Multiparameter BC_n-Kostka-Foulkes Polynomials

Ben Goodberry

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Daniel Orr, Chair
Nicholas A. Loehr
Mark Shimozono

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(ABSTRACT)

The Kostka-Foulkes polynomials describe the change of basis between Schur polynomials and Hall-Littlewood polynomials. In this paper, we extend this idea to the family of BC_n Macdonald spherical functions, with multiparameter Kostka-Foulkes polynomials acting as the change of basis from the BC_n spherical functions to the type C_n Schur polynomials. We develop a Kato-Lusztig formula that describes the multiparameter BC_n-Kostka-Foulkes polynomials.

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(GENERAL AUDIENCE ABSTRACT)

The work done in [11] gives a formula to calculate Kostka-Foulkes polynomials that convert between two other forms of polynomials. However, this only applies in specific instances. In this paper, we generalize those ideas to allow for more parameters, and find that a similar formula still holds.

Contents

Chapter 1

Introduction

Classically, the change of basis polynomials between Macdonald spherical functions and Schur functions for reduced irreducible root systems are given by the Kostka-Foulkes polynomials. In [5], Lascoux and Schützenberger give a combinatorial formula for the Kostka-Foulkes polynomials in the type A_n case, but no such formula has been found for other types. And for reduced root systems of any type, the Kostka-Foulkes polynomials are known to be positive, which was proved first in [8]. Additionally, recent work by and Lecouvey and Lenart in [6] contains combinatorial formulas for certain type C_n Kostka-Foulkes polynomials with equal parameters.

Here we consider a root system that is not reduced, namely the type BC_n root system, and find a similar result to the Kato-Lusztig formula for reduced root system Kostka-Foulkes polynomials. To reach this equation, we build the necessary structure for the type BC_n root system, an extended affine Hecke algebra, then the related Macdonald spherical functions and multiparameter BC_n-Kostka-Foulkes polynomials. We will then give several inner product relations that allow us to construct a Kato-Lusztig formula for the polynomials. In the final chapter, we give some examples using this equation.

In order to prove the formula for BC_n-Kostka-Foulkes polynomials, we follow the work of Nelsen and Ram in [11] closely. The structure of the proofs given here is similar to their work, though there are some substantial technical differences caused by the non-reduced root system. The majority of these differences are found in the proofs of Chapter 4.

Chapter 2

Root Systems & Weyl Groups

In this chapter, we will define the objects necessary to construct the affine Hecke algebra. We will also discuss the type B_n, C_n, and BC_n root systems which will be the systems used in the remaining chapters. Here we follow the construction of root systems in [3].

2.1 Root Systems

Begin with a real vector space V equipped with a symmetric, nondegenerate, bilinear form $\langle , \rangle$. Then a reduced root system R is a subset of V, whose elements $\alpha \in R$ are roots. For each root α, define a reflection s_α that sends α to $-\alpha$ and fixes the hyperplane $H_\alpha = \{\beta \in V \mid \langle \beta, \alpha \rangle = 0\}$. This set is a reduced root system if it satisfies the following axioms:

1. R is finite, $0 \notin R$, and $V = \mathbb{R}\text{-span}(R)$.
2. If $\alpha \in R$, the only multiples of α contained in R are $\pm\alpha$.
3. If $\alpha \in R$, then the set R is invariant under the reflection s_α.
4. If $\alpha, \beta \in R$, then $\beta - \dfrac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha \in \mathbb{Z}$.

We will also consider root systems that are not reduced, which follow the above axioms excluding 2. Additionally, we have a coroot $\alpha^\vee$ for each root α, defined by $\alpha^\vee = \dfrac{2\alpha}{\langle \alpha, \alpha \rangle}$. This gives us the weight lattice P:

$$P = \{\beta \in V \mid \langle \beta, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in R\}.$$

And we also get the Weyl group:

$$W = \langle s_\alpha \mid \alpha \in R \rangle.$$

From the axioms, this means that for any root system, W is finite, and also the action of any $w \in W$ permutes the roots in R.

We now find the positive roots R^+ in R, defined together with the fundamental chamber C. Choosing one defines the other uniquely:

$$R^+ = \{\alpha \in R \mid \langle \beta, \alpha^\vee \rangle > 0 \text{ for all } \beta \in C\}$$

$$C = \{\beta \in V \mid \langle \beta, \alpha^\vee \rangle > 0 \text{ for all } \alpha \in R^+\}.$$

The simple roots are defined as the roots in R^+ that cannot be written as a sum of two other roots in R^+. These form a basis for V. We also define the set of dominant integral weights P^+ given by

$$P^+ = \{\beta \in V \mid \langle \beta, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in R^+\}.$$

There is one dominant integral weight which will appear often in later computations, namely

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha.$$

We can now define a partial order on the weight lattice P, called the dominance order, where

$$\lambda \leq \mu \quad \text{if} \quad \mu = \lambda + \sum_{i=1}^{n} \alpha_i \quad \text{for some} \quad \alpha_i \in R^+.$$

The Weyl group elements can also be given lengths in the following way. Define

$$R^- = \{-\alpha \mid \alpha \in R^+\}.$$

Then from this, if there is a Weyl group element $w \in W$, we can define the inversion set of w and its length respectively to be:

$$\operatorname{Inv}(w) = \{\alpha \in R^+ \mid w\alpha \in R^-\} \qquad\qquad \ell(w) = |\operatorname{Inv}(w)|.$$

Finally, we can distinguish between the lengths of roots. The length of a root is simply its length as a vector in the underlying vector space V. In root systems of type B_n or C_n as we will discuss later, there are two root lengths, so we will distinguish roots as either "short" or "long" roots.

In the remainder of the paper, we will focus on the type B_n, C_n, and BC_n root systems, which will be used to define our affine Hecke algebra.

2.2 B_n and C_n Root Systems

We begin by defining the type B_n system. Begin with the vector space $V = \mathbb{R}^n$. If the standard basis vectors for V are written as $\{e_i\}_{i=1}^n$, then if we let $i, j \in \{1, \ldots, n\}$, the root system R in type B_n is given by

$$R = \{\pm e_i \pm e_j \mid i < j\} \cup \{\pm e_i\}.$$

Then the canonical choice for the positive roots is

$$R^+ = \{e_i \pm e_j \mid i < j\} \cup \{e_i\}.$$

The roots in this case are of length 1 and $\sqrt{2}$ in $\mathbb{R}^n$, which gives us that

$$\{\pm e_i\} \text{ are short roots, and } \{\pm e_i \pm e_j \mid i < j\} \text{ are long roots.}$$

Our weight lattice and dominant integral weights are then:

$$P = \mathbb{Z}\text{- span}\left\{\sum_{i=1}^{k} e_i \mid 1 \leq k < n\right\} \cup \left\{\frac{1}{2}\sum_{i=1}^{n} e_i\right\}$$

$$P^+ = \mathbb{Z}_{\geq 0}\text{- span}\left\{\sum_{i=1}^{k} e_i \mid 1 \leq k < n\right\} \cup \left\{\frac{1}{2}\sum_{i=1}^{n} e_i\right\}.$$

Now we define the type C_n system, which has a very similar structure to type B_n. Taking the same vector space $\mathbb{R}^n$, and again for $i, j \in \{1, \ldots, n\}$, we obtain a root system for C_n:

$$R = \{\pm e_i \pm e_j \mid i < j\} \cup \{\pm 2e_i\}.$$

The choice for the positive roots is again similar,

$$R^+ = \{e_i \pm e_j \mid i < j\} \cup \{2e_i\}.$$

This gives that the roots are of lengths $\sqrt{2}$ and 2 respectively in the above union, separating our roots into lengths,

$$\{\pm e_i \pm e_j \mid i < j\} \text{ are short roots, and} \{\pm 2e_i\} \text{ are long roots.}$$

Notice that the long roots in type B_n are exactly the short roots in type C_n, and that the long roots in C_n are twice the short roots in type B_n. The weight lattice in type C_n is simply $\mathbb{Z}^n$, and finally, we get the set of dominant integral weights:

$$P^+ = \mathbb{Z}_{\geq 0}\text{- span}\left\{\sum_{i=1}^{k} e_i \mid 1 \leq k \leq n\right\}.$$

2.3 The BC_n Root System

The primary root system of interest in this paper is the type BC_n root system. It is defined as:

$$R = \{\pm e_i \pm e_j \mid i < j\} \cup \{\pm e_i\} \cup \{\pm 2e_i\} \text{ for } i, j \in \{1, \ldots, n\}.$$

So R is the union of the root systems of types B_n and C_n. This is not a reduced root system, but still satisfies axioms 1, 3, and 4. The positive roots are:

$$R^+ = \{e_i \pm e_j \mid i < j\} \cup \{e_i\} \cup \{2e_i\}.$$

For clarity, we will write this as $R^+(BC_n)$ to distinguish from the positive type B_n or C_n roots. The weight lattice and set of dominant integral weights coincide with the type B_n case. Here we will take $\rho = (n, n-1, \ldots, 1)$, the type C_n value for ρ.

The Weyl group is the same in types B_n, C_n, and BC_n. For an element $w \in W$, and for positive type B_n roots R^+, the inversions of w can be written as a disjoint union as follows:

$$\mathrm{Inv}_s(w) = \{\alpha \in R^+ \mid w\alpha = -\alpha \text{ and } \alpha \text{ is a short root}\}$$

$$\mathrm{Inv}_\ell(w) = \{\alpha \in R^+ \mid w\alpha = -\alpha \text{ and } \alpha \text{ is a long root}\}$$

Additionally, the weight lattice of the BC_n root system is the same as the type C_n weight lattice. The same is true of the dominant integral weights. We will denote the weight lattice and dominant integral weights as $P(BC_n)$ and $P^+(BC_n)$ respectively.

There is a dominance partial order on weights in the type BC_n weight lattice, which is the same as the order on type B_n weights. The relation $\leq$ will refer to the BC_n dominance partial order unless otherwise noted.

Chapter 3

Affine Hecke Algebra

We now construct the affine Hecke algebra that contains the polynomials which form the basis for the remainder of the paper.

3.1 Affine Hecke algebra definition

Let the ring $\mathbb{K}$ be defined by $\mathbb{K} = \mathbb{Q}[v_l^{\pm 1}, v_d^{\pm 1}, v_s^{\pm 1}]$ for some variable v_l, v_d, v_s. For roots α_i, α_j in a given root system, let m_{ij} be the integer such that π/m_{ij} is the angle between the hyperplanes H_{α_i} and H_{α_j}. Then we define the extended affine Hecke algebra of type BC_n, H, as the algebra over $\mathbb{K}$ with generators T_i, $i \in \{1, \ldots, n\}$, and x^λ for $\lambda \in P$, whose relations are given by:

$$T_iT_jT_i \cdots = T_jT_iT_j \ldots \text{ where each side has } m_{ij} \text{ elements,}$$

$$x^\lambda x^\mu = x^\mu x^\lambda = x^{\lambda+\mu},$$

the quadratic relations,

$$(T_i - v_l)(T_i + v_l^{-1}) = 0 \quad \text{if} \quad 1 \leq i < n,$$

$$(T_n - v_s)(T_n + v_s^{-1}) = 0,$$

and the Bernstein relations,

$$x^\lambda T_i = T_i x^{s_i\lambda} + (v_l - v_l^{-1})\frac{x^\lambda - x^{s_i\lambda}}{1 - x^{-\alpha_i}} \quad \text{if} \quad 1 \leq i < n,$$

$$x^\lambda T_n = T_n x^{s_n\lambda} + \frac{(v_s - v_s^{-1}) + (v_d - v_d^{-1})x^{-\alpha_n}}{1 - x^{-2\alpha_n}}(x^\lambda - x^{s_n\lambda}),$$

where $\alpha_n = e_n$, the simple root of type B_n. Then from H we define parameters t, a, and b:

$$t = v_l^2 \qquad a = v_s v_d \qquad b = -v_s/v_d.$$

3.2 Elements in the affine Hecke algebra

There are many elements in the affine Hecke algebra that are necessary to define the multi-parameter BC_n-Kostka-Foulkes polynomials and prove their formula.

First we define the stabilizer of $\mu \in P$, denoted W_μ, by

$$W_\mu = \{w \in W \mid w\mu = \mu\},$$

and the orbit of μ, written $W\mu$, is

$$W\mu = \{w\mu \mid w \in W\}.$$

Next, we define the term $W_\mu(t, a, b)$,

$$W_\mu(t, a, b) = \sum_{w \in W_\mu} t^{|\mathrm{Inv}_\ell(w)|}(-ab)^{|\mathrm{Inv}_s(w)|}.$$

We can now define the monomial symmetric polynomials m_λ and alternating polynomials a_λ:

$$m_\lambda = \sum_{\gamma \in W\lambda} x^\gamma \qquad\qquad a_\lambda = \sum_{w \in W} (-1)^{\ell(w)} x^{w\lambda}.$$

And from a_λ we construct the Schur polynomials of type C_n, s_λ,

$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho},$$

where $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$ as defined before. It should be noted that the denominator a_ρ can also be found via the Weyl denominator formula (see, e.g., [11]), which will be used in later substitutions:

$$a_\rho = x^\rho \prod_{\alpha \in R^+} (1 - x^{-\alpha}).$$

From [11], we also take the following straightening law for Schur functions. If we take $\mu \in P$ and $w \in W$, then:

$$(-1)^{\ell(w)} a_\mu = a_{w(\mu)}.$$

For the elements x^μ in the affine Hecke algebra, we will generally write $x^\mu = x_1^{\mu_1} x_2^{\mu_2} \ldots x_k^{\mu_k}$ with to indicate the element x^μ where $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \in \mathbb{Z}^n$. Specifically, then, the values $\{x^\alpha\}$ where α are the positive roots in C_n would be written as

$$\{x_i x_j^{\pm 1} \mid i < j\} \cup \{x_i^2\}, \qquad \text{where} \qquad i, j \in \{1, \ldots, n\}.$$

Additionally, if there is a sum or product of some roots, a power written as ± 1 indicates the product should be taken twice, once where the term is positive, and once where the term is negative. For example:

$$\prod_{i<j} (1 - t x_i x_j^{\pm 1}) = \prod_{i<j} (1 - t x_i x_j)(1 - t x_i x_j^{-1}).$$

3.3 Multiparameter BC_n-Kostka-Foulkes polynomials

We first define a family of polynomials $P_\lambda(x;t,a,b)$, the BC_n Macdonald spherical functions, which will be central to the remainder of the paper. Taking W to be the Weyl group for a type C_n root system, for any weight $\lambda \in P(BC_n)$, define $P_\lambda(x;t,a,b)$ as:

$$P_\lambda(x;t,a,b) = \frac{1}{W_\lambda(t,a,b)} \sum_{w\in W} w\left(x^\lambda \prod_{i<j} \frac{1-tx_i^{-1}x_j^{\pm 1}}{1-x_i^{-1}x_j^{\pm 1}} \prod_i \frac{(1-ax_i^{-1})(1-bx_i^{-1})}{1-x_i^{-2}} \right).$$

Then for the Schur function s_λ and Macdonald spherical function $P_\mu(x;t,a,b)$, for dominant integral weights $P^+(BC_n)$, define a change of basis $K_{\lambda\mu}(t,a,b)$ satisfying:

$$s_\lambda = \sum_{\mu\in P^+} K_{\lambda\mu}(t,a,b) P_\mu(x;t,a,b).$$

We call these $K_{\lambda\mu}(t,a,b)$ the multiparameter BC_n-Kostka-Foulkes polynomials. For certain relations between t, a, and b, we can use the above formulas to recover equal and unequal parameter Macdonald spherical functions and Kostka-Foulkes polynomials in types B_n and C_n, which will be discussed in detail in chapter 6.

Where it is possible, we will write $K_{\lambda\mu}$ and P_λ in place of $K_{\lambda\mu}(t,a,b)$ and $P_\lambda(x;t,a,b)$ respectively.

Chapter 4

Changes of Basis

4.1 Lower triangular expansion

We begin with four lemmas that will be used to show that the expansion of $P_\lambda(x;t,a,b)$ into s_μ is a lower triangular matrix with diagonal entries 1 and each entry in the matrix contained in $\mathbb{Z}[t,a,b]$.

Lemma 4.1. *If $\lambda \in P^+$ and $v \in W$, then $\lambda - v^{-1}(\lambda) \geq 0$.*

Proof. See Lemma 1.7 in [12]. ■

Lemma 4.2. *For any $v \in W$, there exists a bijection $\varphi : \mathrm{Inv}(v) \to \mathrm{Inv}(v^{-1})$ given by the map $\varphi(\alpha) = -v(\alpha)$.*

Proof. Let $v \in W$, and let $\alpha \in \mathrm{Inv}(v)$. By definition, $\alpha \in R^+$ and $v(\alpha) \in R^-$. Thus $-v(\alpha) \in R^+$. Then we get that

$$v^{-1}(-v(\alpha)) = -v^{-1}(v(\alpha)) = -\alpha \in R^-,$$

which means $-v(\alpha) \in \mathrm{Inv}(v^{-1})$. The inverse of this map is given by $\varphi(\alpha) = -v^{-1}(\alpha)$, which satisfies the same arguments as above by considering v^{-1} as the original element in W. ■

Lemma 4.3. *For a root system R^+ and any $v \in W$, we have the equality:*

$$\rho - v^{-1}(\rho) = \sum_{\alpha \in \mathrm{Inv}(v)} \alpha.$$

Proof. Applying the bijection from Lemma 4.2, we have the following equalities:

$$\begin{aligned}
\rho - v^{-1}(\rho) &= \frac{1}{2}\sum_{\alpha\in R^+}\alpha - v^{-1}\left(\frac{1}{2}\sum_{\alpha\in R^+}\alpha\right)\\
&= \frac{1}{2}\sum_{\alpha\in R^+}\alpha - \frac{1}{2}\sum_{\alpha\in R^+}v^{-1}(\alpha)\\
&= \frac{1}{2}\sum_{\alpha\in R^+}\alpha - \frac{1}{2}\left(\sum_{\alpha\in R^+\setminus\mathrm{Inv}(v^{-1})}v^{-1}(\alpha) + \sum_{\alpha\in\mathrm{Inv}(v^{-1})}v^{-1}(\alpha)\right),
\end{aligned}$$

and since v^{-1} permutes the positive roots in $R^+\setminus\mathrm{Inv}(v^{-1})$,

$$\begin{aligned}
&= \frac{1}{2}\sum_{\alpha\in R^+}\alpha - \frac{1}{2}\left(\sum_{\alpha\in R^+\setminus\mathrm{Inv}(v^{-1})}\alpha + \sum_{\alpha\in\mathrm{Inv}(v^{-1})}v^{-1}(\alpha)\right)\\
&= \frac{1}{2}\sum_{\alpha\in R^+}\alpha - \frac{1}{2}\left(\sum_{\alpha\in R^+}\alpha - \sum_{\alpha\in\mathrm{Inv}(v^{-1})}\alpha + \sum_{\alpha\in\mathrm{Inv}(v^{-1})}v^{-1}(\alpha)\right)\\
&= \frac{1}{2}\left(\sum_{\alpha\in\mathrm{Inv}(v^{-1})}\alpha - \sum_{\alpha\in\mathrm{Inv}(v^{-1})}v^{-1}(\alpha)\right)\\
&= \frac{1}{2}\left(\sum_{\alpha\in\mathrm{Inv}(v^{-1})}-v^{-1}(\alpha) - \sum_{\alpha\in\mathrm{Inv}(v^{-1})}v^{-1}(\alpha)\right)\\
&= \sum_{\alpha\in\mathrm{Inv}(v^{-1})}-v^{-1}(\alpha)\\
&= \sum_{\alpha\in\mathrm{Inv}(v)}\alpha.
\end{aligned}$$

■

Lemma 4.4. *For a subset $E \subseteq R^+(BC_n)$, if E contains at most one of β and $\beta/2$ for each long type C_n root β, and $\sum_{\alpha\in E}\alpha = \sum_{\alpha\in\mathrm{Inv}(v)}\alpha$, then $E = \mathrm{Inv}(v)$ where $\mathrm{Inv}(v)$ is a subset of positive type C_n roots.*

Proof. Macdonald proves in Lemma 2.14 of [9] that the statement is true if E only contains type C_n roots. So it is sufficient to show that if the sums are equal, E cannot contain any short type B_n roots. We follow a similar process to that in [9] and proceed by induction on $\ell(v)$.

Assume $E \subseteq R^+(BC_n)$ with $\sum_{\alpha\in E} \alpha = \sum_{\alpha\in \mathrm{Inv}(v)} \alpha$. If $\ell(v) = 0$, then $v = 1$ and so $E = \mathrm{Inv}(v) = \emptyset$. So assume $\ell(v) > 0$ and that the result holds for all $w \in W$ where $\ell(w) = \ell(v) - 1$. Let $B \subseteq R^+(BC_n)$ be the set of simple type C_n roots. We can then write $v = v's_\beta$ for some $\beta \in B$, where $\ell(v') = \ell(v) - 1$. Note that $\beta \notin \mathrm{Inv}(v')$ since otherwise we would not have $\ell(v') < \ell(v)$.

Since Macdonald's proof applies when $\beta \in E$ and when neither β nor $\beta/2$ is in E for a long type C_n root β, we are left to prove the induction holds when $\beta/2 \in E$. So assume $\beta/2 \in E$, and then by assumption, we have that $\beta \notin E$. We also know that:

$$\mathrm{Inv}(v) = s_\beta \,\mathrm{Inv}(v') \cup \{\beta\}.$$

So if we define a set $E' = \{\beta/2\} \cup s_\beta(E\backslash\{\beta/2\})$, this is a subset of $R^+(BC_n)$. Denote $|E| = \sum_{\alpha\in E} \alpha$ for any set $E \subseteq R^+(BC_n)$. Then the following holds:

$$\begin{aligned}
|E'| &= \frac{\beta}{2} + s_\beta \left(|E| - \frac{\beta}{2} \right) \\
&= \frac{\beta}{2} + s_\beta \left(|\mathrm{Inv}(v)| - \frac{\beta}{2} \right) \\
&= \frac{\beta}{2} + s_\beta \left(|s_\beta \,\mathrm{Inv}(v') \cup \{\beta\}| - \frac{\beta}{2} \right) \\
&= \frac{\beta}{2} + s_\beta \left(s_\beta |\mathrm{Inv}(v')| + \beta - \frac{\beta}{2} \right) \\
&= \frac{\beta}{2} + |\mathrm{Inv}(v')| - \frac{\beta}{2} \\
&= |\mathrm{Inv}(v')|.
\end{aligned}$$

Because $\ell(v') = \ell(v) - 1$, the inductive hypothesis gives us that $E' = \mathrm{Inv}(v')$. However, $\beta/2 \in E'$ but $\beta/2 \notin \mathrm{Inv}(v')$, and thus we've reached a contradiction. Hence $\beta/2 \notin E$, so E contains only type C_n roots, and the result follows. ■

Define commuting variables $t_{e\pm e_j}, a_{e_i}$, and b_{e_i} indexed by the associated positive roots of type B_n. Then if $\lambda \in P^+(BC_n)$, define the following:

$$R_\lambda(x; t_{e_i\pm e_j}, a_{e_i}, b_{e_i}) = \sum_{w\in W} w \left(x^\lambda \prod_{i<j} \frac{1 - t_{e_i\pm e_j} x_i^{-1} x_j^{\mp 1}}{1 - x_i^{-1} x_j^{\mp 1}} \prod_i \frac{(1 - a_{e_i} x_i^{-1})(1 - b_{e_i} x_i^{-1})}{1 - x_i^{-2}} \right).$$

We will write this simply as R_λ. And for $\mathrm{Inv}_s(w)$ and $\mathrm{Inv}_\ell(w)$ as defined in chapter 2.2, define:

$$W_\lambda(t_\alpha, a_\beta, b_\beta) = \sum_{w\in W_\lambda} \left(\prod_{\alpha\in \mathrm{Inv}_\ell(w)} t_\alpha \prod_{\beta\in \mathrm{Inv}_s(w)} (-a_\beta b_\beta) \right).$$

Theorem 4.1. *With R_λ as defined above, and for $\lambda \in P^+(BC_n)$,*

(a) $R_\lambda = \sum_{\mu \in P^+(BC_n)} u_{\lambda\mu} s_\mu$ *with* $u_{\lambda\mu} \in \mathbb{Z}[t_{e_i \pm e_j}, a_{e_i}, b_{e_i}]$, $u_{\lambda\mu} = 0$ *except when* $\mu \leq \lambda$, *and* $u_{\lambda\lambda} = W_\lambda(t_\alpha, a_\beta, b_\beta)$.

(b) $P_\lambda = \sum_{\mu \in P^+(BC_n)} c_{\lambda\mu} s_\mu$ *where* $c_{\lambda\mu} \in \mathbb{Z}[t, a, b]$, $c_{\lambda\mu} = 0$ *except when* $\mu \leq \lambda$, *and* $c_{\lambda\lambda} = 1$.

Proof. (a) First for any $E \subseteq R^+(BC_n)$, we define $\alpha_E = \sum_{\alpha \in E} \alpha$. Recall that ρ is the weight from type C_n. Then we manipulate R_λ:

$$
\begin{aligned}
R_\lambda &= \sum_{w \in W} w \left(x^\lambda \prod_{i<j} \frac{1 - t_{e_i \pm e_j} x_i^{-1} x_j^{\mp 1}}{1 - x_i^{-1} x_j^{\mp 1}} \prod_i \frac{(1 - a_{e_i} x_i^{-1})(1 - b_{e_i} x_i^{-1})}{1 - x_i^{-2}} \right) \\
&= \sum_{w \in W} w \left(x^{\lambda+\rho} \frac{1}{x^\rho} \prod_{i<j} \frac{1 - t_{e_i \pm e_j} x_i^{-1} x_j^{\mp 1}}{1 - x_i^{-1} x_j^{\mp 1}} \prod_i \frac{(1 - a_{e_i} x_i^{-1})(1 - b_{e_i} x_i^{-1})}{1 - x_i^{-2}} \right) \\
&= \frac{1}{a_\rho} \sum_{w \in W} (-1)^{\ell(w)} w \left(x^{\lambda+\rho} \prod_{i<j} (1 - t_{e_i \pm e_j} x_i^{-1} x_j^{\mp 1}) \prod_i (1 - a_{e_i} x_i^{-1})(1 - b_{e_i} x_i^{-1}) \right) \\
&= \frac{1}{a_\rho} \sum_{w \in W} (-1)^{\ell(w)} w \left(x^{\lambda+\rho} \prod_{i<j} (1 - t_{e_i \pm e_j} x_i^{-1} x_j^{\mp 1}) \prod_i (1 - (a_{e_i} + b_{e_i}) x_i^{-1} + a_{e_i} b_{e_i} x_i^{-2}) \right) \\
&= \frac{1}{a_\rho} \sum_{w \in W} (-1)^{\ell(w)} w \left(x^{\lambda+\rho} \sum_{E \subseteq R^+(BC_n)} f_E x^{-\alpha_E} \right).
\end{aligned}
$$

Here $f_E \in \mathbb{Z}[t_\alpha, a_\beta, b_\beta]$ is a constant we will calculate explicitly later. Note that in this sum, for any subset $E \subseteq R^+(BC_n)$, in the roots of type e_i and $2e_i$, at most one of e_i and $2e_i$ can be in E for any particular i, since only one of the terms in the product,

$$\prod_i (1 - (a_{e_i} + b_{e_i}) x_i^{-1} + a_{e_i} b_{e_i} x_i^{-2}),$$

can be chosen for any given i. We rewrite R_λ further:

$$\begin{aligned} R_\lambda &= \frac{1}{a_\rho} \sum_{w \in W} (-1)^{\ell(w)} w \left(x^{\lambda+\rho} \sum_{E \subseteq R^+(BC_n)} f_E x^{-\alpha_E} \right) \\ &= \frac{1}{a_\rho} \sum_{E \subseteq R^+(BC_n)} f_E \sum_{w \in W} (-1)^{\ell(w)} w(x^{\lambda+\rho-\alpha_E}) \\ &= \frac{1}{a_\rho} \sum_{E \subseteq R^+(BC_n)} f_E a_{\lambda+\rho-\alpha_E} \end{aligned}$$

So by the straightening law for a_λ as proved in [11], either $a_{\lambda+\rho-\alpha_E} = 0$ or $a_{\lambda+\rho-\alpha_E} = (-1)^{\ell(v)} a_{\mu+\rho}$ for some $v \in W$ and $\mu \in P^+$, which satisfy $\mu + \rho = v^{-1}(\lambda + \rho - \alpha_E)$. We now use this fact to show that the only case where a term is nonzero is when $\mu \leq \lambda$ in the type BC_n dominance order.

We find that the following inequalities are equivalent to $\mu \leq \lambda$:

$$\begin{aligned} \lambda - v^{-1}(\lambda) + \rho - v^{-1}(\rho) + v^{-1}(\alpha_E) &\geq 0 \\ \lambda + \rho - v^{-1}(\lambda + \rho - \alpha_E) &\geq 0 \\ (\lambda + \rho) - (\mu + \rho) &\geq 0 \\ \lambda + \rho &\geq \mu + \rho \end{aligned}$$

Using this equivalence, we will show that $\lambda - v^{-1}(\lambda) \geq 0$ and $\rho - v^{-1}(\rho) + v^{-1}(\alpha_E) \geq 0$, which is sufficient to prove that $\mu \leq \lambda$. From Lemma 4.1, we have already that $\lambda - v^{-1}(\lambda) \geq 0$, so it remains to show that for any $E \subseteq R^+(BC_n)$, the second inequality holds.

Fix a set $E \subseteq R^+(BC_n)$. We wish to show for any $\beta \in E$, where β is either a type C_n root or some root of the form e_i, that if $v^{-1}(\beta) < 0$, then there is an associated term in $\rho - v^{-1}(\rho)$ that will cancel the negative $v^{-1}(\beta)$. Note again that at most one of e_i and $2e_i$ is contained in E, and since v permutes the roots of type C_n, we can consider the nonnegativity of $\rho - v^{-1}(\rho) + v^{-1}(\beta)$ separately for each element in $R^+(BC_n)$. We then split into two cases.

Case 1: Suppose β is a type C_n root. Then either $v^{-1}(\beta) < 0$ or $v^{-1}(\beta) > 0$. If the latter holds, we are done. And if $v^{-1}(\beta) < 0$, then $\beta \in \mathrm{Inv}(v^{-1})$. So by the bijection in Lemma 4.2, $-v^{-1}(\beta) \in \mathrm{Inv}(v)$. Recall from Lemma 4.3 that $\rho - v^{-1}(\rho) = \sum_{\alpha \in \mathrm{Inv}(v)} \alpha$. So $-v^{-1}(\beta)$ appears in a term in the sum $\sum_{\alpha \in \mathrm{Inv}(v)} \alpha$, with $-v^{-1}(\beta) + v^{-1}(\beta) = 0$.

Case 2: Suppose $\beta = e_i$, a short type B_n root. We know already that $2e_i \notin E$. So similarly to case 1, if $v^{-1}(\beta) > 0$, then we are done, and if $v^{-1}(e_i) < 0$, then $2e_i \in \mathrm{Inv}(v^{-1})$, or equivalently, $-v^{-1}(2e_i) \in \mathrm{Inv}(v)$. Then the sum $\rho - v^{-1}(\rho)$ contains the term $-v^{-1}(2e_i)$ which causes $-v^{-1}(2e_i) + v^{-1}(e_i) = -v^{-1}(e_i) > 0$. And because $2e_i \notin E$, this positive root is accounted for uniquely in the sum.

Because each term in the sum $\rho - v^{-1}(\rho) + v^{-1}(\alpha_E)$ is nonnegative, their sum must also be nonnegative. Hence we conclude that in the expansion $R_\lambda = \sum_{\mu \in P^+(BC_n)} u_{\lambda\mu} s_\mu$, the coefficients $u_{\lambda\mu} = 0$ unless $\mu \leq \lambda$.

Using the previous inequalities, we can find conditions that occur if $\mu = \lambda$, which will allow us to compute $u_{\lambda\lambda}$ explicitly. First, since $\lambda - v^{-1}(\lambda) \geq 0$, equality holds exactly when $v^{-1}(\lambda) = \lambda$, in which case $v \in W_\lambda$. We also need that $\rho - v^{-1}(\rho - \alpha_E) = 0$, which we use to compute the set E as follows:

$$\begin{aligned} \rho - v^{-1}(\rho - \alpha_E) &= 0 \\ \rho &= v^{-1}(\rho - \alpha_E) \\ v(\rho) &= \rho - \alpha_E \\ \rho - v^{-1}(\rho) &= \alpha_E \\ \sum_{\alpha \in \mathrm{Inv}(v)} \alpha &= \alpha_E. \end{aligned}$$

The final equality holds by Lemma 4.3, where $\mathrm{Inv}(v)$ is a subset of type C_n roots since ρ is the type C_n constant. And by Lemma 4.4, we get that $E = \mathrm{Inv}(v)$. In particular, note that if $\lambda = \mu$, the set E cannot contain any short roots of type B_n, as the set $\mathrm{Inv}(v)$ contains only type C_n inversions.

To compute the coefficients $u_{\lambda\lambda}$, we partition the set $E \subseteq R^+(BC_n)$ into disjoint sets, $E = A \sqcup B \sqcup C$ in the following way:

$$\begin{aligned} A &\subseteq \{\alpha \in R^+ \mid \alpha = x_i x_j^{\pm 1}, i < j\}, \\ B &\subseteq \{\alpha \in R^+ \mid \alpha = x_i\}, \\ C &\subseteq \{\alpha \in R^+ \mid \alpha = x_i^2\}. \end{aligned}$$

Additionally, define the following constants:

$$\begin{aligned} t_A &= \prod_{\alpha \in A} t_\alpha \\ (a+b)_B &= \prod_{\beta \in B} (a_\beta + b_\beta) \\ (ab)_C &= \prod_{\gamma \in C} (a_{\gamma/2} b_{\gamma/2}). \end{aligned}$$

Then using the previously defined f_E, and splitting E into disjoint pieces $E = A \sqcup B \sqcup C$:

$$\begin{aligned}\sum_{E\subseteq R^+(BC_n)} f_E x^{-\alpha_E} &= \prod_{i<j}(1 - t_{e_i\pm e_j}x_i^{-1}x_j^{\mp 1})\prod_i(1-(a_{e_i}+b_{e_i})x_i^{-1}+a_{e_i}b_{e_i}x_i^{-2}) \\ &= \sum_{E\subseteq R^+(BC_n)}(-1)^{|A|}t_A(-1)^{|B|}(a+b)_B(ab)_C x^{-\alpha_E} \\ f_E &= (-1)^{|A|}t_A(-1)^{|B|}(a+b)_B(ab)_C.\end{aligned}$$

Since only type C_n roots appear when $\lambda = \mu$, we get that E contains no roots in the set B, in which case,

$$\begin{aligned}f_E &= (-1)^{|A|}t_A(ab)_C \\ &= (-1)^{|A|+|C|}t_A(-ab)_C \\ &= (-1)^{\ell(v)}t_A(-ab)_C.\end{aligned}$$

Thus our coefficient for s_λ in the expansion is the sum of the terms f_E when $v \in W_\lambda$ and $E = \mathrm{Inv}(v)$, so we rewrite R_λ once more:

$$\begin{aligned}R_\lambda &= \frac{1}{a_\rho}\sum_{E\subseteq R^+(BC_n)} f_E a_{\lambda+\rho-\alpha_E} \\ &= \frac{1}{a_\rho}\sum_{E\subseteq R^+(BC_n)} f_E(-1)^{\ell(v)}a_{\mu+\rho} \\ &= \sum_{E\subseteq R^+(BC_n)} f_E(-1)^{\ell(v)}s_\mu.\end{aligned}$$

And using our previous results, if $\lambda = \mu$, the $(-1)^{\ell(v)}$ in the sum cancels with the same term in f_E, and so the coefficient of s_λ in R_λ is:

$$u_{\lambda\lambda} = \sum_{v\in W_\lambda} t_A(-ab)_C = W_\lambda(t_\alpha, a_\beta, b_\beta).$$

(b) First we have that when $\lambda = 0$, we can apply part (a) and find:

$$\begin{aligned}R_0 &= \sum_{w\in W} w\left(\prod_{i<j}\frac{1-t_{e_i\pm e_j}x_i^{-1}x_j^{\mp 1}}{1-x_i^{-1}x_j^{\mp 1}}\prod_i\frac{(1-a_{e_i}x^i)(1-b_{e_i}x^i)}{1-x_i^{-2i}}\right) \\ &= W_0(t_\alpha, a_\beta, b_\beta).\end{aligned}$$

Define W^λ to be the set of coset representatives of W/W_λ of minimal length. Every element $w \in W$ can be written uniquely in the form $w = uv$ where $u \in W^\lambda$, $v \in W_\lambda$ (as proved for example in [1]). Then define the following sets associated with the long and short roots of type B_n, with R the type B_n root system:

$$Z_\ell(\lambda) = \{\alpha \in R^+ \mid \alpha \text{ is a long root and } \langle \lambda, \alpha^\vee \rangle = 0\},$$
$$Z_\ell(\lambda)^c = \{\alpha \in R^+ \mid \alpha \text{ is a long root and } \langle \lambda, \alpha^\vee \rangle \neq 0\},$$

$$Z_s(\lambda) = \{\alpha \in R^+ \mid \alpha \text{ is a short root and } \langle \lambda, \alpha^\vee \rangle = 0\},$$
$$Z_s(\lambda)^c = \{\alpha \in R^+ \mid \alpha \text{ is a short root and } \langle \lambda, \alpha^\vee \rangle \neq 0\}.$$

Using these, we then define the following products:

$$t_\lambda = \prod_{\alpha \in Z_\ell(\lambda)} \frac{1 - t_\alpha x^{-\alpha}}{1 - x^{-\alpha}} \qquad ab_\lambda = \prod_{\beta \in Z_s(\lambda)} \frac{(1 - a_\beta x^{-\beta})(1 - b_\beta x^{-\beta})}{1 - x^{-2\beta}}$$
$$t^c_\lambda = \prod_{\alpha \in Z_\ell(\lambda)^c} \frac{1 - t_\alpha x^{-\alpha}}{1 - x^{-\alpha}} \qquad ab^c_\lambda = \prod_{\beta \in Z_s(\lambda)^c} \frac{(1 - a_\beta x^{-\beta})(1 - b_\beta x^{-\beta})}{1 - x^{-2\beta}}.$$

Then we can rewrite R_λ as:

$$R_\lambda = \sum_{u \in W^\lambda} u \left(x^\lambda \sum_{v \in W_\lambda} v \left(t_\lambda \cdot t^c_\lambda \cdot ab_\lambda \cdot ab^c_\lambda \right) \right).$$

Since $v \in W_\lambda$ will permute the elements in $Z_s(\lambda)^c$ and the elements in $Z_\ell(\lambda)^c$, we obtain:

$$R_\lambda = \sum_{u \in W^\lambda} u \left(x^\lambda \cdot t^c_\lambda \cdot ab^c_\lambda \sum_{v \in W_\lambda} v \left(t_\lambda \cdot ab_\lambda \right) \right).$$

The terms in the second sum of this expression can be considered independently as a root system $Z(\lambda)$ with Weyl group W_λ (see Proposition 1.10 in [4]). Hence we can apply the equation $R_0 = W_0(t_\alpha, a_\beta, b_\beta)$ from above, where in this case the term corresponding to $W_0(t_\alpha, a_\beta, b_\beta)$ is actually $W_\lambda(t_\alpha, a_\beta, b_\beta)$ (since W_λ is the entire Weyl group), and our sum becomes:

$$\begin{aligned}
R_\lambda &= \sum_{u \in W^\lambda} u \left(x^\lambda \cdot t^c_\lambda \cdot ab^c_\lambda \sum_{w \in W_\lambda} v(t_\lambda \cdot ab_\lambda) \right) \\
&= \sum_{u \in W^\lambda} u \left(x^\lambda \cdot t^c_\lambda \cdot ab^c_\lambda \cdot W_\lambda(t_\alpha, a_\beta, b_\beta) \right) \\
&= W_\lambda(t_\alpha, a_\beta, b_\beta) \sum_{u \in W^\lambda} u \left(x^\lambda \cdot t^c_\lambda \cdot ab^c_\lambda \cdot \right).
\end{aligned}$$

Then because earlier we found the equality:

$$R_\lambda = \frac{1}{a_\rho} \sum_{E \subseteq R^+(BC_n)} f_E a_{\lambda + \rho - \alpha_E} = \sum_{E \subseteq R^+(BC_n)} f_E s_{\lambda - \alpha_E},$$

we see R_λ is the sum of symmetric polynomials and therefore is itself a symmetric polynomial whose coefficients lie in $\mathbb{Z}[t_\alpha, a_\beta, b_\beta]$. If we define $\mathbb{F}$ to be the field of fractions of $\mathbb{Z}[t_\alpha, a_\beta, b_\beta]$, then there exists some $P_\lambda(x; t_\alpha, a_\beta, b_\beta) \in \mathbb{F}[P]$ such that:

$$R_\lambda = W_\lambda(t_\alpha, a_\beta, b_\beta) P_\lambda(x; t_\alpha, a_\beta, b_\beta).$$

Thus $P_\lambda(x; t_\alpha, a_\beta, b_\beta)$ must be a symmetric function with coefficients in $\mathbb{F}[P]$. And because in its explicit form,

$$P_\lambda(x; t_\alpha, a_\beta, b_\beta) = \sum_{u \in W^\lambda} u \left(x^\lambda \prod_{\alpha \in Z_\ell(\lambda)^c} \frac{1 - t_\alpha x^{-\alpha}}{1 - x^{-\alpha}} \prod_{\beta \in Z_s(\lambda)^c} \frac{(1 - a_\beta x^{-\beta})(1 - b_\beta x^{-\beta})}{1 - x^{-2\beta}} \right),$$

$P_\lambda(x; t_\alpha, a_\beta, b_\beta)$ has terms t_α, a_β, and b_β in the numerators, it can be considered as a symmetric polynomial with coefficients in $\mathbb{Z}[t_\alpha, a_\beta, b_\beta]$. Thus the coefficients $u_{\lambda\mu}$ in (a) must be divisible by $W_\lambda(t_\alpha, a_\beta, b_\beta)$, and using the result in (a),

$$P_\lambda(x; t_\alpha, a_\beta, b_\beta) = \sum_{\mu \in P^+} \frac{1}{W_\lambda(t_\alpha, a_\beta, b_\beta)} u_{\lambda\mu} s_\mu.$$

If we write $P_\lambda(x; t_\alpha, a_\beta, b_\beta)$ as

$$P_\lambda(x; t_\alpha, a_\beta, b_\beta) = \sum_{\mu \in P^+} c_{\lambda\mu} s_\mu,$$

this means $c_{\lambda\mu} = 0$ except when $\mu \leq \lambda$, and

$$c_{\lambda\lambda} = \frac{1}{W_\lambda(t_\alpha, a_\beta, b_\beta)} u_{\lambda\lambda} = 1.$$

Then by setting $t_\alpha = t$, $a_\beta = a$, and $b_\beta = b$ for all α and β, the polynomials $P_\lambda(x; t_\alpha, a_\beta, b_\beta)$ become the BC_n Macdonald spherical functions, and we obtain the desired result. ■

4.2 Upper triangular expansion

Theorem 4.2. *The expansion of the series given by*

$$P_\mu W_\mu(t, a, b) \prod_{i<j} \frac{1 - x_i^{\pm 1} x_j^{\pm 1}}{1 - t x_i^{\pm 1} x_j^{\pm 1}} \prod_i \frac{1 - x_i^{\pm 2}}{(1 - a x_i^{\pm 1})(1 - b x_i^{\pm 1})}$$

into m_γ is given by a matrix $B_{\mu\gamma}$ indexed by $P(BC_n)$, where $B_{\mu\gamma}$ is upper triangular with respect to the dominance order of type BC_n weights, and $B_{\mu\mu} = |W_\mu|$.

Proof. First we rewrite the given polynomial in explicit terms:
$$P_\mu W_\mu(t,a,b)\prod_{i<j}\frac{1-x_i^{\pm1}x_j^{\pm1}}{1-tx_i^{\pm1}x_j^{\pm1}}\prod_i\frac{1-x_i^{\pm2}}{(1-ax_i^{\pm1})(1-bx_i^{\pm1})}$$

$$=\sum_{w\in W}w\left(x^\mu\prod_{i<j}\frac{(1-tx_ix_j^{\pm1})}{(1-x_ix_j^{\pm1})}\frac{(1-x_i^{\pm1}x_j^{\pm1})}{(1-tx_i^{\pm1}x_j^{\pm1})}\prod_i\frac{(1-ax_i^{-1})(1-bx_i^{-1})}{(1-x_i^{-2})}\frac{(1-x_i^{\pm2})}{(1-ax_i^{\pm1})(1-bx_i^{\pm1})}\right)$$
$$=\sum_{w\in W}w\left(x^\mu\prod_{i<j}\frac{1-x_ix_j^{\pm1}}{1-tx_ix_j^{\pm1}}\prod_i\frac{1-x_i^2}{(1-ax_i)(1-bx_i)}\right).$$

Then we can find identities for each term in the first product,

$$\begin{aligned}\frac{1-x_ix_j^{\pm1}}{1-tx_ix_j^{\pm1}}-1&=\frac{1-x_ix_j^{\pm1}-1+tx_ix_j^{\pm1}}{1-tx_ix_j^{\pm1}}\\&=(x_ix_j^{\pm1})(t-1)\frac{1}{1-tx_ix_j^{\pm1}}\\&=\sum_{r\geq0}(x_ix_j^{\pm1})(t-1)t^r(x_ix_j^{\pm1})^r\\&=\sum_{r>0}(t-1)t^{r-1}(x_ix_j^{\pm1})^r\\\frac{1-x_ix_j^{\pm1}}{1-tx_ix_j^{\pm1}}&=1+\sum_{r>0}(t-1)t^{r-1}(x_ix_j^{\pm1})^r,\end{aligned}$$

and similarly for the terms in the second product,

$$\begin{aligned}\frac{1-x_i^2}{(1-ax_i)(1-bx_i)}-1&=\frac{-x_i^2+(a+b)x_i-abx_i^2}{(1-ax_i)(1-bx_i)}\\&=x_i(a+b-(ab+1)x_i)\sum_{r\geq0}(ax_i)^r\sum_{q\geq0}(bx_i)^q\\&=x_i(a+b-(ab+1)x_i)\sum_{n\geq0}\sum_{k=0}^{n}(a^kb^{n-k})x_i^n\\&=(a+b-(ab+1)x_i)\sum_{n>0}\sum_{k=0}^{n-1}(a^kb^{n-k-1})x_i^n\\\frac{1-x_i^2}{(1-ax_i)(1-bx_i)}&=1+(a+b-(ab+1)x_i)\sum_{n>0}\sum_{k=0}^{n-1}(a^kb^{n-k-1})x_i^n.\end{aligned}$$

Define Q^+ to be the set of all nonnegative integral linear combinations of positive roots in B_n. Then applying these two identities gives the result:

$$
\begin{aligned}
P_\mu W_\mu(t) \prod_{i<j} \frac{1-x_i^{\pm 1}x_j^{\pm 1}}{1-tx_i^{\pm 1}x_j^{\pm 1}} \prod_i \frac{1-x_i^{\pm 2}}{(1-ax_i^{\pm 1})(1-bx_i^{\pm 1})} \\
&= \sum_{w\in W} w\left(x^\mu \prod_{i<j}\left(1+\sum_{r>0}(t-1)t^{r-1}(x_i x_j^{\pm 1})^r\right)\right. \\
&\qquad \left.\cdot \prod_i \left(1+(a+b-(ab+1)x_i)\sum_{n>0}\sum_{k=0}^{n-1}(a^k b^{n-k-1})x_i^n\right)\right) \\
&= \sum_{w\in W} w\left(\sum_{\gamma\in Q^+} c_\gamma x^{\mu+\gamma}\right) \\
&= \sum_{\gamma\in Q^+} c_\gamma \left(\sum_{w\in W} w(x^{\mu+\gamma})\right),
\end{aligned}
$$

with $c_0 = 1$ and $c_\gamma \in \mathbb{Z}[t,a,b]$ for all $\gamma \in Q^+$. From the definition of m_μ, and using the fact that $c_0 = 1$, we can now rewrite our original series with γ a type BC_n weight,

$$
\begin{aligned}
P_\mu W_\mu(t,a,b) \prod_{i<j} \frac{1-x_i^{\pm 1}x_j^{\pm 1}}{1-tx_i^{\pm 1}x_j^{\pm 1}} \prod_i \frac{1-x_i^{\pm 2}}{(1-ax_i^{\pm 1})(1-bx_i^{\pm 1})} &= |W_\mu| m_\mu + \sum_{\gamma>\mu} B_{\mu\gamma} m_\gamma \\
&= \sum_{\gamma\geq\mu} B_{\mu\gamma} m_\gamma,
\end{aligned}
$$

such that $B_{\mu\gamma} \in \mathbb{Z}[t,a,b]$ and $B_{\mu\mu} = |W_\mu|$, as desired. ■

4.3 Inner products

Let $\mathbb{K} = \mathbb{Z}[t,a,b]$, and let $P(BC_n)$ and $P^+(BC_n)$ be the type BC_n weight lattice and set of dominant integral weights respectively. For $f \in \mathbb{K}[P(BC_n)]$, if we can write

$$
f = \sum_{\lambda\in P(BC_n)} f_\lambda x^\lambda \text{ with } f_\lambda \in \mathbb{K},
$$

define $\overline{f}$ by

$$
\overline{f} = \sum_{\lambda\in P(BC_n)} f_\lambda x^{-\lambda}.
$$

Then the following gives a symmetric bilinear form:

$$[f]_1 = \text{ coefficient of 1 in } f$$

$$\langle f, g\rangle_{t,a,b} = \frac{1}{|W|}\left[f\overline{g}\prod_{i<j}\frac{1-x_i^{\pm1}x_j^{\pm1}}{1-tx_i^{\pm1}x_j^{\pm1}}\prod_i\frac{1-x_i^{\pm2}}{(1-ax_i^{\pm1})(1-bx_i^{\pm1})}\right]_1.$$

We find that $\langle m_\lambda, m_\mu\rangle_{1,1,-1}$ coincides with the single-parameter inner product given in [11] where $t=1$, and likewise $\langle s_\lambda, s_\mu\rangle_{0,0,0}$ coincides with the single-parameter case when $t=0$. We use this and the proofs in [11] to obtain the equalities:

$$\langle m_\lambda, m_\mu\rangle_{1,1,-1} = \frac{1}{|W_\lambda|}\delta_{\lambda\mu} \qquad\qquad \langle s_\lambda, s_\mu\rangle_{0,0,0} = \delta_{\lambda\mu}.$$

Theorem 4.3. $\langle P_\lambda, P_\mu\rangle_{t,a,b} = \dfrac{1}{W_\lambda(t,a,b)}\delta_{\lambda\mu}$

Proof. Recall that we can write $P_\lambda = \sum_{\mu\in P^+} K^{-1}_{\lambda\mu}s_\mu$ where K^{-1} is an upper triangular matrix with $K^{-1}_{\lambda\mu}\in\mathbb{Z}[t,a,b]$ and $K^{-1}_{\lambda\lambda}=1$. Then note that

$$\begin{aligned} P_\lambda(x;1,1,-1) &= \frac{1}{|W_\lambda|}\sum_{w\in W} w\left(x^\lambda\prod_{i<j}\frac{1-x_i^{-1}x_j^{\pm1}}{1-x_i^{-1}x_j^{\pm1}}\prod_i\frac{(1-x_i^{-1})(1+x_i^{-1})}{1-x_i^{-2}}\right)\\ &= \frac{1}{|W_\lambda|}\sum_{w\in W} w(x^\lambda)\\ &= m_\lambda, \end{aligned}$$

so if we have a matrix $k^{-1}_{\lambda\mu}$ such that $m_\lambda = \sum_{\mu\in P^+} k^{-1}_{\lambda\mu}s_\mu$, then the matrix k^{-1} is a specialization of $K^{-1}_{\lambda\mu}$ where $t=1, a=1, b=-1$. And since $K^{-1}_{\lambda\mu}$ has entries in $\mathbb{Z}[t,a,b]$ and is lower triangular with $K^{-1}_{\lambda\lambda}=1$, the same applies to k^{-1}.

Now if we consider the matrix $A = K^{-1}k^{-1}$ to give a change of basis between R_λ and and the monomial symmetric functions m_μ, given by:

$$P_\lambda = \sum_{\nu\leq\lambda} A_{\lambda\nu}m_\nu.$$

The matrix A retains the properties of K^{-1} and k^{-1}, namely that $A_{\lambda\mu}\in\mathbb{Z}[t,a,b]$, $A_{\lambda\lambda}=1$, and A is lower triangular.

For the following computation, we assume that λ and μ are comparable in the dominance order for type BC_n weights, and that $\lambda \leq \mu$ without loss of generality. We then apply the above expansion of P_λ into m_μ, along with Theorem 4.2:

$$\begin{aligned}
\langle P_\lambda, P_\mu \rangle_{t,a,b} &= \left\langle P_\lambda, P_\mu \prod_{i<j} \frac{1 - x_i^{\pm 1} x_j^{\pm 1}}{1 - t x_i^{\pm 1} x_j^{\pm 1}} \prod_i \frac{1 - x_i^{\pm 2}}{(1 - a x_i^{\pm 1})(1 - b x_i^{\pm 1})} \right\rangle_{1,1,-1} \\
&= \frac{1}{W_\mu(t,a,b)} \left\langle P_\lambda, P_\mu W_\mu(t,a,b) \prod_{i<j} \frac{1 - x_i^{\pm 1} x_j^{\pm 1}}{1 - t x_i^{\pm 1} x_j^{\pm 1}} \prod_i \frac{1 - x_i^{\pm 2}}{(1 - a x_i^{\pm 1})(1 - b x_i^{\pm 1})} \right\rangle_{1,1,-1} \\
&= \frac{1}{W_\mu(t,a,b)} \left\langle \sum_{\nu \leq \lambda} A_{\lambda\nu} m_\nu, \sum_{\gamma \geq \mu} B_{\mu\gamma} m_\gamma \right\rangle_{1,1,-1}.
\end{aligned}$$

Then using the bilinearity of the inner product, and the fact that $\langle m_\lambda, m_\mu \rangle_{1,1,-1} = \frac{1}{|W_\lambda|} \delta_{\lambda\mu}$,

$$\begin{aligned}
\langle P_\lambda, P_\mu \rangle_{t,a,b} &= \frac{1}{W_\mu(t,a,b)} \langle m_\lambda, |W_\mu| m_\mu \rangle_{1,1,-1} \\
&= \frac{1}{W_\mu(t,a,b)} \frac{|W_\mu|}{|W_\lambda|} \delta_{\lambda\mu} \\
&= \frac{1}{W_\lambda(t,a,b)} \delta_{\lambda\mu}.
\end{aligned}$$

Note that the last equality holds since the multiplication by $\delta_{\lambda\mu}$ means that $\lambda = \mu$. ■

Chapter 5

Kato-Lusztig Formula for $K_{\lambda\mu}(t, a, b)$

We have now developed the tools needed to prove a Kato-Lusztig formula that computes the values for $K_{\lambda\mu}(t, a, b)$ explicitly. To do so, let Q^+ be as defined in Chapter 4.2, the set of nonnegative integral linear combinations of type positive type B_n roots. Then for a given type BC_n weight $\gamma \in P$, define $E(\gamma; t, a, b)$ by the following:

$$\sum_{\gamma \in Q^+} E(\gamma; t, a, b) x^\gamma = \prod_{i<j} \frac{1}{1 - t x_i x_j^{\pm 1}} \prod_i \frac{1}{(1 - a x_i)(1 - b x_i)}$$

$$\text{and } E(\gamma; t, a, b) = 0 \text{ if } \gamma \notin Q^+.$$

Theorem 5.1. $K_{\lambda\mu} = \sum_{w \in W} (-1)^{\ell(w)} E(w(\lambda + \rho) - (\mu + \rho); t, a, b).$

Proof. First, using the fact that $\langle P_\lambda, P_\mu \rangle_{t,a,b} = \frac{1}{W_\mu(t, a, b)} \delta_{\lambda\mu}$, we can rewrite $K_{\lambda\mu}$:

$$\begin{aligned} W_\mu(t, a, b) \langle s_\lambda, P_\mu \rangle_{t,a,b} &= W_\mu(t, a, b) \langle \sum_{\nu \in P^+(BC_n)} K_{\lambda\nu} P_\nu, P_\mu \rangle_{t,a,b} \\ &= W_\mu(t, a, b) \langle K_{\lambda\mu} P_\mu, P_\mu \rangle_{t,a,b} \\ &= K_{\lambda\mu}. \end{aligned}$$

Then we can consider the following product:

$$P_\mu W_\mu(t,a,b)\prod_{i<j}\frac{1}{1-tx_i^{\pm 1}x_j^{\pm 1}}\prod_i\frac{1}{(1-ax_i^{\pm 1})(1-bx_i^{\pm 1})}$$

$$= \sum_{w\in W} w\left(x^\mu\prod_{i<j}\frac{1-tx_i^{-1}x_j^{\pm 1}}{1-x_i^{-1}x_j^{\pm 1}}\prod_i\frac{(1-ax_i^{-1})(1-bx_i^{-1})}{1-x_i^{-2}}\prod_{i<j}\frac{1}{1-tx_i^{\pm 1}x_j^{\pm 1}}\prod_i\frac{1}{(1-ax_i^{\pm 1})(1-bx_i^{\pm 1})}\right)$$

$$= \sum_{w\in W} w\left(x^\mu\prod_{i<j}\frac{1}{1-x_i^{-1}x_j^{\pm 1}}\prod_i\frac{1}{1-x_i^{-2}}\prod_{i<j}\frac{1}{1-tx_ix_j^{\pm 1}}\prod_i\frac{1}{(1-ax_i)(1-bx_i)}\right)$$

$$= \sum_{w\in W} w\left(x^{\mu+\rho}\frac{1}{x^\rho\prod_{\alpha\in R^+}(1-x^{-\alpha})}\prod_{i<j}\frac{1}{1-tx_ix_j^{\pm 1}}\prod_i\frac{1}{(1-ax_i)(1-bx_i)}\right)$$

$$= \frac{1}{a_\rho}\sum_{w\in W}(-1)^{\ell(w)}w\left(x^{\mu+\rho}\prod_{i<j}\frac{1}{1-tx_ix_j^{\pm 1}}\prod_i\frac{1}{(1-ax_i)(1-bx_i)}\right)$$

Combining these identities, we find:

$$\begin{aligned}
K_{\lambda\mu} &= \langle s_\lambda, W_\mu(t,a,b)P_\mu\rangle_{t,a,b}\\
&= \left\langle s_\lambda, W_\mu(t,a,b)P_\mu\prod_{i<j}\frac{1}{1-tx_i^{\pm 1}x_j^{\pm 1}}\prod_i\frac{1}{(1-ax_i^{\pm 1})(1-bx_i^{\pm 1})}\right\rangle_{0,0,0}\\
&= \text{coefficient of } s_\lambda \text{ in}\\
&\qquad \frac{1}{a_\rho}\sum_{w\in W}(-1)^{\ell(w)}w\left(x^{\mu+\rho}\prod_{i<j}\frac{1}{1-tx_ix_j^{\pm 1}}\prod_i\frac{1}{(1-ax_i)(1-bx_i)}\right)\\
&= \text{coefficient of } a_{\lambda+\rho} \text{ in}\\
&\qquad \sum_{w\in W}(-1)^{\ell(w)}w\left(x^{\mu+\rho}\prod_{i<j}\frac{1}{1-tx_ix_j^{\pm 1}}\prod_i\frac{1}{(1-ax_i)(1-bx_i)}\right)\\
&= \text{coefficient of } x^{\lambda+\rho} \text{ in}\\
&\qquad \sum_{w\in W}(-1)^{\ell(w)}w\left(x^{\mu+\rho}\sum_{\gamma\in Q^+}E(\gamma;t,a,b)x^\gamma\right).
\end{aligned}$$

The element in Q^+ that gives the coefficient of $x^{\lambda+\rho}$ will be γ such that $w(\gamma+\mu+\rho) = \lambda+\rho$, so explicitly $\gamma = w^{-1}(\lambda+\rho) - (\mu+\rho)$. Thus we conclude:

$$\begin{aligned}
K_{\lambda\mu} &= \text{coefficient of } x^{\lambda+\rho} \text{ in } \sum_{w\in W}(-1)^{\ell(w)} w\left(x^{\mu+\rho}\sum_{\gamma\in Q^+} E(\gamma;t,a,b)x^{\gamma}\right) \\
&= \sum_{w\in W}(-1)^{\ell(w)} E(w^{-1}(\lambda+\rho)-(\mu+\rho);t,a,b) \\
&= \sum_{w\in W}(-1)^{\ell(w^{-1})} E(w^{-1}(\lambda+\rho)-(\mu+\rho);t,a,b) \\
&= \sum_{w^{-1}\in W}(-1)^{\ell(w)} E(w(\lambda+\rho)-(\mu+\rho);t,a,b) \\
&= \sum_{w\in W}(-1)^{\ell(w)} E(w(\lambda+\rho)-(\mu+\rho);t,a,b). \qquad \blacksquare
\end{aligned}$$

Chapter 6

Examples of $K_{\lambda\mu}(t,a,b)$

To compute the polynomials $K_{\lambda\mu}(t,a,b)$, we modified the code from [2] which was originally used to calculate the Kostka-Foulkes polynomials. To do so, we use the rational function,

$$\prod_{i<j} \frac{1}{1 - tx_i x_j^{\pm 1}} \prod_i \frac{1}{(1-ax_i)(1-bx_i)}.$$

Expanding these products into geometric sums allows us to find $E(\gamma;t,a,b)$ for each appropriate γ, and from there $K_{\lambda\mu}(t,a,b)$ by applying Theorem 5.1. This is then extended to more parameters, with the result that the polynomials are invariant under permutation of the parameters excluding t.

We will additionally consider reduction of the multiparameter BC_n-Kostka-Foulkes polynomials to the type B_n and C_n Kostka-Foulkes polynomials with unequal parameter affine Hecke algebras as defined in [7]. From there, we further reduce to the standard equal parameter cases.

It follows from Theorem 2.15 in [10] that $K_{\lambda\mu}(t,a,b) \in \mathbb{Z}_{\geq 0}[t,a,b]$ for any λ and μ in $P^+(BC_n)$, so the coefficients we compute will be positive. But the unequal parameter type C_n cases can have negative coefficients, which is demonstrated in Example 2.

Example 1: We look first at the case where there are additional parameters, so we compute $K_{\lambda\mu}(t,a,b,c)$ with $\lambda = [2,0]$, $\mu = [0,0]$:

$$\begin{aligned} K_{\lambda\mu}(t,a,b,c) =& a^2t^2 + abt^2 + b^2t^2 + act^2 + bct^2 + c^2t^2 + a^2t + 2abt + \\ & b^2t + 2act + 2bct + c^2t + a^2 + ab + b^2 + ac + bc + c^2 \\ K_{\lambda\mu}(t,a,b,0) =& a^2t^2 + abt^2 + b^2t^2 + a^2t + 2abt + b^2t + a^2 + ab + b^2 \\ =& K_{\lambda\mu}(t,a,b) \end{aligned}$$

In this example, adding an extra parameter and specializing when $c = 0$ reduces to the previous case with only t, a, and b as parameters.

By imposing relations between the parameters v_s, v_d, and v_l, the BC_n Macdonald spherical functions $P_\lambda(x;t,a,b)$ can be reduced to the equal parameter spherical functions $P_\lambda(x;t)$ associated with the B_n and C_n root systems. Consider $P_\lambda(x;t,a,b)$ written in the following way, using the relations given in chapter 3:

$$P_\lambda(x;t,a,b) = \frac{1}{W_\lambda(t,a,b)} \sum_{w\in W} w\left(x^\lambda \prod_{i<j} \frac{1-v_\ell^2 x_i^{-1}x_j^{\pm 1}}{1-x_i^{-1}x_j^{\pm 1}} \prod_i \frac{(1-(v_s v_d)x_i^{-1})(1+(v_s/v_d)x_i^{-1})}{1-x_i^{-2}} \right).$$

To specify to the B_n unequal parameter Macdonald spherical functions, we set $v_s = v_d$ (equivalently $b=-1$). This makes the spherical functions:

$$P_\lambda(x;t,a,b) = \frac{1}{W_\lambda(t,a,b)} \sum_{w\in W} w\left(x^\lambda \prod_{i<j} \frac{1-v_\ell^2 x_i^{-1}x_j^{\pm 1}}{1-x_i^{-1}x_j^{\pm 1}} \prod_i \frac{1-v_s^2 x_i^{-1}}{1-x_i^{-1}} \right).$$

This has the parameter v_s associated with the short type B_n roots and v_ℓ with the long roots. However, reducing this to the B_n case and applying the Kato-Lusztig formula fails to compute the Kostka-Foulkes polynomials, since $K_{\lambda\mu}(t,a,b)$ is a change of basis into the type C_n Schur polynomials.

To reach the type C_n spherical functions, we set $v_d = 1$ (equivalently $a=-b$) and obtain:

$$P_\lambda(x;t,a,b) = \frac{1}{W_\lambda(t,a,b)} \sum_{w\in W} w\left(x^\lambda \prod_{i<j} \frac{1-v_\ell^2 x_i^{-1}x_j^{\pm 1}}{1-x_i^{-1}x_j^{\pm 1}} \prod_i \frac{1-v_s^2 x_i^{-2}}{1-x_i^{-2}} \right)$$

Here v_s is associated with the long type C_n roots and v_ℓ with the short roots. Again, setting $v_s = v_\ell$ (equivalently $a=\sqrt{t}$) brings this to the equal parameter case.

The following example shows how a multiparameter BC_n-Kostka-Foulkes polynomial reduces in the ways described above to C_n Kostka-Foulkes polynomials. In this example, the unequal parameter case contains negative coefficients.

Example 2: Let $\lambda = [2,2]$ and $\mu = [0,0]$. Then the multiparameter BC_n-Kostka-Foulkes polynomial for λ and μ is:

$$\begin{aligned} K_{\lambda\mu}(t,a,b) &= a^4t^2 + a^3bt^2 + a^2b^2t^2 + ab^3t^2 + b^4t^2 + a^3bt \\ &\quad + a^2b^2t + ab^3t + a^2b^2 + a^2t^2 + abt^2 + b^2t^2 + abt + t^2. \end{aligned}$$

First we set $b=-a$ and find:

$$K_{\lambda\mu}(t,a,-a) = a^4t^2 - a^4t + a^4 + a^2t^2 - a^2t + t^2.$$

This is the unequal parameter type C_n case with negative coefficients, but if we then set $a=\sqrt{t}$:

$$K_{\lambda\mu}(t,\sqrt{t},-\sqrt{t}) = t^4 + t^2,$$

which matches the type C_n Kostka-Foulkes polynomial $K_{\lambda\mu}(t)$ as expected.

Chapter 7

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