The Moment Graph for Bott-Samelson Varieties and Applications to Quantum Cohomology

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(ABSTRACT)

We give a description of the moment graph for Bott-Samelson varieties in arbitrary Lie type. We use this, along with curve neighborhoods and explicit moduli space computations, to compute a presentation for the small quantum cohomology ring of a particular Bott-Samelson variety in Type A.

The Moment Graph for Bott-Samelson Varieties and Applications to Quantum Cohomology

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(GENERAL AUDIENCE ABSTRACT)

Since the early 1990's, the study of *quantum cohomology* has been a fascinating, and fruitful field of research with connections to physics, representation theory, and combinatorics. The quantum cohomology of a space X encodes enumerative information about how many curves intersect certain subspaces of X; these counts are called *Gromov-Witten invariants*. For some spaces X, including the class of spaces we consider here, this count is only "virtual" and negative Gromov-Witten invariants may arise.

In this dissertation, we study the quantum cohomology of *Bott-Samelson varieties*. These spaces arise frequently in applications to representation theory and combinatorics, however their quantum cohomology was previously unexplored. The first of our three main theorems describes the *moment graph* for Bott-Samelson varieties. This is a description of what all the possible curves, stable under certain symmetries, exist in a Bott-Samelson variety. Our second main theorem is a technical result which enables us to compute some Gromov-Witten invariants directly. Finally, our third main theorem is a description of the quantum cohomology for a certain three-dimensional Bott-Samelson variety.

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Chapter 1

Introduction

The (small) quantum cohomology of homogeneous varieties has been studied extensively due to its connection with questions in enumerative geometry. For example, the Severi problem of counting all curves of a given degree d passing through $3d-1$ points in the projective plane admits a beautiful solution in terms of the Gromov-Witten invariants of $\mathbb{P}^2$; see the introduction in [FP97] for a discussion of these results and their relation to the quantum cohomology of the projective plane. The key to obtaining a presentation for the quantum cohomology ring is to quantize the relations in the ordinary cohomology ring. For flag manifolds G/B, this quantization involves the Toda lattice; see [Kim99].

More generally, the quantum cohomology of toric varieties is well understood (originally due to Batyrev [Bat93]; see [AK06] for a modern discussion, Proposition 2.5 in particular.) Vakil in [Vak00] computed Gromov-Witten invariants for Hirzebruch surfaces and (all but two) del Pezzo surfaces in all genera, and shows these invariants are enumerative; see Section 8 in [Vak00] for a discussion of curve counts for Hirzebruch surfaces. Little is known about the quantum cohomology of varieties which are not convex and/or not toric. For example, see [Pec13] and [MS17]. Bott-Samelson varieties are generally not convex in dimension greater than one, and are generally not toric in dimension greater than two, however they are intimately related to homogeneous spaces G/B, so it is natural consider their quantum cohomology.

We will now give an overview of the results contained in this thesis. Reading Chapter 2 and returning to the introduction is recommended for those unfamilar with the geometry of algebraic groups and homogeneous spaces.

In this thesis, we describe the moment graph of Bott-Samelson varieties with a view towards describing the (small) quantum cohomology ring of Bott-Samelson varieties. If X is a variety on which an algebraic torus T acts with finitely many fixed points, the moment graph is defined as follows: the vertices are the set of fixed points X^T, and two vertices $x, y \in X^T$ are connected by an edge if there is a T-stable curve containing both x, y.

In order to describe our results more concretely, we recall some constructions and fix some notation. Let G be a semisimple linear algebraic group over $\mathbb{C}$ (e.g. $G = SL_n(\mathbb{C})$). Fix a maximal torus contained in a Borel subgroup $T \subset B \subset G$ (T is the subgroup of diagonal matrices, and B the subgroup of upper triangular matrices for $SL_n(\mathbb{C})$); the Weyl group is denoted $W := N_G(T)/T$ (so $W = S_n$ the symmetric group on n-letters for $SL_n(\mathbb{C})$). The associated root system is denoted $\Phi = \Phi(G, T)$, the base corresponding to the fixed Borel subgroup is denoted $\Delta \subset \Phi$, and the set of positive roots is denoted Φ^+ (these are described for type A in Chapter 2). The minimal parabolic subgroup corresponding to $\alpha \in \Delta$ is denoted P_α.

For a sequence of simple reflections $(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) \in W^n$, the corresponding Bott-Samelson variety is denoted $Z = Z(\alpha_1, \ldots, \alpha_n)$. A Bott-Samelson variety is a tower of $\mathbb{P}^1$-bundles

$$Z(\alpha_1, \ldots, \alpha_n) \xrightarrow{\pi_n} Z(\alpha_1, \ldots, \alpha_{n-1}) \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} Z(\alpha_1) \xrightarrow{\pi_1} \{pt\}$$

where each bundle has a natural section $s_k : Z(\alpha_1, \ldots, \alpha_{k-1}) \to Z(\alpha_1, \ldots, \alpha_k)$, and each Bott-Samelson variety has a morphism $\theta_k : Z(\alpha_1, \ldots, \alpha_k) \to G/B$.

Given a Bott-Samelson variety Z, we will denote the previous Bott-Samelson variety in the tower by Z', and the section $Z' \to Z$ will be denoted by s.

Bott-Samelson varieties are T-varieties (this is described in Chapter 3.) The fixed point set Z^T is easy to describe; the T-fixed points correspond to subsequences of $(s_{\alpha_1}, \ldots, s_{\alpha_n})$. The combinatorial object which corresponds to the T-fixed points are $\varepsilon \in \{0,1\}^n$; for $x \in Z^T$, we will denote the binary n-tuple corresponding to x by ε_x.

The n-tuple ε_x can be described inductively as follows: ε_x is obtained from the $(n-1)$-tuple $\varepsilon_{\pi(x)}$ by appending either a zero or one according as $x \in s(Z')$ or $x \notin s(Z')$ respectively. The next definition comes from [Wil04, Section 2]; we have slightly modified the notation.

Definition 1.1. For $\varepsilon \in \{0,1\}^n$, denote by $\pi_+(\varepsilon)$ the set of entries i such that $\varepsilon_i = 1$. Define

$$w_k(\varepsilon) = \prod_{\substack{1 \le i \le k \\ i \in \pi_+(\varepsilon)}} s_{\alpha_i}$$

($w_k(\varepsilon) = 1$ if $\{1 \leq i \leq k, i \in \pi_+(\varepsilon)\} = \varnothing$), set $w(\varepsilon) = w_n(\varepsilon)$, and define $\varepsilon(\alpha_k) = w_k(\varepsilon)\alpha_k \in \Phi$ for each $1 \leq k \leq n$.

The next definition gives us a notation for the fixed points which lie on the same fiber as a given fixed point.

Definition 1.2. For $x \in Z^T$, define ε_x^0 and ε_x^∞ by adjoining either a 0 or 1 respectively to $\varepsilon_{\pi(x)}$. Note, ε_x^0 corresponds to the T-fixed point in $s(Z')$ which is also contained in the fiber of $\pi : Z \to Z'$ containing x, and ε_x^∞ corresponds to the other T-fixed point in that fiber.

Our main theorem is an inductive characterization of the set of T-stable curves in Z. Since any T-stable curve is a union of irreducible T-stable curves, we characterize the points $x, y \in Z^T$ which are joined by irreducible T-stable curves.

Theorem 1.3. *Let $x, y \in Z^T$, and let k be the first index where $\varepsilon_x, \varepsilon_y$ differ. If $k = n$, then x, y are joined by a T-stable fiber of $\pi : Z \to Z'$. Otherwise, $k < n$ and we suppose that $\pi(x), \pi(y)$ are joined by an irreducible T-stable curve. Let C be one such T-stable curve joining $\pi(x), \pi(y)$, and let h be the class of the fiber of π.*

There are four possibilities for the restriction of the moment graph to $\{\varepsilon_x^0, \varepsilon_x^\infty, \varepsilon_y^0, \varepsilon_y^\infty\}$ (see Figure 1.1).

The cases are characterized as follows:

I. $\varepsilon_x^0(\alpha_k) \neq \varepsilon_x^0(\alpha_n)$ *and* $\varepsilon_y^0(\alpha_k) \neq \varepsilon_y^0(\alpha_n)$*;*

II. $\varepsilon_x^0(\alpha_k) = \varepsilon_x^0(\alpha_n)$ *and* $\varepsilon_y^0(\alpha_k) = \varepsilon_y^0(\alpha_n)$*;*

III. $\varepsilon_x^0(\alpha_k) = \varepsilon_x^0(\alpha_n)$ *and* $\varepsilon_y^0(\alpha_k) \neq \varepsilon_y^0(\alpha_n)$*;*

IV. $\varepsilon_x^0(\alpha_k) \neq \varepsilon_x^0(\alpha_n)$ *and* $\varepsilon_y^0(\alpha_k) = \varepsilon_y^0(\alpha_n)$*;*

In each case, the unlabeled curves (see Figure 1.1) have the same homology class, which are as follows:

I.

$$\begin{cases} s_*[C] - (\alpha_k, \alpha_n^\vee)h, & w(\varepsilon_x^0) \neq w(\varepsilon_y^0) \\ s_*[C], & w(\varepsilon_x^0) = w(\varepsilon_y^0) \end{cases}$$

II.

$$s_*[C] - h$$

III.

$$s_*[C] + h$$

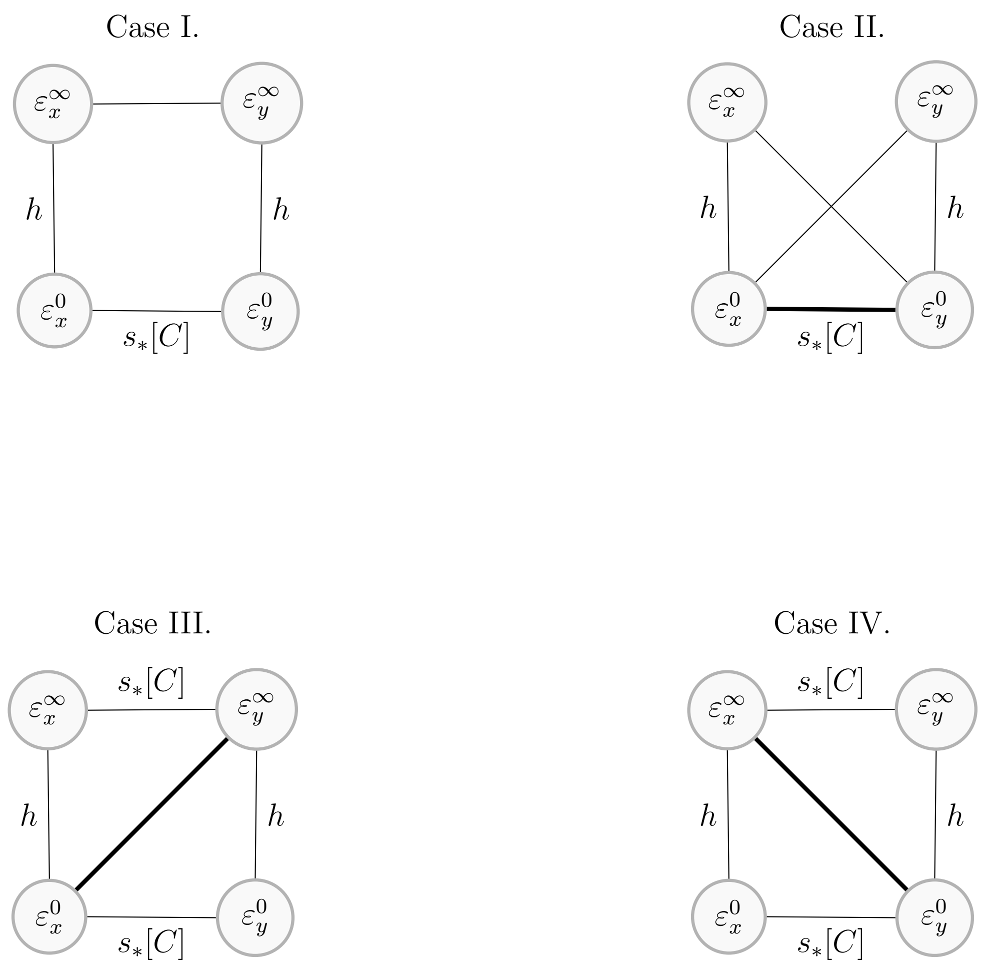
IV.

$$s_*[C] + h$$

In cases II, III, and IV, the bold line indicates there is a one-dimensional family of T-stable curves joining those fixed points.

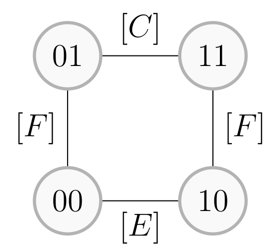
Remark 1.4. Since $\pi : Z \to Z'$ is proper, if x, y are joined by a T-stable curve, then $\pi(x), \pi(y)$ are either joined by a T-stable curve, or $\pi(x) = \pi(y)$. In particular, the inductive hypothesis in Theorem 1.3 is a necessary condition for x, y to be joined by a T-stable curve.

Figure 1.1: The "restricted" moment graph types.



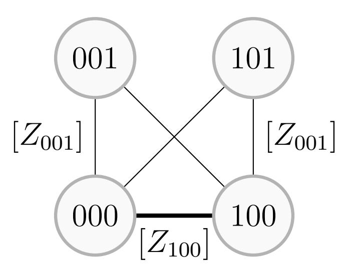
Moreover, since $\pi : Z \to Z'$ is a $\mathbb{P}^1$-bundle with a section s, the push-forward $s_* : H_*(Z') \to H_*(Z)$ is the inclusion $H_*(Z') \subset H_*(Z)$. Thus, the homology classes of T-stable curves are characterized inductively by our theorem.

Example 1.5. In type A_2, the Bott-Samelson variety $Z(\alpha_1, \alpha_2)$ is the Hirzebruch surface $\mathbb{F}_1$, and hence the moment graph is



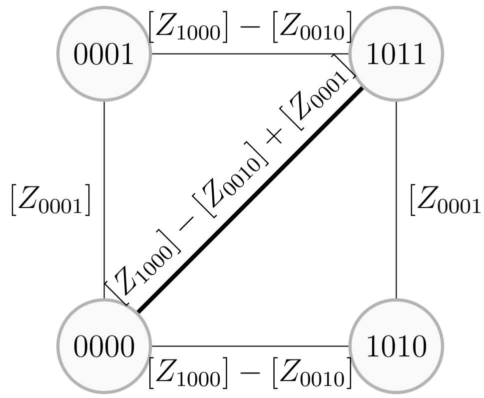
where F is the fiber, E is the exceptional divisor (i.e. the curve with self-intersection -1,) and C is related to E and F in Pic $\mathbb{F}_1$ by $C = E + F$.

In the Bott-Samelson variety $Z(\alpha_1, \alpha_2, \alpha_1)$, the restriction of the moment graph to the fixed points $\{000, 100, 001, 101\}$ is:



where the diagonal curves have homology class $[Z_{100}] - [Z_{001}]$. The complete moment graph for this Bott-Samelson variety is depicted in Figure 4.1.

In the Bott-Samelson variety $Z(\alpha_1, \alpha_2, \alpha_1, \alpha_2)$, the moment graph restricted to the fixed points $\{0000, 1010, 0001, 1011\}$ is:



All homology classes have been expressed in the Bott-Samelson subvariety basis, which we describe in Chapter 3.

In order to obtain a presentation for the quantum cohomology of Bott-Samelson varieties, we need to show that certain Gromov-Witten invariants vanish, and we need to compute some nonzero Gromov-Witten invariants. We address the non-zero invariant calculations first. (For an introduction to quantum cohomology, see the beginning of Chapter 7.)

The Gromov-Witten invariants $I_\beta(\gamma_1, \ldots, \gamma_n)$ are defined by intersection theory in the moduli space. However, in general we cannot control the geometry of the moduli space $\overline{M}_{0,n}(Z, \beta)$. Under some conditions on the curve class β and the algebraic group G, we are able to prove the moduli space $\overline{M}_{0,1}(Z, \beta)$ is smooth, and so we are able to perform the intersection theory calculations directly to determine certain Gromov-Witten invariants.

Theorem 1.6. *Suppose β is indecomposable and effective, and suppose G is of simply laced* *type. Then $\overline{M}_{0,1}(Z,\beta)$ is unobstructed; that is, $\overline{M}_{0,1}(Z,\beta)$ is smooth, irreducible, and has the expected dimension*

$$\dim \overline{M}_{0,1}(Z,\beta) = \dim Z + \int_\beta c_1(T_Z) - 2.$$

Theorem 1.6 has an important corollary which allows us to carry out the necessary calculations.

Corollary 1.7. *If $h \in H_2(Z)$ is the class of the fiber of $\pi : Z \to Z'$, then $ev : \overline{M}_{0,1}(Z,h) \to Z$ is an isomorphism.*

Corollary 1.7 lets us convert intersection theory calculations in the moduli space, into intersection theory calculations on the Bott-Samelson variety Z.

With the explicit calculations described above, and the vanishing of certain Gromov-Witten invariants that will be discussed in a few paragraphs, the final ingredient in obtaining a presentation for the (small) quantum cohomology of $Z(\alpha_1,\alpha_2,\alpha_1)$ is a brute-force calculation. Curve neighborhood techniques and the moduli space results allow us to compute some of the necessary Gromov-Witten invariants to quantize the relations in the ordinary cohomology. The remaining unknown invariants (of which, there are 111), save for one, can be computed simply by imposing the relations that the quantum cohomology ring is commutative (this is a system of 192 (generally) nonlinear equations in 111 unknowns). The final Gromov-Witten invariant is computed using an observation of Manolache [Man12, Remark 5.7].

Theorem 1.8. *Let $Z = Z(\alpha_1,\alpha_2,\alpha_1)$. The (small) quantum cohomology ring $QH^*(Z)$ is* *isomorphic to a quotient of* $\mathbb{Z}[\sigma_{100}, \sigma_{010}, \sigma_{001}, q_1, q_2, q_3]$*, subject to the following relations:*

$$\sigma_{100}^2 = q_1 q_3 - q_3\sigma_{100} + q_3\sigma_{010}$$

$$\sigma_{010}^2 = q_1 q_3 + 2q_1\sigma_{100} - q_1\sigma_{010} + q_1\sigma_{001} + \sigma_{110}$$

$$\sigma_{001}^2 = q_1 q_3 + q_2 - q_3\sigma_{100} + q_3\sigma_{010} - 2\sigma_{101} + \sigma_{011}$$

Under this isomorphism, the generators $\sigma_{100}, \sigma_{010}, \sigma_{001}$ are Poincaré dual to certain Bott-Samelson subvarieties in Z. For example, σ_{001} is dual to the fiber of $\pi : Z \to Z'$. The other two classes $\sigma_{100}, \sigma_{010}$ arise in a similar way.

The quantum parameters q_1, q_2, q_3 correspond curve classes $\beta_1, \beta_2, \beta_3$ which generate the cone of effective 1-cycles.

The connection between Theorem 1.3 and quantum cohomology is given by curve neighborhoods, which will allow us to show that certain Gromov-Witten invariants vanish; curve neighborhoods were used to study the quantum cohomology and quantum K-theory of homogeneous spaces and "almost homogeneous" spaces (see [BM15], [BCMP16], [MS17], [MM14]).

Given an effective curve class β, and a closed subvariety $\Omega \subset Z$, the curve neighborhood $\Gamma_\beta(\Omega)$ is the union of all curves of class β which intersect Ω; our definition later will be written in terms of the moduli space of stable maps, but is equivalent to the one just given. It will be clear from the alternative definition that curve neighborhoods are closed. Moreover, it is clear if Ω is B-stable, then $\Gamma_\beta(\Omega)$ is also B-stable. Finding all B-stable subvarieties in a Bott-Samelson variety is more difficult than finding all B-stable subvarieties in a homogeneous space. However, as we will show in Chapter 6, the unexpected B-stable curves collapse under

$\theta : Z \to G/B$. So, with some *ad hoc* arguments, we can use the moment graph to describe curve neighborhoods for Bott-Samelson varieties.

Example 1.9. The T-stable fiber of π joining (000) and (100) in $Z = Z(\alpha_1, \alpha_2, \alpha_1)$ is a Bott-Samelson subvariety denoted Z_{100}. The curve neighborhood $\Gamma_{\beta_1}(Z_{100}) = Z_{110}$, the image of the canonical section $s : Z(\alpha_1, \alpha_2) \to Z$.

On the other hand, the fiber $\theta^{-1}(x_e)$ is B-stable, so the curve neighborhood (of the B-fixed point) $\Gamma_{\beta_3}(x_{000}) = \theta^{-1}(x_e)$, which is not a Bott-Samelson subvariety; this computation is explained in detail in Example 6.3. However, $\Gamma_{\beta_3}(Z_{100}) = Z_{101}$ which is a Bott-Samelson subvariety. (The construction for this Bott-Samelson subvariety will be described later.) The reason for this is that Z_{101} is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ where the first projection is $\pi|_{Z_{101}}$, the second projection is $\theta|_{Z_{101}}$; the fibers of the second projection have homology class β_3.

In fact, curve neighborhoods need not be irreducible. For example,

$$\Gamma_{\beta_3}(Z_{010}) = Z_{010} \cup \theta^{-1}(x_e).$$

We now outline the body of this thesis. Chapter 2 covers the background on homogeneous spaces and projective bundles we will be using throughout.

Chapter 3 covers the background on Bott-Samelson varieties we will need. In particular, we describe the (ordinary) cohomology of Bott-Samelson varieties, the T-action on Bott-Samelson varieties and Bott-Samelson subvarieties, and describe the cone of effective one-cycles in a Bott-Samelson variety.

In Chapter 4, we prove Theorem 1.3; we show that, when restricted to T-stable curves, the morphisms θ, π restrict to isomorphisms or collapse the curve to a point. We also compute some examples of T-stable curves in various Bott-Samelson varieties, and provide some of the details for the construction of Figure 4.1.

In Chapter 5, we prove Theorem 1.6 and its corollary using obstruction theory. We also do explicit calculations which will be used in Gromov-Witten calculations.

In Chapters 6 and 7, we piece together the moment graph and moduli space results to compute the presentation of $QH^*(Z)$ given in Theorem 1.8. We also verify that conjecture $\mathcal{O}$ holds for this Bott-Samelson variety (Theorem 7.8).

Chapter 2

Preliminaries

2.1 Notation

The symmetric group on n letters is denoted by S_n. Our convention is that S_n acts on the set $\{1, 2, \ldots, n\}$ on the right. That is, if $\sigma, \tau \in S_n$ and $A \subset \{1, 2, \ldots, n\}$, then

$$A(\sigma\tau) = (A\sigma)\tau.$$

We will sometimes write permutations in "one-line notation." So the simple reflection s_1 interchanging 1 and 2 in S_3 is denoted (2 1 3).

For example, in our conventions the permutation $s_1 s_2 \in S_3$ can be written in one-line notation as (2 3 1).

The *inversion set* of a permutation w, denoted $Inv(w)$, is defined

$$Inv(w) = \{(i, j) : i < j \text{ and } w(i) > w(j)\}$$

The cardinality of the inversion set is the *length* of the permutation, denoted $\ell(w)$.

The *fundamental weights* ω^j $(1 \leq j \leq n-1)$ are simply the ordered $(n-1)$-tuples

$$\omega_j = (1, 1, \ldots, 1, 0, \ldots, 0)$$

(the first j entries being populated by ones, the rest by zeros).

2.2 Flag Varieties

The *flag variety* $F\ell(n)$ is the variety of complete flags in $\mathbb{C}^n$. That is, the points in $F\ell(n)$ are chains of vector spaces

$$0 = F_0 \subset F_1 \subset \ldots \subset F_{n-1} \subset F_n = \mathbb{C}^n$$

where $\dim F_i = i$. We denote the "standard flag" by $E_\bullet$:

$$E_i = \mathrm{Span}_{\mathbb{C}}\{e_1, \ldots, e_i\}$$

where $e_i \in \mathbb{C}^n$ are the standard basis vectors.

The flag variety is equipped with a tautological sequence of vector bundles

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n = \mathbb{C}^n_{F\ell(n)},$$

where the fiber of $\mathcal{F}_i$ over the flag $(0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n)$ is F_i.

By omitting some of the indices i, one obtains a *partial flag variety* $F\ell(n; i_1, \dots, i_k)$ (where the indices $i_1, \dots, i_k$ are omitted). If a single index i is omitted, the full flag variety $F\ell(n)$ is a $\mathbb{P}^1$-bundle over the partial flag variety $F\ell(n; i)$; one vector bundle on $F\ell(n; i)$ which realizes $F\ell(n)$ is $\mathcal{F}_{i+1}/\mathcal{F}_{i-1}$.

Given a permutation $w \in S_n$, there are two natural subvarieties of $F\ell(n)$: the Schubert variety $X(w)$ and the opposite Schubert variety $Y(w)$. They are defined (in our current set-up) as follows:

$$X(w) = \{F_\bullet \in F\ell(n) : \dim(F_p \cap E_q) \geq |\{i \leq p : w(i) \leq q\}| \, \forall p, q\},$$

$$Y(w) = \{F_\bullet \in F\ell(n) : \dim(F_p \cap E_q) \geq |\{i \leq p : w(i) > n - q\}| \, \forall p, q\}.$$

They can also be described in terms of the $SL_n(\mathbb{C})$-action on $F\ell(n)$. Namely, $X(w)$ is the closure of the B-orbit of the permutation matrix corresponding to w, whereas $Y(w)$ is the closure of the B^--orbit of the permutation matrix corresponding to w. (B^- is the *opposite Borel subgroup*; in our set-up, B^- is the subgroup of lower triangular matrices.)

The (singular) cohomology of the flag variety $F\ell(n)$ can be described in multiple ways; we will focus on two. The first is a presentation which involves indeterminants $x_1, x_2, \dots, x_n$ which are identified with the Chern roots of $\mathcal{F}_n^\vee$.

Proposition 2.1. *The cohomology of $F\ell(n)$ has the presentation*

$$H^*(F\ell(n)) = \mathbb{Z}[x_1, x_2, \dots, x_n]/(e_1(x_1, \dots, x_n), \dots, e_n(x_1, \dots, x_n)),$$

where $e_i(x_1, \dots, x_n)$ is the ith elementary symmetric function.

The second description uses the fact that $F\ell(n)$ has a cell decomposition in terms of the Schubert cells. In particular, $H^*(F\ell(n))$ is generated by the classes of the opposite Schubert varieties $\sigma_w = [Y(w)]$. Combined with a Chevalley formula, which computes the product a Schubert divisor by any Schubert class, this gives another description of $H^*(F\ell(n))$ (see [Bri05, Proposition 1.4.3]).

Proposition 2.2. *The cohomology of $F\ell(n)$ is generated by the classes $\{\sigma_w : w \in S_n\}$. The product of any Schubert class σ_v by σ_{s_k} ($1 \leq k \leq n-1$) is given by*

$$\sigma_{s_k} \cdot \sigma_v = \sum (\omega_i - \omega_j)\sigma_{vs_{ij}}$$

where $\omega = \omega^k$ is the kth fundamental weight, and the sum is over the pairs (i, j) such that $1 \leq i < j \leq n$, $vs_{ij} > v$, and $\ell(vs_{ij}) = \ell(v) + 1$.

Remark 2.3. It should be noted, the class σ_w is Poincaré dual to the Schubert class $[X(w)]$. The transition between the two bases is explicit:

$$\sigma_w = [X(w_0 w)].$$

2.3 Projective Bundles

If X is a (complex) variety with vector bundle $V \to X$, by $\mathbb{P}_X(V)$ we mean the projective bundle of lines in V. (Note: this is dual to the convention in [Har77].)

We begin by recording the well-known presentation of the (singular) cohomology of $\mathbb{P}_X(V)$ in terms of $H^*(X)$ and the Chern classes of V.

Proposition 2.4. *Suppose V is a rank r vector bundle over X, and let $p : \mathbb{P}_X(V) \to X$ denote the natural projection morphism. There is a natural ring isomorphism*

$$H^*(\mathbb{P}_X(V)) \simeq H^*(X)[\zeta]/(\zeta^r + c_1(V)\zeta^{r-1} + \cdots + c_{r-1}(V)\zeta + c_r(V)),$$

which identifies ζ with $c_1(\mathcal{O}_V(1))$, and commutes with the pull back p^*.

We also record the relative version of the Euler sequence for projective space.

Proposition 2.5. *Let $p : \mathbb{P}_X(V) \to X$ be a projective bundle, and let T_p denote the relative tangent bundle on $\mathbb{P}_X(V)$. The following pair of sequences of vector bundles on $\mathbb{P}_X(V)$ are exact:*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_X(V)} \longrightarrow p^*(V) \otimes \mathcal{O}_V(1) \longrightarrow T_p \longrightarrow 0$$

$$0 \longrightarrow T_p \longrightarrow T_{\mathbb{P}_X(V)} \longrightarrow p^*(T_X) \longrightarrow 0.$$

Proof. This is a combination of the B.2.7 and B.5.8 in [Ful98]. □

If the projective bundle $p : \mathbb{P}_X(V) \to X$ is equipped with a section $s : X \to \mathbb{P}_X(V)$, there is also a relation between their groups of 1-cycles.

Proposition 2.6. *Let $s : X \to \mathbb{P}_X(V)$ be a splitting of the projective bundle $p : \mathbb{P}_X(V) \to X$, and let f denote the class in $A_1(\mathbb{P}_X(V))$ of a curve contained in the fiber of p. Then the section s induces an isomorphism*

$$A_1(\mathbb{P}_X(V)) \simeq A_1(X) \oplus f\,\mathbb{Z}.$$

2.4 Algebraic Groups

An *algebraic group* is a group G which also posses a variety structure, such that the multiplication $m : G \times G \to G$ and inversion $\iota : G \to G$ maps are morphisms of varieties. The *character group* of G is denoted $X(G) = \{\chi : G \to \mathbb{C}\}$.

Denote the Lie algebra associated with G by $\mathfrak{g}$. We will restrict ourselves to algebraic groups G for which $\mathfrak{g}$ is semisimple. For example, $G = SL_n(\mathbb{C})$ is a semisimple linear algebraic group over $\mathbb{C}$.

In what follows, we will use the structure theory of semisimple linear algebraic groups. We will outline this only in type A here (i.e. for $G = SL_n(\mathbb{C})$). The informed reader can extend to the general case without any difficulty.

We denote by T the subgroup of diagonal matrices in $SL_n(\mathbb{C})$, and let $X = X(T)$, the character group of this torus.

The Lie algebra of G has a decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where $\Phi \subset X$ is the set of *roots*; these are the weights of the adjoint representation of T in $\mathfrak{g}$.

For $SL_n(\mathbb{C})$, the roots correspond to the transpositions $s_{i,j} \in S_n$. The roots which correspond to the simple reflections $s_{i,i+1}$ are called *simple roots*. We will denote the simple root corresponding to $s_{i,i+1}$ by α_i, the set of all simple roots by Δ, and the subset of Φ in the positive cone of the simple roots by Φ^+; these are called *positive roots*. Given $\alpha \in \Phi$, we will denote the transposition it corresponds to by s_α.

Remark 2.7. The correspondence between roots and transpositions is two-to-one. The roots $\alpha, -\alpha$ both correspond to the same transposition.

Another important construction is the set of coroots. These are elements of the cocharacter group $X^*(T)$. Before defining coroots, we first make some remarks on the relationship between $X(T)$ and $X^*(T)$.

We first fix a basis for $X^*(T) \otimes \mathbb{R}$. We shift to $G = GL_n(\mathbb{C})$ to avoid some annoying complications. Write $E_i \in \mathfrak{t}$ for the matrix whose only nonzero entry is 1 in the (i, i)th entry. Write $e_i \in X(T)$ for the characteristic function of E_i. The set $\{e_i : 1 \leq i \leq n\}$ form a basis for $X(T) \otimes \mathbb{R}$. In this basis, the simple roots α_i can be written as

$$\alpha_i = e_i - e_{i+1}.$$

Using this basis, we fix an isomorphism $X(T) \simeq X^*(T)$. Write $f_i \in X^*(T)$ for the image of e_i under this isomorphism.

We also have the natural evaluation pairing $X(T) \otimes X^*(T) \to \mathbb{C}$:

$$\alpha \otimes \lambda \mapsto (\alpha, \lambda) = \lambda(\alpha).$$

Note, that this pairing does not induce the isomorphism described above. Namely, $(\alpha_i, \alpha_i) = 2$ for all simple roots.

The *coroot* $\alpha^\vee$ corresponding to α is defined:

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)} \in X^*(T).$$

Remark 2.8. In type A (which we've been describing), a simple root is equal (under the isomorphism described above) to its coroot. This does not hold in general for other types.

Between the torus T and the algebraic group G, there is a maximal, solvable subgroup of G containing T, called a *Borel subgroup.* The Borel subgroup B is the subgroup corresponding to the following Lie subalgebra of $\mathfrak{g}$:

$$\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

In our particular case, the Borel subgroup B is the subgroup of upper triangular matrices in $SL_n(\mathbb{C})$.

Given a root $\alpha \in \Phi$, we can form the corresponding *(minimal) parabolic subgroup* $P_\alpha \subset G$. This subgroup is defined by

$$P_\alpha = B \cup B s_\alpha B.$$

These subgroups will play a prominent role in what follows.

Chapter 3

Bott-Samelson Varieties

3.1 Definitions and Primary Results

Our main reference in this chapter is [BK05, Chapter 2]. Given a sequence of simple reflections $\mathfrak{w} = (s_{\beta_1}, s_{\beta_2}, \ldots, s_{\beta_k})$, consider the space $P_{\mathfrak{w}} := P_{\beta_1} \times \cdots \times P_{\beta_k}$ equipped with the B^k-action

$$(b_1, \ldots, b_k) \odot (p_1, \ldots, p_r) = (p_1 b_1^{-1}, b_1^{-1} p_2 b_2, \ldots, b_{k-1}^{-1} p_k b_k)$$

Definition 3.1. The *Bott-Samelson variety* $Z_{\mathfrak{w}}$ is the coset space

$$Z_{\mathfrak{w}} := P_{\mathfrak{w}} / B^k$$

The points in $Z_{\mathfrak{w}}$ will be denoted by $[p_1, \ldots, p_k]$.

There is a natural B-action given by

$$b.[p_1, p_2, \ldots, p_k] = [bp_1, p_2, \ldots, p_k]$$

$Z_{\mathfrak{w}}$ contains an affine open cell $Z^{\circ}_{\mathfrak{w}}$ defined by

$$Z^{\circ}_{\mathfrak{w}} := Bs_{\beta_1}B \times \cdots \times Bs_{\beta_k}B/B^k$$

As in Chapter 1, we index the subwords of $\mathfrak{w}$ by binary k-tuples $\varepsilon \in \{0,1\}^k$. For example, if $\mathfrak{w} = (s_1, s_2, s_1)$ and $\varepsilon = (1,1,0)$, then $\mathfrak{w}(\varepsilon) = (s_1, s_2)$. For the same $\mathfrak{w}$, if $\varepsilon = (1,0,1)$, then $\mathfrak{w}(\varepsilon) = (s_1, s_1)$.

For any subword $\mathfrak{w}(\varepsilon)$, there is a natural morphism $\pi_\varepsilon : Z_{\mathfrak{w}} \to Z_{\mathfrak{w}(\varepsilon)}$; if $\mathfrak{w}(\varepsilon)$ is the initial subword of length m, we will denote $Z_{\mathfrak{w}(\varepsilon)}$ by $Z_{\mathfrak{w}[m]}$ and π_ε by π_m.

Recall in Chapter 1 we defined $\pi_+(\varepsilon)$ to be the set of indices i such that $\varepsilon_i = 1$ (Definition 1.1). Define the *length* of ε, denoted $\ell(\varepsilon)$, to be the cardinality of $\pi_+(\varepsilon)$. If $\ell(\varepsilon) = 1$, we will denote $\varepsilon = (i)$ where i is the nonzero entry of ε. When $\varepsilon, \varepsilon'$ have no common components, we say they are *transverse*, denoted $\varepsilon \perp \varepsilon'$.

For each word $\mathfrak{w}$, the product of the simple reflections (in order) is an element of the Weyl group which we denote by $w(\mathfrak{w})$; see Definition 1.1.

Proposition 3.2. *Let $Z_{\mathfrak{w}}$ be a Bott-Samelson variety, $X = G/B$ the flag variety, and let $\mathfrak{w}(\varepsilon)$ be a subword of $\mathfrak{w}$.*

(a) $Z_{\mathfrak{w}}$ is a smooth, projective variety.

(b) The natural morphism $\pi : Z_{\mathfrak{w}} \to Z_{\mathfrak{w}[k-1]}$ defined by

$$\pi([p_1, \ldots, p_k]) = [p_1, \ldots, p_{k-1}]$$

is B-equivariant, and realizes $Z_{\mathfrak{w}}$ as a $\mathbb{P}^1$-bundle over $Z_{\mathfrak{w}[k-1]}$.

(c) *The map* $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to X$ *defined by*

$$\theta_{\mathfrak{w}}([p_1, \ldots, p_k]) = (p_1 p_2 \cdots p_k)B$$

is a B*-equivariant morphism. Moreover, if* $s_{\beta_1} s_{\beta_2} \cdots s_{\beta_k} = w(\mathfrak{w})$ *is a reduced word decomposition, then* $\theta_{\mathfrak{w}}$ *is a birational equivalence:* $Z^{\circ}_{\mathfrak{w}} \xrightarrow{\simeq} X(w(\mathfrak{w}))^{\circ}$.

(d) *The map* $j_{\varepsilon} : Z_{\mathfrak{w}(\varepsilon)} \to Z_{\mathfrak{w}}$ *defined by*

$$j_{\varepsilon}([p_1, \ldots, p_\ell]) = [1, 1, \ldots, p_1, 1, \ldots, p_\ell, 1, \ldots, 1]$$

(where ones are placed in the components where ε *is zero) is a* B*-equivariant closed immersion.*

For the $(k-1)$*-initial subword, the morphism will be denoted by* $s_{\mathfrak{w}} : Z_{\mathfrak{w}[k-1]} \to Z_{\mathfrak{w}}$ *and is a section of* $\pi_{\mathfrak{w}}$.

(e) *The natural commutative diagram*

$$\begin{tikzcd} Z_{\mathfrak{w}} \arrow[r, "\theta_{\mathfrak{w}}"] \arrow[d, "\pi_{\mathfrak{w}}"'] & X \arrow[d, "p_{\beta_k}"] \\ Z_{\mathfrak{w}[k-1]} \arrow[r, "p_{\beta_k}\theta_{\mathfrak{w}[k-1]}"] & G/P_{\beta_k} \end{tikzcd}$$

is Cartesian; that is, $Z_{\mathfrak{w}}$ *is the fiber product* $Z_{\mathfrak{w}[k-1]} \times_{G/P_{\beta_k}} X$*. The* B*-action on the fiber product is diagonal.*

Proof. As mentioned before the statement of the proposition, parts (a)-(d) are discussed in [BK05, pp. 64-67]. Part (e) is Exercise 2.2.E.1 in [BK05], with the exception of the B-action statement. This follows easily since each of the maps $\pi_{\mathfrak{w}}$ and $\theta_{\mathfrak{w}}$ are B-equivariant. $\square$

3.2 Magyar's Bott-Samelson construction

In [Mag98], a realization of Bott-Samelson varieties as configuration spaces is given in the case $G = GL(n, \mathbb{C})$. Since our main example is in type A, and since the construction is quite explicit, we will (briefly) describe the construction; more details can be found in [Mag98, Section 2].

Let $[i] = \{1, 2, \ldots, i\}$ and let $e_1, e_2, \ldots, e_n$ be the standard basis for $\mathbb{C}^n$. Given a fixed decomposition $s_{i_1} s_{i_2} \cdots s_{i_k} = w \in S_n$, one defines the *full chamber family*

$$D_{\mathfrak{w}}^+ = \{s_{i_1}[i_1], s_{i_1}s_{i_2}[i_2], \ldots, w[i_k]\} \cup \{[1], [2], \ldots, [n]\}$$

Associated to the full chamber family, is a configuration diagram with nodes $\{V_I : I \in D_{\mathfrak{w}}^+\}$ subject to the following conditions:

(a) If $I = [i]$ for some $1 \leq i \leq n$, then $V_I = \mathrm{Span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_i\}$.

(b) If $I \subset J$ and $\#J = \#I + 1$, then there is a line joining V_I and V_J.

The points in the Bott-Samelson variety $Z_{\mathfrak{w}}$ are precisely collections of vector spaces $\{V_I : I \in D_{\mathfrak{w}}^+\}$ where $\dim V_I = \#I$ and the vector spaces satisfy the incidence conditions given by the configuration diagram, i.e. if V_I and V_J are joined with $\dim V_I < \dim V_J$, then $V_I \subset V_J$.

Example 3.3. Given the reduced word decomposition $w_0 = (3\,2\,1) = s_1 s_2 s_1 \in S_3$, we form the full chamber family

$$D_{\mathfrak{w}}^+ = \{\{2\}, \{2, 3\}, \{3\}, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$$

and the associated configuration diagram

$$\begin{tikzcd}
 & \mathbb{C}^3 \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{C}^2 \arrow[d, dash] \arrow[dr, dash] & & V_{23} \arrow[dl, dash] \arrow[d, dash] \\
\mathbb{C}^1 \arrow[dr, dash] & V_2 \arrow[d, dash] & V_3 \arrow[dl, dash] \\
 & (0) &
\end{tikzcd}$$

The Bott-Samelson variety $Z_{\mathfrak{w}}$ (as a set) consists of points (V_2, V_3, V_{23}) (all vector spaces in $\mathbb{C}^3$) where $\dim V_2 = \dim V_3 = 1$, $\dim V_{23} = 2$, and V_2, V_3, V_{23} satisfy the incidence conditions given by the diagram. That is, $V_2 \subset \mathbb{C}^2 \cap V_{23}$ and $V_3 \subset V_{23}$.

In this construction, the map to the flag, $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to F\ell(3)$, sends (V_2, V_3, V_{23}) to $V_\bullet : (0) \subset V_3 \subset V_{23} \subset \mathbb{C}^3$. That is, the map to the flag is given by the rightmost flag in the configuration diagram.

3.3 Cohomology of Bott-Samelson Varieties

The cells $Z_\varepsilon^\circ := j_\varepsilon(Z(\mathfrak{w}(\varepsilon))^\circ)$ form an affine cell decomposition of $Z_{\mathfrak{w}}$:

$$Z_{\mathfrak{w}} = \bigcup_{\varepsilon} Z_\varepsilon^\circ$$

In particular, $\{[Z_\varepsilon] : \varepsilon \in \{0,1\}^k\}$ is an additive basis for $H^*(Z_{\mathfrak{w}})$; the dual basis (under the Poincare pairing) is denoted $\{\sigma_\varepsilon : \varepsilon \in \{0,1\}^k\}$.

A presentation for the (ordinary) cohomology of Bott-Samelson varieties was obtained in [Dua05, Lemma 4.5].

Proposition 3.4. *The cohomology of $Z_{\mathfrak{w}}$ is generated by $\{\sigma_\varepsilon : \varepsilon \in \{0,1\}^k\}$ with relations*

$$\sigma_\varepsilon \sigma_{\varepsilon'} = \sigma_{\varepsilon+\varepsilon'}, \qquad \text{if } \varepsilon \perp \varepsilon'$$

$$\sigma_{(j)}^2 = \sum_{i<j} -(\alpha_i, \alpha_j^\vee)\sigma_{(i)+(j)}$$

where $(\alpha, \beta^\vee)$ is the usual root/coroot pairing.

Remark 3.5. It is known, though the author was unable to find a reference, that the dual classes σ_ε are preserved under pullback along the morphisms $\pi_{\mathfrak{w}[r]} : Z_{\mathfrak{w}} \to Z_{\mathfrak{w}[r]}$ $(1 \leq r \leq k-1)$. We will write formulas such as: $\pi^*_{\mathfrak{w}[r]}(\sigma_\varepsilon) = \sigma_\varepsilon$. One should interpret $\varepsilon \in \{0,1\}^r$ as a binary k-tuple $(k > r)$ by appending zeros to the end of ε.

We include a proof that the dual classes are preserved under pull-back for completeness.

Proof. It suffices to show that $\sigma_{(i)}$ is preserved under pullback for all i. Let $i < k$ and consider $\sigma_{(i)} \in H^*(Z')$. Then

$$\int_Z \pi^* \sigma_{(i)} \cdot [Z_\varepsilon] = \int_{Z'} \sigma_{(i)} \cdot \pi_*[Z_\varepsilon].$$

However, Z_ε is either the preimage under π of a Bott-Samelson subvariety in Z', in which case $\pi_*[Z_\varepsilon] = 0$, or Z_ε is the image of Z'_ε under the canonical section $s : Z' \to Z$ and $\pi_*[Z_\varepsilon] = [Z'_\varepsilon]$. Therefore, $\pi^*\sigma_{(i)}$ is Poincaré dual to $Z_{(i)}$, as claimed. $\square$

3.4 The Cone of Effective Curves

To conclude this chapter, we define the cone of effective curves.

Definition 3.6. The *cone of effective curves* in $H_2(Z)$ is the set of all effective 1-cycles; that is, a positive combination of the fundamental classes of irreducible curves in Z.

In [And15, Lemma 2.1], it is stated that for complete, irreducible varieties X, the cone of effective k-cycles on X is generated by the classes of B-invariant k-cycles. Thus, the cone of effective curves for a Bott-Samelson variety is generated by the classes of the B-stable curves. We characterize those curves in the next section, and we give a quick proof of [And15, Lemma 2.1].

Proof. Let $d \in N_k(X)_{\mathbb{R}}$ be an irreducible, effective k-cycle on X, and let C_d^X denote the Chow variety for X of degree d. The B-action on X naturally lifts to C_d^X, and by the Borel fixed point theorem, there is a B-fixed point in C_d^X. This fixed point corresponds to a B-invariant k-cycle on X with degree d.

Therefore, every k-cycle on X can be written as a sum of B-invariant k-cycles. $\square$

Chapter 4

The Moment Graph

Bott-Samelson varieties are sympletic varieties with respect to the given torus action, and thus are equipped with a *moment map* $Z \to \mathfrak{t}^*$; see [Esc14, Section 4.1] for more details on the moment map for Bott-Samelson varieties, along with a description of the images of the T-fixed points under this map.

The image of the 1-skeleton of Z under this map (that is, the image of the T-fixed points and T-stable curves) is called the *moment graph* for Z. The T-fixed points in a Bott-Samelson variety were discussed in Chapter 3, so it remains to describe the T-stable curves.

To begin, we characterize the T-stable curves on a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Let $X(T)$ the character group of T, and suppose $p : \Sigma \to \mathbb{P}^1$ is a T-equivariant $\mathbb{P}^1$-bundle. Moreover, we assume the T-actions on $\mathbb{P}^1$ and Σ are nontrivial. For $x \in \Sigma^T$, the *weights* of Σ at x are $\chi, \psi \in X(T)$

where

$$t.v = \chi(t)v, \quad v \in T_{p(x)}\mathbb{P}^1$$

$$t.w = \psi(t)w, \quad w \in T_xF$$

and F is the (geometric) fiber $p^{-1}(x)$.

Lemma 4.1. *There are infinitely many T-stable curves passing through $x \in \Sigma^T$ if and only if the weights at x, χ and ψ, are equal. Otherwise, there are exactly two (irreducible) T-stable curves passing through x.*

Proof. Since Σ is smooth and projective, there is a T-stable affine open neighborhood of x which is T-isomorphic to $T_x\Sigma$ ([ByB73, Theorem 2.5].) Choose local coordinates X, Y so that $\mathbb{C}[T_x\Sigma] \simeq \mathbb{C}[X, Y]$ with $t.X = \chi(t)X$ and $t.Y = \psi(t)Y$.

If $\chi = \psi$, then it is clear that the lines $V(X - \alpha Y) \subset T_x\Sigma$ ($\alpha \in \mathbb{C}$) are T-stable curves; in particular, there are infinitely many T-stable curves in Σ passing through x.

If $\chi \neq \psi$, the characters are linearly independent. Therefore, the only T-stable curves passing through x are (in local coordinates) $V(X), V(Y)$. $\square$

If there is a T-fixed point $x \in \Sigma$ with repeated weights, it is also easy to show there are exactly two T-fixed points with repeated weights. Moreover, there are additional T-stable curves in Σ.

Lemma 4.2. *Let $\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$ (where each $\mathbb{P}^1$ is equipped with a nontrivial T-action). If the T-fixed points in Σ all have distinct weights, then the T-stable fibers of the two projections* *are the only T-stable curves. Otherwise, the images of the sections $s_t : \mathbb{P}^1 \to \Sigma$ ($t \in T$), defined by $s_t(z) = (z, t.z)$, are also T-stable, and these exhaust the set of T-stable curves in Σ.*

Proof. Let $x \in \Sigma^T$. By the previous lemma, the only case that requires analysis is when there are repeated weights in $T_x\Sigma$. Since $T_x\Sigma$ has equal weights, the T-actions on each factor $\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$ are equal. Thus,

$$\begin{aligned} s_t(t'.z) = (t'.z, t.(t'.z)) &= (t'.z, t'.(t.z)) \\ &= t'.(z, t.z) \end{aligned}$$

that is, s_t is a T-equivariant section of $p_1 : \Sigma \to \mathbb{P}^1$.

These curves exhaust the set of (irreducible) T-stable curves in Σ since any other T-stable curve C, aside from the T-stable fibers, intersects one of the sections s_t in a point not fixed by T. Therefore, C shares a dense open orbit with the section s_t and so is equal to the image $s_t(\mathbb{P}^1)$. □

Example 4.3. In Section 3.2, a configuration variety interpretation of Bott-Samelson varieties was discussed in type A. For the Bott-Samelson variety $Z(\alpha_1, \alpha_2, \alpha_1)$, the points correspond to configuration diagrams

$$\begin{tikzcd}
& \mathbb{C}^3 \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{C}^2 \arrow[d, dash] \arrow[dr, dash] & & V_{23} \arrow[dl, dash] \arrow[d, dash] \\
\mathbb{C}^1 \arrow[dr, dash] & V_2 \arrow[d, dash] & V_3 \arrow[dl, dash] \\
& (0) &
\end{tikzcd}$$

where V_2, V_3, V_{23} are vector subspaces of $\mathbb{C}^3$, $\dim V_2 = \dim V_3 = 1$ and $\dim V_{23} = 2$, with a line between two subspaces meaning inclusion (so, $V_2 \subset V_{23}$ and $V_2 \subset \mathbb{C}^2$ in the standard basis).

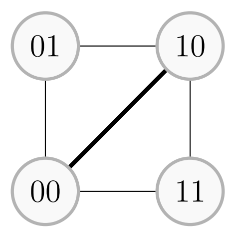
The sub Bott-Samelson variety Z_{101} (note this subword is not reduced) is characterized by $V_{23} = \mathbb{C}^2$, so the configuration diagram is

$$\begin{tikzcd}
& \mathbb{C}^3 \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{C}^2 \arrow[d, dash] \arrow[dr, dash] & & \mathbb{C}^2 \arrow[dl, dash] \arrow[d, dash] \\
\mathbb{C}^1 \arrow[dr, dash] & V_2 \arrow[d, dash] & V_3 \arrow[dl, dash] \\
& (0) &
\end{tikzcd}$$

Thus $Z_{101} \simeq \mathbb{P}^1 \times \mathbb{P}^1$; this isomorphism is the natural morphism

$$\begin{tikzcd}
Z_{101} \arrow[dr, dashed] \arrow[drr, "\theta"] \arrow[ddr, "\pi"'] & & \\
& \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r] \arrow[d] & X(s_1) \\
& Z(s_1) \simeq X(s_1) &
\end{tikzcd}$$

and is T-equivariant for the diagonal T-action on $X(s_1) \times X(s_1)$. The moment graph for Z_{101} is



where the bold line is an infinite family of T-stable curves (given by the images of the morphisms s_t).

Before proving our main theorem, we prove a critical lemma.

Lemma 4.4. *Let $C \subset Z$ be an irreducible T-stable curve.*

(a) If C is not contained in a fiber of $\theta : Z \to G/B$, then $\theta|_C$ is an isomorphism.

(b) If C is not a fiber of $\pi : Z \to Z'$, then $\pi|_C$ is an isomorphism.

In particular, $C \simeq \mathbb{P}^1$.

Proof. We'll prove (a); the proof of (b) is similar. First, observe that $\theta|_C$ is an isomorphism if C is a fiber of π, thus we may assume C is not a fiber of π; let $C' = \pi(C)$. By induction on $\dim Z$, $\theta'|_{C'}$ is an isomorphism. There are two possibilities:

1. $(p_{\beta_k}\theta')|_{C'}$ is an isomorphism. In this case, the preimage $\pi^{-1}(C')$ is isomorphic (under θ) to a surface in G/B. In particular, $\theta|_C$ is an isomorphism.

2. $(p_{\beta_k}\theta')(C')$ is a T-fixed point in G/P_{β_k}. In this case, the preimage $\pi^{-1}(C')$ is T-equivariantly isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ with the diagonal T-action (coming from G/B). Hence, C is a section of θ over $X(s_{\beta_k})$ by Lemma 4.2. In particular, $\theta|_C$ is an isomorphism.

From (b), any T-stable curve C is isomorphic to a fiber, and hence is isomorphic to $\mathbb{P}^1$. $\square$

Lemma 4.5. *Let $C \subset Z(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a T-stable curve, and k the largest integer so that $\dim \pi_k(C) = 0$. Then, if $x \in C^T$, the weight at x is $\varepsilon_x(\alpha_k)$.*

Proof. If C is a fiber of π, then the result follows since $\theta|_C$ is an isomorphism. Otherwise, by Lemma 4.4, $\pi|_C$ is an isomorphism and the result follows by induction on $n = \dim Z$. $\square$

We are now ready to prove our main theorem, Theorem 1.3, stated in the introduction.

Proof of Theorem 1.3. For clarity, we will describe the moment graph pictures as Diagrams I-IV, and the characterizations in terms of $\varepsilon(\alpha_k)$ as Cases I-IV.

We only consider the case where $\pi(x), \pi(y)$ are joined by a T-stable curve C, since the other case is obvious. Let Σ be defined as the pull-back $\Sigma = \pi^{-1}(C)$ as in the fiber diagram

$$\begin{tikzcd} \Sigma \arrow[r, hook] \arrow[d, "\pi|_\Sigma"'] & Z \arrow[d, "\pi"] \\ C \arrow[r, hook] & Z' \end{tikzcd}$$

In particular, Σ is the fiber product

$$\begin{tikzcd} \Sigma \arrow[r, "\theta|_\Sigma"] \arrow[d, "\pi|_\Sigma"'] & G/B \arrow[d, "p_{\alpha_n}"] \\ C \arrow[r, "p_{\alpha_n}\theta'|_C"'] & G/P_{\alpha_n} \end{tikzcd}$$

Lemma 4.4 shows $p_{\alpha_n}\theta'|_C$ is either an isomorphism or a point map. If an isomorphism, then Σ is isomorphic (under θ) to a surface in G/B. In particular, the moment graph restricted to Σ is diagram I.

Since the only curve whose homology class is not clear is the curve joining $\varepsilon_x^\infty, \varepsilon_y^\infty$, we compute the homology class of that curve only. From Lemma 4.5, the weight along C at $\pi(x)$ is $\varepsilon_{\pi(x)}(\alpha_k)$, and the weight at $\pi(y)$ is $\varepsilon_{\pi(y)}(\alpha_k)$. Since $\theta|_{s(C)} = \theta'|_C$ is an isomorphism, the weights are preserved, and since $\theta(C)$ is a translate of $X(s_{\alpha_k})$, we have $(\theta s)_*[C] = [X(s_{\alpha_k})]$. In particular, $w(\varepsilon_x^0) = w(\varepsilon_y^0)s_{\alpha_k}$. Since $p_{\alpha_n}\theta'|_C$ is an isomorphism, $\alpha_k \neq \alpha_n$, so case I holds.

We can then compute the relation between $w(\varepsilon_x^\infty), w(\varepsilon_y^\infty)$:

$$\begin{aligned} w(\varepsilon_x^\infty) = w(\varepsilon_x^0)s_{\alpha_n} &= (w(\varepsilon_y^0)s_{\alpha_k})s_{\alpha_n} \\ &= (w(\varepsilon_y^0)s_{\alpha_n})s_\beta = w(\varepsilon_y^\infty)s_\beta \end{aligned}$$

where $\beta = s_{\alpha_n}(\alpha_k) = \alpha_k - (\alpha_k, \alpha_n^\vee)\alpha_n$. Thus, the degree of the image of the curve joining $\varepsilon_x^\infty, \varepsilon_y^\infty$ is $[X(s_{\alpha_k})] - (\alpha_k, \alpha_n^\vee)[X(s_{\alpha_n})]$. Note, that since $\alpha_k \neq \alpha_n$, $(\alpha_k, \alpha_n^\vee) \leq 0$.

Moreover, the curve joining $\varepsilon_x^\infty, \varepsilon_y^\infty$ has homology class $s_*[C] + ah$ for some constant a. Equating the push-forward calculations, we obtain $a = -(\alpha_k, \alpha_n^\vee)$. Therefore, the homology class of the curve joining $\varepsilon_x^\infty, \varepsilon_y^\infty$ is $s_*[C] - (\alpha_k, \alpha_n^\vee)h$.

We now consider the case where $p_{\alpha_n}\theta'|_C$ is a point map. In this case, Σ is the trivial $\mathbb{P}^1$-bundle $\Sigma = C \times X(s_{\alpha_n})$. According to Lemma 4.4, there are cases corresponding to whether all T-fixed points in Σ have distinct weights or not.

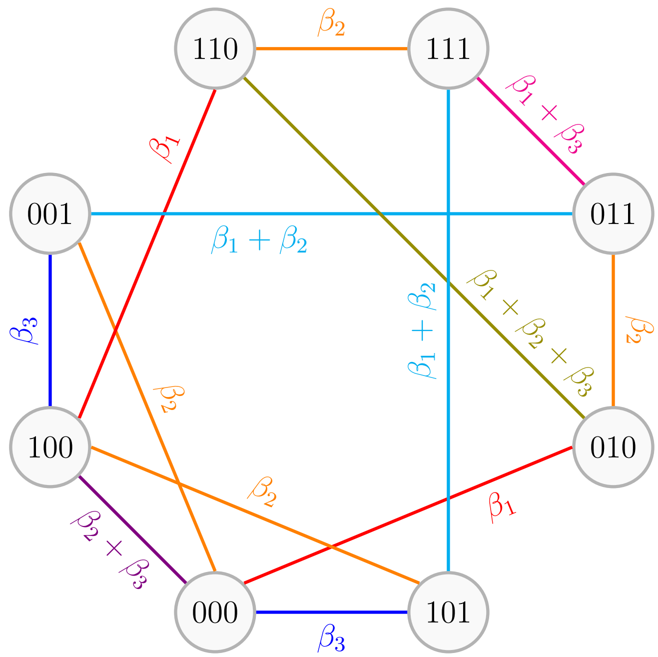
As above, the weights along C at $\varepsilon_x^0, \varepsilon_y^0$ are $\varepsilon_x^0(\alpha_k), \varepsilon_y^0(\alpha_k)$ respectively. Thus, there are repeated weights in cases II, III, and IV. Indeed, there can only be exactly two fixed points with repeated weights, and they cannot be $\varepsilon_x^\infty, \varepsilon_y^\infty$, otherwise there could not be a T-stable curve joining $\varepsilon_x^0, \varepsilon_y^0$. By Lemma 4.2, there are infinitely many T-stable curves joining the two fixed points with repeated weights, hence we get the moment graph pictures which correspond to diagrams II, III, and IV.

If all fixed points have distinct weights, we get the moment graph in diagram I, and the curve joining $\varepsilon_x^\infty, \varepsilon_y^\infty$ is a fiber of the second projection. Therefore, if $w(\varepsilon_x^0) = w(\varepsilon_y^0)$ and case I holds, the homology class of the unlabeled curve is $s_*[C]$.

If $\varepsilon_x^0, \varepsilon_y^0$ are the fixed points with repeated weights, then the moment graph is given by diagram II. Moreover, $w(\varepsilon_x^0) = w(\varepsilon_y^0)s_{\alpha_n}$ so case II holds. From this, $w(\varepsilon_x^0) = w(\varepsilon_y^\infty)$ and vice-versa. Therefore, the homology class of the diagonal curves is $s_*[C] - h$.

If $w(\varepsilon_x^0) = w(\varepsilon_y^0)$, and there are two points with repeated weights, they must be diagonally adjacent since both $\varepsilon_x^0, \varepsilon_y^0$ cannot have repeated weights, in particular cases III and IV hold.

Figure 4.1: The moment graph for $Z(\alpha_1, \alpha_2, \alpha_1)$.



Diagrams III and IV are the cases where ε_x^0 and ε_y^0 respectively have repeated weights. In both cases, the diagonal family of curves have pushforward $[X(s_{\alpha_n})]$, while $(\theta s)_*[C] = 0$. Therefore, the homology class of the diagonal curves is $s_*[C] + h$. $\square$

Example 4.6. Consider the Bott-Samelson variety $Z = Z(\alpha_1, \alpha_2, \alpha_1)$ in type A_2. As stated in Section 3.4, the cone of effective 1-cycles is generated by the B-stable 1-cycles (see [And15, Lemma 2.1].) A basis for the cone of effective 1-cycles is given by $\beta_1 = [Z_{010}]$, $\beta_2 = [Z_{001}]$, $\beta_3 = [Z_{100}] - [Z_{001}]$.

The entire moment graph for the Bott-Samelson variety Z is depicted in Figure 4.1:

Chapter 5

The Moduli Space of Stable Maps

We now turn to our computation of the quantum cohomology of the Bott-Samelson variety $Z = Z(\alpha_1, \alpha_2, \alpha_1)$. In order to compute a presentation $QH^*(Z)$, we need to compute many Gromov-Witten invariants. Some will vanish using curve neighborhoods, however we will need to compute some nonzero invariants in order to use brute-force calculations to finish the presentation. This section provides the tools we need to compute these nonzero invariants. We start with a definition.

Definition 5.1. We say an effective curve class $\beta \in H_2(Z)$ is *indecomposable* if β cannot be expressed: $\beta = \beta_1 + \beta_2$, where $\beta_1, \beta_2 \in H_2(Z)$ are effective.

Example 5.2. In the threefold $Z = Z(\alpha_1, \alpha_2, \alpha_1)$ in type A_2, the classes $[Z_{010}]$ and $[Z_{001}]$ are indecomposable, but

$$[Z_{100}] = [Z_{001}] + \beta_3$$

where $\beta_3 = [C]$, the fiber of $\theta : Z \to F\ell(3)$ over the identity.

Indeed, since C is a fiber of θ, $\theta_*[C] = 0$. Furthermore, C is a T-stable curve which is not contained in the fiber of $\pi : Z(\alpha_1, \alpha_2, \alpha_1) \to Z(\alpha_1, \alpha_2)$. So by Lemma 4.4, $\pi_*\beta_3 = [Z_{10}]$. Since $\pi_*\beta_3 = \pi_*([Z_{100}] - [Z_{001}])$ and $\theta_*\beta_3 = \theta_*([Z_{100}] - [Z_{001}])$, and Z is a fiber product, we have $\beta_3 = [Z_{100}] - [Z_{001}]$

In fact, the generators $\beta_1, \beta_2, \beta_3$ of the effective cone for $Z(\alpha_1, \alpha_2, \alpha_1)$ are all indecomposable.

Recall, the moduli space of stable maps $\overline{M}_{0,n}(Z, \beta)$ consists of stable maps $f : C \to Z$, where C is decorated with n non-singular marked points $p_1, \ldots, p_n \in C$, and $f_*[C] = \beta$. (The stability condition is equivalent to the automorphism group of the marked curve C being finite.)

Proposition 5.3. *If $\beta \in H_2(Z)$ is indecomposable, and the Dynkin diagram of G is simply laced, then the moduli space $\overline{M}_{0,1}(Z, \beta)$ is unobstructed; that is, $\overline{M}_{0,1}(Z, \beta)$ is smooth, irreducible, and has the expected dimension*

$$\dim \overline{M}_{0,1}(Z, \beta) = \dim Z + \int_\beta c_1(T_Z) - 2.$$

Proof. The proof is by induction. Let $[g : C \to Z] \in \overline{M}_{0,1}(Z, \beta)$. Since β is indecomposable and C has a single marked point, $C \simeq \mathbb{P}^1$. There are two cases:

(a) β is the class of the fiber of $\pi : Z \to Z'$. Since $g^*\pi^*T_{Z'}$ is a trivial line bundle, $H^1(C, g^*\pi^*T_{Z'}) = 0$. Moreover, $T_\pi = \theta^*T_{p_{\alpha_n}}$ and $\theta_*\beta = [X(s_{\alpha_n})]$ which implies $g^*T_\pi \simeq \mathcal{O}_{\mathbb{P}^1}(2)$ (since $c_1(T_{p_{\alpha_n}}) \cap [X(s_{\alpha_n})] = (\alpha_n, \alpha_n^\vee)$). Thus $H^1(C, g^*T_\pi) = 0$, and therefore $H^1(C, g^*T_Z) = 0$.

(b) Otherwise, $\pi_*\beta \neq 0$ (and effective). Moreover, $\pi_*\beta$ is indecomposable, otherwise $\pi_*\beta = \beta_1' + \beta_2'$ which implies $\beta - s_*\beta_1'$ is effective. By induction, $\overline{M}_{0,1}(Z', \pi_*\beta)$ is unobstructed, thus $H^1(C, g^*\pi^*T_{Z'}) = 0$. To compute $H^1(C, g^*T_\pi)$ there are two cases:

1. If $\theta_*\beta = 0$, then g^*T_π is the trivial line bundle on C since

$$\int_C c_1(g^*T_\pi) \cap [C] = \int_Z c_1(T_\pi) \cap \beta = \int_{G/B} c_1(T_{p_{\alpha_n}}) \cap \theta_*\beta.$$

   Thus, $H^1(C, g^*T_\pi) = 0$.

2. If $\theta_*\beta \neq 0$, then since β is represented by a B-stable curve, the image of that curve under θ is a Schubert curve. In particular, there is a unique α' for which the integral

$$\int_{G/B} \theta_*\beta \cdot [Y(s_{\alpha'})] = \int_Z \beta \cdot \theta^*[Y(s_{\alpha'})] = \sum \int_Z \beta \cdot \sigma_\varepsilon$$

   has value equal to 1, since β is indecomposable. Therefore, $g^*T_\pi \simeq \mathcal{O}_{\mathbb{P}^1}(d)$ where $d = (\alpha, \alpha') \geq -1$ since the Dynkin diagram for G is simply laced, and so $H^1(C, g^*T_\pi) = 0$.

In either case, we have $H^1(C, g^*T_\pi) = 0$ and $H^1(C, g^*\pi^*T_{Z'}) = 0$, therefore $H^1(C, g^*T_Z) = 0$. □

Corollary 5.4. *If β is indecomposable and Z is the disjoint union of curves of class β (e.g. if Z is a β-fibration), then $ev : \overline{M}_{0,1}(Z, \beta) \to Z$ is an isomorphism.*

Proof. Since β is indecomposable, $\overline{M}_{0,1}(Z, \beta)$ is smooth. Moreover, every point in Z lies on a unique curve of class β, therefore $ev : \overline{M}_{0,1}(Z, \beta) \to Z$ is a bijection. Since both are varieties over $\mathbb{C}$, ev is an isomorphism. □

Example 5.5. For $Z = Z(\alpha_1, \alpha_2, \alpha_1)$ and $\beta = \beta_3$

$$ev : \overline{M}_{0,1}(Z, \beta) \to Z_{101}$$

is an isomorphism from Corollary 5.4.

If h is the class of the fiber for any Bott-Samelson variety Z, $ev : \overline{M}_{0,1}(Z, h) \to Z$ is an isomorphism since h is indecomposable.

Chapter 6

Curve Neighborhoods

As we stated in the introduction, the connection between our moment graph result and quantum cohomology is given by curve neighborhoods. Background on curve neighborhoods, particularly in the context of homogeneous spaces, can be found in [BM15]. The main difference here is that curve neighborhoods need not "grow."

Definition 6.1. Let X be a variety, Y a subset of X, and $\beta \in A_1(X)$ an effective curve class. The curve neighborhood $\Gamma_\beta(Y)$ is defined by

$$\Gamma_\beta(Y) := ev_1(ev_2^{-1}(Y))$$

where $ev_i : \overline{M}_{0,2}(X, \beta) \to X$ $(i = 1, 2)$ are the evaluation morphisms; $\Gamma_\beta(Y)$ is given the reduced scheme structure.

Observe, if X is a G-variety for an algebraic group G, and Y is a G-stable closed subvariety of X, then $\Gamma_\beta(Y)$ is a G-stable closed subvariety of X. Also, note that since the morphism

$ev : \overline{M}_{0,2}(X, \beta) \to X$ is proper, $\Gamma_\beta(Y)$ can be realized as the closure of the union of all curves C of class β passing through Y.

Proposition 6.2. *For any B-stable subvariety $\Omega \subset Z(\alpha_1, \alpha_2, \alpha_1)$, the curve neighborhood $\Gamma_{\beta_3}(\Omega) \subset Z_{101}$. Furthermore, if $\theta(\Omega) = X(s_1)$, then $\Gamma_{\beta_3}(\Omega) = Z_{101}$.*

Proof. Note that any curve of class β_3 collapses under the map $\theta : Z(\alpha_1, \alpha_2, \alpha_1) \to G/B$. Since θ is an isomorphism outside the locally closed set Z°_{101}, any curve of class β_3 is contained in the sub Bott-Samelson variety Z_{101}. Since Z_{101} is a closed, B-stable variety in $Z(\alpha_1, \alpha_2, \alpha_1)$, the first claim follows.

For the second claim, since $Z_{101} \simeq \mathbb{P}^1 \times \mathbb{P}^1$ where β_3 is the class of the fiber of the second projection, every point $x \in Z_{101}$ has a unique curve of class β_3 passing through it which also intersects Ω. $\square$

Lemma 6.3. *The following are curve neighborhoods for the Bott-Samelson variety $Z(\alpha_1, \alpha_2, \alpha_1)$:*

$$\Gamma_{\beta_3}(Z_{100}) = Z_{101}, \quad \Gamma_{\beta_3}(Z_{010}) = \theta^{-1}(x_e), \quad \Gamma_{\beta_3}(Z_{001}) = Z_{101}, \quad \Gamma_{\beta_1}(Z_{100}) = Z_{110}.$$

Proof. We prove each equality in the order specified above.

$\Gamma_{\beta_3}(Z_{100})$: Note that $Z_{100} \subset Z_{101}$. In fact, the image of Z_{100} under $\theta : Z(\alpha_1, \alpha_1) \to X(s_{\alpha_1})$ is $X(s_{\alpha_1})$. Therefore, the union of the fibers of $pr_2 : Z(\alpha_1, \alpha_1) \to X(s_{\alpha_1})$ is the curve neighborhood $\Gamma_{\beta_3}(Z_{100})$. However this union is the whole space. Therefore, $\Gamma_{\beta_3}(Z_{100}) = Z_{101}$.

$\Gamma_{\beta_3}(Z_{010})$: Since $Z_{010} \cap Z_{101} = x_{000}$, and since there is a unique curve of class β_3 passing through that point (the curve joining x_{000} to x_{101}) the curve neighborhood is $\Gamma_{\beta_3}(Z_{010}) = C$, the fiber of θ over $x_e \in G/B$.

$\Gamma_{\beta_3}(Z_{001})$: A similar analysis to $\Gamma_{\beta_3}(Z_{100})$ shows that $\Gamma_{\beta_3}(Z_{001}) = Z_{101}$ (since the image of Z_{001} under $\theta : Z(\alpha_1, \alpha_1) \to X(s_{\alpha_1})$ is $X(s_{\alpha_1})$).

$\Gamma_{\beta_1}(Z_{100})$: There is a T-stable curve of class β_1 joining x_{100} to x_{110}. Since the B-action on the cell Z°_{110} is transitive ($\theta|_{Z_{110}}$ is an isomorphism,) and since Z_{100} is B-stable, we see that $\Gamma_{\beta_1}(Z_{100}) = Z_{110}$.

$\square$

Remark 6.4. In general, it is clear that curve neighborhoods will be connected. However, it would be useful to have a condition for when a particular curve neighborhood is irreducible. For example, for Schubert varieties, it is known that all such curve neighborhoods are irreducible (see [BCMP13]).

Chapter 7

Quantum Cohomology

7.1 Introduction

Let X be a smooth, projective $\mathbb{C}$-variety. Fix a homogeneous basis $\{\gamma_j\}$ for $H^*(X)$, and a basis $\{\beta_k\}$ for the cone of effective curves in $H_2(X)$. As a $\mathbb{Q}$-vector space, the (small) quantum cohomology $QH^*(X) = H^*(X;\mathbb{Q}) \otimes \mathbb{Q}[q^{\beta_k}]$, where there is one quantum parameter q^{β_k} for each generator of the effective cone. The quantum product of two classes $x, y \in H^*(X)$ is defined by

$$x * y = \sum_{\beta, j} I_\beta(x, y, \gamma_j^\vee) q^\beta \gamma_j$$

where $\gamma_j^\vee$ is Poincaré dual to γ_j, and the *Gromov-Witten invariant* $I_\beta(x, y, \gamma_j^\vee)$ is defined by

$$I_\beta(x, y, \gamma_j^\vee) = \int_{[\overline{M}_{0,3}(X,\beta)]^{\mathrm{virt}}} ev_1^*(x) \cdot ev_2^*(y) \cdot ev_3^*(\gamma_j^\vee)$$

where $[\overline{M}_{0,3}(X,\beta)]^{\text{virt}}$ is the *virtual fundamental class*, a generalization of the fundamental class which is necessary since $\overline{M}_{0,3}(X,\beta)$ is not generally irreducible or even equidimensional. The class $[\overline{M}_{0,3}(X,\beta)]^{\text{virt}}$ has the "expected dimension" of the moduli space (see Proposition 5.3).

In general, for $\sigma_1,\ldots,\sigma_n \in H^*(X)$, Gromov-Witten invariants are defined on $\overline{M}_{0,n}(X,\beta)$ as

$$I_\beta(\sigma_1,\ldots,\sigma_n) = \int_{[\overline{M}_{0,n}(X,\beta)]^{\text{virt}}} \prod_j ev_j^*(\sigma_j).$$

For us, there are two important properties of Gromov-Witten invariants:

1. The Gromov-Witten invariant $I_\beta(\sigma_1,\ldots,\sigma_n) = 0$ unless

$$\sum_j \operatorname{codim} \sigma_j = \dim X + \int_\beta c_1(T_X) + n - 3;$$

   this property is called the *codimension condition*.

2. If $\sigma_1 \in H^2(X)$ (i.e. if σ_1 is a divisor class,) then

$$I_\beta(\sigma_1,\sigma_2,\ldots,\sigma_n) = \left(\int_\beta \sigma_1\right) I_\beta(\sigma_2,\ldots,\sigma_n);$$

   this property is called the *divisor axiom*.

As a result of the codimension condition, $QH^*(X)$ is equipped with a grading compatible with the grading on $H^*(X)$:

$$\deg q^\beta = \int_\beta c_1(T_X).$$

7.2 First Calculations for $Z(\alpha_1, \alpha_2, \alpha_1)$

Lemma 7.1. *For $Z = Z(\alpha_1, \alpha_2, \alpha_1)$, the degrees of the quantum parameters are*

$$\deg q^{\beta_1} = 1, \quad \deg q^{\beta_2} = 2, \quad \deg q^{\beta_3} = 1.$$

Proof. Let Z' denote the Bott-Samelson variety $Z(\alpha_1, \alpha_2)$. From the exact sequence of tangent bundles on Z

$$0 \longrightarrow T_\pi \longrightarrow T_Z \longrightarrow \pi^* T_{Z'} \longrightarrow 0$$

we get $c_1(T_Z) = c_1(T_\pi) + c_1(\pi^* T_{Z'})$, where T_π denotes the relative tangent bundle (see Proposition 2.5). Therefore, from the projection formula

$$\begin{aligned}\int_\beta c_1(T_Z) &= \int_Z c_1(T_Z) \cdot \beta \\ &= \int_{G/B} c_1(T_{p_{\alpha_1}}) \cdot \theta_* \beta + \int_{Z'} c_1(T_{Z'}) \cdot \pi_* \beta.\end{aligned}$$

For $\beta = \beta_2$, since $\beta_2 = \pi^*[pt]$ the second integral vanishes. Thus

$$\deg q^{\beta_2} = \int_{\beta_2} c_1(T_Z) = \int_{G/B} c_1(T_{p_{\alpha_1}}) \cdot [X(s_{\alpha_1})] = (\alpha_1, \alpha_1^\vee) = 2.$$

For $\beta = \beta_3$, the pushforward $\theta_* \beta_3 = 0$, which implies the first integral vanishes. Thus

$$\begin{aligned}\deg q^{\beta_3} = \int_{\beta_3} c_1(T_Z) &= \int_{Z'} c_1(T_{Z'}) \cdot [Z'_{10}] \\ &= \int_{G/B} c_1(T_{p_{\alpha_2}}) \cdot [X(s_{\alpha_1})] + \int_{\mathbb{P}^1} c_1(T_{\mathbb{P}^1}) \cdot [pt] \\ &= (\alpha_2, \alpha_1^\vee) + 2 = 1.\end{aligned}$$

For $\beta = \beta_1$, we have

$$\deg q^{\beta_1} = \int_{\beta_1} c_1(T_Z) = \int_{G/B} c_1(T_{p_{\alpha_1}}) \cdot [X(s_{\alpha_2})] + \int_{Z'} c_1(T_{Z'}) \cdot [Z'_{01}]$$
$$= (\alpha_1, \alpha_2^\vee) + 2 = 1.$$

□

Remark 7.2. In the course of proving the previous Lemma, we showed $c_1(T_Z) = 3\sigma_{100} + \sigma_{010} + 2\sigma_{001}$.

In [LT04], a basis for the ample cone on a Bott-Samelson variety is given; the $\mathcal{O}_w(1)$ basis. These line bundles are pullbacks of the line bundles $L_{\omega_{\alpha_k}}$ corresponding to a dominant fundamental weight ω_{α_k}. Since $c_1(L_{\omega_{\alpha_k}}) = [Y(s_{\alpha_k})]$, the line bundle $\mathcal{O}_{\mathfrak{w}}(1)$ has Chern class $\sigma_{100} + \sigma_{001}$ (for $\mathfrak{w} = (1, 2, 1)$). The other two generators have Chern classes

$$c_1(\mathcal{O}_{1,2}(1)) = \sigma_{010}$$
$$c_1(\mathcal{O}_1(1)) = \sigma_{100}$$

Therefore, we can write

$$c_1(T_Z) = c_1(\mathcal{O}_1(1)) + c_1(\mathcal{O}_{1,2}(1)) + 2c_1(\mathcal{O}_{1,2,1}(1))$$

Thus $-K_Z$ is ample, that is Z is Fano.

From here on, Z denotes the Bott-Samelson variety $Z(\alpha_1, \alpha_2, \alpha_1)$ in type A_2. In order to compute a presentation for the small quantum cohomology $QH^*(Z)$, it suffices to quantize the relations in the ordinary cohomology $H^*(Z)$; see [FP97, Proposition 11] for more details.

Since the relations in $H^*(Z)$ are all in codimension one, and since it is necessary for the approach we use, we will compute all products $\sigma_{(j)} * \sigma_\varepsilon$. We organize this data in what we call Chevalley matrices.

Since one of the terms of our quantum products will always be a divisor class, the divisor axiom will always be used to reduced three-point Gromov-Witten invariants to two-point invariants. For two-point invariants, we have the following lemma which relates curve neighborhoods to the vanishing of Gromov-Witten invariants.

Lemma 7.3. *Let X be a smooth, projective $\mathbb{C}$-variety, $\Omega \subset X$ a closed subvariety, $\gamma \in H^*(X)$, and $\beta \in H_2(X)$ an effective curve class. Suppose*

$$\mathit{codim}\,\gamma + \mathit{codim}\,[\Omega] = \dim X + \int_\beta c_1(T_X) - 1$$

(that is, the codimension condition is satisfied.) Denote the irreducible components of $\Gamma_\beta(\Omega)$ by Γ_i, that is:

$$\Gamma_\beta(\Omega) = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k.$$

1. *If $\dim \Gamma_\beta(\Omega) < \mathit{codim}\,\gamma$, then $I_\beta(\gamma, [\Omega]) = 0$.*
2. *If $\dim \Gamma_\beta(\Omega) = \mathit{codim}\,\gamma$ and $\int_X \gamma \cdot [\Gamma_i] = 0$ for each $1 \le i \le k$, then $I_\beta(\gamma, [\Omega]) = 0$.*

Proof. Using the projection formula, we have

$$\begin{aligned} I_\beta(\gamma, [\Omega]) &= \int_{[\overline{M}_{0,2}(X,\beta)]^{\mathrm{virt}}} ev_1^*(\gamma) \cdot ev_2^*[\Omega] \\ &= \int_X \gamma \cdot ev_{1*}(ev_2^*[\Omega] \cdot [\overline{M}_{0,2}(X,\beta)]^{\mathrm{virt}}) \end{aligned}$$

where, by definition of curve neighborhoods,

$$ev_{1*}(ev_2^*[\Omega] \cdot [\overline{M}_{0,2}(X,\beta)]^{\mathrm{virt}}) = \sum_{i=1}^{k} m_i\,[\Gamma_i];$$

the constants $m_i \geq 0$ are zero when $\dim \Gamma_i < \dim ev_2^*(\Omega)$, otherwise they are the degrees of ev_1 restriced to each irreducible component.

The desired conclusion follows in both cases since $\int_X \gamma \cdot [\Gamma_i] = 0$ for all i. $\square$

7.3 A Chevalley Formula

Using the moment graph to compute curve neighborhoods for $Z(\alpha_1, \alpha_2, \alpha_1)$, and Lemma 7.3, we are able to show that certain Gromov-Witten invariants vanish. Combining this with the explicit moduli space results obtained in Chapter 5, we are able to compute many of the Gromov-Witten invariants needed to compute $QH^*(Z)$. However, there are still some unknown invariants that are needed.

Using a computer, and the fact that $QH^*(Z)$ is a commutative ring, we are able to solve for all of the unknowns in the Chevalley matrices. In each of the following subsections, we record the calculations necessary to produce the Chevalley matrices, the matrix A corresponding to quantum multiplication by σ_{100}, B corresponding to quantum multiplication by σ_{010}, and C corresponding to quantum multiplication by σ_{001}.

7.3.1 Chevalley matrix A

Given any $\varepsilon \in \{0,1\}^3$, we have

$$\sigma_{100} * \sigma_\varepsilon = \sum_{\beta,\varepsilon'} I_\beta(\sigma_{100}, \sigma_\varepsilon, [Z_{\varepsilon'}])\, q^\beta \sigma_{\varepsilon'}$$

We will calculate the quantum product $\sigma_{100} * \sigma_\varepsilon$ for each $\varepsilon \in \{0,1\}^3$.

- For $\varepsilon = 000$, since $1 = [Z] \in H^*(Z)$ is also the identity in $QH^*(Z)$ we have

$$\sigma_{100} * 1 = \sigma_{100} \tag{7.1}$$

- For $\varepsilon = 100$, 010, or 001, we can use the divisor axiom twice to reduce the three-point Gromov-Witten invariants to one-point invariants.

  Using the codimension condition, for the curve classes $\beta \neq 0$ which need to be considered, q^β is either degree 1 or 2. Therefore, the curve classes β for which $I_\beta(\sigma_{100}, \sigma_\varepsilon, [Z_{\varepsilon'}])$ is possibly nonzero are

$$\beta_3, \beta_1 + \beta_3, 2\beta_3$$

  Corollary 5.4 implies $ev : \overline{M}_{0,1}(Z, \beta_3) \to Z_{101}$ is an isomorphism. Thus

$$I_{\beta_3}([Z_{100}]) = \int_{Z_{101}} [Z_{100}] = -1$$
$$I_{\beta_3}([Z_{010}]) = \int_{Z_{101}} [Z_{010}] = 1$$
$$I_{\beta_3}([Z_{001}]) = \int_{Z_{101}} [Z_{001}] = 0$$

We will assign variables for the remaining Gromov-Witten invariants; we compute the values of these unknowns using brute force after analyzing all three Chevalley matrices.

$$I_{\beta_1+\beta_3}([pt]) = x_1, \quad I_{2\beta_3}([pt]) = x_2$$

We can now write the quantum products as follows:

$$\sigma_{100} * \sigma_{100} = -q_3\sigma_{100} + q_3\sigma_{010} + (q_1q_3x_1 + 4q_3^2x_2) \tag{7.2}$$

$$\sigma_{100} * \sigma_{010} = \sigma_{110} + q_1q_3x_1 \tag{7.3}$$

$$\sigma_{100} * \sigma_{001} = \sigma_{101} + q_3\sigma_{100} - q_3\sigma_{010} + (-q_1q_3x_1 - 4q_3^2x_2) \tag{7.4}$$

- For $\varepsilon = 110, 101, 011$, we can only use the divisor axiom once to reduce the three-point Gromov-Witten invariant to a two-point invariant. Moreover, using the codimension condition and the divisor axiom, we can reduce the curve classes β which need to be considered to the following:

$$\beta_3, \beta_1 + \beta_3, 2\beta_3, 2\beta_1 + \beta_3, \beta_1 + 2\beta_3, \beta_2 + \beta_3, 3\beta_3$$

From the codimension condition, if $\ell(\varepsilon) = 2$, $I_{\beta_3}(\sigma_{100}, \sigma_\varepsilon, [Z_{\varepsilon'}]) = 0$ unless $\ell(\varepsilon') = 2$. Moreover, the curve neighborhoods for class β_3 have been considered in Chapter 6:

$$\Gamma_{\beta_3}(Z_{110}) = Z_{101}, \quad \Gamma_{\beta_3}(Z_{101}) = Z_{101}, \quad \Gamma_{\beta_3}(Z_{011}) = Z_{101}$$

Using Lemma 7.3, the curve neighborhoods show $I_{\beta_3}(\sigma_\varepsilon, [Z_{\varepsilon'}]) = 0$ unless $\varepsilon = 101$. Thus we have shown that six Gromov-Witten invariants vanish, leaving three more (for curve class β_3) which are unknown.

We assign variables for the remaining Gromov-Witten invariants.

$$
\begin{array}{ccc}
I_{\beta_3}(\sigma_{101},[Z_{110}]) = x_3, & I_{\beta_3}(\sigma_{101},[Z_{101}]) = x_4, & I_{\beta_3}(\sigma_{101},[Z_{011}]) = x_5, \\
I_{\beta_1+\beta_3}(\sigma_{110},[Z_{100}]) = x_6, & I_{\beta_1+\beta_3}(\sigma_{110},[Z_{010}]) = x_7, & I_{\beta_1+\beta_3}(\sigma_{110},[Z_{001}]) = x_8, \\
I_{\beta_1+\beta_3}(\sigma_{101},[Z_{100}]) = x_9, & I_{\beta_1+\beta_3}(\sigma_{101},[Z_{010}]) = x_{10}, & I_{\beta_1+\beta_3}(\sigma_{101},[Z_{001}]) = x_{11}, \\
I_{\beta_1+\beta_3}(\sigma_{011},[Z_{100}]) = x_{12}, & I_{\beta_1+\beta_3}(\sigma_{011},[Z_{010}]) = x_{13}, & I_{\beta_1+\beta_3}(\sigma_{011},[Z_{001}]) = x_{14}, \\
I_{2\beta_3}(\sigma_{110},[Z_{100}]) = x_{15}, & I_{2\beta_3}(\sigma_{110},[Z_{010}]) = x_{16}, & I_{2\beta_3}(\sigma_{110},[Z_{001}]) = x_{17}, \\
I_{2\beta_3}(\sigma_{101},[Z_{100}]) = x_{18}, & I_{2\beta_3}(\sigma_{101},[Z_{010}]) = x_{19}, & I_{2\beta_3}(\sigma_{101},[Z_{001}]) = x_{20}, \\
I_{2\beta_3}(\sigma_{011},[Z_{100}]) = x_{21}, & I_{2\beta_3}(\sigma_{011},[Z_{010}]) = x_{22}, & I_{2\beta_3}(\sigma_{011},[Z_{001}]) = x_{23}, \\
I_{2\beta_1+\beta_3}(\sigma_{110},[pt]) = x_{24}, & I_{2\beta_1+\beta_3}(\sigma_{101},[pt]) = x_{25}, & I_{2\beta_1+\beta_3}(\sigma_{011},[pt]) = x_{26}, \\
I_{\beta_1+2\beta_3}(\sigma_{110},[pt]) = x_{27}, & I_{\beta_1+2\beta_3}(\sigma_{101},[pt]) = x_{28}, & I_{\beta_1+2\beta_3}(\sigma_{011},[pt]) = x_{29}, \\
I_{\beta_2+\beta_3}(\sigma_{110},[pt]) = x_{30}, & I_{\beta_2+\beta_3}(\sigma_{101},[pt]) = x_{31}, & I_{\beta_2+\beta_3}(\sigma_{011},[pt]) = x_{32}, \\
I_{3\beta_3}(\sigma_{110},[pt]) = x_{33}, & I_{3\beta_3}(\sigma_{101},[pt]) = x_{34}, & I_{3\beta_3}(\sigma_{011},[pt]) = x_{35}
\end{array}
$$

We now record these quantum products:

$$\sigma_{100} * \sigma_{110} = (q_1q_3x_6 + 2q_3^2x_{15})\sigma_{100} + (q_1q_3x_7 + 2q_3^2x_{16})\sigma_{010} + (q_1q_3x_8 + 2q_3^2x_{17})\sigma_{001} + (q_1^2q_3x_{24} + 2q_1q_3^2x_{27} + q_2q_3x_{30} + 3q_3^3x_{33}) \tag{7.5}$$

$$\sigma_{100} * \sigma_{101} = q_3x_3\sigma_{110} + q_3x_4\sigma_{101} + q_3x_5\sigma_{011} + (q_1q_3x_9 + 2q_3^2x_{18})\sigma_{100} + (q_1q_3x_{10} + 2q_3^2x_{19})\sigma_{010} + (q_1q_3x_{11} + 2q_3^2x_{20})\sigma_{001} + q_1^2q_3x_{25} + 2q_1q_3^2x_{28} + q_2q_3x_{31} + 3q_3^3x_{34} \tag{7.6}$$

$$\sigma_{100} * \sigma_{011} = [pt] + (q_1q_3x_{12} + 2q_3^2x_{21})\sigma_{100} + (q_1q_3x_{13} + 2q_3^2x_{22})\sigma_{010} + (q_1q_3x_{14} + 2q_3^2x_{23})\sigma_{001} + q_1^2q_3x_{26} + 2q_1q_3^2x_{29} + q_2q_3x_{32} + 3q_3^3x_{35} \tag{7.7}$$

- For $\varepsilon = 111$, the curve classes which possibly contribute with non-zero Gromov-Witten invariants are

$$\beta_3, \beta_1 + \beta_3, 2\beta_3, 2\beta_1 + \beta_3, \beta_1 + 2\beta_3, \beta_2 + \beta_3, 3\beta_3,$$
$$3\beta_1 + \beta_3, 2\beta_1 + 2\beta_3, \beta_1 + 3\beta_3, \beta_2 + 2\beta_3, 4\beta_3, \beta_1 + \beta_2 + \beta_3$$

Using the fundamental class axiom for Gromov-Witten invariants,

$$I_{\beta_3}(\sigma_{100}, [pt], [Z]) = 0.$$

We assign variables for the remaining Gromov-Witten invariants.

$$I_{\beta_1+\beta_3}([pt],[Z_{110}]) = x_{36}, \quad I_{\beta_1+\beta_3}([pt],[Z_{101}]) = x_{37}, \quad I_{\beta_1+\beta_3}([pt],[Z_{011}]) = x_{38},$$

$$I_{2\beta_3}([pt],[Z_{110}]) = x_{39}, \quad I_{2\beta_3}([pt],[Z_{101}]) = x_{40}, \quad I_{2\beta_3}([pt],[Z_{011}]) = x_{41},$$

$$I_{2\beta_1+\beta_3}([pt],[Z_{100}]) = x_{42}, \quad I_{2\beta_1+\beta_3}([pt],[Z_{010}]) = x_{43}, \quad I_{2\beta_1+\beta_3}([pt],[Z_{001}]) = x_{44},$$

$$I_{\beta_1+2\beta_3}([pt],[Z_{100}]) = x_{45}, \quad I_{\beta_1+2\beta_3}([pt],[Z_{010}]) = x_{46}, \quad I_{\beta_1+2\beta_3}([pt],[Z_{001}]) = x_{47},$$

$$I_{\beta_2+\beta_3}([pt],[Z_{100}]) = x_{48}, \quad I_{\beta_2+\beta_3}([pt],[Z_{010}]) = x_{49}, \quad I_{\beta_2+\beta_3}([pt],[Z_{001}]) = x_{50},$$

$$I_{3\beta_3}([pt],[Z_{100}]) = x_{51}, \quad I_{3\beta_3}([pt],[Z_{010}]) = x_{52}, \quad I_{3\beta_3}([pt],[Z_{001}]) = x_{53},$$

$$I_{3\beta_1+\beta_3}([pt],[pt]) = x_{54}, \quad I_{2\beta_1+2\beta_3}([pt],[pt]) = x_{55}, \quad I_{\beta_1+3\beta_3}([pt],[pt]) = x_{56},$$

$$I_{\beta_2+2\beta_3}([pt],[pt]) = x_{57}, \quad I_{4\beta_3}([pt],[pt]) = x_{58}, \quad I_{\beta_1+\beta_2+\beta_3}([pt],[pt]) = x_{59}$$

We now record the final quantum product for the Chevalley matrix A:

$$\sigma_{100} * [pt] = (q_1q_3x_{36} + 2q_3^2x_{39})\sigma_{110} + (q_1q_3x_{37} + 2q_3^2x_{40})\sigma_{101} + (q_1q_3x_{38} + 2q_3^2x_{41})\sigma_{011} \tag{7.8}$$

$$+ (q_1^2q_3x_{42} + 2q_1q_3^2x_{45} + q_2q_3x_{48} + 3q_3^3x_{51})\sigma_{100}$$

$$+ (q_1^2q_3x_{43} + 2q_1q_3^2x_{46} + q_2q_3x_{49} + 3q_3^3x_{52})\sigma_{010}$$

$$+ (q_1^2q_3x_{44} + 2q_1q_3^2x_{47} + q_2q_3x_{50} + 3q_3^3x_{53})\sigma_{010}$$

$$+ q_1^3q_3x_{54} + 2q_1^2q_3^2x_{55} + 3q_1q_3^3x_{56} + 2q_2q_3^2x_{57} + 4q_3^4x_{58} + q_1q_2q_3x_{59}$$

7.3.2 Chevalley matrix B

As in the previous section, we can write the quantum product as

$$\sigma_{010} * \sigma_\varepsilon = \sum_{\beta,\varepsilon'} I_\beta(\sigma_{010}, \sigma_\varepsilon, [Z_{\varepsilon'}])\, q^\beta \sigma_{\varepsilon'}$$

- As before, $1 = [Z] \in H^*(Z)$ is also the identity in $QH^*(Z)$. Therefore,

$$\sigma_{010} * 1 = \sigma_{010} \tag{7.9}$$

- For $\varepsilon = 100, 010, 001$, we can use the divisor axiom twice to reduce the three-point Gromov-Witten invariants to one-point invariants. The curve classes which give possibly non-zero Gromov-Witten invariants are

$$\beta_1, \beta_1 + \beta_3, 2\beta_1$$

  We previously defined $I_{\beta_1+\beta_3}([pt]) = x_1$, so we only need the following new invariants:

$$I_{\beta_1}([Z_{100}]) = y_1, \quad I_{\beta_1}([Z_{010}]) = y_2, \quad I_{\beta_1}([Z_{001}]) = y_3, \quad I_{2\beta_1}([pt]) = y_4$$

  We can then write the quantum products as

$$\sigma_{010} * \sigma_{100} = \sigma_{110} + q_1q_3x_1 \tag{7.10}$$

$$\sigma_{010} * \sigma_{010} = \sigma_{110} + q_1y_1\sigma_{100} + q_1y_2\sigma_{010} + q_1y_3\sigma_{001} + (q_1q_3x_1 + 2q_1^2y_4) \tag{7.11}$$

$$\sigma_{010} * \sigma_{001} = \sigma_{011} - q_1q_3x_1 \tag{7.12}$$

- For $\varepsilon = 110, 101, 011$, we use the divisor axiom to reduce the three-point Gromov-Witten invariants to two-point invariants. The curve classes which give possibly non-zero Gromov-Witten invariants are

$$\beta_1, \beta_1 + \beta_3, 2\beta_1, 2\beta_1 + \beta_3, \beta_1 + 2\beta_3, \beta_1 + \beta_2, 3\beta_1$$

  We assign variables for the remaining Gromov-Witten invariants (note: many of these

Gromov-Witten invariants were already considered in the Chevalley matrix A.)

$$
\begin{array}{lll}
I_{\beta_1}(\sigma_{110}, [Z_{110}]) = y_5, & I_{\beta_1}(\sigma_{110}, [Z_{101}]) = y_6, & I_{\beta_1}(\sigma_{110}, [Z_{011}]) = y_7, \\
I_{\beta_1}(\sigma_{101}, [Z_{110}]) = y_8, & I_{\beta_1}(\sigma_{101}, [Z_{101}]) = y_9, & I_{\beta_1}(\sigma_{101}, [Z_{011}]) = y_{10}, \\
I_{\beta_1}(\sigma_{011}, [Z_{110}]) = y_{11}, & I_{\beta_1}(\sigma_{011}, [Z_{101}]) = y_{12}, & I_{\beta_1}(\sigma_{011}, [Z_{011}]) = y_{13}, \\
I_{\beta_1+\beta_3}(\sigma_{110}, [Z_{100}]) = x_6, & I_{\beta_1+\beta_3}(\sigma_{110}, [Z_{010}]) = x_7, & I_{\beta_1+\beta_3}(\sigma_{110}, [Z_{001}]) = x_8, \\
I_{\beta_1+\beta_3}(\sigma_{101}, [Z_{100}]) = x_9, & I_{\beta_1+\beta_3}(\sigma_{101}, [Z_{010}]) = x_{10}, & I_{\beta_1+\beta_3}(\sigma_{101}, [Z_{001}]) = x_{11}, \\
I_{\beta_1+\beta_3}(\sigma_{011}, [Z_{100}]) = x_{12}, & I_{\beta_1+\beta_3}(\sigma_{011}, [Z_{010}]) = x_{13}, & I_{\beta_1+\beta_3}(\sigma_{011}, [Z_{001}]) = x_{14}, \\
I_{2\beta_1}(\sigma_{110}, [Z_{100}]) = y_{14}, & I_{2\beta_1}(\sigma_{110}, [Z_{010}]) = y_{15}, & I_{2\beta_1}(\sigma_{110}, [Z_{001}]) = y_{16}, \\
I_{2\beta_1}(\sigma_{101}, [Z_{100}]) = y_{17}, & I_{2\beta_1}(\sigma_{101}, [Z_{010}]) = y_{18}, & I_{2\beta_1}(\sigma_{101}, [Z_{001}]) = y_{19}, \\
I_{2\beta_1}(\sigma_{011}, [Z_{100}]) = y_{20}, & I_{2\beta_1}(\sigma_{011}, [Z_{010}]) = y_{21}, & I_{2\beta_1}(\sigma_{011}, [Z_{001}]) = y_{22}, \\
I_{2\beta_1+\beta_3}(\sigma_{110}, [pt]) = x_{24}, & I_{2\beta_1+\beta_3}(\sigma_{101}, [pt]) = x_{25}, & I_{2\beta_1+\beta_3}(\sigma_{011}, [pt]) = x_{26}, \\
I_{\beta_1+2\beta_3}(\sigma_{110}, [pt]) = x_{27}, & I_{\beta_1+2\beta_3}(\sigma_{101}, [pt]) = x_{28}, & I_{\beta_1+2\beta_3}(\sigma_{011}, [pt]) = x_{29}, \\
I_{\beta_1+\beta_2}(\sigma_{110}, [pt]) = y_{23}, & I_{\beta_1+\beta_2}(\sigma_{101}, [pt]) = y_{24}, & I_{\beta_1+\beta_2}(\sigma_{011}, [pt]) = y_{25}, \\
I_{3\beta_1}(\sigma_{110}, [pt]) = y_{26}, & I_{3\beta_1}(\sigma_{101}, [pt]) = y_{27}, & I_{3\beta_1}(\sigma_{011}, [pt]) = y_{28}
\end{array}
$$

We can then write the quantum products:

$$\begin{aligned}\sigma_{010} * \sigma_{110} = {} & q_1y_5\sigma_{110} + q_1y_6\sigma_{101} + q_1y_7\sigma_{011} + (q_1q_3x_6 + 2q_1^2y_{14})\sigma_{100} \\ & + (q_1q_3x_7 + 2q_1^2y_{15})\sigma_{010} + (q_1q_3x_8 + 2q_1^2y_{16})\sigma_{001} \\ & + 2q_1^2q_3x_{24} + q_1q_3^2x_{27} + q_1q_2y_{23} + 3q_1^3y_{26}\end{aligned} \tag{7.13}$$

$$\begin{aligned}\sigma_{010} * \sigma_{101} = {} & [pt] + q_1y_8\sigma_{110} + q_1y_9\sigma_{101} + q_1y_{10}\sigma_{011} + (q_1q_3x_9 + 2q_1^2y_{17})\sigma_{100} \\ & + (q_1q_3x_{10} + 2q_1^2y_{18})\sigma_{010} + (q_1q_3x_{11} + 2q_1^2y_{19})\sigma_{001} \\ & + 2q_1^2q_3x_{25} + q_1q_3^2x_{28} + q_1q_2y_{24} + 3q_1^3y_{27}\end{aligned} \tag{7.14}$$

$$\begin{aligned}\sigma_{010} * \sigma_{011} = {} & [pt] + q_1y_{11}\sigma_{110} + q_1y_{12}\sigma_{101} + q_1y_{13}\sigma_{011} + (q_1q_3x_{12} + 2q_1^2y_{20})\sigma_{100} \\ & + (q_1q_3x_{13} + 2q_1^2y_{21})\sigma_{010} + (q_1q_3x_{14} + 2q_1^2y_{22})\sigma_{001} \\ & + 2q_1^2q_3x_{26} + q_1q_3^2x_{29} + q_1q_2y_{25} + 3q_1^3y_{28}\end{aligned} \tag{7.15}$$

- For $\varepsilon = 111$, the curve classes which contribute with possibly non-zero Gromov-Witten invariants are

$$\beta_1, \beta_1+\beta_3, 2\beta_1, 2\beta_1+\beta_3, \beta_1+2\beta_3, \beta_1+\beta_2, 3\beta_1$$

$$3\beta_1+\beta_3, 2\beta_1+2\beta_3, \beta_1+3\beta_3, 2\beta_1+\beta_2, 4\beta_1, \beta_1+\beta_2+\beta_3$$

Using the fundamental class axiom, $I_{\beta_1}(\sigma_{010}, [pt], [Z]) = 0$. We record the remaining

Gromov-Witten invariants.

$$I_{\beta_1+\beta_3}([pt],[Z_{110}]) = x_{36}, \quad I_{\beta_1+\beta_3}([pt],[Z_{101}]) = x_{37}, \quad I_{\beta_1+\beta_3}([pt],[Z_{011}]) = x_{38},$$

$$I_{2\beta_1}([pt],[Z_{110}]) = y_{29}, \quad I_{2\beta_1}([pt],[Z_{101}]) = y_{30}, \quad I_{2\beta_1}([pt],[Z_{011}]) = y_{31},$$

$$I_{2\beta_1+\beta_3}([pt],[Z_{100}]) = x_{42}, \quad I_{2\beta_1+\beta_3}([pt],[Z_{010}]) = x_{43}, \quad I_{2\beta_1+\beta_3}([pt],[Z_{001}]) = x_{44},$$

$$I_{\beta_1+2\beta_3}([pt],[Z_{100}]) = x_{45}, \quad I_{\beta_1+2\beta_3}([pt],[Z_{010}]) = x_{46}, \quad I_{\beta_1+2\beta_3}([pt],[Z_{001}]) = x_{47},$$

$$I_{\beta_1+\beta_2}([pt],[Z_{100}]) = y_{32}, \quad I_{\beta_1+\beta_2}([pt],[Z_{010}]) = y_{33}, \quad I_{\beta_1+\beta_2}([pt],[Z_{001}]) = y_{34},$$

$$I_{3\beta_1}([pt],[Z_{100}]) = y_{35}, \quad I_{3\beta_1}([pt],[Z_{010}]) = y_{36}, \quad I_{3\beta_1}([pt],[Z_{001}]) = y_{37},$$

$$I_{3\beta_1+\beta_3}([pt],[pt]) = x_{54}, \quad I_{2\beta_1+2\beta_3}([pt],[pt]) = x_{55}, \quad I_{\beta_1+3\beta_3}([pt],[pt]) = x_{56},$$

$$I_{2\beta_1+\beta_2}([pt],[pt]) = y_{38}, \quad I_{4\beta_1}([pt],[pt]) = y_{39}, \quad I_{\beta_1+\beta_2+\beta_3}([pt],[pt]) = x_{59}$$

We record the final quantum product for the matrix B:

$$\begin{aligned}\sigma_{010} * [pt] = {} & (q_1q_3x_{36} + 2q_1^2y_{29})\sigma_{110} + (q_1q_3x_{37} + 2q_1^2y_{30})\sigma_{101} + (q_1q_3x_{38} + 2q_1^2y_{31})\sigma_{011} \\ & + (2q_1^2q_3x_{42} + q_1q_3^2x_{45} + q_1q_2y_{32} + 3q_1^3y_{35})\sigma_{100} \\ & + (2q_1^2q_3x_{43} + q_1q_3^2x_{46} + q_1q_2y_{33} + 3q_1^3y_{36})\sigma_{010} \\ & + (2q_1^2q_3x_{44} + q_1q_3^2x_{47} + q_1q_2y_{34} + 3q_1^3y_{37})\sigma_{001} \\ & + 3q_1^3q_3x_{54} + 2q_1^2q_3^2x_{55} + q_1q_3^3x_{56} + 2q_1^2q_2y_{38} + 4q_1^4y_{39} + q_1q_2q_3x_{59}\end{aligned} \tag{7.16}$$

7.3.3 Chevalley matrix C

As in the previous two sections, we can express the quantum product as follows

$$\sigma_{001} * \sigma_\varepsilon = \sum_{\beta,\varepsilon'} I_\beta(\sigma_{001}, \sigma_\varepsilon, [Z_{\varepsilon'}])\, q^\beta \sigma_{\varepsilon'}$$

- For $\varepsilon = 000$, since $1 = [Z] \in H^*(Z)$ is also the identity in $QH^*(Z)$, we have

$$\sigma_{001} * 1 = \sigma_{001} \tag{7.17}$$

- For $\varepsilon = 100, 010, 001$, we use the divisor axiom twice to reduce three-point Gromov-Witten invariants to one-point invariants. The applicable curve classes are

$$\beta_3, \beta_1 + \beta_3, \beta_2, 2\beta_3$$

In the first subsection, we observed that the one-point invariants for curve class β_3 are explicitly computable:

$$I_{\beta_3}([Z_{100}]) = \int_{Z_{101}} [Z_{100}] = -1$$
$$I_{\beta_3}([Z_{010}]) = \int_{Z_{101}} [Z_{010}] = 1$$
$$I_{\beta_3}([Z_{001}]) = \int_{Z_{101}} [Z_{001}] = 0$$

In fact, all the relevant Gromov-Witten invariants (except $I_{\beta_2}([pt])$) for these curve classes were already considered in the subsection for matrix A. Using Corollary 5.4, we compute $I_{\beta_2}([pt]) = 1$. We record the quantum products

$$\sigma_{001} * \sigma_{100} = \sigma_{101} + q_3\sigma_{100} - q_3\sigma_{010} + (-q_1q_3x_1 - 4q_3^2x_2) \tag{7.18}$$

$$\sigma_{001} * \sigma_{010} = \sigma_{011} - q_1q_3x_1 \tag{7.19}$$

$$\sigma_{001} * \sigma_{001} = \sigma_{011} - 2\sigma_{101} - q_3\sigma_{100} + q_3\sigma_{010} + (q_1q_3x_1 + 4q_3^2x_2 + q_2) \tag{7.20}$$

- For $\varepsilon = 110, 101, 011$, we use the divisor axiom to reduce three-point Gromov-Witten invariants to two-point invariants. The applicable curve classes are very similar to those in matrix A, however $\beta_2+\beta_3 = [Z_{100}]$. So, by the divisor axiom $I_{\beta_2+\beta_3}(\sigma_{001}, \sigma_\varepsilon, [Z_{\varepsilon'}]) = 0$. The applicable curve classes are

$$\beta_3, \beta_1+\beta_3, \beta_2, 2\beta_3, 2\beta_1+\beta_3, \beta_1+2\beta_3, \beta_1+\beta_2, 3\beta_3$$

Recall, curve neighborhoods are used to show $I_{\beta_3}(\sigma_\varepsilon, [Z_{\varepsilon'}]) = 0$ unless $\varepsilon = 101$. We record the remaining Gromov-Witten invariants here

$$\begin{array}{ccc}
I_{\beta_3}(\sigma_{101}, [Z_{110}]) = x_3, & I_{\beta_3}(\sigma_{101}, [Z_{101}]) = x_4, & I_{\beta_3}(\sigma_{101}, [Z_{011}]) = x_5, \\
I_{\beta_1+\beta_3}(\sigma_{110}, [Z_{100}]) = x_6, & I_{\beta_1+\beta_3}(\sigma_{110}, [Z_{010}]) = x_7, & I_{\beta_1+\beta_3}(\sigma_{110}, [Z_{001}]) = x_8, \\
I_{\beta_1+\beta_3}(\sigma_{101}, [Z_{100}]) = x_9, & I_{\beta_1+\beta_3}(\sigma_{101}, [Z_{010}]) = x_{10}, & I_{\beta_1+\beta_3}(\sigma_{101}, [Z_{001}]) = x_{11}, \\
I_{\beta_1+\beta_3}(\sigma_{011}, [Z_{100}]) = x_{12}, & I_{\beta_1+\beta_3}(\sigma_{011}, [Z_{010}]) = x_{13}, & I_{\beta_1+\beta_3}(\sigma_{011}, [Z_{001}]) = x_{14}, \\
I_{\beta_2}(\sigma_{110}, [Z_{100}]) = z_1, & I_{\beta_2}(\sigma_{110}, [Z_{010}]) = z_2, & I_{\beta_2}(\sigma_{110}, [Z_{001}]) = z_3, \\
I_{\beta_2}(\sigma_{101}, [Z_{100}]) = z_4, & I_{\beta_2}(\sigma_{101}, [Z_{010}]) = z_5, & I_{\beta_2}(\sigma_{101}, [Z_{001}]) = z_6, \\
I_{\beta_2}(\sigma_{011}, [Z_{100}]) = z_7, & I_{\beta_2}(\sigma_{011}, [Z_{010}]) = z_8, & I_{\beta_2}(\sigma_{011}, [Z_{001}]) = z_9, \\
I_{2\beta_3}(\sigma_{110}, [Z_{100}]) = x_{15}, & I_{2\beta_3}(\sigma_{110}, [Z_{010}]) = x_{16}, & I_{2\beta_3}(\sigma_{110}, [Z_{001}]) = x_{17}, \\
I_{2\beta_3}(\sigma_{101}, [Z_{100}]) = x_{18}, & I_{2\beta_3}(\sigma_{101}, [Z_{010}]) = x_{19}, & I_{2\beta_3}(\sigma_{101}, [Z_{001}]) = x_{20}, \\
I_{2\beta_3}(\sigma_{011}, [Z_{100}]) = x_{21}, & I_{2\beta_3}(\sigma_{011}, [Z_{010}]) = x_{22}, & I_{2\beta_3}(\sigma_{011}, [Z_{001}]) = x_{23}, \\
I_{2\beta_1+\beta_3}(\sigma_{110}, [pt]) = x_{24}, & I_{2\beta_1+\beta_3}(\sigma_{101}, [pt]) = x_{25}, & I_{2\beta_1+\beta_3}(\sigma_{011}, [pt]) = x_{26}, \\
I_{\beta_1+2\beta_3}(\sigma_{110}, [pt]) = x_{27}, & I_{\beta_1+2\beta_3}(\sigma_{101}, [pt]) = x_{28}, & I_{\beta_1+2\beta_3}(\sigma_{011}, [pt]) = x_{29}, \\
I_{\beta_1+\beta_2}(\sigma_{110}, [pt]) = y_{23}, & I_{\beta_1+\beta_2}(\sigma_{101}, [pt]) = y_{24}, & I_{\beta_1+\beta_2}(\sigma_{011}, [pt]) = y_{25}, \\
I_{3\beta_3}(\sigma_{110}, [pt]) = x_{33}, & I_{3\beta_3}(\sigma_{101}, [pt]) = x_{34}, & I_{3\beta_3}(\sigma_{011}, [pt]) = x_{35}
\end{array}$$

And the quantum products are

$$\sigma_{001} * \sigma_{110} = [pt] + (-q_1q_3x_6 + q_2z_1 - 2q_3^2x_{15})\sigma_{100} + (-q_1q_3x_7 + q_2z_2 - 2q_3^2x_{16})\sigma_{010} \tag{7.21}$$

$$+ (-q_1q_3x_8 + q_2z_3 - 2q_3^2x_{17})\sigma_{001} + (-q_1^2q_3x_{24} - 2q_1q_3^2x_{27} + q_1q_2y_{23} - 3q_3^3x_{33})$$

$$\sigma_{001} * \sigma_{101} = [pt] - q_3x_3\sigma_{110} - q_3x_4\sigma_{101} - q_3x_5\sigma_{011} \tag{7.22}$$

$$+ (-q_1q_3x_9 + q_2z_4 - 2q_3^2x_{18})\sigma_{100} + (-q_1q_3x_{10} + q_2z_5 - 2q_3^2x_{19})\sigma_{010}$$

$$+ (-q_1q_3x_{11} + q_2z_6 - 2q_3^2x_{20})\sigma_{001} + (-q_1^2q_3x_{25} - 2q_1q_3^2x_{28} + q_1q_2y_{24} - 3q_3^3x_{34})$$

$$\sigma_{001} * \sigma_{011} = -[pt] + (-q_1q_3x_{12} + q_2z_7 - 2q_3^2x_{21})\sigma_{100} + (-q_1q_3x_{13} + q_2z_8 - 2q_3^2x_{22})\sigma_{010} \tag{7.23}$$

$$+ (-q_1q_3x_{14} + q_2z_9 - 2q_3^2x_{23})\sigma_{001} + (-q_1^2q_3x_{26} - 2q_1q_3^2x_{29} + q_1q_2y_{25} - 3q_3^3x_{35})$$

- For $\varepsilon = 111$, the curve classes which contribute with possibly nonzero Gromov-Witten invariants are:

$$\beta_3, \beta_1 + \beta_3, \beta_2, 2\beta_3, 2\beta_1 + \beta_3, \beta_1 + 2\beta_3, \beta_1 + \beta_2, 3\beta_3$$

$$3\beta_1 + \beta_3, 2\beta_1 + 2\beta_3, \beta_1 + 3\beta_3, 2\beta_2, \beta_2 + 2\beta_3, 4\beta_3$$

Using the fundamental class axiom, $I_{\beta_3}(\sigma_{001}, [pt], [Z]) = 0$. We record the remaining unknown Gromov-Witten invariants.

$$\begin{array}{ccc}
I_{\beta_1+\beta_3}([pt],[Z_{110}]) = x_{36}, & I_{\beta_1+\beta_3}([pt],[Z_{101}]) = x_{37}, & I_{\beta_1+\beta_3}([pt],[Z_{011}]) = x_{38}, \\
I_{\beta_2}([pt],[Z_{110}]) = z_{10}, & I_{\beta_2}([pt],[Z_{101}]) = z_{11}, & I_{\beta_2}([pt],[Z_{011}]) = z_{12}, \\
I_{2\beta_3}([pt],[Z_{110}]) = x_{39}, & I_{2\beta_3}([pt],[Z_{101}]) = x_{40}, & I_{2\beta_3}([pt],[Z_{011}]) = x_{41}, \\
I_{2\beta_1+\beta_3}([pt],[Z_{100}]) = x_{42}, & I_{2\beta_1+\beta_3}([pt],[Z_{010}]) = x_{43}, & I_{2\beta_1+\beta_3}([pt],[Z_{001}]) = x_{44}, \\
I_{\beta_1+2\beta_3}([pt],[Z_{100}]) = x_{45}, & I_{\beta_1+2\beta_3}([pt],[Z_{010}]) = x_{46}, & I_{\beta_1+2\beta_3}([pt],[Z_{001}]) = x_{47}, \\
I_{\beta_1+\beta_2}([pt],[Z_{100}]) = y_{32}, & I_{\beta_1+\beta_2}([pt],[Z_{010}]) = y_{33}, & I_{\beta_1+\beta_2}([pt],[Z_{001}]) = y_{34}, \\
I_{3\beta_3}([pt],[Z_{100}]) = x_{51}, & I_{3\beta_3}([pt],[Z_{010}]) = x_{52}, & I_{3\beta_3}([pt],[Z_{001}]) = x_{53}, \\
I_{3\beta_1+\beta_3}([pt],[pt]) = x_{54}, & I_{2\beta_1+2\beta_3}([pt],[pt]) = x_{55}, & I_{\beta_1+3\beta_3}([pt],[pt]) = x_{56}, \\
I_{2\beta_2}([pt],[pt]) = z_{13}, & I_{\beta_2+2\beta_3}([pt],[pt]) = x_{57}, & I_{4\beta_3}([pt],[pt]) = x_{58}
\end{array}$$

The final quantum product is

$$\begin{aligned}
\sigma_{001} * [pt] = {} & (-q_1q_3x_{36} + q_2z_{10} - 2q_3^2x_{39})\sigma_{110} + (-q_1q_3x_{37} + q_2z_{11} - 2q_3^2x_{40})\sigma_{101} \\
& + (-q_1q_3x_{38} + q_2z_{12} - 2q_3^2x_{41})\sigma_{011} + (-q_1^2q_3x_{42} - 2q_1q_3^2x_{45} + q_1q_2y_{32} - 3q_3^3x_{51})\sigma_{100} \\
& + (-q_1^2q_3x_{43} - 2q_1q_3^2x_{46} + q_1q_2y_{33} - 3q_3^3x_{52})\sigma_{010} \\
& + (-q_1^2q_3x_{44} - 2q_1q_3^2x_{47} + q_1q_2y_{34} - 3q_3^3x_{53})\sigma_{001} \\
& + -q_1^3q_3x_{54} - 2q_1^2q_3^2x_{55} - 3q_1q_3^3x_{56} + 2q_2^2z_{13} - 2q_2q_3^2x_{57} - 4q_3^4x_{58}
\end{aligned} \tag{7.24}$$

7.3.4 Brute force

The unknown Gromov-Witten invariants can be computed by imposing the relations

$$[A,B]=0, \quad [A,C]=0, \quad [B,C]=0;$$

these are the relations that quantum multiplication commutes. This gives relations among the remaining unknown invariants which can then be solved using brute force. This gives values for all but a single Gromov-Witten invariant

$$y_3 = I_{\beta_1}([Z_{001}])$$

We record the matrices here: (A is the matrix obtained from multiplication by σ_{100}, B from multiplication by σ_{010}, and C from multiplication by σ_{001})

$$A = \begin{pmatrix} 0 & q_1q_3y_3 & q_1q_3y_3 & -q_1q_3y_3 & 0 & 0 & 0 & q_1q_2q_3y_3 \\ 1 & -q_3 & 0 & q_3 & q_1q_3y_3 & q_1q_3y_3 & 0 & 0 \\ 0 & q_3 & 0 & -q_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_1q_3y_3 & q_1q_3y_3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & q_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -q_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_3 & 0 & q_1q_3y_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & q_1q_3y_3 & q_1q_3y_3 & -q_1q_3y_3 & 0 & 0 & q_1q_2y_3 & q_1q_2q_3y_3 \\ 0 & 0 & 2q_1y_3 & 0 & q_1q_3y_3 & q_1q_3y_3 & 0 & q_1q_2y_3 \\ 1 & 0 & -q_1y_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_1y_3 & 0 & q_1q_3y_3 & q_1q_3y_3 & 0 & 0 \\ 0 & 1 & 1 & 0 & -q_1y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_1y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & q_1q_3y_3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & -q_1q_3y_3 & -q_1q_3y_3 & q_1q_3y_3+q_2 & 0 & 0 & q_1q_2y_3 & 0 \\ 0 & q_3 & 0 & -q_3 & -q_1q_3y_3 & -q_1q_3y_3+q_2 & 0 & q_1q_2y_3 \\ 0 & -q_3 & 0 & q_3 & 0 & 0 & q_2 & 0 \\ 1 & 0 & 0 & 0 & -q_1q_3y_3 & -q_1q_3y_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -q_3 & 0 & q_2 \\ 0 & 1 & 0 & -2 & 0 & q_3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -q_3 & 0 & -q_1q_3y_3 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{pmatrix}$$

In order to compute y_3, we use [Man12, Remark 5.7]. In particular, this remark immediately implies the following result.

Proposition 7.4. *Let Z denote the Bott-Samelson variety $Z(\alpha_1, \alpha_2, \alpha_1)$, and $Z' = Z(\alpha_1, \alpha_2)$.*

The following commutative diagram is Cartesian:

$$\begin{tikzcd} \overline{M}_{0,1}(Z,\beta_1) \arrow[r, "\overline{\theta}"] \arrow[d, "\overline{\pi}"] & \overline{M}_{0,1}(G/B,[X(s_{\alpha_2})]) \arrow[d, "\overline{p_{\alpha_1}}"] \\ \overline{M}_{0,1}(Z',[Z'_{01}]) \arrow[r, "\overline{p_{\alpha_1}\theta'}"] & \overline{M}_{0,1}(G/P_{\alpha_1},[X(s_{\alpha_2})]) \end{tikzcd}$$

In particular, since $\overline{p_{\alpha_1}}$ is an isomorphism, $\overline{\pi} : \overline{M}_{0,1}(Z,\beta_1) \to \overline{M}_{0,1}(Z',[Z'_{01}])$ is also an isomorphism.

Combined with Corollary 5.4, we can compute y_3:

$$\begin{aligned} y_3 = I_{\beta_1}([Z_{001}]) &= \int ev^*(\sigma_{110}) \cdot [\overline{M}_{0,1}(Z,\beta_1)] \\ &= \int ev^*([pt]) \cdot [\overline{M}_{0,1}(Z',[Z'_{01}]) = \int_{Z'} [pt] = 1. \end{aligned}$$

Proposition 7.5. *The Gromov-Witten invariant $y_3 = I_{\beta_1}([Z_{001}]) = 1$.*

One can read the entire Chevalley formula for $Z = Z(\alpha_1, \alpha_2, \alpha_1)$ from the Chevalley matrices now that y_3 has been computed; we record the presentation for $QH^*(Z)$ so obtained here:

Theorem 7.6. *$QH^*(Z)$ is generated by $\sigma_{100}, \sigma_{010}, \sigma_{001}, q_1, q_2, q_3$ subject to the following relations:*

$$\sigma_{100}^2 = q_1q_3 - q_3\sigma_{100} + q_3\sigma_{010}$$

$$\sigma_{010}^2 = q_1q_3 + 2q_1\sigma_{100} - q_1\sigma_{010} + q_1\sigma_{001} + \sigma_{110}$$

$$\sigma_{001}^2 = q_1q_3 + q_2 - q_3\sigma_{100} + q_3\sigma_{010} - 2\sigma_{101} + \sigma_{011}$$

We can also record the "Giambelli formula," the representation of the vector space generators $\sigma_{000}, \sigma_{100}, \ldots, \sigma_{011}, \sigma_{111}$ as polynomials in the algebra generators $\sigma_{100}, \sigma_{010}, \sigma_{001}$.

Corollary 7.7. *In $QH^*(Z)$, the Giambelli formulae for the classes*

$$\sigma_{110}, \sigma_{101}, \sigma_{011}, \sigma_{111}$$

are as follows:

$$\sigma_{110} = \sigma_{100}\sigma_{010} - q_1q_3$$

$$\sigma_{101} = \sigma_{100}\sigma_{001} - q_3\sigma_{100} + q_3\sigma_{010} + q_1q_3$$

$$\sigma_{011} = \sigma_{010}\sigma_{001} + q_1q_3$$

$$\sigma_{111} = \sigma_{100}\sigma_{010}\sigma_{001} + q_1q_3\sigma_{100}$$

Using these formulas, we are able to verify that the ring $QH^*(Z)$ is indeed associative.

7.4 Conjecture $\mathcal{O}$

Consider the operator $\widehat{c_1} : H^*(Z) \to H^*(Z)$ defined as follows: let $c_1 : QH^*(Z) \to QH^*(Z)$ be the operator defined by multiplication by $c_1(-K_Z)$, and let $\widehat{c_1}$ denote the specialization of c_1 with all quantum parameters set equal to one.

Conjecture $\mathcal{O}$, which is related to the Gamma conjectures of Galkin, Golyshev, and Iritani (see [CL17]), is a statement concerning the eigenvalues of $\widehat{c_1}$. Namely, Conjecture $\mathcal{O}$ states that if Z is Fano, then the following properties hold:

1. Let δ_0 denote the maximum modulus among the eigenvalues of $\widehat{c_1}$. Then δ_0 is one of the eigenvalues of $\widehat{c_1}$, and occurs with multiplicity one;

2. If δ is any eigenvalue of $\widehat{c_1}$ with $|\delta| = \delta_0$, then there is an rth root of unity ζ such that $\delta = \delta_0\zeta$, where r is the Fano index of Z.

Since the Bott-Samelson variety $Z = Z(\alpha_1, \alpha_2, \alpha_1)$ is Fano (see Remark 7.2), and using the Chevalley matrices from the last section, we can write the matrix for $\widehat{c_1}$:

$$\widehat{c_1} = \begin{pmatrix} 0 & 2 & 2 & -2 & 0 & 0 & 3 & 4 \\ 3 & -1 & 2 & 1 & 2 & 4 & 0 & 3 \\ 1 & 1 & -1 & -1 & 0 & 0 & 2 & 0 \\ 2 & 0 & 1 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 4 & 0 & -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 3 & 2 & 0 \end{pmatrix}$$

This matrix has eight distinct eigenvalues which are approximated numerically, the eigenvalue with the largest eigenvalue is real, and all other eigenvalues have strictly smaller modulus. In particular, Conjecture $\mathcal{O}$ holds for the Bott-Samelson variety $Z(\alpha_1, \alpha_2, \alpha_1)$.

Theorem 7.8. *The Conjecture $\mathcal{O}$ holds for the Bott-Samelson variety $Z = Z(\alpha_1, \alpha_2, \alpha_1)$ (in Type A_2).*

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