC = X#/C;
CP = PD#/C;
XP = CP'//CP'//CP'//CP'//CP'//CP'//CP'//CP'//CP'//CP'//
  CP'//CP'//CP'//CP'//CP'//CP'//CP'//CP'//CP'//CP'//
  CP'//CP'//CP'//CP'//CP'//CP'//CP'//CP'//CP'//CP';
XPU = XP#UU;
XPS = XPU(  , + );
XPSS = XPS( + , );
XPSUM = XPS( + , );
PD2 = (D2#X)#/C;
DX = XD';
NDX = N#XD;NDS = NDX( , + );
USUM = UU( , + );
P2 = X#X;
PP = P2'//P2'//P2'//P2'//P2'//P2'//P2'//P2'//P2'//P2'//
  P2'//P2'//P2'//P2'//P2'//P2'//P2'//P2'//P2'//P2'//
  P2'//P2'//P2'//P2'//P2'//P2'//P2'//P2'//P2'//P2';
ND = N*D;
ND2 = N*D2;
ATS1 = ND(16, + );
```

```
NND=N#XD;*PRINT NND;
NS=X'*ND;
XD2=0.5*(X#D2)#/C;
XND=ND#XD2;
PD2=(D2#X)#/C;
DO2=D#/2;
DN=N'*D;
   ND'//ND'//ND'//ND'//ND'//ND'//ND'//ND'//ND'//ND'//
   ND'//ND'//ND'//ND'//ND'//ND'//ND'//ND'//ND'//ND';
DO L=1 TO 30;
DO KL=1 TO 15;
MND(L,KL) =0;
END;
END;
NDU=MND#UU';
NDUS=NDU( ,+);
DO L=1 TO 30;
DS(L, )=A(L, )#D(L,1);
END;
EY=0.5*(D2'*X)#/C;
XSP=XPS;
DO L=1 TO 15;
XPS(L,1)=0;
END;
ES=XPS'*ND;
EYZ=(CP#DO2)'*(NDUS);* PRINT EYZ EY ES;
COVYZ=EYZ-EY*ES;
ATS=ES+EY;
```

```
ESV = XSP#ND ;
DO L = 1 TO 15;
ESV(L,1) = 0;
END;
EVS = XSP#ND ;
ESV2 = XSP#ND2;
NN = N;
DO L =  1 TO 30;
DO KL = 2 TO 16;
NN(L,1) = NN(L,1) + NN(L,KL);
END;
END;
DO L = 1 TO 30;
DO KL = 2 TO 16;
NN(L,KL) = NN(L,1);
END;
END;
DSN = DS#NN;
DSN1 = N*DS;  DE1 = 2*DSN1*ND ; LL1 = ND2 + DE1;LL2 = ND#ND;
LL1 = LL1(1,1);LL2 = LL2(1,1);
DE = 2*DSN*EVS ;* PRINT DSN DS;
DO L = 1 TO 15;
DE(L,1) = 0;
END;
L1 = ESV2 + DE;
L2 = ESV#ESV;
VARZ = M'*(L1-L2);
VARZ1 = LL1-LL2; SD1 = SQRT(VARZ1);
```

```
CV=SD1#/ATS1;

VARY=D3#PC#/3 -(D2#PC#/2)#(D2#PC#/2);

VARY=M'*VARY;

VARYZ=VARY +VARZ + 2*COVYZ;

SDY=SQRT(VARY);

SDZ=SQRT(VARZ);

SDYZ=SQRT(VARYZ);

CVYZ=SDYZ#/ATS;

******************************************************************************

* ATS   = ADJUSTED AVERAGE TIME TO SIGNAL

* SDY   = STANDARD DEVIATION OF Y

* SDZ   = STANDARD DEVIATION OF Z

* SDYZ  = STANDARD DEVIATION OF THE ADJUSTED TIME TO SIGNAL

* CVYZ  = COVARIANCE OF THE ADJUSTED TIME TO SIGNAL

* ATS1  = UNADJUSTED AVERAGE TIME TO SIGNAL

* SD1   = STANDARD DEVIATION OF THE TIME TO SIGNAL

* CV    = COEFFICIENT OF VARIATION OF THE TIME TO SIGNAL

* ASN   = AVERAGE NUMBER OF SAMPLES TO SIGNAL

******************************************************************************

V(NM,1)=Z ;

V(NM,2)=ES;

V(NM,3)=ATS;

V(NM,4)=NS;

V(NM,5)=VARZ;

V(NM,6)=COVYZ;

V(NM,7)=SDY;

V(NM,8)=SDZ;

V(NM,9)=SDYZ;
```

```
V(NM,10)=ATSVB;
V(NM,11)=ASNB;
V(NM,12)=EY;
V(NM,13)=ASN;
V(NM,14)=ATS1;
V(NM,15)=VARZ1;
V(NM,16)=SD1;
V(NM,17)=CV;
V(NM,18)=CVYZ;
V(NM,19)=INTERVAL;
END;
OUTPUT V OUT=STAT;
PROC PRINT;  VAR COL1-COL19;
```

The vita has been removed from the scanned document