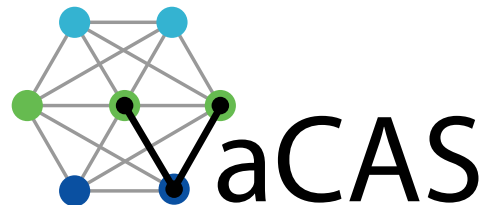


# UNDER-ACTUATED CONTROLLABILITY FOR SPACECRAFT RENDEZVOUS

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## Summary

In this report, we examine the controllability of a particular form of the equations of motion for spacecraft formation flying. These equations, the Tschauner-Hempel equations, rescale the formation flying equations to a domain in which the true anomaly is the independent variable. Using this form, we are able to compute an explicit, closed-form Gramian matrix for the period of one full orbit at arbitrary eccentricity. We do this for two cases: 1) the case in which there are three inputs to the system as well as 2) the restricted case where authority only exists in the in-track and cross-track directions. This Gramian is invertible and as a result the system is controllable for both cases. Since the transformation between the time-domain, linear equations of motion and the Tschauner-Hempel equations is bijective, we conclude that the linear equations of motion are also controllable.

# Contents

1	Introduction	1
2	Tschauner-Hempel Equations	2
3	State Transition Matrix	5
4	Controllability for Linear, Time-Varying Systems	7
5	Fully-Actuated Controllability	8
6	Under-Actuated Controllability	13
7	Conclusions	21

# List of Figures

1	Two satellites near each other in arbitrarily elliptic orbits . . . . .	2
2	Effect of eccentricity on the relative dynamics for five different eccentricities	5

# List of Tables

1	Initial conditions for three parameterizations of equations of motion . . . . .	5
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# 1 Introduction

In this report, the controllability of the Tschauner-Hempel equations is evaluated. The Tschauner-Hempel equations (abbreviated as TH) are a specific representation of the equations of relative motion for two satellites flying in formation.

The analysis in this report supports the use of linear-quadratic, optimal control design for two particular rendezvous scenarios:

1. Fully-actuated
2. Under-actuated

For the fully-actuated case (1), we model the controller as having authority in all three directions, and for the under-actuated case (2) we model the controller as having authority only in the in-track and cross-track directions.

It has been claimed in the literature [9], [8], [5] that the controllability Gramian of the Tschauner-Hempel equations exists, is unique and is invertible. However, to the authors' knowledge, no resource has explicitly computed this matrix for either the fully-actuated case or the under-actuated case. In [7], the authors use LTI theory and the Hill-Clohessy-Wiltshire equations (abbreviated as HCW) to show complete controllability for both cases 1 and 2, and we use the brief analysis in that paper as our starting point.

The report has the following structure. In Section 2, we present the TH equations and their system properties. In Section 3 the state transition matrix and the influence matrices for both actuations are presented. The theory leading to the development of the TH equations is not presented, and we defer to [8], [5], [6], and [10] for excellent presentations on their derivation.

In Section 4 we review the conditions for linear, time-varying controllability. In Section 5, we present the controllability Gramian over the horizon of one orbit,  $f \in [0, 2\pi]$  for the fully-actuated case. In Section 6, we present the same analysis, however in this case the radial thrust component is removed. The critical result in this report that is presented in Sections 5 and 6 is that for both of these cases, the Gramian is invertible, and thus the system is completely controllable. In Section 7 we review the conclusions made throughout the analysis.

## 2 Tschauner-Hempel Equations

Consider the relative trajectory of two satellites flying near to each other in arbitrarily elliptic orbits, this geometry can be seen in Figure 1. The vector  $\boldsymbol{\rho}$  represents the relative config-

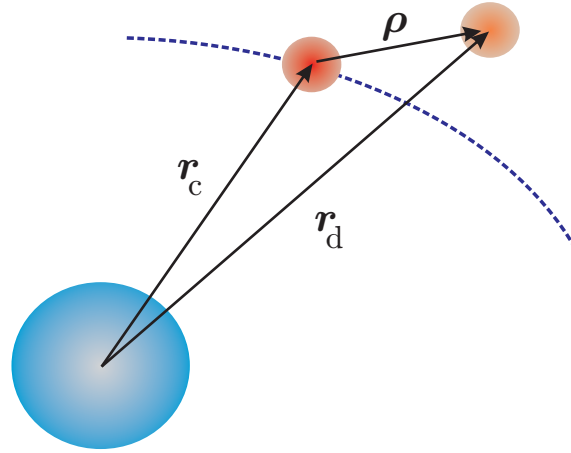


Figure 1: Two satellites near each other in arbitrarily elliptic orbits

uration of the deputy to the chief. The configuration variables and their time-derivatives are

$$\begin{bmatrix} \boldsymbol{\rho} \\ \dot{\boldsymbol{\rho}} \end{bmatrix} = [x \ y \ z \ \dot{x} \ \dot{y} \ \dot{z}]^T \quad (1)$$

where  $x$  represents radial position,  $y$  represents in-track position,  $z$  represents cross-track motion their time-derivatives give the rates of each one of these variables. The nonlinear equations of the relative motion (NERM) are [1]

$$\begin{aligned} \ddot{x} &= 2\dot{\theta}\dot{y} + \ddot{\theta}y + \dot{\theta}^2x + \frac{\mu}{r_c^2} - \frac{\mu(x + r_c)}{[(r_c + x)^2 + y^2 + z^2]^{3/2}} \\ \ddot{y} &= -2\dot{\theta}\dot{x} - \ddot{\theta}x + \dot{\theta}^2y - \frac{\mu y}{[(r_c + x)^2 + y^2 + z^2]^{3/2}} \\ \ddot{z} &= -\frac{\mu z}{[(r_c + x)^2 + y^2 + z^2]^{3/2}} \end{aligned} \quad (2)$$

A special approximation of the potential forces [8] yields the linear equations of the relative motion (LERM):

$$\begin{aligned} \ddot{x} &= 2\dot{\theta}\dot{y} + \left(\dot{\theta}^2 + 2\frac{\mu}{r_c^3}\right)x + \ddot{\theta}y \\ \ddot{y} &= -2\dot{\theta}\dot{x} - \ddot{\theta}x + \left(\dot{\theta}^2 - \frac{\mu}{r_c^3}\right)y \\ \ddot{z} &= -\frac{\mu}{r_c^3}z \end{aligned} \quad (3)$$



The time-varying coefficients' time histories are computed by the polar form of the orbit equation for the chief satellite

$$\ddot{r}_c = r_c \dot{\theta}^2 - \frac{\mu}{r_c^2} \quad \ddot{\theta} = -\frac{2\dot{r}_c \dot{\theta}}{r_c} \quad (4)$$

where  $\theta = \omega + f$  is the argument of true latitude,  $\omega$  is the argument of periapsis (constant for a Keplerian orbit) and  $f$  is the true anomaly. Equation 3 is a six-dimensional, linear system with time-varying coefficients. A closed-form state transition matrix exists, which was shown by Broucke [4], however it must be noted that a solution to Kepler's equation must be determined at each time-step. Kepler's equation is

$$M = n(t - t_0) = E - e \sin E \quad (5)$$

where  $M$  is the mean-anomaly,  $n$  is the mean angular motion,  $e$  is the eccentricity and  $E$  is the eccentric anomaly. Kepler's equation is transcendental in the eccentric anomaly, which makes computing the true anomaly difficult/ impossible analytically. The relationship between the eccentric and true anomaly is given by

$$\tan \frac{f}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \quad (6)$$

Rather than working in the time-domain and numerically evaluating Kepler's equation, we can form a similarity transformation which will rescale Equation 3 and change the independent variable to true anomaly rather than time. This similarity transformation is

$$\bar{x} = (1 + e \cos f)x(t) \quad (7)$$

And let the derivative of  $\bar{x}$  with respect to the anomaly,  $f$ , be denoted by  $(\cdot)'$ . This derivative will be

$$\bar{x}' = -e \sin f x + (1 + e \cos f)x' \quad (8)$$

Where we define

$$x' = \frac{\dot{x}}{\dot{f}} = \frac{\dot{x}r^2}{h} = \frac{p^2}{h(1 + e \cos f^2)} \quad (9)$$

We can construct a linear transformation  $\mathbf{T}(f)$  which will transform the configuration space and dynamic variable  $t$  into another configuration space with the dynamic variable  $f$ . This transformation is

$$\mathbf{T}(f) = \begin{bmatrix} (1 + e \cos f)\mathbb{I}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ -e \sin f \mathbb{I}_{3 \times 3} & \frac{p^2}{h(1 + e \cos f)} \mathbb{I}_{3 \times 3} \end{bmatrix} \quad (10)$$

The vector transformation is carried out according to  $\bar{\mathbf{x}} = \mathbf{T}(f)\mathbf{x}$ , where, now,  $\bar{\mathbf{x}}$  is the transformed state vector and  $\mathbf{x}$  is the original state vector. Furthermore, the transformation is smoothly invertible for all eccentricities  $e \in [0, 1)$ , and its inverse is

$$\mathbf{T}^{-1}(f) = \begin{bmatrix} \frac{1}{1 + e \cos f} \mathbb{I}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \frac{he}{p^2} \sin f \mathbb{I}_{3 \times 3} & \frac{h}{p^2} (1 + e \cos f) \mathbb{I}_{3 \times 3} \end{bmatrix} \quad (11)$$

Applying this transformation to Equation 3 the following linear system is obtained:

$$\begin{aligned} \bar{\mathbf{x}}' &= \mathbf{A}(f)\bar{\mathbf{x}} \\ &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{3}{1+e \cos f} & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \bar{\mathbf{x}} \end{aligned} \quad (12)$$

which are the Tschauner-Hempel equations.

Note that in the similarity transformation, the independent variable,  $t$ , was changed and rescaled to be the true anomaly,  $f$ . This is a bijective linear transformation which means that each point in the original configuration space can be uniquely represented by a point in the TH space and vice-versa. This has no effect on the essential structure of the system, and as a result, we can use this significantly simpler time-varying system to draw structural conclusions about Equation 3.

In the limit that the eccentricity is zero, the LERM become the HCW equations which represent the degenerate case of a perfectly circular orbit:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 0 & 2n & 0 \\ 0 & 0 & 0 & -2n & 0 & 0 \\ 0 & 0 & -n^2 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} \quad (13)$$

These equations are in the time domain, but if the eccentricity in the TH equations is zero, then the plant matrix is the same structure, albeit with different scaling.

One interesting property to note immediately is the null space of the matrix  $\mathbf{A}(f)$  which is

$$\mathcal{N}(\mathbf{A}(f)) = [0 \ 1 \ 0 \ 0 \ 0 \ 0]^T \quad (14)$$

which implies that the in-track component of the position has no effect on the actual relative motion. The trajectory defined by a given set of initial conditions can have an in-track component either in front ( $y_0 > 0$ ) or behind ( $y_0 < 0$ ), but the resulting motion will be the same. Using the bijectivity property, this result can be extrapolated to the LERM; the in-track position at  $t_0$  will not affect the relative motion.

Another important property to note is the role that the initial conditions play in defining the trajectory. A bounded, relative orbit exists for the correct ratio of initial radial position to in-track rate. A comparison of the similarities between the initial conditions is given in Table 1.

In Figure 2 we can see the effect that eccentricity has on the relative motion. Most importantly, it warps the relative orbit and stretches it; if a relative orbit begins in the radial/ in-track plane, the eccentricity/ weak coupling of the cross-track direction will force the satellite to have cross-track components of motion.

System	Indep. Variable	Ratio
LERM	time, $t$	$\frac{\dot{y}_0}{x_0} = -\frac{n(2+e)}{(1+e)^{\frac{1}{2}}(1-e)^{\frac{3}{2}}}$
HCW	time, $t$	$\frac{\dot{y}_0}{x_0} = -2n$
TH	true anomaly, $f$	$\frac{\dot{y}_0}{x_0} = -\frac{2+e}{1+e}$

Table 1: Initial conditions for three parameterizations of equations of motion

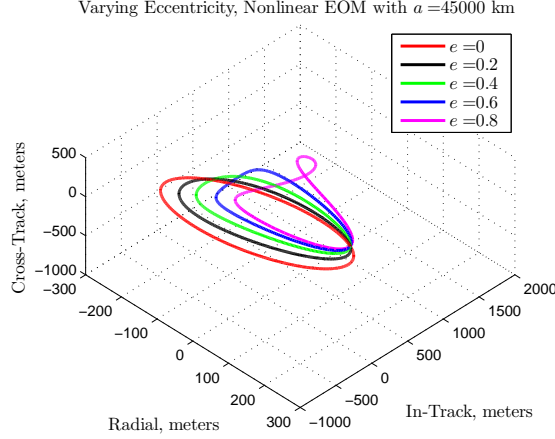


Figure 2: Effect of eccentricity on the relative dynamics for five different eccentricities

### 3 State Transition Matrix

Using the approach provided by Yamanaka and Ankersen [10], we can explicitly compute the state transition matrix for the TH equations. Using the bijective transformation in Equation 10 and 11, we can map this state transition matrix into the time domain. The fundamental solution matrix for the TH equations is given by

$$\phi(f) = \begin{bmatrix} ck(f) & sk(f) & \frac{2}{\eta^2} \left[ 1 - \frac{3e}{2\eta^3} sk(f)I(f) \right] & 0 & 0 & 0 \\ -s(2+ec) & c(2+ec) & -\frac{3}{\eta^3} k(f)I(f) & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & s \\ -(s+es^2) & c+e\cos 2f & -\frac{3e}{\eta^2} \left[ \frac{s}{k(f)} + \frac{1}{\eta^3} (c+e\cos 2f)I(f) \right] & 0 & 0 & 0 \\ -(2c+e\cos 2f) & -(2s+e\sin 2f) & -\frac{3}{\eta^2} \left[ 1 - \frac{e}{\eta^3} (2s+e\sin 2f)I(f) \right] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -s & c \end{bmatrix} \quad (15)$$

Which is nonsingular, and in fact has a determinant equal to  $\det(\phi(f)) = \sqrt{1-e^2}$ . We let  $\eta = \sqrt{1-e^2}$ ,  $k(f) = 1+e\cos f$  and  $I(f)$  be defined by

$$I(f) = \frac{h(t-\tau)}{p^2} = \int_{f_0}^f \frac{1}{k(f)^2} df \quad (16)$$

Furthermore,  $s = \sin f$  and  $c = \cos f$ . The inverse of the fundamental solution matrix evaluated at  $f_0$  is

$$\phi^{-1}(f_0) = \frac{1}{\eta^2} \begin{bmatrix} -3(e + cf_0) & 0 & 0 & -sf_0k(f_0) & -(2cf_0 + e + ec^2f_0) & 0 \\ -\frac{3(sf_0(1+ecf_0+e^2))}{k(f_0)} & 0 & 0 & cf_0 - 2e + ec^2f_0 & sf_0(2 + ecf_0) & 0 \\ \eta^2(2 + 3ecf_0 + e^2) & 0 & 0 & \eta^2esf_0k(f_0) & \eta^2k(f_0)^2 & 0 \\ -(2 + ecf_0)\frac{3esf_0}{k(f_0)} & 1 & 0 & \eta^2k(f_0)^2 & -(2 + ecf_0)esf_0 & 0 \\ 0 & 0 & \eta^2cf_0 & 0 & 0 & -\eta^2sf_0 \\ 0 & 0 & \eta^2sf_0 & 0 & 0 & \eta^2cf_0 \end{bmatrix} \quad (17)$$

where  $cf_0 = \cos f_0$  and  $sf_0 = \sin f_0$ . Here,  $t$ , is the time which corresponds to the particular instance of  $f$ , which is related to the mean and eccentric anomalies through Kepler's equation. Recall Equation 5, we can express the eccentric anomaly as

$$E = 2 \tan^{-1} \left[ \sqrt{\frac{1-e}{1+e}} \tan \frac{f}{2} \right] \quad (18)$$

Inserting this into Kepler's equation, we return the transformation between true anomaly and time:

$$t = t_0 + \frac{1}{n} \left[ \left( 2 \tan^{-1} \left[ \sqrt{\frac{1-e}{1+e}} \tan \frac{f}{2} \right] \right) - e \sin \left( 2 \tan^{-1} \left[ \sqrt{\frac{1-e}{1+e}} \tan \frac{f}{2} \right] \right) \right] \quad (19)$$

Furthermore,  $t_0$  is an arbitrary reference time; this is typically taken as the time since the last periapsis passage.

The state transition matrix for the TH domain is simply the matrix product of the fundamental solution matrix with its inverse which is evaluated at  $f_0$ :

$$\mathbf{x}(f) = \phi(f)\mathbf{k} \text{ and } \mathbf{k} = \phi^{-1}(f_0)\mathbf{x}(f_0) \quad (20)$$

which yields the well-known expression for the transition matrix

$$\mathbf{x}(f) = \underbrace{\phi(f)\phi^{-1}(f_0)}_{\Phi(f,f_0)} \mathbf{x}(f_0) \quad (21)$$

In the time domain, we can show that

$$\mathbf{x}(t) = \mathbf{T}^{-1}(f)\Phi(f, f_0)\mathbf{T}(f_0)\mathbf{x}(t_0) \quad (22)$$

In the LERM, a fully-actuated influence matrix would take the form

$$\mathbf{B} = \begin{bmatrix} \mathbf{0}_{3 \times 3} \\ \mathbb{I}_{3 \times 3} \end{bmatrix} \quad (23)$$

This needs to be rescaled for use with the TH equations. Using the bijective transformation, we can find a coefficient that premultiplies the influence matrix [5], this takes the form

$$\mathbf{B}(f) = \frac{h^6}{\mu^4(1 + e \cos f)^3} \begin{bmatrix} \mathbf{0}_{3 \times 3} \\ \mathbb{I}_{3 \times 3} \end{bmatrix} \quad (24)$$

Using definitions from astrodynamics [2], we know that  $h = \sqrt{\mu a(1 - e^2)}$ . Normalizing the motion<sup>1</sup> by the semi-major axis  $a$  and gravitational parameter  $\mu$ , the influence matrix becomes

$$\mathbf{B}(f) = \frac{\eta^3}{(1 + e \cos f)^3} \begin{bmatrix} \mathbf{0}_{3 \times 3} \\ \mathbb{I}_{3 \times 3} \end{bmatrix} \quad (25)$$

To represent the under-actuated influence, we set the element  $b_{41} = 0$ .

## 4 Controllability for Linear, Time-Varying Systems

A system is controllable if and only if it can be steered by an admissible control from a state  $(\mathbf{x}_0, t_0)$  to a state  $(\mathbf{x}_1, t_1)$  with  $t_1 > t_0$  and  $t_1 < \infty$ . It is an intrinsic property of the system, and in this section we consider the controllability of systems with the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \quad (26)$$

In the limiting case in which  $\mathbf{A}(t) = \mathbf{A}$  and  $\mathbf{B}(t) = \mathbf{B}$ , i.e. all elements are constant for all time, the controllability of the system is evaluated using the rank condition. A linear, time-invariant system is controllable if and only if

$$\text{rank}(\mathcal{C}(\mathbf{A}, \mathbf{B})) = n \quad (27)$$

where  $n$  is the dimensionality of the system and  $\mathcal{C}(\mathbf{A}, \mathbf{B})$  is the controllability matrix

$$\mathcal{C}(\mathbf{A}, \mathbf{B}) = [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \mathbf{A}^2\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}] \quad (28)$$

If a linear, time-invariant system is controllable on an interval  $t \in [t_0, t_1]$ , then it is controllable for all time.

Controllability of linear, time-varying systems requires more analysis. First, we define the controllability Gramian

$$\mathbf{W}(t_1, t_0) = \int_{t_0}^{t_1} \Phi(\tau, t_0) \mathbf{B}(\tau) \mathbf{B}^T(\tau) \Phi^T(\tau, t_0) d\tau \quad (29)$$

For a given interval  $t \in [t_0, t_1]$  and initial and terminal set  $\mathbf{x}(t_0), \mathbf{x}(t_1)$ , the linear, time-varying system is controllable if and only if [3]

$$[\mathbf{x}(t_1) - \Phi(t_1, t_0)\mathbf{x}(t_0)] \in \text{range}(\mathbf{W}(t_1, t_0)) \quad (30)$$

If the system is controllable, then the following open-loop control signal

$$\mathbf{u}^*(t) = -\mathbf{B}^T(t) \Phi(t, t_0) \mathbf{W}^{-1}(t_1, t_0) \mathbf{x}(t_0) \quad (31)$$

minimizes the  $L^2$  energy

$$\langle \mathbf{u}(t), \mathbf{u}(t) \rangle = \int_{t_0}^{t_1} \mathbf{u}^T(t) \mathbf{u}(t) dt \quad (32)$$

---

<sup>1</sup>Note that normalizing the distances/ speeds won't change the state vector or the plant matrix in the TH equations

Therefore, a system is controllable on the interval  $t \in [t_0, t_1]$  if and only if the Gramian is invertible on that interval. The Gramian is a symmetric matrix which is positive, semidefinite on the interval  $(t_0, t_1)$ . It satisfies the differential equation

$$\dot{\mathbf{W}}(t, t_1) = \mathbf{A}(t)\mathbf{W}(t, t_0) + \mathbf{W}(t, t_0)\mathbf{A}^T(t) - \mathbf{B}(t)\mathbf{B}^T(t), \quad \mathbf{W}(t_1, t_1) = \mathbf{0}_{n \times n} \quad (33)$$

as well as the functional equation

$$\mathbf{W}(t_1, t_0) = \mathbf{W}(t, t_0) + \Phi(t, t_0)\mathbf{W}(t_1, t)\Phi^T(t, t_0) \quad (34)$$

## 5 Fully-Actuated Controllability

For the fully-actuated system, the controller has authority in radial, in-track and cross-track directions. First, we examine the rank condition for the HCW equations. The plant and influence matrices are

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 0 & 2n & 0 \\ 0 & 0 & 0 & -2n & 0 & 0 \\ 0 & 0 & -n^2 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbf{0}_{3 \times 3} \\ \mathbb{I}_{3 \times 3} \end{bmatrix} \quad (35)$$

It's a simple task to verify that the rank condition holds, and the system is completely controllable.

The TH equations present a significantly more challenging computation to evaluate controllability. It *must* be noted that the indefinite form of the controllability Gramian is extremely large and complex, and due to space consideration, we do not include it.

We are particularly interested in whether or not the system is controllable on the horizon  $f \in [0, 2\pi]$ , so we compute the definite integral over this horizon. For this particular case, the elements of the controllability Gramian are given:

$$\begin{aligned} \mathbf{W}(f, f_0)_{(1,1)} = & \frac{9\pi e^2 t^2}{(e-1)^4 \sqrt{e^2-1}} - \frac{6\pi e^2}{\sqrt{1-e^2}} - \frac{16\pi e^2}{(e-1)^4 \sqrt{e^2-1}} + \frac{32\pi e}{(e-1)^4 \sqrt{e^2-1}} + \\ & \frac{\pi}{\sqrt{1-e^2}} + \frac{12\pi}{(e-1)^4 \sqrt{e^2-1}} - \frac{\pi e^{10}}{(e-1)^4 \sqrt{e^2-1}} + \frac{2\pi e^9}{(e-1)^4 \sqrt{e^2-1}} + \\ & \frac{4\pi e^8}{\sqrt{1-e^2}} - \frac{17\pi e^8}{(e-1)^4 \sqrt{e^2-1}} - \frac{38\pi e^7}{(e-1)^4 \sqrt{e^2-1}} - \frac{12\pi e^6}{\sqrt{1-e^2}} + \\ & \frac{45\pi e^6}{(e-1)^4 \sqrt{e^2-1}} + \frac{102\pi e^5}{(e-1)^4 \sqrt{e^2-1}} + \frac{13\pi e^4}{\sqrt{1-e^2}} - \frac{23\pi e^4}{(e-1)^4 \sqrt{e^2-1}} - \\ & \frac{98\pi e^3}{(e-1)^4 \sqrt{e^2-1}} \end{aligned} \quad (36)$$

$$\mathbf{W}(f, f_0)_{(1,2)} = -\frac{3\pi \sqrt{e+1} e^4 t}{(1-e)^{7/2}} - \frac{3\pi \sqrt{e+1} e^3 t}{2(1-e)^{7/2}} - \frac{36\pi \sqrt{e+1} e t}{(1-e)^{7/2}} - \frac{12\pi \sqrt{e+1} t}{(1-e)^{7/2}} \quad (37)$$

$$\mathbf{W}(f, f_0)_{(1,3)} = 0 \quad (38)$$

$$\begin{aligned} \mathbf{W}(f, f_0)_{(1,4)} = & \frac{2\pi\sqrt{e+1}e^8}{(1-e)^{7/2}} - \frac{7\pi\sqrt{e+1}e^7}{(1-e)^{7/2}} - \frac{3\pi\sqrt{e+1}e^6}{(1-e)^{7/2}} + \frac{18\pi\sqrt{e+1}e^5}{(1-e)^{7/2}} + \\ & \frac{15\pi\sqrt{e+1}e^4t}{4(1-e)^{7/2}} + \frac{30\pi\sqrt{e+1}e^3t}{(1-e)^{7/2}} - \frac{15\pi\sqrt{e+1}e^3}{(1-e)^{7/2}} + \frac{45\pi\sqrt{e+1}e^2t}{(1-e)^{7/2}} + \\ & \frac{\pi\sqrt{e+1}e^2}{(1-e)^{7/2}} + \frac{4\pi\sqrt{e+1}e}{(1-e)^{7/2}} \end{aligned} \quad (39)$$

$$\begin{aligned} \mathbf{W}(f, f_0)_{(1,5)} = & \frac{8\pi e^{14}}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{32\pi e^{13}}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{14\pi e^{12}}{(1-e)^{11/2}(e+1)^{3/2}} + \\ & \frac{185\pi e^{11}}{2(1-e)^{11/2}(e+1)^{3/2}} - \frac{161\pi e^{10}}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{127\pi e^9}{(1-e)^{11/2}(e+1)^{3/2}} - \\ & \frac{3\pi e^8t}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{345\pi e^8}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{15\pi e^7t}{2(1-e)^{11/2}(e+1)^{3/2}} + \\ & \frac{200\pi e^7}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{12\pi e^6t}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{307\pi e^6}{(1-e)^{11/2}(e+1)^{3/2}} - \\ & \frac{15\pi e^5t}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{283\pi e^5}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{9\pi e^4t^2}{2(1-e)^{11/2}(e+1)^{3/2}} - \\ & \frac{15\pi e^4t}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{83\pi e^4}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{15\pi e^3t}{2(1-e)^{11/2}(e+1)^{3/2}} + \\ & \frac{407\pi e^3}{2(1-e)^{11/2}(e+1)^{3/2}} - \frac{18\pi e^2t^2}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{6\pi e^2t}{(1-e)^{11/2}(e+1)^{3/2}} + \\ & \frac{40\pi e^2}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{54\pi e}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{22\pi}{(1-e)^{11/2}(e+1)^{3/2}} \end{aligned} \quad (40)$$

$$\mathbf{W}(f, f_0)_{(1,6)} = 0 \quad (41)$$

$$\begin{aligned}
 \mathbf{W}(f, f_0)_{(2,2)} = & \frac{15\pi e^{28}}{4(1-e)^{23/2}(e+1)^{15/2}} + \frac{12\pi e^{27}}{(1-e)^{23/2}(e+1)^{15/2}} - \frac{51\pi e^{26}}{(1-e)^{23/2}(e+1)^{15/2}} - \\
 & \frac{165\pi e^{25}}{(1-e)^{23/2}(e+1)^{15/2}} + \frac{707\pi e^{24}}{2(1-e)^{23/2}(e+1)^{15/2}} + \frac{2023\pi e^{23}}{2(1-e)^{23/2}(e+1)^{15/2}} - \\
 & \frac{1642\pi e^{22}}{(1-e)^{23/2}(e+1)^{15/2}} - \frac{3563\pi e^{21}}{(1-e)^{23/2}(e+1)^{15/2}} + \frac{21809\pi e^{20}}{4(1-e)^{23/2}(e+1)^{15/2}} + \\
 & \frac{15209\pi e^{19}}{2(1-e)^{23/2}(e+1)^{15/2}} - \frac{50397\pi e^{18}}{4(1-e)^{23/2}(e+1)^{15/2}} - \frac{9527\pi e^{17}}{(1-e)^{23/2}(e+1)^{15/2}} + \\
 & \frac{19395\pi e^{16}}{(1-e)^{23/2}(e+1)^{15/2}} + \frac{5726\pi e^{15}}{(1-e)^{23/2}(e+1)^{15/2}} + \frac{27\pi t^2 e^{14}}{(1-e)^{23/2}(e+1)^{15/2}} - \\
 & \frac{18679\pi e^{14}}{(1-e)^{23/2}(e+1)^{15/2}} + \frac{555\pi e^{13}}{(1-e)^{23/2}(e+1)^{15/2}} - \frac{144\pi t^2 e^{12}}{(1-e)^{23/2}(e+1)^{15/2}} + \\
 & \frac{37633\pi e^{12}}{4(1-e)^{23/2}(e+1)^{15/2}} - \frac{2432\pi e^{11}}{(1-e)^{23/2}(e+1)^{15/2}} + \frac{297\pi t^2 e^{10}}{(1-e)^{23/2}(e+1)^{15/2}} - \\
 & \frac{21\pi e^{10}}{2(1-e)^{23/2}(e+1)^{15/2}} - \frac{322\pi e^9}{(1-e)^{23/2}(e+1)^{15/2}} - \frac{270\pi t^2 e^8}{(1-e)^{23/2}(e+1)^{15/2}} - \\
 & \frac{5669\pi e^8}{2(1-e)^{23/2}(e+1)^{15/2}} + \frac{4725\pi e^7}{2(1-e)^{23/2}(e+1)^{15/2}} + \frac{45\pi t^2 e^6}{(1-e)^{23/2}(e+1)^{15/2}} + \\
 & \frac{1371\pi e^6}{(1-e)^{23/2}(e+1)^{15/2}} - \frac{1756\pi e^5}{(1-e)^{23/2}(e+1)^{15/2}} + \frac{108\pi t^2 e^4}{(1-e)^{23/2}(e+1)^{15/2}} - \\
 & \frac{353\pi e^4}{4(1-e)^{23/2}(e+1)^{15/2}} + \frac{1127\pi e^3}{2(1-e)^{23/2}(e+1)^{15/2}} - \frac{81\pi t^2 e^2}{(1-e)^{23/2}(e+1)^{15/2}} - \\
 & \frac{405\pi e^2}{4(1-e)^{23/2}(e+1)^{15/2}} - \frac{70\pi e}{(1-e)^{23/2}(e+1)^{15/2}} + \frac{18\pi t^2}{(1-e)^{23/2}(e+1)^{15/2}} + \\
 & \frac{22\pi}{(1-e)^{23/2}(e+1)^{15/2}}
 \end{aligned} \tag{42}$$

$$\mathbf{W}(f, f_0)_{(2,3)} = 0 \tag{43}$$



$$\begin{aligned}
 \mathbf{W}(f, f_0)_{(2,4)} = & \frac{9\pi e^{15}}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{2\pi e^{14}}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{167\pi e^{13}}{2(1-e)^{11/2}(e+1)^{3/2}} + \\
 & \frac{303\pi e^{12}}{4(1-e)^{11/2}(e+1)^{3/2}} + \frac{991\pi e^{11}}{4(1-e)^{11/2}(e+1)^{3/2}} - \frac{1683\pi e^{10}}{4(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{177\pi e^9}{4(1-e)^{11/2}(e+1)^{3/2}} + \frac{3\pi e^8 t}{2(1-e)^{11/2}(e+1)^{3/2}} + \frac{1949\pi e^8}{4(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{3\pi e^7 t}{2(1-e)^{11/2}(e+1)^{3/2}} - \frac{1127\pi e^7}{4(1-e)^{11/2}(e+1)^{3/2}} + \frac{3\pi e^6 t}{(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{187\pi e^6}{4(1-e)^{11/2}(e+1)^{3/2}} - \frac{3\pi e^5 t}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{443\pi e^5}{4(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{45\pi e^4 t^2}{4(1-e)^{11/2}(e+1)^{3/2}} - \frac{21\pi e^4 t}{2(1-e)^{11/2}(e+1)^{3/2}} - \frac{126\pi e^4}{(1-e)^{11/2}(e+1)^{3/2}} + \\
 & \frac{21\pi e^3 t}{2(1-e)^{11/2}(e+1)^{3/2}} + \frac{84\pi e^3}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{45\pi e^2 t^2}{(1-e)^{11/2}(e+1)^{3/2}} + \\
 & \frac{6\pi e^2 t}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{45\pi e^2}{2(1-e)^{11/2}(e+1)^{3/2}} - \frac{6\pi e t}{(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{42\pi e}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{10\pi}{(1-e)^{11/2}(e+1)^{3/2}}
 \end{aligned} \tag{44}$$

$$\begin{aligned}
 \mathbf{W}(f, f_0)_{(2,5)} = & -\frac{15\pi e^8 t}{4(1-e)^{11/2}(e+1)^{3/2}} + \frac{21\pi e^7 t}{2(1-e)^{11/2}(e+1)^{3/2}} + \frac{51\pi e^6 t}{(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{21\pi e^5 t}{2(1-e)^{11/2}(e+1)^{3/2}} - \frac{9\pi e^4 t^2}{4(1-e)^{11/2}(e+1)^{3/2}} + \frac{3\pi e^4 t}{4(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{84\pi e^3 t}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{9\pi e^2 t^2}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{66\pi e^2 t}{(1-e)^{11/2}(e+1)^{3/2}} + \\
 & \frac{84\pi e t}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{18\pi t}{(1-e)^{11/2}(e+1)^{3/2}}
 \end{aligned} \tag{45}$$

$$\mathbf{W}(f, f_0)_{(2,6)} = 0 \tag{46}$$

$$\mathbf{W}(f, f_0)_{(3,3)} = \frac{1}{4}\pi\sqrt{e^2-1}e^2 - \pi\sqrt{e^2-1} + \frac{3}{4}\pi\sqrt{e^2-1}e^4 \tag{47}$$

$$\mathbf{W}(f, f_0)_{(3,4)} = 0 \tag{48}$$

$$\mathbf{W}(f, f_0)_{(3,5)} = 0 \tag{49}$$

$$\mathbf{W}(f, f_0)_{(3,6)} = 0 \tag{50}$$

$$\begin{aligned}
 \mathbf{W}(f, f_0)_{(4,4)} = & \frac{29\pi e^{14}}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{116\pi e^{13}}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{695\pi e^{12}}{8(1-e)^{11/2}(e+1)^{3/2}} + \\
 & \frac{423\pi e^{11}}{2(1-e)^{11/2}(e+1)^{3/2}} - \frac{3343\pi e^{10}}{8(1-e)^{11/2}(e+1)^{3/2}} + \frac{93\pi e^9}{2(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{24\pi e^8 t}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{3737\pi e^8}{8(1-e)^{11/2}(e+1)^{3/2}} + \frac{24\pi e^7 t}{(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{581\pi e^7}{2(1-e)^{11/2}(e+1)^{3/2}} + \frac{48\pi e^6 t}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{1095\pi e^6}{8(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{48\pi e^5 t}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{303\pi e^5}{2(1-e)^{11/2}(e+1)^{3/2}} + \frac{261\pi e^4 t^2}{4(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{24\pi e^4 t}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{119\pi e^4}{2(1-e)^{11/2}(e+1)^{3/2}} + \frac{24\pi e^3 t}{(1-e)^{11/2}(e+1)^{3/2}} + \\
 & \frac{21\pi e^3}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{9\pi e^2 t^2}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{105\pi e^2}{4(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{24\pi e}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{5\pi}{(1-e)^{11/2}(e+1)^{3/2}}
 \end{aligned} \tag{51}$$

$$\begin{aligned}
 \mathbf{W}(f, f_0)_{(4,5)} = & -\frac{5\pi e^{12}}{2(1-e)^{11/2}(e+1)^{3/2}} + \frac{19\pi e^{11}}{2(1-e)^{11/2}(e+1)^{3/2}} + \frac{16\pi e^{10}}{(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{58\pi e^9}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{27\pi e^8 t}{4(1-e)^{11/2}(e+1)^{3/2}} - \frac{25\pi e^8}{(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{63\pi e^7 t}{2(1-e)^{11/2}(e+1)^{3/2}} + \frac{123\pi e^7}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{213\pi e^6 t}{4(1-e)^{11/2}(e+1)^{3/2}} + \\
 & \frac{4\pi e^6}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{105\pi e^5 t}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{116\pi e^5}{(1-e)^{11/2}(e+1)^{3/2}} + \\
 & \frac{18\pi e^4 t^2}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{132\pi e^4 t}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{31\pi e^4}{2(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{147\pi e^3 t}{2(1-e)^{11/2}(e+1)^{3/2}} + \frac{95\pi e^3}{2(1-e)^{11/2}(e+1)^{3/2}} - \frac{72\pi e^2 t}{(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{8\pi e^2}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{6\pi e}{(1-e)^{11/2}(e+1)^{3/2}}
 \end{aligned} \tag{52}$$

$$\mathbf{W}(f, f_0)_{(4,6)} = 0 \tag{53}$$

$$\begin{aligned}
 \mathbf{W}(f, f_0)_{(5,5)} = & \frac{16\pi e^{16}}{(1-e)^{13/2}(e+1)^{5/2}} - \frac{64\pi e^{15}}{(1-e)^{13/2}(e+1)^{5/2}} + \frac{33\pi e^{14}}{4(1-e)^{13/2}(e+1)^{5/2}} + \\
 & \frac{249\pi e^{13}}{(1-e)^{13/2}(e+1)^{5/2}} - \frac{1445\pi e^{12}}{4(1-e)^{13/2}(e+1)^{5/2}} - \frac{364\pi e^{11}}{(1-e)^{13/2}(e+1)^{5/2}} - \\
 & \frac{15\pi e^{10}t}{2(1-e)^{13/2}(e+1)^{5/2}} + \frac{2009\pi e^{10}}{2(1-e)^{13/2}(e+1)^{5/2}} + \frac{21\pi e^9t}{(1-e)^{13/2}(e+1)^{5/2}} + \\
 & \frac{468\pi e^9}{(1-e)^{13/2}(e+1)^{5/2}} + \frac{123\pi e^8t}{2(1-e)^{13/2}(e+1)^{5/2}} - \frac{2437\pi e^8}{2(1-e)^{13/2}(e+1)^{5/2}} - \\
 & \frac{84\pi e^7t}{(1-e)^{13/2}(e+1)^{5/2}} - \frac{846\pi e^7}{(1-e)^{13/2}(e+1)^{5/2}} - \frac{27\pi e^6t^2}{4(1-e)^{13/2}(e+1)^{5/2}} - \\
 & \frac{237\pi e^6t}{2(1-e)^{13/2}(e+1)^{5/2}} + \frac{2973\pi e^6}{4(1-e)^{13/2}(e+1)^{5/2}} + \frac{105\pi e^5t}{(1-e)^{13/2}(e+1)^{5/2}} + \\
 & \frac{979\pi e^5}{(1-e)^{13/2}(e+1)^{5/2}} + \frac{135\pi e^4t^2}{4(1-e)^{13/2}(e+1)^{5/2}} + \frac{165\pi e^4t}{2(1-e)^{13/2}(e+1)^{5/2}} - \\
 & \frac{665\pi e^4}{4(1-e)^{13/2}(e+1)^{5/2}} - \frac{42\pi e^3t}{(1-e)^{13/2}(e+1)^{5/2}} - \frac{518\pi e^3}{(1-e)^{13/2}(e+1)^{5/2}} + \\
 & \frac{36\pi e^2t^2}{(1-e)^{13/2}(e+1)^{5/2}} - \frac{18\pi e^2t}{(1-e)^{13/2}(e+1)^{5/2}} - \frac{64\pi e^2}{(1-e)^{13/2}(e+1)^{5/2}} + \\
 & \frac{96\pi e}{(1-e)^{13/2}(e+1)^{5/2}} + \frac{38\pi}{(1-e)^{13/2}(e+1)^{5/2}}
 \end{aligned} \tag{54}$$

$$\mathbf{W}(f, f_0)_{(5,6)} = 0 \tag{55}$$

$$\mathbf{W}(f, f_0)_{(6,6)} = \frac{41}{4}\pi\sqrt{1-e^2}e^2 + \pi\sqrt{1-e^2} + \frac{9}{2}\pi\sqrt{1-e^2}e^4 \tag{56}$$

By the symmetric property of the Gramian matrix, there will be at most  $\frac{n(n+1)}{2}$  unique elements for the  $n \times n$  matrix. This is why we only present the 21 unique elements of the Gramian.

The reader may also notice that the variable  $t$  appears in a number of the elements of the matrix. It is generally not possible to completely eliminate the time variable from the system because of the intimate and transcendental coupling between time and true anomaly.

The most important result for this section is that, although the elements are extremely complex and difficult to compute, this system is completely controllable on the interval  $f \in [0, 2\pi]$ . The Gramian has full rank and it is nonsingular.

## 6 Under-Actuated Controllability

We now turn our attention to a more interesting case; we now set the radial component of the controller to be  $b_{41} = 0 \implies u_x = 0$ . Examining once more the HCW plant with the

following influence matrix

$$\mathbf{B} = \begin{bmatrix} \mathbf{0}_{4 \times 3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (57)$$

we can compute the controllability matrix. Once again, we find that eliminating the radial component still yields a completely controllable system! Returning to the TH equations, and using the same control horizon, we find the elements of the controllability Gramian to be

$$\begin{aligned} \mathbf{W}(f, f_0)_{(1,1)} = & \frac{9\pi e^2 t^2}{(e-1)^4 \sqrt{e^2-1}} - \frac{16\pi e^2}{(e-1)^4 \sqrt{e^2-1}} + \frac{32\pi e}{(e-1)^4 \sqrt{e^2-1}} + \\ & \frac{12\pi}{(e-1)^4 \sqrt{e^2-1}} - \frac{\pi e^{10}}{(e-1)^4 \sqrt{e^2-1}} + \frac{2\pi e^9}{(e-1)^4 \sqrt{e^2-1}} - \\ & \frac{17\pi e^8}{(e-1)^4 \sqrt{e^2-1}} - \frac{38\pi e^7}{(e-1)^4 \sqrt{e^2-1}} + \frac{45\pi e^6}{(e-1)^4 \sqrt{e^2-1}} + \\ & \frac{102\pi e^5}{(e-1)^4 \sqrt{e^2-1}} - \frac{23\pi e^4}{(e-1)^4 \sqrt{e^2-1}} - \frac{98\pi e^3}{(e-1)^4 \sqrt{e^2-1}} \end{aligned} \quad (58)$$

$$\mathbf{W}(f, f_0)_{(1,2)} = -\frac{3\pi\sqrt{e+1}e^4 t}{(1-e)^{7/2}} - \frac{3\pi\sqrt{e+1}e^3 t}{2(1-e)^{7/2}} - \frac{36\pi\sqrt{e+1}et}{(1-e)^{7/2}} - \frac{12\pi\sqrt{e+1}t}{(1-e)^{7/2}} \quad (59)$$

$$\mathbf{W}(f, f_0)_{(1,3)} = 0 \quad (60)$$

$$\begin{aligned} \mathbf{W}(f, f_0)_{(1,4)} = & \frac{2\pi\sqrt{e+1}e^8}{(1-e)^{7/2}} - \frac{7\pi\sqrt{e+1}e^7}{(1-e)^{7/2}} - \frac{3\pi\sqrt{e+1}e^6}{(1-e)^{7/2}} + \frac{18\pi\sqrt{e+1}e^5}{(1-e)^{7/2}} + \\ & \frac{15\pi\sqrt{e+1}e^4 t}{4(1-e)^{7/2}} + \frac{30\pi\sqrt{e+1}e^3 t}{(1-e)^{7/2}} - \frac{15\pi\sqrt{e+1}e^3}{(1-e)^{7/2}} + \frac{45\pi\sqrt{e+1}e^2 t}{(1-e)^{7/2}} + \\ & \frac{\pi\sqrt{e+1}e^2}{(1-e)^{7/2}} + \frac{4\pi\sqrt{e+1}e}{(1-e)^{7/2}} \end{aligned} \quad (61)$$

$$\begin{aligned}
 \mathbf{W}(f, f_0)_{(1,5)} = & -\frac{2\pi e^{12}}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{7\pi e^{11}}{2(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{27\pi e^{10}}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{55\pi e^9}{(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{3\pi e^8 t}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{115\pi e^8}{(1-e)^{11/2}(e+1)^{3/2}} + \\
 & \frac{15\pi e^7 t}{2(1-e)^{11/2}(e+1)^{3/2}} + \frac{248\pi e^7}{(1-e)^{11/2}(e+1)^{3/2}} + \\
 & \frac{12\pi e^6 t}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{143\pi e^6}{(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{15\pi e^5 t}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{379\pi e^5}{(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{9\pi e^4 t^2}{2(1-e)^{11/2}(e+1)^{3/2}} - \frac{15\pi e^4 t}{(1-e)^{11/2}(e+1)^{3/2}} + \\
 & \frac{39\pi e^4}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{15\pi e^3 t}{2(1-e)^{11/2}(e+1)^{3/2}} + \\
 & \frac{503\pi e^3}{2(1-e)^{11/2}(e+1)^{3/2}} - \frac{18\pi e^2 t^2}{(1-e)^{11/2}(e+1)^{3/2}} + \\
 & \frac{6\pi e^2 t}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{38\pi e^2}{(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{62\pi e}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{20\pi}{(1-e)^{11/2}(e+1)^{3/2}}
 \end{aligned} \tag{62}$$

$$\mathbf{W}(f, f_0)_{(1,6)} = 0 \tag{63}$$

$$\begin{aligned}
 \mathbf{W}(f, f_0)_{(2,2)} = & -\frac{2\pi e^{12}}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{4\pi e^{11}}{(1-e)^{11/2}(e+1)^{3/2}} + \\
 & \frac{15\pi e^{10}}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{34\pi e^9}{(1-e)^{11/2}(e+1)^{3/2}} - \\
 & \frac{38\pi e^8}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{110\pi e^7}{(1-e)^{11/2}(e+1)^{3/2}} + \\
 & \frac{28\pi e^6}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{166\pi e^5}{(1-e)^{11/2}(e+1)^{3/2}} + \\
 & \frac{24\pi e^4}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{118\pi e^3}{(1-e)^{11/2}(e+1)^{3/2}} + \\
 & \frac{27\pi e^2 t^2}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{43\pi e^2}{(1-e)^{11/2}(e+1)^{3/2}} + \\
 & \frac{18\pi t^2}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{32\pi e}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{16\pi}{(1-e)^{11/2}(e+1)^{3/2}}
 \end{aligned} \tag{64}$$

$$\mathbf{W}(f, f_0)_{(2,3)} = 0 \tag{65}$$

$$\begin{aligned} \mathbf{W}(f, f_0)_{(2,4)} = & \frac{19\pi e^{12}}{4(1-e)^{11/2}(e+1)^{3/2}} - \frac{23\pi e^{11}}{4(1-e)^{11/2}(e+1)^{3/2}} - \\ & \frac{83\pi e^{10}}{4(1-e)^{11/2}(e+1)^{3/2}} + \frac{99\pi e^9}{4(1-e)^{11/2}(e+1)^{3/2}} + \\ & \frac{3\pi e^8 t}{2(1-e)^{11/2}(e+1)^{3/2}} + \frac{55\pi e^8}{4(1-e)^{11/2}(e+1)^{3/2}} - \\ & \frac{3\pi e^7 t}{2(1-e)^{11/2}(e+1)^{3/2}} - \frac{47\pi e^7}{4(1-e)^{11/2}(e+1)^{3/2}} + \\ & \frac{3\pi e^6 t}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{111\pi e^6}{4(1-e)^{11/2}(e+1)^{3/2}} - \\ & \frac{3\pi e^5 t}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{223\pi e^5}{4(1-e)^{11/2}(e+1)^{3/2}} - \\ & \frac{45\pi e^4 t^2}{4(1-e)^{11/2}(e+1)^{3/2}} - \frac{21\pi e^4 t}{2(1-e)^{11/2}(e+1)^{3/2}} - \\ & \frac{59\pi e^4}{2(1-e)^{11/2}(e+1)^{3/2}} + \frac{21\pi e^3 t}{2(1-e)^{11/2}(e+1)^{3/2}} + \\ & \frac{153\pi e^3}{2(1-e)^{11/2}(e+1)^{3/2}} - \frac{45\pi e^2 t^2}{(1-e)^{11/2}(e+1)^{3/2}} + \\ & \frac{6\pi e^2 t}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{4\pi e^2}{(1-e)^{11/2}(e+1)^{3/2}} - \\ & \frac{6\pi e t}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{28\pi e}{(1-e)^{11/2}(e+1)^{3/2}} + \\ & \frac{8\pi}{(1-e)^{11/2}(e+1)^{3/2}} \end{aligned} \tag{66}$$

$$\begin{aligned} \mathbf{W}(f, f_0)_{(2,5)} = & - \frac{15\pi e^8 t}{4(1-e)^{11/2}(e+1)^{3/2}} + \frac{21\pi e^7 t}{2(1-e)^{11/2}(e+1)^{3/2}} + \\ & \frac{51\pi e^6 t}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{21\pi e^5 t}{2(1-e)^{11/2}(e+1)^{3/2}} - \\ & \frac{9\pi e^4 t^2}{4(1-e)^{11/2}(e+1)^{3/2}} + \frac{3\pi e^4 t}{4(1-e)^{11/2}(e+1)^{3/2}} - \\ & \frac{84\pi e^3 t}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{9\pi e^2 t^2}{(1-e)^{11/2}(e+1)^{3/2}} - \\ & \frac{66\pi e^2 t}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{84\pi e t}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{18\pi t}{(1-e)^{11/2}(e+1)^{3/2}} \end{aligned} \tag{67}$$

$$\mathbf{W}(f, f_0)_{(2,6)} = 0 \tag{68}$$

$$\mathbf{W}(f, f_0)_{(3,3)} = \frac{1}{4}\pi\sqrt{e^2 - 1}e^2 - \pi\sqrt{e^2 - 1} + \frac{3}{4}\pi\sqrt{e^2 - 1}e^4 \quad (69)$$

$$\mathbf{W}(f, f_0)_{(3,4)} = 0 \quad (70)$$

$$\mathbf{W}(f, f_0)_{(3,5)} = 0 \quad (71)$$

$$\mathbf{W}(f, f_0)_{(3,6)} = 0 \quad (72)$$

$$\begin{aligned} \mathbf{W}(f, f_0)_{(4,4)} = & -\frac{33\pi e^{12}}{8(1-e)^{11/2}(e+1)^{3/2}} - \frac{9\pi e^{11}}{2(1-e)^{11/2}(e+1)^{3/2}} - \\ & \frac{249\pi e^{10}}{8(1-e)^{11/2}(e+1)^{3/2}} + \frac{87\pi e^9}{2(1-e)^{11/2}(e+1)^{3/2}} - \\ & \frac{24\pi e^8 t}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{697\pi e^8}{8(1-e)^{11/2}(e+1)^{3/2}} + \\ & \frac{24\pi e^7 t}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{167\pi e^7}{2(1-e)^{11/2}(e+1)^{3/2}} + \\ & \frac{48\pi e^6 t}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{299\pi e^6}{8(1-e)^{11/2}(e+1)^{3/2}} - \\ & \frac{48\pi e^5 t}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{69\pi e^5}{2(1-e)^{11/2}(e+1)^{3/2}} + \\ & \frac{261\pi e^4 t^2}{4(1-e)^{11/2}(e+1)^{3/2}} - \frac{24\pi e^4 t}{(1-e)^{11/2}(e+1)^{3/2}} - \\ & \frac{75\pi e^4}{2(1-e)^{11/2}(e+1)^{3/2}} + \frac{24\pi e^3 t}{(1-e)^{11/2}(e+1)^{3/2}} + \\ & \frac{30\pi e^3}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{9\pi e^2 t^2}{(1-e)^{11/2}(e+1)^{3/2}} + \\ & \frac{19\pi e^2}{(1-e)^{11/2}(e+1)^{3/2}} - \frac{20\pi e}{(1-e)^{11/2}(e+1)^{3/2}} + \frac{4\pi}{(1-e)^{11/2}(e+1)^{3/2}} \end{aligned} \quad (73)$$

$$\begin{aligned}
 \mathbf{W}(f, f_0)_{(4,5)} = & -\frac{72\pi e^2 t}{(e-1)^4(1-e^2)^{3/2}} - \frac{8\pi e^2}{(e-1)^4(1-e^2)^{3/2}} - \frac{6\pi e}{(e-1)^4(1-e^2)^{3/2}} - \\
 & \frac{5\pi e^{12}}{2(e-1)^4(1-e^2)^{3/2}} + \frac{19\pi e^{11}}{2(e-1)^4(1-e^2)^{3/2}} + \frac{16\pi e^{10}}{(e-1)^4(1-e^2)^{3/2}} - \\
 & \frac{58\pi e^9}{(e-1)^4(1-e^2)^{3/2}} - \frac{27\pi e^8 t}{4(e-1)^4(1-e^2)^{3/2}} - \frac{25\pi e^8}{(e-1)^4(1-e^2)^{3/2}} - \\
 & \frac{63\pi e^7 t}{2(e-1)^4(1-e^2)^{3/2}} + \frac{123\pi e^7}{(e-1)^4(1-e^2)^{3/2}} - \frac{213\pi e^6 t}{4(e-1)^4(1-e^2)^{3/2}} + \\
 & \frac{4\pi e^6}{(e-1)^4(1-e^2)^{3/2}} + \frac{105\pi e^5 t}{(e-1)^4(1-e^2)^{3/2}} - \frac{116\pi e^5}{(e-1)^4(1-e^2)^{3/2}} + \\
 & \frac{18\pi e^4 t^2}{(e-1)^4(1-e^2)^{3/2}} + \frac{132\pi e^4 t}{(e-1)^4(1-e^2)^{3/2}} + \frac{31\pi e^4}{2(e-1)^4(1-e^2)^{3/2}} - \\
 & \frac{147\pi e^3 t}{2(e-1)^4(1-e^2)^{3/2}} + \frac{95\pi e^3}{2(e-1)^4(1-e^2)^{3/2}}
 \end{aligned} \tag{74}$$

$$\mathbf{W}(f, f_0)_{(4,6)} = 0 \tag{75}$$



$$\begin{aligned}
 \mathbf{W}(f, f_0)_{(5,5)} = & -\frac{31\pi e^{14}}{4(1-e)^{13/2}(e+1)^{5/2}} - \frac{7\pi e^{13}}{(1-e)^{13/2}(e+1)^{5/2}} - \\
 & \frac{245\pi e^{12}}{4(1-e)^{13/2}(e+1)^{5/2}} - \frac{28\pi e^{11}}{(1-e)^{13/2}(e+1)^{5/2}} - \\
 & \frac{15\pi e^{10}t}{2(1-e)^{13/2}(e+1)^{5/2}} + \frac{553\pi e^{10}}{2(1-e)^{13/2}(e+1)^{5/2}} + \\
 & \frac{21\pi e^9t}{(1-e)^{13/2}(e+1)^{5/2}} + \frac{420\pi e^9}{(1-e)^{13/2}(e+1)^{5/2}} + \\
 & \frac{123\pi e^8t}{2(1-e)^{13/2}(e+1)^{5/2}} - \frac{861\pi e^8}{2(1-e)^{13/2}(e+1)^{5/2}} - \\
 & \frac{84\pi e^7t}{(1-e)^{13/2}(e+1)^{5/2}} - \frac{1134\pi e^7}{(1-e)^{13/2}(e+1)^{5/2}} - \\
 & \frac{27\pi e^6t^2}{4(1-e)^{13/2}(e+1)^{5/2}} - \frac{237\pi e^6t}{2(1-e)^{13/2}(e+1)^{5/2}} + \\
 & \frac{1309\pi e^6}{4(1-e)^{13/2}(e+1)^{5/2}} + \frac{105\pi e^5t}{(1-e)^{13/2}(e+1)^{5/2}} + \\
 & \frac{1267\pi e^5}{(1-e)^{13/2}(e+1)^{5/2}} + \frac{135\pi e^4t^2}{4(1-e)^{13/2}(e+1)^{5/2}} + \\
 & \frac{165\pi e^4t}{2(1-e)^{13/2}(e+1)^{5/2}} - \frac{329\pi e^4}{4(1-e)^{13/2}(e+1)^{5/2}} - \\
 & \frac{42\pi e^3t}{(1-e)^{13/2}(e+1)^{5/2}} - \frac{630\pi e^3}{(1-e)^{13/2}(e+1)^{5/2}} + \\
 & \frac{36\pi e^2t^2}{(1-e)^{13/2}(e+1)^{5/2}} - \frac{18\pi e^2t}{(1-e)^{13/2}(e+1)^{5/2}} - \\
 & \frac{56\pi e^2}{(1-e)^{13/2}(e+1)^{5/2}} + \frac{112\pi e}{(1-e)^{13/2}(e+1)^{5/2}} + \frac{34\pi}{(1-e)^{13/2}(e+1)^{5/2}}
 \end{aligned} \tag{76}$$

$$\mathbf{W}(f, f_0)_{(5,6)} = 0 \tag{77}$$

$$\mathbf{W}(f, f_0)_{(6,6)} = \frac{41}{4}\pi\sqrt{1-e^2}e^2 + \pi\sqrt{1-e^2} + \frac{9}{2}\pi\sqrt{1-e^2}e^4 \tag{78}$$

Once again, only 21 unique elements need to be computed. Furthermore, this matrix is nonsingular and has full rank, so the system is completely controllable.

Physically, this means that we need only apply thrust in the in-track and cross-track directions to obtain complete authority over the linear system. This is related to the conservation of angular momentum. It is well-known from Lambert guidance [2] that in order to raise an orbit, the simplest way in which to do this is thrust in the velocity direction. The speed increases and since angular momentum must be conserved, this raises the satellite to a higher orbit.

While the problem is concerned primarily with the relative motion, we must stress that the chief and deputy are still both in Keplerian orbits, so the essential results from the Kepler problem (i.e. conserved quantities) should still hold in a macroscopic sense. While this system is expressed in a rotating frame, the controllability of the formation flying equations using only in-track/ cross-track thrust gives immediate insight to the underlying Keplerian structure of the system.

## 7 Conclusions

In this report, we have discussed the essential systems used to model spacecraft formation flying/ rendezvous. In the first half of the report, we showed the three primary models and how they are related using simple transformations/ assumptions. In addition to this, we showed how results in one system, namely the TH equations, could be extrapolated back to the LERM and HCW equations. We also presented the state-transition matrix and stressed why it was much easier/ more advantageous to analyze the state-transition matrix for the TH equations than for the LERM. This was a result of needing to use Kepler's equation in the time-domain, and the numerical/ analytical difficulties this presents.

In the second half of the report, we performed a controllability analysis of the TH system. We did this noting that the results could be mapped to the LERM using a similarity transformation. After a brief review of controllability for linear, time-invariant and time-varying systems, we showed using both the HCW equations and the TH equations that the system is completely controllable when fully actuated. We then showed the interesting result that the in-track/ cross-track controlled TH equations are also completely controllable, and then we also discussed the physical significance of this result.

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