Computationally Driven Algorithms for Distributed Control of Complex Systems

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(ABSTRACT)

This dissertation studies the model reduction and distributed control problems for interconnected systems, i.e., systems that consist of multiple interacting agents/subsystems. The study of the analysis and synthesis problems for interconnected systems is motivated by the multiple applications that can benefit from the design and implementation of distributed controllers. These applications include automated highway systems and formation flight of unmanned aircraft systems.

The systems of interest are modeled using arbitrary directed graphs, where the subsystems correspond to the nodes, and the interconnections between the subsystems are described using the directed edges. In addition to the states of the subsystems, the adopted frameworks also model the interconnections between the subsystems as spatial states. Each agent/subsystem is assumed to have its own actuating and sensing capabilities. These capabilities are leveraged in order to design a controller subsystem for each plant subsystem. In the distributed control paradigm, the controller subsystems interact over the same interconnection structure as the plant subsystems.

The models assumed for the subsystems are linear time-varying or linear parameter-varying. Linear time-varying models are useful for describing nonlinear equations that are linearized about prespecified trajectories, and linear parameter-varying models allow for capturing the nonlinearities of the agents, while still being amenable to control using linear techniques.

It is clear from the above description that the size of the model for an interconnected system increases with the number of subsystems and the complexity of the interconnection structure. This motivates the development of model reduction techniques to rigorously reduce the size of the given model. In particular, this dissertation presents structure-preserving techniques for model reduction, i.e., techniques that guarantee that the interpretation of each state is retained in the reduced order system. Namely, the sought reduced order system is an interconnected system formed by reduced order subsystems that are interconnected over the same interconnection structure as that of the full order system. Model reduction is important for reducing the computational complexity of the system analysis and control synthesis problems.

In this dissertation, interior point methods are extensively used for solving the semidefinite programming problems that arise in analysis and synthesis.

Computationally Driven Algorithms for Distributed Control of Complex Systems

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(GENERAL AUDIENCE ABSTRACT)

The work in this dissertation is motivated by the numerous applications in which multiple agents interact and cooperate to perform a coordinated task. Examples of such applications include automated highway systems and formation flight of unmanned aircraft systems. For instance, one can think of the hazardous conditions created by a fire in a building and the benefits of using multiple interacting multirotors to deal with this emergency situation and reduce the risks on humans. This dissertation develops mathematical tools for studying and dealing with these complex systems. Namely, it is shown how controllers can be designed to ensure that such systems perform in the desired way, and how the models that describe the systems of interest can be systematically simplified to facilitate performing the tasks of mathematical analysis and control design.

Dedication

To my parents, sister, and brother:

Joseph, Roula, Maya, and Sami.

Lord, make me an instrument of your peace;
where there is hatred, let me sow love; where there is injury, pardon;
where there is doubt, faith; where there is despair, hope;
where there is darkness, light; and where there is sadness, joy.
O Divine Master, grant that I may not so much seek
to be consoled, as to console;
to be understood, as to understand; to be loved, as to love;
for it is in giving that we receive, it is in pardoning that we are pardoned,
and it is in dying that we are born to Eternal Life. Amen.
Attributed to Saint Francis of Assisi.

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I would like to thank my family members for their love, support, prayers, and frequent visits to Blacksburg, VA.

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The material in this dissertation is based on work supported by the National Science Foundation (NSF) under Grant Number CMMI-1333785. I have also done work on robustness analysis of control systems using integral quadratic constraints (IQCs) as part of my PhD studies, which is not included in this dissertation. My work on IQC-based robustness analysis was supported by NSF under Grant Number CMMI-1351640 and the Center for Unmanned Aircraft Systems (C-UAS), an NSF sponsored industry/university cooperative research center (I/UCRC) under NSF Grant Numbers IIP-1539975 and CNS-1650465, along with significant contributions from C-UAS industry members.

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List of Abbreviations

CCF	Contractive coprime factorization
CM	Center of mass
LFT	Linear fractional transformation
LMI	Linear matrix inequality
LPV	Linear parameter-varying
LTI	Linear time-invariant
LTV	Linear time-varying
NCF	Normalized coprime factorization
NSLPV	Nonstationary linear parameter-varying
RCF	Right coprime factorization
SDP	Semidefinite program/programming

Chapter 1

Introduction

1.1 Distributed Control

The study of interconnected systems is of interest to the control community because systems in which multiple agents exhibit a collective and coordinated behavior are commonly observed in nature and in engineering applications [53]. Flocks of birds and schools of fishes are examples of multi-agent systems from the realm of nature.

In recent years, many applications that can benefit from the design of distributed controllers have emerged such as mobile sensor networks, multi-point surveillance, automated highway control and vehicular platoons, aircraft formation flight, and applications based on lumped approximations of partial differential equations, see e.g., [37, 45, 47, 53, 67, 71]. Distributed control approaches assume that each plant subsystem has its own actuating and sensing capabilities. These capabilities are exploited in control design, and the synthesized controller is formed of subsystems that interact over an interconnection structure similar to that of the plant. The field of distributed control has witnessed a boost as a result of the technological advances that have made it possible to implement large distributed sensor and actuator arrays [22]. In general, distributed control approaches are preferred over centralized control approaches because they are simpler, more flexible, and more robust, and they scale better with problems of increasing sizes and complexity [21, 70]. Synthesis techniques grounded in the distributed control paradigm may also be preferred over synthesis techniques based on a decentralized paradigm in applications that have strict performance requirements. Decentralized control approaches also make use of the actuating and sensing capabilities of the plant subsystems, and a controller subsystem is individually designed for each plant subsystem. For the design of the controller subsystems, the effect of the interconnection structure is modeled as noise inputs that affect the plant subsystems [37]. The controller subsystems do not interact in a decentralized controller, which makes a decentralized controller designed for a given plant simpler than a distributed controller designed for the same plant. However, the mathematical guarantees for stability and performance are generally lacking in decentralized control approaches. Specifically, even if each controller subsystem is designed to be optimal for the corresponding plant subsystem, it is not necessary that the decentralized controller is optimal for the overall interconnection [83]. Additionally, distributed controllers apply whenever the overall interconnected system is stabilizable, whereas decentralized controllers require that each plant subsystem is individually stabilizable.

Motivated by the potential advantages and applications of distributed control, the work in this dissertation seeks to establish analysis and synthesis results for interconnected systems in which the constituent subsystems have complex models. Namely, subsystems that have linear time-varying (LTV), linear parameter-varying (LPV), and nonstationary linear parameter-varying (NSLPV) models are considered throughout this dissertation. As compared to linear time-invariant (LTI) models, the considered models allow for capturing the behavior of complex nonlinear systems along trajectories and for augmenting the operation envelope of the synthesized controller.

NSLPV models were introduced and motivated in [33, 34]. They are extensions of standard LPV models in that the state-space matrices depend on a priori known time-varying terms, in addition to their dependence on parameters that are not known a priori, but are assumed to be available for measurement at each time-step. It is assumed that the dependence of the state-space matrices on these parameters is rational so as to allow for formulating the subsystems in an linear fractional transformation (LFT) framework. This assumption is not generally restrictive as nonlinear functions that are not rational can frequently be approximated by rational ones. An NSLPV model formulated in an LFT framework is basically an interconnection of a nominal LTV model and a Δ-operator that consists of all the scheduling parameters. Thus, analysis results for NSLPV models provide robustness conditions for LTV models against static time-varying uncertainties [38]. Like standard LPV models [59, 66], NSLPV models allow for capturing the nonlinearities of the studied system, while being amenable to control using linear techniques. In general, when it comes to describing time-varying nonlinear systems using parameter-varying models, NSLPV models are far less conservative than stationary LPV models, and in some cases, the only stabilizable models attainable are NSLPV [34]. In the context of multiple interconnected agents, a distributed NSLPV system can be formed by an interconnection of NSLPV subsystems and/or combinations of LTV, LPV, and LTI subsystems.

Several works have appeared in the literature that address the problem of distributed control for interconnected LPV systems and interconnected uncertain systems, e.g., [20, 29, 44, 57, 81, 87]. These works can be classified based upon various criteria. For instance, the classification criterion can be the interconnection and uncertainty structures of the controller. In [29, 44, 57, 81, 87], the sought controller inherits the interconnection structure of the plant. On the other hand, in [20], the structure and the order of the controller are design inputs. The controller subsystems in [81] are assumed to have LTI models, whereas, in [29, 44, 57, 87], the subsystems of the controller are parameter-dependent and are described similarly to the subsystems of the plant. In [29], the controller subsystems depend on their own local parameters as well as parameters received from other subsystems. A second classification criterion is the type of Lyapunov function used in the derivation of the analysis and synthesis results.

For example, in [20, 29, 57], the synthesis results are derived using a parameter-dependent Lyapunov function, whereas, in [44, 81, 87], a parameter-independent Lyapunov function is used. The use of various types of Lyapunov functions bares consequences on the convexity, tractability, and conservativeness of the derived analysis and synthesis results. Third, the classification can be based upon the complexity of the interconnection structure and the heterogeneity of the subsystems. Namely, the works of [20, 57, 87] consider subsystems that have identical models and are interconnected over an infinite lattice. The work of [44] considers heterogeneous groups of subsystems. Within each group, the subsystems have identical models and the interconnections between the subsystems are undirected. Among different groups, subsystems can have different models, and the interconnections can be directed. Heterogeneous subsystems and arbitrary graphs are considered in [29, 81]. The work of [29] further allows for directed interconnections, and accounts for communication latency between the subsystems. A fourth possible classification criterion is the modeling of the interconnections between the subsystems. Specifically, the works of [20, 29, 57, 81, 87] use spatial states to model the interconnections between the subsystems, in addition to the states associated with the subsystems, whereas in [44], the possibly time-varying interconnection topology is modeled using a feedback operator in an LFT framework.

The interested reader is also referred to the related work on distributed control of interconnected LTI or LTV systems reported in [13, 21, 22, 35, 51, 68, 74].

The work in Chapter 4 constitutes the first work on the distributed control of interconnected NSLPV systems. First, an operator theoretic framework is developed in the context of robust control tools for working with distributed NSLPV systems. Then, by leveraging features of the developed framework, and following a parameter-independent Lyapunov function approach, analysis and synthesis results are derived for interconnected NSLPV systems. Specifically, the aim in Chapter 4 is to construct a distributed controller that renders the closed-loop system asymptotically stable, and further guarantees some ℓ_2-gain performance level, i.e., an upper bound on the ℓ_2-induced norm of the closed-loop input-output map for all permissible parameter trajectories. The sought controller inherits the interconnection structure of the plant, and the controller subsystems have NSLPV models, are formulated in an LFT framework, and are scheduled by the same parameters as their corresponding plant subsystems.

In addition to dealing with the more general class of interconnected NSLPV systems, one important contribution of Chapter 4 is the development of the operator theoretic framework for describing distributed NSLPV systems. This framework builds on previous ones developed for single NSLPV systems [34] and distributed LTV systems [35] and has the following characteristics: 1) it allows for heterogeneous (nonidentical) subsystem models, 2) it models the interconnections between the subsystems as spatial states, 3) it allows for the interconnection structure to be described by an arbitrary directed graph, 4) it accounts for a delay of one time-step on the information transfer between the subsystems, 5) it assumes that the subsystems have NSLPV models and are formulated in a LFT framework. The main advantage of this framework is that it allows for representing the state-space equations of the complex distributed system under consideration in a compact operator form so that the distributed system equations look formally identical to the equations of standard LPV-LFT state-space systems. As a result of this formal analogy, the derivations and proofs of standard analysis and synthesis results [23, 39, 66] can be adapted to NSLPV systems interconnected over arbitrary graphs, with many inherently complex manipulations becoming transparent. However, there are some intricacies that have to be addressed to make sure that these transparent manipulations go through. These intricacies include appropriately characterizing causal and memoryless operators with special structures, and imposing a desired structure on analysis solutions with no added conservatism. When extended to the distributed system setting, the standard results acquire new interpretations and characteristics. For instance, the nominal part of the system is an interconnection of time-varying subsystems, and there is an uncertainty operator locally affecting each subsystem. In the single system setting, one distinguishes between the states of the nominal part of the system and the signals introduced by the LFT formulation, which will be called the parameter states for simplicity and ease of reference. On the other hand, in the distributed system setting, one distinguishes between the standard states of the subsystems, which are referred to as the temporal states, the parameter states that are due to formulating the subsystems in an LFT framework, and the spatial states that are associated with the interconnections between the subsystems. Similarly, the designed distributed NSLPV controller inherits both the interconnection structure and the LFT structure of the plant, and so on.

1.2 Structure-Preserving Model Reduction

After discussing the distributed control problem, the focus in now shifted to the second major topic in this dissertation: the model reduction problem for interconnected systems. As noted before, various biological and engineering systems can be described as an interconnection of multiple interacting agents. However, such a description can lead to mathematical models with a very large number of states, especially if the models of the constituent agents are large and the graph describing the interconnection structure is complex. Thus arises the need for model reduction techniques that reduce the order of such interconnected systems. Namely, the analysis problem for the reduced order system obtained from model reduction is less computationally intensive than the analysis problem for the full order system, which can result in cost and time savings if the analysis is to be repeated using the reduced order system. The control synthesis problem can also benefit from model reduction since it is larger than the analysis problem. Specifically, if the synthesis conditions are computationally intractable for the full order system, model reduction becomes necessary to obtain a reduced order system for which the synthesis conditions are computationally feasible. Moreover, even if the controller is designed for the full order system, it might still be advantageous to apply model reduction techniques to obtain a reduced order controller.

As with centralized control design, it is possible to treat an interconnection of agents as one global augmented system for the purpose of model reduction. The global system is then reduced by applying standard model reduction tools such as the balanced truncation and coprime factors reduction methods [11, 26, 41, 43, 62, 63].

Some of the features of these standard methods are now summarized. To apply the balanced truncation method, a balanced realization of the system is needed, i.e., a realization in which the most controllable state variables are also the most observable ones, and the least controllable state variables are also the least observable ones. The concept of a balanced realization is expressed mathematically by requiring the controllability and observability gramians of the system to be diagonal and equal to each other. It can be thus seen that that the classical balanced truncation method applies to stable systems, since it requires the existence of unique positive definite solutions to the Lyapunov equations. The nice feature of the balanced truncation method is that it guarantees the stability of the reduced order system and an a priori bound on the norm of the error system that results from model reduction. The coprime factors reduction method extends the balanced truncation method to systems that are stabilizable and detectable. Namely, a coprime factorization of the system under consideration is computed. This factorization is then used to construct a stable augmented system that can be reduced by the application of the balanced truncation method. The coprime factors reduction method can be thought of as a model reduction technique in the closed-loop sense because the associated error bound can be interpreted in terms of robust stability of the closed-loop system.

The approach that consists of dealing with an interconnected system as one global system may not be always desirable, as it does not guarantee the preservation of the underlying structure of the system. The problem is precisely the following: when a state transformation is applied to construct a balanced realization for the system, the structure of the problem may be destroyed and the interpretation of the states lost. For example, in [72, 82], a global system with block-diagonal state-space matrices is used to describe the interconnection of multiple subsystems, and it is desirable to preserve the block-diagonal structure of the state-space matrices after applying the balancing transformation. However, this cannot be guaranteed if generic/non-structured state transformations are allowed.

As an alternative to dealing with the interconnected system as one global system, it is possible to first reduce the subsystems individually in the open-loop and then form the interconnected system using the reduced order subsystems. However, such an approach is also not recommended, since the interconnections might excite modes in the subsystems that are otherwise not observed [72].

In light of the above discussion, it can be concluded that any model reduction method to be applied to an interconnected system must 1) take into account the interconnection structure of the system and its effect on the subsystems, and 2) guarantee that the structure of the system is preserved during model reduction. Model reduction methods that satisfy these criteria are said to be structure-preserving.

For the balanced truncation method to be structure-preserving, the balancing transformation has to be block-diagonal and partitioned according to the structure of the system under consideration. In the classical balanced truncation method for single systems, the balancing transformation is constructed from the controllability and observability gramians, i.e., the unique solutions to the Lyapunov equations. In the work of [43], it is proposed to use the Lyapunov inequalities and their (non-unique) solutions, which are referred to as the generalized gramians, in the construction of the balancing transformation and the balanced realization of the system. In [17], it is further proposed to impose a block-diagonal structure on the generalized gramians in order to preserve the uncertainty structure of the system considered therein during model reduction. Namely, by forcing the generalized gramians to be block-diagonal, it is ensured that the constructed balancing transformation is also block-diagonal and appropriately structured, and so the system structure is preserved during the balancing and truncation procedures. However, it should be noted that the existence of structured solutions to the Lyapunov inequalities is in general only a sufficient, but not necessary, condition for stability. And so, the structure-preserving balanced truncation method requires more than the stability of the system to be reduced. Namely, the system to be reduced has to be stable and has to possess appropriately structured solutions to the Lyapunov inequalities. This stronger notion of stability is referred to as strong stability.

The idea of imposing a block-diagonal structure on solutions to linear matrix inequalities (LMIs) has since been exploited to derive balanced truncation and coprime factors model reduction techniques for positive systems [69] and Markov jump linear systems [30, 48, 49]. More relevant to the methods developed in this dissertation are the works on structure-preserving balanced truncation and coprime factors reduction for systems with an uncertainty structure or an interconnection structure [14, 16, 18, 33, 54, 55, 72]. For sake of completion, it is noted here that a heuristic is proposed in [72, 82] for constructing a structured balancing transformation without imposing a block-diagonal structure on the generalized gramians. However, this heuristic may fail in the sense that it may result in an unstable reduced order system.

In the context of model reduction for interconnected systems, the work of [55] deals with systems in which a given partition of the system states is to be preserved during model reduction. In [72], the interconnected system is modeled using two transfer function matrices: one transfer function matrix is block-diagonal and represents the subsystems, and the other transfer function matrix models the possibly dynamical interconnection structure. The structure-preserving balanced truncation method therein allows for truncating the states of the subsystems, but not the states associated with the interconnection structure.

Chapters 2, 3, 5, and 6 of this dissertation deal with the problem of structure-preserving model reduction for interconnected systems. The adopted frameworks model the interconnections between the subsystems as spatial states and allow for heterogeneous subsystems that are interconnected over arbitrary directed graphs. Chapters 2 and 3 assume that the subsystems have LTV models. The balanced truncation and coprime factors reduction methods proposed therein allow for individually truncating the temporal state of each subsystem and the spatial state associated with each interconnection. Even when the subsystems have LTI models, the methods of these chapters remain novel by virtue of the adopted framework. Additionally, one important novelty of the methods of Chapters 2 and 3 is that they are structure-simplifying in addition to being structure-preserving. That is, the dimension of a particular spatial state may be reduced to zero for all time-steps, and so the associated interconnection is completely removed from the interconnection structure during model reduction. Thus, the proposed methods can be used to quantify the importance of each interconnection in the interconnected system, and accordingly, the interconnections can be classified into critical and negligible. In Chapters 5 and 6, the structure-preserving/structure-simplifying model reduction methods of Chapters 2 and 3 are further extended to allow for subsystems with NSLPV models that are formulated in an LFT framework. If the signals introduced by formulating the subsystems in an LFT framework are called the parameter states for simplicity and ease of reference, then the methods of Chapters 5 and 6 allow for individually truncating each temporal, spatial, and parameter state. Moreover, if the dimension of a parameter state is reduced to zero for all time-steps, then the corresponding channel is removed from the uncertainty structure during model reduction. By virtue of the adopted framework, the methods of Chapter 5 and 6 also remain novel when the subsystems have standard LPV models, i.e., have time-invariant nominal parts.

The structure-preserving balanced truncation methods of Chapters 2 and 5 apply to strongly stable systems, i.e., stable systems with appropriately structured generalized gramians. The resulting reduced order systems are shown to be strongly stable, and a priori bounds on the norm of the error system are derived. Namely, the classical 'twice the sum of distinct truncated entries in the balanced generalized gramian' error bound is generalized to the distributed LTV and NSLPV settings in Chapters 2 and 5, respectively. Moreover, since the subsystems considered here are time-varying or have a time-varying nominal part, then error bound expressions are also derived that generalize their counterparts for single LTV and NSLPV systems in [32, 33, 73].

The coprime factors reduction method is also extended to the distributed system setting in Chapters 3 and 6. The coprime factors reduction method is of interest as it partly addresses the conservatism of the balanced truncation method by applying to systems that are strongly stabilizable and strongly detectable, but not necessarily strongly stable. To apply this method, the notion of a coprime factorization is extended to distributed LTV systems, i.e., systems with an interconnection structure, and distributed NSLPV systems, i.e., systems with an interconnection and an uncertainty structures. A strongly stabilizable and strongly detectable distributed system is then shown to admit a strongly stable coprime factorization. In addition to being necessary for the development of the coprime factors reduction method, proving the existence of coprime factorizations for interconnected systems is of independent interest, since coprime factorizations find various applications in robust control theory as explained in [36, 88]. The constructed coprime factorization is used to form an augmented strongly stable system that is reduced by the application of the balanced truncation method. This results in a reduced order coprime factorization from which the desired reduced order system is constructed. This reduced order system is strongly stabilizable and strongly detectable. Chapter 6 also discusses how to interpret the error bound from the coprime factors reduction method in terms of robust feedback stability. The discussion in Chapter 6 extends the discussions on robust feedback stability in [16, 54] to interconnected NSLPV systems controlled by distributed NSLPV controllers. Finally, Chapter 6 contains three sections pertaining to contractive coprime factorizations, which will be motivated in due course.

Before concluding, it is pointed out that all the analysis and synthesis conditions discussed in this dissertation are convex. Since the subsystems have models that are time-varying or have time-varying nominal parts, then these conditions are in general expressed in terms of infinite sequences of LMIs. Namely, there is one LMI sequence associated with each subsystem, and the LMI sequences associated with various subsystems are coupled. Eventually time-periodic subsystems are subsystems with state-space matrices that become time-periodic after some initial finite time-horizon. Eventually time-periodic subsystems include finite-horizon, time-periodic, and time-invariant subsystems as special cases. For eventually time-periodic subsystems that are interconnected over finite graphs, i.e., graphs with finite sets of vertices and edges, it can be shown that the analysis and synthesis problems can be reduced into finite dimensional semidefinite programs (SDPs) [23, 28, 31]. Namely, the aforementioned LMI sequences are feasible if and only if they admit eventually time-periodic solutions. The formulated SDPs can then be solved using efficient interior-point methods [65]. Indeed, the development of these efficient methods have made the use of LMIs in control theory possible and popular, see for example the survey paper [75]. SDP solvers based on interior-point methods include SDPT-3 [79], SeDuMi [78], MOSEK [64], and Yalmip [58].

1.3 List of Publications

The work in this dissertation is based on material published in the following conference and journal papers:

- Dany Abou Jaoude and Mazen Farhood. Balanced truncation of linear systems interconnected over arbitrary graphs with communication latency. In 2015 54th IEEE Conference on Decision and Control (CDC), pages 5346–5351, 2015.
- Dany Abou Jaoude and Mazen Farhood. Coprime factors model reduction of linear systems interconnected over arbitrary graphs with communication latency. In 2016 American Control Conference (ACC), pages 3656–3661, 2016.
- Dany Abou Jaoude and Mazen Farhood. An LFT approach for distributed control of nonstationary LPV systems. In 2016 American Control Conference (ACC), pages 3704–3709, 2016.
- Dany Abou Jaoude and Mazen Farhood. Balanced truncation of spatially distributed nonstationary LPV systems. In 2017 American Control Conference (ACC), pages 3470–3475, 2017.
- Dany Abou Jaoude and Mazen Farhood. Balanced truncation model reduction of nonstationary systems interconnected over arbitrary graphs. Automatica, 85:405–411, 2017.
- Dany Abou Jaoude and Mazen Farhood. Coprime factors model reduction of spatially distributed LTV systems over arbitrary graphs. IEEE Transactions on Automatic Control, 62(10):5254–5261, 2017.

- Dany Abou Jaoude and Mazen Farhood. Distributed control of nonstationary LPV systems over arbitrary graphs. Systems and Control Letters, 108:23–32, 2017.

- Dany Abou Jaoude and Mazen Farhood. Model reduction of distributed nonstationary LPV systems. European Journal of Control, 40:27–39, 2018.

- Dany Abou Jaoude and Mazen Farhood. Coprime factors reduction of distributed nonstationary LPV systems. International Journal of Control, 2018. doi: 10.1080/00207179.2018.1453614.

1.4 Summary of Chapters

The contents of the chapters are briefly summarized next. Chapters 2 and 3 are based on work that is published in [1, 2, 5, 6]. These chapter deal with the model reduction problem for interconnected LTV systems. Specifically, the balanced truncation and coprime factors reduction methods are articulated for this class of systems in Chapters 2 and 3, respectively.

The work in Chapter 4 is based on work that is published in [3, 7]. This chapter develops an operator theoretic framework for the representation of distributed NSLPV systems along with the corresponding analysis and synthesis results.

Chapters 5 and 6 are based on work that is published in [4, 8, 9]. These chapters treat the model reduction problem for the distributed NSLPV systems described in Chapter 4. Chapter 5 treats the balanced truncation method. Chapter 6 discusses the coprime factors reduction method along with two possible variations on the method that ensure the contractiveness of the factorizations.

The dissertation concludes with Chapter 7.

Finally, it is noted that the framework adopted in Chapters 2 and 3 for describing distributed LTV systems uses the notation introduced in [29], whereas Chapters 4, 5, and 6 employ an operator theoretic framework that extends the one in [35] to allow for the description of distributed NSLPV systems. While the results of Chapters 2 and 3 can be represented using the operator theoretic framework of [35], see for instance the work in [6], it is the author's view that the two frameworks are complementary and there is benefit in working with both frameworks and showcasing their characteristics. Namely, the operator theoretic framework allows for an elegant and compact presentation of the results and the proofs, whereas the other framework is more intuitive and does not rely heavily on operator theoretic machinery.

Chapter 2

Balanced Truncation of Interconnected Linear Time-Varying Systems

2.1 Chapter Overview

This chapter deals with the structure-preserving balanced truncation method for distributed systems that are formed of heterogeneous, linear time-varying (LTV) subsystems interconnected over finite arbitrary directed graphs. It is assumed throughout the chapter that the information transfer between the subsystems is subjected to a delay of one time-step. That is, the information sent by a given subsystem at a certain time-step reaches the target subsystem at the next time-step. The notation adopted throughout this chapter and Chapter 3 is introduced in Section 2.2. Section 2.3 introduces the framework that is adopted to represent the interconnected LTV systems of interest. This framework uses notation from [29]. In Section 2.4, the analysis results for distributed LTV systems from [35] are summarized and are expressed in the framework of this chapter. The balanced truncation method is detailed in Section 2.5, and two expressions for the balanced truncation error bound are derived in Section 2.6. The first expression generalizes the standard balanced truncation error bound to the distributed LTV setting. The second expression holds when the truncation sequences are monotonic in time. Since the subsystems can have general LTV models, the analysis problems are expressed in terms of infinite sequences of linear matrix inequalities (LMIs). Section 2.7 deals with the specialized class of eventually time-periodic subsystems, i.e., subsystems with state-space matrices that become time-periodic after a certain initial finite time-horizon. It is shown that if the subsystems are eventually time-periodic, then the LMI sequences in the analysis problems admit eventually time-periodic solutions. That is, the analysis problems can be reduced to finite dimensional semidefinite programs (SDPs) in the case of eventually time-periodic subsystems. Section 2.8 applies the proposed balanced truncation method to an illustrative example. This example showcases novel features of the proposed method such as allowing for the individual, non-uniform in time, reduction of each temporal and each spatial state. This example also illustrates the use of a novel heuristic for improving on the error bound computation.

2.2 Notation

The sets of nonnegative integers, real numbers, and $n \times n$ symmetric matrices are denoted by $\mathbb{N}_0$, $\mathbb{R}$, and $\mathbb{S}^n$, respectively. Let S be an ordered subset of $\mathbb{N}_0$. $(v_i)_{i \in S}$ denotes the vector-valued sequence associated with S and $\operatorname{vec}(v_i)_{i \in S}$ denotes the vertical concatenation of the elements of $(v_i)_{i \in S}$. The elements in $(v_i)_{i \in S}$ are ordered conformably with the elements in S. As an example, let $S = \{1, 2, 4\}$. Then, $(v_i)_{i \in S} = (v_1, v_2, v_4)$ and $\operatorname{vec}(v_i)_{i \in S} = \begin{bmatrix} v_1^T & v_2^T & v_4^T \end{bmatrix}^T$, where T denotes the transposition operation. Similarly, $(M_i)_{i \in S}$ denotes the matrix-valued sequence associated with S and $\operatorname{diag}(M_i)_{i \in S}$ denotes the block-diagonal augmentation of the elements of $(M_i)_{i \in S}$. In the considered example, $(M_i)_{i \in S} = (M_1, M_2, M_4)$ and $\operatorname{diag}(M_i)_{i \in S} =$

$\operatorname{diag}(M_1, M_2, M_4) = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_4 \end{bmatrix}$. $0_{i\times j}$ denotes the $i \times j$ zero matrix and I_i denotes the $i \times i$ identity matrix. The simplified denotations 0 and I will be frequently used when the dimensions are clear from context. $X \prec 0$ (respectively $\succ 0$) means that the symmetric matrix $X = X^T$ is negative definite (respectively positive definite).

$\mathcal{G}(V, E)$ refers to a directed graph with set of vertices V and set of directed edges E. It is assumed throughout this chapter and Chapter 3 that the directed graph under consideration is finite. That is, both V and E are finite sets. Let N denote the finite number of vertices. Then, $V = \{1, \ldots, N\}$ for simplicity. The ordered pair (i, j) is in E if there exists a directed edge from $i \in V$ to $j \in V$. For each $k \in V$, define the set of vertices with an outgoing edge to k as $E_{\text{in}}^{(k)} = \{i \in V \mid (i, k) \in E\}$, and denote its cardinality by $m(k)$. Similarly, define the set of vertices with an incoming edge from k as $E_{\text{out}}^{(k)} = \{j \in V \mid (k, j) \in E\}$, and denote its cardinality by $p(k)$. The elements in these sets are ordered in an increasing fashion. For example, consider the directed graph in Figure 2.1. For $k = 1$, $E_{\text{in}}^{(1)} = \{2, 3, 4\}$, $m(1) = 3$, $E_{\text{out}}^{(1)} = \{2, 3, 5\}$, $p(1) = 3$, and so on.

$\ell(\{\mathbb{R}^{\eta(t)}\})$ is defined as the vector space of mappings $w : t \in \mathbb{N}_0 \to w(t) \in \mathbb{R}^{\eta(t)}$. The Hilbert space $\ell_2(\{\mathbb{R}^{\eta(t)}\})$ is the subspace of $\ell(\{\mathbb{R}^{\eta(t)}\})$ that consists of mappings w with a finite ℓ_2-norm $\|w\|_{\ell_2} = \left(\sum_{t\in\mathbb{N}_0} (w(t))^T w(t)\right)^{\frac{1}{2}}$. The abbreviated symbols ℓ and ℓ_2 will be frequently used when the dimensions $\eta(t)$ are clear from context. The Hilbert space direct sum $\ell_2(\{\mathbb{R}^{\eta_1(t)}\}) \oplus \cdots \oplus \ell_2(\{\mathbb{R}^{\eta_d(t)}\})$ consists of elements $(w_1, \ldots, w_d)$, where each $w_i \in \ell_2(\{\mathbb{R}^{\eta_i(t)}\})$ for $i = 1, \ldots, d$. This Hilbert space will be simply denoted by ℓ_2 regardless of its structure and the associated dimensions.

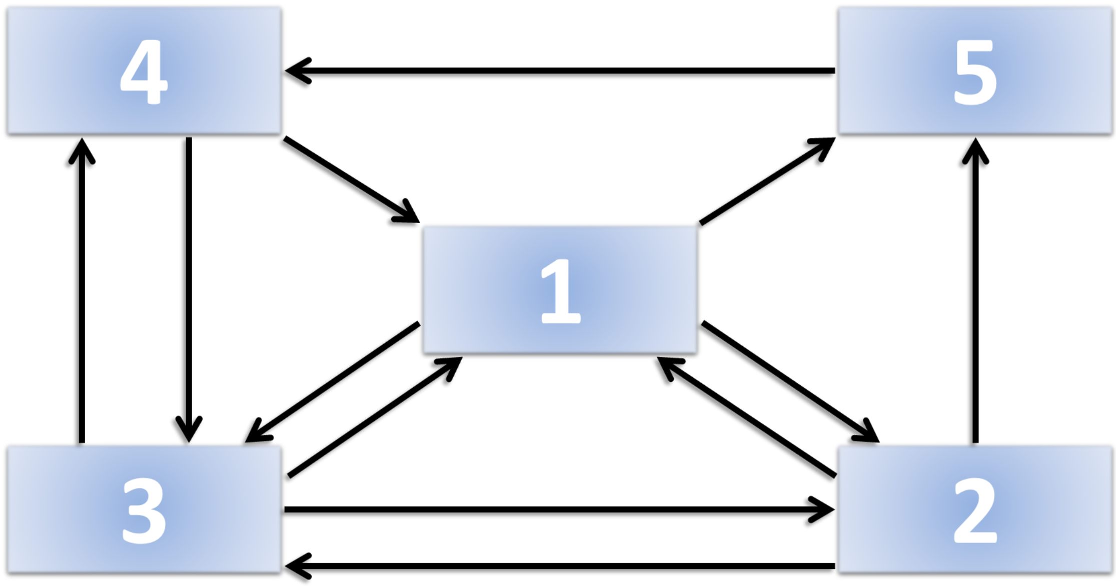

Figure 2.1: This figure shows an example of an arbitrary directed graph that can be used to represent an interconnected system. In the figure, each node in $V = \{1, 2, 3, 4, 5\}$ corresponds to a subsystem, and the interconnections between the subsystems are represented using the directed edges.

2.3 State-Space Representation

This section gives the state-space equations for a distributed system G, formed by discrete-time, heterogeneous, LTV subsystems interconnected over finite arbitrary directed graphs and subjected to a communication latency. The interconnection structure of G is represented using a directed graph $\mathcal{G}(V, E)$, where each subsystem $G^{(k)}$ corresponds to a vertex $k \in V$, and the interconnections between the subsystems are described by the directed edges in E. The adopted framework uses notation similar to the one introduced in [29]. Let t denote discrete time. Each subsystem $G^{(k)}$ is assumed to be described using a discrete-time LTV model, with state vector $x^{(k)}(t)$, input vector $u^{(k)}(t)$, and output vector $y^{(k)}(t)$. The state vectors associated with the subsystems are referred to as temporal states. The interconnections between the subsystems are also modeled using states, which are referred to as spatial states. Namely, a state vector $x^{(ij)}(t)$ is associated with each edge $(i,j) \in E$. Due to the communication latency, the information sent from subsystem $G^{(i)}$ at time-step t reaches the target subsystem $G^{(j)}$ at the next time-step $t+1$. For each subsystem $G^{(k)}$, define the vectors

$$x_{\text{in}}^{(k)}(t) = \text{vec}(x^{(ik)}(t))_{i \in E_{\text{in}}^{(k)}}, \qquad\qquad x_{\text{out}}^{(k)}(t) = \text{vec}(x^{(kj)}(t))_{j \in E_{\text{out}}^{(k)}},$$

which are partitioned into $m(k)$ and $p(k)$ vector-valued channels, respectively. These vectors represent the total information received and sent by subsystem $G^{(k)}$ at time-step t. When all subsystems are considered, and since the interconnection input to a subsystem is an output to another subsystem, both $x_{\text{out}}^{(k)}$ and $x_{\text{in}}^{(k)}$ contain all spatial states $x^{(ij)}$. Zero initial conditions are assumed for the temporal and the spatial states, i.e., $x^{(k)}(0) = 0$ and $x^{(ij)}(0) = 0$ for all $k \in V$ and $(i,j) \in E$. Then, for all $(t,k) \in \mathbb{N}_0 \times V$, the state-space equations are given by

$$\begin{aligned}
\begin{bmatrix} x^{(k)}(t+1) \\ x_{\text{out}}^{(k)}(t+1) \end{bmatrix} &= A^{(k)}(t) \begin{bmatrix} x^{(k)}(t) \\ x_{\text{in}}^{(k)}(t) \end{bmatrix} + B^{(k)}(t)\, u^{(k)}(t), & x^{(k)}(0) &= 0, \\
y^{(k)}(t) &= C^{(k)}(t) \begin{bmatrix} x^{(k)}(t) \\ x_{\text{in}}^{(k)}(t) \end{bmatrix} + D^{(k)}(t)\, u^{(k)}(t), & x_{\text{in}}^{(k)}(0) &= 0.
\end{aligned} \tag{2.1}$$

The matrix-valued sequences of state-space matrices, i.e., $A^{(k)}(t)$, $B^{(k)}(t)$, $C^{(k)}(t)$, and $D^{(k)}(t)$ are known a priori and are assumed to be uniformly bounded. The dimensions of $x^{(k)}(t)$, $u^{(k)}(t)$, $y^{(k)}(t)$, and $x^{(ij)}(t)$ can vary with t, and k or (i,j), and are denoted by $n^{(k)}(t)$, $n_u^{(k)}(t)$, $n_y^{(k)}(t)$, and $n^{(ij)}(t)$, respectively, for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$. The realization of system G is denoted by the quadruple $\left(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t)\right)$.

For each $(t,k) \in \mathbb{N}_0 \times V$, the state-space matrices are naturally partitioned conformably with the partitioning of $\begin{bmatrix} \left(x^{(k)}(t+1)\right)^T & \left(x_{\text{out}}^{(k)}(t+1)\right)^T \end{bmatrix}^T$ and $\begin{bmatrix} \left(x^{(k)}(t)\right)^T & \left(x_{\text{in}}^{(k)}(t)\right)^T \end{bmatrix}^T$. For example, consider the distributed system G that has a realization $\big(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t)\big)$ and whose interconnection structure is represented in Figure 2.1. Then, the state-space matrices of subsystem $G^{(1)}$ are partitioned as follows:

$$A^{(1)}(t) = \begin{bmatrix} A_{00}^{(1)}(t) & A_{02}^{(1)}(t) & A_{03}^{(1)}(t) & A_{04}^{(1)}(t) \\ A_{20}^{(1)}(t) & A_{22}^{(1)}(t) & A_{23}^{(1)}(t) & A_{24}^{(1)}(t) \\ A_{30}^{(1)}(t) & A_{32}^{(1)}(t) & A_{33}^{(1)}(t) & A_{34}^{(1)}(t) \\ A_{50}^{(1)}(t) & A_{52}^{(1)}(t) & A_{53}^{(1)}(t) & A_{54}^{(1)}(t) \end{bmatrix}, \qquad B^{(1)}(t) = \begin{bmatrix} B_0^{(1)}(t) \\ B_2^{(1)}(t) \\ B_3^{(1)}(t) \\ B_5^{(1)}(t) \end{bmatrix},$$
$$C^{(1)}(t) = \begin{bmatrix} C_0^{(1)}(t) & C_2^{(1)}(t) & C_3^{(1)}(t) & C_4^{(1)}(t) \end{bmatrix}, \tag{2.2}$$

where $A_{00}^{(1)}(t)$ is an $n^{(1)}(t+1) \times n^{(1)}(t)$ matrix, $A_{02}^{(1)}(t)$ is an $n^{(1)}(t+1) \times n^{(21)}(t)$ matrix, $A_{20}^{(1)}(t)$ is an $n^{(12)}(t+1) \times n^{(1)}(t)$ matrix, $B_0^{(1)}(t)$ is an $n^{(1)}(t+1) \times n_u^{(1)}(t)$ matrix, $B_2^{(1)}(t)$ is an $n^{(12)}(t+1) \times n_u^{(1)}(t)$ matrix, $C_0^{(1)}(t)$ is an $n_y^{(1)}(t) \times n^{(1)}(t)$ matrix, $C_2^{(1)}(t)$ is an $n_y^{(1)}(t) \times n^{(21)}(t)$ matrix, etc.

2.4 Analysis Results

This section summarizes the analysis results of [35] that will be used in deriving and proving the results in this chapter and Chapter 3. The analysis results in [35] are originally written using the notation and framework adopted therein. Here, these results are rewritten using the notation and framework of Section 2.3, see also [29].

Consider a distributed LTV system G that is described using the equations in (2.1) for all $(t, k) \in \mathbb{N}_0 \times V$ and has a realization denoted by $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$. System G can be viewed as a map from $u = \big(u^{(1)}, \ldots, u^{(N)}\big)$ to $y = \big(y^{(1)}, \ldots, y^{(N)}\big)$. Since the state-space equations have zero initial conditions and the state-space matrices are defined for $t \in \mathbb{N}_0$, it can equivalently assumed that the state-space matrices are zeros for $t < 0$. Then, it can be shown that system G under consideration is well-posed [22]; see Definition 2.1 for the formal definition of well-posedness.

Definition 2.1. Consider a distributed LTV system G that has a realization denoted by $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$. System G is said to be well-posed if given inputs in $u^{(k)} \in \ell$ for all $k \in V$, the state-space equations (2.1) admit unique solutions $x^{(k)}$, $x^{(ij)}$, $y^{(k)}$ in ℓ for all $k \in V$ and $(i,j) \in E$ and further define a linear causal mapping on ℓ.

Definition 2.2. Consider a distributed LTV system G that has a realization denoted by $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$. System G is said to be stable if it is well-posed and if, given inputs $u^{(k)} \in \ell_2$ for all $k \in V$, the unique solutions $x^{(k)}$, $x^{(ij)}$, $y^{(k)}$ to the state-space equations (2.1) are in ℓ_2 for all $k \in V$ and $(i,j) \in E$. That is, system G is said to be stable if the system equations define a linear causal mapping on ℓ_2.

The following result from [35] gives a Lyapunov-based test to check if the distributed LTV system G under consideration is stable. The result constitutes the basis of the balanced truncation method developed in this chapter.

Lemma 2.3. *Consider a distributed LTV system G that has a realization denoted by $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$. System G is stable if there exist $\mu > 0$ and uniformly bounded and positive definite matrix-valued functions $X^{(k)}(t) \in \mathbb{S}^{n^{(k)}(t)}$ and $X^{(ij)}(t) \in \mathbb{S}^{n^{(ij)}(t)}$ such that, for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$, $X^{(k)}(t) \succ \mu I$, $X^{(ij)}(t) \succ \mu I$, and*

$$(A^{(k)}(t))^T \begin{bmatrix} X^{(k)}(t+1) & 0 \\ 0 & X^{(k)}_{\text{out}}(t+1) \end{bmatrix} A^{(k)}(t) - \begin{bmatrix} X^{(k)}(t) & 0 \\ 0 & X^{(k)}_{\text{in}}(t) \end{bmatrix} \prec -\mu I, \tag{2.3}$$

where $X^{(k)}_{\text{in}}(t) = \operatorname{diag}\left(X^{(ik)}(t)\right)_{i \in E^{(k)}_{\text{in}}}$ and $X^{(k)}_{\text{out}}(t) = \operatorname{diag}\left(X^{(kj)}(t)\right)_{j \in E^{(k)}_{\text{out}}}$.

The solutions to (2.3) can be classified into temporal terms $X^{(k)}(t)$ for all $k \in V$ and spatial terms $X^{(ij)}(t)$ for all $(i, j) \in E$. Since the subsystems have LTV models, there is an infinite sequence of LMIs associated with each subsystem. Moreover, the LMI sequence associated with a given subsystem is coupled with the LMI sequences of the other subsystems through the spatial terms. The μI terms in (2.3) are small quantities added to ensure that the matrix-valued sequences on the left-hand-side do not converge to singular matrices as t approaches infinity. Even though they are implied by (2.3), the (redundant) conditions $X^{(k)}(t) \succ \mu I$ and $X^{(ij)}(t) \succ \mu I$ are explicitly given in the statement of the lemma to stress that the terms $X^{(k)}(t)$ and $X^{(ij)}(t)$ do not approach singular matrices as t approaches infinity. Subsequently, the dimensions of the matrices $X^{(k)}(t)$ and $X^{(ij)}(t)$ and of similar matrices will not be specified.

Distributed LTV systems that satisfy the conditions in Lemma 2.3 are called strongly stable. Since Lemma 2.3 provides a sufficient condition for stability of distributed LTV systems, then strong stability implies stability, but the converse may not be always true. Specifically, strongly stable systems are stable systems that have the required structured solutions to (2.3). The proposed balanced truncation method exploits this imposed structure on the solutions of (2.3) to preserve the meaning of the temporal and spatial states during model reduction, which makes the proposed method structure-preserving. However, since the method requires the existence of structured solutions to (2.3), then it is only applicable to strongly stable systems.

Remark 2.4. In [22], it is pointed out that it might be a very difficult task to sharply and quantitatively assess the conservatism introduced by imposing a block-diagonal structure on the solutions to LMIs. In [77], sufficient conditions are given for the existence of structured solutions to the LMI therein, and in [80], a set of systems is identified with guaranteed structured solutions to the Lyapunov inequalities.

Recall that a distributed LTV system G can be viewed as a map from $u = \left(u^{(1)}, \ldots, u^{(N)}\right)$ to $y = \left(y^{(1)}, \ldots, y^{(N)}\right)$. For a stable system G mapping $u \in \ell_2$ to $y \in \ell_2$ and starting from zero initial conditions, the ℓ_2-induced norm is defined as

$$\|G\| = \sup_{0 \neq u \in \ell_2} \frac{\|y\|_{\ell_2}}{\|u\|_{\ell_2}},$$

where

$$\|u\|_{\ell_2} = \left(\sum_{(t,k) \in \mathbb{N}_0 \times V} \left(u^{(k)}(t)\right)^T u^{(k)}(t) \right)^{\frac{1}{2}},$$

$$\|y\|_{\ell_2} = \left(\sum_{(t,k) \in \mathbb{N}_0 \times V} \left(y^{(k)}(t)\right)^T y^{(k)}(t) \right)^{\frac{1}{2}}.$$

Lemma 2.5. *Consider a distributed LTV system G that has a realization denoted by $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$. System G is strongly stable and satisfies $\|G\| < \gamma$, for some $\gamma > 0$, if there exist $\mu > 0$ and uniformly bounded and positive definite matrix-valued functions $X^{(k)}(t) \succ \mu I$ and $X^{(ij)}(t) \succ \mu I$ such that*

$$\begin{bmatrix} A^{(k)}(t) & B^{(k)}(t) \\ C^{(k)}(t) & D^{(k)}(t) \end{bmatrix}^T \begin{bmatrix} X^{(k)}(t+1) & 0 & 0 \\ 0 & X^{(k)}_{\text{out}}(t+1) & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A^{(k)}(t) & B^{(k)}(t) \\ C^{(k)}(t) & D^{(k)}(t) \end{bmatrix} - \begin{bmatrix} X^{(k)}(t) & 0 & 0 \\ 0 & X^{(k)}_{\text{in}}(t) & 0 \\ 0 & 0 & \gamma^2 I \end{bmatrix} \prec -\mu I, \quad (2.4)$$

for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$, where $X^{(k)}_{\text{in}}(t) = \operatorname{diag}\left(X^{(ik)}(t)\right)_{i \in E^{(k)}_{\text{in}}}$ and $X^{(k)}_{\text{out}}(t) = \operatorname{diag}\left(X^{(kj)}(t)\right)_{j \in E^{(k)}_{\text{out}}}$.

2.5 Balanced Truncation Model Reduction

2.5.1 Balanced Realization

Consider a distributed LTV system G that is described using the equations in (2.1) for all $(t,k) \in \mathbb{N}_0 \times V$ and has a realization denoted by $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$. Furthermore, assume that system G is strongly stable as defined in Section 2.4. The present section begins by defining the notion of a balanced realization for system G. First, the generalized Lyapunov inequalities and generalized gramians [17, 43] are defined for the distributed LTV systems treated in this chapter. Namely, the generalized controllability gramians, denoted by $X^{(k)}(t)$ and $X^{(ij)}(t)$, and the generalized observability gramians, denoted by $Y^{(k)}(t)$ and $Y^{(ij)}(t)$, are uniformly bounded and positive definite matrix-valued functions such that $X^{(k)}(t) \succ \mu I$, $X^{(ij)}(t) \succ \mu I$, $Y^{(k)}(t) \succ \mu I$, $Y^{(ij)}(t) \succ \mu I$, and

$$
\begin{aligned}
A^{(k)}(t)\mathrm{diag}\left(X^{(k)}(t), X^{(k)}_{\mathrm{in}}(t)\right)(A^{(k)}(t))^T - \mathrm{diag}\left(X^{(k)}(t+1), X^{(k)}_{\mathrm{out}}(t+1)\right)& \\
+ B^{(k)}(t)(B^{(k)}(t))^T \prec -\mu I,& \quad (2.5)
\end{aligned}
$$

$$
\begin{aligned}
(A^{(k)}(t))^T\mathrm{diag}\left(Y^{(k)}(t+1), Y^{(k)}_{\mathrm{out}}(t+1)\right)A^{(k)}(t) - \mathrm{diag}\left(Y^{(k)}(t), Y^{(k)}_{\mathrm{in}}(t)\right)& \\
+ (C^{(k)}(t))^T C^{(k)}(t) \prec -\mu I,& \quad (2.6)
\end{aligned}
$$

for some scalar $\mu > 0$ and all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$, where

$$
\begin{aligned}
X^{(k)}_{\mathrm{in}}(t) &= \mathrm{diag}\left(X^{(ik)}(t)\right)_{i \in E^{(k)}_{\mathrm{in}}}, & X^{(k)}_{\mathrm{out}}(t) &= \mathrm{diag}\left(X^{(kj)}(t)\right)_{j \in E^{(k)}_{\mathrm{out}}}, \\
Y^{(k)}_{\mathrm{in}}(t) &= \mathrm{diag}\left(Y^{(ik)}(t)\right)_{i \in E^{(k)}_{\mathrm{in}}}, & Y^{(k)}_{\mathrm{out}}(t) &= \mathrm{diag}\left(Y^{(kj)}(t)\right)_{j \in E^{(k)}_{\mathrm{out}}}.
\end{aligned}
$$

The sequences of inequalities defined in (2.5) and (2.6) are called the generalized Lyapunov inequalities.

It can shown, e.g., see the proof of Theorem 2.15, that the existence of solutions to (2.3) is equivalent to the existence of solutions to (2.5) and (2.6), respectively. Thus, the generalized gramians are only defined for strongly stable systems, which justifies the assumption made in this section that system G is strongly stable. The generalized Lyapunov inequalities and the generalized gramians allow for the definition of a balanced realization for system G as given next.

Definition 2.6. Consider a strongly stable, distributed LTV system G that has a realization denoted by $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$. The realization of system G is said to be balanced if there exist $\mu > 0$ and uniformly bounded, *diagonal*, and positive definite matrix-valued functions $\Sigma^{(k)}(t) \succ \mu I$ and $\Sigma^{(ij)}(t) \succ \mu I$ that simultaneously satisfy (2.5) and (2.6) for all $(t, k) \in \mathbb{N}_0 \times V$ and $(i, j) \in E$, i.e., $\Sigma^{(k)}(t) = X^{(k)}(t) = Y^{(k)}(t)$, $\Sigma^{(ij)}(t) = X^{(ij)}(t) = Y^{(ij)}(t)$, and $\Sigma^{(k)}(t)$ and $\Sigma^{(ij)}(t)$ are diagonal matrices. If they exist, $\Sigma^{(k)}(t)$ and $\Sigma^{(ij)}(t)$ are called the balanced generalized gramians.

Consider a strongly stable system G, and let $X^{(k)}(t)$ and $X^{(ij)}(t)$ be solutions to (2.5) and $Y^{(k)}(t)$ and $Y^{(ij)}(t)$ be solutions to (2.6), respectively. The following algorithm shows how to use these generalized gramians to construct a balanced realization and balanced generalized gramians $\Sigma^{(k)}(t)$ and $\Sigma^{(ij)}(t)$ for system G.

Algorithm 2.7. Consider a strongly stable system G that has a realization denoted by $(A^{(k)}(t),\ B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$. Given generalized gramians $X^{(k)}(t)$, $X^{(ij)}(t)$ and $Y^{(k)}(t)$, $Y^{(ij)}(t)$, a balanced realization $(A^{(k)}_{\text{bal}}(t), B^{(k)}_{\text{bal}}(t), C^{(k)}_{\text{bal}}(t), D^{(k)}(t))$ and balanced generalized gramians $\Sigma^{(k)}(t)$ and $\Sigma^{(ij)}(t)$ can be constructed for system G as follows.

1. For all $(t, k) \in \mathbb{N}_0 \times V$, compute the Cholesky factorizations

$X^{(k)}(t) = \left(R^{(k)}(t)\right)^T R^{(k)}(t)$ and $Y^{(k)}(t) = \left(H^{(k)}(t)\right)^T H^{(k)}(t)$.

2. For all $(t,k) \in \mathbb{N}_0 \times V$, perform the singular value decomposition

   $H^{(k)}(t)(R^{(k)}(t))^T = U^{(k)}(t)\Sigma^{(k)}(t)\left(V^{(k)}(t)\right)^T$.

3. For all $(t,k) \in \mathbb{N}_0 \times V$, define the temporal blocks of the balancing transformations

$$\begin{aligned} T^{(k)}(t) &= \left(\Sigma^{(k)}(t)\right)^{-\frac{1}{2}} \left(U^{(k)}(t)\right)^T H^{(k)}(t), \\ \left(T^{(k)}(t)\right)^{-1} &= \left(R^{(k)}(t)\right)^T V^{(k)}(t) \left(\Sigma^{(k)}(t)\right)^{-\frac{1}{2}}. \end{aligned}$$

4. For all $t \in \mathbb{N}_0$ and $(i,j) \in E$, repeat similar steps to steps 1-3 for the spatial terms.

5. For all $(t,k) \in \mathbb{N}_0 \times V$, form the balancing transformations

   $T^{(k)}_{\text{pre}}(t) = \operatorname{diag}\left(T^{(k)}(t), T^{(k)}_{\text{out}}(t)\right)$ and $T^{(k)}_{\text{post}}(t) = \operatorname{diag}\left(\left(T^{(k)}(t)\right)^{-1}, \left(T^{(k)}_{\text{in}}(t)\right)^{-1}\right)$,

   where $T^{(k)}_{\text{in}}(t) = \operatorname{diag}\left(T^{(ik)}(t)\right)_{i \in E^{(k)}_{\text{in}}}$ and $T^{(k)}_{\text{out}}(t) = \operatorname{diag}\left(T^{(kj)}(t)\right)_{j \in E^{(k)}_{\text{out}}}$.

6. Define $A^{(k)}_{\text{bal}}(t) = T^{(k)}_{\text{pre}}(t+1)A^{(k)}(t)T^{(k)}_{\text{post}}(t)$, $B^{(k)}_{\text{bal}}(t) = T^{(k)}_{\text{pre}}(t+1)B^{(k)}(t)$,

   and $C^{(k)}_{\text{bal}}(t) = C^{(k)}(t)T^{(k)}_{\text{post}}(t)$.

An alternative algorithm can be also derived by replacing steps 1-3 in Algorithm 2.7 by the following steps.

1. For all $(t,k) \in \mathbb{N}_0 \times V$, perform the singular value decomposition

   $\left(X^{(k)}(t)\right)^{1/2} Y^{(k)}(t) \left(X^{(k)}(t)\right)^{1/2} = U^{(k)}(t) \left(\Sigma^{(k)}(t)\right)^2 \left(U^{(k)}(t)\right)^T$.

2. For all $(t,k) \in \mathbb{N}_0 \times V$, define the temporal blocks of the balancing transformations

   $T^{(k)}(t) = \left(\Sigma^{(k)}(t)\right)^{1/2} \left(U^{(k)}(t)\right)^T \left(X^{(k)}(t)\right)^{-1/2}$.

From the preceding discussion, it can be thus seen that the balanced realization of a strongly stable system is not unique since it depends on the algorithm used for the construction of the balanced realization as well as the (non-unique) solutions to (2.5) and (2.6) used in the algorithm. To obtain useful results for model reduction, namely, entries in the balanced generalized gramians that yield reasonable error bounds, a trace minimization heuristic is used [18, 72]. Specifically, generalized gramians with minimum sum of traces are usually sought, e.g., the generalized controllability gramians $X^{(k)}(t)$ and $X^{(ij)}(t)$ that solve (2.5) and minimize the following objective function are computed:

$$\min \sum_{t\in\mathbb{N}_0} \left(\sum_{k\in V} \operatorname{trace} X^{(k)}(t) + \sum_{(i,j)\in E} \operatorname{trace} X^{(ij)}(t) \right). \tag{2.7}$$

In Section 2.8 and Algorithm 5.8, an improved heuristic that builds on the trace minimization heuristic is proposed and discussed in more details.

2.5.2 Balanced Truncation

Consider a strongly stable distributed LTV system G that is described using the equations in (2.1) for all $(t,k) \in \mathbb{N}_0 \times V$ and has a balanced realization denoted by $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$ as per Definition 2.6. Without loss of generality, it can be assumed that the diagonal entries of the balanced generalized gramians are ordered in a decreasing fashion, i.e., for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$, the diagonal entries of $\Sigma^{(k)}(t)$ and $\Sigma^{(ij)}(t)$ are ordered from largest to smallest with the largest entry in the $(1,1)$ position. The essence of the balanced truncation method is to truncate the state variables associated with the negligible entries in the balanced generalized gramians. Namely, since the realization of the system is balanced, and the generalized controllability gramians are equal to the generalized observability gramians, then the state variables with negligible entries in the balanced generalized gramians are simultaneously the least observable and the least controllable state variables, and so they can be truncated without introducing a large error. Moreover, since the balanced generalized gramians can be classified into temporal terms $\Sigma^{(k)}(t)$ and spatial terms $\Sigma^{(ij)}(t)$, then both types of state vectors can be truncated using the proposed method while ensuring that the interpretation of any given state vector is retained in the resulting reduced order system. Specifically, the gramians are partitioned into two blocks each: one corresponding to the non-truncated state variables and the other corresponding to the truncated state variables. The partitioning process is illustrated for the temporal terms. Given integers $r^{(k)}(t)$ such that $0 \leq r^{(k)}(t) \leq n^{(k)}(t)$, each $\Sigma^{(k)}(t)$ is partitioned as $\Sigma^{(k)}(t) = \operatorname{diag}\left(\Gamma^{(k)}(t), \Omega^{(k)}(t)\right)$, where $\Gamma^{(k)}(t) \in \mathbb{S}^{r^{(k)}(t)}$ is a reduced order gramian associated with the non-truncated variables in the temporal state vector $x^{(k)}(t)$ at time-step t, and $\Omega^{(k)}(t)$ corresponds to the truncated variables in the same temporal state vector at the same time-step. By allowing $r^{(k)}(t)$ to be equal to 0 or $n^{(k)}(t)$ for some (t, k), the method allows that either all or no variables be truncated from the corresponding temporal state vector $x^{(k)}(t)$. This results in either $\Gamma^{(k)}(t)$ or $\Omega^{(k)}(t)$ having a zero dimension, which is a slight abuse of notation. Given integers $r^{(ij)}(t)$ such that $0 \leq r^{(ij)}(t) \leq n^{(ij)}(t)$, the spatial terms $\Sigma^{(ij)}(t)$ are partitioned in a similar way.

At this point, the various novel features of the proposed balanced truncation method can be highlighted: 1) for all $k \in V$ and $(i, j) \in E$, the temporal state $x^{(k)}$ of each subsystem $G^{(k)}$ and the spatial state $x^{(ij)}$ associated with each interconnection (i, j) can be truncated individually in a unique way; 2) the proposed method allows for time-varying reduced order dimensions $r^{(k)}(t)$ and $r^{(ij)}(t)$ even if the corresponding full order dimensions $n^{(k)}(t)$ and $n^{(ij)}(t)$ are constants with respect to time; 3) the proposed method is structure-preserving, since the interpretation of the state vectors and gramians is retained during model reduction; and 4) if $r^{(i_0 j_0)}(t) = 0$ for all $t \in \mathbb{N}_0$, then the corresponding interconnection (i_0, j_0) is removed altogether from the interconnection structure during model reduction, and so the proposed method can also be said to be structure-simplifying.

The next step is to partition the blocks of the state-space matrices in accordance with the partitioning of the blocks of diag $\left(\Sigma^{(k)}(t+1), \Sigma_{\text{out}}^{(k)}(t+1)\right)$ and diag $\left(\Sigma^{(k)}(t), \Sigma_{\text{in}}^{(k)}(t)\right)$, where $\Sigma_{\text{in}}^{(k)}(t) = \text{diag}\left(\Sigma^{(ik)}(t)\right)_{i \in E_{\text{in}}^{(k)}}$ and $\Sigma_{\text{out}}^{(k)}(t) = \text{diag}\left(\Sigma^{(kj)}(t)\right)_{j \in E_{\text{out}}^{(k)}}$. To illustrate the partitioning procedure, assume that the interconnection structure of system G is described by the graph shown in Figure 2.1. Recall the partitioning of the state-space matrices of subsystem $G^{(1)}$ shown in (2.2). $A_{00}^{(1)}(t)$ is partitioned according to the partitioning of $\Sigma^{(1)}(t+1) = \text{diag}\left(\Gamma^{(1)}(t+1),\, \Omega^{(1)}(t+1)\right)$ and $\Sigma^{(1)}(t) = \text{diag}\left(\Gamma^{(1)}(t),\, \Omega^{(1)}(t)\right)$ as in

$$A_{00}^{(1)}(t) = \begin{bmatrix} \hat{A}_{00}^{(1)}(t) & A_{00_{12}}^{(1)}(t) \\ A_{00_{21}}^{(1)}(t) & A_{00_{22}}^{(1)}(t) \end{bmatrix},$$

where $\hat{A}_{00}^{(1)}(t)$ is an $r^{(1)}(t+1) \times r^{(1)}(t)$ matrix. $B_0^{(1)}(t)$ is partitioned conformably with the partitioning of $\Sigma^{(1)}(t+1) = \text{diag}\left(\Gamma^{(1)}(t+1),\, \Omega^{(1)}(t+1)\right)$ as in

$$B_0^{(1)}(t) = \begin{bmatrix} \hat{B}_0^{(1)}(t) \\ B_{0_2}^{(1)}(t) \end{bmatrix},$$

where $\hat{B}_0^{(1)}(t)$ is an $r^{(1)}(t+1) \times n_u^{(1)}(t)$ matrix. Likewise, $C_0^{(1)}(t)$ is partitioned according to the partitioning of $\Sigma^{(1)}(t) = \text{diag}\left(\Gamma^{(1)}(t),\, \Omega^{(1)}(t)\right)$, namely,

$$C_0^{(1)}(t) = \begin{bmatrix} \hat{C}_0^{(1)}(t) & C_{0_2}^{(1)}(t) \end{bmatrix},$$

where $\hat{C}_0^{(1)}(t)$ is an $n_y^{(1)}(t) \times r^{(1)}(t)$ matrix, and so on.

The realization $(A_r^{(k)}(t), B_r^{(k)}(t), C_r^{(k)}(t), D^{(k)}(t))$ of the reduced order system G_r can now be formed: for all $(t,k) \in \mathbb{N}_0 \times V$, $A_r^{(k)}(t)$, $B_r^{(k)}(t)$, and $C_r^{(k)}(t)$ are respectively constructed from the partitions of the blocks of $A^{(k)}(t)$, $B^{(k)}(t)$, and $C^{(k)}(t)$ that are marked with a hat, i.e., the block partitions that correspond to the non-truncated state variables. For instance, the state-space matrices of subsystem $G_r^{(1)}$ in the reduced order system of the previously considered example are given by

$$A_r^{(1)}(t) = \begin{bmatrix} \hat{A}_{00}^{(1)}(t) & \hat{A}_{02}^{(1)}(t) & \hat{A}_{03}^{(1)}(t) & \hat{A}_{04}^{(1)}(t) \\ \hat{A}_{20}^{(1)}(t) & \hat{A}_{22}^{(1)}(t) & \hat{A}_{23}^{(1)}(t) & \hat{A}_{24}^{(1)}(t) \\ \hat{A}_{30}^{(1)}(t) & \hat{A}_{32}^{(1)}(t) & \hat{A}_{33}^{(1)}(t) & \hat{A}_{34}^{(1)}(t) \\ \hat{A}_{50}^{(1)}(t) & \hat{A}_{52}^{(1)}(t) & \hat{A}_{53}^{(1)}(t) & \hat{A}_{54}^{(1)}(t) \end{bmatrix}, \qquad B_r^{(1)}(t) = \begin{bmatrix} \hat{B}_0^{(1)}(t) \\ \hat{B}_2^{(1)}(t) \\ \hat{B}_3^{(1)}(t) \\ \hat{B}_5^{(1)}(t) \end{bmatrix},$$
$$C_r^{(1)}(t) = \begin{bmatrix} \hat{C}_0^{(1)}(t) & \hat{C}_2^{(1)}(t) & \hat{C}_3^{(1)}(t) & \hat{C}_4^{(1)}(t) \end{bmatrix}.$$

It will be useful to permute the original state-space matrices and balanced gramians in order to group together the non-truncated blocks. Namely,

$$A_b^{(k)}(t) = \begin{bmatrix} A_r^{(k)}(t) & \overline{A}_{12}^{(k)}(t) \\ \overline{A}_{21}^{(k)}(t) & \overline{A}_{22}^{(k)}(t) \end{bmatrix}, \qquad \Gamma_k^{\text{in}}(t) = \begin{bmatrix} \Gamma^{(k)}(t) & 0 \\ 0 & \Gamma_{\text{in}}^{(k)}(t) \end{bmatrix},$$
$$\Gamma_k^{\text{out}}(t) = \begin{bmatrix} \Gamma^{(k)}(t) & 0 \\ 0 & \Gamma_{\text{out}}^{(k)}(t) \end{bmatrix}, \qquad B_b^{(k)}(t) = \begin{bmatrix} B_r^{(k)}(t) \\ \overline{B}_2^{(k)}(t) \end{bmatrix},$$
$$C_b^{(k)}(t) = \begin{bmatrix} C_r^{(k)}(t) & \overline{C}_2^{(k)}(t) \end{bmatrix}, \qquad \Omega_k^{\text{in}}(t) = \begin{bmatrix} \Omega^{(k)}(t) & 0 \\ 0 & \Omega_{\text{in}}^{(k)}(t) \end{bmatrix},$$
$$\Omega_k^{\text{out}}(t) = \begin{bmatrix} \Omega^{(k)}(t) & 0 \\ 0 & \Omega_{\text{out}}^{(k)}(t) \end{bmatrix},$$
$$\Gamma_{\text{in}}^{(k)}(t) = \operatorname{diag}\left(\Gamma^{(ik)}(t)\right)_{i\in E_{\text{in}}^{(k)}}, \qquad \Omega_{\text{in}}^{(k)}(t) = \operatorname{diag}\left(\Omega^{(ik)}(t)\right)_{i\in E_{\text{in}}^{(k)}},$$
$$\Gamma_{\text{out}}^{(k)}(t) = \operatorname{diag}\left(\Gamma^{(kj)}(t)\right)_{j\in E_{\text{out}}^{(k)}}, \qquad \Omega_{\text{out}}^{(k)}(t) = \operatorname{diag}\left(\Omega^{(kj)}(t)\right)_{j\in E_{\text{out}}^{(k)}}, \tag{2.8}$$

for all $(t,k) \in \mathbb{N}_0 \times V$ and appropriately defined matrix-valued functions $\overline{A}_{12}^{(k)}(t)$, $\overline{A}_{21}^{(k)}(t)$, $\overline{A}_{22}^{(k)}(t)$, $\overline{B}_{2}^{(k)}(t)$, and $\overline{C}_{2}^{(k)}(t)$.

Lemma 2.8. *Consider a strongly stable distributed LTV system G that has a given balanced realization $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$ and balanced generalized gramians $\Sigma^{(k)}(t)$ and $\Sigma^{(ij)}(t)$ for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$. Let G_r be the reduced order distributed LTV system obtained by applying the balanced truncation method to reduce system G, and denote the obtained realization of system G_r by $(A_r^{(k)}(t), B_r^{(k)}(t), C_r^{(k)}(t), D^{(k)}(t))$. Then, system G_r is strongly stable, and its given realization is balanced with balanced generalized gramians $\Gamma^{(k)}(t)$ and $\Gamma^{(ij)}(t)$ for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$.*

Proof. Since the realization of system G is balanced, then the balanced generalized gramians $\Sigma^{(k)}(t)$ and $\Sigma^{(ij)}(t)$ simultaneously satisfy (2.5) and (2.6) for all $(t,k) \in \mathbb{N}_0 \times V$, where the generalized Lyapunov inequalities are expressed for the given balanced realization of system G. By applying appropriate permutations to (2.5) and (2.6), and using the definitions in (2.8), the following are obtained:

$$A_b^{(k)}(t)\mathrm{diag}(\Gamma_k^{\mathrm{in}}(t), \Omega_k^{\mathrm{in}}(t))(A_b^{(k)}(t))^T - \mathrm{diag}(\Gamma_k^{\mathrm{out}}(t+1), \Omega_k^{\mathrm{out}}(t+1)) + B_b^{(k)}(t)(B_b^{(k)}(t))^T \prec -\mu I, \tag{2.9}$$

$$(A_b^{(k)}(t))^T\mathrm{diag}(\Gamma_k^{\mathrm{out}}(t+1), \Omega_k^{\mathrm{out}}(t+1))A_b^{(k)}(t) - \mathrm{diag}(\Gamma_k^{\mathrm{in}}(t), \Omega_k^{\mathrm{in}}(t)) + (C_b^{(k)}(t))^T C_b^{(k)}(t) \prec -\mu I. \tag{2.10}$$

It can then be inferred that

$$A_r^{(k)}(t)\,\mathrm{diag}(\Gamma^{(k)}(t), \Gamma_{\mathrm{in}}^{(k)}(t))(A_r^{(k)}(t))^T - \mathrm{diag}(\Gamma^{(k)}(t+1), \Gamma_{\mathrm{out}}^{(k)}(t+1)) + B_r^{(k)}(t)(B_r^{(k)}(t))^T \prec -\mu I,$$

$$\left(A_r^{(k)}(t)\right)^T \operatorname{diag}\left(\Gamma^{(k)}(t+1), \Gamma_{\text{out}}^{(k)}(t+1)\right) A_r^{(k)}(t) - \operatorname{diag}\left(\Gamma^{(k)}(t), \Gamma_{\text{in}}^{(k)}(t)\right) \\ + \left(C_r^{(k)}(t)\right)^T C_r^{(k)}(t) \prec -\mu I,$$

hold for all $(t, k) \in \mathbb{N}_0 \times V$. Thus, system G_r is strongly stable and its given realization is balanced with balanced generalized gramians $\Gamma^{(k)}(t)$ and $\Gamma^{(ij)}(t)$. □

2.6 Balanced Truncation Error Bounds

Consider a strongly stable distributed LTV system G that has a given balanced realization $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$ and balanced generalized gramians $\Sigma^{(k)}(t)$ and $\Sigma^{(ij)}(t)$ for all $(t, k) \in \mathbb{N}_0 \times V$ and $(i, j) \in E$. Denote by G_r the strongly stable reduced order distributed LTV system obtained by applying the balanced truncation method to system G, by $(A_r^{(k)}(t), B_r^{(k)}(t), C_r^{(k)}(t), D^{(k)}(t))$ the obtained balanced realization of system G_r, and by $\Gamma^{(k)}(t)$ and $\Gamma^{(ij)}(t)$ the corresponding reduced order balanced generalized gramians for all $(t, k) \in \mathbb{N}_0 \times V$ and $(i, j) \in E$. In this section, upper bounds on the ℓ_2-induced norm of the error system $(G - G_r)$ are derived that generalize their counterparts for single LTV systems in [73] and single NSLPV systems in [33]. Specifically, Theorem 2.10 generalizes the standard 'twice the sum of truncated entries in the balanced generalized gramians' error bound to the class of distributed LTV systems, and Theorem 2.12 gives a specialized error bound expression that applies when the sequences of truncated entries in the balanced generalized gramians are monotonic in time.

For each $t \in \mathbb{N}_0$, define $\hat{\Omega}(t) = \operatorname{diag}(\Omega^{(k)}(t))_{k \in V}$, and let $\tilde{\Omega}(t)$ be the block-diagonal augmentation of $\Omega^{(ij)}(t)$, where $(i, j) \in E$. The specific ordering of the diagonal blocks in $\tilde{\Omega}(t)$ is inconsequential for the purposes of this section, e.g., $\tilde{\Omega}(t) = \operatorname{diag}\left(\operatorname{diag}\left(\Omega^{(ik)}(t)\right)_{i \in E_{\text{in}}^{(k)}}\right)_{k \in V}$

or $\tilde{\Omega}(t) = \operatorname{diag}\left(\operatorname{diag}\left(\Omega^{(kj)}(t)\right)_{j\in E_{\text{out}}^{(k)}}\right)_{k\in V}$. Then, define

$$\overline{\Omega}(t) = \operatorname{diag}(\hat{\Omega}(t), \tilde{\Omega}(t)) \qquad \text{and} \qquad \Omega = \operatorname{diag}(\overline{\Omega}(t))_{t\in\mathbb{N}_0}. \tag{2.11}$$

Depending on the state variables that are to be truncated, some diagonal blocks of Ω may have zero dimensions and, hence, be nonexistent. For instance, if $r^{(k_0)}(t_0) = n^{(k_0)}(t_0)$ and $r^{(i_0j_0)}(t_0) = n^{(i_0j_0)}(t_0)$ for some $(t_0, k_0) \in \mathbb{N}_0 \times V$ and $(i_0, j_0) \in E$, then $\Omega^{(k_0)}(t_0)$ and $\Omega^{(i_0j_0)}(t_0)$ have zero dimensions and do not appear in Ω. On the other hand, if $r^{(k)}(t) \neq n^{(k)}(t)$ and $r^{(ij)}(t) \neq n^{(ij)}(t)$ only for $t = t_0$, $k = k_0$, and $(i,j) = (i_0, j_0)$, then $\Omega = \operatorname{diag}(\Omega^{(k_0)}(t_0), \Omega^{(i_0j_0)}(t_0))$.

Theorem 2.9. *Consider a strongly stable distributed LTV system G that has a given balanced realization $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$ and balanced generalized gramians $\Sigma^{(k)}(t)$ and $\Sigma^{(ij)}(t)$ for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$. Denote by G_r the strongly stable reduced order system obtained by applying the balanced truncation method to system G, by $(A_r^{(k)}(t), B_r^{(k)}(t), C_r^{(k)}(t), D^{(k)}(t))$ the obtained balanced realization of system G_r, and by $\Gamma^{(k)}(t)$ and $\Gamma^{(ij)}(t)$ the corresponding reduced order balanced generalized gramians for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$. If $\Omega^{(k)}(t) = I$ and $\Omega^{(ij)}(t) = I$ for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$, where $\Omega^{(k)}(t)$ and $\Omega^{(ij)}(t)$ are defined from $\Sigma^{(k)}(t) = \operatorname{diag}(\Gamma^{(k)}(t), \Omega^{(k)}(t))$ and $\Sigma^{(ij)}(t) = \operatorname{diag}(\Gamma^{(ij)}(t), \Omega^{(ij)}(t))$, then $\|(G - G_r)\| < 2$.*

Proof. This result is proved by constructing structured solutions to (2.4) for a realization of the error system $\frac{1}{2}(G - G_r)$ and $\gamma = 1$. Since G and G_r are strongly stable systems as per Lemma 2.8, then so is the error system $\frac{1}{2}(G - G_r)$. Applying the Schur complement formula twice to (2.9) and invoking (2.10), it can be shown that

$$\left(K^{(k)}(t)\right)^T (R_2^{(k)}(t+1))^{-1} K^{(k)}(t) - R_1^{(k)}(t) \prec -\mu I, \tag{2.12}$$

holds for all $(t,k) \in \mathbb{N}_0 \times V$, where

$$
\begin{aligned}
K^{(k)}(t) &= \begin{bmatrix} 0 & 0 & A_b^{(k)}(t) \\ 0 & 0 & C_b^{(k)}(t) \\ A_b^{(k)}(t) & B_b^{(k)}(t) & 0 \end{bmatrix}, \\
R_1^{(k)}(t) &= \operatorname{diag}\left((\Gamma_k^{\text{in}}(t))^{-1}, (\Omega_k^{\text{in}}(t))^{-1}, I_{n_u^{(k)}(t)}, \Gamma_k^{\text{in}}(t),\, \Omega_k^{\text{in}}(t)\right), \\
R_2^{(k)}(t+1) &= \operatorname{diag}\left((\Gamma_k^{\text{out}}(t+1))^{-1}, (\Omega_k^{\text{out}}(t+1))^{-1}, I_{n_y^{(k)}(t)}, \Gamma_k^{\text{out}}(t+1),\, \Omega_k^{\text{out}}(t+1)\right).
\end{aligned}
$$

The constituent blocks that define $K^{(k)}(t)$, $R_1^{(k)}(t)$, and $R_2^{(k)}(t+1)$ are given in (2.8). To simplify the algebraic manipulations when applying the Schur complement formula to (2.9), the $-\mu I$ term in (2.9) is provisionally set to zero. Afterwards, some $-\mu I$ term is added to the right-hand-side of (2.12) to emphasize that the sequences of matrices on the left-hand-side are uniformly negative definite. (2.12) is now pre- and post-multiplied by $(P^{(k)}(t))^T$ and $P^{(k)}(t)$, respectively, and $(L^{(k)}(t))^T L^{(k)}(t) = I$ is appropriately invoked to obtain

$$
\begin{aligned}
\left(L^{(k)}(t)K^{(k)}(t)P^{(k)}(t)\right)^T \left(L^{(k)}(t)\left(R_2^{(k)}(t+1)\right)^{-1}(L^{(k)}(t))^T\right)\left(L^{(k)}(t)K^{(k)}(t)P^{(k)}(t)\right)& \\
-(P^{(k)}(t))^T R_1^{(k)}(t)P^{(k)}(t) \prec -\mu I&,
\end{aligned}
$$

where $P^{(k)}(t)$ and $L^{(k)}(t)$ are defined as

$$
P^{(k)}(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & I \\ 0 & 0 & 0 & \sqrt{2}I_{n_u^{(k)}(t)} & 0 \\ -I & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & -I \end{bmatrix}, \quad L^{(k)}(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} -I & 0 & 0 & I & 0 \\ I & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & I \\ 0 & 0 & \sqrt{2}I_{n_y^{(k)}(t)} & 0 & 0 \\ 0 & -I & 0 & 0 & I \end{bmatrix}.
$$

The matrix products $(P^{(k)}(t))^T R_1^{(k)}(t) P^{(k)}(t)$ and $L^{(k)}(t)(R_2^{(k)}(t+1))^{-1}(L^{(k)}(t))^T$ have a similar structure. Namely,

$$(P^{(k)}(t))^T R_1^{(k)}(t) P^{(k)}(t) = \frac{1}{2}\times$$
$$\begin{bmatrix} (\Gamma_k^{\text{in}}(t))^{-1} + \Gamma_k^{\text{in}}(t) & (\Gamma_k^{\text{in}}(t))^{-1} - \Gamma_k^{\text{in}}(t) & 0 & 0 & 0 \\ (\Gamma_k^{\text{in}}(t))^{-1} - \Gamma_k^{\text{in}}(t) & (\Gamma_k^{\text{in}}(t))^{-1} + \Gamma_k^{\text{in}}(t) & 0 & 0 & 0 \\ 0 & 0 & (\Omega_k^{\text{in}}(t))^{-1} + \Omega_k^{\text{in}}(t) & 0 & (\Omega_k^{\text{in}}(t))^{-1} - \Omega_k^{\text{in}}(t) \\ 0 & 0 & 0 & 2I_{n_u^{(k)}(t)} & 0 \\ 0 & 0 & (\Omega_k^{\text{in}}(t))^{-1} - \Omega_k^{\text{in}}(t) & 0 & (\Omega_k^{\text{in}}(t))^{-1} + \Omega_k^{\text{in}}(t) \end{bmatrix},$$

$$L^{(k)}(t)(R_2^{(k)}(t+1))^{-1}(L^{(k)}(t))^T =$$
$$\frac{1}{2}\operatorname{diag}\left(\begin{bmatrix} (\Gamma_k^{\text{out}}(t+1))^{-1} + \Gamma_k^{\text{out}}(t+1) & (\Gamma_k^{\text{out}}(t+1))^{-1} - \Gamma_k^{\text{out}}(t+1) \\ (\Gamma_k^{\text{out}}(t+1))^{-1} - \Gamma_k^{\text{out}}(t+1) & (\Gamma_k^{\text{out}}(t+1))^{-1} + \Gamma_k^{\text{out}}(t+1) \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} (\Omega_k^{\text{out}}(t+1))^{-1} + \Omega_k^{\text{out}}(t+1) & 0 & (\Omega_k^{\text{out}}(t+1))^{-1} - \Omega_k^{\text{out}}(t+1) \\ 0 & 2I_{n_y^{(k)}(t)} & 0 \\ (\Omega_k^{\text{out}}(t+1))^{-1} - \Omega_k^{\text{out}}(t+1) & 0 & (\Omega_k^{\text{out}}(t+1))^{-1} + \Omega_k^{\text{out}}(t+1) \end{bmatrix} \right).$$

Let $M^{(k)}(t) = \left[\begin{array}{cc|c} A_r^{(k)}(t) & 0 & \frac{1}{\sqrt{2}} B_r^{(k)}(t) \\ 0 & A_b^{(k)}(t) & \frac{1}{\sqrt{2}} B_b^{(k)}(t) \\ \hline \frac{-1}{\sqrt{2}} C_r^{(k)}(t) & \frac{1}{\sqrt{2}} C_b^{(k)}(t) & 0 \end{array}\right]$. Then by performing the matrix multiplications, it can be seen that $L^{(k)}(t)K^{(k)}(t)P^{(k)}(t) = \begin{bmatrix} M^{(k)}(t) & N_{12}^{(k)}(t) \\ N_{21}^{(k)}(t) & \overline{A}_{22}^{(k)}(t) \end{bmatrix}$ for appropriately defined $N_{12}^{(k)}(t)$ and $N_{21}^{(k)}(t)$. The matrices in $M^{(k)}(t)$ can be used to describe the error system $\frac{1}{2}(G - G_r)$. However, the resultant system equations will not be in the form of the equations defined in (2.1), and so appropriate permutations need to be used to equivalently express the resultant system equations in the desired form. Since $\Omega^{(k)}(t) = I$ and $\Omega^{(ij)}(t) = I$ for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$, then it is not difficult to see that the following inequality holds for all $(t,k) \in \mathbb{N}_0 \times V$:

$$(M^{(k)}(t))^T \mathrm{diag}\left(V_2^{(k)}(t+1), I\right) M^{(k)}(t) - \mathrm{diag}\left(V_1^{(k)}(t), I\right) \prec -\mu I,$$

where

$$V_2^{(k)}(t+1) = \frac{1}{2}\begin{bmatrix} (\Gamma_k^{\mathrm{out}}(t+1))^{-1} + \Gamma_k^{\mathrm{out}}(t+1) & (\Gamma_k^{\mathrm{out}}(t+1))^{-1} - \Gamma_k^{\mathrm{out}}(t+1) & 0 \\ (\Gamma_k^{\mathrm{out}}(t+1))^{-1} - \Gamma_k^{\mathrm{out}}(t+1) & (\Gamma_k^{\mathrm{out}}(t+1))^{-1} + \Gamma_k^{\mathrm{out}}(t+1) & 0 \\ 0 & 0 & 2I \end{bmatrix} \succ \mu I,$$

$$V_1^{(k)}(t) = \frac{1}{2}\begin{bmatrix} (\Gamma_k^{\mathrm{in}}(t))^{-1} + \Gamma_k^{\mathrm{in}}(t) & (\Gamma_k^{\mathrm{in}}(t))^{-1} - \Gamma_k^{\mathrm{in}}(t) & 0 \\ (\Gamma_k^{\mathrm{in}}(t))^{-1} - \Gamma_k^{\mathrm{in}}(t) & (\Gamma_k^{\mathrm{in}}(t))^{-1} + \Gamma_k^{\mathrm{in}}(t) & 0 \\ 0 & 0 & 2I \end{bmatrix} \succ \mu I.$$

$V_1^{(k)}(t)$ and $V_2^{(k)}(t+1)$ correspond to the upper-left-corner blocks of $\left(P^{(k)}(t)\right)^T R_1^{(k)}(t) P^{(k)}(t)$ and $L^{(k)}(t)(R_2^{(k)}(t+1))^{-1}(L^{(k)}(t))^T$, respectively. By applying the aforementioned appropriate permutations and invoking Lemma 2.5 with $\gamma = 1$, it is concluded that $\|\frac{1}{2}(G - G_r)\| < 1$. $\square$

Theorem 2.10. *Consider a strongly stable distributed LTV system G that has a given balanced realization $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$ and balanced generalized gramians $\Sigma^{(k)}(t)$ and $\Sigma^{(ij)}(t)$ for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$. Denote by G_r the strongly stable reduced order system obtained by applying the balanced truncation method to system G, by $(A_r^{(k)}(t), B_r^{(k)}(t), C_r^{(k)}(t), D^{(k)}(t))$ the obtained balanced realization of system G_r, and by $\Gamma^{(k)}(t)$ and $\Gamma^{(ij)}(t)$ the corresponding reduced order balanced generalized gramians for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$. Then, the error system $(G - G_r)$ satisfies $\|(G - G_r)\| < 2\zeta(\Omega)$,* *where $\zeta(X)$ denotes the sum of distinct diagonal entries of a square, possibly infinite dimensional, matrix X, Ω is defined in (2.11), and $\Omega^{(k)}(t)$ and $\Omega^{(ij)}(t)$ therein are defined from $\Sigma^{(k)}(t) = \operatorname{diag}(\Gamma^{(k)}(t), \Omega^{(k)}(t))$ and $\Sigma^{(ij)}(t) = \operatorname{diag}(\Gamma^{(ij)}(t), \Omega^{(ij)}(t))$ for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$.*

Proof. Since the realization of a reduced order system obtained by the application of the balanced truncation method is balanced by Lemma 2.8, the truncation procedure can be implemented in multiple steps. For each step i, the smallest diagonal entry in Ω is found and is denoted by q_i. Then, all the state variables with a corresponding diagonal entry in Ω equal to q_i are truncated, and Ω is updated by removing all the entries equal to q_i. The resulting reduced order system obtained at each step i is denoted by $G_{r,i}$. Suppose that, in the first step, $\Omega^{(k)}(t) = q_1 I$ for some $(t,k) \in \mathbb{N}_0 \times V$ and/or $\Omega^{(ij)}(t) = q_1 I$ for some $(i,j) \in E$ and (perhaps different) $t \in \mathbb{N}_0$. It is to be shown that $\|(G - G_{r,1})\| < 2q_1$. To do so, a scaled system G_{new} is constructed with realization $(A^{(k)}(t), \frac{1}{\sqrt{q_1}} B^{(k)}(t), \frac{1}{\sqrt{q_1}} C^{(k)}(t), \frac{1}{q_1} D^{(k)}(t))$. It is not difficult to verify that the corresponding $\Omega^{(k)}_{\text{new}}(t) = I$ and/or $\Omega^{(ij)}_{\text{new}}(t) = I$. Denote by $G_{\text{new},r,1}$ the reduced order system obtained by applying the balanced truncation method to G_{new}. From Theorem 2.9, $\|(G_{\text{new}} - G_{\text{new},r,1})\| < 2$. Since $\|(G_{\text{new}} - G_{\text{new},r,1})\| = \frac{1}{q_1}\|(G - G_{r,1})\|$, it follows that $\|(G - G_{r,1})\| < 2q_1$. A similar procedure is applied in the second step to obtain $\|(G_{r,1} - G_{r,2})\| < 2q_2$. Then, by the triangle inequality, $\|(G - G_{r,2})\| = \|(G - G_{r,1} + G_{r,1} - G_{r,2})\| \leq \|(G - G_{r,1})\| + \|(G_{r,1} - G_{r,2})\| < 2(q_1 + q_2)$, and so on. □

The upper bound on $\|(G - G_r)\|$ derived in Theorem 2.10 may not always be finite as there may be infinitely many distinct entries in Ω. Theorem 2.12 gives a tighter expression for the error bound that holds when the diagonal entries of $\Omega^{(k)}(t)$ and $\Omega^{(ij)}(t)$ define monotonic sequences in time. Define the subsets of time at which truncation occurs as $\mathcal{F}_k = \{t \in \mathbb{N}_0 \,|\, r^{(k)}(t) \neq n^{(k)}(t)\}$ and $\mathcal{F}_{i,j} = \{t \in \mathbb{N}_0 \,|\, r^{(ij)}(t) \neq n^{(ij)}(t)\}$ for all $k \in V$ and $(i,j) \in E$.

Definition 2.11 below is from [33].

Definition 2.11. Let α_t be a scalar sequence defined on a subset $\mathcal{W}$ of $\mathbb{N}_0$, and define $t_{\min} = \min\{t \,|\, t \in \mathcal{W}\}$. The domain of definition of α_t is extended to all $t \in \mathbb{N}_0$ as follows:

$$\alpha_t = \begin{cases} \alpha_{t_{\min}} & \text{if } 0 \leq t \leq t_{\min}, \\ \alpha_d & \text{if } t_{\min} < t, \text{ where } d = \max\{\tau \leq t \,|\, \tau \in \mathcal{W}\}. \end{cases}$$

Theorem 2.12. *Consider a strongly stable distributed LTV system G that has a given balanced realization $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$ and balanced generalized gramians $\Sigma^{(k)}(t)$ and $\Sigma^{(ij)}(t)$ for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$. Denote by G_r the strongly stable reduced order system obtained by applying the balanced truncation method to system G, by $(A_r^{(k)}(t), B_r^{(k)}(t), C_r^{(k)}(t), D^{(k)}(t))$ the obtained balanced realization of system G_r, and by $\Gamma^{(k)}(t)$ and $\Gamma^{(ij)}(t)$ the corresponding reduced order balanced generalized gramians for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$. Define $\Omega^{(k)}(t)$ and $\Omega^{(ij)}(t)$ from $\Sigma^{(k)}(t) = \operatorname{diag}(\Gamma^{(k)}(t), \Omega^{(k)}(t))$ and $\Sigma^{(ij)}(t) = \operatorname{diag}(\Gamma^{(ij)}(t), \Omega^{(ij)}(t))$ for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$. If $\Omega^{(k)}(t) = w^{(k)}(t)I$ for $t \in \mathcal{F}_k$ and all $k \in V$, $\Omega^{(ij)}(t) = w^{(ij)}(t)I$ for $t \in \mathcal{F}_{i,j}$ and all $(i,j) \in E$, and the sequences $w^{(k)}(t)$ and $w^{(ij)}(t)$ are monotonic in time, then*

$$\|(G - G_r)\| < 2 \left(\sum_{k \in V} \sup_{t \in \mathcal{F}_k} w^{(k)}(t) + \sum_{(i,j) \in E} \sup_{t \in \mathcal{F}_{i,j}} w^{(ij)}(t) \right).$$

Proof. The result needs to be proved for the special case where only one temporal state or one spatial state is truncated, i.e., there is only one nonempty set $\mathcal{F}_{k_0}$ or $\mathcal{F}_{i_0,j_0}$. The general case considered in the theorem then follows by repeated application of the result for this special case and the triangle inequality. Namely, for a fixed $k_0 \in V$, assume that the temporal state $x^{(k_0)}$ is the only truncated state. It can be assumed without loss of generality that $w^{(k_0)}(t) \leq 1$ for all $t \in \mathcal{F}_{k_0}$, as this can be always achieved by scaling. The domain of definition of $w^{(k_0)}(t)$

is extended to all $t \in \mathbb{N}_0$ using the rule in Definition 2.11. Clearly, the extended sequence is still monotonic in time. The case where $w^{(k_0)}(t)$ is monotone non-decreasing is considered in the first part of the proof. For all $(t,k) \in \mathbb{N}_0 \times V$, the state-space transformations $T_{\mathrm{pre}}^{(k)}(t) = (w^{(k_0)}(t))^{-\frac{1}{2}} I$ and $T_{\mathrm{post}}^{(k)}(t) = (w^{(k_0)}(t))^{\frac{1}{2}} I$ are defined. Since $\Sigma^{(k_0)}(t) \succ \mu I$ for some $\mu > 0$ and all $t \in \mathbb{N}_0$ by definition, then $T_{\mathrm{pre}}^{(k)}(t)$ is bounded for all $(t,k) \in \mathbb{N}_0 \times V$. From these transformations, a new realization $(A_{\mathrm{new}}^{(k)}(t), B_{\mathrm{new}}^{(k)}(t), C_{\mathrm{new}}^{(k)}(t), D^{(k)}(t))$ is constructed for system G as follows:

$$A_{\mathrm{new}}^{(k)}(t) = T_{\mathrm{pre}}^{(k)}(t+1) A^{(k)}(t) T_{\mathrm{post}}^{(k)}(t), \qquad B_{\mathrm{new}}^{(k)}(t) = T_{\mathrm{pre}}^{(k)}(t+1) B^{(k)}(t),$$
$$C_{\mathrm{new}}^{(k)}(t) = C^{(k)}(t) T_{\mathrm{post}}^{(k)}(t).$$

To simplify the discussion, system G will continue to refer to the system with the original realization $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$, whereas system G_{new} will be used to refer to the system with the new realization $(A_{\mathrm{new}}^{(k)}(t), B_{\mathrm{new}}^{(k)}(t), C_{\mathrm{new}}^{(k)}(t), D^{(k)}(t))$. The realization of system G_{new} is now shown to be balanced. Since the realization of system G is balanced, then $\Sigma^{(k)}(t)$ and $\Sigma^{(ij)}(t)$ satisfy (2.5) and (2.6) for all $(t,k) \in \mathbb{N}_0 \times V$. By pre- and post-multiplying (2.5) by $T_{\mathrm{pre}}^{(k)}(t+1)$, invoking $T_{\mathrm{post}}^{(k)}(t)(T_{\mathrm{post}}^{(k)}(t))^{-1} = (T_{\mathrm{post}}^{(k)}(t))^{-1} T_{\mathrm{post}}^{(k)}(t) = I$, and defining $\Sigma_{\mathrm{new}}^{(k)}(t) = (w^{(k_0)}(t))^{-1}\Sigma^{(k)}(t)$, $\Sigma_{\mathrm{new}}^{(ij)}(t) = (w^{(k_0)}(t))^{-1}\Sigma^{(ij)}(t)$, $\Sigma_{\mathrm{new,in}}^{(k)}(t) = \operatorname{diag}\left(\Sigma_{\mathrm{new}}^{(ik)}(t)\right)_{i \in E_{\mathrm{in}}^{(k)}}$, and $\Sigma_{\mathrm{new,out}}^{(k)}(t) = \operatorname{diag}\left(\Sigma_{\mathrm{new}}^{(kj)}(t)\right)_{j \in E_{\mathrm{out}}^{(k)}}$, it can be shown that

$$A_{\mathrm{new}}^{(k)}(t) \operatorname{diag}\left(\Sigma_{\mathrm{new}}^{(k)}(t), \Sigma_{\mathrm{new,in}}^{(k)}(t)\right) (A_{\mathrm{new}}^{(k)}(t))^T - \operatorname{diag}\left(\Sigma_{\mathrm{new}}^{(k)}(t+1), \Sigma_{\mathrm{new,out}}^{(k)}(t+1)\right) + B_{\mathrm{new}}^{(k)}(t)(B_{\mathrm{new}}^{(k)}(t))^T \prec -\mu I,$$

for all $(t,k) \in \mathbb{N}_0 \times V$. Similarly, (2.6) is pre- and post-multiplied by $T_{\mathrm{post}}^{(k)}(t)$, and $(T_{\mathrm{pre}}^{(k)}(t+1))^{-1} T_{\mathrm{pre}}^{(k)}(t+1) = T_{\mathrm{pre}}^{(k)}(t+1)(T_{\mathrm{pre}}^{(k)}(t+1))^{-1} = I$ are invoked to show that

$$w^{(k_0)}(t+1)(A^{(k)}_{\text{new}}(t))^T\text{diag}\left(\Sigma^{(k)}(t+1),\Sigma^{(k)}_{\text{out}}(t+1)\right)A^{(k)}_{\text{new}}(t)-w^{(k_0)}(t)\text{diag}\left(\Sigma^{(k)}(t),\Sigma^{(k)}_{\text{in}}(t)\right)$$
$$+(C^{(k)}_{\text{new}}(t))^TC^{(k)}_{\text{new}}(t)\prec-\mu I,$$

for all $(t,k)\in\mathbb{N}_0\times V$. Since $0<w^{(k_0)}(t)\le 1$ by assumption, then $(w^{(k_0)}(t))^{-1}\ge w^{(k_0)}(t)$. Also, since $w^{(k_0)}(t)$ is a monotone non-decreasing sequence in t, then $w^{(k_0)}(t)\le w^{(k_0)}(t+1)$ and $(w^{(k_0)}(t))^{-1}\ge(w^{(k_0)}(t+1))^{-1}$ for all $t\in\mathbb{N}_0$. With this in mind, it is not difficult to verify that the following inequality holds for all $(t,k)\in\mathbb{N}_0\times V$:

$$(w^{(k_0)}(t+1))^{-1}(A^{(k)}_{\text{new}}(t))^T\text{diag}\left(\Sigma^{(k)}(t+1),\Sigma^{(k)}_{\text{out}}(t+1)\right)A^{(k)}_{\text{new}}(t)$$
$$-(w^{(k_0)}(t))^{-1}\text{diag}\left(\Sigma^{(k)}(t),\Sigma^{(k)}_{\text{in}}(t)\right)+(C^{(k)}_{\text{new}}(t))^TC^{(k)}_{\text{new}}(t)\prec-\mu I.$$

Thus, $\Sigma^{(k)}_{\text{new}}(t)$ and $\Sigma^{(ij)}_{\text{new}}(t)$ satisfy (2.5) and (2.6) for all $(t,k)\in\mathbb{N}_0\times V$ and $(i,j)\in E$, where the generalized Lyapunov inequalities are expressed for the realization of G_{new}. In other words, the realization of system G_{new} is balanced and can be reduced by the application of the balanced truncation method. Denote by $G_{\text{new},r}$ the reduced order system obtained by truncating the temporal state of subsystem $G^{(k_0)}_{\text{new}}$. Since $\Omega^{(k_0)}(t)=w^{(k_0)}(t)I$, then $\Omega^{(k_0)}_{\text{new}}(t)=I$ for $t\in\mathcal{F}_{k_0}$. By Theorem 2.9, $\|(G_{\text{new}}-G_{\text{new},r})\|<2$. However, system $(G_{\text{new}}-G_{\text{new},r})$ is equivalent to $(G-G_r)$ because of the special structure of $T^{(k)}_{\text{post}}(t)$ and $T^{(k)}_{\text{pre}}(t)$, and so $\|(G-G_r)\|<2$. The second part of the proof treats the case where $w^{(k_0)}(t)$ is a monotone non-increasing sequence in time and uses a similar argument to the one used in the first of the proof. In this case, the state-space transformations are defined as $T^{(k)}_{\text{pre}}(t)=(w^{(k_0)}(t))^{\frac{1}{2}}I$ and $T^{(k)}_{\text{post}}(t)=(w^{(k_0)}(t))^{-\frac{1}{2}}I$ for all $(t,k)\in\mathbb{N}_0\times V$. $\square$

As an example, assume that the only truncated state is $x^{(k_0)}$, the truncation only occurs at three time-steps t_1, t_2, t_3, and $\Omega^{(k_0)}(t_1)=\text{diag}(7,3,2)$, $\Omega^{(k_0)}(t_2)=\text{diag}(4,2,2)$, and $\Omega^{(k_0)}(t_3)=\text{diag}(6,5,2)$. Theorem 2.10 gives an error bound of $2\times(7+6+5+4+3+2)=54$, whereas Theorem 2.12 gives an error bound of $2 \times (2+5+7) = 28$. As can be seen below, in the latter case, the truncation sequences are $(2, \mathrm{diag}(2,2), 2)$, $(3,4,5)$, and $(7,6)$. That is, the terms in squares are truncated first, followed by the terms in circles and the terms in triangles, respectively. In the first truncation sequence, all truncated entries are equal to 2. The second and third truncation sequences are monotone increasing and monotone decreasing, respectively, with the largest entries equal to 5 and 7. This gives the error bound $2 \times (2+5+7) = 28$ computed above.

$$\Omega^{(k_0)}(t_1) = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \boxed{2} \end{bmatrix}, \Omega^{(k_0)}(t_2) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & \boxed{2} & 0 \\ 0 & 0 & \boxed{2} \end{bmatrix}, \Omega^{(k_0)}(t_3) = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & \boxed{2} \end{bmatrix}.$$

2.7 Eventually Time-Periodic Systems

This section is concerned with the special case of eventually time-periodic subsystems, i.e., subsystems with state-space matrices that become time-periodic after some initial finite time-horizon. It is shown that, for eventually time-periodic subsystems, the bound given in Theorem 2.10 reduces to a finite sum.

Definition 2.13. Consider a distributed LTV system G that has a realization denoted by $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$. System G is said to be (h,q)-eventually time-periodic for some integers $h \geq 0$ and $q > 0$, if the state-space matrices of the subsystems are (h,q)-eventually time-periodic, e.g., $A^{(k)}(t+h+zq) = A^{(k)}(t+h)$, for all $t, z \in \mathbb{N}_0$ and $k \in V$. Eventually time-periodic subsystems include time-periodic subsystems ($(0,q)$-eventually time-periodic), time-invariant subsystems ($(0,1)$-eventually time-periodic), and finite time-horizon subsystems ($(h,1)$-eventually time-periodic with zero state-space matrices for $t \geq h$) as special cases.

Lemma 2.14. *Consider a strongly stable, q time-periodic, distributed LTV system G that has a realization denoted by $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$. Then, there exist $\mu > 0$ and q time-periodic, uniformly bounded, and positive definite matrix-valued functions $W_{\text{per}}^{(k)}(t) \succ \mu I$ and $W_{\text{per}}^{(ij)}(t) \succ \mu I$ that satisfy (2.3) for all $(t, k) \in \mathbb{N}_0 \times V$ and $(i, j) \in E$.*

Proof. Since G is a strongly stable system, then there exist uniformly bounded and positive definite solutions to (2.3), which are denoted by $W^{(k)}(t)$ and $W^{(ij)}(t)$ for all $(t, k) \in \mathbb{N}_0 \times V$ and $(i, j) \in E$. From these solutions, the desired q time-periodic solutions $W_{\text{per}}^{(k)}(t)$ and $W_{\text{per}}^{(ij)}(t)$ are constructed. Namely, averaging techniques are used that are similar to the ones used in [23, 28]. Since system G is q time-periodic, then $A^{(k)}(t + zq) = A^{(k)}(t)$ for all $t, z \in \mathbb{N}_0$ and $k \in V$. For each pair $(t, k) \in \mathbb{N}_0 \times V$, evaluate (2.3) at the time-steps $t + zq$ for $z = 0, \ldots, \lambda - 1$, where λ is any integer such that $\lambda \geq 1$, and average the resulting inequalities to obtain

$$(A^{(k)}(t))^T \text{diag}\left(Y_\lambda^{(k)}(t+1), Y_{\lambda,\text{out}}^{(k)}(t+1)\right) A^{(k)}(t) - \text{diag}\left(Y_\lambda^{(k)}(t), Y_{\lambda,\text{in}}^{(k)}(t)\right) \prec -\mu I,$$

for all $(t, k) \in \mathbb{N}_0 \times V$, where $Y_\lambda^{(k)}(t) = \frac{1}{\lambda}\sum_{z=0}^{\lambda-1} W^{(k)}(t+zq)$, $Y_\lambda^{(ij)}(t) = \frac{1}{\lambda}\sum_{z=0}^{\lambda-1} W^{(ij)}(t+zq)$, $Y_{\lambda,\text{in}}^{(k)}(t) = \text{diag}\left(Y_\lambda^{(ik)}(t)\right)_{i \in E_{\text{in}}^{(k)}}$, and $Y_{\lambda,\text{out}}^{(k)}(t) = \text{diag}\left(Y_\lambda^{(kj)}(t)\right)_{j \in E_{\text{out}}^{(k)}}$. Since the matrix-valued sequences $W^{(k)}(t)$ and $W^{(ij)}(t)$ (indexed by t) are uniformly bounded for all $k \in V$ and $(i, j) \in E$, then the matrix-valued sequences $Y_\lambda^{(k)}(t)$ and $Y_\lambda^{(ij)}(t)$ (indexed by λ) are also uniformly bounded for all $(t, k) \in \mathbb{N}_0 \times V$ and $(i, j) \in E$, and there exist weakly convergent subsequences $Y_{\lambda_c}^{(k)}(t)$ and $Y_{\lambda_c}^{(ij)}(t)$ with limits $L^{(k)}(t)$ and $L^{(ij)}(t)$ as $\lambda_c \to \infty$, respectively. The reader is referred to [50] for further details on convergence in weak topology. By construction, the limits are positive definite and satisfy $L^{(k)}(t) \succ \mu I$ and $L^{(ij)}(t) \succ \mu I$ for all $(t, k) \in \mathbb{N}_0 \times V$ and $(i, j) \in E$. It remains to be shown that these limits are q time-periodic. This is done for $L^{(k)}(t)$ only, as the proof for $L^{(ij)}(t)$ follows similarly. Namely, for any $(t,k) \in \mathbb{N}_0 \times V$,

$$\begin{aligned} L^{(k)}(t+q) - L^{(k)}(t) &= \lim_{\lambda_c \to \infty} \frac{1}{\lambda_c} \sum_{z=0}^{\lambda_c - 1} \left(W^{(k)}(t+(z+1)q) - W^{(k)}(t+zq) \right) \\ &= \lim_{\lambda_c \to \infty} \frac{1}{\lambda_c} \left(W^{(k)}(t+\lambda_c\, q) - W^{(k)}(t) \right) = 0. \end{aligned}$$

The proof is concluded by setting $W_{\text{per}}^{(k)}(t) = L^{(k)}(t)$ and $W_{\text{per}}^{(ij)}(t) = L^{(ij)}(t)$ for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$. □

Theorem 2.15. *Consider a strongly stable, (h,q)-eventually time-periodic, distributed LTV system G that has a realization denoted by $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$. Then, there exist $\mu > 0$ and (h,q)-eventually time-periodic, uniformly bounded, and positive definite matrix-valued functions $X_{\text{eper}}^{(k)}(t) \succ \mu I$, $X_{\text{eper}}^{(ij)}(t) \succ \mu I$ that satisfy (2.5) for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$. There also exist $\mu > 0$ and (h,q)-eventually time-periodic, uniformly bounded, and positive definite matrix-valued functions $Y_{\text{eper}}^{(k)}(t) \succ \mu I$, and $Y_{\text{eper}}^{(ij)}(t) \succ \mu I$ that satisfy (2.6) for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$.*

Proof. Since G is a strongly stable system, then there exist uniformly bounded, positive definite solutions to (2.3), which are denoted by $W^{(k)}(t)$ and $W^{(ij)}(t)$ for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$. From these solutions, (h,q)-eventually time-periodic solutions $W_{\text{eper}}^{(k)}(t)$ and $W_{\text{eper}}^{(ij)}(t)$ to (2.3) are constructed that are also uniformly bounded and positive definite. The LMIs that correspond to $t \geq h-1$ are only needed in the proof, and so it is assumed that $h = 1$ without loss of generality. For all $k \in V$, the following hold:

$$\begin{aligned} &(A^{(k)}(0))^T \text{diag}\left(W^{(k)}(1), W_{\text{out}}^{(k)}(1) \right) A^{(k)}(0) - \text{diag}\left(W^{(k)}(0), W_{\text{in}}^{(k)}(0) \right) \prec -\mu I, \\ &(A^{(k)}(t))^T \text{diag}\left(W^{(k)}(t+1), W_{\text{out}}^{(k)}(t+1) \right) A^{(k)}(t) - \text{diag}\left(W^{(k)}(t), W_{\text{in}}^{(k)}(t) \right) \prec -\mu I, \quad t \geq 1, \end{aligned} \tag{2.13}$$

where $W_{\text{in}}^{(k)}(t) = \operatorname{diag}\left(W^{(ik)}(t)\right)_{i\in E_{\text{in}}^{(k)}}$ and $W_{\text{out}}^{(k)}(t) = \operatorname{diag}\left(W^{(kj)}(t)\right)_{j\in E_{\text{out}}^{(k)}}$.

By Lemma 2.14, there exist q time-periodic solutions to (2.13) that are also uniformly bounded and positive definite. Denote these solutions by $W_{\text{per}}^{(k)}(t)$ and $W_{\text{per}}^{(ij)}(t)$ for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$. However, $\alpha > 0$ can always be chosen such that $\alpha\, W_{\text{per}}^{(k)}(1) \prec W^{(k)}(1)$ and $\alpha\, W_{\text{per}}^{(ij)}(1) \prec W^{(ij)}(1)$. Then, the following inequality holds for all $k \in V$:

$$(A^{(k)}(0))^T \operatorname{diag}\left(\alpha W_{\text{per}}^{(k)}(1), \alpha W_{\text{per,out}}^{(k)}(1)\right) A^{(k)}(0) - \operatorname{diag}\left(W^{(k)}(0), W_{\text{in}}^{(k)}(0)\right) \prec -\mu I,$$

where $W_{\text{per,out}}^{(k)}(1) = \operatorname{diag}\left(W_{\text{per}}^{(kj)}(1)\right)_{j\in E_{\text{out}}^{(k)}}$. Thus, the desired (h,q)-eventually time-periodic solutions to (2.3) are defined as follows: $W_{\text{eper}}^{(k)}(0) = W^{(k)}(0)$, $W_{\text{eper}}^{(ij)}(0) = W^{(ij)}(0)$, $W_{\text{eper}}^{(k)}(t) = \alpha\, W_{\text{per}}^{(k)}(t)$, $W_{\text{eper}}^{(ij)}(t) = \alpha\, W_{\text{per}}^{(ij)}(t)$ for all $k \in V$, $(i,j) \in E$, and $t \geq 1$.

Since $W_{\text{eper}}^{(k)}(t)$ and $W_{\text{eper}}^{(ij)}(t)$ are (h,q)-eventually time-periodic solutions to (2.3) and the sequences of state-spaces matrices are assumed to be uniformly bounded, then $\eta > 0$ can be chosen such that $Y_{\text{eper}}^{(k)}(t) = \eta W_{\text{eper}}^{(k)}(t)$ and $Y_{\text{eper}}^{(ij)}(t) = \eta W_{\text{eper}}^{(ij)}(t)$ are the desired (h,q)-eventually time-periodic solutions to (2.6). Moreover, by applying the Schur complement formula twice to (2.3), it can be seen that $\xi > 0$ can be chosen such that $X_{\text{eper}}^{(k)}(t) = \xi(W_{\text{eper}}^{(k)}(t))^{-1}$ and $X_{\text{eper}}^{(ij)}(t) = \xi(W_{\text{eper}}^{(ij)}(t))^{-1}$ are the desired (h,q)-eventually time-periodic solutions to (2.5). $\square$

Corollary 2.16. *A strongly stable, (h,q)-eventually time-periodic system G has an (h,q)-eventually time-periodic balanced realization, i.e., a realization with state-space matrices that are (h,q)-eventually time-periodic, and (h,q)-eventually time-periodic balanced generalized gramians $\Sigma_{\text{eper}}^{(k)}(t) = \operatorname{diag}(\Gamma_{\text{eper}}^{(k)}(t), \Omega_{\text{eper}}^{(k)}(t))$ and $\Sigma_{\text{eper}}^{(ij)}(t) = \operatorname{diag}(\Gamma_{\text{eper}}^{(ij)}(t), \Omega_{\text{eper}}^{(ij)}(t))$ for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$. Moreover, system G_r that results from the application of the balanced truncation method to system G has an (h,q)-eventually time-periodic balanced realization and (h,q)-eventually time-periodic balanced generalized gramians $\Gamma_{\text{eper}}^{(k)}(t)$ and $\Gamma_{\text{eper}}^{(ij)}(t)$ for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$, and satisfies $\|(G - G_r)\| < 2\zeta(\Omega_{\text{eper}})$, where Ω_{eper} is* *is obtained from (2.11) by restricting* $0 \le t \le h+q-1$.

The error bound in Corollary 2.16 may be further improved by using Theorem 2.12 to compute the bound due to the states truncated over the finite time-horizon, i.e., for $0 \le t < h$.

2.8 Illustrative Example

In this section, the balanced truncation method is applied to reduce a distributed LTV system G formed by $N = 5$ agents that are interconnected as shown in Figure 2.1. The purpose of this example is to illustrate that 1) the proposed method allows for truncating both types of state variables, i.e., temporal and spatial state variables; 2) the truncation does not need to be uniform in time, i.e, different number of state variables can be truncated at different time-steps; 3) each state vector is truncated individually in a unique way; and 4) heuristics can be used to improve on the error bound computation in Corollary 2.16.

The temporal states and the spatial states are of constant dimensions $n_T = 6$ and $n_S = 3$, respectively. There are two sets of building blocks for the state-space matrices: one for the odd-numbered subsystems and another for the even-numbered subsystems. All the blocks of the state-space matrices are constants, except for the $A_{00}^{(k)}(t)$ terms that are time-periodic with period $q = 28$, i.e., the $A_{00}^{(k)}(t)$ terms satisfy $A_{00}^{(k)}(t+28\,z) = A_{00}^{(k)}(t)$ for all $t, z \in \mathbb{N}_0$ and $k \in V$. Namely,

$$A_{00}^{(k)}(t) = A_{TT} \text{ for } t = 0, \dots, 6, \qquad A_{00}^{(k)}(t) = \mathcal{M} A_{TT} \mathcal{M}^T \text{ for } t = 7, \dots, 13,$$
$$A_{00}^{(k)}(t) = \mathcal{M}^2 A_{TT} (\mathcal{M}^T)^2 \text{ for } t = 14, \dots, 20, \quad A_{00}^{(k)}(t) = \mathcal{M}^3 A_{TT} (\mathcal{M}^T)^3 \text{ for } t = 21, \dots, 27,$$

where A_{TT} and $\mathcal{M}$ are building blocks. For odd-numbered subsystems, the building blocks are given by

$$A_{TT} = 0.1 \begin{bmatrix} \begin{bmatrix} -9 & -7 \\ -5 & -5 \end{bmatrix} & \begin{bmatrix} 0.1 & 0.3 & -0.1 & 0.2 \\ 0.3 & 0.2 & 0.1 & -0.2 \end{bmatrix} \\ \begin{bmatrix} \mathrm{diag}(1,-2) \\ -I_2 \end{bmatrix} & 0.01\mathrm{diag}(-5,1,-3,2) \end{bmatrix}, \qquad A_{SS} = 0_{3\times 3},$$

$$A_{TS} = 0.1 \begin{bmatrix} \begin{bmatrix} -0.5 & 0.5 & 0.01 \\ 0.5 & -0.5 & -0.02 \end{bmatrix} \\ 0_{4\times 3} \end{bmatrix}, \qquad B_T = 0.2 \begin{bmatrix} I_2 \\ 0_{4\times 2} \end{bmatrix},$$

$$A_{ST} = 0.1 \begin{bmatrix} \mathrm{diag}(1,-2,0.1) & \mathrm{diag}(0.5,0.4,0.2) \end{bmatrix}, \qquad B_S = 0.1 \begin{bmatrix} I_2 \\ 0_{1\times 2} \end{bmatrix},$$

$$C_T = \begin{bmatrix} I_2 & 0_{2\times 4} \end{bmatrix}, \qquad C_S = 0_{2\times 3}.$$

The subscripts T and S are used to distinguish the temporal building blocks from the spatial building blocks, respectively. For instance, for subsystem $G^{(1)}$ and all $t \in \mathbb{N}_0$, the following hold:

$$A_{02}^{(1)}(t) = A_{03}^{(1)}(t) = A_{04}^{(1)}(t) = A_{TS}, \qquad B_0^{(1)}(t) = B_T,$$

$$B_2^{(1)}(t) = B_3^{(1)}(t) = B_5^{(1)}(t) = B_S, \qquad C_2^{(1)}(t) = C_3^{(1)}(t) = C_4^{(1)}(t) = C_S,$$

$$A_{20}^{(1)}(t) = A_{30}^{(1)}(t) = A_{50}^{(1)}(t) = A_{ST}, \qquad C_0^{(1)}(t) = C_T,$$

and so on. Finally, the building block $\mathcal{M}$ is given by $\mathcal{M} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$.

For even-numbered subsystems, the building blocks are given by

$$A_{SS} = 0_{3\times 3}, \qquad B_S = 0_{3\times 2},$$

$$C_S = \begin{bmatrix} -I_2 & 0_{2\times 1} \end{bmatrix}, \qquad A_{TS} = 0.1 \begin{bmatrix} \mathrm{diag}(-0.5, 0.1, -0.1) \\ \mathrm{diag}(-0.5, -0.2, 0.3) \end{bmatrix},$$

$$A_{ST} = \begin{bmatrix} 0.2I_3 & -0.03I_3 \end{bmatrix}, \qquad C_T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$B_T = 0.1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_{TT} = 0.1 \begin{bmatrix} 1 & -4 & -0.3 & 0.1 & 0.5 & 0.3 \\ 3 & -5 & 0.2 & 0.1 & -0.2 & 0.3 \\ 0.1 & -0.3 & -0.5 & 0.2 & 0.1 & 0.1 \\ -0.2 & 0 & 0 & -0.1 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0.15 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & -0.1 \end{bmatrix},$$

$$\mathcal{M} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

For all $k \in V$ and $t \in \mathbb{N}_0$, $D^{(k)}(t) = 0_{2\times 2}$. By applying Lemma 2.5, the $q = 28$ time-periodic distributed LTV system G thus formed is shown to be strongly stable, and an upper bound on $\|G\|$ is computed by solving the following SDP:

$$\text{minimize } \gamma^2 \text{ subject to: } P^{(k)}(t) \succ \mu I,\ \ P^{(ij)}(t) \succ \mu I,\ (2.4),\ P^{(k)}(q) = P^{(k)}(0), \text{ and}$$
$$P^{(ij)}(q) = P^{(ij)}(0) \text{ for some } \mu > 0 \text{ and all } t = 0, ..., q-1,\ k \in V, \text{ and } (i,j) \in E. \tag{2.14}$$

In the SDP defined in (2.14), μ is a chosen small positive quantity that ensures that the LMIs are strict. The result, $\gamma = 3.47$, helps in assessing the upper bound on $\|(G - G_r)\|$ and in choosing how many temporal and spatial state variables to truncate.

Then, q time-periodic generalized gramians $X^{(k)}(t)$, $X^{(ij)}(t)$ and $Y^{(k)}(t)$, $Y^{(ij)}(t)$ are computed by solving the following SDPs:

$$\text{minimize } \textstyle\sum_{t=0}^{q} \left(\sum_{k=1}^{5} \operatorname{trace} X^{(k)}(t) + \sum_{(i,j)\in E} \operatorname{trace} X^{(ij)}(t)\right) \text{ subject to:}$$
$$X^{(k)}(t) \succ \mu I,\ \ X^{(ij)}(t) \succ \mu I,\ (2.5),\ X^{(k)}(q) = X^{(k)}(0), \text{ and } X^{(ij)}(q) = X^{(ij)}(0),$$
$$\text{for some } \mu > 0 \text{ and all } t = 0, ..., q-1,\ k \in V, \text{ and } (i,j) \in E. \tag{2.15}$$
$$\text{minimize } \textstyle\sum_{t=0}^{q} \left(\sum_{k=1}^{5} \operatorname{trace} Y^{(k)}(t) + \sum_{(i,j)\in E} \operatorname{trace} Y^{(ij)}(t)\right) \text{ subject to:}$$
$$Y^{(k)}(t) \succ \mu I,\ \ Y^{(ij)}(t) \succ \mu I,\ (2.6),\ Y^{(k)}(q) = Y^{(k)}(0), \text{ and } Y^{(ij)}(q) = Y^{(ij)}(0)$$
$$\text{for some } \mu > 0 \text{ and all } t = 0, ..., q-1,\ k \in V, \text{ and } (i,j) \in E. \tag{2.16}$$

The computational complexity of these SDPs is determined by formulating the corresponding dual problems; see [6] for a detailed discussion. Let $N_I = 12$ be the number of interconnections and $n_u = 2$ be the number of inputs to each subsystem. For the SDP defined in (2.14), the dimension of the SDP variable is $q\,(N(2n_T + n_u) + 2N_I\, n_S) = 3976$, the dimension of the linear variable is 1, the number of SDP blocks is $q\,(2N + N_I) = 616$, and the number of constraints is $\frac{1}{2}q\left(N\,n_T\,(n_T+1)+N_I\,n_S\,(n_S+1)\right)+1\,=\,4957$. As for the SDPs defined in (2.15) and (2.16), respectively, the dimension of the SDP variable is $2\,q\,(N\,n_T+N_I\,n_S)=3696$, the number of SDP blocks is $q\,(2N+N_I)=616$, and the number of constraints is $\frac{1}{2}q\left(N\,n_T\,(n_T+1)+N_I\,n_S\,(n_S+1)\right)\,=\,4956$. Yalmip [58] is used to model these problems, and SDPT3 [79] is used to solve them. The computations are carried out in Matlab 7.10.0.499 on a Hewlett-Packard laptop with 2 Intel Cores, 2.30 GHz processors, and 4 GB of RAM running Windows 7. The most time consuming problem is the SDP defined in (2.14) with a corresponding total elapsed time of 30 seconds and a CPU time of 24 seconds.

Then, a q time-periodic balanced realization for system G is constructed using these generalized gramians by following the steps in Algorithm 2.7. To obtain useful error bounds, heuristics are employed as explained next. The reader is referred to Algorithm 5.8 for more details. Namely, the generalized Lyapunov inequalities (2.5) and (2.6) are expressed for the balanced realization of system G just constructed. Then, q time-periodic balanced generalized gramians are computed, i.e., diagonal matrices $\Sigma^{(k)}(t)\succeq\epsilon I$ and $\Sigma^{(ij)}(t)\succeq\epsilon I$ for all $k\in V$ and $(i,j)\in E$, such that the following objective function is minimized:

$$\sum_{t=0}^{27}\left(\sum_{k=1}^{5}\left\|\mathrm{vect}\left(\Sigma^{(k)}(t)-\epsilon I\right)\right\|_1+\sum_{(i,j)\in E}\left\|\mathrm{vect}\left(\Sigma^{(ij)}(t)-\epsilon I\right)\right\|_1\right)+a_1\times\epsilon,$$

where $\mathrm{vect}(M)$ is the vector formed by the diagonal entries of the matrix M, $\|v\|_1$ is the 1-norm of vector v, and $a_1=750$ is the weight given to ϵ in the cost function. This value of a_1 gives the best trade-off between the following two competing objectives. The first objective is to minimize the first term in the cost function, where the 1-norm is used as a heuristic for finding diagonal gramians with many entries equal to ϵ. This is done with the intention of truncating all the temporal and spatial state variables whose corresponding entries in the gramians are equal to ϵ. The second objective is to minimize ϵ, since $\|(G-G_r)\|<2\epsilon$

by Corollary 2.16. A value of $\epsilon = 0.034$ is obtained, i.e., $\|(G - G_r)\| < 2\%\gamma$. Figure 2.2 shows the first and second diagonal entries of $\Sigma^{(21)}(t)$ for $0 \leq t < 28$. This figure illustrates the usefulness of the proposed heuristic in forcing many entries in the balanced generalized gramians to be equal to ϵ. Namely, all the state variables with a corresponding value of ϵ in the balanced generalized gramians are truncated and are only accounted for once in the computed error bound. The dimension of the state vector $x^{(21)}(t)$ varies between 0 and 1 in the reduced order system G_r in contrast to 3 in the full order system G. As can be seen in Figure 2.3, between 14 and 18 temporal state variables and between 20 and 25 spatial state variables are truncated at each time-step. Clearly, truncation need not be uniform in time even if the dimensions of the states in the full order system are constants. The full order system G and the reduced order system G_r are simulated using the same set of applied inputs, and their responses are plotted in Figure 2.4. The inputs vary randomly between -10 and 10 for the first 100 time-steps and then are set equal to zero. As predicted by the small error bound, the responses of systems G and G_r are very close.

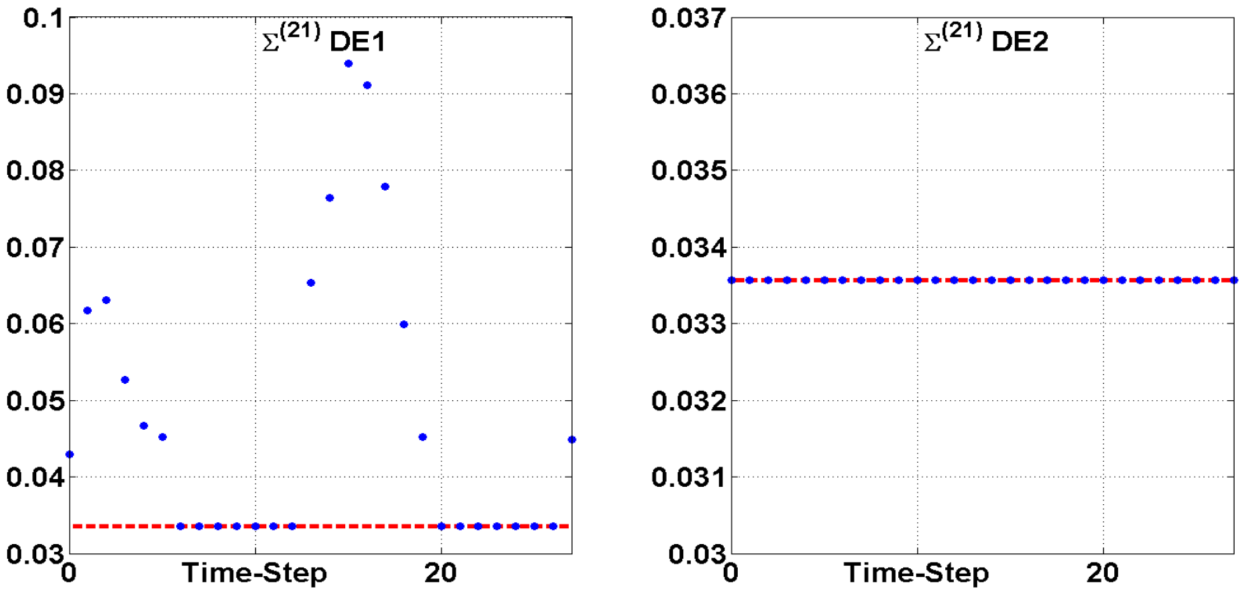

Figure 2.2: The figure plots the first and second diagonal entries (DE) of $\Sigma^{(21)}(t)$ for $t = 0, \ldots, 27$. The figure illustrates the usefulness of the heuristics detailed in Algorithm 5.8 in forcing many entries in the balanced generalized gramians to be equal to the truncation cut-off value shown using the red dashed line.

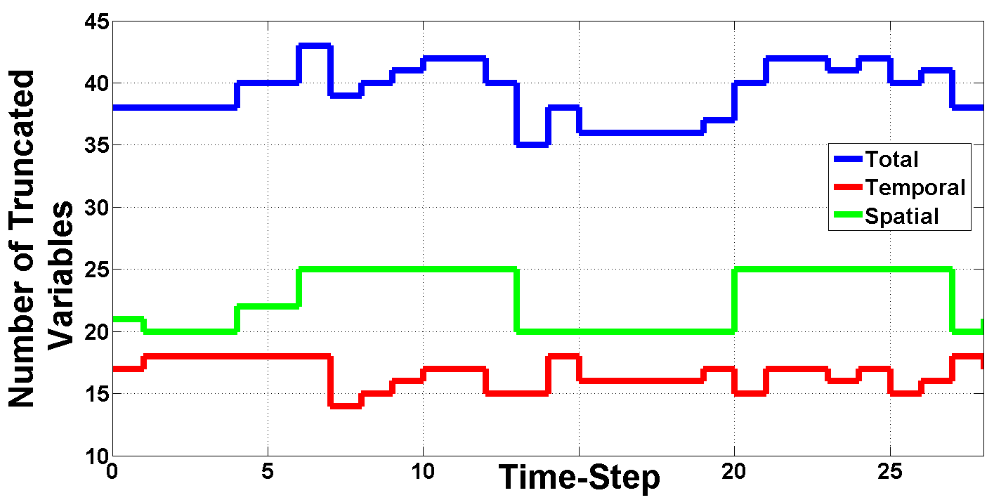

Figure 2.3: This plot shows the total number of truncated state variables from each type: temporal state variables and spatial state variables. This plot also shows that the truncation of state variables does not need to be uniform in time, i.e., different number of state variables can be truncated at different time-steps.

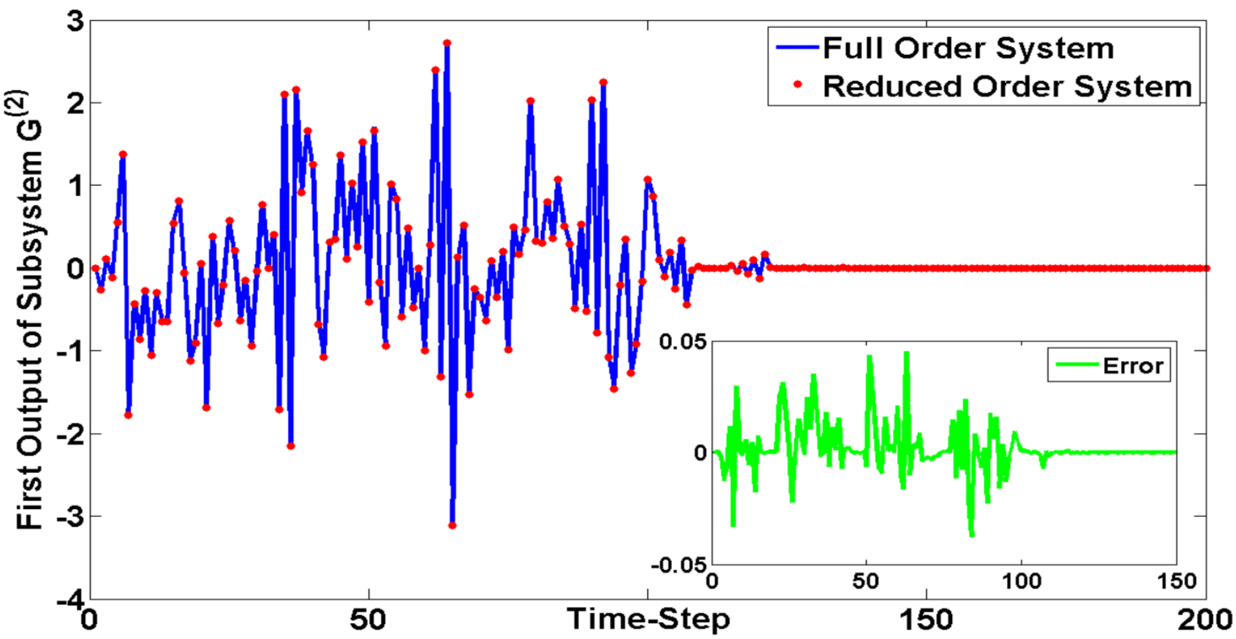

Figure 2.4: This figure plots the responses of the full order system G and the reduced order system G_r that results from the application of the balanced truncation method.

Chapter 3

Coprime Factors Reduction of Interconnected Linear Time-Varying Systems

3.1 Chapter Overview

This chapter applies the coprime factors reduction method for the model reduction of the interconnected LTV systems discussed in Chapter 2. Specifically, this chapter extends the balanced truncation method of Chapter 2 to systems that are not necessarily strongly stable, but are strongly stabilizable and strongly detectable. In Section 3.2, the notion of strong stabilizability is defined, and a convex characterization of strong stabilizability is given. In Section 3.3, the notion of a coprime factorization of a distributed LTV system is defined. It is then shown how to construct a strongly stable coprime factorization for a given strongly stabilizable and strongly detectable system. Section 3.4 details the coprime factors reduction method for interconnected LTV systems. Specifically, it is shown how to form an augmented strongly stable system from the coprime factorization computed in Section 3.3, and how this system is then reduced by the application of the balanced truncation method. This chapter concludes with an example in Section 3.5. This example showcases the novelty of the proposed model reduction methods in allowing for the simplification of the interconnection structure. Namely, it is shown how the proposed methods can be used to quantify the importance of each interconnection for the interconnected system, and how to truncate the interconnections that are deemed negligible.

3.2 Strong Stabilizability

This section starts with the definition of strong stabilizability. Then, a Lyapunov-like test is given for checking whether a given distributed LTV system is strongly stabilizable or not. This result generalizes its counterpart for uncertain systems given in [59].

Definition 3.1. A well-posed distributed LTV system G that has a realization denoted by $(A^{(k)}(t), B^{(k)}(t),\ C^{(k)}(t), D^{(k)}(t))$ is said to be strongly stabilizable if there exist uniformly bounded, $n_u^{(k)}(t) \times \left(n^{(k)}(t) + \sum_{i \in E_{\text{in}}^{(k)}} n^{(ik)}(t)\right)$ matrix-valued functions $F^{(k)}(t)$ for all $k \in V$ such that the resulting closed-loop system is strongly stable. In other words, system G is well-posed if there exist a positive scalar $\mu > 0$, uniformly bounded matrix-valued functions $F^{(k)}(t)$, and uniformly bounded and positive definite matrix-valued functions $P^{(k)}(t) \succ \mu I$ and $P^{(ij)}(t) \succ \mu I$ that satisfy

$$\begin{aligned}\left(A^{(k)}(t) + B^{(k)}(t)F^{(k)}(t)\right) \operatorname{diag}\left(P^{(k)}(t), P_{\text{in}}^{(k)}(t)\right) \left(A^{(k)}(t) + B^{(k)}(t)F^{(k)}(t)\right)^T \\ - \operatorname{diag}\left(P^{(k)}(t+1), P_{\text{out}}^{(k)}(t+1)\right) \prec -\mu I, \quad (3.1)\end{aligned}$$

for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$, where $P_{\text{in}}^{(k)}(t) = \operatorname{diag}\left(P^{(ik)}(t)\right)_{i \in E_{\text{in}}^{(k)}}$ and $P_{\text{out}}^{(k)}(t) = \operatorname{diag}\left(P^{(kj)}(t)\right)_{j \in E_{\text{out}}^{(k)}}$.

In a similar way, a well-posed distributed LTV system G that has a realization denoted by $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t),\ D^{(k)}(t))$ is said to be strongly detectable if there exist uniformly bounded matrix-valued functions $K^{(k)}(t)$ for all $k \in V$, of appropriate dimensions, such that the resulting closed-loop system is strongly stable. In other words, system G is strongly detectable if there exist a positive scalar $\mu > 0$, uniformly bounded matrix-valued functions $K^{(k)}(t)$, and uniformly bounded and positive definite matrix-valued functions $Q^{(k)}(t) \succ \mu I$ and $Q^{(ij)}(t) \succ \mu I$ that satisfy

$$\left(A^{(k)}(t) + K^{(k)}(t)C^{(k)}(t)\right)^T \operatorname{diag}\left(Q^{(k)}(t+1), Q^{(k)}_{\text{out}}(t+1)\right)\left(A^{(k)}(t) + K^{(k)}(t)C^{(k)}(t)\right) \\ - \operatorname{diag}\left(Q^{(k)}(t), Q^{(k)}_{\text{in}}(t)\right) \prec -\mu I,$$

for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$, where $Q^{(k)}_{\text{in}}(t) = \operatorname{diag}\left(Q^{(ik)}(t)\right)_{i \in E^{(k)}_{\text{in}}}$ and $Q^{(k)}_{\text{out}}(t) = \operatorname{diag}\left(Q^{(kj)}(t)\right)_{j \in E^{(k)}_{\text{out}}}$.

Theorem 3.2. *Consider a well-posed distributed LTV system G that has a realization denoted by $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$. System G is strongly stabilizable if and only if there exist a positive scalar $\mu > 0$ and uniformly bounded and positive definite matrix-valued functions $P^{(k)}(t) \succ \mu I$ and $P^{(ij)}(t) \succ \mu I$ that satisfy*

$$A^{(k)}(t) \operatorname{diag}\left(P^{(k)}(t), P^{(k)}_{\text{in}}(t)\right)(A^{(k)}(t))^T - \operatorname{diag}\left(P^{(k)}(t+1), P^{(k)}_{\text{out}}(t+1)\right) \\ - B^{(k)}(t)(B^{(k)}(t))^T \prec -\mu I, \quad (3.2)$$

for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$. Furthermore, provided that the given quantities are well-defined and uniformly bounded for all $(t,k) \in \mathbb{N}_0 \times V$, the feedback gains $F^{(k)}(t)$ can be chosen as follows in order to render the closed-loop system strongly stable:

$$F^{(k)}(t) = -\left((B^{(k)}(t))^T \operatorname{diag}\left(P^{(k)}(t+1), P^{(k)}_{\text{out}}(t+1)\right)^{-1} B^{(k)}(t)\right)^{-1} \\ \times (B^{(k)}(t))^T \operatorname{diag}\left(P^{(k)}(t+1), P^{(k)}_{\text{out}}(t+1)\right)^{-1} A^{(k)}(t).$$

Proof. Without loss of generality, it is assumed that $\operatorname{rank} B^{(k)}(t) = n_u^{(k)}(t) < n^{(k)}(t+1) + \sum_{j\in E_{\text{out}}^{(k)}} n^{(kj)}(t+1)$ for all $(t,k) \in \mathbb{N}_0 \times V$. Namely, if $\operatorname{rank} B^{(k)}(t) < n_u^{(k)}(t)$, then there exist redundant controls that can be easily removed. As for the case of a square and nonsingular $B^{(k)}(t)$, the proof follows immediately from the proof given here. Then, for all $k \in V$, there exist uniformly bounded matrix-valued functions $B_\perp^{(k)}(t)$ such that, for all $t \in \mathbb{N}_0$, $(B_\perp^{(k)}(t))^T B_\perp^{(k)}(t) = I$, $(B^{(k)}(t))^T B_\perp^{(k)}(t) = 0$, and the inverses of $\begin{bmatrix} B^{(k)}(t) & B_\perp^{(k)}(t) \end{bmatrix}$ exist and are uniformly bounded. By Definition 3.1, system G is strongly stabilizable if and only if there exist a positive scalar $\mu > 0$, uniformly bounded matrix-valued functions $F^{(k)}(t)$, and uniformly bounded and positive definite matrix-valued functions $P^{(k)}(t) \succ \mu I$ and $P^{(ij)}(t) \succ \mu I$ that satisfy (3.1) for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$. By applying the Schur complement formula to (3.1), the following inequality is obtained:

$$\overbrace{\begin{bmatrix} -\operatorname{diag}\left(P^{(k)}(t), P_{\text{in}}^{(k)}(t)\right)^{-1} & (A^{(k)}(t))^T \\ A^{(k)}(t) & -\operatorname{diag}\left(P^{(k)}(t+1), P_{\text{out}}^{(k)}(t+1)\right) + \mu I \end{bmatrix}}^{\Psi^{(k)}(t)} + \begin{bmatrix} 0 \\ B^{(k)}(t) \end{bmatrix} \underbrace{F^{(k)}(t)}_{\Theta^{(k)}(t)} \underbrace{\begin{bmatrix} I & 0 \end{bmatrix}}_{M^{(k)}(t)} + \begin{bmatrix} I \\ 0 \end{bmatrix} (F^{(k)}(t))^T \underbrace{\begin{bmatrix} 0 & (B^{(k)}(t))^T \end{bmatrix}}_{N^{(k)}(t)} \prec 0,$$

for all $(t,k) \in \mathbb{N}_0 \times V$. By [39], the previous inequality has a solution $\Theta^{(k)}(t)$ for each $(t,k) \in \mathbb{N}_0 \times V$ if and only if $(W_M^{(k)}(t))^T \Psi^{(k)}(t) W_M^{(k)}(t) \prec 0$ and $(W_N^{(k)}(t))^T \Psi^{(k)}(t) W_N^{(k)}(t) \prec 0$ for any matrices $W_M^{(k)}(t)$ and $W_N^{(k)}(t)$ whose columns form bases for the null spaces of $M^{(k)}(t)$ and $N^{(k)}(t)$, respectively. In this proof, $W_M^{(k)}(t)$ and $W_N^{(k)}(t)$ are chosen as $W_M^{(k)}(t) = \begin{bmatrix} 0 & I \end{bmatrix}^T$ and $W_N^{(k)}(t) = \operatorname{diag}(I, B_\perp^{(k)}(t))$, respectively. The condition $(W_M^{(k)}(t))^T \Psi^{(k)}(t) W_M^{(k)}(t) \prec 0$ is trivially satisfied for all $(t,k) \in \mathbb{N}_0 \times V$, and so the discussion is restricted to the condition $(W_N^{(k)}(t))^T \Psi^{(k)}(t) W_N^{(k)}(t) \prec 0$. By applying the Schur complement formula to this condition, the following inequality is obtained:

$$(B_{\perp}^{(k)}(t))^T \left(-\mathrm{diag}\left(P^{(k)}(t+1), P_{\text{out}}^{(k)}(t+1)\right) + \mu I \right.$$
$$\left. + A^{(k)}(t)\,\mathrm{diag}\left(P^{(k)}(t), P_{\text{in}}^{(k)}(t)\right)(A^{(k)}(t))^T\right) B_{\perp}^{(k)}(t) \prec 0,$$

for all $(t,k) \in \mathbb{N}_0 \times V$, which is equivalent to (3.2) by Finsler's Lemma.

It remains to be shown that the given choices for $F^{(k)}(t)$ render the closed-loop system strongly stable. First, it is noted that if the matrix-valued functions $(B^{(k)}(t))^T B^{(k)}(t)$ are uniformly invertible, then matrix-valued functions $F^{(k)}(t)$ are well-defined and uniformly bounded. This can be ensured by removing all redundant controls and properly perturbing the matrix-valued functions $B^{(k)}(t)$, if necessary, to ensure that the aforementioned product has a uniform full rank. The Schur complement formula is now applied twice to (3.2). For simplicity, the $-\mu I$ term in (3.2) is provisionally set to zero when applying the Schur complement formula; however, it is stressed that the resulting sequences of matrices on the left-hand-side are uniformly negative definite. The following inequality is obtained:

$$(A^{(k)}(t))^T \left(\mathrm{diag}\left(P^{(k)}(t+1), P_{\text{out}}^{(k)}(t+1)\right) + B^{(k)}(t)(B^{(k)}(t))^T\right)^{-1} A^{(k)}(t)$$
$$- \mathrm{diag}\left(P^{(k)}(t), P_{\text{in}}^{(k)}(t)\right)^{-1} \prec 0,$$

for all $(t,k) \in \mathbb{N}_0 \times V$. Using the matrix inversion lemma, it can then be verified that the following inequality holds for all $(t,k) \in \mathbb{N}_0 \times V$:

$$\left(A^{(k)}(t) + B^{(k)}(t)F^{(k)}(t)\right)^T \mathrm{diag}\left(P^{(k)}(t+1), P_{\text{out}}^{(k)}(t+1)\right)^{-1} \left(A^{(k)}(t) + B^{(k)}(t)F^{(k)}(t)\right)$$
$$- \mathrm{diag}\left(P^{(k)}(t), P_{\text{in}}^{(k)}(t)\right)^{-1} \prec 0.$$

By applying the Schur complement formula twice to the previous inequality, (3.1) is retrieved, which concludes the proof. □

A result similar to Theorem 3.2 can also be derived for checking if a given well-posed distributed LTV system G is strongly detectable.

3.3 Coprime Factorizations

In the following, the notion of a right coprime factorization (RCF) is extended to the class of distributed LTV systems. It is shown that a strongly stabilizable and strongly detectable distributed LTV system admits a strongly stable RCF. It is useful to recall at this point that a distributed LTV system G can be viewed as a map from $u = \left(u^{(1)}, \ldots, u^{(N)}\right)$ to $y = \left(y^{(1)}, \ldots, y^{(N)}\right)$, where $u^{(k)}$ and $y^{(k)}$ are the input to and the output from subsystem $G^{(k)}$, respectively.

Definition 3.3. Two linear causal mappings on ℓ_2, M and N, are said to be right coprime if there exist two linear causal mappings on ℓ_2, X and Y, such that $YM + XN = I$, where I is the identity map on ℓ_2.

Definition 3.4. The pair (N, M) of stable distributed LTV systems is said to be an RCF for the distributed LTV system G if M has a causal inverse on ℓ, M and N are right coprime, and $G = NM^{-1}$.

Lemma 3.5. *Consider a strongly stabilizable and strongly detectable distributed LTV system G that has a realization denoted by $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$. Then, system G admits an RCF (N, M), where the distributed LTV systems N and M are strongly stable and have the realizations $(A^{(k)}(t) + B^{(k)}(t)F^{(k)}(t), B^{(k)}(t), C^{(k)}(t) + D^{(k)}(t)F^{(k)}(t),\ D^{(k)}(t))$ and*

$(A^{(k)}(t) + B^{(k)}(t)F^{(k)}(t), B^{(k)}(t), F^{(k)}(t), I)$, *respectively, where* $F^{(k)}(t)$ *are any uniformly bounded feedback gains that render the resulting closed-loop system strongly stable.*

Proof. By construction, the distributed LTV systems N and M are strongly stable and define linear causal mappings on ℓ_2. To show that system M has a causal inverse on ℓ, consider the distributed LTV system R that has a realization denoted by $(A^{(k)}(t), B^{(k)}(t), -F^{(k)}(t), I)$. As per Definition 2.1, system R is well-posed and defines a linear causal mapping on ℓ, since throughout Chapters 2 and 3 the system equations are defined for time-steps $t \in \mathbb{N}_0$, i.e., the state-space matrices are zeros for $t < 0$. So, it is sufficient to show that $MR = RM = I$, where I is the identity operator on ℓ. The proof of $MR = I$ is given, and the proof of $RM = I$ follows similarly. Namely, the state-space equations for systems M and R are written similarly to the equations in (2.1) with the appropriate subscripts M and R, and it is assumed that $u_M \equiv y_R$. Then, for all $(t, k) \in \mathbb{N}_0 \times V$, the following equations hold:

$$\begin{bmatrix} x_M^{(k)}(t+1) \\ x_{M,\text{out}}^{(k)}(t+1) \end{bmatrix} - \begin{bmatrix} x_R^{(k)}(t+1) \\ x_{R,\text{out}}^{(k)}(t+1) \end{bmatrix} = \left(A^{(k)}(t) + B^{(k)}(t)F^{(k)}(t)\right) \left(\begin{bmatrix} x_M^{(k)}(t) \\ x_{M,\text{in}}^{(k)}(t) \end{bmatrix} - \begin{bmatrix} x_R^{(k)}(t) \\ x_{R,\text{in}}^{(k)}(t) \end{bmatrix} \right),$$

$$y_M^{(k)}(t) = F^{(k)}(t) \left(\begin{bmatrix} x_M^{(k)}(t) \\ x_{M,\text{in}}^{(k)}(t) \end{bmatrix} - \begin{bmatrix} x_R^{(k)}(t) \\ x_{R,\text{in}}^{(k)}(t) \end{bmatrix} \right) + u_R^{(k)}(t).$$

Since systems M and R start from zero initial conditions, i.e., $x_M^{(k)}(0) = 0$, $x_M^{(ij)}(0) = 0$, $x_R^{(k)}(0) = 0$, and $x_R^{(ij)}(0) = 0$ for all $k \in V$ and $(i, j) \in E$, it follows from the first equation that $x_M^{(k)}(t) = 0$, $x_M^{(ij)}(t) = 0$, $x_R^{(k)}(t) = 0$, and $x_R^{(ij)}(t) = 0$ for all $(t, k) \in \mathbb{N}_0 \times V$ and $(i, j) \in E$. Moreover, the second equation simplifies to $y_M^{(k)}(t) = u_R^{(k)}(t)$ for all $(t, k) \in \mathbb{N}_0 \times V$, i.e., $MR = I$. It is then shown in a similar way that $G = NM^{-1} = NR$. Specifically, the state-space equations for systems G, N, and R are written similarly to the equations in (2.1) with the appropriate subscripts G, N, and R. Then, assuming that $u_N \equiv y_R$ and $u_R \equiv u_G$, it is concluded that $y_G \equiv y_N$. The final step is to prove that the mappings N and M are right coprime. To this end, the strongly stable distributed LTV systems Y and X are defined with the state-space realizations $(A^{(k)}(t)+K^{(k)}(t)C^{(k)}(t), B^{(k)}(t)+K^{(k)}(t)D^{(k)}(t), -F^{(k)}(t), I)$ and $(A^{(k)}(t)+K^{(k)}(t)C^{(k)}(t), K^{(k)}(t), F^{(k)}(t), 0)$, respectively. Indeed, since system G is strongly detectable, there exist uniformly bounded observer gains $K^{(k)}(t)$ such that X and Y are strongly stable systems and define bounded linear causal mappings. To prove that $YM + XN = I$, the state-space equations of the systems $YM : u \to y_1$ and $XN : u \to y_2$ are written, and it is shown that $y = u$, where $y = y_1 + y_2$. For all $(t,k) \in \mathbb{N}_0 \times V$, the relevant equations are as follows:

$$
\begin{aligned}
y_1^{(k)}(t) &= -F^{(k)}(t)\begin{bmatrix} x_Y^{(k)}(t) \\ x_{Y,\text{in}}^{(k)}(t) \end{bmatrix} + F^{(k)}(t)\begin{bmatrix} x_M^{(k)}(t) \\ x_{M,\text{in}}^{(k)}(t) \end{bmatrix} + u^{(k)}(t), \\
y_2^{(k)}(t) &= F^{(k)}(t)\begin{bmatrix} x_X^{(k)}(t) \\ x_{X,\text{in}}^{(k)}(t) \end{bmatrix}, \\
y^{(k)}(t) &= F^{(k)}(t)\left(\begin{bmatrix} x_M^{(k)}(t) \\ x_{M,\text{in}}^{(k)}(t) \end{bmatrix} - \begin{bmatrix} x_Y^{(k)}(t) \\ x_{Y,\text{in}}^{(k)}(t) \end{bmatrix} + \begin{bmatrix} x_X^{(k)}(t) \\ x_{X,\text{in}}^{(k)}(t) \end{bmatrix}\right) + u^{(k)}(t).
\end{aligned}
$$

By an argument similar to the one used in the proof of $MR = I$, it can be shown that $x_M^{(k)}(t) = x_N^{(k)}(t)$, $x_M^{(ij)}(t) = x_N^{(ij)}(t)$, and

$$
\begin{aligned}
&\begin{bmatrix} x_M^{(k)}(t+1) \\ x_{M,\text{out}}^{(k)}(t+1) \end{bmatrix} - \begin{bmatrix} x_Y^{(k)}(t+1) \\ x_{Y,\text{out}}^{(k)}(t+1) \end{bmatrix} + \begin{bmatrix} x_X^{(k)}(t+1) \\ x_{X,\text{out}}^{(k)}(t+1) \end{bmatrix} = \\
&\qquad \left(A^{(k)}(t)+K^{(k)}(t)C^{(k)}(t)\right)\left(\begin{bmatrix} x_M^{(k)}(t) \\ x_{M,\text{in}}^{(k)}(t) \end{bmatrix} - \begin{bmatrix} x_Y^{(k)}(t) \\ x_{Y,\text{in}}^{(k)}(t) \end{bmatrix} + \begin{bmatrix} x_X^{(k)}(t) \\ x_{X,\text{in}}^{(k)}(t) \end{bmatrix}\right),
\end{aligned}
$$

for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$. Since the initial conditions of systems M, N, X, and

Y are zeros, then the quantity in parentheses remains equal to zero for all time-steps, which implies that $y \equiv u$ and concludes the proof. □

3.4 Coprime Factors Model Reduction Algorithm

This section presents the coprime factors model reduction method for distributed LTV systems. An outline of the method is first given, then the detailed steps are listed in Algorithm 3.6. Let G be a strongly stabilizable and strongly detectable distributed LTV system, and consider a strongly stable RCF (N, M) of system G. Such an RCF is guaranteed to exist by Lemma 3.5. From the RCF (N, M), the augmented strongly stable system H is constructed such that $H = \begin{bmatrix} N \\ M \end{bmatrix}$. Since H is a strongly stable system, then it is reduced by the application of the balanced truncation method of Chapter 2. Denote the resulting reduced order system by $H_r = \begin{bmatrix} N_r \\ M_r \end{bmatrix}$. The distributed LTV systems N_r and M_r that are defined from H_r are strongly stable and right coprime. Moreover, (N_r, M_r) forms an RCF for the reduced order system G_r that approximates the full order system G, i.e., $G_r = N_r M_r^{-1}$. For distributed LTV systems, the reduced order system G_r is always well-posed, since all the state-space matrices are assumed to be zeros for $t < 0$, and the information transfer between the subsystems is subjected to a delay of one time-step. Theorems 2.10 and 2.12 guarantee upper bounds on $\|(H - H_r)\|$ that can be used as guidelines for the reduction process.

Algorithm 3.6. Consider a strongly stabilizable and strongly detectable distributed LTV system G that has a realization denoted by $(A^{(k)}(t), B^{(k)}(t), C^{(k)}(t), D^{(k)}(t))$. This algorithm shows how to reduce system G by the application of the coprime factors reduction method. The reduced order system is denoted by G_r, and its realization is given by $(A_r^{(k)}(t), B_r^{(k)}(t), C_r^{(k)}(t), D^{(k)}(t))$.

1. Solve for uniformly bounded and positive definite matrix-valued functions $P^{(k)}(t) \succ \mu I$ and $P^{(ij)}(t) \succ \mu I$ that satisfy (3.2) for all $(t, k) \in \mathbb{N}_0 \times V$ and $(i, j) \in E$.

2. For all $(t, k) \in \mathbb{N}_0 \times V$, define the feedback gains $F^{(k)}(t)$ as in Theorem 3.2.

- Remove all control redundancies and properly perturb the product $(B^{(k)}(t))^T B^{(k)}(t)$ when needed to ensure that the feedback gains are well-defined and uniformly bounded.

3. Construct the augmented strongly stable system $H = \begin{bmatrix} N \\ M \end{bmatrix}$.

- System H has a realization denoted by $\left(A_H^{(k)}(t), B_H^{(k)}(t), C_H^{(k)}(t), D_H^{(k)}(t)\right)$, where

$$A_H^{(k)}(t) = A^{(k)}(t) + B^{(k)}(t)F^{(k)}(t), \qquad B_H^{(k)}(t) = B^{(k)}(t),$$
$$C_H^{(k)}(t) = \begin{bmatrix} C^{(k)}(t) + D^{(k)}(t)F^{(k)}(t) \\ F^{(k)}(t) \end{bmatrix}, \qquad D_H^{(k)}(t) = \begin{bmatrix} D^{(k)}(t) \\ I \end{bmatrix}.$$

4. Solve for the generalized gramians of system H that have the minimum sum of traces as follows:

a) minimize $\sum_{t\in\mathbb{N}_0} \left(\sum_{k\in V} \operatorname{trace} X^{(k)}(t) + \sum_{(i,j)\in E} \operatorname{trace} X^{(ij)}(t)\right)$ subject to:

$X^{(k)}(t) \succ \mu I$, $X^{(ij)}(t) \succ \mu I$, and

$$A_H^{(k)}(t) \operatorname{diag}\left(X^{(k)}(t), X_{\text{in}}^{(k)}(t)\right) \left(A_H^{(k)}(t)\right)^T - \operatorname{diag}\left(X^{(k)}(t+1), X_{\text{out}}^{(k)}(t+1)\right) + B_H^{(k)}(t) \left(B_H^{(k)}(t)\right)^T \prec -\mu I,$$

for some $\mu > 0$ and all $(t, k) \in \mathbb{N}_0 \times V$ and $(i, j) \in E$.

b) minimize $\sum_{t\in\mathbb{N}_0} \left(\sum_{k\in V} \operatorname{trace} Y^{(k)}(t) + \sum_{(i,j)\in E} \operatorname{trace} Y^{(ij)}(t)\right)$ subject to:

$Y^{(k)}(t) \succ \mu I$, $Y^{(ij)}(t) \succ \mu I$, and

$$\left(A_H^{(k)}(t)\right)^T \operatorname{diag}\left(Y^{(k)}(t+1), Y_{\text{out}}^{(k)}(t+1)\right) A_H^{(k)}(t) - \operatorname{diag}\left(Y^{(k)}(t), Y_{\text{in}}^{(k)}(t)\right) + \left(C_H^{(k)}(t)\right)^T C_H^{(k)}(t) \prec -\mu I,$$

for some $\mu > 0$ and all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$.

- Alternatively follow the heuristic discussed in Algorithm 5.8.

5. Apply Algorithm 2.7 to obtain a balanced realization for system H and balanced generalized gramians $\Sigma^{(k)}(t)$ and $\Sigma^{(ij)}(t)$ for all $(t,k) \in \mathbb{N}_0 \times V$ and $(i,j) \in E$.

6. Apply the balanced truncation method of Section 2.5 to reduce system H.

- See Sections 3.5 and 6.7 for discussions on choosing the temporal and spatial state variables to be truncated.

- Denote the reduced order system by $H_r = \begin{bmatrix} N_r \\ M_r \end{bmatrix}$.

- Compute an upper bound on $\|(H - H_r)\|$ using Theorems 2.10 and 2.12.

- Denote the realization of system H_r by $\left(A_{H,r}^{(k)}(t), B_{H,r}^{(k)}(t), C_{H,r}^{(k)}(t), D_H^{(k)}(t)\right)$.

7. Define the state-space matrices of system G_r and the reduced order feedback gains $F_r^{(k)}(t)$ from the following equations:

$$A_{H,r}^{(k)}(t) = A_r^{(k)}(t) + B_r^{(k)}(t) F_r^{(k)}(t), \qquad B_{H,r}^{(k)}(t) = B_r^{(k)}(t),$$

$$C_{H,r}^{(k)}(t) = \begin{bmatrix} C_r^{(k)}(t) + D^{(k)}(t) F_r^{(k)}(t) \\ F_r^{(k)}(t) \end{bmatrix}.$$

- The feedback gains $F_r^{(k)}(t)$ strongly stabilize the reduced order system G_r.

- N_r and M_r are strongly stable systems with respective realizations

$$\begin{aligned} N_r: &\quad \left(A_r^{(k)}(t)+B_r^{(k)}(t)F_r^{(k)}(t), B_r^{(k)}(t), C_r^{(k)}(t)+D^{(k)}(t)F_r^{(k)}(t), D^{(k)}(t)\right), \\ M_r: &\quad \left(A_r^{(k)}(t)+B_r^{(k)}(t)F_r^{(k)}(t), B_r^{(k)}(t), F_r^{(k)}(t), I\right). \end{aligned}$$

- (N_r, M_r) is a strongly stable RCF for the reduced order system G_r.

3.5 Illustrative Example

In this section, the coprime factors reduction method is applied to an illustrative example. The purpose of this example is to showcase an important novel feature of the proposed balanced truncation and coprime factors reduction methods. Specifically, the subsystems are chosen to have LTI models, and the focus is placed on evaluating the importance of each spatial state variable for the overall interconnected system. It is shown that the proposed methods allow for simplifying the interconnection structure of the distributed system: interconnections that are deemed negligible are removed from the interconnection structure during the reduction process.

Consider a distributed LTI system G formed by 5 agents that are interconnected as shown in Figure 2.1. Each subsystem is a discrete-time double integrator. The equations of the subsystems are given by

$$\begin{aligned} r_k\left((t+1)T\right) &= r_k(tT)+v_k(tT)T, \\ v_k\left((t+1)T\right) &= v_k(tT)+w_k(tT)T, \end{aligned}$$

where k is the subsystem index, t denotes the nonnegative discrete time-steps, $T = 0.05$ seconds is the sampling time, and r_k, v_k, w_k in $\mathbb{R}$ denote the position, velocity, and input of agent k, respectively. If an interconnection exists between agents i and j, i.e, $(i, j) \in E$, then agent i sends its position and velocity to agent j. Due to the assumed communication latency, the information is received by agent j at the following time-step. The input w_k to agent k includes the consensus control protocol [56] and the exogenous input d_k. Namely,

$$w_k(tT) = -p_{0k}\, v_k(tT) + p_{1k} \sum_{i \in E_{\text{in}}^{(k)}} a_{ik}\ (r_i(tT - T) - r_k(tT)) \\ + p_{2k} \sum_{i \in E_{\text{in}}^{(k)}} a_{ik}\ (v_i(tT - T) - v_k(tT)) + d_k(tT),$$

where p_{0k}, p_{1k}, p_{2k} are positive control gains, and $a_{ik} > 0$ is the weight associated with the edge $(i, k) \in E$. This control law is used to achieve consensus between the agents. All the weights of the edges are assumed to be equal to 1 except for the weights a_{15}, a_{43}, a_{23}, a_{32} that are set equal to 0.0001. The latter four weights are purposefully chosen to be very small compared to the remaining weights. This is done in order to demonstrate the ability of the proposed methods to 1) identify the corresponding interconnections, i.e., $(1, 5)$, $(4, 3)$, $(2, 3)$, and $(3, 2)$, as negligible for the overall interconnection structure; and 2) remove these negligible interconnections without introducing a significant error. Finally, the control gains are set as follows: $p_{01} = 15$, $p_{02} = p_{03} = p_{04} = p_{05} = 2$, $p_{13} = 1.5$, $p_{15} = p_{23} = p_{25} = 0.1$. The remaining control gains are equal to 1.

The problem is reformulated in the framework of Section 2.3. The temporal state of subsystem $G^{(k)}$ is defined as the position and velocity of agent k, i.e., $x^{(k)}(t) = (r_k(tT), v_k(tT))$ for all $k \in V$. The spatial state associated with interconnection (i, j) is defined as the position and velocity of the agent that corresponds to the source (node i) of the edge (i, j), i.e.,

$x^{(ij)}(t) = (r_i(tT), v_i(tT))$ for all $(i, j) \in E$. The input to each agent reduces to the exogenous input acting on it, i.e., $u^{(k)}(t) = d_k(tT)$ for all $k \in V$. This is because the other terms in the control law are expressed in terms of the temporal and spatial states. The measured outputs of each agent are its position and velocity, i.e., $y^{(k)}(t) = x^{(k)}(t) = (r_k(tT), v_k(tT))$ for all $k \in V$. Since the double integrator equations and the employed control law do not explicitly depend on t, the constructed distributed system is time-invariant. The state-space matrices of subsystem $G^{(k)}$ are denoted by $A^{(k)}$, $B^{(k)}$, $C^{(k)}$, and $D^{(k)}$, i.e., the dependence of the state-space matrices on t is dropped to stress that they are time-invariant. Note that $D^{(k)} = 0$ for all $k \in V$.

The distributed LTI system G thus formed is not strongly stable, and so cannot be reduced by the application of the balanced truncation method. However, system G is strongly stabilizable and strongly detectable. The structure-preserving coprime factors reduction method detailed in Algorithm 3.6 is thus used to reduce system G.

First, the following SDP is solved:

$$
\begin{aligned}
&\text{find} && P^{(k)} \text{ and } P^{(ij)} \\
&\text{subject to:} && P^{(k)} \succ 0, \;\; P^{(ij)} \succ 0, \\
& && P_{\text{in}}^{(k)} = \operatorname{diag}\left(P^{(ik)}\right)_{i \in E_{\text{in}}^{(k)}}, \;\; P_{\text{out}}^{(k)} = \operatorname{diag}\left(P^{(kj)}\right)_{j \in E_{\text{out}}^{(k)}}, \\
& && A^{(k)} \operatorname{diag}\left(P^{(k)}, P_{\text{in}}^{(k)}\right)(A^{(k)})^T - \operatorname{diag}\left(P^{(k)}, P_{\text{out}}^{(k)}\right) - B^{(k)}(B^{(k)})^T \prec 0, \\
& && \text{for all } k \in V \text{ and } (i, j) \in E.
\end{aligned}
$$

The sought time-invariant solutions $P^{(k)}$ and $P^{(ij)}$ to the above SDP are proved to exist using an argument similar to the one in the proof of Lemma 2.14. These solutions are then used to compute the feedback gains $F^{(k)}$ according to Theorem 3.2. For example,

$$F^{(5)} = -\left(\left(B^{(5)}\right)^T \operatorname{diag}\left(P^{(5)}, P^{(54)}\right)^{-1} B^{(5)}\right)^{-1} \left(B^{(5)}\right)^T \operatorname{diag}\left(P^{(5)}, P^{(54)}\right)^{-1} A^{(5)}.$$

Second, the augmented strongly stable system H is formed. The realization of system H is denoted by $\left(A_H^{(k)}, B_H^{(k)}, C_H^{(k)}, D_H^{(k)}\right)$ and is defined as follows:

$$A_H^{(k)} = A^{(k)} + B^{(k)}F^{(k)}, \qquad B_H^{(k)} = B^{(k)},$$

$$C_H^{(k)} = \begin{bmatrix} C^{(k)} + D^{(k)}F^{(k)} \\ F^{(k)} \end{bmatrix}, \qquad D_H^{(k)} = \begin{bmatrix} D^{(k)} \\ I \end{bmatrix},$$

where $D^{(k)} = 0$ as explained before.

Third, generalized controllability gramians $X^{(k)}$ and $X^{(ij)}$, and generalized observability gramians $Y^{(k)}$ and $Y^{(ij)}$ are computed for system H by solving the following SDPs, respectively:

$$\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in E} \operatorname{trace} X^{(ij)} \\
\text{subject to:} \quad & X^{(k)} \succ 0, \;\; X^{(ij)} \succ 0, \\
& X_{\text{in}}^{(k)} = \operatorname{diag}\left(X^{(ik)}\right)_{i\in E_{\text{in}}^{(k)}}, \;\; X_{\text{out}}^{(k)} = \operatorname{diag}\left(X^{(kj)}\right)_{j\in E_{\text{out}}^{(k)}}, \\
& A_H^{(k)} \operatorname{diag}\left(X^{(k)}, X_{\text{in}}^{(k)}\right) \left(A_H^{(k)}\right)^T - \operatorname{diag}\left(X^{(k)}, X_{\text{out}}^{(k)}\right) + B_H^{(k)} \left(B_H^{(k)}\right)^T \prec 0, \\
& \text{for all } k \in V \text{ and } (i,j) \in E.
\end{aligned}$$

$$\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in E} \operatorname{trace} Y^{(ij)} \\
\text{subject to:} \quad & Y^{(k)} \succ 0, \;\; Y^{(ij)} \succ 0, \\
& Y_{\text{in}}^{(k)} = \operatorname{diag}\left(Y^{(ik)}\right)_{i\in E_{\text{in}}^{(k)}}, \;\; Y_{\text{out}}^{(k)} = \operatorname{diag}\left(Y^{(kj)}\right)_{j\in E_{\text{out}}^{(k)}},
\end{aligned}$$

$$\left(A_H^{(k)}\right)^T \operatorname{diag}\left(Y^{(k)}, Y_{\text{out}}^{(k)}\right) A_H^{(k)} - \operatorname{diag}\left(Y^{(k)}, Y_{\text{in}}^{(k)}\right) + \left(C_H^{(k)}\right)^T C_H^{(k)} \prec 0,$$

for all $k \in V$ and $(i,j) \in E$.

In the above SDPs, the sum of the traces of the gramians associated with the spatial states is minimized, since the objective of the example is to evaluate the importance of the spatial state variables and to simplify the interconnection structure of the distributed system accordingly.

Fourth, using the computed generalized gramians, Algorithm 2.7 is implemented to obtain a balanced realization for system H and balanced generalized gramians $\Sigma^{(k)}$ and $\Sigma^{(ij)}$ for all $k \in V$ and $(i,j) \in E$. Then, the balanced truncation method of Chapter 2 is applied to reduce system H. At this point, the ℓ_2-induced norm of system H, $\|H\|$, is computed to help in assessing and quantifying the error bound predicted from Corollary 2.16. $\|H\| \approx 1.9985$ is computed in two steps. An upper bound on $\|H\|$ is found using Lemma 2.5, and a lower bound on $\|H\|$ is found by treating the distributed system H as a single global system with 5 inputs and 15 outputs, and applying the standard Kalman-Yakubovich-Popov (KYP) Lemma for discrete-time LTI systems. In this case, the structure of the distributed system is dropped/neglected, and so, the computed H_∞-norm of the (unstructured) global system gives a lower bound on the ℓ_2-induced norm of the distributed system. The lower and upper bounds are almost equal, and so the lower bound is taken as $\|H\|$.

Denote the reduced order system that results from reducing system H by H_r, and denote the realization of H_r by $\left(A_{H,r}^{(k)}, B_{H,r}^{(k)}, C_{H,r}^{(k)}, D_H^{(k)}\right)$. By looking at the values of the diagonal entries in the balanced generalized gramians and comparing their relative orders, and by computing the predicted bound on $\|(H - H_r)\|$, it is decided to eliminate the interconnections $(1,5)$, $(4,3)$, $(2,3)$, and $(3,2)$ altogether from the interconnection structure, and truncate the second state variable in the remaining spatial states, i.e., reduce the dimension of the remaining spatial states from 2 to 1 each. For example, consider the computed balanced generalized gramians $\Sigma^{(15)} = 10^{-3}\mathrm{diag}(0.1461, 0.0190)$, $\Sigma^{(25)} = \mathrm{diag}(0.0107, 0.0001)$, and $\Sigma^{(21)} = \mathrm{diag}(0.1856, 0.0001)$. One can immediately see that both entries of $\Sigma^{(15)}$, the second entry in $\Sigma^{(25)}$, and the second entry in $\Sigma^{(21)}$ are negligible compared to the other entries, and hence the corresponding state variables are to be truncated. When the negligible entries in all the gramians $\Sigma^{(ij)}$ are considered, Corollary 2.16 gives the following error bound $\|(H - H_r)\| < 0.0094 = 0.4712\,\%\|H\|$.

Finally, the state-space matrices of system G_r and the reduced order feedback gains $F_r^{(k)}$ are extracted from the state-space matrices of system H_r as follows:

$$B_r^{(k)} = B_{H,r}^{(k)}, \qquad \begin{bmatrix} C_r^{(k)} + D^{(k)} F_r^{(k)} \\ F_r^{(k)} \end{bmatrix} = C_{H,r}^{(k)},$$

$$A_r^{(k)} = A_{H,r}^{(k)} - B_r^{(k)} F_r^{(k)}.$$

Figure 3.1 shows the interconnection structure of the full order system and the interconnection structure of the reduced order system. A light blue color is used for the interconnections of the reduced order system to indicate that the corresponding spatial states are of reduced dimensions. It is also noted that some interconnections are completely removed from the interconnection structure during model reduction, and so, the interconnection structure of the reduced order system is simpler than the interconnection structure of the full order system.

The a priori bound on $\|(H - H_r)\|$ obtained from the coprime factors reduction method is negligible. To see if system G_r closely approximates system G, both systems are simulated using the same exogenous inputs. Namely, the systems are subjected to the same mix of sinusoidal and exponential exogenous inputs for 20 seconds, and then the systems are left to evolve on their own. The position of agent 1 in systems G and G_r is plotted in Figure 3.2. As can be seen from the figure, the error is negligible. However, the error in the position

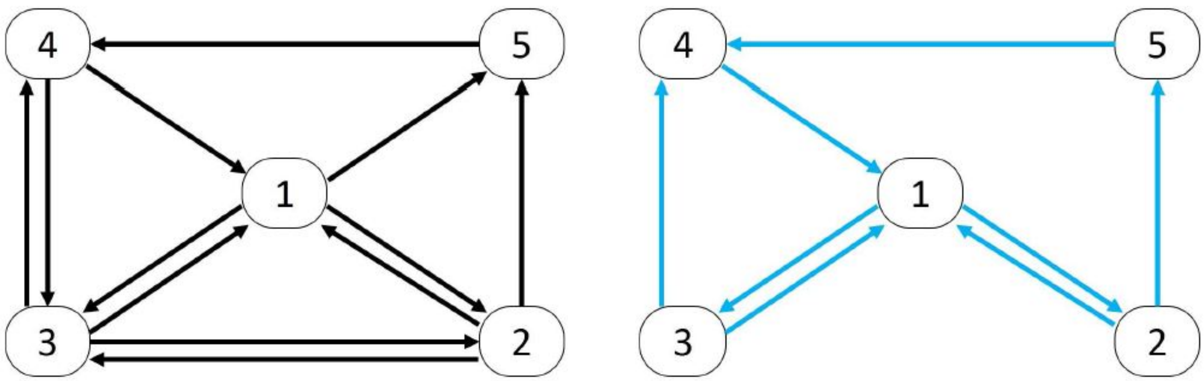

Figure 3.1: The figure on the left shows the interconnection structure of the full order system and the figure on the right shows the interconnection structure of the reduced order system. Note that some interconnections are completely removed from the interconnection structure during model reduction. The dimensions of the remaining spatial states in the reduced order system are also reduced, which is depicted by the use of the light blue color in the figure on the right.

between the agents of G and G_r slowly increases as time elapses because the agents of G_r do not reach consensus in position. They only reach a nonzero consensus in velocity, with a consensus value of the order 10^{-3}. So, while the full order system G is stable (though not strongly stable) in the sense that the zero-input response converges to a common state, system G_r is not. This remark does not contradict the theory of coprime factors model reduction, which only guarantees the strong stabilizability and strong detectability of the reduced order system G_r.

Finally, by looking at the first entry of $\Sigma^{(25)}$, namely, 0.0107, one may choose to remove interconnection $(2, 5)$ altogether from the interconnection structure when reducing system H, i.e., truncate the first variable of the balanced spatial state associated with the interconnection $(2, 5)$ in addition to the state variables truncated before. Denote the corresponding reduced order system by H_{r1}. In this case, the bound on $\|(H - H_{r1})\|$ remains negligible, namely, $1.5435\,\%\|H\|$. However, the response of the corresponding reduced order system G_{r1} will be significantly different from the response of system G. This is because, in this

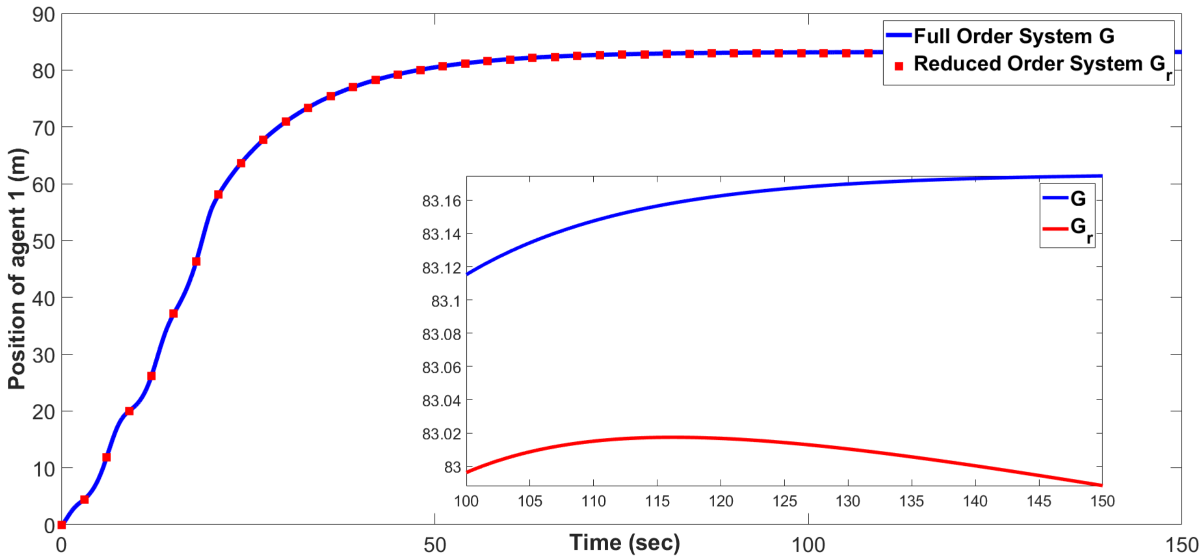

Figure 3.2: The figure plots the position of agent 1 in the full order system G and the reduced order system G_r.

case, there will no more be interconnections feeding into agent 5. However, if one groups the entries in the balanced generalized gramians according to their orders of magnitude, one sees that the first entry of $\Sigma^{(25)}$ does not belong to the group of the previously truncated entries. In fact, it is one order of magnitude larger. Also, the new error bound $(1.5435\,\%\|H\|)$, while still small, is 3 times larger than the previous bound $(0.4712\,\%\|H\|)$, i.e., truncating one additional state variable beyond what was truncated previously results in a jump in the error bound. Weighing in these observations, one concludes that the state variable associated with the first entry in $\Sigma^{(25)}$ should not be truncated.

Chapter 4

Distributed Control of Interconnected Nonstationary LPV Systems

4.1 Chapter Overview

This chapter deals with the distributed control problem for interconnected nonstationary linear parameter-varying (NSLPV) systems. Specifically, the interconnection structure of the plant is described by an arbitrary directed graph, and a communication latency of one time-step is assumed between the subsystems. The plant subsystems have discrete-time NSLPV models [33, 34] that are formulated in a linear fractional transformation (LFT) framework. NSLPV models consist of a nominal linear time-varying (LTV) part and a block-diagonal uncertainty operator in LFT form, where each block is a repeated, scalar, time-varying uncertainty. That is, NSLPV models extend standard linear parameter-varying (LPV) models formulated in an LFT framework in the sense that the nominal part of the model is time-varying. The problem addressed in this chapter is to design a distributed controller that inherits the interconnection structure and LFT structure of the plant. Namely, the interconnection structure of the controller is the same as that of the plant, and the controller subsystems have NSLPV models, are formulated in an LFT framework, and are scheduled by the same parameters as their corresponding plant subsystems.

The notation adopted throughout this chapter and Chapters 5 and 6 as well as some preliminary material are presented in Section 4.2. The operator theoretic framework that allows for a compact representation of interconnected NSLPV systems is developed in Section 4.3. Using the developed framework, and by following a parameter-independent Lyapunov function approach, analysis and synthesis results for interconnected NSLPV systems are derived in Sections 4.4 and 4.5, respectively. Section 4.6 briefly discusses how the analysis and synthesis problems become finite dimensional SDPs in the special case of eventually time-periodic subsystems that are interconnected over finite graphs. The proposed control synthesis approach is applied to an illustrative example in Section 4.7.

4.2 Preliminaries

Some of the notation of Chapters 2 and 3 carries over to Chapters 4, 5, and 6. The sets of nonnegative integers, integers, real numbers, and $n \times n$ symmetric matrices are denoted by $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{S}^n$, respectively. $0_{i\times j}$, 0_i, and I_i denote the $i \times j$ zero matrix, $i \times i$ zero matrix, and $i \times i$ identity matrix, respectively. $\operatorname{diag}(M_i)$ is the block-diagonal augmentation of the elements of the sequence of operators M_i.

The notation and operator theoretic machinery [35] needed to derive the results of Chapters 4, 5, and 6 are now presented. Consider a directed graph with a countable set of vertices V and a set of directed edges E. An element in E directed from $i \in V$ to $j \in V$ is denoted by the ordered pair (i, j). The vertex degree $v(k)$, i.e., the maximum between the indegree and the outdegree of vertex k, is assumed to be uniformly bounded. Without loss of generality, the directed graph under consideration is assumed to be d-regular, i.e., for each $k \in V$, the indegree and the outdegree are equal to d. This assumption is made because an arbitrary directed graph with a uniformly bounded vertex degree can be turned into a d-regular directed graph, where $d = \max_{k\in V} v(k)$, by the addition of the necessary virtual edges and/or nodes. With this assumption, d permutations, $\rho_1, \ldots, \rho_d$, of the set of vertices are defined such that if $(i, j) \in E$, then one $e \in \{1, \ldots, d\}$ satisfies $\rho_e(i) = j$ and $\rho_e^{-1}(j) = i$. The reader is referred to the work of [35] for more details about regular graphs, how to turn an arbitrary directed graph into a regular directed one, and how to appropriately define the permutations. As an example, Figure 4.1 shows a directed graph and the same graph rendered d = 2-regular after the addition of the needed virtual edges. The permutations ρ_1 and ρ_2 are defined as follows: $\rho_1(1) = 2$, $\rho_1(2) = 3$, $\rho_1(3) = 4$, $\rho_1(4) = 1$, $\rho_2(1) = 3$, $\rho_2(3) = 1$, $\rho_2(2) = 4$, and $\rho_2(4) = 2$.

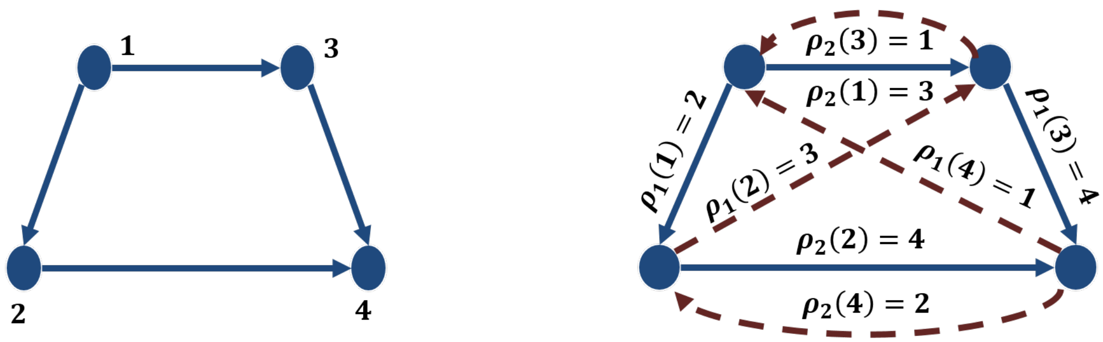

Figure 4.1: The figure on the left shows a direct graph. The figure on the right shows the same directed graph rendered 2-regular by the addition of the necessary virtual edges. The added virtual edges are shown as dashed red arrows. The permutations ρ_1 and ρ_2 are also defined in the figure.

$\operatorname{Im} T$ and $\ker T$ denote the image and the kernel of a linear operator T. $J_1 \oplus J_2$ denotes the vector space direct sum of the vector spaces J_1 and J_2. Let H and F be Hilbert spaces. The inner product and norm associated with H are denoted by $\langle \cdot, \cdot \rangle_H$ and $\| \cdot \|_H$, respectively. The subscript is suppressed when H is clear from context. The spaces of bounded linear operators and bounded linear causal operators mapping H to F are denoted by $\mathcal{L}(H, F)$ and $\mathcal{L}_c(H, F)$, respectively. The symbols simplify to $\mathcal{L}(H)$ and $\mathcal{L}_c(H)$ when $H = F$. For $X \in \mathcal{L}(H, F)$, $\|X\|$ refers to the H to F induced norm of X, and X^* denotes the adjoint of X. A self-adjoint operator $X \in \mathcal{L}(H)$ is said to be negative definite, $X \prec 0$, if there exists $\mu > 0$ such that $\langle x, Xx \rangle < -\mu \|x\|^2$ for all nonzero $x \in H$.

Given the sequence $n : (t,k) \in \mathbb{Z} \times V \to n(t,k) \in \mathbb{N}_0$, $\ell\left(\{\mathbb{R}^{n(t,k)}\}\right)$ is defined as the vector space of mappings $w : (t,k) \in \mathbb{Z} \times V \to w(t,k) \in \mathbb{R}^{n(t,k)}$. The Hilbert space $\ell_2\left(\{\mathbb{R}^{n(t,k)}\}\right)$ is the subspace of $\ell\left(\{\mathbb{R}^{n(t,k)}\}\right)$ consisting of mappings w with finite norm $\|w\|_{\ell_2} = \sqrt{\sum_{(t,k)\in\mathbb{Z}\times V} w^*(t,k)w(t,k)}$. $\ell_{2e}\left(\{\mathbb{R}^{n(t,k)}\}\right)$ is the subspace of $\ell\left(\{\mathbb{R}^{n(t,k)}\}\right)$ consisting of mappings w that satisfy $\sum_{k\in V} w(t,k)^* w(t,k) < \infty$ for each $t \in \mathbb{Z}$. The symbols ℓ, ℓ_2, and ℓ_{2e} are used when the dimensions are clear from context. Given $\overline{n} = (n_1, \ldots, n_f)$, where $n_1, \ldots, n_f$ are integer-valued sequences defined similarly to n, define $\ell^{\overline{n}} = \oplus_{i=1}^{f} \ell\left(\{\mathbb{R}^{n_i(t,k)}\}\right)$. Similar definitions apply for $\ell_2^{\overline{n}}$ and $\ell_{2e}^{\overline{n}}$.

An operator Q on ℓ_2 is said to be *graph-diagonal* if there exists a uniformly bounded matrix-valued sequence $Q(t,k)$ such that $(Qw)(t,k) = Q(t,k)w(t,k)$ for all $(t,k) \in \mathbb{Z} \times V$. An operator $W = [W_{ij}]$, where each W_{ij} is a constituent block of W, is said to be *partitioned graph-diagonal* if each block W_{ij} is a graph-diagonal operator. The graph-diagonal representation of a partitioned graph-diagonal operator W is denoted by $[\![W]\!]$ and satisfies $[\![W]\!](t,k) = [W_{ij}(t,k)]$ for all $(t,k) \in \mathbb{Z} \times V$. Specifically, the mapping $[\![\cdot]\!]$ is a homomorphism from the space of partitioned graph-diagonal operators to the space of graph-diagonal operators. This mapping is isometric and preserves products, addition, and ordering, i.e., $\|W_1\| = \|[\![W_1]\!]\|$, $[\![W_1 W_2]\!] = [\![W_1]\!][\![W_2]\!]$, $[\![W_1 + W_2]\!] = [\![W_1]\!] + [\![W_2]\!]$, where W_1 and W_2 are compatible partitioned graph-diagonal operators. If W_1 is self-adjoint, then $W_1 \succ 0$ if and only if $[\![W_1]\!] \succ 0$ if and only if $[\![W_1]\!](t,k) \succ \mu I$ for all $(t,k) \in \mathbb{Z} \times V$ and some scalar $\mu > 0$. The definition of graph-diagonal operators extend to ℓ and ℓ_{2e}.

I^q denotes the graph-diagonal identity operator such that $I^q(t,k) = I_{q(t,k)}$, and $0^{e\times h}$ denotes the graph-diagonal zero operator such that $0^{e\times h}(t,k) = 0_{e(t,k)\times h(t,k)}$ for all $(t,k) \in \mathbb{Z} \times V$. Then, $I^{(q_1,\ldots,q_m)} = \operatorname{diag}\left(I^{q_1}, \ldots, I^{q_m}\right)$ and $0^{(n_1,\ldots,n_f)\times(m_1,\ldots,m_g)} = [0^{n_i \times m_j}]_{i=1,\ldots,f; j=1,\ldots,g}$ denote the partitioned graph-diagonal identity and zero operators that have the shown dimensions, respectively. If the dimensions are not pertinent to the discussion, the partitioned graph-diagonal identity and zero operators are simply denoted by I and 0, respectively. Finally, the unitary temporal-shift operator S_0 and the unitary spatial-shift operators S_i, for $i = 1, \dots, d$, are defined as follows:

$$
\begin{aligned}
S_0 &: \ell_2 \to \ell_2, &\quad (S_0 v)(t,k) &= v(t-1,k), &\quad (S_0^* v)(t,k) &= v(t+1,k),\\
S_i &: \ell_2 \to \ell_2, &\quad (S_i v)(t,k) &= v(t,\rho_i^{-1}(k)), &\quad (S_i^* v)(t,k) &= v(t,\rho_i(k)).
\end{aligned}
$$

These definitions extend to ℓ and ℓ_{2e}. Note that no distinction will be made between the shift operators for different vector spaces $\ell(\{\mathbb{R}^{n(t,k)}\})$.

4.3 Operator Theoretic Framework

In this section, an operator theoretic framework is developed to describe distributed NSLPV systems. Namely, consider a distributed NSLPV system $\mathcal{G}_\delta$ whose interconnection structure is represented by a d-regular directed graph. Each subsystem $G^{(k)}$ corresponds to a vertex $k \in V$, and the interconnections between the subsystems are described by the directed edges in E. The interconnection from subsystem $G^{(i)}$ to subsystem $G^{(j)}$ corresponds to the directed edge $(i,j) \in E$. The subsystems have discrete-time NSLPV models that are formulated in an LFT framework. That is, each subsystem consists of an LTV nominal part and a static LTV uncertainty operator. NSLPV models [33, 34] generalize standard LPV models in the sense that the nominal part of the system can be time-varying. The adopted framework allows for heterogeneous subsystem models and accounts for a delay of one time-step on the information transfer between the subsystems. Let t denote discrete time. For all $(t,k) \in \mathbb{Z} \times V$, the state-space equations of system $\mathcal{G}_\delta$ are given by

$$\left[\begin{array}{c} x_T\left(t+1,k\right) \\ \hline x_1\left(t+1,\rho_1(k)\right) \\ \vdots \\ x_d\left(t+1,\rho_d(k)\right) \\ \hline \alpha\left(t,k\right) \\ \hline z\left(t,k\right) \\ \hline y\left(t,k\right) \end{array}\right] = \left[\begin{array}{c|c|c|c|c} \overline{A}_{TT}(t,k) & \overline{A}_{TS}(t,k) & \overline{A}_{TP}(t,k) & \overline{B}_{T1}(t,k) & \overline{B}_{T2}(t,k) \\ \hline \overline{A}_{ST}(t,k) & \overline{A}_{SS}(t,k) & \overline{A}_{SP}(t,k) & \overline{B}_{S1}(t,k) & \overline{B}_{S2}(t,k) \\ \hline \overline{A}_{PT}(t,k) & \overline{A}_{PS}(t,k) & \overline{A}_{PP}(t,k) & \overline{B}_{P1}(t,k) & \overline{B}_{P2}(t,k) \\ \hline \overline{C}_{1T}(t,k) & \overline{C}_{1S}(t,k) & \overline{C}_{1P}(t,k) & \overline{D}_{11}(t,k) & \overline{D}_{12}(t,k) \\ \hline \overline{C}_{2T}(t,k) & \overline{C}_{2S}(t,k) & \overline{C}_{2P}(t,k) & \overline{D}_{21}(t,k) & \overline{D}_{22}(t,k) \end{array}\right] \left[\begin{array}{c} x_T\left(t,k\right) \\ \hline x_1\left(t,k\right) \\ \vdots \\ x_d\left(t,k\right) \\ \hline \beta\left(t,k\right) \\ \hline w\left(t,k\right) \\ \hline u\left(t,k\right) \end{array}\right],$$

$$\beta\left(t,k\right) = \begin{bmatrix} \delta_1(t,k) I_{n_1^P(t,k)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \delta_m(t,k) I_{n_m^P(t,k)} \end{bmatrix} \alpha\left(t,k\right) = \underline{\Delta}(t,k)\ \alpha\left(t,k\right). \tag{4.1}$$

$x_T(t,k)$ denotes the state vector associated with subsystem $G^{(k)}$. The dimension of $x_T(t,k)$ is denoted by $n_T(t,k)$ for all $(t,k) \in \mathbb{Z} \times V$. The states associated with the subsystems are referred to as the temporal states. The interconnections between the subsystems are also modeled as states that are called spatial states. The spatial state vector $x_i(t,k)$ is associated with the edge $(\rho_i^{-1}(k),k)$ and has a dimension denoted by $n_i^S(t,k)$. The spatial state vector $x_i(t,\rho_i(k))$ is associated with the edge $(k,\rho_i(k))$ and has a dimension denoted by $n_i^S(t,\rho_i(k))$. In other words, $x_i(t,k)$ is associated with the incoming edge to vertex k along the permutation ρ_i, whereas $x_i(t,\rho_i(k))$ is associated with the outgoing edge from vertex k along the permutation ρ_i. Due to the assumed communication latency, the information sent by a subsystem at time-step t reaches the target subsystem at $t+1$. The spatial states corresponding to the virtual interconnections and their corresponding blocks in the state-space matrices are of zero dimensions, i.e., nonexistent. This is because the virtual edges are not present in the actual interconnection structure of system $\mathcal{G}_\delta$ and are only added to render the directed graph that describes the interconnection structure d-regular, as required for the development of the operator theoretic framework.

β and α are the internal signals introduced by formulating the subsystems in an LFT framework. In addition to being related by the state-space equations for the nominal parts of the subsystems, $\beta(t,k)$ and $\alpha(t,k)$ are related by the equation $\beta(t,k) = \underline{\Delta}(t,k)\alpha(t,k)$ for all $(t,k) \in \mathbb{Z} \times V$. Specifically, $\underline{\Delta}(t,k) = \operatorname{diag}\left(\delta_1(t,k)I_{n_1^P(t,k)}, \ldots, \delta_m(t,k)I_{n_m^P(t,k)}\right)$, where $\delta_j(t,k)$ are scalar-valued functions for $j = 1, \ldots, m$. The parameters $\delta_j(t,k)$ are not known a priori but are assumed to be measurable at each time-step t. β and α will be hereafter called the parameter states for simplicity and ease of reference. The vectors $\beta(t,k)$ and $\alpha(t,k)$ are partitioned into m vector-valued channels conformably with the partitioning of $\underline{\Delta}(t,k)$, i.e., $\beta(t,k) = [\beta_1^*(t,k)\ \beta_2^*(t,k)\ \cdots\ \beta_m^*(t,k)]^*$ and $\alpha(t,k) = [\alpha_1^*(t,k)\ \alpha_2^*(t,k)\ \cdots\ \alpha_m^*(t,k)]^*$, where the dimension of $\alpha_j(t,k)$ and $\beta_j(t,k)$ is denoted by $n_j^P(t,k)$. The framework allows for heterogeneous subsystems and for a local dependence of the subsystems on the parameters. That is, different subsystems may depend on different parameters, and even if two subsystems are affected by the same parameters, the framework assumes that the evolution of the parameters is independent in each subsystem. Let m_k be the number of parameters affecting subsystem $G^{(k)}$. Then, $m = \max_{k \in V} m_k$. If $m_{k_0} < m$ for some $k_0 \in V$, then $n_j^P(t,k_0) = 0$ for all $t \in \mathbb{Z}$ and $j = m_{k_0} + 1, \ldots, m$.

Each subsystem has its own actuating and sensing capabilities. $w(t,k)$, $z(t,k)$, $u(t,k)$, and $y(t,k)$ are the vectors of exogenous disturbances, performance outputs, control inputs, and measurement outputs associated with subsystem $G^{(k)}$, respectively. Their corresponding dimensions are denoted by $n_w(t,k)$, $n_z(t,k)$, $n_u(t,k)$, and $n_y(t,k)$. Unless otherwise stated, it is assumed hereafter that $w \in \ell_2$. As an example, consider Figure 4.2 that shows a distributed NSLPV system whose interconnection structure is described by the graph shown in Figure 4.1.

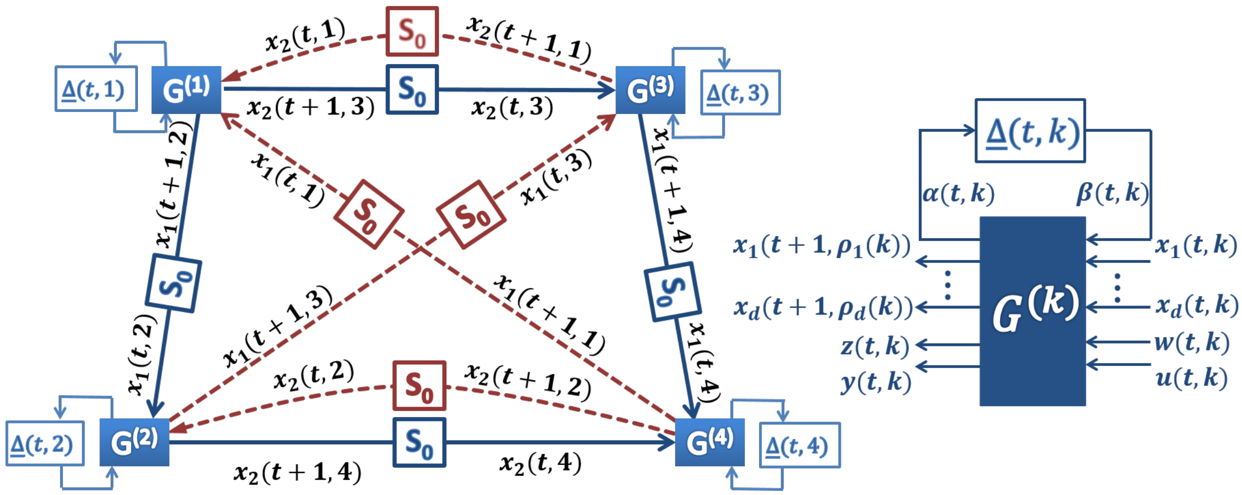

Figure 4.2: The figure on the left shows a distributed system whose interconnection structure is described by the graph shown in Figure 4.1. The subsystems have NSLPV models and are formulated in an LFT framework. The spatial states are defined, and the operator S_0 is used to mark the communication latency between the subsystems. The figure on the right shows a close-up view of subsystem $G^{(k)}$.

The state-space matrices are known a priori, are assumed to be uniformly bounded, and are partitioned according to the permutations $\rho_1, \ldots, \rho_d$ and the blocks in $\underline{\Delta}(t,k)$. Namely,

$$\overline{A}_{ST}(t,k) = \begin{bmatrix} A_1^{ST}(t,k) \\ \vdots \\ A_d^{ST}(t,k) \end{bmatrix}, \qquad \overline{A}_{PT}(t,k) = \begin{bmatrix} A_1^{PT}(t,k) \\ \vdots \\ A_m^{PT}(t,k) \end{bmatrix},$$

$$\overline{B}_{Sg}(t,k) = \begin{bmatrix} B_{1g}^{S}(t,k) \\ \vdots \\ B_{dg}^{S}(t,k) \end{bmatrix}, \qquad \overline{B}_{Pg}(t,k) = \begin{bmatrix} B_{1g}^{P}(t,k) \\ \vdots \\ B_{mg}^{P}(t,k) \end{bmatrix},$$

$$\overline{C}_{gS}(t,k) = \begin{bmatrix} C_{g1}^{S}(t,k) & \cdots & C_{gd}^{S}(t,k) \end{bmatrix}, \qquad \overline{C}_{gP}(t,k) = \begin{bmatrix} C_{g1}^{P}(t,k) & \cdots & C_{gm}^{P}(t,k) \end{bmatrix},$$

$$\overline{A}_{TS}(t,k) = \begin{bmatrix} A_1^{TS}(t,k) & \cdots & A_d^{TS}(t,k) \end{bmatrix}, \qquad \overline{A}_{TP}(t,k) = \begin{bmatrix} A_1^{TP}(t,k) & \cdots & A_m^{TP}(t,k) \end{bmatrix},$$

$$\overline{A}_{SS}(t,k) = \left[A_{ie}^{SS}(t,k)\right]_{i=1,\ldots,d;e=1,\ldots,d}, \qquad \overline{A}_{SP}(t,k) = \left[A_{ij}^{SP}(t,k)\right]_{i=1,\ldots,d;j=1,\ldots,m},$$

$$\overline{A}_{PS}(t,k) = \left[A_{ji}^{PS}(t,k)\right]_{j=1,\ldots,m;i=1,\ldots,d}, \qquad \overline{A}_{PP}(t,k) = \left[A_{jf}^{PP}(t,k)\right]_{j=1,\ldots,m;f=1,\ldots,m},$$

for $g = 1, 2$, where $A_i^{ST}(t,k)$ is an $n_i^S(t+1,\rho_i(k)) \times n_T(t,k)$ matrix, $A_j^{PT}(t,k)$ is an $n_j^P(t,k) \times n_T(t,k)$ matrix, $A_i^{TS}(t,k)$ is an $n_T(t+1,k) \times n_i^S(t,k)$ matrix, $A_j^{TP}(t,k)$ is an $n_T(t+1,k) \times n_j^P(t,k)$ matrix, and so on. The matrix-valued sequence obtained from each block/partition of the state-space matrices defines a graph-diagonal operator. For instance, the matrix-valued sequence $A_i^{TS}(t,k)$ defines the graph-diagonal operator A_i^{TS}. These graph-diagonal operators are then augmented to form partitioned graph-diagonal operators as follows:

$$
\begin{aligned}
A_{TS} &= \begin{bmatrix} A_1^{TS} & \cdots & A_d^{TS} \end{bmatrix}, & A_{TP} &= \begin{bmatrix} A_1^{TP} & \cdots & A_m^{TP} \end{bmatrix}, \\
A_{ST} &= \begin{bmatrix} A_1^{ST} \\ \vdots \\ A_d^{ST} \end{bmatrix}, & A_{PT} &= \begin{bmatrix} A_1^{PT} \\ \vdots \\ A_m^{PT} \end{bmatrix}, \\
A_{SS} &= \left[A_{ie}^{SS} \right]_{i=1,\ldots,d;e=1,\ldots,d}, & A_{SP} &= \left[A_{ij}^{SP} \right]_{i=1,\ldots,d;j=1,\ldots,m}, \\
A_{PS} &= \left[A_{ji}^{PS} \right]_{j=1,\ldots,m;i=1,\ldots,d}, & A_{PP} &= \left[A_{jf}^{PP} \right]_{j=1,\ldots,m;f=1,\ldots,m}, \\
B_{Sg} &= \begin{bmatrix} B_{1g}^S \\ \vdots \\ B_{dg}^S \end{bmatrix}, & B_{Pg} &= \begin{bmatrix} B_{1g}^P \\ \vdots \\ B_{mg}^P \end{bmatrix}, \\
C_{gS} &= \begin{bmatrix} C_{g1}^S & \cdots & C_{gd}^S \end{bmatrix}, & C_{gP} &= \begin{bmatrix} C_{g1}^P & \cdots & C_{gm}^P \end{bmatrix},
\end{aligned}
$$

for $g = 1, 2$. That is, for all $(t,k) \in \mathbb{Z} \times V$, the following hold:

$$
\begin{aligned}
[\![A_{TS}]\!](t,k) &= \overline{A}_{TS}(t,k), & [\![A_{TP}]\!](t,k) &= \overline{A}_{TP}(t,k), & [\![A_{ST}]\!](t,k) &= \overline{A}_{ST}(t,k), \\
[\![A_{PT}]\!](t,k) &= \overline{A}_{PT}(t,k), & [\![A_{SS}]\!](t,k) &= \overline{A}_{SS}(t,k), & [\![A_{SP}]\!](t,k) &= \overline{A}_{SP}(t,k), \\
[\![A_{PS}]\!](t,k) &= \overline{A}_{PS}(t,k), & [\![A_{PP}]\!](t,k) &= \overline{A}_{PP}(t,k), & [\![B_{Sg}]\!](t,k) &= \overline{B}_{Sg}(t,k), \\
[\![B_{Pg}]\!](t,k) &= \overline{B}_{Pg}(t,k), & [\![C_{gS}]\!](t,k) &= \overline{C}_{gS}(t,k), & [\![C_{gP}]\!](t,k) &= \overline{C}_{gP}(t,k).
\end{aligned}
$$

The matrix-valued sequences $\overline{A}_{TT}(t,k)$, $\overline{B}_{T1}(t,k)$, $\overline{B}_{T2}(t,k)$, $\overline{C}_{1T}(t,k)$, $\overline{D}_{11}(t,k)$, $\overline{D}_{12}(t,k)$, $\overline{C}_{2T}(t,k)$, $\overline{D}_{21}(t,k)$, and $\overline{D}_{22}(t,k)$ also define graph-diagonal operators denoted by A_{TT}, B_{T1}, B_{T2}, C_{1T}, D_{11}, D_{12}, C_{2T}, D_{21}, and D_{22}, respectively. That is, for all $(t,k) \in \mathbb{Z} \times V$, the following hold:

$$
\begin{aligned}
[\![A_{TT}]\!](t,k) &= A_{TT}(t,k) = \overline{A}_{TT}(t,k), & [\![B_{T1}]\!](t,k) &= B_{T1}(t,k) = \overline{B}_{T1}(t,k),\\
[\![B_{T2}]\!](t,k) &= B_{T2}(t,k) = \overline{B}_{T2}(t,k), & [\![C_{1T}]\!](t,k) &= C_{1T}(t,k) = \overline{C}_{1T}(t,k),\\
[\![D_{11}]\!](t,k) &= D_{11}(t,k) = \overline{D}_{11}(t,k), & [\![D_{12}]\!](t,k) &= D_{12}(t,k) = \overline{D}_{12}(t,k),\\
[\![C_{2T}]\!](t,k) &= C_{2T}(t,k) = \overline{C}_{2T}(t,k), & [\![D_{21}]\!](t,k) &= D_{21}(t,k) = \overline{D}_{21}(t,k),\\
[\![D_{22}]\!](t,k) &= D_{22}(t,k) = \overline{D}_{22}(t,k).
\end{aligned}
$$

The augmented partitioned graph-diagonal operators A, B_1, B_2, B, C_1, C_2, C, and D are then defined as follows:

$$
A = \begin{bmatrix} A_{TT} & A_{TS} & A_{TP} \\ A_{ST} & A_{SS} & A_{SP} \\ A_{PT} & A_{PS} & A_{PP} \end{bmatrix}, \qquad
B = \begin{bmatrix} B_1 & B_2 \end{bmatrix} = \begin{bmatrix} B_{T1} & B_{T2} \\ B_{S1} & B_{S2} \\ B_{P1} & B_{P2} \end{bmatrix},
$$

$$
C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} C_{1T} & C_{1S} & C_{1P} \\ C_{2T} & C_{2S} & C_{2P} \end{bmatrix}, \qquad
D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}.
$$

For all $(t,k) \in \mathbb{Z} \times V$, these operators satisfy

$$
[\![A]\!](t,k) = \begin{bmatrix} \overline{A}_{TT}(t,k) & \overline{A}_{TS}(t,k) & \overline{A}_{TP}(t,k) \\ \overline{A}_{ST}(t,k) & \overline{A}_{SS}(t,k) & \overline{A}_{SP}(t,k) \\ \overline{A}_{PT}(t,k) & \overline{A}_{PS}(t,k) & \overline{A}_{PP}(t,k) \end{bmatrix}, \quad
[\![B]\!](t,k) = \begin{bmatrix} \overline{B}_{T1}(t,k) & \overline{B}_{T2}(t,k) \\ \overline{B}_{S1}(t,k) & \overline{B}_{S2}(t,k) \\ \overline{B}_{P1}(t,k) & \overline{B}_{P2}(t,k) \end{bmatrix},
$$

$$[\![C]\!](t,k) = \begin{bmatrix} \overline{C}_{1T}(t,k) & \overline{C}_{1S}(t,k) & \overline{C}_{1P}(t,k) \\ \overline{C}_{2T}(t,k) & \overline{C}_{2S}(t,k) & \overline{C}_{2P}(t,k) \end{bmatrix}, \qquad [\![D]\!](t,k) = \begin{bmatrix} \overline{D}_{11}(t,k) & \overline{D}_{12}(t,k) \\ \overline{D}_{21}(t,k) & \overline{D}_{22}(t,k) \end{bmatrix}.$$

Graph-diagonal operators Δ_j are defined such that $\Delta_j(t,k) = \delta_j(t,k) I_{n_j^P(t,k)}$ for all $(t,k) \in \mathbb{Z} \times V$ and $j = 1, \dots, m$. These operators are block-diagonally augmented to construct the partitioned graph-diagonal operator $\Delta_P = \operatorname{diag}(\Delta_1, \dots, \Delta_m)$ that satisfies $[\![\Delta_P]\!](t,k) = \underline{\Delta}(t,k)$ for all $(t,k) \in \mathbb{Z} \times V$. Define

$$\begin{aligned} n_T^+ &= S_0^* n_T S_0, & \overline{n}_S^+ &= \left(S_1^* S_0^* n_1^S S_0 S_1, \dots, S_d^* S_0^* n_d^S S_0 S_d\right), \\ \overline{n}_S &= \left(n_1^S, \dots, n_d^S\right), & \overline{n}_P &= \left(n_1^P, \dots, n_m^P\right), \end{aligned} \tag{4.2}$$

where $(S_0^* n_T S_0)(t,k) = n_T(t+1,k)$ and $(S_i^* S_0^* n_i^S S_0 S_i)(t,k) = n_i^S(t+1, \rho_i(k))$ for $i = 1, \dots, d$. Then, the composite-shift operator S is defined as $S = \operatorname{diag}(S_0, S_0 S_1, \dots, S_0 S_d, I^{\overline{n}_P})$ and the operator Δ is defined as $\Delta = \operatorname{diag}\left(I^{(n_T, \overline{n}_S)}, \Delta_P\right)$. The equations in (4.1) can thus be rewritten in compact operator form as follows:

$$\begin{bmatrix} x \\ \alpha \end{bmatrix} = SA \begin{bmatrix} x \\ \beta \end{bmatrix} + SB \begin{bmatrix} w \\ u \end{bmatrix}, \quad \begin{bmatrix} z \\ y \end{bmatrix} = C \begin{bmatrix} x \\ \beta \end{bmatrix} + D \begin{bmatrix} w \\ u \end{bmatrix}, \qquad \begin{bmatrix} x \\ \beta \end{bmatrix} = \Delta \begin{bmatrix} x \\ \alpha \end{bmatrix}, \tag{4.3}$$

$$\begin{bmatrix} x \\ \beta \end{bmatrix} = \Delta SA \begin{bmatrix} x \\ \beta \end{bmatrix} + \Delta SB \begin{bmatrix} w \\ u \end{bmatrix}, \qquad \begin{bmatrix} z \\ y \end{bmatrix} = C \begin{bmatrix} x \\ \beta \end{bmatrix} + D \begin{bmatrix} w \\ u \end{bmatrix}, \tag{4.4}$$

where $x = [x_T^* \; x_1^* \; \cdots \; x_d^*]^*$, $\alpha = [\alpha_1^* \; \cdots \; \alpha_m^*]^*$, and $\beta = [\beta_1^* \; \cdots \; \beta_m^*]^*$.

It is assumed throughout Chapters 4, 5, and 6 that Δ is restricted to $\boldsymbol{\Delta} = \{\Delta \,|\, \|\Delta\| \leq 1\}$. For every $\Delta \in \boldsymbol{\Delta}$, and assuming that the inverses in question exist, the input-output map of system $\mathcal{G}_\delta$ can be expressed as

$$G_\delta = C(I - \Delta SA)^{-1}\Delta SB + D = C\Delta(I - SA\Delta)^{-1}SB + D. \tag{4.5}$$

The distributed NSLPV system $\mathcal{G}_\delta$ is then defined as $\mathcal{G}_\delta = \{G_\delta \,|\, \Delta \in \boldsymbol{\Delta}\}$.

The realization of system $\mathcal{G}_\delta$ is denoted hereafter by the quintuple $(A, B, C, D, \boldsymbol{\Delta})$. For an a priori known and fixed Δ, system $\mathcal{G}_\delta$ reduces to a distributed LTV system such as the ones described in [35]. If only one subsystem is considered, then system $\mathcal{G}_\delta$ reduces to a single NSLPV system such as the ones described in [33, 34].

To simplify the presentation of the results in the subsequent sections of this chapter, the following operators are defined to group the temporal and spatial blocks of A, B, and C:

$$\hat{A}_{11} = \begin{bmatrix} A_{TT} & A_{TS} \\ A_{ST} & A_{SS} \end{bmatrix}, \qquad \hat{A}_{12} = \begin{bmatrix} A_{TP} \\ A_{SP} \end{bmatrix}, \qquad \hat{A}_{21} = \begin{bmatrix} A_{PT} & A_{PS} \end{bmatrix},$$
$$\hat{B}_g = \begin{bmatrix} B_{Tg} \\ B_{Sg} \end{bmatrix}, \qquad \hat{C}_g = \begin{bmatrix} C_{gT} & C_{gS} \end{bmatrix}, \qquad g = 1, 2. \tag{4.6}$$

Finally, it is noted that the following convention is adopted throughout Chapters 4, 5, and 6. If $\mathcal{H}_\delta$ denotes a distributed NSLPV system, then the input-output map of system $\mathcal{H}_\delta$ is denoted by H_δ for all $\Delta \in \boldsymbol{\Delta}$. Similarly, the input-output map of a distributed NSLPV system $\mathcal{K}_\delta$ is denoted by K_δ for all $\Delta \in \boldsymbol{\Delta}$, and so on.

4.4 Analysis Results

Consider a distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. This section gives the analysis results for system $\mathcal{G}_\delta$. For simplicity, the control input u and the measurement output y are dropped from the system equations. First, the equations in (4.1) are rewritten so as to eliminate $\alpha(t,k)$ and $\beta(t,k)$. Clearly, for the resulting equations to be well-defined, $I - \overline{A}_{PP}(t,k)\underline{\Delta}(t,k)$ must be invertible for all $(t,k) \in \mathbb{Z} \times V$.

Definition 4.1. Consider a distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. System $\mathcal{G}_\delta$ is said to be well-posed if (i) given an input $w \in \ell_{2e}$, the equations (4.4) admit a unique solution $(x, \beta) \in \ell_{2e}^{(n_T, \overline{n}_S, \overline{n}_P)}$, and (ii) G_δ defines a linear causal mapping on ℓ_{2e} for all $\Delta \in \mathbf{\Delta}$.

Definition 4.2. Consider a distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. System $\mathcal{G}_\delta$ is said to be stable if (i) it is well-posed, (ii) given an input $w \in \ell_2$, the equations (4.4) admit a unique solution $(x, \beta) \in \ell_2^{(n_T, \overline{n}_S, \overline{n}_P)}$, and (iii) G_δ defines a bounded linear causal mapping for all $\Delta \in \mathbf{\Delta}$, i.e., $G_\delta \in \mathcal{L}_c\left(\ell_2\left(\{\mathbb{R}^{n_w(t,k)}\}\right), \ell_2\left(\{\mathbb{R}^{n_z(t,k)}\}\right)\right)$ for all permissible parameter trajectories.

Lemma 4.3. *Consider a distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. If $[\![A]\!](t,k) = 0$ for $t < 0$ and all $k \in V$, and $I - \Delta_P A_{PP}$ has a causal inverse on $\ell_{2e}^{\overline{n}_P}$ for all $\Delta \in \mathbf{\Delta}$, then $I - \Delta S A$ has a causal inverse on $\ell_{2e}^{(n_T, \overline{n}_S, \overline{n}_P)}$ for all $\Delta \in \mathbf{\Delta}$, and system $\mathcal{G}_\delta$ is well-posed.*

Proof. The proof parallels the proofs found in [22, 33]. Since the set of linear causal mappings on ℓ_{2e} is an algebra, the well-posedness of system $\mathcal{G}_\delta$ is equivalent to the existence of a causal inverse for $I - \Delta S A$ on $\ell_{2e}^{(n_T, \overline{n}_S, \overline{n}_P)}$ for all $\Delta \in \mathbf{\Delta}$. Using the definitions in (4.6), and since

$I - \Delta_P A_{PP}$ is assumed to have a causal inverse on $\ell_{2e}^{\overline{n}_P}$, $I - \Delta SA$ can be factored as

$$
\begin{aligned}
I - \Delta SA &= \begin{bmatrix} I - \hat{S}_0 \hat{S} \hat{A}_{11} & -\hat{S}_0 \hat{S} \hat{A}_{12} \\ -\Delta_P \hat{A}_{21} & I - \Delta_P A_{PP} \end{bmatrix} \\
&= \begin{bmatrix} I & -\hat{S}_0 \hat{S} \hat{A}_{12} (I - \Delta_P A_{PP})^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} I - \hat{S}_0 \hat{S} R & 0 \\ -\Delta_P \hat{A}_{21} & I - \Delta_P A_{PP} \end{bmatrix},
\end{aligned}
$$

for all $\Delta \in \mathbf{\Delta}$, where $\hat{S}_0$=diag$(S_0, \ldots, S_0)$ ($d+1$ blocks), $\hat{S}$=diag$\left(I^{n_T}, S_1, \ldots, S_d\right)$, and $R = \hat{A}_{11} + \hat{A}_{12}(I - \Delta_P A_{PP})^{-1} \Delta_P \hat{A}_{21}$. Thus, $I - \Delta SA$ has a causal inverse on $\ell_{2e}^{(n_T, \overline{n}_S, \overline{n}_P)}$ if $I - \hat{S}_0 \hat{S} R$ has a causal inverse on $\ell_{2e}^{(n_T, \overline{n}_S)}$. By generalizing the characterizations of causality and memorylessness to partitioned operators mapping multiple inputs to multiple outputs, i.e., operators mapping $\oplus_{i=1}^{N_1} \ell_{2e}$ to $\oplus_{i=1}^{N_2} \ell_{2e}$ for some positive integers N_1, N_2, and by extending [22, Lemma 6] to this class of operators, it can be shown that since R is a linear memoryless operator and $[\![A]\!](t,k) = 0$ for $t < 0$ and all $k \in V$, then $(I - \hat{S}_0 \hat{S} R)^{-1}$ exists and is causal for each $\Delta \in \mathbf{\Delta}$, which concludes the proof. $\square$

Hereafter, it is assumed that $[\![A]\!](t,k) = 0$, $[\![B]\!](t,k) = 0$, $[\![C]\!](t,k) = 0$, and $[\![D]\!](t,k) = 0$ for all $t < 0$ and $k \in V$. To check for the well-posedness of system $\mathcal{G}_\delta$, it becomes sufficient to check that $I - \Delta_P A_{PP}$ is invertible on $\ell_{2e}^{\overline{n}_P}$ for all $\Delta \in \mathbf{\Delta}$. The inverse, when existent, is memoryless (and so, it is causal) since $I - \Delta_P A_{PP}$ is a memoryless operator, and the set of memoryless mappings is closed under the inversion operation.

The following result gives a sufficient condition for the stability of system $\mathcal{G}_\delta$, i.e., the validity of the said condition implies that $I - \Delta SA$ has a bounded causal inverse for all $\Delta \in \mathbf{\Delta}$. Let

$$
\mathcal{X} = \Big\{ X \mid X = \text{diag}\left(X_T, X_1^S, \ldots, X_d^S, X_1^P, \ldots, X_m^P\right) = X^* \in \mathcal{L}_c\left(\ell_2^{(n_T, \overline{n}_S, \overline{n}_P)}\right),
$$

where X_T, X_i^S, and X_j^P are graph-diagonal operators for $i = 1, \ldots, d$ and $j = 1, \ldots, m$,

$$X \succ 0, \text{ and } X^{-1} \in \mathcal{L}_c\left(\ell_2^{(n_T,\overline{n}_S,\overline{n}_P)}\right)\bigg\}. \tag{4.7}$$

Clearly, the set $\mathcal{X}$ is a commutant of $\mathbf{\Delta}$. The symbol $\mathcal{X}$ is used to denote similarly defined sets regardless of the associated dimensions.

Lemma 4.4. *Consider a distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. If there exists $X \in \mathcal{X}$, where the set $\mathcal{X}$ is defined in (4.7), such that*

$$A^* S^* X S A - X \prec 0, \tag{4.8}$$

then the system $\mathcal{G}_\delta$ is stable.

Proof. The proof of this result parallels the proof of the counterpart result for single LPV and uncertain systems found in [16]. Since $\mathcal{L}_c(\ell_2, \ell_2)$ is an algebra, the stability of system $\mathcal{G}_\delta$ is equivalent to $(I - \Delta SA)^{-1}$ being in $\mathcal{L}_c\left(\ell_2^{(n_T,\overline{n}_S,\overline{n}_P)}\right)$. From (4.8), it follows that $\|X^{\frac{1}{2}} SAX^{-\frac{1}{2}}\| < 1$. Since $X^{\frac{1}{2}} \in \mathcal{X}$, then $X^{\frac{1}{2}}$ commutes with every $\Delta \in \mathbf{\Delta}$, and so, $\|X^{\frac{1}{2}} \Delta SAX^{-\frac{1}{2}}\| < 1$ by the sub-multiplicative property. In other words, ΔSA has a spectral radius less than 1. Thus, $(I - \Delta SA)^{-1} = \sum_{i=0}^{\infty} (\Delta SA)^i$ is a well-defined quantity in $\mathcal{L}\left(\ell_2^{(n_T,\overline{n}_S,\overline{n}_P)}\right)$. This quantity is causal since each term in the summation is causal. □

Since $X \in \mathcal{X}$, it follows that $S^* X S$ is a partitioned graph-diagonal operator with a block-diagonal structure similar to the structure of X, i.e.,

$$S^* X S = \operatorname{diag}\left(S_0^* X_T S_0,\ S_1^* S_0^* X_1^S S_0 S_1,\ \ldots,\ S_d^* S_0^* X_d^S S_0 S_d,\ X_1^P,\ \ldots,\ X_m^P\right).$$

The condition $X \succ 0$ can be expressed as $X_T(t,k) \succ \mu I$, $X_i^S(t,k) \succ \mu I$, and $X_j^P(t,k) \succ \mu I$ for some scalar $\mu > 0$ and all $(t,k) \in \mathbb{Z} \times V$, $i = 1, \ldots, d$, and $j = 1, \ldots, m$. Similarly, inequality (4.8) can be expressed in terms of equivalent sequences of LMIs, namely,

$$\begin{aligned}&[\![A^*]\!](t,k)\operatorname{diag}\left(X_T\left(t+1,k\right),X_1^S\left(t+1,\rho_1(k)\right),\ldots,X_d^S\left(t+1,\rho_d(k)\right),X_1^P(t,k),\ldots,X_m^P(t,k)\right)\\&\qquad\times[\![A]\!](t,k)-\operatorname{diag}\left(X_T(t,k),X_1^S(t,k),\ldots,X_d^S(t,k),X_1^P(t,k),\ldots,X_m^P(t,k)\right)\prec-\mu I,\end{aligned}$$

for all $(t,k)\in\mathbb{Z}\times V$. One sequence of LMIs is associated with every subsystem $G^{(k)}$. These sequences are in general infinite dimensional because of the explicit dependence on time in the state-space equations, i.e., the nominal parts of the subsystems can have LTV models. The sequences associated with the various subsystems are coupled through the spatial terms $X_i^S(t,k)$. The parameter terms $X_j^P(t,k_0)$ only appear in the LMI sequence associated with subsystem $G^{(k_0)}$. This highlights the local dependence of the state-space matrices on the parameters. Note that the spatial terms associated with the virtual interconnections have zero dimensions and do not appear in the LMIs. Similarly, if $m_{k_0}<m$ for some $k_0\in V$, then $X_j^P(t,k_0)$ has zero dimensions for $j=m_{k_0}+1,\ldots,m$ and all $t\in\mathbb{Z}$. The μI terms ensure that the matrix-valued sequences on the left-hand-side do not converge to singular matrices as t approaches infinity. Since the state-space matrices are assumed to be zeros for $t<0$, the LMI sequences are trivial for $t<0$, and so t is restricted to $\mathbb{N}_0$.

Systems that satisfy the previous condition are said to be strongly stable. Strong stability implies stability, but the converse is not necessarily true. Namely, a strongly stable system is a stable system that possesses a solution $X\in\mathcal{X}$ to (4.8). However, there exist stable systems that are not strongly stable. The reader is referred to Remark 2.4 for a related discussion.

The following result gives a condition that ensures the strongly stable of system $\mathcal{G}_\delta$ and guarantees an upper bound on the ℓ_2-induced norm of the input-output map G_δ defined in (4.5) for all permissible parameter trajectories.

Lemma 4.5. *Consider a distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. If there exists $X \in \mathcal{X}$, where the set $\mathcal{X}$ is defined in (4.7), such that*

$$\begin{bmatrix} SA & SB \\ C & D \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} SA & SB \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & \gamma^2 I \end{bmatrix} \prec 0, \tag{4.9}$$

then system $\mathcal{G}_\delta$ is strongly stable, and its input-output map G_δ, defined in (4.5), satisfies $\|G_\delta\| < \gamma$ for all $\Delta \in \mathbf{\Delta}$.

Proof. The proof of this result extends the proof of the counterpart result for interconnected LTV systems given in [22]. The inequality defined in (4.8) is implied by the inequality defined in (4.9), and so system $\mathcal{G}_\delta$ is strongly stable and $I - SA\Delta$ has a bounded causal inverse for all $\Delta \in \mathbf{\Delta}$. (4.9) is pre- and post-multiplied by $\begin{bmatrix} B^*S^*(I - \Delta^* A^* S^*)^{-1}\Delta^* & I \end{bmatrix}$ and $\begin{bmatrix} \Delta(I - SA\Delta)^{-1}SB \\ I \end{bmatrix}$, respectively. Since $X \in \mathcal{X}$ commutes with every $\Delta \in \mathbf{\Delta}$ and $\|\Delta\| < 1$ for all $\Delta \in \mathbf{\Delta}$, then $X - \Delta^* X \Delta = (I - \Delta^*\Delta)X$ and $\Delta^*\Delta \prec I$, and so it can be concluded that $G_\delta^* G_\delta - \gamma^2 I \prec 0$, i.e., $\|G_\delta\| < \gamma$ for all $\Delta \in \mathbf{\Delta}$. $\square$

Lemmas 4.4 and 4.5 require $X \in \mathcal{X}$. In fact, X only needs to be positive definite and in the commutant of $\mathbf{\Delta}$. It is shown in the next result that the condition $X \in \mathcal{X}$ can be imposed without adding conservatism. A similar result for single NSLPV systems is given in [34].

Theorem 4.6. *Inequality (4.9) admits a solution $\overline{X} \succ 0$ in the commutant of $\mathbf{\Delta}$ if and only if it admits a solution $X \in \mathcal{X}$, where the set $\mathcal{X}$ is defined in (4.7).*

Proof. The proof of the 'if' direction is trivial, since the set $\mathcal{X}$ is a commutant of $\mathbf{\Delta}$. The 'only if' direction is proved next. Namely, the desired solution $X \in \mathcal{X}$ is constructed from the given solution $\overline{X}$.

For this purpose, define the operator $E_{(\tau,\zeta)} : \mathbb{R}^g \to \ell_2\left(\{\mathbb{R}^{n(t,k)}\}\right)$ for some mapping n that satisfies $n(\tau,\zeta) = g$ as follows: if $E_{(\tau,\zeta)}e = v \in \ell_2$, then $v(t,k) = e$ if $(t,k) = (\tau,\zeta)$ and $v(t,k) = 0$ otherwise. The adjoint operator $E^*_{(\tau,\zeta)} : \ell_2\left(\{\mathbb{R}^{n(t,k)}\}\right) \to \mathbb{R}^{n(\tau,\zeta)}$ satisfies $E^*_{(\tau,\zeta)}v = v(\tau,\zeta) \in \mathbb{R}^{n(\tau,\zeta)}$. No distinction will be made between the operators $E_{(\tau,\zeta)}$ defined for various mappings n. Thus, any $v \in \ell_2$ can be written as $v = \sum_{(t,k)\in\mathbb{Z}\times V} E_{(t,k)}v(t,k)$, and it follows that $Qv = \sum_{(t,k)\in\mathbb{Z}\times V} QE_{(t,k)}v(t,k)$, for any operator Q on ℓ_2.

Since $\overline{X}$ is in the commutant of $\boldsymbol{\Delta}$, then $\overline{X}$ must have the block-diagonal structure

$$\overline{X} = \operatorname{diag}\left(\overline{X}_T, \overline{X}^S_1, \ldots, \overline{X}^S_d, \overline{X}^P_1, \ldots, \overline{X}^P_m\right).$$

Since the operators Δ_j are graph-diagonal for $j = 1,\ldots,m$, then the blocks $\overline{X}^P_j$ must also be graph-diagonal operators as they must satisfy $\overline{X}^P_j \Delta_j v = \Delta_j \overline{X}^P_j v$ for all $v \in \ell_2$ and $j = 1,\ldots,m$. To see this, both sides of the previous equation are evaluated at (τ,ζ), i.e., pre-multiplied by $E^*_{(\tau,\zeta)}$. By the above discussion, the following equation is obtained:

$$\sum_{(t,k)\in\mathbb{Z}\times V} \delta_j(t,k) E^*_{(\tau,\zeta)} \overline{X}^P_j E_{(t,k)} v(t,k) = \delta_j(\tau,\zeta) \sum_{(t,k)\in\mathbb{Z}\times V} E^*_{(\tau,\zeta)} \overline{X}^P_j E_{(t,k)} v(t,k).$$

For this equality to hold for an arbitrary $v \in \ell_2$, $\overline{X}^P_j$ must be graph-diagonal. That is, there exists a sequence of uniformly bounded matrices $\overline{X}^P_j(t,k)$ such that $E^*_{(\tau,\zeta)}\overline{X}^P_j E_{(t,k)} = \overline{X}^P_j(\tau,\zeta)$ for $(t,k) = (\tau,\zeta)$, and $E^*_{(\tau,\zeta)}\overline{X}^P_j E_{(t,k)} = 0$ for $(t,k) \neq (\tau,\zeta)$. However, a similar conclusion cannot be made about $\overline{X}_T$ and $\overline{X}^S_i$ for $i = 1,\ldots,d$. Since $\overline{X}$ satisfies (4.9), then the following inequality holds for some $\mu > 0$:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} S^*\overline{X}S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} \overline{X} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \prec -\mu I.$$

Define the operator $\hat{E}_{(\tau,\zeta)} = \operatorname{diag}\left(E_{(\tau,\zeta)}, E_{(\tau,\zeta)}, \ldots, E_{(\tau,\zeta)}\right)$ $(1+d+m$ blocks) that satisfies

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \hat{E}_{(\tau,\zeta)} & 0 \\ 0 & E_{(\tau,\zeta)} \end{bmatrix} = \begin{bmatrix} \hat{E}_{(\tau,\zeta)} & 0 \\ 0 & E_{(\tau,\zeta)} \end{bmatrix} \left[\!\!\left[\begin{matrix} A & B \\ C & D \end{matrix} \right]\!\!\right] (\tau,\zeta).$$

Then, by pre- and post-multiplying the previous inequality by $\operatorname{diag}\left(\hat{E}_{(\tau,\zeta)}, E_{(\tau,\zeta)}\right)^*$ and its adjoint, respectively, and using the fact that $E^*_{(\tau,\zeta)}E_{(\tau,\zeta)} = I$, the following inequality is obtained:

$$\left[\!\!\left[\begin{matrix} A & B \\ C & D \end{matrix} \right]\!\!\right]^* (\tau,\zeta) \begin{bmatrix} \hat{E}^*_{(\tau,\zeta)} S^* \overline{X} S \hat{E}_{(\tau,\zeta)} & 0 \\ 0 & I \end{bmatrix} \left[\!\!\left[\begin{matrix} A & B \\ C & D \end{matrix} \right]\!\!\right] (\tau,\zeta) - \begin{bmatrix} \hat{E}^*_{(\tau,\zeta)} \overline{X} \hat{E}_{(\tau,\zeta)} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \prec -\mu I. \tag{4.10}$$

The previous inequality can be shown to hold for all $(\tau,\zeta) \in \mathbb{Z} \times V$. From the definitions of the temporal-shift operator S_0, the spatial-shift operators S_i, and the operators $E_{(\tau,\zeta)}$ and $E^*_{(\tau,\zeta)}$, it can be verified that

$$\begin{aligned} & E^*_{(\tau,\zeta)} S_0^* \overline{X}_T S_0 E_{(\tau,\zeta)} = E^*_{(\tau+1,\zeta)} \overline{X}_T E_{(\tau+1,\zeta)}, \\ & E^*_{(\tau,\zeta)} S_i^* S_0^* \overline{X}_i^S S_0 S_i E_{(\tau,\zeta)} = E^*_{(\tau+1,\rho_i(\zeta))} \overline{X}_i^S E_{(\tau+1,\rho_i(\zeta))}, \qquad \text{for } i = 1,\ldots,d. \end{aligned} \tag{4.11}$$

If graph-diagonal operators X_T and X_i^S for $i = 1,\ldots,d$ are defined such that $X_T(t,k) = E^*_{(t,k)} \overline{X}_T E_{(t,k)} \succ \mu I$ and $X_i^S(t,k) = E^*_{(t,k)} \overline{X}_i^S E_{(t,k)} \succ \mu I$ for all $(t,k) \in \mathbb{Z} \times V$, then it can be seen from (4.10) and (4.11) that $X = \operatorname{diag}\left(X_T, X_1^S, \ldots, X_d^S, \overline{X}_1^P, \ldots, \overline{X}_m^P\right) \in \mathcal{X}$ satisfies the sequences of LMIs that are equivalent to (4.9) for all $(t,k) \in \mathbb{Z} \times V$. Thus, a solution $X \in \mathcal{X}$ that satisfies (4.9) has been found, which concludes the proof. $\square$

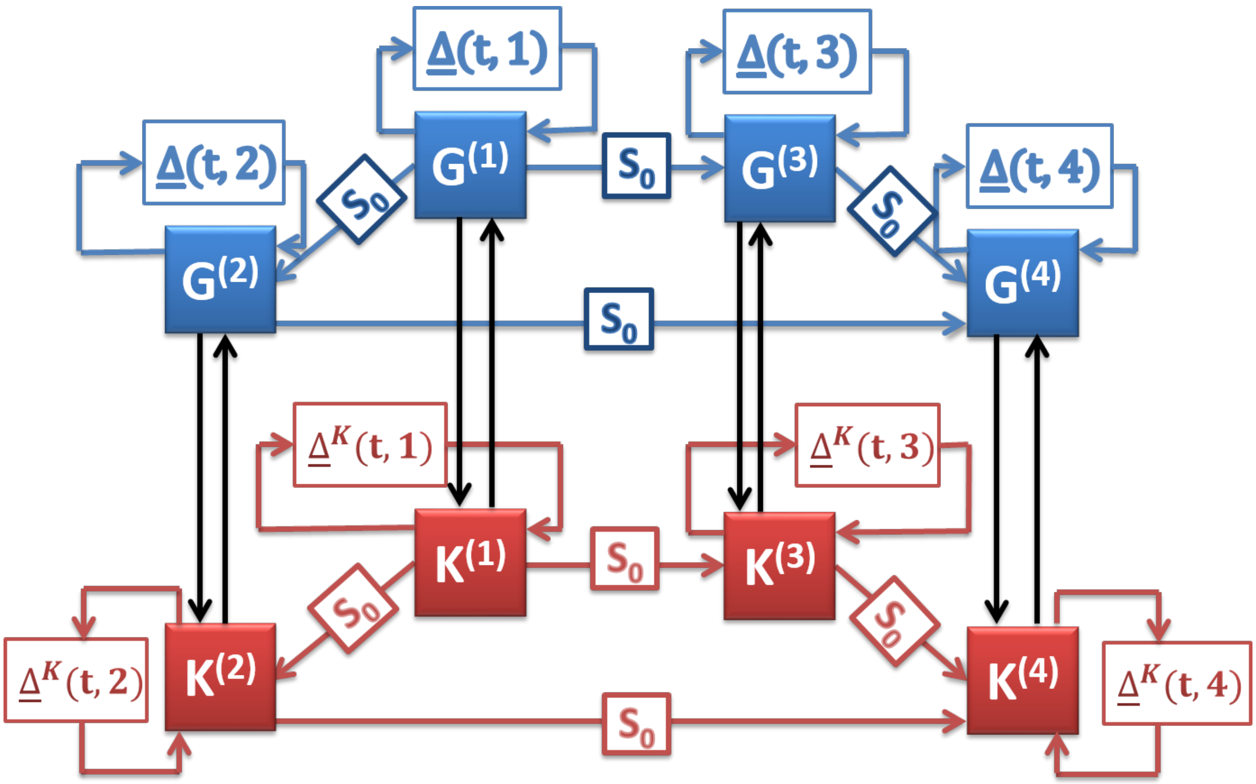

Figure 4.3: The figure shows a distributed NSLPV controller $\mathcal{K}_\delta$ that is applied to the distributed NSLPV plant $\mathcal{G}_\delta$ shown in Figure 4.2. The shown controller inherits the interconnection and uncertainty structures of the plant. The information exchange between a controller subsystem and the corresponding plant subsystem is depicted using the black arrows.

4.5 Synthesis Results

This section addresses the control synthesis problem for distributed NSLPV systems. Namely, let the plant $\mathcal{G}_\delta$ be a distributed NSLPV system, and denote its realization by $(A, B, C, D, \mathbf{\Delta})$. The plant $\mathcal{G}_\delta$ is assumed to be a well-posed system as per Definition 4.1, and the state-space matrices of the plant are assumed to be zeros for $t < 0$. Without loss of generality, it is further assumed that $\overline{D}_{22}(t, k) = 0$ for all $(t, k) \in \mathbb{Z} \times V$. The sought controller $\mathcal{K}_\delta$ is a distributed NSLPV system that has the same interconnection structure as that of the plant, and the controller subsystems have LFT structures that are similar to the LFT structures of the plant subsystems. That is, the controller consists of NSLPV subsystems that are formulated in an LFT framework and interconnected over the same interconnection structure as that of the plant. The information transfer between the controller subsystems is subjected to a one-step time-delay. For all $k \in V$, the controller subsystem $K^{(k)}$ is scheduled using the same parameters that affect the plant subsystem $G^{(k)}$. Thus, the controller equations are in the form of (4.1) with input $y(t,k)$ and output $u(t,k)$. The controller state-space matrices are denoted using the same symbols as the state-space matrices of the plant with the additional superscript K. The controller dimensions are denoted by $r_T(t,k)$, $r_i^S(t,k)$, $r_j^P(t,k)$ for all $(t,k) \in \mathbb{Z} \times V$, $i = 1,\dots,d$, and $j = 1,\dots,m$. These dimensions are not known a priori, but are determined from the synthesis solutions as explained in Algorithm 4.12. The symbols r_T^+, $\overline{r}_S$, $\overline{r}_S^+$, and $\overline{r}_P$ are defined similarly to the symbols n_T^+, $\overline{n}_S$, $\overline{n}_S^+$, and $\overline{n}_P$ defined in (4.2), respectively. That is,

$$
\begin{aligned}
r_T^+ &= S_0^* r_T S_0, & \overline{r}_S &= (r_1^S, \dots, r_d^S), \\
\overline{r}_S^+ &= (S_1^* S_0^* r_1^S S_0 S_1, \dots, S_d^* S_0^* r_d^S S_0 S_d), & \overline{r}_P &= (r_1^P, \dots, r_m^P).
\end{aligned}
$$

For all $(t,k) \in \mathbb{Z} \times V$, the controller equations are thus given by

$$
\left[\begin{array}{c}
x_T^K(t+1,k) \\ \hline
x_1^K(t+1,\rho_1(k)) \\ \vdots \\ x_d^K(t+1,\rho_d(k)) \\ \hline
\alpha^K(t,k) \\ \hline
u(t,k)
\end{array}\right]
=
\left[\begin{array}{c|c|c|c}
\overline{A}_{TT}^K(t,k) & \overline{A}_{TS}^K(t,k) & \overline{A}_{TP}^K(t,k) & \overline{B}_T^K(t,k) \\ \hline
\overline{A}_{ST}^K(t,k) & \overline{A}_{SS}^K(t,k) & \overline{A}_{SP}^K(t,k) & \overline{B}_S^K(t,k) \\ \hline
\overline{A}_{PT}^K(t,k) & \overline{A}_{PS}^K(t,k) & \overline{A}_{PP}^K(t,k) & \overline{B}_P^K(t,k) \\ \hline
\overline{C}_T^K(t,k) & \overline{C}_S^K(t,k) & \overline{C}_P^K(t,k) & \overline{D}^K(t,k)
\end{array}\right]
\left[\begin{array}{c}
x_T^K(t,k) \\ \hline
x_1^K(t,k) \\ \vdots \\ x_d^K(t,k) \\ \hline
\beta^K(t,k) \\ \hline
y(t,k)
\end{array}\right],
$$

$$
\beta^K(t,k) = \operatorname{diag}\left(\delta_1(t,k) I_{r_1^P(t,k)}, \dots, \delta_m(t,k) I_{r_m^P(t,k)}\right) \alpha^K(t,k) = \underline{\Delta}^K(t,k)\ \alpha^K(t,k),
$$

where $\alpha^K(t,k)$ and $\beta^K(t,k)$ are partitioned as in $\alpha^K(t,k) = \left[(\alpha_1^K(t,k))^* \cdots (\alpha_m^K(t,k))^*\right]^*$ and $\beta^K(t,k) = \left[(\beta_1^K(t,k))^* \cdots (\beta_m^K(t,k))^*\right]^*$, respectively.

Figure 4.3 shows the closed-loop system formed by the distributed NSLPV plant shown in Figure 4.2 and the corresponding distributed NSLPV controller.

The controller state-space matrices are partitioned according to the permutations $\rho_1, \ldots, \rho_d$ and the blocks in $\underline{\Delta}^K(t,k)$. The matrix-valued sequence obtained from each block/partition of the controller state-space matrices defines a graph-diagonal operator. These graph-diagonal operators are in turn augmented to form partitioned graph-diagonal operators A_{TS}^K, A_{TP}^K, A_{ST}^K, A_{PT}^K, A_{SS}^K, A_{SP}^K, A_{PS}^K, A_{PP}^K, B_S^K, B_P^K, C_S^K, C_P^K that satisfy

$$
\begin{aligned}
&[\![A_{TS}^K]\!](t,k) = \overline{A}_{TS}^K(t,k), && [\![A_{TP}^K]\!](t,k) = \overline{A}_{TP}^K(t,k), && [\![A_{ST}^K]\!](t,k) = \overline{A}_{ST}^K(t,k),\\
&[\![A_{PT}^K]\!](t,k) = \overline{A}_{PT}^K(t,k), && [\![A_{SS}^K]\!](t,k) = \overline{A}_{SS}^K(t,k), && [\![A_{SP}^K]\!](t,k) = \overline{A}_{SP}^K(t,k),\\
&[\![A_{PS}^K]\!](t,k) = \overline{A}_{PS}^K(t,k), && [\![A_{PP}^K]\!](t,k) = \overline{A}_{PP}^K(t,k), && [\![B_{S}^K]\!](t,k) = \overline{B}_{S}^K(t,k),\\
&[\![B_{P}^K]\!](t,k) = \overline{B}_{P}^K(t,k), && [\![C_{S}^K]\!](t,k) = \overline{C}_{S}^K(t,k), && [\![C_{P}^K]\!](t,k) = \overline{C}_{P}^K(t,k),
\end{aligned}
$$

for all $(t,k) \in \mathbb{Z} \times V$. Moreover, the matrix-valued sequences $\overline{A}_{TT}^K(t,k)$, $\overline{B}_T^K(t,k)$, $\overline{C}_T^K(t,k)$, and $\overline{D}^K(t,k)$ define graph-diagonal operators that are denoted by A_{TT}^K, B_T^K, C_T^K, and D^K, respectively. For all $(t,k) \in \mathbb{Z} \times V$, these operators satisfy

$$
\begin{aligned}
&[\![A_{TT}^K]\!](t,k) = A_{TT}^K(t,k) = \overline{A}_{TT}^K(t,k), && [\![B_{T}^K]\!](t,k) = B_{T}^K(t,k) = \overline{B}_{T}^K(t,k),\\
&[\![C_{T}^K]\!](t,k) = C_{T}^K(t,k) = \overline{C}_{T}^K(t,k), && [\![D^K]\!](t,k) = D^K(t,k) = \overline{D}^K(t,k).
\end{aligned}
$$

The augmented partitioned graph-diagonal operators A^K, B^K, and C^K can then be defined as follows:

$$A^K = \begin{bmatrix} A^K_{TT} & A^K_{TS} & A^K_{TP} \\ A^K_{ST} & A^K_{SS} & A^K_{SP} \\ A^K_{PT} & A^K_{PS} & A^K_{PP} \end{bmatrix}, \qquad B^K = \begin{bmatrix} B^K_T \\ B^K_S \\ B^K_P \end{bmatrix}, \qquad C^K = \begin{bmatrix} C^K_T & C^K_S & C^K_P \end{bmatrix}.$$

For all $(t,k) \in \mathbb{Z} \times V$, these operators satisfy

$$[\![A^K]\!](t,k) = \begin{bmatrix} \overline{A}^K_{TT}(t,k) & \overline{A}^K_{TS}(t,k) & \overline{A}^K_{TP}(t,k) \\ \overline{A}^K_{ST}(t,k) & \overline{A}^K_{SS}(t,k) & \overline{A}^K_{SP}(t,k) \\ \overline{A}^K_{PT}(t,k) & \overline{A}^K_{PS}(t,k) & \overline{A}^K_{PP}(t,k) \end{bmatrix}, \qquad [\![B^K]\!](t,k) = \begin{bmatrix} \overline{B}^K_T(t,k) \\ \overline{B}^K_S(t,k) \\ \overline{B}^K_P(t,k) \end{bmatrix},$$
$$[\![C^K]\!](t,k) = \begin{bmatrix} \overline{C}^K_T(t,k) & \overline{C}^K_S(t,k) & \overline{C}^K_P(t,k) \end{bmatrix}.$$

The graph-diagonal operators Δ^K_j are defined such that $\Delta^K_j(t,k) = \delta_j(t,k) I_{r^P_j(t,k)}$ for all $j = 1, \ldots, m$ and $(t,k) \in \mathbb{Z} \times V$. It is stressed that the parameters $\delta_j(t,k)$ are the same as the plant parameters. These operators are block-diagonally augmented to form the partitioned graph-diagonal operator $\Delta^K_P = \operatorname{diag}\left(\Delta^K_1, \ldots, \Delta^K_m\right)$ that satisfies $[\![\Delta^K_P]\!](t,k) = \underline{\Delta}^K(t,k)$ for all $(t,k) \in \mathbb{Z} \times V$. Grouping the temporal and spatial blocks of A^K, B^K, and C^K, the following operators are defined:

$$\hat{A}^K_{11} = \begin{bmatrix} A^K_{TT} & A^K_{TS} \\ A^K_{ST} & A^K_{SS} \end{bmatrix}, \qquad \hat{A}^K_{12} = \begin{bmatrix} A^K_{TP} \\ A^K_{SP} \end{bmatrix}, \qquad \hat{A}^K_{21} = \begin{bmatrix} A^K_{PT} & A^K_{PS} \end{bmatrix},$$
$$\hat{B}^K = \begin{bmatrix} B^K_T \\ B^K_S \end{bmatrix}, \qquad \hat{C}^K = \begin{bmatrix} C^K_T & C^K_S \end{bmatrix}.$$

These operators are similar to the plant operators defined in (4.6). Using all these definitions, the controller equations are expressed as follows:

$$\begin{bmatrix} x_K \\ \alpha_K \\ u \end{bmatrix} = \begin{bmatrix} \hat{S}_0\hat{S}\hat{A}^K_{11} & \hat{S}_0\hat{S}\hat{A}^K_{12} & \hat{S}_0\hat{S}\hat{B}^K \\ \hat{A}^K_{21} & A^K_{PP} & B^K_P \\ \hat{C}^K & C^K_P & D^K \end{bmatrix} \begin{bmatrix} x_K \\ \beta_K \\ y \end{bmatrix},$$
$$\begin{bmatrix} x_K \\ \beta_K \end{bmatrix} = \begin{bmatrix} I^{(r_T,\overline{r}_S)} & 0 \\ 0 & \Delta^K_P \end{bmatrix} \begin{bmatrix} x_K \\ \alpha_K \end{bmatrix} = \Delta^K \begin{bmatrix} x_K \\ \alpha_K \end{bmatrix},$$

where $x_K = \left[(x^K_T)^* \, (x^K_1)^* \, \cdots \, (x^K_d)^*\right]^*$, $\alpha_K = \left[(\alpha^K_1)^* \, \cdots \, (\alpha^K_m)^*\right]^*$, and $\beta_K = \left[(\beta^K_1)^* \, \cdots \, (\beta^K_m)^*\right]^*$. The definitions of $\hat{S}_0$ and $\hat{S}$ are similar to their definitions in the proof of Lemma 4.3, i.e., $\hat{S}_0 = \mathrm{diag}(S_0, \ldots, S_0)$ ($d+1$ blocks) and $\hat{S} = \mathrm{diag}\left(I^{r_T}, S_1, \ldots, S_d\right)$. The controller state-space matrices are assumed to be zeros for $t < 0$ and all $k \in V$. Then, by Lemma 4.3, the distributed NSLPV controller $\mathcal{K}_\delta$ is well-posed if $I - \Delta^K_P A^K_{PP}$ has a causal inverse on $\ell^{\overline{r}_P}_{2e}$ for all $\Delta^K \in \boldsymbol{\Delta}^K = \{\Delta^K \mid \|\Delta^K\| \le 1\}$.

Consider a distributed NSLPV plant $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \boldsymbol{\Delta})$. Let $\mathcal{K}_\delta$ be a distributed NSLPV controller such as the ones defined above, and denote the realization of $\mathcal{K}_\delta$ by $(A^K, B^K, C^K, D^K, \boldsymbol{\Delta}^K)$, where $\boldsymbol{\Delta}^K = \{\Delta^K \mid \|\Delta^K\| \le 1\}$. The equations for the closed-loop system obtained by applying the controller $\mathcal{K}_\delta$ to the plant $\mathcal{G}_\delta$ are derived by combining the plant equations and the controller equations. Let

$$\tilde{S} = \mathrm{diag}\left(S, S\right) = \mathrm{diag}\left(\hat{S}_0\hat{S}, I^{\overline{n}_P}, \hat{S}_0\hat{S}, I^{\overline{r}_P}\right), \qquad x_{\mathrm{CL}} = \begin{bmatrix} x \\ \beta \\ x_K \\ \beta_K \end{bmatrix},$$

$$\tilde{\Delta} = \mathrm{diag}\left(\Delta, \Delta^K\right) = \mathrm{diag}\left(I^{(n_T,\overline{n}_S)}, \Delta_P, I^{(r_T,\overline{r}_S)}, \Delta^K_P\right).$$

Then, the closed-loop system equations are given by

$$x_{\mathrm{CL}} = \tilde{\Delta}\tilde{S}A_{\mathrm{CL}}x_{\mathrm{CL}} + \tilde{\Delta}\tilde{S}B_{\mathrm{CL}}w, \qquad z = C_{\mathrm{CL}}x_{\mathrm{CL}} + D_{\mathrm{CL}}w,$$

where the partitioned graph-diagonal operators A_{CL}, B_{CL}, C_{CL}, and D_{CL} are defined as follows:

$$\begin{aligned}
A_{\mathrm{CL}} &= \overline{\overline{A}} + \underline{\underline{B}}J\underline{\underline{C}}, & B_{\mathrm{CL}} &= \overline{\overline{B}} + \underline{\underline{B}}J\underline{\underline{D}}_{21}, \\
C_{\mathrm{CL}} &= \overline{\overline{C}} + \underline{\underline{D}}_{12}J\underline{\underline{C}}, & D_{\mathrm{CL}} &= D_{11} + \underline{\underline{D}}_{12}J\underline{\underline{D}}_{21}, \\
\overline{\overline{A}} &= \begin{bmatrix} A & 0 \\ 0 & 0^{(r_T^+,\overline{r}_S^+,\overline{r}_P)\times(r_T,\overline{r}_S,\overline{r}_P)} \end{bmatrix}, & \overline{\overline{B}} &= \begin{bmatrix} B_1 \\ 0^{(r_T^+,\overline{r}_S^+,\overline{r}_P)\times n_w} \end{bmatrix}, \\
\underline{\underline{B}} &= \begin{bmatrix} 0 & B_2 \\ I^{(r_T^+,\overline{r}_S^+,\overline{r}_P)} & 0 \end{bmatrix}, & \underline{\underline{C}} &= \begin{bmatrix} 0 & I^{(r_T,\overline{r}_S,\overline{r}_P)} \\ C_2 & 0 \end{bmatrix}, \\
J &= \begin{bmatrix} \hat{A}_{11}^K & \hat{A}_{12}^K & \hat{B}^K \\ \hat{A}_{21}^K & A_{PP}^K & B_P^K \\ \hat{C}^K & C_P^K & D^K \end{bmatrix}, & \overline{\overline{C}} &= \begin{bmatrix} C_1 & 0^{n_z\times(r_T,\overline{r}_S,\overline{r}_P)} \end{bmatrix}, \\
\underline{\underline{D}}_{12} &= \begin{bmatrix} 0^{n_z\times(r_T^+,\overline{r}_S^+,\overline{r}_P)} & D_{12} \end{bmatrix}, & \underline{\underline{D}}_{21} &= \begin{bmatrix} 0^{(r_T,\overline{r}_S,\overline{r}_P)\times n_w} \\ D_{21} \end{bmatrix}.
\end{aligned} \tag{4.12}$$

Since the controller operators can be extracted from the operator J, then the given parameterization of the closed-loop operators A_{CL}, B_{CL}, C_{CL}, and D_{CL} is said to be affine in the controller realization. This parameterization allows the derivation of a condition to check whether a given controller $\mathcal{K}_\delta$ ensures the stability of the resulting closed-loop system, and further guarantees an upper bound γ on the ℓ_2-induced norm of the mapping $w \to z$ for all permissible parameter trajectories.

The above equations describe the closed-loop system, but are not in the form of (4.4). Define the partitioned graph-diagonal operator $\Delta_P^L = \operatorname{diag}\left(\Delta_1^L, \ldots, \Delta_m^L\right)$, where the graph-diagonal operators Δ_j^L satisfy $\Delta_j^L(t,k) = \delta_j(t,k) I_{n_j^P(t,k)+r_j^P(t,k)}$ for all $(t,k) \in \mathbb{Z} \times V$ and $j = 1, \ldots, m$. Let P be an appropriately defined partitioned graph-diagonal operator such that

$$P^*P = I, \qquad PP^* = I, \qquad P^*\tilde{\Delta}P = \operatorname{diag}\left(I^{n_T+r_T}, I^{\bar{n}_S+\bar{r}_S}, \Delta_P^L\right) = \Delta^L. \tag{4.13}$$

A constructive proof for the existence of an operator similar to P is given in [8, Appendix A]. Then, define the partitioned graph-diagonal operators A^L, B^L, C^L, and D^L as follows:

$$SA^L = P^*\tilde{S}A_{\mathrm{CL}}P, \qquad SB^L = P^*\tilde{S}B_{\mathrm{CL}}, \qquad C^L = C_{\mathrm{CL}}P, \qquad D^L = D_{\mathrm{CL}}. \tag{4.14}$$

The closed-loop system equations can now be written in the desired form of (4.4) as follows:

$$\begin{bmatrix} x^L \\ \beta^L \end{bmatrix} = \Delta^L S A^L \begin{bmatrix} x^L \\ \beta^L \end{bmatrix} + \Delta^L S B^L w, \qquad z = C^L \begin{bmatrix} x^L \\ \beta^L \end{bmatrix} + D^L w, \tag{4.15}$$

where the operator Δ^L defined in (4.13) is in $\mathbf{\Delta}^L = \{\Delta^L \mid \|\Delta^L\| \leq 1\}$, and

$$\begin{aligned} &x^L = \left[(x_T^L)^* \, (x_1^L)^* \, \cdots \, (x_d^L)^*\right]^*, \quad \beta^L = \left[(\beta_1^L)^* \, \ldots \, (\beta_m^L)^*\right]^*, \\ &x_T^L(t,k) \in \mathbb{R}^{n_T(t,k)+r_T(t,k)}, \qquad x_i^L(t,k) \in \mathbb{R}^{n_i^S(t,k)+r_i^S(t,k)}, \quad \beta_j^L(t,k) \in \mathbb{R}^{n_j^P(t,k)+r_j^P(t,k)}, \end{aligned}$$

for all $(t,k) \in \mathbb{Z} \times V$, $i = 1, \ldots, d$ and $j = 1, \ldots, m$.

Definition 4.7. Consider a distributed NSLPV plant $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. A distributed NSLPV controller $\mathcal{K}_\delta$ that has a realization denoted by $(A^K, B^K, C^K, D^K, \mathbf{\Delta}^K)$ is said to be a γ-admissible synthesis for the plant $\mathcal{G}_\delta$ if the resulting closed-loop system described by the equations in (4.15) is stable and its input-output map satisfies

$$\|w \to z\| = \left\| C^L \left(I - \Delta^L S A^L\right)^{-1} \Delta^L S B^L + D^L \right\| < \gamma,$$

for all $\Delta^L \in \mathbf{\Delta}^L$, where A^L, B^L, C^L, and D^L are defined in (4.14) and Δ^L is defined in (4.13).

Without loss of generality, the discussion is restricted to the case $\gamma = 1$ because a γ-admissible synthesis for a plant $\mathcal{G}_\delta$ is a 1-admissible synthesis for the scaled system $\mathcal{G}_{\text{scaled},\delta}$ in which γ is absorbed into the system operators, namely, $C_{\text{scaled},1} = \frac{1}{\gamma}C_1$, $D_{\text{scaled},11} = \frac{1}{\gamma}D_{11}$, and $D_{\text{scaled},12} = \frac{1}{\gamma}D_{12}$. More details on this issue are provided in Section 4.7.

Theorem 4.8. *Consider a distributed NSLPV plant $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$ and a given distributed NSLPV controller $\mathcal{K}_\delta$ that has a realization denoted by $(A^K, B^K, C^K, D^K, \mathbf{\Delta}^K)$. Let the operators $\overline{\overline{A}}$, $\overline{\overline{B}}$, $\overline{\overline{C}}$, $\underline{\underline{B}}$, $\underline{\underline{C}}$, $\underline{\underline{D}}_{12}$, $\underline{\underline{D}}_{21}$, and J be defined as in (4.12). If there exists an operator $X^L \in \mathcal{X}$ such that*

$$H + Q^* J^* R + R^* J Q \prec 0, \tag{4.16}$$

where

$$R = \begin{bmatrix} \underline{\underline{B}}^* & 0 & 0 & \underline{\underline{D}}_{12}^* \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & \underline{\underline{C}} & \underline{\underline{D}}_{21} & 0 \end{bmatrix},$$

$$X_P^L = P X^L P^*, \qquad H = \begin{bmatrix} -\tilde{S}^* \left(X_P^L\right)^{-1} \tilde{S} & \overline{\overline{A}} & \overline{\overline{B}} & 0 \\ \left(\overline{\overline{A}}\right)^* & -X_P^L & 0 & \left(\overline{\overline{C}}\right)^* \\ \left(\overline{\overline{B}}\right)^* & 0 & -I & D_{11}^* \\ 0 & \overline{\overline{C}} & D_{11} & -I \end{bmatrix}, \tag{4.17}$$

the operator P is defined as in (4.13), and the set $\mathcal{X}$ is defined as in (4.7) with appropriate dimensions, then the controller $\mathcal{K}_\delta$ is a 1-admissible synthesis for the plant $\mathcal{G}_\delta$.

Proof. By Lemma 4.5, if there exists $X^L \in \mathcal{X}$ that satisfies (4.9) for $\gamma = 1$, where (4.9) is expressed for the closed-loop system realization $\left(A^L, B^L, C^L, D^L, \mathbf{\Delta}^L\right)$ with A^L, B^L, C^L, and D^L defined as in (4.14) and Δ^L defined as in (4.13), then $\mathcal{K}_\delta$ is a 1-admissible synthesis for $\mathcal{G}_\delta$. If (4.9) is pre- and post-multiplied by $\operatorname{diag}(P, I)$ and its adjoint, respectively, the following equivalent inequality is obtained:

$$\begin{bmatrix} A_{\mathrm{CL}} & B_{\mathrm{CL}} \\ C_{\mathrm{CL}} & D_{\mathrm{CL}} \end{bmatrix}^* \begin{bmatrix} \tilde{S}^* X_P^L \tilde{S} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{\mathrm{CL}} & B_{\mathrm{CL}} \\ C_{\mathrm{CL}} & D_{\mathrm{CL}} \end{bmatrix} - \begin{bmatrix} X_P^L & 0 \\ 0 & I \end{bmatrix} \prec 0,$$

where A_{CL}, B_{CL}, C_{CL}, and D_{CL} are defined as in (4.12). Inequality (4.16) is retrieved by applying the Schur complement formula to the previous inequality, and appropriately rearranging the blocks in the resulting inequality. □

Consider the partitioned graph-diagonal operator $X^L \in \mathcal{X}$ defined in the statement of Theorem 4.8. X^L has the block-diagonal structure $X^L = \operatorname{diag}\left(X_T, X_1^S, \ldots, X_d^S, X_1^P, \ldots, X_m^P\right)$. For all $(t,k) \in \mathbb{Z} \times V$, $X_T(t,k)$ can be partitioned as in $X_T(t,k) = \left[X_{T,fg}(t,k)\right]_{f=1,2;g=1,2}$, where $X_{T,11}(t,k) \in \mathbb{S}^{n_T(t,k)}$, $X_{T,22}(t,k) \in \mathbb{S}^{r_T(t,k)}$, and $X_{T,21}(t,k) = X_{T,12}^*(t,k)$. The matrix-valued sequences $X_{T,11}(t,k)$, $X_{T,12}(t,k)$, and $X_{T,22}(t,k)$ define graph-diagonal operators that are denoted by $X_{T,11}$, $X_{T,12}$, and $X_{T,22}$, respectively. The partitioning process is repeated for all $X_i^S(t,k)$ and $X_j^P(t,k)$. Namely,

$$X_i^S(t,k) = \left[X_{i,fg}^S(t,k)\right]_{f=1,2;g=1,2}, \qquad X_j^P(t,k) = \left[X_{j,fg}^P(t,k)\right]_{f=1,2;g=1,2},$$

where $X_{i,11}^S(t,k) \in \mathbb{S}^{n_i^S(t,k)}$, $X_{i,22}^S(t,k) \in \mathbb{S}^{r_i^S(t,k)}$, $X_{i,21}^S(t,k) = (X_{i,12}^S)^*(t,k)$, $X_{j,11}^P(t,k) \in \mathbb{S}^{n_j^P(t,k)}$, $X_{j,22}^P(t,k) \in \mathbb{S}^{r_j^P(t,k)}$, $X_{j,21}^P(t,k) = (X_{j,12}^P)^*(t,k)$ for all $(t,k) \in \mathbb{Z} \times V$, $i = 1, \ldots, d$, and $j = 1, \ldots, m$. The matrix-valued sequences $X_{i,11}^S(t,k)$, $X_{i,12}^S(t,k)$, $X_{i,22}^S(t,k)$, $X_{j,11}^P(t,k)$, $X_{j,12}^P(t,k)$, and $X_{j,22}^P(t,k)$ define graph-diagonal operators that are denoted by $X_{i,11}^S$, $X_{i,12}^S$, $X_{i,22}^S$, $X_{j,11}^P$, $X_{j,12}^P$, and $X_{j,22}^P$, respectively, for all $i = 1, \ldots, d$ and $j = 1, \ldots, m$. Using these graph-diagonal operators, it is possible to construct the partitioned graph-diagonal operators $\hat{X}_{11}$, $\hat{X}_{12}$, and $\hat{X}_{22}$ as follows:

$$\hat{X}_{11} = \operatorname{diag}\left(X_{T,11}, X^S_{1,11}, \ldots, X^S_{d,11}, X^P_{1,11}, \ldots, X^P_{m,11}\right) \in \mathcal{X}, \tag{4.18}$$

$$\hat{X}_{12} = \operatorname{diag}\left(X_{T,12}, X^S_{1,12}, \ldots, X^S_{d,12}, X^P_{1,12}, \ldots, X^P_{m,12}\right), \tag{4.19}$$

$$\hat{X}_{22} = \operatorname{diag}\left(X_{T,22}, X^S_{1,22}, \ldots, X^S_{d,22}, X^P_{1,22}, \ldots, X^P_{m,22}\right) \in \mathcal{X}. \tag{4.20}$$

Define the operator $X^L_P = PX^LP^*$, where the operator P is defined as in (4.13). Then, one can see that X^L_P has the structure shown in (4.21). Moreover, since $\left(X^L\right)^{-1} \in \mathcal{X}$ by the definition of the set $\mathcal{X}$ in (4.7), then $\left(X^L_P\right)^{-1} = P\left(X^L\right)^{-1}P^*$ has a similar structure to X^L_P. Namely,

$$X^L_P = \begin{bmatrix} \hat{X}_{11} & \hat{X}_{12} \\ \hat{X}^*_{12} & \hat{X}_{22} \end{bmatrix}, \qquad \left(X^L_P\right)^{-1} = \begin{bmatrix} \hat{Y}_{11} & \hat{Y}_{12} \\ \hat{Y}^*_{12} & \hat{Y}_{22} \end{bmatrix}, \tag{4.21}$$

where $\hat{Y}_{11}$, $\hat{Y}_{12}$, and $\hat{Y}_{22}$ are defined similarly to $\hat{X}_{11}$, $\hat{X}_{12}$, and $\hat{X}_{22}$ in (4.18), (4.19), and (4.20), respectively.

Given some $\hat{X}_{11}$ and $\hat{Y}_{11}$ in $\mathcal{X}$ structured as shown in (4.18), the next result gives necessary and sufficient conditions for the existence of an operator X^L_P such that X^L_P and its inverse $(X^L_P)^{-1}$ are structured as shown in (4.21).

Lemma 4.9. *Let $\hat{X}_{11}$ and $\hat{Y}_{11}$ in $\mathcal{X}$ be two partitioned graph-diagonal operators structured as shown in (4.18). Given $r_T(t,k)$, $r^S_i(t,k)$, and $r^P_j(t,k)$ for all $(t,k) \in \mathbb{Z} \times V$, $i = 1, \ldots, d$, and $j = 1, \ldots, m$, there exists an operator $X^L_P \succ 0$ such that X^L_P and its inverse $(X^L_P)^{-1}$ are* *structured as shown in (4.21) if and only if the following conditions hold:*

$$
\begin{aligned}
&\begin{bmatrix} \hat{X}_{11} & I \\ I & \hat{Y}_{11} \end{bmatrix} \succeq 0, \\
\operatorname{rank} &\begin{bmatrix} X_{T,11}(t,k) & I \\ I & Y_{T,11}(t,k) \end{bmatrix} \leq n_T(t,k) + r_T(t,k), \\
\operatorname{rank} &\begin{bmatrix} X^S_{i,11}(t,k) & I \\ I & Y^S_{i,11}(t,k) \end{bmatrix} \leq n^S_i(t,k) + r^S_i(t,k), \\
\operatorname{rank} &\begin{bmatrix} X^P_{j,11}(t,k) & I \\ I & Y^P_{j,11}(t,k) \end{bmatrix} \leq n^P_j(t,k) + r^P_j(t,k),
\end{aligned} \tag{4.22}
$$

for all $(t,k) \in \mathbb{Z} \times V$, $i = 1,\ldots,d$, and $j = 1,\ldots,m$.

The proof of Lemma 4.9 includes a procedure for constructing the appropriately structured operators X^L_P and $(X^L_P)^{-1}$ from the given $\hat{X}_{11}$ and $\hat{Y}_{11}$. This proof is omitted here as it is an immediate generalization of the proof of [66, Lemma 6.2].

The first inequality in (4.22) can alternatively be expressed in terms of its equivalent sequences of LMIs as follows:

$$
\begin{bmatrix} X_{T,11}(t,k) & I \\ I & Y_{T,11}(t,k) \end{bmatrix} \succeq 0, \quad \begin{bmatrix} X^S_{i,11}(t,k) & I \\ I & Y^S_{i,11}(t,k) \end{bmatrix} \succeq 0, \quad \begin{bmatrix} X^P_{j,11}(t,k) & I \\ I & Y^P_{j,11}(t,k) \end{bmatrix} \succeq 0,
$$

for all $(t,k) \in \mathbb{Z} \times V$, $i = 1,\ldots,d$, and $j = 1,\ldots,m$.

By Theorem 4.8, if there exists $X^L \in \mathcal{X}$ such that (4.16) holds, then the given controller $\mathcal{K}_\delta$ is a 1-admissible synthesis for the plant $\mathcal{G}_\delta$. Theorem 4.11 complements Theorem 4.8 by giving sufficient conditions for the existence of a 1-admissible synthesis for the plant $\mathcal{G}_\delta$.

Moreover, the solutions to these conditions can be used along with Lemma 4.9 to construct the realization of the desired controller.

Lemma 4.10. *Given partitioned graph-diagonal operators $H = H^*$, R, and Q, there exists a partitioned graph-diagonal operator J with a compatible structure to R and Q such that $H + Q^*J^*R + R^*JQ \prec 0$ if and only if $W_R^* H W_R \prec 0$ and $W_Q^* H W_Q \prec 0$, where the partitioned graph-diagonal operators W_R and W_Q satisfy $\mathrm{Im} W_R = \ker R$, $W_R^* W_R = I$, $\mathrm{Im} W_Q = \ker Q$, and $W_Q^* W_Q = I$.*

Lemma 4.10 is a generalization of [39, Lemma 3.1] and [23, Lemmas 16, 17] to the class of partitioned graph-diagonal operators and is given here without proof. This lemma is needed in the proof of Theorem 4.11 below, and is also needed in the proof of Theorem 6.2 in Chapter 6.

Theorem 4.11. *Consider a distributed NSLPV plant $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \boldsymbol{\Delta})$. There exists a 1-admissible synthesis $\mathcal{K}_\delta$ for the plant $\mathcal{G}_\delta$ such that the controller dimensions satisfy $r_T(t,k) \le n_T(t,k)$, $r_i^S(t,k) \le n_i^S(t,k)$, and $r_j^P(t,k) \le n_j^P(t,k)$ for all $(t,k) \in \mathbb{Z} \times V$, $i = 1, \ldots, d$, and $j = 1, \ldots, m$, if there exist $\hat{X}_{11}$ and $\hat{Y}_{11}$ in $\mathcal{X}$ structured as shown in (4.18) such that*

$$\begin{bmatrix} V_1^* & V_2^* \end{bmatrix} \left(\begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} \begin{bmatrix} \hat{Y}_{11} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}^* - \begin{bmatrix} S^*\hat{Y}_{11}S & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \prec 0, \tag{4.23}$$

$$\begin{bmatrix} U_1^* & U_2^* \end{bmatrix} \left(\begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}^* \begin{bmatrix} S^*\hat{X}_{11}S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} - \begin{bmatrix} \hat{X}_{11} & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \prec 0, \tag{4.24}$$

$$\begin{bmatrix} \hat{X}_{11} & I \\ I & \hat{Y}_{11} \end{bmatrix} \succeq 0. \tag{4.25}$$

The partitioned graph-diagonal operators U_1 and V_1 and the graph-diagonal operators U_2 and

V_2 in (4.23) and (4.24) satisfy

$$\operatorname{Im}\begin{bmatrix} V_1^* & V_2^* \end{bmatrix}^* = \ker\begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix}, \qquad V_1^* V_1 + V_2^* V_2 = I,$$
$$\operatorname{Im}\begin{bmatrix} U_1^* & U_2^* \end{bmatrix}^* = \ker\begin{bmatrix} C_2 & D_{21} \end{bmatrix}, \qquad U_1^* U_1 + U_2^* U_2 = I.$$

Namely,

$$U_1 = \begin{bmatrix} (U_T)^* & \left(U_1^S\right)^* & \cdots & \left(U_d^S\right)^* & \left(U_1^P\right)^* & \cdots & \left(U_m^P\right)^* \end{bmatrix}^*,$$
$$V_1 = \begin{bmatrix} (V_T)^* & \left(V_1^S\right)^* & \cdots & \left(V_d^S\right)^* & \left(V_1^P\right)^* & \cdots & \left(V_m^P\right)^* \end{bmatrix}^*,$$

and for all $(t,k) \in \mathbb{Z} \times V$, $i = 1, \dots, d$, and $j = 1, \dots, m$, the graph-diagonal operators U_2, V_2, U_T, U_i^S, U_j^P, V_T, V_i^S, and V_j^P satisfy

$$U_T(t,k) \in \mathbb{R}^{n_T(t,k) \times ?}, \qquad U_i^S(t,k) \in \mathbb{R}^{n_i^S(t,k) \times ?}, \qquad U_j^P(t,k) \in \mathbb{R}^{n_j^P(t,k) \times ?},$$
$$V_T(t,k) \in \mathbb{R}^{n_T(t+1,k) \times ?}, \qquad V_i^S(t,k) \in \mathbb{R}^{n_i^S(t+1,\rho_i(k)) \times ?}, \qquad V_j^P(t,k) \in \mathbb{R}^{n_j^P(t,k) \times ?},$$
$$U_2(t,k) \in \mathbb{R}^{n_w(t,k) \times ?}, \qquad V_2(t,k) \in \mathbb{R}^{n_z(t,k) \times ?}.$$

Proof. $\hat{X}_{11}$ and $\hat{Y}_{11}$ in $\mathcal{X}$ are structured as shown in (4.18) and satisfy (4.25). Then, by invoking Lemma 4.9 with some $r_T(t,k) \leq n_T(t,k)$, $r_i^S(t,k) \leq n_i^S(t,k)$, and $r_j^P(t,k) \leq n_j^P(t,k)$ for all $(t,k) \in \mathbb{Z} \times V$, $i = 1, \dots, d$, and $j = 1, \dots, m$ (the controller dimensions $r_T(t,k)$, $r_i^S(t,k)$, and $r_j^P(t,k)$ are determined as in Algorithm 4.12), an operator $X_P^L \succ 0$ can be constructed such that X_P^L and its inverse $(X_P^L)^{-1}$ are structured as shown in (4.21).

Using X_P^L and $(X_P^L)^{-1}$, a well-defined operator H can be constructed as in (4.17). The operators R and Q used in (4.16) are defined in (4.17), and so the variable in (4.16) is now the operator J. Specifically, for the computed X_P^L and $(X_P^L)^{-1}$, if there exists an operator

J that satisfies (4.16), then there exists a 1-admissible synthesis for the plant $\mathcal{G}_\delta$ as per Theorem 4.8. In fact, the controller operators can be extracted from J.

By Lemma 4.10, the inequality defined in (4.16) admits a solution J if and only if $W_R^* H W_R \prec 0$ and $W_Q^* H W_Q \prec 0$, where W_R and W_Q are partitioned graph-diagonal operators that satisfy $\mathrm{Im} W_R = \ker R$, $W_R^* W_R = I$, $\mathrm{Im} W_Q = \ker Q$, and $W_Q^* W_Q = I$. It can be verified that the desired W_R and W_Q can be chosen as follows:

$$W_R = \left[\begin{array}{c|c|c} V_1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & I^{(n_T,\overline{n}_S,\overline{n}_P,r_T,\overline{r}_S,\overline{r}_P)} & 0 \\ \hline 0 & 0 & I^{n_w} \\ \hline V_2 & 0 & 0 \end{array}\right], \qquad W_Q = \left[\begin{array}{c|c|c} 0 & I^{(n_T^+,\overline{n}_S^+,\overline{n}_P,r_T^+,\overline{r}_S^+,\overline{r}_P)} & 0 \\ \hline U_1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline U_2 & 0 & 0 \\ \hline 0 & 0 & I^{n_z} \end{array}\right].$$

To see that the inequalities $W_R^* H W_R \prec 0$ and $W_Q^* H W_Q \prec 0$ hold, the left-hand-side of each inequality is expanded, and the Schur complement formula is applied to each resulting inequality. The following inequalities are obtained:

$$\begin{aligned}(V_1^* A + V_2^* C_1)\,\hat{Y}_{11}\,(A^* V_1 + C_1^* V_2) + (V_1^* B_1 + V_2^* D_{11})\,(B_1^* V_1 + D_{11}^* V_2)& \\ - V_1^* S^* \hat{Y}_{11} S V_1 - V_2^* V_2 \prec 0&,\end{aligned}$$

$$\begin{aligned}(U_1^* A^* + U_2^* B_1^*)\, S^* \hat{X}_{11} S\,(A U_1 + B_1 U_2) + (U_1^* C_1^* + U_2^* D_{11}^*)\,(C_1 U_1 + D_{11} U_2)& \\ - U_1^* \hat{X}_{11} U_1 - U_2^* U_2 \prec 0&,\end{aligned}$$

which correspond to (4.23) and (4.24), respectively. □

Algorithm 4.12 shows how to use the solutions $\hat{X}_{11}$ and $\hat{Y}_{11}$ to the synthesis conditions (4.23), (4.24), and (4.25) to construct the desired controller $\mathcal{K}_\delta$.

Algorithm 4.12. Consider a distributed NSLPV plant $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. Denote the dimensions of the plant by $n_T(t,k)$, $n_i^S(t,k)$, and $n_j^P(t,k)$ for all $(t,k) \in \mathbb{Z} \times V$, $i = 1, \ldots, d$, and $j = 1, \ldots, m$. Denote by $\mathcal{K}_\delta$ the desired 1-admissible synthesis for the plant $\mathcal{G}_\delta$. Denote the computed realization of $\mathcal{K}_\delta$ by $(A^K, B^K, C^K, D^K, \mathbf{\Delta}^K)$ and the computed controller dimensions by $r_T(t,k)$, $r_i^S(t,k)$, and $r_j^P(t,k)$ for all $(t,k) \in \mathbb{Z} \times V$, $i = 1, \ldots, d$, and $j = 1, \ldots, m$. Given $\hat{X}_{11}$ and $\hat{Y}_{11}$ in $\mathcal{X}$ structured as shown in (4.18) that satisfy (4.23), (4.24), and (4.25), the distributed NSLPV controller $\mathcal{K}_\delta$ is constructed as follows:

1. Determine the controller dimensions as follows:

$$
\begin{aligned}
r_T(t,k) &= \operatorname{rank}\left(Y_{T,11}(t,k) - (X_{T,11}(t,k))^{-1}\right), \\
r_i^S(t,k) &= \operatorname{rank}\left(Y_{i,11}^S(t,k) - (X_{i,11}^S(t,k))^{-1}\right), \\
r_j^P(t,k) &= \operatorname{rank}\left(Y_{j,11}^P(t,k) - (X_{j,11}^P(t,k))^{-1}\right).
\end{aligned}
$$

2. Use Lemma 4.9 to construct $X_P^L \succ 0$ such that X_P^L and its inverse $(X_P^L)^{-1}$ have the structure shown in (4.21).

3. Construct H as defined in (4.17), and solve the inequality defined in (4.16) for J.

4. Extract the controller operators A^K, B^K, C^K, and D^K from J as in (4.12).

4.6 Finite Dimensional Semidefinite Programs

This section contains a brief discussion on distributed NSLPV systems that are formed by eventually time-periodic subsystems interconnected over finite graphs, i.e., graphs in which the set of vertices V and the set of edges E are finite. It is shown that, for such systems, the analysis and synthesis problems are finite dimensional semidefinite programs (SDPs). It is assumed in this section that the graph representing the interconnection structure is finite.

Definition 4.13. Consider a distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. A subsystem $G^{(k)}$ of $\mathcal{G}_\delta$ with zero state-space matrices for $t < 0$ is said to be (h_k, q_k)-eventually time-periodic for some integers $h_k \geq 0$ and $q_k > 0$ if the corresponding state-space matrices become time-periodic with period q_k after some initial finite time-horizon h_k, i.e., $[\![W]\!](t + h_k + z\, q_k, k) = [\![W]\!](t + h_k, k)$ for all $W = A,\ B,\ C,\ D$ and $t, z \in \mathbb{N}_0$. Furthermore, system $\mathcal{G}_\delta$ is said to be (h, q)-eventually time-periodic if all the subsystems are (h, q)-eventually time-periodic. Namely, if each subsystem $G^{(k)}$ is (h_k, q_k)-eventually time-periodic, then system $\mathcal{G}_\delta$ is (h, q)-eventually time-periodic, where $h = \max_{k \in V} h_k$ and q is the least common multiple of q_k for all $k \in V$.

Definition 4.14. A partitioned graph-diagonal operator W is said to be (h, q)-eventually time-periodic if $[\![W]\!](t + h + zq, k) = [\![W]\!](t + h, k)$ for all $t, z \in \mathbb{N}_0$ and $k \in V$.

An important special case of an eventually time-periodic NSLPV subsystem is a standard LPV subsystem, i.e., a subsystem in which the nominal part is time-invariant. Specifically, a standard LPV subsystem is a $(0, 1)$-eventually time-periodic NSLPV subsystem, and so the analysis and synthesis results derived in this chapter apply to distributed LPV systems in which all the subsystems have standard LPV models.

Theorem 4.15. *Consider an (h, q)-eventually time-periodic distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$, and let the set $\mathcal{X}$ be defined as in (4.7). Then, the following statements hold:*

1. *Inequality (4.9) (respectively, inequality (4.8)) admits a solution $X \in \mathcal{X}$ if and only*

if it admits an (M,q)-eventually time-periodic (respectively, an (h,q)-eventually time-periodic) solution $X_{\mathrm{eper}} \in \mathcal{X}$ for some integer $M \geq h$.

2. *The synthesis conditions (4.23), (4.24), and (4.25) admit solutions $\hat{X}_{11}$ and $\hat{Y}_{11}$ in $\mathcal{X}$ if and only if they admit (M,q)-eventually time-periodic solutions $\hat{X}_{11,\mathrm{eper}}$ and $\hat{Y}_{11,\mathrm{eper}}$ in $\mathcal{X}$ for some integer $M \geq h$.*

3. *If $h = 0$ in statements 1 and 2 of this theorem, then M can also be taken equal to 0.*

The proof of this result uses averaging techniques such as the ones discussed in [23, 28, 31]. The proof is omitted as it parallels the proof of [35, Proposition 21]; see also the discussion in Section 2.7 on eventually time-periodic distributed LTV systems.

Thus, for an (h,q)-eventually time-periodic distributed NSLPV system, t is restricted to some finite time-horizon $M \geq h$ and one time-period, i.e., $0 \leq t \leq M + q - 1$, when expressing the analysis and synthesis conditions in terms of their equivalent sequences of LMIs. Moreover, since the graph that describes the interconnection structure of the system is assumed to be finite, then the analysis and synthesis problems become finite dimensional SDPs.

4.7 Illustrative Example

4.7.1 Problem Formulation

Consider a distributed NSLPV plant $\mathcal{G}_\delta$ formed by four subsystems that are interconnected as shown in Figure 4.2. The leader $G^{(1)}$ is a two-thruster hovercraft described in [34], and the followers $G^{(2)}$, $G^{(3)}$, and $G^{(4)}$ are non-holonomic vehicles described in [42]. The leader and subsystem $G^{(2)}$ are depicted in Figure 4.4. The leader is to track the eventually time-periodic reference trajectory defined in [34], and the followers are to track the leader while keeping formation. For simplicity, the four vehicles are assumed to be initially on top of each other; any desired formation can then be implemented by applying appropriate translations when the controller is executed.

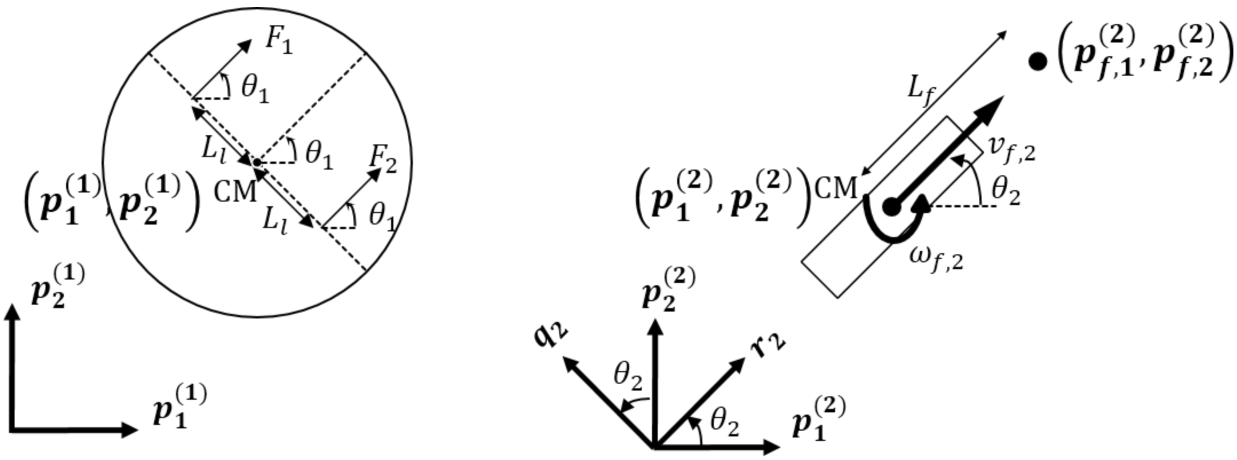

Figure 4.4: The figure on the left depicts the two-thruster hovercraft that corresponds to the leader subsystem $G^{(1)}$. The figure on the right depicts the non-holonomic vehicle corresponding to subsystem $G^{(2)}$.

Let $(p_1^{(1)}, p_2^{(1)})$ be the position of the center of mass (CM) of the hovercraft, θ_1 be its orientation with respect to the $p_1^{(1)}$-axis, and F_1 and F_2 be the force control inputs applied at a distance $L_l = 0.15\,\mathrm{m}$ from the CM. The forces take values between 0 and 2.5 N. Let $\nu = \left[p_1^{(1)}\ p_2^{(1)}\ \theta_1\ \dot{p}_1^{(1)}\ \dot{p}_2^{(1)}\ \dot{\theta}_1\right]^*$ and $F = [F_1\ F_2]^*$, then the equations of motion of the hovercraft are given by $\dot{\nu} = f(\nu, F)$, where $f(\nu, F)$ is defined in [34]. These equations are linearized about the reference trajectory (ν_r, F_r) to result in

$$\dot{\overline{\nu}} = A_c(t_c)\overline{\nu} + B_{2c}(t_c)\overline{F}, \qquad \overline{\nu} = \nu - \nu_r,$$

$$\overline{F} = F - F_r, \qquad A_c(t_c) = \left.\frac{\partial f}{\partial \nu}\right|_{(\nu,F)=(\nu_r,F_r)}, \qquad B_{2c}(t_c) = \left.\frac{\partial f}{\partial F}\right|_{(\nu,F)=(\nu_r,F_r)},$$

where t_c denotes continuous time. Then, the effect of the exogenous disturbances $\tilde{w}(t_c)$ is added as in

$$\dot{\overline{\nu}}(t_c) = A_c(t_c)\overline{\nu}(t_c) + B_{1c}\tilde{w}(t_c) + B_{2c}(t_c)\overline{F}(t_c).$$

The disturbances consist of torques and forces in both the $p_1^{(1)}$ and $p_2^{(1)}$ directions. For simplicity, B_{1c} is defined as $B_{1c} = \begin{bmatrix} 0_3 & I_3 \end{bmatrix}^*$. The inputs and disturbances are applied in discrete-time at a frequency of 20Hz. A bilinear transformation [12] is used to obtain a discrete-time trapezoidal approximation for the previous equations. Let $t \in \mathbb{N}_0$ denote discrete time and $\tau = 0.05$ seconds be the sampling period, then the following equations are obtained:

$$\begin{aligned} &x_T(t+1,1) = \overline{A}_{TT}(t,1)x_T(t,1) + \overline{B}_{T1}(t,1)w(t,1) + \overline{B}_{T2}(t,1)u(t,1), \\ &x_T(t,1) = \overline{\nu}(t\,\tau), \quad w(t,1) = \tilde{w}(t\,\tau), \quad u(t,1) = \overline{F}(t\,\tau). \end{aligned}$$

The reference trajectory is $(45,120)$-eventually time-periodic, and so are the state-space matrices $\overline{A}_{TT}(t,1)$, $\overline{B}_{T1}(t,1)$, and $\overline{B}_{T2}(t,1)$. The position and the orientation of the hovercraft are assumed to be measurable at each time-step. Then,

$$\begin{aligned} \overline{C}_{1T}(t,1) &= \begin{bmatrix} \alpha_1 \begin{bmatrix} I_3 & 0_3 \end{bmatrix} \\ 0_{2\times 6} \end{bmatrix}, & \overline{D}_{12}(t,1) &= \begin{bmatrix} 0_{3\times 2} \\ \alpha_2 I_2 \end{bmatrix}, \\ \overline{D}_{11}(t,1) &= 0_{5\times 3}, & \overline{C}_{2T}(t,1) &= \begin{bmatrix} I_3 & 0_3 \end{bmatrix}, \\ \overline{D}_{21}(t,1) &= 0_3, & \overline{D}_{22}(t,1) &= 0_{3\times 2}, \end{aligned}$$

for all $t \in \mathbb{N}_0$. The weights α_1 and α_2 are chosen as $\alpha_1 = 0.3$ and $\alpha_2 = 0.2$, respectively.

Since there are no incoming edges to vertex $k = 1$ and there are no parameters affecting subsystem $G^{(1)}$, then the state-space matrices $\overline{A}_{TS}(t,1)$, $\overline{A}_{SS}(t,1)$, $\overline{A}_{PS}(t,1)$, $\overline{C}_{1S}(t,1)$, $\overline{C}_{2S}(t,1)$ $\overline{A}_{TP}(t,1)$, $\overline{A}_{SP}(t,1)$, $\overline{A}_{PP}(t,1)$, $\overline{C}_{1P}(t,1)$, $\overline{C}_{2P}(t,1)$, $\overline{A}_{PT}(t,1)$, $\overline{B}_{P1}(t,1)$, and $\overline{B}_{P2}(t,1)$ have at least one zero dimension, i.e., are non-existent for all $t \in \mathbb{N}_0$.

The leader sends $\left(p_1^{(1)}, p_2^{(1)}\right) = \left(\overline{p}_1^{(1)} + p_{1,r}^{(1)}, \overline{p}_2^{(1)} + p_{2,r}^{(1)}\right)$ to the followers, i.e., the spatial states that correspond to the outgoing edges from vertex $k = 1$ are defined as $x_1(t,2) = x_2(t,3) = \begin{bmatrix} p_1^{(1)}(t\,\tau) & p_2^{(1)}(t\,\tau) \end{bmatrix}^*$. Since the temporal state vector $x_T(t,1)$ denotes the error between the actual and reference trajectories, and since the information transfer between the subsystems is subjected to a delay of one time-step, the vector of exogenous inputs $w(t,1)$ needs to be augmented by $\begin{bmatrix} 10\,p_{1,r}^{(1)}((t+1)\tau) & 10\,p_{2,r}^{(1)}((t+1)\tau) \end{bmatrix}^*$. If $\mathcal{W}$ is defined as $\mathcal{W} = \begin{bmatrix} I_2 & 0_{2\times 4} \end{bmatrix}$, it then follows that

$$\overline{A}_{ST}(t,1) = \begin{bmatrix} \mathcal{W} \\ \mathcal{W} \end{bmatrix} \overline{A}_{TT}(t,1), \qquad \overline{B}_{S1}(t,1) = \begin{bmatrix} \begin{bmatrix} \mathcal{W} \\ \mathcal{W} \end{bmatrix} \overline{B}_{T1}(t,1) & \begin{bmatrix} 0.1 I_2 \\ 0.1 I_2 \end{bmatrix} \end{bmatrix},$$
$$\overline{B}_{S2}(t,1) = \begin{bmatrix} \mathcal{W} \\ \mathcal{W} \end{bmatrix} \overline{B}_{T2}(t,1),$$

for all $t \in \mathbb{N}_0$. Finally, the state-space matrices $\overline{B}_{T1}(t,1)$, $\overline{D}_{11}(t,1)$, and $\overline{D}_{21}(t,1)$ defined above are augmented by two zero columns to account for the augmentation of the vector of exogenous inputs. This concludes the derivation of the state-space matrices of the leader.

For the followers, the focus is placed on modeling subsystem $G^{(2)}$, since subsystems $G^{(3)}$ and $G^{(4)}$ can be modeled in a similar way. As in [42, 52], a point $\left(p_{f,1}^{(2)}, p_{f,2}^{(2)}\right)$ at a fixed distance $L_f = 0.02\,\mathrm{m}$ ahead of the CM $\left(p_1^{(2)}, p_2^{(2)}\right)$ of the vehicle is considered. Let θ_2 be the orientation of $G^{(2)}$ with respect to the $p_1^{(2)}$-axis, $v_{f,2} = v_{c,2} + v_{d,2}$ be the forward velocity, and $\omega_{f,2} = \omega_{c,2} + \omega_{d,2}$ be the angular velocity, where $v_{c,2}$ and $\omega_{c,2}$ denote the control inputs and $v_{d,2}$ and $\omega_{d,2}$ denote the disturbances. In the (r_2, q_2)-reference frame, the equations of motion of subsystem $G^{(2)}$ are given by

$$\dot{r}_2 = v_{f,2} + \omega_{f,2} q_2,$$

$$\dot{q}_2 = \omega_{f,2} L_f - \omega_{f,2} r_2,$$
$$\dot{\theta}_2 = \omega_{f,2}.$$

As in [42], the third equation is disregarded during control design. If one assumes that $\omega_{f,2} \in [-5, 5]$ rad/sec and defines $\eta_2 = \frac{1}{5}\omega_{f,2}$, then the previous equations can be expressed as

$$\begin{bmatrix} \dot{r}_2 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 5\eta_2 \\ -5\eta_2 & 0 \end{bmatrix} \begin{bmatrix} r_2 \\ q_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & L_f \end{bmatrix} \begin{bmatrix} v_{d,2} \\ \omega_{d,2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & L_f \end{bmatrix} \begin{bmatrix} v_{c,2} \\ \omega_{c,2} \end{bmatrix}.$$

These equations are written in LFT form by taking η_2 as the parameter. In the modeling step, η_2 is chosen as $\eta_2 = \frac{1}{5} w_{f,2}$, where $w_{f,2} = w_{c,2} + w_{d,2}$. However, in Section 4.7.3, when the closed-loop system is simulated using the designed controller, η_2 is implemented as $\eta_2 = w_{c,2}$, since the disturbance $w_{d,2}$ is not measurable. Since the sampling period τ is sufficiently small, the bilinear transformation in [12] can be used to obtain a discrete-time trapezoidal approximation of these equations. When applying the bilinear transformation, the parameter states, i.e., the signals introduced by the LFT formulation, are treated as exogenous inputs, and it is assumed that the parameters and the parameter states are constants over each time-interval $[t\,\tau\,,\,(t+1)\,\tau)$ for all $t \in \mathbb{N}_0$. The following terms are thus defined:

$$x_T(t,2) = [r_2(t\,\tau) \;\; q_2(t\,\tau)]^*\,, \qquad w(t,2) = [v_{d,2}(t\,\tau) \;\; \omega_{d,2}(t\,\tau)]^*\,,$$
$$u(t,2) = [v_{c,2}(t\,\tau) \;\; \omega_{c,2}(t\,\tau)]^*\,, \qquad \delta_1(t,2) = \eta_2\,(t\,\tau)\,.$$

Subsystem $G^{(2)}$ receives $x_1(t,2) = \left[p_1^{(1)}(t\,\tau) \;\; p_2^{(1)}(t\,\tau)\right]^*$, i.e., the spatial state associated with the incoming edge to vertex $k = 2$ contains the leader information. Subsystem $G^{(2)}$ computes the received coordinates in the (r_2, q_2)-reference frame. The output measurements and performance outputs of subsystem $G^{(2)}$ are thus defined as follows:

$$y(t,2) = \begin{bmatrix} r_2(t\,\tau) - L_f - \left(p_1^{(1)}(t\,\tau)\cos\theta_2(t\,\tau) + p_2^{(1)}(t\,\tau)\sin\theta_2(t\,\tau)\right) \\ q_2(t\,\tau) - \left(-p_1^{(1)}(t\,\tau)\sin\theta_2(t\,\tau) + p_2^{(1)}(t\,\tau)\cos\theta_2(t\,\tau)\right) \end{bmatrix},$$

$$z(t,2) = \begin{bmatrix} \alpha_3\, y(t,2) \\ \alpha_4\, u(t,2) \end{bmatrix},$$

where the weights α_3 and α_4 are chosen as $\alpha_3 = 0.3$ and $\alpha_4 = 0.5$, respectively. The equations for $y(t,2)$ and $z(t,2)$ are written in LFT form by defining the parameters $\delta_2(t,2) = \cos(\theta_2(t\,\tau))$ and $\delta_3(t,2) = \sin(\theta_2(t\,\tau))$. The choice of cosine and sine functions as scheduling parameters may be conservative, but in the considered problem, this choice reduces the computational complexity significantly. In light of the above, $\underline{\Delta}(t,2)$ is defined as

$$\underline{\Delta}(t,2) = \operatorname{diag}\left(\delta_1(t,2)I_2, \delta_2(t,2)I_2, \delta_3(t,2)I_2\right),$$

for all $t \in \mathbb{N}_0$. Since L_f appears as an exogenous input in $y(t,2)$ and $z(t,2)$ as shown above, then the vector $w(t,2)$ is correspondingly augmented by $10L_f$.

Subsystem $G^{(2)}$ sends its coordinates (r_2, q_2) to subsystem $G^{(4)}$, and so the spatial state vector that corresponds to the outgoing edge from vertex $k = 2$ is defined as $x_2(t,4) = [r_2(t\,\tau) \quad q_2(t\,\tau)]^*$ for all $t \in \mathbb{N}_0$. The modeling of subsystem $G^{(2)}$ is now concluded by explicitly giving the state-space matrices that were constructed as per the above discussion. Namely, for all $t \in \mathbb{N}_0$, the state-space matrices of subsystem $G^{(2)}$ are given by

$$\overline{C}_{2T}(t,2) = I_2, \qquad \overline{C}_{2S}(t,2) = 0_2, \qquad \overline{C}_{2P}(t,2) = \begin{bmatrix} 0_2 & I_2 & I_2 \end{bmatrix},$$

$$\overline{C}_{1T}(t,2) = \begin{bmatrix} \alpha_3\, I_2 \\ 0_2 \end{bmatrix}, \qquad \overline{C}_{1S}(t,2) = 0_{4\times 2}, \qquad \overline{C}_{1P}(t,2) = \begin{bmatrix} \alpha_3 \begin{bmatrix} 0_2 & I_2 & I_2 \end{bmatrix} \\ 0_{2\times 6} \end{bmatrix},$$

$$\overline{D}_{12}(t,2) = \begin{bmatrix} 0_2 \\ \alpha_4\, I_2 \end{bmatrix}, \qquad \overline{D}_{22}(t,2) = 0_2, \qquad \overline{D}_{11}(t,2) = \begin{bmatrix} 0_{4\times 2} & \begin{bmatrix} -0.1\,\alpha_3 \\ 0_{3\times 1} \end{bmatrix} \end{bmatrix},$$

$$\overline{A}_{PT}(t,2) = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ 0_{4\times 2} \end{bmatrix}, \qquad \overline{A}_{PP}(t,2) = 0_6, \qquad \overline{D}_{21}(t,2) = \begin{bmatrix} 0_2 & \begin{bmatrix} -0.1 \\ 0 \end{bmatrix} \end{bmatrix},$$

$$\overline{A}_{TS}(t,2) = 0_2, \qquad \overline{A}_{SS}(t,2) = 0_2, \qquad \overline{A}_{PS}(t,2) = \begin{bmatrix} 0_2 \\ - \begin{bmatrix} I_2 \\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{bmatrix} \end{bmatrix},$$

$$\overline{B}_{P1}(t,2) = 0_{6\times 3}, \qquad \overline{B}_{P2}(t,2) = 0_{6\times 2},$$

$$\overline{A}_{TT}(t,2) = \overline{A}_{ST}(t,2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \overline{A}_{TP}(t,2) = \overline{A}_{SP}(t,2) = \begin{bmatrix} \begin{bmatrix} 5\,\tau & 0 \\ 0 & 5\,\tau \end{bmatrix} & 0_{2\times 4} \end{bmatrix},$$

$$\overline{B}_{T2}(t,2) = \overline{B}_{S2}(t,2) = \begin{bmatrix} \tau & 0 \\ 0 & \tau\, L_f \end{bmatrix}, \qquad \overline{B}_{T1}(t,2) = \overline{B}_{S1}(t,2) = \begin{bmatrix} \tau & 0 & 0 \\ 0 & \tau\, L_f & 0 \end{bmatrix}.$$

These state-space matrices are time-invariant, i.e., $(0,1)$-eventually time-periodic. In other words, the followers $G^{(2)}$, $G^{(3)}$, and $G^{(4)}$ have standard LPV models. The state-space matrices for subsystems $G^{(3)}$ and $G^{(4)}$ are constructed in a similar way. For subsystem $G^{(4)}$, it is noted that

$$\underline{\Delta}(t,4) = \operatorname{diag}\left(\delta_1(t,4)I_2, \delta_2(t,4)I_2, \delta_3(t,4)I_2, \delta_4(t,4)I_2, \delta_5(t,4)I_2\right),$$

$$z(t,4) = \begin{bmatrix} 0.1\, y(t,4) \\ \alpha_4\, u(t,4) \end{bmatrix},$$

$$y(t,4) = \begin{bmatrix} r_4(t\,\tau) - (\ \ r_3(t\,\tau)\delta_2(t,4) + q_3(t\,\tau)\delta_3(t,4)) \\ q_4(t\,\tau) - (-r_3(t\,\tau)\delta_3(t,4) + q_3(t\,\tau)\delta_2(t,4)) \\ r_4(t\,\tau) - (\ \ r_2(t\tau)\delta_4(t,4) + q_2(t\,\tau)\delta_5(t,4)) \\ q_4(t\,\tau) - (-r_2(t\,\tau)\delta_5(t,4) + q_2(t\,\tau)\delta_4(t,4)) \end{bmatrix},$$

$$\begin{bmatrix} \delta_1(t,4) \\ \delta_2(t,4) \\ \delta_3(t,4) \\ \delta_4(t,4) \\ \delta_5(t,4) \end{bmatrix} = \begin{bmatrix} \eta_4(t\tau) \\ \cos(\theta_4(t\,\tau) - \theta_3(t\,\tau)) \\ \sin(\theta_4(t\,\tau) - \theta_3(t\,\tau)) \\ \cos(\theta_4(t\,\tau) - \theta_2(t\,\tau)) \\ \sin(\theta_4(t\,\tau) - \theta_2(t\,\tau)) \end{bmatrix},$$

for all $t \in \mathbb{N}_0$.

4.7.2 Design of Distributed NSLPV Controller

The plant $\mathcal{G}_\delta$ just defined is a distributed NSLPV system, since the leader $G^{(1)}$ has an LTV model and the followers $G^{(2)}$, $G^{(3)}$, and $G^{(4)}$ have standard LPV models. A γ-admissible synthesis $\mathcal{K}_\delta$ for the distributed NSLPV plant $\mathcal{G}_\delta$ is now designed using the technique developed in this chapter. In the derivation of the synthesis results in Section 4.5, it is assumed that $\gamma = 1$. This assumption is made without loss of generality because a γ-admissible synthesis for a plant $\mathcal{G}_\delta$ is 1-admissible for the scaled plant $\mathcal{G}_{\text{scaled},\delta}$ in which γ is absorbed into the system operators, namely, $C_{\text{scaled},1} = \frac{1}{\gamma}C_1$, $D_{\text{scaled},11} = \frac{1}{\gamma}D_{11}$, and $D_{\text{scaled},12} = \frac{1}{\gamma}D_{12}$. Using the scaled operators, γ is incorporated into the synthesis conditions of Theorem 4.11; and using the Schur complement formula, the resulting conditions are reformulated so that they are linear in γ^2; see [27, Remark 10] for the details. Denote by $\gamma_{\min}$ the minimum value of γ for which there exist $(45, 120)$-eventually time-periodic solutions to the synthesis conditions.

Specifically, $\gamma_{\min}$ is obtained by solving the following SDP:

$$\begin{aligned}
&\text{minimize} && \gamma^2 \\
&\text{subject to:} && \mathcal{E}_1\left(Y_{\text{in}}(t,k), Y_{\text{out}}(t,k), \gamma, t, k\right) \prec 0, \qquad \mathcal{E}_2\left(X_{\text{in}}(t,k), X_{\text{out}}(t,k), \gamma, t, k\right) \prec 0,
\end{aligned}$$

$$\begin{bmatrix} Y_T(t,1) & I_6 \\ I_6 & X_T(t,1) \end{bmatrix} \succeq 0, \qquad \begin{bmatrix} Y_T(t,k_0) & I_2 \\ I_2 & X_T(t,k_0) \end{bmatrix} \succeq 0,$$

$$\begin{bmatrix} Y_1^S(t,2) & I_2 \\ I_2 & X_1^S(t,2) \end{bmatrix} \succeq 0, \qquad \begin{bmatrix} Y_1^S(t,4) & I_2 \\ I_2 & X_1^S(t,4) \end{bmatrix} \succeq 0,$$

$$\begin{bmatrix} Y_2^S(t,3) & I_2 \\ I_2 & X_2^S(t,3) \end{bmatrix} \succeq 0, \qquad \begin{bmatrix} Y_2^S(t,4) & I_2 \\ I_2 & X_2^S(t,4) \end{bmatrix} \succeq 0,$$

$$\begin{bmatrix} Y_1^P(t,k_0) & I_2 \\ I_2 & X_1^P(t,k_0) \end{bmatrix} \succeq 0, \qquad \begin{bmatrix} Y_2^P(t,k_0) & I_2 \\ I_2 & X_2^P(t,k_0) \end{bmatrix} \succeq 0,$$

$$\begin{bmatrix} Y_3^P(t,k_0) & I_2 \\ I_2 & X_3^P(t,k_0) \end{bmatrix} \succeq 0, \qquad \begin{bmatrix} Y_4^P(t,4) & I_2 \\ I_2 & X_4^P(t,4) \end{bmatrix} \succeq 0,$$

$$\begin{bmatrix} Y_5^P(t,4) & I_2 \\ I_2 & X_5^P(t,4) \end{bmatrix} \succeq 0,$$

for all $t = 0, 1, \ldots, 164$, $k = 1, \ldots, 4$, $k_0 = 2, \ldots, 4$, and $Z = X$, Y, where

$$\begin{aligned}
&Z_T(165,k) = Z_T(45,k), && Z_1^S(165,2) = Z_1^S(45,2), \\
&Z_1^S(165,4) = Z_1^S(45,4), && Z_2^S(165,3) = Z_2^S(45,3), \\
&Z_2^S(165,4) = Z_2^S(45,4).
\end{aligned}$$

$$
\begin{aligned}
Z_{\mathrm{in}}(t,1) &= Z_T(t,1),\\
Z_{\mathrm{out}}(t,1) &= \operatorname{diag}\left(Z_T(t+1,1), Z_1^S(t+1,2), Z_2^S(t+1,3)\right),\\
Z_{\mathrm{in}}(t,2) &= \operatorname{diag}\left(Z_T(t,2), Z_1^S(t,2), Z_1^P(t,2), Z_2^P(t,2), Z_3^P(t,2)\right),\\
Z_{\mathrm{out}}(t,2) &= \operatorname{diag}\left(Z_T(t+1,2), Z_2^S(t+1,4), Z_1^P(t,2), Z_2^P(t,2), Z_3^P(t,2)\right),\\
Z_{\mathrm{in}}(t,3) &= \operatorname{diag}\left(Z_T(t,3), Z_2^S(t,3), Z_1^P(t,3), Z_2^P(t,3), Z_3^P(t,3)\right),\\
Z_{\mathrm{out}}(t,3) &= \operatorname{diag}\left(Z_T(t+1,3), Z_1^S(t+1,4), Z_1^P(t,3), Z_2^P(t,3), Z_3^P(t,3)\right),\\
Z_{\mathrm{in}}(t,4) &= \operatorname{diag}\left(Z_T(t,4), Z_1^S(t,4), Z_2^S(t,4), Z_1^P(t,4), Z_2^P(t,4), Z_3^P(t,4), Z_4^P(t,4), Z_5^P(t,4)\right),\\
Z_{\mathrm{out}}(t,4) &= \operatorname{diag}\left(Z_T(t+1,4), Z_1^P(t,4), Z_2^P(t,4), Z_3^P(t,4), Z_4^P(t,4), Z_5^P(t,4)\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}_1(P,Q,\gamma,t,k) = (F(t,k))^* \; P \; F(t,k) - (V_1(t,k))^* \; Q \; V_1(t,k)&\\
+ (M(t,k))^* M(t,k) - \gamma^2 \; (V_2(t,k))^* V_2(t,k)&,
\end{aligned}
$$

$$
\mathcal{E}_2(P,Q,\gamma,t,k) = \begin{bmatrix} (W(t,k))^* Q \, W(t,k) - (U_1(t,k))^* P \, U_1(t,k) - (U_2(t,k))^* U_2(t,k) & (N(t,k))^* \\ N(t,k) & -\gamma^2 \; I \end{bmatrix},
$$

$$
\begin{aligned}
&\operatorname{Im}\begin{bmatrix} V_1(t,k) \\ V_2(t,k) \end{bmatrix} = \ker \begin{bmatrix} [\![B_2^*]\!](t,k) & (\overline{D}_{12}(t,k))^* \end{bmatrix},\\
&(V_1(t,k))^* V_1(t,k) + (V_2(t,k))^* V_2(t,k) = I,\\
&\operatorname{Im}\begin{bmatrix} U_1(t,k) \\ U_2(t,k) \end{bmatrix} = \ker \begin{bmatrix} [\![C_2]\!](t,k) & \overline{D}_{21}(t,k) \end{bmatrix},\\
&(U_1(t,k))^* U_1(t,k) + (U_2(t,k))^* U_2(t,k) = I,
\end{aligned}
$$

$$
\begin{aligned}
F(t,k) &= [\![A^*]\!](t,k)V_1(t,k) + [\![C_1^*]\!](t,k)V_2(t,k),\\
M(t,k) &= [\![B_1^*]\!](t,k)V_1(t,k) + (\overline{D}_{11}(t,k))^*V_2(t,k),\\
W(t,k) &= [\![A]\!](t,k)U_1(t,k) + [\![B_1]\!](t,k)U_2(t,k),\\
N(t,k) &= [\![C_1]\!](t,k)U_1(t,k) + \overline{D}_{11}(t,k)U_2(t,k).
\end{aligned}
$$

This SDP is modeled using Yalmip [58] and solved using SDPT-3 [79]. The total number of constraints is 24751, the dimension of the SDP variable is 33000, the dimension of the linear variable is 1, and the number of SDP blocks is 4455. The computations are carried out in Matlab R2016a on a Dell computer with 4 Intel Cores, 3.07 GHz processors, and 6 GB of RAM running Windows 10. The wall-clock time is 378 sec (CPU time 67 sec). The obtained optimal value of γ is $\gamma_{\min} \approxeq 64.86$. This value is relaxed to 66, and the problem is re-solved as a feasibility problem.

4.7.3 Closed-Loop System Simulation

Using the obtained synthesis solutions, the controller realization is constructed by following the procedure in Algorithm 4.12. The Matlab function ‘basiclmi’ is used in step 3 of the algorithm. The designed distributed NSLPV controller is applied to the nonlinear distributed plant, and the resulting closed-loop system is simulated. The subsystems start with their CM at $(0, 0.825)$m. The leader is subjected to random force and torque disturbances in $[-0.5, +0.5]$N and $[-0.05, 0.05]$ N.m, respectively. The followers are subjected to random disturbances that lie within $\pm15\%$ of the nominal input. The reference trajectory and the position of the CM of each of the four agents is shown in Figure 4.5. From the sample simulation shown in the figure, it is seen that the leader and the followers are able to track the reference trajectory reasonably well even in the presence of disturbances. Figure 4.6

shows plots of various performance outputs.

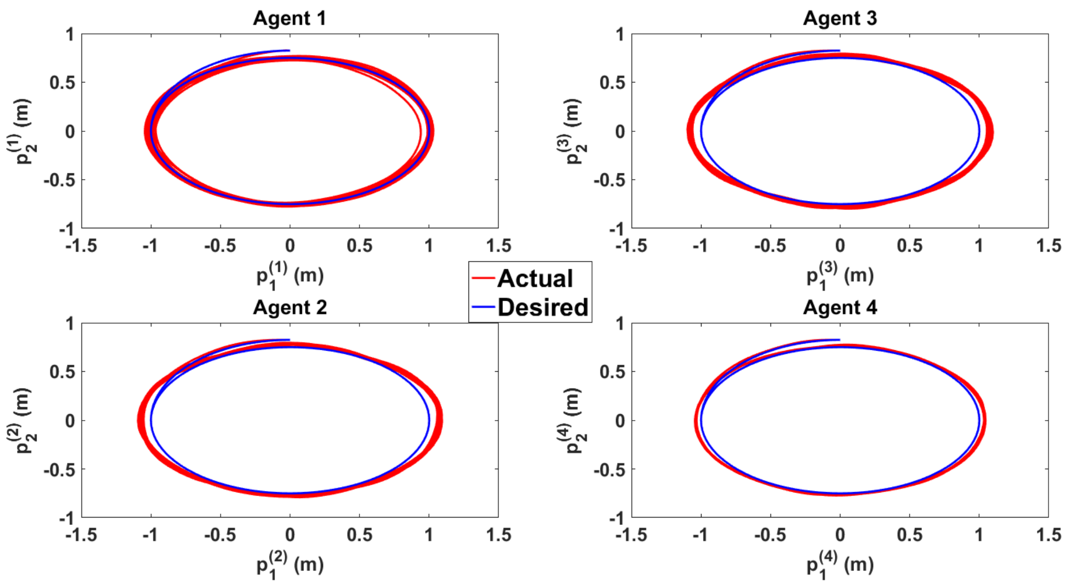

Figure 4.5: The figure shows the outcome of a sample simulation of the closed-loop system formed by the nonlinear distributed plant and the designed distributed NSLPV controller.

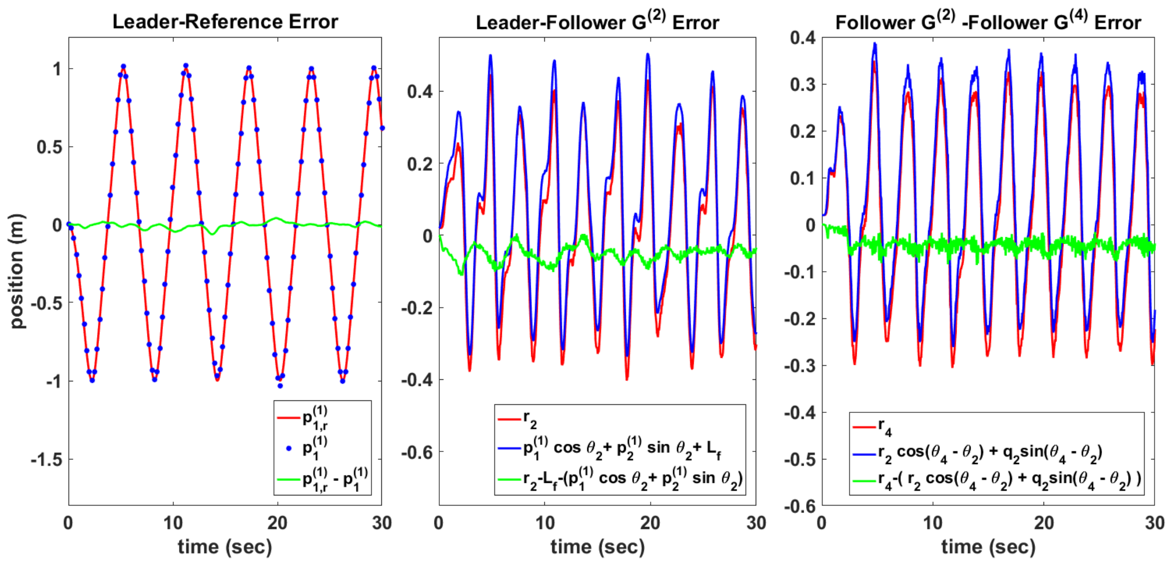

Figure 4.6: The figure shows plots of various performance outputs. The left plot shows a leader-reference error plot, the middle plot shows a leader-follower error plot, and the right plot shows a follower-follower error plot.

Chapter 5

Balanced Truncation of Interconnected Nonstationary LPV Systems

5.1 Chapter Overview

Model reduction is desirable for the interconnected NSLPV systems described in Chapter 4, since the sizes of the distributed system model and the analysis and synthesis problems scale with the numbers and dimensions of the temporal, spatial, and parameter states. This chapter applies the balanced truncation method for the model reduction of strongly stable interconnected NSLPV systems. To make the presentation in Chapters 5 and 6 self-contained, the operator theoretic framework developed in Chapter 4 is summarized in Section 5.2. The balanced truncation method is discussed in Section 5.3, and two expressions for the balanced truncation error bound are derived in Section 5.4. The first expression, given in Theorem 5.7, generalizes the standard balanced truncation error bound to the distributed NSLPV setting. The second expression, given in Theorem 5.11, extends the error bound expression given in Theorem 2.12. Namely, Theorem 2.12 applies whenever the truncation sequences are monotonic in time, whereas Theorem 5.11 applies for general truncation sequences. The application of the various error bound expressions is illustrated in Section 5.5.

5.2 Preliminaries

The notation and operator theoretic machinery defined in Chapter 4 are used in the present chapter, see Section 4.2 for the details. Some additional definitions are needed in this chapter. Let $X \succ 0$ be a positive definite graph-diagonal operator such that $X(t,k)$ is a diagonal matrix for all $(t,k) \in \mathbb{Z} \times V$. $\phi(X)$ denotes the sum of distinct diagonal entries of X, i.e., $\phi(X)$ is the sum of the distinct diagonal entries in $\operatorname{diag}\left(X(t,k)\right)_{(t,k)\in\mathbb{Z}\times V}$. For example, assume that $X(t,k) = 0$ for all (t,k) except for some (t_0,k_0), (t_0,k_1), and (t_1,k_1), where $X(t_0,k_0) = \operatorname{diag}(w_1,w_1,w_2,w_2)$, $X(t_0,k_1) = \operatorname{diag}(w_1,w_3,w_4)$, and $X(t_1,k_1) = \operatorname{diag}(w_3,w_4)$. Then, $\phi(X) = w_1 + w_2 + w_3 + w_4$. Consider a positive definite partitioned graph-diagonal operator $W = \operatorname{diag}(W_i) \succ 0$, where W_i are graph-diagonal operators such that $W_i(t,k)$ are diagonal matrices for all $(t,k) \in \mathbb{Z} \times V$. $\Phi(W)$ denotes the sum of distinct diagonal entries of W, i.e., $\Phi(W) = \phi(\llbracket W \rrbracket)$.

5.2.1 Operator Theoretic Framework

This section summarizes the operator theoretic framework of Section 4.3 along with the relevant analysis results from Section 4.4. Consider a distributed NSLPV system $\mathcal{G}_\delta$ that is described using the following state-space equations:

$$\left[\begin{array}{c} x_T\left(t+1,k\right) \\ \hline x_1\left(t+1,\rho_1(k)\right) \\ \vdots \\ x_d\left(t+1,\rho_d(k)\right) \\ \hline \alpha\left(t,k\right) \\ \hline y\left(t,k\right) \end{array}\right] = \left[\begin{array}{c|c|c|c} \overline{A}_{TT}(t,k) & \overline{A}_{TS}(t,k) & \overline{A}_{TP}(t,k) & \overline{B}_{T}(t,k) \\ \hline \overline{A}_{ST}(t,k) & \overline{A}_{SS}(t,k) & \overline{A}_{SP}(t,k) & \overline{B}_{S}(t,k) \\ \hline \overline{A}_{PT}(t,k) & \overline{A}_{PS}(t,k) & \overline{A}_{PP}(t,k) & \overline{B}_{P}(t,k) \\ \hline \overline{C}_{T}(t,k) & \overline{C}_{S}(t,k) & \overline{C}_{P}(t,k) & \overline{D}(t,k) \end{array}\right] \left[\begin{array}{c} x_T\left(t,k\right) \\ \hline x_1\left(t,k\right) \\ \vdots \\ x_d\left(t,k\right) \\ \hline \beta\left(t,k\right) \\ \hline u\left(t,k\right) \end{array}\right],$$

$$\beta\,(t,k) = \operatorname{diag}\left(\delta_1(t,k) I_{n_1^P(t,k)}, \ldots, \delta_m(t,k) I_{n_m^P(t,k)}\right) \alpha\,(t,k) = \underline{\Delta}(t,k)\ \alpha\,(t,k)\,, \tag{5.1}$$

for all $(t,k) \in \mathbb{Z} \times V$. $x_T(t,k)$ denotes the temporal state vector associated with subsystem $G^{(k)}$. $x_T(t,k)$ has a possibly time-varying dimension denoted by $n_T(t,k)$. $x_i(t,k)$ denotes the spatial state vector associated with the edge $(\rho_i^{-1}(k), k)$, and its dimension is denoted by $n_i^S(t,k)$. Similarly, $x_i(t, \rho_i(k))$ denotes the spatial state vector with dimension $n_i^S(t, \rho_i(k))$ that is associated with the edge $(k, \rho_i(k))$. The spatial states corresponding to the virtual interconnections that are added to ensure the regularity of the graph describing the interconnection structure, and their corresponding blocks in the state-space matrices are of zero dimensions. Due to the assumed communication latency, the information sent by a subsystem at time-step t reaches the target subsystem at $t+1$. The parameter states β and α are the internal signals introduced by formulating the subsystems in an LFT framework. The parameters $\delta_j(t,k)$ in $\underline{\Delta}(t,k)$ are not known a priori but are assumed to be measurable at each time-step t. The parameter state vectors $\beta(t,k)$ and $\alpha(t,k)$ are partitioned into m vector-valued channels conformably with the partitioning of $\underline{\Delta}(t,k)$, i.e., $\beta(t,k) = [\beta_1^*(t,k)\ \beta_2^*(t,k)\ \cdots\ \beta_m^*(t,k)]^*$ and $\alpha(t,k) = [\alpha_1^*(t,k)\ \alpha_2^*(t,k)\ \cdots\ \alpha_m^*(t,k)]^*$, where the dimension of $\alpha_j(t,k)$ and $\beta_j(t,k)$ is denoted by $n_j^P(t,k)$. $u(t,k)$ and $y(t,k)$ are the vectors of inputs and outputs associated with subsystem $G^{(k)}$, respectively. The dimensions of $u(t,k)$ and $y(t,k)$ are denoted by $n_u(t,k)$ and $n_y(t,k)$, respectively.

The state-space matrices are known a priori, are assumed to be uniformly bounded, and are partitioned according to the permutations $\rho_1, \ldots, \rho_d$ and the blocks in $\underline{\Delta}(t,k)$. Namely,

$$\overline{A}_{ST}(t,k) = \begin{bmatrix} A_1^{ST}(t,k) \\ \vdots \\ A_d^{ST}(t,k) \end{bmatrix}, \qquad \overline{A}_{PT}(t,k) = \begin{bmatrix} A_1^{PT}(t,k) \\ \vdots \\ A_m^{PT}(t,k) \end{bmatrix},$$

$$\overline{B}_S(t,k) = \begin{bmatrix} B_1^S(t,k) \\ \vdots \\ B_d^S(t,k) \end{bmatrix}, \qquad \overline{B}_P(t,k) = \begin{bmatrix} B_1^P(t,k) \\ \vdots \\ B_m^P(t,k) \end{bmatrix},$$

$$\overline{C}_S(t,k) = \begin{bmatrix} C_1^S(t,k) & \cdots & C_d^S(t,k) \end{bmatrix}, \qquad \overline{C}_P(t,k) = \begin{bmatrix} C_1^P(t,k) & \cdots & C_m^P(t,k) \end{bmatrix},$$

$$\overline{A}_{TS}(t,k) = \begin{bmatrix} A_1^{TS}(t,k) & \cdots & A_d^{TS}(t,k) \end{bmatrix}, \qquad \overline{A}_{TP}(t,k) = \begin{bmatrix} A_1^{TP}(t,k) & \cdots & A_m^{TP}(t,k) \end{bmatrix},$$

$$\overline{A}_{SS}(t,k) = \left[A_{ie}^{SS}(t,k)\right]_{i=1,\ldots,d;e=1,\ldots,d}, \qquad \overline{A}_{SP}(t,k) = \left[A_{ij}^{SP}(t,k)\right]_{i=1,\ldots,d;j=1,\ldots,m},$$

$$\overline{A}_{PS}(t,k) = \left[A_{ji}^{PS}(t,k)\right]_{j=1,\ldots,m;i=1,\ldots,d}, \qquad \overline{A}_{PP}(t,k) = \left[A_{jf}^{PP}(t,k)\right]_{j=1,\ldots,m;f=1,\ldots,m},$$

where $A_i^{TS}(t,k)$ is an $n_T(t+1,k) \times n_i^S(t,k)$ matrix, $A_{ie}^{SS}(t,k)$ is an $n_i^S(t+1,\rho_i(k)) \times n_e^S(t,k)$ matrix, $A_j^{TP}(t,k)$ is an $n_T(t+1,k) \times n_j^P(t,k)$ matrix, $A_{jf}^{PP}(t,k)$ is an $n_j^P(t,k) \times n_f^P(t,k)$ matrix, and so on.

The matrix-valued sequence obtained from each block/partition of the state-space matrices defines a graph-diagonal operator. These operators are then augmented to form the following partitioned graph-diagonal operators:

$$A_{TS} = \begin{bmatrix} A_1^{TS} & \cdots & A_d^{TS} \end{bmatrix}, \qquad A_{TP} = \begin{bmatrix} A_1^{TP} & \cdots & A_m^{TP} \end{bmatrix},$$

$$A_{ST} = \begin{bmatrix} A_1^{ST} \\ \vdots \\ A_d^{ST} \end{bmatrix}, \qquad A_{PT} = \begin{bmatrix} A_1^{PT} \\ \vdots \\ A_m^{PT} \end{bmatrix},$$

$$A_{SS} = \left[A_{ie}^{SS}\right]_{i=1,\ldots,d;e=1,\ldots,d}, \qquad A_{SP} = \left[A_{ij}^{SP}\right]_{i=1,\ldots,d;j=1,\ldots,m},$$

$$A_{PS} = \left[A_{ji}^{PS}\right]_{j=1,\ldots,m;i=1,\ldots,d}, \qquad A_{PP} = \left[A_{jf}^{PP}\right]_{j=1,\ldots,m;f=1,\ldots,m},$$

$$B_S = \begin{bmatrix} B_1^S \\ \vdots \\ B_d^S \end{bmatrix}, \qquad B_P = \begin{bmatrix} B_1^P \\ \vdots \\ B_m^P \end{bmatrix},$$

$$C_S = \begin{bmatrix} C_1^S & \cdots & C_d^S \end{bmatrix}, \qquad C_P = \begin{bmatrix} C_1^P & \cdots & C_m^P \end{bmatrix}.$$

Namely, for all $(t,k) \in \mathbb{Z} \times V$, $[\![A_{TS}]\!](t,k) = \overline{A}_{TS}(t,k)$, $[\![A_{TP}]\!](t,k) = \overline{A}_{TP}(t,k)$, and so on. The matrix-valued sequences $\overline{A}_{TT}(t,k)$, $\overline{B}_T(t,k)$, $\overline{C}_T(t,k)$, and $\overline{D}(t,k)$ also define graph-diagonal operators that are denoted by A_{TT}, B_T, C_T, and D, respectively.

The augmented partitioned graph-diagonal operators A, B, and C are then defined as follows:

$$A = \begin{bmatrix} A_{TT} & A_{TS} & A_{TP} \\ A_{ST} & A_{SS} & A_{SP} \\ A_{PT} & A_{PS} & A_{PP} \end{bmatrix}, \qquad B = \begin{bmatrix} B_T \\ B_S \\ B_P \end{bmatrix}, \qquad C = \begin{bmatrix} C_T & C_S & C_P \end{bmatrix}.$$

For all $(t,k) \in \mathbb{Z} \times V$, these operators satisfy

$$[\![A]\!](t,k) = \begin{bmatrix} \overline{A}_{TT}(t,k) & \overline{A}_{TS}(t,k) & \overline{A}_{TP}(t,k) \\ \overline{A}_{ST}(t,k) & \overline{A}_{SS}(t,k) & \overline{A}_{SP}(t,k) \\ \overline{A}_{PT}(t,k) & \overline{A}_{PS}(t,k) & \overline{A}_{PP}(t,k) \end{bmatrix}, \qquad [\![B]\!](t,k) = \begin{bmatrix} \overline{B}_T(t,k) \\ \overline{B}_S(t,k) \\ \overline{B}_P(t,k) \end{bmatrix},$$

$$[\![C]\!](t,k) = \begin{bmatrix} \overline{C}_T(t,k) & \overline{C}_S(t,k) & \overline{C}_P(t,k) \end{bmatrix}.$$

The graph-diagonal operators Δ_j are defined such that $\Delta_j(t,k) = \delta_j(t,k) I_{n_j^P(t,k)}$ for all $(t,k) \in \mathbb{Z} \times V$ and $j = 1, \ldots, m$. These operators are block-diagonally augmented to construct the partitioned graph-diagonal operator $\Delta_P = \operatorname{diag}(\Delta_1, \ldots, \Delta_m)$ that satisfies $[\![\Delta_P]\!](t,k) = \underline{\Delta}(t,k)$ for all $(t,k) \in \mathbb{Z} \times V$. If the operators S and Δ are defined as in $S = \operatorname{diag}(S_0, S_0 S_1, \ldots, S_0 S_d, I^{\overline{n}_P})$ and $\Delta = \operatorname{diag}\left(I^{(n_T, \overline{n}_S)}, \Delta_P\right)$, respectively, then the equations of the distributed NSLPV system $\mathcal{G}_\delta$ can be written in compact operator form as follows:

$$\begin{bmatrix} x \\ \beta \end{bmatrix} = \Delta S A \begin{bmatrix} x \\ \beta \end{bmatrix} + \Delta S B\, u, \qquad y = C \begin{bmatrix} x \\ \beta \end{bmatrix} + D\, u, \tag{5.2}$$

where $x = [x_T^* \; x_1^* \; \cdots \; x_d^*]^*$, $\beta = [\beta_1^* \; \cdots \; \beta_m^*]^*$, and $\Delta \in \mathbf{\Delta} = \{\Delta \,|\, \|\Delta\| \leq 1\}$.

The quintuple $(A, B, C, D, \mathbf{\Delta})$ is used to denote the realization of the distributed NSLPV system $\mathcal{G}_\delta$. For a fixed $\Delta \in \mathbf{\Delta}$, the input-output map of system $\mathcal{G}_\delta$ is given by

$$G_\delta = \Delta \star \left[\begin{array}{c|c} SA & SB \\ \hline C & D \end{array}\right] = C(I - \Delta S A)^{-1} \Delta S B + D,$$

assuming that the inverse exists. The distributed NSLPV system $\mathcal{G}_\delta$ is then defined as $\mathcal{G}_\delta = \{G_\delta \,|\, \Delta \in \mathbf{\Delta}\}$. The distributed NSLPV system $\mathcal{G}_\delta$ thus defined can be interpreted as an LFT on Δ as shown in Figure 5.1. For all $\Delta \in \mathbf{\Delta}$, the corresponding equations are given by

$$\varphi = \Delta\, \xi, \qquad \xi = SA\, \varphi + SB\, u, \qquad y = C\, \varphi + D\, u. \tag{5.3}$$

5.2.2 Analysis Results

Consider a distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. System $\mathcal{G}_\delta$ is said to be well-posed if $I - \Delta S A$ has a causal inverse on $\ell_{2e}^{(n_T, \overline{n}_S, \overline{n}_P)}$ for all $\Delta \in \mathbf{\Delta}$. By Lemma 4.3, if $[\![A]\!](t,k) = 0$ for $t < 0$ and all $k \in V$, and $I - \Delta_P A_{PP}$ has a causal inverse on $\ell_{2e}^{\overline{n}_P}$ for all $\Delta \in \mathbf{\Delta}$, then system $\mathcal{G}_\delta$ is well-posed. Hereafter, it is assumed that $[\![A]\!](t,k) = 0$, $[\![B]\!](t,k) = 0$, $[\![C]\!](t,k) = 0$, and $[\![D]\!](t,k) = 0$ for all $t < 0$ and $k \in V$.

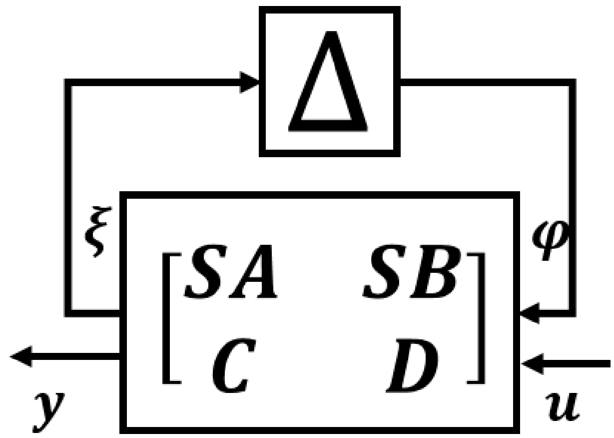

Figure 5.1: This figure shows how to interpret a distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \boldsymbol{\Delta})$ as an LFT on Δ. The operator S is the composite-shift operator, i.e., $S = \operatorname{diag}(S_0, S_0 S_1, \ldots, S_0 S_d, I^{\overline{n}_P})$.

Thus, to check for the well-posedness of system $\mathcal{G}_\delta$, it is only checked that $I - \Delta_P A_{PP}$ is invertible on $\ell_{2e}^{\overline{n}_P}$ for all $\Delta \in \boldsymbol{\Delta}$. If the inverse exists, then it is memoryless (and causal) since $I - \Delta_P A_{PP}$ is a memoryless operator.

System $\mathcal{G}_\delta$ is said to be stable if $I - \Delta S A$ has a bounded causal inverse for all $\Delta \in \boldsymbol{\Delta}$. The following result gives a sufficient condition for the stability of distributed NSLPV systems. A system that satisfies this condition is said to be strongly stable.

Define the set $\mathcal{T}$ of allowable transformations as follows:

$$\mathcal{T} = \Big\{ X \mid X = \operatorname{diag}\left(X_T, X_1^S, \ldots, X_d^S, X_1^P, \ldots, X_m^P\right) \in \mathcal{L}_c\left(\ell_2^{(n_T, \overline{n}_S, \overline{n}_P)}\right),$$
$$\text{where } X_T,\ X_i^S,\ X_j^P \text{ are graph-diagonal operators for } i = 1, \ldots, d \text{ and } j = 1, \ldots, m,$$
$$X^{-1} \in \mathcal{L}_c\left(\ell_2^{(n_T, \overline{n}_S, \overline{n}_P)}\right) \Big\}. \tag{5.4}$$

Then, the set $\mathcal{X}$ defined in (4.7) can be equivalently defined as

$$\mathcal{X} = \{ X \mid X = X^* \in \mathcal{T},\ X \succ 0 \}. \tag{5.5}$$

The sets $\mathcal{T}$ and $\mathcal{X}$ are in the commutant of $\mathbf{\Delta}$. The symbol $\mathcal{X}$ is used to denote similarly defined sets regardless of the associated dimensions.

Lemma 5.1. *Consider a distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. System $\mathcal{G}_\delta$ is strongly stable if and only if there exists $X \in \mathcal{X}$, where the set $\mathcal{X}$ is defined in (5.5), such that*

$$A^* S^* X S A - X \prec 0. \tag{5.6}$$

The next result is used in deriving the balanced truncation error bounds in Section 5.4.

Lemma 5.2. *Consider a distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. System $\mathcal{G}_\delta$ is strongly stable and its input-output map G_δ satisfies $\|G_\delta\| < \gamma$ for all $\Delta \in \mathbf{\Delta}$ if there exists $X \in \mathcal{X}$, where the set $\mathcal{X}$ is defined in (5.5), such that*

$$\begin{bmatrix} SA & SB \\ C & D \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} SA & SB \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & \gamma^2 I \end{bmatrix} \prec 0. \tag{5.7}$$

Since $X \in \mathcal{X}$, then by an application of the Schur complement formula, inequality (5.7) can be equivalently written as

$$\begin{bmatrix} -\begin{bmatrix} X & 0 \\ 0 & \gamma^2 I \end{bmatrix} & \begin{bmatrix} SA & SB \\ C & D \end{bmatrix}^* \\ \begin{bmatrix} SA & SB \\ C & D \end{bmatrix} & -\begin{bmatrix} X^{-1} & 0 \\ 0 & I \end{bmatrix} \end{bmatrix} \prec 0. \tag{5.8}$$

It can be seen from the preceding discussion that the size of system $\mathcal{G}_\delta$ and the size of the analysis problems increase with the number of subsystems, interconnections, and parameters, as well as the dimensions of the corresponding temporal, spatial, and parameter states. Thus, model reduction may be very useful for reducing the computational complexity of the problems at hand. Specifically, a reduced order system $\mathcal{G}_{r,\delta}$ is sought that approximates the behavior of the full order system $\mathcal{G}_{\delta}$ and preserves its interconnection and uncertainty structures. The sought reduced order system $\mathcal{G}_{r,\delta}$ is a distributed NSLPV system whose interconnection structure is described using the same graph that describes the interconnection structure of $\mathcal{G}_{\delta}$. Moreover, the subsystems in $\mathcal{G}_{r,\delta}$ have NSLPV models that are formulated in an LFT framework, and the reduced order uncertainty operator has the same structure as the full order uncertainty operator Δ.

Model reduction is of particular interest for the distributed control problem treated in Chapter 4. Namely, the synthesis problem involves solving the coupled inequalities (4.23), (4.24), and (4.25) that are of a larger size than the inequalities appearing in the analysis and model reduction problems. Thus, model reduction may be helpful in rendering the control synthesis problem computationally feasible. Moreover, since the distributed control technique of Chapter 4 yields distributed NSLPV controllers that are of a comparable size to the plant and that inherit the plant's interconnection and uncertainty structures, then model reduction can also be used to construct reduced order controllers. Namely, applying model reduction prior to control synthesis ensures that the least controllable and least observable modes of the plant are not reflected/mirrored in the designed controller. Also, since the proposed model reduction methods allow for the evaluation of the importance of each interconnection for the overall interconnection structure, and the possible truncation of the corresponding spatial states, then model reduction may help in simplifying the communication network in a rigorous manner by removing inconsequential communication links.

5.3 Balanced Truncation Model Reduction

This section applies the balanced truncation method for the model reduction of distributed NSLPV systems. The notion of a balanced realization for a distributed NSLPV system is defined, and a strongly stable system is shown to admit a balanced realization. The reduced order system resulting from the application of the balanced truncation method is proved to be strongly stable, and its constructed realization is shown to be balanced.

5.3.1 Balanced Realization

Consider a distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \boldsymbol{\Delta})$. The generalized Lyapunov inequalities associated with system $\mathcal{G}_\delta$ are given by

$$AXA^* - S^*XS + BB^* \prec 0, \tag{5.9}$$

$$A^*S^*YSA - Y + C^*C \prec 0. \tag{5.10}$$

System $\mathcal{G}_\delta$ is strongly stable if and only if there exists a solution $X \in \mathcal{X}$ to (5.9) if and only if there exists a solution $Y \in \mathcal{X}$ to (5.10), see the argument provided in the proof of Lemma 5.4. A solution $X \in \mathcal{X}$ to (5.9) is called a generalized controllability gramian for system $\mathcal{G}_\delta$. Similarly, a solution $Y \in \mathcal{X}$ to (5.10) is called a generalized observability gramian for system $\mathcal{G}_\delta$. The generalized gramians and the generalized Lyapunov inequalities [17, 43] extend the standard control theoretic notions of controllability and observability gramians and the Lyapunov equations, respectively, to systems with an inherent structure. For a distributed NSLPV system, since the generalized gramians X and Y are in $\mathcal{X}$, where the set $\mathcal{X}$ is defined in (5.5), then X and Y are partitioned graph-diagonal operators with a block-diagonal structure that is derived from the interconnection and uncertainty structures of the system. These gramians contain quantitative information about how controllable and observable a particular temporal, spatial, and parameter state variable is. The notion of a balanced realization for a distributed NSLPV system is defined next.

Definition 5.3. Consider a distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. The realization of system $\mathcal{G}_\delta$ is said to be balanced if there exists a balanced generalized gramian $\Sigma \in \mathcal{X}$ that simultaneously satisfies both generalized Lyapunov inequalities defined in (5.9) and (5.10) such that $[\![\Sigma]\!](t,k)$ is a diagonal matrix for all $(t,k) \in \mathbb{Z} \times V$.

The next result shows that a strongly stable distributed NSLPV system admits a balanced realization which can be constructed from any given generalized gramians X and Y in $\mathcal{X}$.

Lemma 5.4. *Consider a strongly stable distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. Then, system $\mathcal{G}_\delta$ admits a balanced realization that is denoted by $(A_{\mathrm{bal}}, B_{\mathrm{bal}}, C_{\mathrm{bal}}, D, \mathbf{\Delta})$.*

Proof. Since system $\mathcal{G}_\delta$ is strongly stable, then there exists a solution in $\mathcal{X}$ to (5.6). By homogeneity and scalability of (5.6), there equivalently exist solutions $X \in \mathcal{X}$ and $Y \in \mathcal{X}$ to (5.9) and (5.10), respectively. As discussed next, the generalized gramians X and Y are used to construct a balanced realization for system $\mathcal{G}_\delta$ and a balanced generalized gramian Σ, see also Algorithm 2.7. First, the Cholesky factorizations $X = R^*R$ and $Y = H^*H$ are performed. Namely, for all $(t,k) \in \mathbb{Z} \times V$, $i = 1, \ldots, d$, and $j = 1, \ldots, m$, the following Cholesky factorizations are performed:

$$
\begin{aligned}
X_T(t,k) &= (R_T(t,k))^* R_T(t,k), & \qquad Y_T(t,k) &= (H_T(t,k))^* H_T(t,k),\\
X_i^S(t,k) &= (R_i^S(t,k))^* R_i^S(t,k), & \qquad Y_i^S(t,k) &= (H_i^S(t,k))^* H_i^S(t,k),\\
X_j^P(t,k) &= (R_j^P(t,k))^* R_j^P(t,k), & \qquad Y_j^P(t,k) &= (H_j^P(t,k))^* H_j^P(t,k).
\end{aligned}
$$

Then, the singular value decomposition $HR^* = U\Sigma V^*$ is performed, where U and V are in $\mathcal{T}$, and Σ is the balanced generalized gramian satisfying the properties in Definition 5.3. Namely, for all $(t,k) \in \mathbb{Z} \times V$, $i = 1, \ldots, d$, and $j = 1, \ldots, m$, the following singular value decompositions are performed:

$$H_T(t,k)(R_T(t,k))^* = U_T(t,k)\Sigma_T(t,k)(V_T(t,k))^*,$$
$$H_i^S(t,k)(R_i^S(t,k))^* = U_i^S(t,k)\Sigma_i^S(t,k)(V_i^S(t,k))^*,$$
$$H_j^P(t,k)(R_j^P(t,k))^* = U_j^P(t,k)\Sigma_j^P(t,k)(V_j^P(t,k))^*.$$

Then, the balancing transformation is defined as $T = \Sigma^{-1/2}U^*H \in \mathcal{T}$, and its inverse is obtained from $T^{-1} = R^*V\Sigma^{-1/2} \in \mathcal{T}$. That is, the blocks of the balancing transformation and its inverse are obtained from

$$T_T(t,k)=(\Sigma_T(t,k))^{-\frac{1}{2}}(U_T(t,k))^*H_T(t,k), \quad (T_T(t,k))^{-1}=(R_T(t,k))^*V_T(t,k)(\Sigma_T(t,k))^{-\frac{1}{2}},$$
$$T_i^S(t,k)=(\Sigma_i^S(t,k))^{-\frac{1}{2}}(U_i^S(t,k))^*H_i^S(t,k), \quad (T_i^S(t,k))^{-1}=(R_i^S(t,k))^*V_i^S(t,k)(\Sigma_i^S(t,k))^{-\frac{1}{2}},$$
$$T_j^P(t,k)=(\Sigma_j^P(t,k))^{-\frac{1}{2}}(U_j^P(t,k))^*H_j^P(t,k), \quad (T_j^P(t,k))^{-1}=(R_j^P(t,k))^*V_j^P(t,k)(\Sigma_j^P(t,k))^{-\frac{1}{2}},$$

for all $(t,k) \in \mathbb{Z} \times V$, $i = 1, \ldots, d$, and $j = 1, \ldots, m$. It is not difficult to verify that Σ can be expressed as $\Sigma = TXT^* = (T^{-1})^*YT^{-1}$. This can be shown by performing component-wise computations similar to the ones shown above. If the operators A_{bal}, B_{bal}, and C_{bal} are defined as $A_{\text{bal}} = (S^*TS)AT^{-1}$, $B_{\text{bal}} = (S^*TS)B$, and $C_{\text{bal}} = CT^{-1}$, then Σ can be shown to simultaneously satisfy (5.9) and (5.10), where the inequalities are expressed for the realization $(A_{\text{bal}}, B_{\text{bal}}, C_{\text{bal}}, D, \boldsymbol{\Delta})$. That is, $(A_{\text{bal}}, B_{\text{bal}}, C_{\text{bal}}, D, \boldsymbol{\Delta})$ constitutes a balanced realization for system $\mathcal{G}_\delta$, which concludes the proof. $\square$

5.3.2 Balanced Truncation

Consider a strongly stable distributed NSLPV system $\mathcal{G}_\delta$ that has a balanced realization denoted by $(A, B, C, D, \mathbf{\Delta})$ and a balanced generalized gramian denoted by Σ, and assume that system $\mathcal{G}_\delta$ is to be reduced by applying the balanced truncation method. To determine which state variables to truncate, one looks at the diagonal entries of $[\![\Sigma]\!](t,k)$ for all $(t,k) \in \mathbb{Z} \times V$ and their relative order, the value of γ obtained from Lemma 5.2, and the error bounds from Theorems 5.7 and 5.11. Since Σ has a block-diagonal structure and contains temporal terms $\Sigma_T(t,k)$, spatial terms $\Sigma_i^S(t,k)$, and parameter terms $\Sigma_j^P(t,k)$, then the balanced truncation method allows for the truncation of each temporal, spatial, and parameter state individually, and the truncation need not be uniform in time, i.e., one may truncate different numbers of variables from a particular state at different time-steps.

The blocks of Σ are partitioned into truncated and non-truncated parts. The partitioning process is illustrated for the temporal blocks $\Sigma_T(t,k)$ for all $(t,k) \in \mathbb{Z} \times V$. A similar partitioning process is repeated for $\Sigma_i^S(t,k)$ and $\Sigma_j^P(t,k)$ for all $(t,k) \in \mathbb{Z} \times V$, $i = 1, \ldots, d$, and $j = 1, \ldots, m$. For each $(t,k) \in \mathbb{Z} \times V$, $\Sigma_T(t,k)$ is an $n_T(t,k) \times n_T(t,k)$ positive definite diagonal matrix. Without loss of generality, it is assumed that the diagonal entries of $\Sigma_T(t,k)$ are sorted in a decreasing order with the largest value in the first entry. Denote the dimensions of the temporal state vectors in the reduced order system by $r_T(t,k)$, where $0 \leq r_T(t,k) \leq n_T(t,k)$. Then, $\Sigma_T(t,k)$ is partitioned into two blocks as in $\Sigma_T(t,k) = \operatorname{diag}(\Gamma_T(t,k), \Omega_T(t,k))$, where $\Gamma_T(t,k)$ is an $r_T(t,k) \times r_T(t,k)$ matrix. If $r_T(t_0,k_0) = 0$ for some $(t_0,k_0) \in \mathbb{Z} \times V$, then $\Gamma_T(t_0,k_0)$ is nonexistent. Similarly, if $r_T(t_0,k_0) = n_T(t_0,k_0)$, then $\Omega_T(t_0,k_0)$ is nonexistent. The matrix-valued sequences $\Gamma_T(t,k)$ and $\Omega_T(t,k)$ define graph-diagonal operators that are denoted by Γ_T and Ω_T, respectively.

In addition to truncating the temporal states, the proposed balanced truncation method allows for truncating the spatial states and the parameter states. The dimensions of the spatial and parameter state vectors in the reduced order system are given by $r_i^S(t,k)$ and $r_j^P(t,k)$, respectively, where $0 \leq r_i^S(t,k) \leq n_i^S(t,k)$ and $0 \leq r_j^P(t,k) \leq n_j^P(t,k)$ for all $(t,k) \in \mathbb{Z}\times V$, $i = 1,\ldots,d$, and $j = 1,\ldots,m$. If $r_{i_0}^S(t,k_0) = 0$ for all time-steps $t \in \mathbb{Z}$, then the corresponding interconnection $(\rho_{i_0}^{-1}(k_0), k_0)$ is altogether removed from the interconnection structure during model reduction. Similarly, if $r_{j_0}^P(t,k_0) = 0$ for all time-steps $t \in \mathbb{Z}$, then the corresponding block $\Delta_{j_0}(t,k_0)$ is removed from $\underline{\Delta}(t,k_0)$ during model reduction. The matrix-valued sequences $\Gamma_i^S(t,k)$, $\Gamma_j^P(t,k)$, $\Omega_i^S(t,k)$, and $\Omega_j^P(t,k)$ obtained from partitioning $\Sigma_i^S(t,k)$ as in $\Sigma_i^S(t,k) = \operatorname{diag}\left(\Gamma_i^S(t,k), \Omega_i^S(t,k)\right)$ and partitioning $\Sigma_j^P(t,k)$ as in $\Sigma_j^P(t,k) = \operatorname{diag}\left(\Gamma_j^P(t,k), \Omega_j^P(t,k)\right)$ define graph-diagonal operators that are denoted by Γ_i^S, Γ_j^P, Ω_i^S, and Ω_j^P, respectively, for all $i = 1,\ldots,d$ and $j = 1,\ldots,m$.

The graph-diagonal operators thus defined are augmented as follows:

$$\Gamma = \operatorname{diag}\left(\Gamma_T, \Gamma_1^S, \ldots, \Gamma_d^S, \Gamma_1^P, \ldots, \Gamma_m^P\right), \tag{5.11}$$

$$\Omega = \operatorname{diag}\left(\Omega_T, \Omega_1^S, \ldots, \Omega_d^S, \Omega_1^P, \ldots, \Omega_m^P\right), \tag{5.12}$$

where Γ and Ω are associated with the non-truncated and truncated blocks of Σ, respectively. For all $(t,k) \in \mathbb{Z} \times V$, the state-space matrices are partitioned into non-truncated portions, which are marked with a hat, and truncated portions. The partitioning is conformable with the partitioning of the blocks of Σ, namely,

$$\begin{aligned}
\Sigma_T(t,k) &= \llbracket \operatorname{diag}\left(\Gamma_T, \Omega_T\right) \rrbracket (t,k) = \operatorname{diag}\left(\Gamma_T(t,k), \Omega_T(t,k)\right), \\
\Sigma_i^S(t,k) &= \left\llbracket \operatorname{diag}\left(\Gamma_i^S, \Omega_i^S\right) \right\rrbracket (t,k) = \operatorname{diag}\left(\Gamma_i^S(t,k), \Omega_i^S(t,k)\right), \\
\Sigma_j^P(t,k) &= \left\llbracket \operatorname{diag}\left(\Gamma_j^P, \Omega_j^P\right) \right\rrbracket (t,k) = \operatorname{diag}\left(\Gamma_j^P(t,k), \Omega_j^P(t,k)\right).
\end{aligned}$$

For instance,

$$
\begin{aligned}
\overline{A}_{TS}(t,k) &= \begin{bmatrix} A_1^{TS}(t,k) & \cdots & A_d^{TS}(t,k) \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} \hat{A}_1^{TS}(t,k) & A_{1,12}^{TS}(t,k) \\ A_{1,21}^{TS}(t,k) & A_{1,22}^{TS}(t,k) \end{bmatrix} & \cdots & \begin{bmatrix} \hat{A}_d^{TS}(t,k) & A_{d,12}^{TS}(t,k) \\ A_{d,21}^{TS}(t,k) & A_{d,22}^{TS}(t,k) \end{bmatrix} \end{bmatrix}, \\
\overline{A}_{SS}(t,k) &= \left[A_{ie}^{SS}(t,k)\right]_{i=1,\ldots,d;e=1,\ldots,d} = \begin{bmatrix} \begin{bmatrix} \hat{A}_{ie}^{SS}(t,k) & A_{ie,12}^{SS}(t,k) \\ A_{ie,21}^{SS}(t,k) & A_{ie,22}^{SS}(t,k) \end{bmatrix} \end{bmatrix}_{i=1,\ldots,d;e=1,\ldots,d}, \\
\overline{A}_{TP}(t,k) &= \begin{bmatrix} A_1^{TP}(t,k) & \cdots & A_m^{TP}(t,k) \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} \hat{A}_1^{TP}(t,k) & A_{1,12}^{TP}(t,k) \\ A_{1,21}^{TP}(t,k) & A_{1,22}^{TP}(t,k) \end{bmatrix} & \cdots & \begin{bmatrix} \hat{A}_m^{TP}(t,k) & A_{m,12}^{TP}(t,k) \\ A_{m,21}^{TP}(t,k) & A_{m,22}^{TP}(t,k) \end{bmatrix} \end{bmatrix}, \\
\overline{A}_{PP}(t,k) &= \left[A_{jf}^{PP}(t,k)\right]_{j=1,\ldots,m;f=1,\ldots,m} = \begin{bmatrix} \begin{bmatrix} \hat{A}_{jf}^{PP}(t,k) & A_{jf,12}^{PP}(t,k) \\ A_{jf,21}^{PP}(t,k) & A_{jf,22}^{PP}(t,k) \end{bmatrix} \end{bmatrix}_{j=1,\ldots,m;f=1,\ldots,m},
\end{aligned}
$$

where $\hat{A}_i^{TS}(t,k)$ is an $r_T(t+1,k) \times r_i^S(t,k)$ matrix, $\hat{A}_{ie}^{SS}(t,k)$ is an $r_i^S(t+1,\rho_i(k)) \times r_e^S(t,k)$ matrix, $\hat{A}_j^{TP}(t,k)$ is an $r_T(t+1,k) \times r_j^P(t,k)$ matrix, and $\hat{A}_{jf}^{PP}(t,k)$ is an $r_j^P(t,k) \times r_f^P(t,k)$ matrix. That is, the partitioning of the state-space matrices is performed at the level of their constituent blocks, e.g., the blocks $A_i^{TS}(t,k)$ for $i = 1,\ldots,d$ (and not $\overline{A}_{TS}(t,k)$) are each partitioned into a non-truncated block $\hat{A}_i^{TS}(t,k)$ and other truncated blocks.

The matrix-valued sequence obtained from each non-truncated block, e.g., $\hat{A}_{TT}(t,k)$, $\hat{A}_i^{TS}(t,k)$, defines a graph-diagonal operator, e.g., $A_{r,TT}$, $\hat{A}_i^{TS}$. These graph-diagonal operators are then augmented to form the operators A_r, B_r, and C_r of the reduced order system. Namely,

$$A_r = \begin{bmatrix} A_{r,TT} & A_{r,TS} & A_{r,TP} \\ A_{r,ST} & A_{r,SS} & A_{r,SP} \\ A_{r,PT} & A_{r,PS} & A_{r,PP} \end{bmatrix} = \begin{bmatrix} A_{r,TT} & \begin{bmatrix} \hat{A}_1^{TS} & \cdots & \hat{A}_d^{TS} \end{bmatrix} & \begin{bmatrix} \hat{A}_1^{TP} & \cdots & \hat{A}_m^{TP} \end{bmatrix} \\ \begin{bmatrix} \hat{A}_1^{ST} \\ \vdots \\ \hat{A}_d^{ST} \end{bmatrix} & \begin{bmatrix} \hat{A}_{11}^{SS} & \cdots & \hat{A}_{1d}^{SS} \\ \vdots & \ddots & \vdots \\ \hat{A}_{d1}^{SS} & \cdots & \hat{A}_{dd}^{SS} \end{bmatrix} & \begin{bmatrix} \hat{A}_{11}^{SP} & \cdots & \hat{A}_{1m}^{SP} \\ \vdots & \ddots & \vdots \\ \hat{A}_{d1}^{SP} & \cdots & \hat{A}_{dm}^{SP} \end{bmatrix} \\ \begin{bmatrix} \hat{A}_1^{PT} \\ \vdots \\ \hat{A}_m^{PT} \end{bmatrix} & \begin{bmatrix} \hat{A}_{11}^{PS} & \cdots & \hat{A}_{1d}^{PS} \\ \vdots & \ddots & \vdots \\ \hat{A}_{m1}^{PS} & \cdots & \hat{A}_{md}^{PS} \end{bmatrix} & \begin{bmatrix} \hat{A}_{11}^{PP} & \cdots & \hat{A}_{1m}^{PP} \\ \vdots & \ddots & \vdots \\ \hat{A}_{m1}^{PP} & \cdots & \hat{A}_{mm}^{PP} \end{bmatrix} \end{bmatrix},$$

$$B_r = \begin{bmatrix} B_{r,T} \\ B_{r,S} \\ B_{r,P} \end{bmatrix} = \begin{bmatrix} B_{r,T} \\ \begin{bmatrix} \hat{B}_1^S \\ \vdots \\ \hat{B}_d^S \end{bmatrix} \\ \begin{bmatrix} \hat{B}_1^P \\ \vdots \\ \hat{B}_m^P \end{bmatrix} \end{bmatrix},$$

$$C_r = \begin{bmatrix} C_{r,T} & C_{r,S} & C_{r,P} \end{bmatrix} = \begin{bmatrix} C_{r,T} & \begin{bmatrix} \hat{C}_1^S & \cdots & \hat{C}_d^S \end{bmatrix} & \begin{bmatrix} \hat{C}_1^P & \cdots & \hat{C}_m^P \end{bmatrix} \end{bmatrix}.$$

For $j = 1, \ldots, m$, define the graph-diagonal operators $\hat{\Delta}_j$ such that $\hat{\Delta}_j(t,k) = \delta_j(t,k) I_{r_j^P(t,k)}$ for all $(t,k) \in \mathbb{Z} \times V$. These operators are augmented to form $\Delta_{r,P} = \operatorname{diag}\left(\hat{\Delta}_1, \ldots, \hat{\Delta}_m\right)$. Let $\overline{r}_S = (r_1^S, \ldots, r_d^S)$ and $\overline{r}_P = (r_1^P, \ldots, r_m^P)$. Then, the reduced order uncertainty operator Δ_r is defined as

$$\Delta_r = \operatorname{diag}\left(I^{(r_T, \overline{r}_S)}, \Delta_{r,P}\right). \tag{5.13}$$

In other words, Δ_r is defined similarly to Δ with the exception that the blocks in Δ_r have reduced dimensions. Finally, the realization of the reduced order system $\mathcal{G}_{r,\delta}$ is defined as $(A_r, B_r, C_r, D, \mathbf{\Delta}_r)$, where $\mathbf{\Delta}_r = \{\Delta_r \,|\, \Delta_r$ is structured as in (5.13) and $\|\Delta_r\| \leq 1\}$.

The balanced truncation method just presented is said to be structure-preserving. Namely, the interconnection structure of the reduced order system $\mathcal{G}_{r,\delta}$ is the same as the interconnection structure of the full order system $\mathcal{G}_\delta$, with the exception that the spatial states in the reduced order system have smaller or equal dimensions than their counterparts in the full order system. Similarly, the structure of the Δ_r-operator in $\mathcal{G}_{r,\delta}$ is the same as the structure of the Δ-operator in $\mathcal{G}_\delta$, with the exception that the parameter states in the reduced order system have smaller or equal dimensions than their counterparts in the full order system. Specifically, by forcing the balancing transformation T constructed in the proof of Lemma 5.4 to be in the set $\mathcal{T}$ defined in (5.4), the interpretation of each state is preserved in the constructed balanced realization of the system, thus allowing for a structure-preserving truncation of the negligible state variables. The interpretation of a state specifies its type, i.e., whether the state is a temporal, spatial, or parameter state, and identifies the subsystem, interconnection, or uncertainty channel with which the state is associated. The presented balanced truncation method can be further thought of as structure-simplifying because the spatial states and the parameter states in the reduced order system $\mathcal{G}_{r,\delta}$ are allowed to have zero dimensions for all time-steps $t \in \mathbb{Z}$.

Lemma 5.5. *Consider a strongly stable distributed NSLPV system $\mathcal{G}_\delta$ that has a balanced realization denoted by $(A, B, C, D, \mathbf{\Delta})$ and a balanced generalized gramian denoted by Σ. Assume that the balanced truncation method is applied to reduce system $\mathcal{G}_\delta$, and denote the resulting reduced order system by $\mathcal{G}_{r,\delta}$ and its obtained realization by $(A_r, B_r, C_r, D, \mathbf{\Delta}_r)$. Then, system $\mathcal{G}_{r,\delta}$ is strongly stable, and its given realization is balanced. Moreover, the operator Γ defined (5.11) is a balanced generalized gramian for the reduced order system.*

Proof. There exists a unique partitioned graph-diagonal operator Q such that $Q^*\Sigma Q =$

$\mathrm{diag}(\Gamma, \Omega)$, $QQ^* = I$, and $Q^*Q = I$, where Γ and Ω are defined in (5.11) and (5.12), respectively. The reader is referred to [8, Appendix A] for the detailed structure of the operator Q. It can also be verified that Q satisfies

$$Q^*SAQ = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} A_r & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix} = \tilde{S}\,\overline{\overline{A}}, \qquad Q^*SB = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} B_r \\ \overline{B}_2 \end{bmatrix} = \tilde{S}\,\overline{\overline{B}},$$
$$CQ = \begin{bmatrix} C_r & \overline{C}_2 \end{bmatrix} = \overline{\overline{C}}, \qquad Q^*\Delta Q = \begin{bmatrix} \Delta_r & 0 \\ 0 & \overline{\Delta}_2 \end{bmatrix} = \overline{\overline{\Delta}}, \tag{5.14}$$

where $\overline{A}_{12}$, $\overline{A}_{21}$, $\overline{A}_{22}$, $\overline{B}_2$, $\overline{C}_2$, and $\overline{\Delta}_2$ are appropriately defined partitioned graph-diagonal operators. Since Σ is a balanced generalized gramian for system $\mathcal{G}_\delta$, then it simultaneously satisfies the generalized Lyapunov inequalities defined in (5.9) and (5.10). Pre- and post-multiplying (5.9) by $\tilde{S}^*Q^*S$ and $S^*Q\tilde{S}$, respectively, and pre- and post-multiplying (5.10) by Q^* and Q, respectively, result in

$$\overline{\overline{A}}\,\mathrm{diag}(\Gamma, \Omega)\left(\overline{\overline{A}}\right)^* - \tilde{S}^*\mathrm{diag}(\Gamma, \Omega)\tilde{S} + \overline{\overline{B}}\left(\overline{\overline{B}}\right)^* \prec 0, \tag{5.15}$$
$$\left(\overline{\overline{A}}\right)^* \tilde{S}^*\mathrm{diag}(\Gamma, \Omega)\tilde{S}\,\overline{\overline{A}} - \mathrm{diag}(\Gamma, \Omega) + \left(\overline{\overline{C}}\right)^* \overline{\overline{C}} \prec 0. \tag{5.16}$$

From (5.15) and (5.16), it can be concluded that

$$A_r \Gamma A_r^* - S^*\Gamma S + B_r B_r^* \prec 0,$$
$$A_r^* S^* \Gamma S A_r - \Gamma + C_r^* C_r \prec 0.$$

In other words, the reduced order system $\mathcal{G}_{r,\delta}$ is strongly stable, and its given realization $(A_r, B_r, C_r, D, \mathbf{\Delta}_r)$ is balanced with Γ being a balanced generalized gramian. $\square$

5.4 Balanced Truncation Error Bounds

Consider a strongly stable distributed NSLPV system $\mathcal{G}_\delta$ that has a balanced realization denoted by $(A, B, C, D, \mathbf{\Delta})$ and a balanced generalized gramian denoted by Σ. Assume that the balanced truncation method is applied to reduce system $\mathcal{G}_\delta$, and denote the resulting reduced order system by $\mathcal{G}_{r,\delta}$ and its obtained realization by $(A_r, B_r, C_r, D, \mathbf{\Delta}_r)$. Define the partitioned graph-diagonal operators Γ and Ω as in (5.11) and (5.12), respectively. By Lemma 5.5, system $\mathcal{G}_{r,\delta}$ is strongly stable, the realization $(A_r, B_r, C_r, D, \mathbf{\Delta}_r)$ is balanced, and Γ is a balanced generalized gramian. Hereafter, the reduced order uncertainty operator Δ_r defined (5.13) and the corresponding set $\mathbf{\Delta}_r$ will not be referred to separately from the full order uncertainty operator Δ and the corresponding set $\mathbf{\Delta}$, because the structure of Δ_r can be derived from the structure of Δ. This section derives two upper bounds on $\|(G_\delta - G_{r,\delta})\|$ for all $\Delta \in \mathbf{\Delta}$.

Theorem 5.6. *Consider a strongly stable distributed NSLPV system $\mathcal{G}_\delta$ that has a balanced realization denoted by $(A, B, C, D, \mathbf{\Delta})$ and a balanced generalized gramian denoted by Σ. Assume that the balanced truncation method is applied to reduce system $\mathcal{G}_\delta$, and denote the resulting reduced order system by $\mathcal{G}_{r,\delta}$ and its obtained realization by $(A_r, B_r, C_r, D, \mathbf{\Delta}_r)$. If the partitioned graph-diagonal operator Ω is defined as in (5.12) and satisfies $\Omega = I$, i.e., $\Omega_T(t,k) = I$, $\Omega_i^S(t,k) = I$, and $\Omega_j^P(t,k) = I$ for all $(t,k) \in \mathbb{Z} \times V$, $i = 1, \ldots, d$, and $j = 1, \ldots, m$, then $\|(G_\delta - G_{r,\delta})\| < 2$ for all $\Delta \in \mathbf{\Delta}$.*

Proof. Since the systems $\mathcal{G}_\delta$ and $\mathcal{G}_{r,\delta}$ are strongly stable, then so is the error system $\mathcal{E}_\delta = \left\{\frac{1}{2}(G_\delta - G_{r,\delta}) \,|\, \Delta \in \mathbf{\Delta}\right\}$. Recalling the denotations $\tilde{S}$, $\overline{\overline{A}}$, $\overline{\overline{B}}$, $\overline{\overline{C}}$, and $\overline{\overline{\Delta}}$ defined in (5.14), one can see that

$$\frac{1}{2}\left(G_\delta - G_{r,\delta}\right) = \begin{bmatrix} \Delta_r & 0 \\ 0 & \overline{\overline{\Delta}} \end{bmatrix} \star \left[\begin{array}{cc|c} SA_r & 0 & \frac{1}{\sqrt{2}}SB_r \\ 0 & \tilde{S}\overline{\overline{A}} & \frac{1}{\sqrt{2}}\tilde{S}\overline{\overline{B}} \\ \hline -\frac{1}{\sqrt{2}}C_r & \frac{1}{\sqrt{2}}\overline{\overline{C}} & 0 \end{array}\right], \text{ for all } \Delta \in \boldsymbol{\Delta}.$$

By Theorem 4.6 and Lemma 5.2, if there exists an operator $\mathcal{V} \succ 0$ that commutes with $\operatorname{diag}\left(\Delta_r, \overline{\overline{\Delta}}\right)$ for all $\Delta \in \boldsymbol{\Delta}$ and satisfies (5.8) with $\gamma = 1$, where inequality (5.8) is expressed for the given realization of system $\mathcal{E}_\delta$, then it follows that $\left\|\frac{1}{2}\left(G_\delta - G_{r,\delta}\right)\right\| < 1$ for all $\Delta \in \boldsymbol{\Delta}$. By direct application of the Schur complement formula twice to (5.15) and (5.16), it can be verified that

$$\begin{bmatrix} -R_1 & K^* \\ K & -S_a^* R_2 S_a \end{bmatrix} \prec 0,$$

where $R_1 = \operatorname{diag}\left(\Gamma^{-1}, \Omega^{-1}, I^{n_u}, \Gamma, \Omega\right)$, $R_2 = \operatorname{diag}\left(\Gamma^{-1}, \Omega^{-1}, I^{n_y}, \Gamma, \Omega\right)$,

$$S_a = \begin{bmatrix} \tilde{S} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \tilde{S} \end{bmatrix}, \qquad K = \begin{bmatrix} 0 & 0 & \overline{\overline{A}} \\ 0 & 0 & \overline{\overline{C}} \\ \overline{\overline{A}} & \overline{\overline{B}} & 0 \end{bmatrix}.$$

Define the operators L and P by

$$L = \frac{1}{\sqrt{2}} \begin{bmatrix} -I & 0 & 0 & I & 0 \\ I & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & I \\ 0 & 0 & \sqrt{2}I^{n_y} & 0 & 0 \\ 0 & -I & 0 & 0 & I \end{bmatrix}, \qquad P = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & I \\ 0 & 0 & 0 & \sqrt{2}I^{n_u} & 0 \\ -I & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & -I \end{bmatrix},$$

respectively, and pre- and post-multiply the previous inequality by $\operatorname{diag}(P^*, L)$ and $\operatorname{diag}(P, L^*)$, respectively, to obtain

$$\begin{bmatrix} -P^*R_1P & P^*K^*L^* \\ LKP & -LS_a^*R_2S_aL^* \end{bmatrix} \prec 0,$$

where

$$LKP = \left[\begin{array}{c|c} M & N_{12} \\ \hline N_{21} & N_{22} \end{array}\right] = \left[\begin{array}{cccc|c} A_r & 0 & 0 & \frac{1}{\sqrt{2}}B_r & \overline{A}_{12} \\ 0 & A_r & \overline{A}_{12} & \frac{1}{\sqrt{2}}B_r & 0 \\ 0 & \overline{A}_{21} & \overline{A}_{22} & \frac{1}{\sqrt{2}}\overline{B}_2 & 0 \\ \frac{-1}{\sqrt{2}}C_r & \frac{1}{\sqrt{2}}C_r & \frac{1}{\sqrt{2}}\overline{C}_2 & 0 & \frac{-1}{\sqrt{2}}\overline{C}_2 \\ \hline \overline{A}_{21} & 0 & 0 & \frac{1}{\sqrt{2}}\overline{B}_2 & \overline{A}_{22} \end{array}\right].$$

Since $\Omega = I$ and $S^*S = I$, it follows that

$$P^*R_1P = \operatorname{diag}\left(\frac{1}{2}\begin{bmatrix} \Gamma^{-1}+\Gamma & \Gamma^{-1}-\Gamma \\ \Gamma^{-1}-\Gamma & \Gamma^{-1}+\Gamma \end{bmatrix}, \operatorname{diag}\left(I, I^{n_u}, I\right) \right),$$

$$LS_a^*R_2S_aL^* = \operatorname{diag}\left(\frac{1}{2}\begin{bmatrix} S^*\left(\Gamma^{-1}+\Gamma\right)S & S^*\left(\Gamma-\Gamma^{-1}\right)S \\ S^*\left(\Gamma-\Gamma^{-1}\right)S & S^*\left(\Gamma^{-1}+\Gamma\right)S \end{bmatrix}, \operatorname{diag}\left(I, I^{n_y}, I\right) \right).$$

Let the desired $\mathcal{V}$ be defined as $\mathcal{V} = \frac{1}{2}\operatorname{diag}\left(\begin{bmatrix} \Gamma^{-1}+\Gamma & \Gamma^{-1}-\Gamma \\ \Gamma^{-1}-\Gamma & \Gamma^{-1}+\Gamma \end{bmatrix}, 2I \right)$. Then, $\mathcal{V}$ is positive definite, commutes with $\operatorname{diag}(\Delta_r, \overline{\overline{\Delta}})$ for all $\Delta \in \boldsymbol{\Delta}$, and satisfies the inequality

$$\begin{bmatrix} -\begin{bmatrix} \mathcal{V} & 0 \\ 0 & I^{n_u} \end{bmatrix} & M^* \\ M & -\begin{bmatrix} \begin{bmatrix} S & 0 \\ 0 & \tilde{S} \end{bmatrix}^* \mathcal{V}^{-1} \begin{bmatrix} S & 0 \\ 0 & \tilde{S} \end{bmatrix} & 0 \\ 0 & I^{n_y} \end{bmatrix} \end{bmatrix} \prec 0,$$

which concludes the proof. □

Recall the function Φ defined in Section 5.2. The following theorem gives the balanced truncation error bound for the case of a general operator Ω in terms of $\Phi(\Omega)$.

Theorem 5.7. *Consider a strongly stable distributed NSLPV system $\mathcal{G}_\delta$ that has a balanced realization denoted by $(A, B, C, D, \mathbf{\Delta})$ and a balanced generalized gramian denoted by Σ. Assume that the balanced truncation method is applied to reduce system $\mathcal{G}_\delta$, and denote the resulting reduced order system by $\mathcal{G}_{r,\delta}$ and its obtained realization by $(A_r, B_r, C_r, D, \mathbf{\Delta}_r)$. If the partitioned graph-diagonal operator Ω is defined as in (5.12), then $\|(G_\delta - G_{r,\delta})\| < 2\Phi(\Omega)$ for all $\Delta \in \mathbf{\Delta}$.*

Proof. The theorem follows by scaling and repeated application of Theorem 5.6 as detailed in the proof of Theorem 2.10. Lemma 5.5 ensures that the balanced truncation method is applicable to the resulting intermediate reduced order systems, since they are known a priori to be strongly stable and their computed realizations are balanced. □

The error bound given in Theorem 5.7 can become infinitely large as the number of distinct entries in Ω increases. Assume now that the graph that represents the interconnection structure of the full order system $\mathcal{G}_\delta$ is finite, i.e., the set of vertices V and the set of edges E are finite. If system $\mathcal{G}_\delta$ is strongly stable and (h, q)-eventually time-periodic as per Definition 4.13, then it admits an (h, q)-eventually time-periodic balanced realization, i.e., a realization with (h, q)-eventually time-periodic state-space matrices, and an (h, q)-eventually time-periodic balanced generalized gramian Σ_{eper}. More details on eventually time-periodic systems can be found in Sections 2.7 and 4.6. The realization of the reduced order system $\mathcal{G}_{r,\delta}$ that results from applying the balanced truncation method to system $\mathcal{G}_\delta$ is also (h, q)-eventually time-periodic. In this case, when evaluating $\Phi(\Omega_{\text{eper}})$ to compute the balanced truncation error bound in Theorem 5.7, where Ω_{eper} is defined from the blocks of Σ_{eper} as in (5.12), t can be restricted to the finite time-horizon h and the first time-period, i.e., $0 \le t \le h+q-1$. Since the sets V and E are assumed to be finite, and t is restricted between 0 and $h+q-1$, then $\Phi(\Omega_{\text{eper}})$ is guaranteed to be finite. In the following algorithm, a heuristic is proposed that can be used in order to increase the number of small and equal entries in Ω_{eper}, thereby increasing the effectiveness of Theorem 5.7.

Algorithm 5.8. Let $\mathcal{G}_\delta$ be a strongly stable (h,q)-eventually time-periodic distributed NSLPV system whose interconnection structure is described using a finite graph, and denote the given realization of system $\mathcal{G}_\delta$ by $(A, B, C, D, \mathbf{\Delta})$. Given a weight $a_1 > 0$, this algorithm proposes a heuristic for finding an (h,q)-eventually time-periodic balanced realization $(A_{\text{bal}}, B_{\text{bal}}, C_{\text{bal}}, D, \mathbf{\Delta})$ for system $\mathcal{G}_\delta$ and an (h,q)-eventually time-periodic balanced generalized gramian Σ_{eper}. In the following, let $\operatorname{vect}(\mathcal{Q})$ denote the vector formed by the diagonal entries of a square matrix $\mathcal{Q}$, and $\|v\|_1$ denote the 1-norm of vector v.

1. minimize

$$\sum_{t=0}^{h+q-1} \sum_{k \in V} \left(\operatorname{trace} X_{\text{eper},T}(t,k) + \sum_{i=1}^{d} \operatorname{trace} X^S_{\text{eper},i}(t,k) + \sum_{j=1}^{m} \operatorname{trace} X^P_{\text{eper},j}(t,k) \right)$$

subject to: $X_{\text{eper},T}(t,k) \succ 0$, $X^S_{\text{eper},i}(t,k) \succ 0$, $X^P_{\text{eper},j}(t,k) \succ 0$,

$X_{\text{eper},T}(h+q,k) = X_{\text{eper},T}(h,k)$, $X^S_{\text{eper},i}(h+q,k) = X^S_{\text{eper},i}(h,k)$, and

$$[\![A]\!](t,k)[\![X_{\text{eper}}]\!](t,k)[\![A^*]\!](t,k) - [\![S^* X_{\text{eper}} S]\!](t,k) + [\![B]\!](t,k)[\![B^*]\!](t,k) \prec 0,$$

for all $t = 0, \ldots, h+q-1$, $i = 1, \ldots, d$, $j = 1, \ldots, m$, and $k \in V$.

2. minimize

$$\sum_{t=0}^{h+q-1}\sum_{k\in V}\left(\operatorname{trace}Y_{\mathrm{eper},T}(t,k)+\sum_{i=1}^{d}\operatorname{trace}Y^{S}_{\mathrm{eper},i}(t,k)+\sum_{j=1}^{m}\operatorname{trace}Y^{P}_{\mathrm{eper},j}(t,k)\right)$$

subject to: $Y_{\mathrm{eper},T}(t,k)\succ 0$, $Y^{S}_{\mathrm{eper},i}(t,k)\succ 0$, $Y^{P}_{\mathrm{eper},j}(t,k)\succ 0$,

$Y_{\mathrm{eper},T}(h+q,k)=Y_{\mathrm{eper},T}(h,k)$, $Y^{S}_{\mathrm{eper},i}(h+q,k)=Y^{S}_{\mathrm{eper},i}(h,k)$, and

$$[\![A^*]\!](t,k)[\![S^*Y_{\mathrm{eper}}S]\!](t,k)[\![A]\!](t,k)-[\![Y_{\mathrm{eper}}]\!](t,k)+[\![C^*]\!](t,k)[\![C]\!](t,k)\prec 0,$$

for all $t=0,\dots,h+q-1$, $i=1,\dots,d$, $j=1,\dots,m$, and $k\in V$.

3. Using the (h,q)-eventually time-periodic generalized gramians X_{eper} and Y_{eper} computed in steps 1 and 2, find the (h,q)-eventually time-periodic balanced realization $(A_{\mathrm{bal}},B_{\mathrm{bal}},C_{\mathrm{bal}},D,\boldsymbol{\Delta})$ for system $\mathcal{G}_\delta$ as in the proof of Lemma 5.4.

4. minimize

$$a_1\times\epsilon+\sum_{t=0}^{h+q-1}\sum_{k\in V}\Bigg(\left\|\operatorname{vect}\left(\Sigma_{\mathrm{eper},T}(t,k)-\epsilon I\right)\right\|_1+\sum_{i=1}^{d}\left\|\operatorname{vect}\left(\Sigma^{S}_{\mathrm{eper},i}(t,k)-\epsilon I\right)\right\|_1 \\ +\sum_{j=1}^{m}\left\|\operatorname{vect}\left(\Sigma^{P}_{\mathrm{eper},j}(t,k)-\epsilon I\right)\right\|_1\Bigg)$$

subject to: $\Sigma_{\mathrm{eper},T}(t,k)$, $\Sigma^{S}_{\mathrm{eper},i}(t,k)$, and $\Sigma^{P}_{\mathrm{eper},j}(t,k)$ are diagonal matrices,

$\epsilon>0$, $\Sigma_{\mathrm{eper},T}(t,k)\succeq\epsilon I$, $\Sigma^{S}_{\mathrm{eper},i}(t,k)\succeq\epsilon I$, $\Sigma^{P}_{\mathrm{eper},j}(t,k)\succeq\epsilon I$,

$\Sigma_{\mathrm{eper},T}(h+q,k)=\Sigma_{\mathrm{eper},T}(h,k)$, $\Sigma^{S}_{\mathrm{eper},i}(h+q,k)=\Sigma^{S}_{\mathrm{eper},i}(h,k)$,

$$[\![A_{\mathrm{bal}}]\!](t,k)[\![\Sigma_{\mathrm{eper}}]\!](t,k)[\![A^*_{\mathrm{bal}}]\!](t,k)-[\![S^*\Sigma_{\mathrm{eper}}S]\!](t,k)+[\![B_{\mathrm{bal}}]\!](t,k)[\![B^*_{\mathrm{bal}}]\!](t,k)\prec 0,$$

$$[\![A^*_{\mathrm{bal}}]\!](t,k)[\![S^*\Sigma_{\mathrm{eper}}S]\!](t,k)[\![A_{\mathrm{bal}}]\!](t,k) - [\![\Sigma_{\mathrm{eper}}]\!](t,k) + [\![C^*_{\mathrm{bal}}]\!](t,k)[\![C_{\mathrm{bal}}]\!](t,k) \prec 0,$$

for all $t = 0, \ldots, h+q-1$, $i = 1, \ldots, d$, $j = 1, \ldots, m$, and $k \in V$.

Algorithm 5.8 is now discussed in more details. In the first two steps, the (h,q)-eventually time-periodic generalized gramians X_{eper} and Y_{eper} with minimum trace are computed, see [18, 72] for comments on the usefulness of computing minimum trace generalized gramians. These gramians are then used to find a balanced realization of the system. To obtain a 'small' value of $\Phi(\Omega_{\mathrm{eper}})$ and a useful error bound, the generalized Lyapunov inequalities are expressed for the obtained balanced realization of the system and are solved again. In this problem, both inequalities are imposed simultaneously and an (h,q)-eventually time-periodic balanced generalized gramian Σ_{eper} is sought. That is, in addition to $\Sigma_{\mathrm{eper}} \in \mathcal{X}$, $[\![\Sigma_{\mathrm{eper}}]\!](t,k)$ is a diagonal matrix for all (t,k). In other words, it is required that $\Sigma_{\mathrm{eper},T}(t,k)$, $\Sigma^S_{\mathrm{eper},i}(t,k)$, and $\Sigma^P_{\mathrm{eper},j}(t,k)$ are diagonal matrices for all $t = 0, \ldots, h+q-1$, $i = 1, \ldots, d$, $j = 1, \ldots, m$, and $k \in V$. Note that Σ_{eper} is guaranteed to exist since the generalized Lyapunov inequalities are now expressed for the (h,q)-eventually time-periodic balanced realization of the system.

The purpose behind Algorithm 5.8 is to obtain a useful error bound from Theorem 5.7. Since $\Phi(\Omega_{\mathrm{eper}})$ consists of summing the distinct diagonal entries in Ω_{eper}, then it is desired that many diagonal entries in Ω_{eper} be equal to each other and to some value denoted by ϵ. Namely, if ϵ is chosen as the truncation cut-off value, i.e., all the state variables that have a corresponding entry in Σ_{eper} equal to ϵ are truncated (and no other state variables are truncated), then $\Phi(\Omega_{\mathrm{eper}}) = \epsilon$. In this case, the value of ϵ is also desired to be small, since Theorem 5.7 gives $\|(G_\delta - G_{r,\delta})\| < 2\epsilon$ for all $\Delta \in \boldsymbol{\Delta}$. In the SDP defined in step 4, these requirements are expressed using the chosen objective function and the constraint $\Sigma_{\mathrm{eper}} \succeq \epsilon I$. Specifically, the chosen objective function consists of two competing costs/components that are to be minimized. On the one hand, it is desired that the cut-off truncation value ϵ be as small as possible, which justifies the first term, ϵ, in the objective function. On the other hand, it is desired to truncate the maximum number of state variables. This justifies the second term (the summation) in the objective function and the imposed constraint $\Sigma_{\text{eper}} \succeq \epsilon I$, wherein the 1-norm heuristic is employed to result in many entries in Σ_{eper} that are equal to ϵ. The two component vector-valued objective function is then scalarized, i.e, the two cost functions are added and the first term is multiplied by the weight $a_1 > 0$. The optimal weight a_1 for the purposes of model reduction is determined by tracing the optimal trade-off curve: the SDP defined in step 4 is solved for multiple values of a_1, and for each choice of a_1, the resulting value of ϵ and the corresponding total number of truncated state variables (number of entries in Σ_{eper} equal to ϵ) are recorded. The optimal weight a_1 is then chosen so as to result in an acceptably small value of ϵ and a large enough number of truncated state variables. The reader is referred to [19] for background material on the convex optimization tools that are leveraged in Algorithm 5.8.

Next, an alternative expression for the balanced truncation error bound is derived that can be less conservative than the expression given in Theorem 5.7 when the subsystems have general (not eventually time-periodic) NSLPV models. This expression generalizes the expression given in Theorem 2.12 in that it applies to non-monotonic truncation sequences. The counterparts of the next result for single LTV systems and single NSLPV systems are found in [73] and [33], respectively. For all $k \in V$, $i = 1, \dots, d$, and $j = 1, \dots, m$, define the sets $\mathcal{F}_T(k) = \{t \in \mathbb{N}_0 \,|\, r_T(t,k) \neq n_T(t,k)\}$, $\mathcal{F}_i^S(k) = \{t \in \mathbb{N}_0 \,|\, r_i^S(t,k) \neq n_i^S(t,k)\}$, and $\mathcal{F}_j^P(k) = \{t \in \mathbb{N}_0 \,|\, r_j^P(t,k) \neq n_j^P(t,k)\}$. The following definition is from [33] and restates Definition 2.11.

Definition 5.9. Consider a scalar sequence α_t defined on a subset $\mathcal{W}$ of $\mathbb{N}_0$, and let $t_{\min} =$

$\min\{t \,|\, t \in \mathcal{W}\}$. The following rule extends the domain of definition of α_t to all $t \in \mathbb{Z}$:

$$\alpha_t = \begin{cases} \alpha_{t_{\min}} & \text{if } t \leq t_{\min}, \\ \alpha_e & \text{if } t > t_{\min}, \text{ where } e = \max\{\tau \leq t : \tau \in \mathcal{W}\}. \end{cases}$$

In the following, the min-max ratio of a sequence is defined [73]. This ratio is very useful for expressing the error bound in Theorem 5.11.

Definition 5.10. Consider a scalar sequence $v = (v_1, v_2, \ldots, v_s)$ for some integer $s \geq 1$, where v_1 cannot be considered as a local maximum and v_s cannot be considered as a local minimum. Then, the sequence v has l local maxima and l local minima for some $l \in \mathbb{N}_0$. If $l \geq 1$, the local maxima and local minima of v are denoted by $v_{\max,e}$ and $v_{\min,e}$, respectively, for $e = 1, \ldots, l$. The min-max ratio S_v of the sequence v is then defined as follows:

$$S_v = \begin{cases} v_1 & \text{if } l = 0, \\ v_1 \prod_{e=1}^{l} \left(\frac{v_{\max,e}}{v_{\min,e}}\right) & \text{if } l > 0. \end{cases} \tag{5.17}$$

Theorem 5.11. *Consider a strongly stable distributed NSLPV system $\mathcal{G}_\delta$ that has a balanced realization denoted by $(A, B, C, D, \boldsymbol{\Delta})$ and a balanced generalized gramian denoted by Σ. Assume that the balanced truncation method is applied to reduce system $\mathcal{G}_\delta$, and denote the resulting reduced order system by $\mathcal{G}_{r,\delta}$ and its obtained realization by $(A_r, B_r, C_r, D, \boldsymbol{\Delta}_r)$. Let the partitioned graph-diagonal operator Ω be defined as in (5.12). If for all $k \in V$, $i = 1, \ldots, d$, and $j = 1, \ldots, m$,*

$$\begin{aligned} \Omega_T(t,k) &= w_T(t,k)I && \textit{for } t \in \mathcal{F}_T(k), \\ \Omega_i^S(t,k) &= w_i^S(t,k)I && \textit{for } t \in \mathcal{F}_i^S(k), \\ \Omega_j^P(t,k) &= w_j^P(t,k)I && \textit{for } t \in \mathcal{F}_j^P(k), \end{aligned}$$

and the sets $\mathcal{F}_T(k)$, $\mathcal{F}_i^S(k)$, and $\mathcal{F}_j^P(k)$ are finite, then

$$\|(G_\delta - G_{r,\delta})\| < 2\sum_{k\in V}\left(S_{w_T(t,k)} + \sum_{i=1}^{d} S_{w_i^S(t,k)} + \sum_{j=1}^{m} S_{w_j^P(t,k)}\right) \quad \textit{for all } \Delta \in \mathbf{\Delta}, \tag{5.18}$$

where the min-max ratio S_v of a sequence v is defined in (5.17).

Proof. The result is proved for the special case where only one state is truncated, e.g., the temporal state associated with subsystem $G^{(k_0)}$. The result for the general case considered in the theorem then follows by repeated application of this specialized result and the triangle inequality. Without loss of generality, it is assumed that $w_T(t,k_0) \le 1$ for all $t \in \mathcal{F}_T(k_0)$. This assumption can be ensured by scaling system $\mathcal{G}_\delta$. Suppose that the sequence $w_T(t,k_0)$ has l local maxima and l local minima for some $l \in \mathbb{N}_0$. As stipulated in Definition 5.10, the first element in the sequence cannot be considered as a local maximum and the last element cannot be considered as a local minimum.

If $l \ge 1$, then the set $\mathcal{F}_T(k_0)$ is of the form $\{t_1, \ldots, t_{\min,1}, \ldots, t_{\max,1}, \ldots, t_{\max,l}, \ldots, t_s\}$. For $e = 1, \ldots, l$, $t_{\min,e}$ and $t_{\max,e}$ denote the time-steps at which $w_T(t,k_0)$ reaches its local minima and local maxima, respectively. The domain of $w_T(t,k_0)$ is extended to all $t \in \mathbb{Z}$ using the rule given in Definition 5.9. Note that $w_T(t,k_0) = w_T(t_1,k_0)$ for $t \le t_1$ and $w_T(t,k_0) = w_T(t_s,k_0)$ for $t \ge t_s$. Define the state transformation $T = T^* \in \mathcal{T}$ as follows:

$$[\![T]\!](t,k) = \begin{cases} (w_T(t,k_0))^{1/2}\ I & \text{for } t \le t_{\min,1}, \\ w_T(t_{\min,1},k_0)\ (w_T(t,k_0))^{-1/2}\ I & \text{for } t_{\min,1}+1 \le t \le t_{\max,1}, \\ w_T(t_{\min,1},k_0)\,(w_T(t_{\max,1},k_0))^{-1}\,(w_T(t,k_0))^{1/2}\, I & \text{for } t_{\max,1}+1 \le t \le t_{\min,2}, \\ \qquad\vdots & \qquad\vdots \\ \eta\ (w_T(t,k_0))^{1/2}\ I & \text{for } t \ge t_{\max,l}+1, \end{cases}$$

for all $(t,k) \in \mathbb{Z}\times V$, where $\eta = \prod_{e=1}^{l} w_T(t_{\min,e},k_0)\ \ (w_T(t_{\max,e},k_0))^{-1}$.

By definition, $\Sigma_T(t,k_0) \succ \mu I$ for some $\mu > 0$ and all $t \in \mathbb{Z}$, and so T and T^{-1} are bounded. The state transformation T is used to construct a new realization $(A_{\text{new}}, B_{\text{new}}, C_{\text{new}}, D, \boldsymbol{\Delta})$ for system $\mathcal{G}_\delta$, where $A_{\text{new}} = (S^*TS)AT^{-1}$, $B_{\text{new}} = (S^*TS)B$, and $C_{\text{new}} = CT^{-1}$. For simplicity, the system with this new realization is referred to as system $\mathcal{G}_{\text{new},\delta}$, whereas system $\mathcal{G}_\delta$ continues to refer to the system with the realization $(A, B, C, D, \boldsymbol{\Delta})$. Let the operators P, Q, and W in $\mathcal{T}$ be defined as follows: $[\![P]\!](t,k) = (w_T(t,k_0))^{1/2} I$, $[\![Q]\!](t,k) = [\![T]\!](t,k)[\![P^{-1}]\!](t,k)$, and $[\![W]\!](t,k) = [\![T]\!](t,k)[\![P]\!](t,k)$ for all $(t,k) \in \mathbb{Z} \times V$. For each $k \in V$, the scalar-valued sequences $\alpha_1(t,k)$, $\alpha_2(t,k)$, and $\alpha_3(t,k)$, where $[\![T]\!](t,k) = \alpha_1(t,k)I$, $[\![Q]\!](t,k) = \alpha_2(t,k)I$, and $[\![W]\!](t,k) = \alpha_3(t,k)I$, are non-increasing in t.

Since the realization $(A, B, C, D, \boldsymbol{\Delta})$ of system $\mathcal{G}_\delta$ is balanced, then the balanced generalized gramian Σ satisfies (5.9) and (5.10). By pre- and post-multiplying (5.9) by S^*TS, and invoking $TT^{-1} = T^{-1}T = I$, the following inequality is obtained:

$$A_{\text{new}}\, T\, \Sigma\, T\, A_{\text{new}}^* - (S^*TS)\,(S^*\Sigma S)\,(S^*TS) + B_{\text{new}} B_{\text{new}}^* \prec 0$$

In other words,

$$\begin{aligned}[\![A_{\text{new}}]\!](t,k)[\![T]\!](t,k)[\![\Sigma]\!](t,k)[\![T]\!](t,k)[\![A_{\text{new}}^*]\!](t,k) - [\![S^*TS]\!](t,k)[\![S^*\Sigma S]\!](t,k)[\![S^*TS]\!](t,k)\\ + [\![B_{\text{new}}]\!](t,k)[\![B_{\text{new}}^*]\!](t,k) \prec -\mu I,\end{aligned}$$

for all $(t,k) \in \mathbb{Z} \times V$ and some $\mu > 0$. Pre- and post-multiplying the previous inequality by $[\![S^*W^{-1}S]\!](t,k) = (\alpha_3(t+1,k))^{-1} I$, where $\alpha_3(t,k) = \alpha_1(t,k)(w_T(t,k_0))^{1/2}$, or equivalently, $(w_T(t,k_0))^{-1/2} = (\alpha_3(t,k))^{-1}\alpha_1(t,k)$, result in

$$\begin{aligned}[\![A_{\text{new}}]\!](t,k)\,\alpha_1(t,k)\,(\alpha_3(t+1,k))^{-1}\,[\![\Sigma]\!](t,k)\,\alpha_1(t,k)\,(\alpha_3(t+1,k))^{-1}\,[\![A_{\text{new}}^*]\!](t,k)\\ - (w_T(t+1,k_0))^{-1}\,[\![S^*\Sigma S]\!](t,k)\end{aligned}$$

$$+ (\alpha_3(t+1,k))^{-1} \llbracket B_{\text{new}} \rrbracket(t,k) \llbracket B^*_{\text{new}} \rrbracket(t,k) (\alpha_3(t+1,k))^{-1} \prec -\mu I.$$

For each $k \in V$, the sequence $\alpha_3(t,k)$ is non-increasing in t, i.e., $0 < \alpha_3(t+1,k) \leq \alpha_3(t,k) \leq w_T(t_1,k_0)$ and $(\alpha_3(t+1,k))^{-1} \geq (\alpha_3(t,k))^{-1} \geq (w_T(t_1,k_0))^{-1}$. Then, one can verify that the following holds for all $(t,k) \in \mathbb{Z} \times V$:

$$\begin{aligned} &\llbracket A_{\text{new}} \rrbracket(t,k) (w_T(t,k_0))^{-1} \llbracket \Sigma \rrbracket(t,k) \llbracket A^*_{\text{new}} \rrbracket(t,k) - (w_T(t+1,k_0))^{-1} \llbracket S^*\Sigma S \rrbracket(t,k) \\ &\qquad + (w_T(t_1,k_0))^{-1} \llbracket B_{\text{new}} \rrbracket(t,k) \llbracket B^*_{\text{new}} \rrbracket(t,k) (w_T(t_1,k_0))^{-1} \prec -\mu I. \end{aligned} \tag{5.19}$$

Similarly, by pre- and post-multiplying (5.10) by T^{-1}, and invoking $(S^*T^{-1}S)(S^*TS) = (S^*TS)(S^*T^{-1}S) = I$, the following inequality is obtained:

$$A^*_{\text{new}}(S^*T^{-1}S)(S^*\Sigma S)(S^*T^{-1}S)A_{\text{new}} - T^{-1}\Sigma T^{-1} + C^*_{\text{new}}C_{\text{new}} \prec 0.$$

In other words,

$$\begin{aligned} &\llbracket A^*_{\text{new}} \rrbracket(t,k) \llbracket S^*T^{-1}S \rrbracket(t,k) \llbracket S^*\Sigma S \rrbracket(t,k) \llbracket S^*T^{-1}S \rrbracket(t,k) \llbracket A_{\text{new}} \rrbracket(t,k) \\ &\qquad - \llbracket T^{-1} \rrbracket(t,k) \llbracket \Sigma \rrbracket(t,k) \llbracket T^{-1} \rrbracket(t,k) + \llbracket C^*_{\text{new}} \rrbracket(t,k) \llbracket C_{\text{new}} \rrbracket(t,k) \prec -\mu I, \end{aligned}$$

for all $(t,k) \in \mathbb{Z} \times V$ and some $\mu > 0$. By pre- and post-multiplying the previous inequality by $\llbracket Q \rrbracket(t,k) = \alpha_2(t,k)I$, and using the fact that $\alpha_2(t,k) = \alpha_1(t,k)(w_T(t,k_0))^{-1/2}$, i.e., $\alpha_2(t,k)(\alpha_1(t,k))^{-1} = (w_T(t,k_0))^{-1/2}$, the following inequality is obtained:

$$\begin{aligned} &\llbracket A^*_{\text{new}} \rrbracket(t,k)\, \alpha_2(t,k) (\alpha_1(t+1,k))^{-1} \llbracket S^*\Sigma S \rrbracket(t,k) (\alpha_1(t+1,k))^{-1} \alpha_2(t,k) \llbracket A_{\text{new}} \rrbracket(t,k) \\ &\qquad - (w_T(t,k_0))^{-1} \llbracket \Sigma \rrbracket(t,k) + \alpha_2(t,k) \llbracket C^*_{\text{new}} \rrbracket(t,k) \llbracket C_{\text{new}} \rrbracket(t,k)\, \alpha_2(t,k) \prec -\mu I. \end{aligned}$$

For each $k \in V$, the sequence $\alpha_2(t,k)$ is non-increasing in t, i.e., $\eta \le \alpha_2(t+1,k) \le \alpha_2(t,k)$, and so it follows that

$$\llbracket A^*_{\text{new}} \rrbracket(t,k)\,(w_T(t+1,k_0))^{-1}\,\llbracket S^*\Sigma S \rrbracket(t,k)\,\llbracket A_{\text{new}} \rrbracket(t,k) \; - (w_T(t,k_0))^{-1}\,\llbracket \Sigma \rrbracket(t,k) \\ + \eta\,\llbracket C^*_{\text{new}} \rrbracket(t,k)\,\llbracket C_{\text{new}} \rrbracket(t,k)\,\eta \prec -\mu I. \quad (5.20)$$

It can be concluded from (5.19) and (5.20) that the scaled realization of system $\mathcal{G}_{\text{new},\delta}$ given by $(A_{\text{new}},\, (w_T(t_1,k_0))^{-1}\,B_{\text{new}},\, \eta\,C_{\text{new}},\, (w_T(t_1,k_0))^{-1}\,\eta\,D,\, \mathbf{\Delta})$ is balanced with a balanced generalized gramian Σ_{new}, where $\llbracket \Sigma_{\text{new}} \rrbracket(t,k) = (w_T(t,k_0))^{-1}\llbracket \Sigma \rrbracket(t,k)$ for all $(t,k) \in \mathbb{Z} \times V$. In particular, $\Omega_{\text{new},T}(t,k_0) = (w_T(t,k_0))^{-1}\,\Omega_T(t,k_0) = I$ for all $t \in \mathcal{F}_T(k_0)$. The balanced truncation method is applied to reduce system $\mathcal{G}_{\text{new},\delta}$ by only truncating the temporal state of subsystem $G^{(k_0)}_{\text{new}}$ at the time-steps $t \in \mathcal{F}_T(k_0)$. If the resulting reduced order system is denoted by $\mathcal{G}_{\text{new},r,\delta}$, it follows from Theorem 5.6 that $(w_T(t_1,k_0))^{-1} \times \eta \times \|(G_{\text{new},\delta} - G_{\text{new},r,\delta})\| < 2$, i.e., $\|(G_{\text{new},\delta} - G_{\text{new},r,\delta})\| < 2\,S_{w_T(t,k_0)}$ for all $\Delta \in \mathbf{\Delta}$. The proof of the theorem for the case $l \ge 1$ is concluded by noting that $\|(G_\delta - G_{r,\delta})\| = \|(G_{\text{new},\delta} - G_{\text{new},r,\delta})\|$ for all $\Delta \in \mathbf{\Delta}$ because of the special structure of the state transformation T.

The proof for the case $l = 0$, i.e., when the sequence $w_T(t,k_0)$ is monotone non-increasing in t, is similar to the proof of Theorem 2.12. In this case, the state transformation T is defined as $\llbracket T \rrbracket(t,k) = (w_T(t,k_0))^{1/2}\,I$ for all $(t,k) \in \mathbb{Z} \times V$, and the realization $(A_{\text{new}}, B_{\text{new}}, C_{\text{new}}, D, \mathbf{\Delta})$ of system $\mathcal{G}_{\text{new},\delta}$ defined as above satisfies

$$\llbracket A_{\text{new}} \rrbracket(t,k)\,w_T(t,k_0)\,\llbracket \Sigma \rrbracket(t,k)\,\llbracket A^*_{\text{new}} \rrbracket(t,k) \; - w_T(t+1,k_0)\,\llbracket S^*\Sigma S \rrbracket(t,k) \\ + \llbracket B_{\text{new}} \rrbracket(t,k)\,\llbracket B^*_{\text{new}} \rrbracket(t,k) \prec -\mu I,$$

$$\llbracket A_{\text{new}}^* \rrbracket(t,k)\,(w_T(t+1,k_0))^{-1}\,\llbracket S^*\Sigma S \rrbracket(t,k)\,\llbracket A_{\text{new}} \rrbracket(t,k)\; - (w_T(t,k_0))^{-1}\,\llbracket \Sigma \rrbracket(t,k) \\ + \llbracket C_{\text{new}}^* \rrbracket(t,k)\,\llbracket C_{\text{new}} \rrbracket(t,k) \prec -\mu I,$$

for all $(t,k) \in \mathbb{Z} \times V$ and some $\mu > 0$. Since $w_T(t,k_0)$ is non-increasing in t, then $0 < w_T(t+1,k_0) \leq w_T(t,k_0)$ and $(w_T(t,k_0))^{-1} \leq (w_T(t+1,k_0))^{-1}$; and since $w_T(t,k_0) \leq 1$ for all $t \in \mathbb{Z}$ by assumption, then $w_T(t,k_0) \leq (w_T(t,k_0))^{-1}$. Thus, one can verify that

$$\llbracket A_{\text{new}} \rrbracket(t,k)\,(w_T(t,k_0))^{-1}\,\llbracket \Sigma \rrbracket(t,k)\,\llbracket A_{\text{new}}^* \rrbracket(t,k)\; - (w_T(t+1,k_0))^{-1}\,\llbracket S^*\Sigma S \rrbracket(t,k) \\ + \llbracket B_{\text{new}} \rrbracket(t,k)\,\llbracket B_{\text{new}}^* \rrbracket(t,k) \prec -\mu I,$$

for all $(t,k) \in \mathbb{Z} \times V$. That is, the realization $(A_{\text{new}}, B_{\text{new}}, C_{\text{new}}, D, \boldsymbol{\Delta})$ of system $\mathcal{G}_{\text{new},\delta}$ is balanced, and Σ_{new} is a balanced generalized gramian, where $\llbracket \Sigma_{\text{new}} \rrbracket(t,k) = (w_T(t,k_0))^{-1}\,\llbracket \Sigma \rrbracket(t,k)$ for all $(t,k) \in \mathbb{Z} \times V$. In particular, $\Omega_{\text{new},T}(t,k_0) = (w_T(t,k_0))^{-1}\Omega_T(t,k_0) = I$ for all $t \in \mathcal{F}_T(k_0)$. The balanced truncation method is applied to reduce the temporal state corresponding to subsystem $G_{\text{new}}^{(k_0)}$ at time-steps $t \in \mathcal{F}_T(k_0)$, and the resulting reduced order system is denoted by $\mathcal{G}_{\text{new},r,\delta}$. From Theorem 5.6, it follows that $\|(G_\delta - G_{r,\delta})\| = \|(G_{\text{new},\delta} - G_{\text{new},r,\delta})\| < 2 = 2w_T(t_1,k_0)$ for all $\Delta \in \boldsymbol{\Delta}$. The left-hand-side equality holds because of the special structure of the state transformation T, and the right-hand-side equality follows from the fact that $w_T(t_1,k_0) = 1$. This is true since the system is scaled to ensure that the monotone non-increasing sequence $w_T(t,k_0)$ satisfies $w_T(t,k_0) \leq 1$ for all $t \in \mathcal{F}_T(k_0)$. $\square$

The balanced truncation error bound explicitly relates to the ℓ_2-induced norm of the error system and can be used in robustness analysis [61], wherein the full order system is replaced by the reduced order system and a perturbation operator whose norm is less than the error bound. Namely, for all $\Delta \in \boldsymbol{\Delta}$, G_δ can be expressed as $G_\delta = G_{r,\delta} + (G_\delta - G_{r,\delta})$, where a bound on $\|(G_\delta - G_{r,\delta})\|$ is known a priori. Thus, deriving and computing tighter error bounds help in better quantifying the said perturbation operator and ultimately in yielding less conservative robustness results. The application of Theorems 5.7 and 5.11 is illustrated in the following section.

5.5 Illustrative Example

This example illustrates the application of Theorems 5.7 and 5.11. Suppose that the only truncated state is the temporal state associated with subsystem $G^{(k_0)}$ and $\Omega_T(t, k_0) = \frac{1}{t}I$, where $t \in \mathcal{F}_T(k_0) = \{1, 2, 3, 4, 5\}$. By Theorem 5.7, $\|(G_\delta - G_{r,\delta})\| < 2 \times (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}) \approx 4.57$ for all $\Delta \in \mathbf{\Delta}$. However, Theorem 5.11 gives $\|(G_\delta - G_{r,\delta})\| < 2 \times 1 = 2$ for all $\Delta \in \mathbf{\Delta}$, since $w_T(t, k_0) = \frac{1}{t}$ is a monotone decreasing sequence. This bound represents a 56% improvement over the bound computed using Theorem 5.7.

For the same set $\mathcal{F}_T(k_0)$, assume now that $\Omega_T(1, k_0) = \text{diag}(2, 1)$, $\Omega_T(2, k_0) = \text{diag}(17, 0.5)$, $\Omega_T(3, k_0) = \text{diag}(4, 2)$, $\Omega_T(4, k_0) = \text{diag}(6, 1)$, $\Omega_T(5, k_0) = \text{diag}(16, 4)$. By Theorem 5.7, $\|(G_\delta - G_{r,\delta})\| < 2 \times (17 + 16 + 6 + 4 + 2 + 1 + 0.5) = 93$ for all $\Delta \in \mathbf{\Delta}$, whereas by Theorem 5.11, $\|(G_\delta - G_{r,\delta})\| < 2 \times (2 \times \frac{17}{2} \times \frac{16}{4} + 1 \times \frac{2}{0.5} \times \frac{4}{1}) = 168$ for all $\Delta \in \mathbf{\Delta}$. As illustrated below, this bound is obtained by first truncating the boxed sequence $(1, 0.5, 2, 1, 4)$ and then truncating the circled sequence $(2, 17, 4, 6, 16)$.

$$\Omega_T(1, k_0) = \begin{bmatrix} 2 & 0 \\ 0 & \boxed{1} \end{bmatrix}, \quad \Omega_T(2, k_0) = \begin{bmatrix} 17 & 0 \\ 0 & \boxed{0.5} \end{bmatrix}, \quad \Omega_T(3, k_0) = \begin{bmatrix} 4 & 0 \\ 0 & \boxed{2} \end{bmatrix},$$
$$\Omega_T(4, k_0) = \begin{bmatrix} 6 & 0 \\ 0 & \boxed{1} \end{bmatrix}, \quad \Omega_T(5, k_0) = \begin{bmatrix} 16 & 0 \\ 0 & \boxed{4} \end{bmatrix}.$$

However, the truncation sequence can be split into more than two sequences as illustrated below:

$$\Omega_T(1,k_0)=\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},\quad \Omega_T(2,k_0)=\begin{bmatrix} 17 & 0 \\ 0 & \boxed{0.5} \end{bmatrix},\quad \Omega_T(3,k_0)=\begin{bmatrix} 4 & 0 \\ 0 & \boxed{2} \end{bmatrix},$$
$$\Omega_T(4,k_0)=\begin{bmatrix} 6 & 0 \\ 0 & \underline{1} \end{bmatrix},\quad \Omega_T(5,k_0)=\begin{bmatrix} 16 & 0 \\ 0 & \boxed{4} \end{bmatrix}.$$

These sequences are truncated in the following order: the underlined sequence (1), the boxed sequence $(0.5, 2, 4)$, the circled sequence $(1, 17, 16)$, and the sequence $(2, 4, 6)$ inside the triangles. In this case, the bound from Theorem 5.11 becomes $2\times(1+0.5\times\frac{4}{0.5}+1\times\frac{17}{1}+2\times\frac{6}{2}) = 56$, which represents a 40% improvement over the bound from Theorem 5.7.

The same upper bound of 56 can be retrieved by considering the truncation sequences $(0.5, 1)$, $(1, 2, 6, 4)$, $(2, 4)$, and $(17, 16)$, respectively. This observation raises the question of how to best apply Theorem 5.11 to obtain the least conservative error bound. Answering this question is indeed a nontrivial task, and future work will focus on developing a fast computational algorithm that effectively applies Theorem 5.11 to compute a useful bound for a given truncation sequence. One possible approach to deal with this problem consists of modeling the truncation sequence as a directed graph, where the vertices correspond to the truncated values and the directed edges are obtained from the allowable truncation sequences. The problem then becomes one of finding the graph partition that minimizes the upper bound expression from Theorem 5.11. The precise details of this modeling approach are still not developed, and other possible approaches still need to be explored. Solutions to the formulated problem can then be computed using heuristics and approximation algorithms [46]. Adopting ideas from Dijkstra's shortest path algorithm [10] and set packing [76], among others, may prove useful in this direction. This example is concluded by noting that Theorem 5.11 can also be used to improve on the error bound due to truncation over the finite time-horizon in the case of eventually time-periodic subsystems.

Chapter 6

Coprime Factors Reduction of Interconnected Nonstationary LPV Systems

6.1 Chapter Overview

This chapter presents the coprime factors reduction method for the model reduction of interconnected NSLPV systems. In Section 6.2, the coprime factors reduction method of Chapter 3 is extended to the distributed NSLPV setting, and it is shown how to apply the balanced truncation method of Chapter 5 to reduce systems that are strongly stabilizable and strongly detectable, but not necessarily strongly stable. The method of Section 6.2 does not address the issue of the contractiveness of the coprime factorizations that are used in the model reduction algorithm. Contractive coprime factorizations (CCFs) and their use in the coprime factors reduction algorithm are the topics of Sections 6.3, 6.4, and 6.5. In Section 6.3, the literature on coprime factors reduction methods that employ CCFs is surveyed. Then, Sections 6.4 and 6.5 give two possible modifications on the coprime factors reduction method of Section 6.2 to ensure the contractiveness of the employed coprime factorizations. Section 6.6 discusses how to interpret the coprime factors reduction error bound in terms of robust stability of the closed-loop system. The three coprime factors reduction methods discussed in this chapter are compared using an illustrative example in Section 6.7.

6.2 Coprime Factors Model Reduction

Section 5.3 applies the balanced truncation method to reduce distributed NSLPV systems that are strongly stable. However, there exist stable systems that are not strongly stable and to which the balanced truncation method does not apply. The present section presents the coprime factors reduction method as a means of extending the range of applicability of the balanced truncation method to distributed NSLPV systems that are not necessarily strongly stable but are strongly stabilizable and strongly detectable. Denote by $\mathcal{G}_\delta$ the strongly stabilizable and strongly detectable distributed NSLPV system to be reduced by the application of the coprime factors reduction method. First, a strongly stable coprime factorization is found for system $\mathcal{G}_\delta$ and is used in forming an augmented system $\mathcal{H}_\delta$. Being strongly stable, system $\mathcal{H}_\delta$ is reduced by applying the balanced truncation method, and the resulting reduced order system is denoted by $\mathcal{H}_{r,\delta}$. System $\mathcal{H}_{r,\delta}$ gives a strongly stable coprime factorization that is used to construct the desired reduced order system $\mathcal{G}_{r,\delta}$.

6.2.1 Strong Stabilizability

Let the set $\mathcal{F}$ be defined as follows:

$$\begin{aligned}\mathcal{F} = \Big\{ F = \begin{bmatrix} F_T & F_1^S & \cdots & F_d^S & F_1^P & \cdots & F_m^P \end{bmatrix} \mid\ & F_T,\ F_i^S, \text{ and } F_j^P \\ & \text{are graph-diagonal operators for } i = 1, \ldots, d \text{ and } j = 1, \ldots, m, \text{ where} \\ & F_T \in \mathcal{L}_c\left(\ell_2\left(\left\{\mathbb{R}^{n_T(t,k)}\right\}\right), \ell_2\left(\left\{\mathbb{R}^{n_u(t,k)}\right\}\right)\right),\ F_i^S \in \mathcal{L}_c\left(\ell_2\left(\left\{\mathbb{R}^{n_i^S(t,k)}\right\}\right), \ell_2\left(\left\{\mathbb{R}^{n_u(t,k)}\right\}\right)\right), \\ & \text{and } F_j^P \in \mathcal{L}_c\left(\ell_2\left(\left\{\mathbb{R}^{n_j^P(t,k)}\right\}\right), \ell_2\left(\left\{\mathbb{R}^{n_u(t,k)}\right\}\right)\right)\Big\}. \end{aligned} \tag{6.1}$$

Definition 6.1. Consider a well-posed distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. System $\mathcal{G}_\delta$ is said to be strongly stabilizable if there exists a feedback operator $F \in \mathcal{F}$, where the set $\mathcal{F}$ is defined in (6.1), such that the resulting closed-loop system is strongly stable, i.e., if and only if there exists $F \in \mathcal{F}$ and $P \in \mathcal{X}$ such that

$$(A + BF)P(A + BF)^* - S^*PS \prec 0. \tag{6.2}$$

The notion of strong detectability is defined as the dual notion to strong stabilizability.

Theorem 6.2. *Consider a well-posed distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. System $\mathcal{G}_\delta$ is strongly stabilizable if and only if there exists $P \in \mathcal{X}$ that satisfies*

$$APA^* - S^*PS - BB^* \prec 0. \tag{6.3}$$

Furthermore, an appropriate choice of $F \in \mathcal{F}$ that renders the resulting closed-loop system strongly stable is given by $F = -(B^*S^*P^{-1}SB)^{-1}B^*S^*P^{-1}SA$, provided that the inverse exists and is bounded.

Proof. For all $(t, k) \in \mathbb{Z} \times V$, assume that rank $[\![B]\!](t, k) = n_u(t, k) < n_T(t+1, k) + \sum_{i=1}^{d} n_i^S(t+1, \rho_i(k)) + \sum_{j=1}^{m} n_j^P(t, k)$. The case rank $[\![B]\!](t, k) < n_u(t, k)$ corresponds to the existence of redundant controls, which can be removed so that the assumption is satisfied. Note that if rank$[\![B]\!](t_0, k_0) = 0$ for some $(t_0, k_0) \in \mathbb{Z} \times V$, e.g., $[\![B]\!](t, k) = 0$ for $t < 0$ and all $k \in V$, then $n_u(t_0, k_0) = 0$ and in this case, all the controls are redundant and removed, and $[\![F]\!](t_0, k_0)$ has a zero dimension and is irrelevant. Also, the proof for square and non-singular $[\![B]\!](t, k)$ follows immediately from the given proof. With the given assumption, a partitioned graph-diagonal operator $B_\perp$ can be constructed with the same structure as B such that $B_\perp^* B_\perp = I$, $B^*B_\perp = 0$, and $\begin{bmatrix} B & B_\perp \end{bmatrix}$ has a bounded inverse. By Definition 6.1, system $\mathcal{G}_\delta$ is strongly stabilizable if and only if there exists $P \in \mathcal{X}$ and $F \in \mathcal{F}$ such that (6.2) holds. Applying the

Schur complement formula to (6.2), the following equivalent inequality is obtained:

$$\begin{bmatrix} -P^{-1} & A^* \\ A & -S^*PS \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} F \begin{bmatrix} I & 0 \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} F^* \begin{bmatrix} 0 & B^* \end{bmatrix} \prec 0, \tag{6.4}$$

which is in the form of the inequality in Lemma 4.10. By matching H, R, and Q defined in Lemma 4.10 with their corresponding terms in (6.4), one obtains

$$H = \begin{bmatrix} -P^{-1} & A^* \\ A & -S^*PS \end{bmatrix}, \qquad R = \begin{bmatrix} I & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & B^* \end{bmatrix}.$$

Thus, it can be verified that $W_R = \begin{bmatrix} 0 & I \end{bmatrix}^*$ and $W_Q = \mathrm{diag}(I, B_\perp)$ satisfy the requirements in Lemma 4.10. Thus, system $\mathcal{G}_\delta$ is strongly stabilizable if and only if $W_R^* H W_R \prec 0$ and $W_Q^* H W_Q \prec 0$. The condition $W_R^* H W_R = -S^*PS \prec 0$ is trivially satisfied, since $P \succ 0$ by the definition of the set $\mathcal{X}$ in (5.5). The condition $W_Q^* H W_Q \prec 0$ is expanded and the Schur complement formula is applied to yield the equivalent condition $B_\perp^*(APA^* - S^*PS)B_\perp \prec 0$. By scaling and a generalization of Finsler's lemma, this inequality is equivalent to (6.3).

It remains to be shown that the given choice of F is in $\mathcal{F}$ and renders the closed-loop system strongly stable. If B^*B is invertible in $\mathcal{L}_c(\ell_2)$, then F is well-defined since $P \succ 0$. This can be achieved by removing the redundant controls and appropriately perturbing B if necessary. Applying the Schur complement formula twice to (6.3), the equivalent inequality $-P^{-1} + A^*(S^*PS + BB^*)^{-1}A \prec 0$ is obtained, where $(S^*PS + BB^*)^{-1}$ is expanded using a generalization of the matrix inversion lemma. Since $P \succ 0$, then $(B^*S^*P^{-1}SB)^{-1} \succ (I + B^*S^*P^{-1}SB)^{-1}$, and it follows that

$$-P^{-1} + A^*S^*P^{-1}SA - A^*S^*P^{-1}SB \left(B^*S^*P^{-1}SB\right)^{-1} B^*S^*P^{-1}SA \prec 0.$$

For the given choice of $F \in \mathcal{F}$, the previous inequality can be rewritten as $-P^{-1} + (A + BF)^* S^* P^{-1} S (A + BF) \prec 0$. By applying the Schur complement formula twice to this inequality, inequality (6.2) is retrieved, which concludes the proof. □

6.2.2 Coprime Factorizations

This section defines the notion of a right coprime factorization (RCF) for a distributed NSLPV system $\mathcal{G}_\delta$. It is shown that if system $\mathcal{G}_\delta$ is strongly stabilizable and strongly detectable as per Definition 6.1, then it admits a strongly stable RCF.

Definition 6.3. Two operators N_δ and M_δ in $\mathcal{L}_c(\ell_2, \ell_2)$ are said to be right coprime if there exist two operators U_δ and V_δ in $\mathcal{L}_c(\ell_2, \ell_2)$ such that $U_\delta N_\delta + V_\delta M_\delta = I$. Two stable distributed NSLPV systems $\mathcal{N}_\delta$ and $\mathcal{M}_\delta$ are said to be right coprime if their input-output maps N_δ and M_δ are right coprime for all $\Delta \in \mathbf{\Delta}$.

Definition 6.4. The pair $(\mathcal{N}_\delta, \mathcal{M}_\delta)$ of stable distributed NSLPV systems is said to be an RCF for the distributed NSLPV system $\mathcal{G}_\delta$ if the systems $\mathcal{N}_\delta$ and $\mathcal{M}_\delta$ are right coprime and, for all $\Delta \in \mathbf{\Delta}$, M_δ has a causal inverse on ℓ_{2e} and $G_\delta = N_\delta M_\delta^{-1}$.

Theorem 6.5. *Consider a strongly stabilizable and strongly detectable system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$, and let $F \in \mathcal{F}$ be any feedback operator that renders the closed-loop system strongly stable. Then, system $\mathcal{G}_\delta$ admits a strongly stable RCF denoted by $(\mathcal{N}_\delta, \mathcal{M}_\delta)$, where the realizations of systems $\mathcal{N}_\delta$ and $\mathcal{M}_\delta$ are given by $(A + BF, B, C + DF, D, \mathbf{\Delta})$ and $(A + BF, B, F, I, \mathbf{\Delta})$, respectively.*

Proof. The distributed NSLPV systems $\mathcal{N}_\delta$ and $\mathcal{M}_\delta$ are strongly stable, since F is a strongly stabilizing feedback operator. To show that $(\mathcal{N}_\delta, \mathcal{M}_\delta)$ is an RCF for system $\mathcal{G}_\delta$, it is shown that the conditions in Definition 6.4 are satisfied.

First, it is shown that M_δ has a causal inverse on ℓ_{2e} for all $\Delta \in \mathbf{\Delta}$. To do so, define

$$R_\delta = \Delta \star \left[\begin{array}{c|c} SA & SB \\ \hline -F & I \end{array}\right].$$

For all $\Delta \in \mathbf{\Delta}$, the mapping R_δ is well-defined and causal on ℓ_{2e}, since system $\mathcal{G}_\delta$ is well-posed, i.e., $I - \Delta SA$ has a causal inverse on $\ell_{2e}^{(n_T,\overline{n}_S,\overline{n}_P)}$ for all $\Delta \in \mathbf{\Delta}$. It remains to verify that $M_\delta R_\delta = R_\delta M_\delta = I$ for all $\Delta \in \mathbf{\Delta}$. For each $\Delta \in \mathbf{\Delta}$, the equations for M_δ and R_δ are written similarly to (5.2) as in

$$\begin{aligned} \zeta_M &= \Delta S(A+BF)\zeta_M + \Delta SBu_M, & y_M &= F\zeta_M + u_M, \\ \zeta_R &= \Delta SA\zeta_R + \Delta SBu_R, & y_R &= -F\zeta_R + u_R, \end{aligned}$$

respectively, where the symbol $\zeta = [x^* \ \beta^*]^*$ is introduced for compactness. In the above equations, u_R and y_R denote the input and output to R_δ, respectively. Similar symbols with subscript M are associated with M_δ, and so on. If $y_M = u_R$, then $(I - \Delta SA)(\zeta_M - \zeta_R) = 0$ and $y_R = F(\zeta_M - \zeta_R) + u_M$. Since $I - \Delta SA$ has a causal inverse on $\ell_{2e}^{(n_T,\overline{n}_S,\overline{n}_P)}$, it follows that $\zeta_M - \zeta_R = 0$ and $y_R = u_M$, i.e., $R_\delta M_\delta = I$. $M_\delta R_\delta = I$ can be proved in a similar way. Thus, R_δ is the inverse of M_δ for all $\Delta \in \mathbf{\Delta}$.

Second, it is shown that $G_\delta = N_\delta M_\delta^{-1} = N_\delta R_\delta$ for all $\Delta \in \mathbf{\Delta}$. Namely,

$$N_\delta R_\delta = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \star \left[\begin{array}{cc|c} S(A+BF) & -SBF & SB \\ 0 & SA & SB \\ \hline C+DF & -DF & D \end{array}\right] = \overline{\overline{\Delta}} \star \left[\begin{array}{c|c} \overline{\overline{A}} & \overline{\overline{B}} \\ \hline \overline{\overline{C}} & \overline{\overline{D}} \end{array}\right].$$

Let $Q = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$, where each identity operator I has a structure that is compatible with the structure of Δ. Then,

$$N_\delta R_\delta = \left(Q^{-1}\overline{\overline{\Delta}}Q\right) \star \left[\begin{array}{c|c} Q^{-1}\overline{\overline{A}}Q & Q^{-1}\overline{\overline{B}} \\ \hline \overline{\overline{C}}Q & \overline{\overline{D}} \end{array}\right] = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \star \left[\begin{array}{cc|c} S(A+BF) & 0 & 0 \\ 0 & SA & SB \\ \hline C+DF & C & D \end{array}\right]$$

$$= C(I-\Delta SA)^{-1}\Delta SB + D = G_\delta.$$

Third, systems $\mathcal{N}_\delta$ and $\mathcal{M}_\delta$ are shown to be right coprime. Since $\mathcal{G}_\delta$ is a strongly detectable system, there exists a bounded, partitioned graph-diagonal operator K that has a structure similar to the structure of F^* and appropriate dimensions, such that when K is applied to system $\mathcal{G}_\delta$, the resulting closed-loop system is strongly stable. For each $\Delta \in \boldsymbol{\Delta}$, consider the operators U_δ and V_δ in $\mathcal{L}_c(\ell_2, \ell_2)$ defined as

$$U_\delta = \Delta \star \left[\begin{array}{c|c} S(A+KC) & SK \\ \hline F & 0 \end{array}\right], \qquad V_\delta = \Delta \star \left[\begin{array}{c|c} S(A+KC) & S(B+KD) \\ \hline -F & I \end{array}\right],$$

respectively. For each $\Delta \in \boldsymbol{\Delta}$, $U_\delta N_\delta + V_\delta M_\delta = I$ is proved by showing that $y_U + y_V = u_M$ whenever $u_N = u_M$, $u_U = y_N$, and $u_V = y_M$. Using these relations, the equations of U_δ, N_δ, V_δ, and M_δ are combined and simplified into

$$\begin{aligned} &(I - \Delta S(A+KC))\left(\zeta_U - \zeta_V + \zeta_M\right) = \Delta SK(C+DF)\left(\zeta_N - \zeta_M\right), \\ &y_U + y_V = F\left(\zeta_U - \zeta_V + \zeta_M\right) + u_M, \\ &(I - \Delta S(A+BF))\left(\zeta_N - \zeta_M\right) = 0. \end{aligned}$$

Since $I - \Delta S(A+BF)$ and $I - \Delta S(A+KC)$ have bounded causal inverses for all $\Delta \in \boldsymbol{\Delta}$, it follows that $\zeta_N - \zeta_M = 0$, $\zeta_U - \zeta_V + \zeta_M = 0$, and $y_U + y_V = u_M$, which concludes the proof. □

6.2.3 Coprime Factors Model Reduction Algorithm

Algorithm 6.6. Consider a strongly stabilizable and strongly detectable distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. This algorithm shows how to apply the coprime factors reduction method to reduce system $\mathcal{G}_\delta$. The resulting reduced order system is denoted by $\mathcal{G}_{r,\delta}$ and its realization is denoted by $(A_r, B_r, C_r, D, \mathbf{\Delta}_r)$.

1. Find $P \in \mathcal{X}$ that satisfies (6.3).

2. Define $F = -(B^*S^*P^{-1}SB)^{-1}B^*S^*P^{-1}SA \in \mathcal{F}$. Ensure that F is well-defined by removing any control redundancies so that $[\![B]\!](t,k)$ has full column rank for all $(t,k) \in \mathbb{Z} \times V$, see the proof of Theorem 6.2.

3. Construct a strongly stable RCF $(\mathcal{N}_\delta, \mathcal{M}_\delta)$ for system $\mathcal{G}_\delta$ as in Theorem 6.5.

- The realization of system $\mathcal{N}_\delta$ is given by $(A + BF, B, C + DF, D, \mathbf{\Delta})$.

- The realization of system $\mathcal{M}_\delta$ is given by $(A + BF, B, F, I, \mathbf{\Delta})$.

4. Form an augmented strongly stable system $\mathcal{H}_\delta = \begin{bmatrix} \mathcal{N}_\delta \\ \mathcal{M}_\delta \end{bmatrix}$ with realization

$$(A_H, B_H, C_H, D_H, \mathbf{\Delta}) = \left(A + BF, B, \begin{bmatrix} C + DF \\ F \end{bmatrix}, \begin{bmatrix} D \\ I \end{bmatrix}, \mathbf{\Delta} \right).$$

5. Find generalized gramians X and Y for system $\mathcal{H}_\delta$.

- Find $X \in \mathcal{X}$ such that $A_H X A_H^* - S^* X S + B_H B_H^* \prec 0$.

- Find $Y \in \mathcal{X}$ such that $A_H^* S^* Y S A_H - Y + C_H^* C_H \prec 0$.

- See Algorithm 5.8 for a possible heuristic for efficiently determining the generalized gramians for model reduction purposes.

6. Construct a balanced realization $(A_{H,\mathrm{bal}}, B_{H,\mathrm{bal}}, C_{H,\mathrm{bal}}, D_H, \mathbf{\Delta})$ for system $\mathcal{H}_\delta$ as in the proof of Lemma 5.4.

- Find a balancing transformation $T \in \mathcal{T}$ and express the balanced generalized gramian as $\Sigma = TXT^* = (T^{-1})^* Y T^{-1}$.

- Define $A_{\mathrm{bal}} = (S^*TS)AT^{-1}$, $B_{\mathrm{bal}} = (S^*TS)B$, $C_{\mathrm{bal}} = CT^{-1}$, and $F_{\mathrm{bal}} = FT^{-1}$.

- Define $A_{H,\mathrm{bal}} = A_{\mathrm{bal}} + B_{\mathrm{bal}}F_{\mathrm{bal}}$, $B_{H,\mathrm{bal}} = B_{\mathrm{bal}}$, and $C_{H,\mathrm{bal}} = \begin{bmatrix} (C_{\mathrm{bal}} + DF_{\mathrm{bal}})^* & F_{\mathrm{bal}}^* \end{bmatrix}^*$.

7. Using this balanced realization, reduce system $\mathcal{H}_\delta$ by applying the balanced truncation method. The resulting reduced order system is denoted by $\mathcal{H}_{r,\delta} = \begin{bmatrix} \mathcal{N}_{r,\delta}^* & \mathcal{M}_{r,\delta}^* \end{bmatrix}^*$.

- Determine the dimensions of the reduced order system based on the following:

a) The upper bound on $\|(H_\delta - H_{r,\delta})\|$, for all $\Delta \in \mathbf{\Delta}$, obtained by judiciously applying Theorems 5.7 and 5.11.

b) The upper bound γ on $\|H_\delta\|$, for all $\Delta \in \mathbf{\Delta}$, obtained by applying Lemma 5.2.

c) The absolute and relative orders of the diagonal entries in Σ.

- Denote the realization of system $\mathcal{H}_{r,\delta}$ by $(A_{H,r}, B_{H,r}, C_{H,r}, D_H, \mathbf{\Delta}_r)$.

- Define the reduced order balanced generalized gramian Γ as in (5.11).

8. Define the operators A_r, B_r, C_r, and F_r as follows:

$B_r = B_{H,r}$, $\begin{bmatrix} (C_r + DF_r)^* & F_r^* \end{bmatrix}^* = C_{H,r}$, and $A_r = A_{H,r} - B_r F_r$.

- With Q defined as in the proof of Lemma 5.5, notice that the previous operators also satisfy $C_{\text{bal}}\, Q = \begin{bmatrix} C_r & \overline{C}_2 \end{bmatrix}$, $F_{\text{bal}}\, Q = \begin{bmatrix} F_r & \overline{F}_2 \end{bmatrix}$,

  $Q^* S A_{\text{bal}} Q = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} A_r & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix}$, and $Q^* S B_{\text{bal}} = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} B_r \\ \overline{B}_2 \end{bmatrix}$,

  where $\overline{A}_{12}$, $\overline{A}_{21}$, $\overline{A}_{22}$, $\overline{B}_2$, $\overline{C}_2$, and $\overline{F}_2$ are appropriately defined partitioned graph-diagonal operators.

9. Define the realization of system $\mathcal{N}_{r,\delta}$ by $(A_r + B_r F_r, B_r, C_r + DF_r, D, \mathbf{\Delta}_r)$.

- Define the realization of system $\mathcal{M}_{r,\delta}$ by $(A_r + B_r F_r, B_r, F_r, I, \mathbf{\Delta}_r)$.

- Note that systems $\mathcal{N}_{r,\delta}$ and $\mathcal{M}_{r,\delta}$ are strongly stable and right coprime.

10. If $I - \Delta_r S A_r$ has a causal inverse on $\ell_{2e}^{(r_T, \overline{r}_S, \overline{r}_P)}$ for all $\Delta_r \in \mathbf{\Delta}_r$, see Remark 6.7, then:

a) the reduced order system $\mathcal{G}_{r,\delta}$ is defined by the realization $(A_r, B_r, C_r, D, \mathbf{\Delta}_r)$ and is well-posed;

b) $(\mathcal{N}_{r,\delta}, \mathcal{M}_{r,\delta})$ is an RCF for system $\mathcal{G}_{r,\delta}$, i.e., $G_{r,\delta} = N_{r,\delta} M_{r,\delta}^{-1}$ for all $\Delta_r \in \mathbf{\Delta}_r$; and

c) F_r is a bounded feedback operator that strongly stabilizes system $\mathcal{G}_{r,\delta}$.

One major difference between Algorithms 3.6 and 6.6 relates to the guarantee on the well-posedness of the reduced order system obtained by the application of the coprime factors reduction method. For distributed LTV systems, the reduced order system is guaranteed to be well-posed. However, for distributed NSLPV systems, the well-posedness of the reduced order system needs to be separately imposed or verified. Remark 6.7 treats this well-posedness issue in more details.

Remark 6.7. Using the dimensions of the reduced order system, define $\bar{r}_S = (r_1^S, \ldots, r_d^S)$ and $\bar{r}_P = (r_1^P, \ldots, r_m^P)$. Recall the definitions of the reduced order operators $A_{r,PP}$, $\Delta_{r,P}$, and Δ_r made in Section 5.3.2. To ensure that $I - \Delta_r S A_r$ has a causal inverse on $\ell_{2e}^{(r_T, \bar{r}_S, \bar{r}_P)}$ for all $\Delta_r \in \boldsymbol{\Delta}_r$, it is sufficient to ensure that $I - \Delta_{r,P} A_{r,PP}$ has a causal inverse on $\ell_{2e}^{\bar{r}_P}$ for all $\Delta_r \in \boldsymbol{\Delta}_r$, since the state-space matrices are zeros for $t < 0$, see Lemma 4.3.

The well-posedness of the reduced order system is guaranteed if the generalized gramian X computed in step 5 of Algorithm 6.6 additionally satisfies the condition

$$A_{PP} \operatorname{diag}\left(X_1^P, \ldots, X_m^P\right) A_{PP}^* - \operatorname{diag}\left(X_1^P, \ldots, X_m^P\right) \prec 0. \tag{6.5}$$

Namely, since $\Sigma = TXT^* \in \mathcal{X}$ and $T \in \mathcal{T}$ have block-diagonal structures, the previous inequality can be equivalently written as

$$A_{\text{bal},PP} \operatorname{diag}\left(\Sigma_1^P, \ldots, \Sigma_m^P\right) A_{\text{bal},PP}^* - \operatorname{diag}\left(\Sigma_1^P, \ldots, \Sigma_m^P\right) \prec 0,$$

where $A_{\text{bal},PP}$ is a partitioned graph-diagonal operator defined similarly to A_{PP} from the blocks of A_{bal}. A procedure similar to the one in the proof of Lemma 5.5 can be followed to deduce that

$$A_{r,PP} \operatorname{diag}\left(\Gamma_1^P, \ldots, \Gamma_m^P\right) A_{r,PP}^* - \operatorname{diag}\left(\Gamma_1^P, \ldots, \Gamma_m^P\right) \prec 0.$$

Using an argument similar to the one in the proof of Lemma 4.4, it can be concluded from the previous inequality that $I - \Delta_{r,P} A_{r,PP}$ has a bounded causal inverse for all $\Delta_r \in \boldsymbol{\Delta}_r$. Namely, the previous inequality implies that $\left\| \operatorname{diag}\left(\Gamma_1^P, \ldots, \Gamma_m^P\right)^{-\frac{1}{2}} A_{r,PP} \operatorname{diag}\left(\Gamma_1^P, \ldots, \Gamma_m^P\right)^{\frac{1}{2}} \right\| < 1$. Since $\operatorname{diag}\left(\Gamma_1^P, \ldots, \Gamma_m^P\right)$ commutes with every permissible $\Delta_{r,P}$, it follows from the sub-multiplicative property that $\Delta_{r,P} A_{r,PP}$ has a spectral radius less than 1 and $I - \Delta_{r,P} A_{r,PP}$ has a bounded causal inverse for all $\Delta_r \in \boldsymbol{\Delta}_r$. A similar condition that ensures the well-posedness of the reduced order system can be alternatively imposed on the generalized gramian Y

computed in step 5 of Algorithm 6.6 and derived based on the fact that $\Sigma = (T^{-1})^* Y T^{-1}$.

6.3 Contractive Coprime Factorizations (CCFs)

Section 6.2 deals with the coprime factors reduction method for distributed NSLPV systems. However, Theorem 6.5 and Algorithm 6.6 therein do not address the issue of the contractiveness of the coprime factorizations under consideration. Contractive coprime factorizations (CCFs) are the topic of the present section and Sections 6.4 and 6.5. Namely, two methods are provided for the construction of a CCF for a given strongly stabilizable and strongly detectable distributed NSLPV system. The coprime factors reduction algorithm is modified to start with a contractive coprime factorization for the full order system, and the resulting coprime factorization of the reduced order system is proved to be contractive.

For standard LTI systems, normalized coprime factorizations (NCFs) are employed in the coprime factors reduction algorithm, since they result in the least conservative robustness conditions when the coprime factors reduction error bound is interpreted in a gap metric sense, see [40, 60, 62, 84, 85] for more details. However, it is difficult to ensure the normalization of coprime factorizations for systems with structural constraints, and so relaxations of NCFs are pursued instead. The work in [86] presents CCFs as the natural extension of NCFs for the class of LPV systems, since the normalization condition may not be satisfied for all permissible parameter trajectories. For the same class of systems, the work in [25] employs CCFs in a unified approach for control synthesis and model reduction. In [15], contractiveness and expansiveness concepts are used to derive stability margins for behavioural systems with uncertainty. The work in [55] treats systems in which the state partitioning is to be preserved during model reduction, and proposes coprime factors reduction methods that use either an expansive or a contractive coprime factorization for the full order system.

Employing an expansive factorization allows for the extension of the standard robustness theorem to the class of systems treated therein, but results in a non-convex optimization problem, since the stability of the factorization needs to be imposed separately. On the other hand, employing a CCF is more computationally attractive, since the factorization can be constructed from the solution of an LMI. Moreover, the stability of the constructed factorization is automatically guaranteed. However, extending the robustness theorem when a CCF is employed requires imposing the difficult condition of some level of expansiveness; and so, a heuristic that makes the factorization approach normalization is proposed instead. The work of [14] on the model reduction of unstable uncertain systems proposes a method for finding a CCF for a given stabilizable and detectable system, as well as an iterative algorithm that ensures a level of expansiveness as close to one as possible. An alternative method for constructing a CCF for an uncertain system is presented in [54] along with a heuristic that makes the factorization approach normalization. It is noted therein that for discrete-time systems, even in the simple LTI case, if the coprime factors reduction method starts with an NCF of the full order system, then the resulting coprime factorization of the reduced order system is only guaranteed to be contractive. Thus, for discrete-time systems, one may as well employ a CCF (instead of an NCF) in the coprime factors reduction algorithm. Finally, CCFs for distributed LTV systems were discussed in [6].

The results in Sections 6.4 and 6.5 extend the results in [6, 14, 54] to the more general class of distributed NSLPV systems. The extension of the results is carried out transparently by virtue of the adopted operator theoretic framework of Chapter 4. Moreover, the results derived in Sections 6.4 and 6.5 gain novel interpretations, since distributed NSLPV systems possess both an interconnection structure and an uncertainty structure, and the nominal parts of the subsystems are time-varying. Before concluding this motivational section on CCFs, it is noted that the two coprime factors reduction methods based on CCFs proposed in Sections 6.4 and 6.5, respectively, are applicable to strongly stabilizable and strongly detectable distributed NSLPV systems. In other words, the contractiveness requirement does not introduce conservatism into the coprime factors reduction method, since the systems that can be reduced by the method of Section 6.2 can also be reduced by the methods of Sections 6.4 and 6.5. However, as will become clear later on, the latter methods are more computationally intensive than the method of Section 6.2.

6.4 First Method for Coprime Factors Reduction using CCFs

Consider a well-posed distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \boldsymbol{\Delta})$, and recall the notion of strong stabilizability given in Definition 6.1 and the definition of the set $\mathcal{F}$ in (6.1). The following result gives a convex characterization of strong stabilizability for distributed NSLPV systems. This characterization is an alternative to the characterization given in Theorem 6.2.

Theorem 6.8. *Consider a well-posed distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \boldsymbol{\Delta})$. System $\mathcal{G}_\delta$ is strongly stabilizable if and only if there exist $P \in \mathcal{X}$ and $W \in \mathcal{F}$ such that*

$$\begin{bmatrix} -P & (AP+BW)^* & W^* & (CP+DW)^* \\ AP+BW & -S^*PS & 0 & 0 \\ W & 0 & -I & 0 \\ CP+DW & 0 & 0 & -I \end{bmatrix} \prec 0. \tag{6.6}$$

Moreover, one choice for a strongly stabilizing feedback operator is $F = WP^{-1} \in \mathcal{F}$.

Proof. From Definition 6.1 and the proof of Lemma 5.4, system $\mathcal{G}_\delta$ is strongly stabilizable if and only if there exist $Y \in \mathcal{X}$ and $F \in \mathcal{F}$ such that

$$(A+BF)^* S^* Y S (A+BF) - Y + \begin{bmatrix} (C+DF)^* & F^* \end{bmatrix} \begin{bmatrix} C+DF \\ F \end{bmatrix} \prec 0. \tag{6.7}$$

Assume that system $\mathcal{G}_\delta$ is strongly stabilizable and (6.7) holds. By pre- and post-multiplying (6.7) by Y^{-1}, and letting $P = Y^{-1} \in \mathcal{X}$ and $W = FY^{-1} \in \mathcal{F}$, one retrieves

$$(AP+BW)^* S^* P^{-1} S (AP+BW) - P + \begin{bmatrix} (CP+DW)^* & W^* \end{bmatrix} \begin{bmatrix} CP+DW \\ W \end{bmatrix} \prec 0,$$

which is equivalent by the Schur Complement Formula to (6.6). Conversely, assume there exist $P \in \mathcal{X}$ and $W \in \mathcal{F}$ such that (6.6) holds. Then, applying the Schur complement formula to (6.6) and pre- and post-multiplying the resulting inequality by P^{-1} yield

$$\begin{aligned} &-P^{-1} + \left(A + BWP^{-1}\right)^* S^* P^{-1} S \left(A + BWP^{-1}\right) + \left(WP^{-1}\right)^* \left(WP^{-1}\right) \\ &\qquad + \left(C + DWP^{-1}\right)^* \left(C + DWP^{-1}\right) \prec 0. \end{aligned}$$

(6.7) is retrieved by defining $F = WP^{-1} \in \mathcal{F}$ and $Y = P^{-1} \in \mathcal{X}$. Thus, system $\mathcal{G}_\delta$ is strongly stabilizable, and F is a strongly stabilizing feedback operator for system $\mathcal{G}_\delta$. □

Lemma 6.9. *Consider a well-posed distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. If $P \in \mathcal{X}$ and $W \in \mathcal{F}$ satisfy (6.6), then P and $W^c \in \mathcal{F}$ satisfy (6.6), where*

$$Z = I + D^*D + B^*S^*P^{-1}SB = Z^*, \qquad W^c = -Z^{-1}(B^*S^*P^{-1}SAP + D^*CP). \tag{6.8}$$

Moreover,

$$F^c = W^c P^{-1} = -Z^{-1}\left(B^* S^* P^{-1} S A + D^* C\right) \in \mathcal{F} \tag{6.9}$$

is a strongly stabilizing feedback operator for system $\mathcal{G}_\delta$.

Proof. By the Schur complement formula and since $P \in \mathcal{X}$, (6.6) is equivalent to

$$-P + (AP + BW)^* \; S^* P^{-1} S \; (AP + BW) + W^* W + (CP + DW)^* \, (CP + DW) \prec 0.$$

By adding and subtracting $(PC^* D + PA^* S^* P^{-1} SB)\, Z^{-1}\, (D^* CP + B^* S^* P^{-1} SAP)$, and after some algebraic manipulations, the previous inequality can be rewritten as

$$\begin{aligned}
&- P + PC^* CP + PA^* S^* P^{-1} SAP \\
&\qquad\qquad - \left(PC^* D + PA^* S^* P^{-1} SB\right) Z^{-1} \left(D^* CP + B^* S^* P^{-1} SAP\right) \\
&\quad + \left(W^* + \left(PC^* D + PA^* S^* P^{-1} SB\right) Z^{-1}\right) Z \left(W + Z^{-1}\left(D^* CP + B^* S^* P^{-1} SAP\right)\right) \prec 0.
\end{aligned}$$

In other words, P and W satisfy (6.6) if and only if they satisfy the above inequality. Since replacing W with W^c in the last term on the left-hand-side of the above inequality makes that term zero, then it is not difficult to see that P and W^c also satisfy the above inequality and, hence, satisfy (6.6). In addition, $F^c = W^c P^{-1} \in \mathcal{F}$ is a strongly stabilizing feedback operator for system $\mathcal{G}_\delta$ as per Theorem 6.8. □

Recall the definition of right coprime distributed NSLPV systems and the definition of an RCF for a distributed NSLPV system given in Definitions 6.3 and 6.4, respectively. Next, the definition of a CCF for a distributed NSLPV system is given, and it is shown how to construct a strongly stable CCF for a given strongly stabilizable and strongly detectable distributed NSLPV system.

Definition 6.10. The pair $(\mathcal{N}_\delta^c, \mathcal{M}_\delta^c)$ of stable distributed NSLPV systems is said to be a CCF for the distributed NSLPV system $\mathcal{G}_\delta$ if $(\mathcal{N}_\delta^c, \mathcal{M}_\delta^c)$ is an RCF for system $\mathcal{G}_\delta$ and $(N_\delta^c)^* N_\delta^c + (M_\delta^c)^* M_\delta^c \preceq I$ for all $\Delta \in \mathbf{\Delta}$.

Theorem 6.11. *Given a strongly stabilizable and strongly detectable distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$, let $P \in \mathcal{X}$ and $W \in \mathcal{F}$ satisfy (6.6), and define Z and $F^c \in \mathcal{F}$ by (6.8) and (6.9), respectively. Then, system $\mathcal{G}_\delta$ admits a strongly stable CCF denoted by $(\mathcal{N}_\delta^c, \mathcal{M}_\delta^c)$, where the realizations of systems $\mathcal{N}_\delta^c$ and $\mathcal{M}_\delta^c$ are given by $\left(A + BF^c, BZ^{-1/2}, C + DF^c, DZ^{-1/2}, \mathbf{\Delta}\right)$ and $\left(A + BF^c, BZ^{-1/2}, F^c, Z^{-1/2}, \mathbf{\Delta}\right)$, respectively.*

Proof. First, it is noted that systems $\mathcal{N}_\delta^c$ and $\mathcal{M}_\delta^c$ are strongly stable as a consequence of Lemma 6.9. The proof of this theorem is twofold. First, the pair $(\mathcal{N}_\delta^c, \mathcal{M}_\delta^c)$ is shown to be an RCF for system $\mathcal{G}_\delta$. Second, this RCF is shown to be contractive.

The factorization $(\mathcal{N}_\delta^c, \mathcal{M}_\delta^c)$ is defined similarly to the RCF given in Theorem 6.5 with the additional scaling factor $Z^{-1/2}$. With minor modifications to the proof of Theorem 6.5, it can be shown that $(\mathcal{N}_\delta^c, \mathcal{M}_\delta^c)$ is an RCF for system $\mathcal{G}_\delta$. For each $\Delta \in \mathbf{\Delta}$, the input-output maps of $\mathcal{N}_\delta^c$ and $\mathcal{M}_\delta^c$ are given by

$$N_\delta^c = (C + DF^c)\left(I - \Delta S\left(A + BF^c\right)\right)^{-1} \Delta S B Z^{-1/2} + D Z^{-1/2},$$
$$M_\delta^c = F^c\left(I - \Delta S\left(A + BF^c\right)\right)^{-1} \Delta S B Z^{-1/2} + Z^{-1/2},$$

respectively. By direct computation, it can be shown that the distributed NSLPV system $\mathcal{R}_\delta$ which has the realization $\left(A, B, -Z^{1/2}F^c, Z^{1/2}, \mathbf{\Delta}\right)$ is the inverse system of $\mathcal{M}_\delta^c$, i.e., $R_\delta M_\delta^c = M_\delta^c R_\delta = I$ for all $\Delta \in \mathbf{\Delta}$, where $R_\delta = -Z^{1/2}F^c\left(I - \Delta S A\right)^{-1} \Delta S B + Z^{1/2}$. System $\mathcal{R}_\delta$ is well-posed since system $\mathcal{G}_\delta$ is well-posed, i.e., $I - \Delta S A$ has a causal inverse on $\ell_{2e}^{(n_T, \overline{n}_S, \overline{n}_P)}$ for all $\Delta \in \mathbf{\Delta}$. It can also be verified that $G_\delta = N_\delta^c\left(M_\delta^c\right)^{-1} = N_\delta^c R_\delta$ for all $\Delta \in \mathbf{\Delta}$.

Since system $\mathcal{G}_\delta$ is strongly detectable, there exists a bounded partitioned graph-diagonal operator K that has a structure similar to the structure of $(F^c)^*$ and appropriate dimensions, such that when K is applied to system $\mathcal{G}_\delta$, the resulting closed-loop system is strongly stable. Using K, define the strongly stable systems $\mathcal{U}_\delta$ and $\mathcal{V}_\delta$ that have the realizations $\left(A + KC, K, Z^{1/2}F^c, 0, \mathbf{\Delta}\right)$ and $\left(A + KC, B + KD, -Z^{1/2}F^c, Z^{1/2}, \mathbf{\Delta}\right)$, respectively, and the following input-output maps for all $\Delta \in \mathbf{\Delta}$:

$$\begin{aligned} U_\delta &= Z^{\frac{1}{2}} F^c \left(I - \Delta S \left(A + KC\right)\right)^{-1} \Delta S K, \\ V_\delta &= -Z^{\frac{1}{2}} F^c \left(I - \Delta S \left(A + KC\right)\right)^{-1} \Delta S \left(B + KD\right) + Z^{\frac{1}{2}}. \end{aligned}$$

These systems are used to show that systems $\mathcal{N}^c_\delta$ and $\mathcal{M}^c_\delta$ are right coprime: some algebraic computations and manipulations allow to verify that $U_\delta N^c_\delta + V_\delta M^c_\delta = I$ for all $\Delta \in \mathbf{\Delta}$. Thus, $(\mathcal{N}^c_\delta, \mathcal{M}^c_\delta)$ is an RCF for system $\mathcal{G}_\delta$, which concludes the first part of the proof.

It remains to show that this RCF is contractive, i.e., $\left(N^c_\delta\right)^* N^c_\delta + \left(M^c_\delta\right)^* M^c_\delta \preceq I$ for all $\Delta \in \mathbf{\Delta}$. Define the augmented system $\mathcal{H}^c_\delta = \begin{bmatrix} (\mathcal{N}^c_\delta)^* & (\mathcal{M}^c_\delta)^* \end{bmatrix}^*$ and denote its realization by $(A^c_H, B^c_H, C^c_H, D^c_H, \mathbf{\Delta})$, where

$$\begin{aligned} A^c_H &= A + BF^c, & B^c_H &= BZ^{-\frac{1}{2}}, \\ C^c_H &= \begin{bmatrix} C + DF^c \\ F^c \end{bmatrix}, & D^c_H &= \begin{bmatrix} DZ^{-\frac{1}{2}} \\ Z^{-\frac{1}{2}} \end{bmatrix}. \end{aligned}$$

The previous condition can be expressed as $\|H^c_\delta\| \leq 1$ for all $\Delta \in \mathbf{\Delta}$. Thus, to prove the contractiveness of the RCF $(\mathcal{N}^c_\delta, \mathcal{M}^c_\delta)$, it is shown that there exists a solution in $\mathcal{X}$ to (5.7) for all $\gamma > 1$, where the inequality is expressed for the realization of $\mathcal{H}^c_\delta$. In the following, it is verified that $P^{-1} \in \mathcal{X}$ is one such solution. Specifically, the left-hand-side of (5.7) is expressed for P^{-1} and the state-space operators of system $\mathcal{H}^c_\delta$, and the necessary algebraic operations are carried out to arrive at a 2-by-2 block operator. The diagonal blocks of this operator are shown to be negative definite for all $\gamma > 1$, and the off-diagonal blocks are shown to be zeros, hence proving that $P^{-1} \in \mathcal{X}$ satisfies (5.7) for the realization of $\mathcal{H}_\delta^c$ and all $\gamma > 1$, i.e., $\|H_\delta^c\| \leq 1$ for all $\Delta \in \mathbf{\Delta}$. To start, the $(1,1)$-block can be expressed as the left-hand-side of (6.7) with $Y = P^{-1}$ and F replaced with F^c. Then, by Lemma 6.9 and following a similar argument to the one used in the proof of the 'if direction' of Theorem 6.8, it can be shown that the $(1,1)$-block is negative definite. The off-diagonal blocks are zeros since

$$
\begin{aligned}
&(A_H^c)^* S^* P^{-1} S B_H^c + (C_H^c)^* D_H^c = \\
&\left((A + BF^c)^* S^* P^{-1} SB + (C + DF^c)^* D + (F^c)^*\right) Z^{-\frac{1}{2}} = \\
&\left(A^* S^* P^{-1} SB + C^* D + (F^c)^* \left(B^* S^* P^{-1} SB + D^* D + I\right)\right) Z^{-\frac{1}{2}} = \\
&\left(A^* S^* P^{-1} SB + C^* D + (ZF^c)^*\right) Z^{-\frac{1}{2}} = \\
&\left(A^* S^* P^{-1} SB + C^* D - A^* S^* P^{-1} SB - C^* D\right) Z^{-\frac{1}{2}} = 0.
\end{aligned}
$$

The $(2,2)$-block can be written as $(B_H^c)^* S^* P^{-1} S B_H^c + (D_H^c)^* D_H^c - \gamma^2 I = (1 - \gamma^2) I$, since $(B_H^c)^* S^* P^{-1} S B_H^c + (D_H^c)^* D_H^c = Z^{-\frac{1}{2}} (B^* S^* P^{-1} SB + D^* D + I) Z^{-\frac{1}{2}} = Z^{-\frac{1}{2}} Z Z^{-\frac{1}{2}} = I$. Thus, the $(2,2)$-block is negative definite for all $\gamma > 1$. $\square$

Consider a strongly stabilizable and strongly detectable distributed NSLPV system $\mathcal{G}_\delta$. Using the strongly stable CCF $(\mathcal{N}_\delta^c, \mathcal{M}_\delta^c)$ of system $\mathcal{G}_\delta$ constructed in Theorem 6.11, an augmented strongly stable system $\mathcal{H}_\delta^c = \begin{bmatrix} \mathcal{N}_\delta^c \\ \mathcal{M}_\delta^c \end{bmatrix}$ is formed that can be reduced by the application of the balanced truncation method of Section 5.3. The resulting reduced order system is denoted by $\mathcal{H}_{r,\delta}^c = \begin{bmatrix} \mathcal{N}_{r,\delta}^c \\ \mathcal{M}_{r,\delta}^c \end{bmatrix}$ and gives the coprime factorization $\left(\mathcal{N}_{r,\delta}^c, \mathcal{M}_{r,\delta}^c\right)$ from which the reduced order system $\mathcal{G}_{r,\delta}$ is constructed. The following algorithm details the procedure to reduce system $\mathcal{G}_\delta$ by the application of the coprime factors reduction method. Specifically, this algorithm modifies Algorithm 6.6 to take into account the CCF defined in Theorem 6.11. The resulting reduced order RCF $\left(\mathcal{N}^c_{r,\delta}, \mathcal{M}^c_{r,\delta}\right)$ is then proved to be contractive.

Algorithm 6.12. Given a strongly stabilizable and strongly detectable distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$, this algorithm shows how to apply the coprime factors reduction method to reduce system $\mathcal{G}_\delta$ while ensuring the contractiveness of the coprime factorizations. The resulting reduced order system is denoted by $\mathcal{G}_{r,\delta}$ and its realization is given by $(A_r, B_r, C_r, D, \mathbf{\Delta}_r)$.

1. Find solutions $P \in \mathcal{X}$ and $W \in \mathcal{F}$ to (6.6).

2. Define $Z = I + D^*D + B^*S^*P^{-1}SB$ and $F^c = -Z^{-1}(B^*S^*P^{-1}SA + D^*C) \in \mathcal{F}$ as in (6.8) and (6.9), respectively.

3. Construct a strongly stable CCF $(\mathcal{N}^c_\delta, \mathcal{M}^c_\delta)$ for system $\mathcal{G}_\delta$ as in Theorem 6.11.

- The realization of $\mathcal{N}^c_\delta$ is given by $\left(A + BF^c, BZ^{-1/2}, C + DF^c, DZ^{-1/2}, \mathbf{\Delta}\right)$.

- The realization of $\mathcal{M}^c_\delta$ is given by $\left(A + BF^c, BZ^{-1/2}, F^c, Z^{-1/2}, \mathbf{\Delta}\right)$.

4. Form the augmented strongly stable system $\mathcal{H}^c_\delta = \begin{bmatrix} \mathcal{N}^c_\delta \\ \mathcal{M}^c_\delta \end{bmatrix}$ with realization

$$(A^c_H, B^c_H, C^c_H, D^c_H, \mathbf{\Delta}) = \left(A + BF^c, BZ^{-\frac{1}{2}}, \begin{bmatrix} C + DF^c \\ F^c \end{bmatrix}, \begin{bmatrix} DZ^{-\frac{1}{2}} \\ Z^{-\frac{1}{2}} \end{bmatrix}, \mathbf{\Delta} \right).$$

5. Find generalized gramians X and Y for system $\mathcal{H}^c_\delta$.

- Find $X \in \mathcal{X}$ such that

$$A_H^c X \left(A_H^c\right)^* - S^* X S + B_H^c \left(B_H^c\right)^* = \left(A + BF^c\right) X \left(A + BF^c\right)^* \\ - S^* X S + B Z^{-1} B^* \prec 0.$$

- Set $Y = P^{-1} \in \mathcal{X}$.
- See Remark 6.14 and Algorithm 5.8 for choosing the objective functions in steps 1 and 5.

6. Construct a balanced realization $(A^c_{H,\text{bal}}, B^c_{H,\text{bal}}, C^c_{H,\text{bal}}, D^c_H, \boldsymbol{\Delta})$ for system $\mathcal{H}^c_\delta$ as in the proof of Lemma 5.4.

- Find a balancing transformation $T \in \mathcal{T}$ and express the balanced generalized gramian as $\Sigma = TXT^* = (T^{-1})^* P^{-1} T^{-1}$.
- Define $A_{\text{bal}} = (S^*TS)AT^{-1}$, $B_{\text{bal}} = (S^*TS)B$, $C_{\text{bal}} = CT^{-1}$, and $F^c_{\text{bal}} = F^c T^{-1}$.
- Define

$$A^c_{H,\text{bal}} = (S^*TS)A^c_H T^{-1} = A_{\text{bal}} + B_{\text{bal}} F^c_{\text{bal}}, \qquad B^c_{H,\text{bal}} = (S^*TS)B^c_H = B_{\text{bal}} Z^{-\frac{1}{2}},$$
$$C^c_{H,\text{bal}} = C^c_H T^{-1} = \begin{bmatrix} C_{\text{bal}} + DF^c_{\text{bal}} \\ F^c_{\text{bal}} \end{bmatrix}. \tag{6.10}$$

7. Using this balanced realization, reduce system $\mathcal{H}^c_\delta$ by applying the balanced truncation method. The resulting reduced order system is denoted by $\mathcal{H}^c_{r,\delta} = \begin{bmatrix} (\mathcal{N}^c_{r,\delta})^* & (\mathcal{M}^c_{r,\delta})^* \end{bmatrix}^*$.

- Determine the dimensions of the reduced order system as discussed in Algorithm 6.6.
- Denote the realization of the reduced order system $\mathcal{H}^c_{r,\delta}$ by $(A^c_{H,r}, B^c_{H,r}, C^c_{H,r}, D^c_H, \boldsymbol{\Delta}_r)$.
- Define the reduced order balanced generalized gramian Γ as in (5.11) and the partitioned graph-diagonal operator Ω as in (5.12).

8. Define the operators A_r, B_r, C_r, and F_r^c as follows:

$$\begin{bmatrix}(C_r + DF_r^c)^* & (F_r^c)^*\end{bmatrix}^* = C_{H,r}^c,\ B_r = B_{H,r}^c\ Z^{\frac{1}{2}},\text{ and } A_r = A_{H,r}^c - B_r F_r^c.$$

- With Q defined as in the proof of Lemma 5.5, notice that the previous operators also satisfy

$$\begin{aligned} QQ^* &= I, & Q^*Q &= I, \\ C_{\text{bal}}\,Q &= \begin{bmatrix} C_r & \overline{C}_2 \end{bmatrix}, & F_{\text{bal}}^c\,Q &= \begin{bmatrix} F_r^c & \overline{F}_2 \end{bmatrix}, \\ Q^* S A_{\text{bal}} Q &= \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} A_r & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix}, & Q^* S B_{\text{bal}} &= \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} B_r \\ \overline{B}_2 \end{bmatrix}, \\ Q^*\,\Sigma\,Q &= \begin{bmatrix} \Gamma & 0 \\ 0 & \Omega \end{bmatrix}, & Q^*\,\Delta\,Q &= \begin{bmatrix} \Delta_r & 0 \\ 0 & \overline{\Delta}_2 \end{bmatrix}, \end{aligned} \tag{6.11}$$

where $\overline{A}_{12}$, $\overline{A}_{21}$, $\overline{A}_{22}$, $\overline{B}_2$, $\overline{C}_2$, $\overline{F}_2$, and $\overline{\Delta}_2$ are appropriately defined partitioned graph-diagonal operators.

9. Define the realization of system $\mathcal{N}_{r,\delta}^c$ by $(A_r + B_r F_r^c, B_r Z^{-\frac{1}{2}}, C_r + DF_r^c, DZ^{-\frac{1}{2}}, \mathbf{\Delta}_r)$.

- Define the realization of system $\mathcal{M}_{r,\delta}^c$ by $(A_r + B_r F_r^c, B_r Z^{-\frac{1}{2}}, F_r^c, Z^{-\frac{1}{2}}, \mathbf{\Delta}_r)$.

- Note that systems $\mathcal{N}_{r,\delta}^c$ and $\mathcal{M}_{r,\delta}^c$ are strongly stable and right coprime.

10. If $I - \Delta_r S A_r$ has a causal inverse on $\ell_{2e}^{(r_T, \overline{r}_S, \overline{r}_P)}$ for all $\Delta_r \in \mathbf{\Delta}_r$, see Remark 6.7, then

a) the reduced order system $\mathcal{G}_{r,\delta}$ is defined by the realization $(A_r, B_r, C_r, D, \mathbf{\Delta}_r)$ and is well-posed;

b) $(\mathcal{N}_{r,\delta}^c, \mathcal{M}_{r,\delta}^c)$ is an RCF for system $\mathcal{G}_{r,\delta}$, i.e., $G_{r,\delta} = N_{r,\delta}^c (M_{r,\delta}^c)^{-1}$ for all $\Delta_r \in \mathbf{\Delta}_r$; and

c) F_r^c is a bounded feedback operator that strongly stabilizes system $\mathcal{G}_{r,\delta}$.

Theorem 6.13. *Consider a strongly stabilizable and strongly detectable distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. Suppose that system $\mathcal{G}_\delta$ is reduced by applying the coprime factors reduction method detailed in Algorithm 6.12, and denote the RCF for the obtained reduced order system $\mathcal{G}_{r,\delta}$ by $(\mathcal{N}^c_{r,\delta}, \mathcal{M}^c_{r,\delta})$. Then, the RCF $(\mathcal{N}^c_{r,\delta}, \mathcal{M}^c_{r,\delta})$ is contractive.*

Proof. As shown in the proof of Theorem 6.11, the following inequality holds for all $\gamma > 1$:

$$\begin{bmatrix} A^c_H & B^c_H \\ C^c_H & D^c_H \end{bmatrix}^* \begin{bmatrix} S^*P^{-1}S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A^c_H & B^c_H \\ C^c_H & D^c_H \end{bmatrix} - \begin{bmatrix} P^{-1} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \prec 0. \tag{6.12}$$

Define the operator Q as in the proof of Lemma 5.5 such that the relations in (6.11) hold. The following relations also hold:

$$Q^*SA^c_{H,\text{bal}}Q = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} A^c_{H,r} & A^c_{H,\text{rem1}} \\ A^c_{H,\text{rem2}} & A^c_{H,\text{rem3}} \end{bmatrix}, \qquad Q^*SB^c_{H,\text{bal}} = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} B^c_{H,r} \\ B^c_{H,\text{rem}} \end{bmatrix},$$
$$C^c_{H,\text{bal}}Q = \begin{bmatrix} C^c_{H,r} & C^c_{H,\text{rem}} \end{bmatrix},$$

where the terms with subscript 'rem' are defined in the obvious way. Pre- and post-multiplying (6.12) by $\operatorname{diag}((T^{-1})^*, I)$ and its adjoint, respectively, where $T \in \mathcal{T}$ is the structured balancing transformation defined in step 6 of Algorithm 6.12, and using (6.10), one gets

$$\begin{bmatrix} SA^c_{H,\text{bal}} & SB^c_{H,\text{bal}} \\ C^c_{H,\text{bal}} & D^c_H \end{bmatrix}^* \begin{bmatrix} \Sigma & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} SA^c_{H,\text{bal}} & SB^c_{H,\text{bal}} \\ C^c_{H,\text{bal}} & D^c_H \end{bmatrix} - \begin{bmatrix} \Sigma & 0 \\ 0 & \gamma^2 I \end{bmatrix} \prec 0 \ \text{ for all } \gamma > 1.$$

This inequality is then pre- and post-multiplied by $\operatorname{diag}(Q^*, I)$ and $\operatorname{diag}(Q, I)$, respectively, to obtain

$$\begin{bmatrix} \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} A^c_{H,r} & A^c_{H,\mathrm{rem1}} \\ A^c_{H,\mathrm{rem2}} & A^c_{H,\mathrm{rem3}} \end{bmatrix} & \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} B^c_{H,r} \\ B^c_{H,\mathrm{rem}} \end{bmatrix} \\ \begin{bmatrix} C^c_{H,r} & C^c_{H,\mathrm{rem}} \end{bmatrix} & D^c_H \end{bmatrix}^* \begin{bmatrix} \begin{bmatrix} \Gamma & 0 \\ 0 & \Omega \end{bmatrix} & 0 \\ 0 & I \end{bmatrix} \times$$

$$\begin{bmatrix} \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} A^c_{H,r} & A^c_{H,\mathrm{rem1}} \\ A^c_{H,\mathrm{rem2}} & A^c_{H,\mathrm{rem3}} \end{bmatrix} & \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} B^c_{H,r} \\ B^c_{H,\mathrm{rem}} \end{bmatrix} \\ \begin{bmatrix} C^c_{H,r} & C^c_{H,\mathrm{rem}} \end{bmatrix} & D^c_H \end{bmatrix} - \begin{bmatrix} \begin{bmatrix} \Gamma & 0 \\ 0 & \Omega \end{bmatrix} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \prec 0 \quad \text{for all } \gamma > 1.$$

Thus, it can be concluded that the following inequality holds for all $\gamma > 1$:

$$\begin{bmatrix} A^c_{H,r} & B^c_{H,r} \\ C^c_{H,r} & D^c_H \end{bmatrix}^* \begin{bmatrix} S^*\Gamma S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A^c_{H,r} & B^c_{H,r} \\ C^c_{H,r} & D^c_H \end{bmatrix} - \begin{bmatrix} \Gamma & 0 \\ 0 & \gamma^2 I \end{bmatrix} \prec 0,$$

where $(A^c_{H,r}, B^c_{H,r}, C^c_{H,r}, D^c_H, \mathbf{\Delta}_r)$ is a realization of the reduced order system $\mathcal{H}^c_{r,\delta}$. In other words, $\|H^c_{r,\delta}\| \le 1$ for all $\Delta_r \in \mathbf{\Delta}_r$, and the RCF $(\mathcal{N}^c_{r,\delta}, \mathcal{M}^c_{r,\delta})$ of the reduced order system $\mathcal{G}_{r,\delta}$ is contractive. □

A brief comparison between Algorithms 6.6 and 6.12 is now given. First, the operator F^c in step 2 of Algorithm 6.12 is always well-defined, which relaxes the assumption in Algorithm 6.6 that $[\![B]\!](t,k)$ has full column rank for all $(t,k) \in \mathbb{Z} \times V$. Second, one does not need to separately solve for the generalized observability gramian $Y \in \mathcal{X}$ in Algorithm 6.12. Instead, Y is set equal to P^{-1} in step 5 of the algorithm, since $(A^c_H)^* S^* Y S A^c_H - Y + (C^c_H)^* C^c_H \prec 0$ corresponds to (6.7) with $Y = P^{-1}$ and F replaced by F^c. Finally, the scaling factor $Z^{-1/2}$, which is used to ensure the contractiveness of the coprime factorizations, must be accounted for when defining the operators of the reduced order system $\mathcal{G}_{r,\delta}$ in step 8 of Algorithm 6.12.

Remark 6.14. Consider a strongly stabilizable and strongly detectable distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. Assume that system $\mathcal{G}_\delta$ is to be reduced by applying the coprime factors reduction method detailed in Algorithm 6.12. In order to obtain useful error bounds for model reduction, and make the derived CCF for system $\mathcal{G}_\delta$ approach normalization, it is proposed to solve the following SDPs, see also the discussions in [54, 55]:

$$\begin{aligned}
&\text{minimize} \quad \sum_{(t,k)\in\mathbb{Z}\times V} \left(\operatorname{trace} U_T(t,k) + \sum_{i=1}^{d} \operatorname{trace} U_i^S(t,k) + \sum_{j=1}^{m} \operatorname{trace} U_j^P(t,k)\right) \\
&\text{subject to: } P \in \mathcal{X},\ W \in \mathcal{F},\ U \in \mathcal{X},\ (6.6), \text{ and } \begin{bmatrix} U & I \\ I & P \end{bmatrix} \succ 0; \text{ and} \qquad (6.13) \\
&\text{minimize} \quad \sum_{(t,k)\in\mathbb{Z}\times V} \left(\operatorname{trace} X_T(t,k) + \sum_{i=1}^{d} \operatorname{trace} X_i^S(t,k) + \sum_{j=1}^{m} \operatorname{trace} X_j^P(t,k)\right) \\
&\text{subject to: } X \in \mathcal{X} \text{ and } (A + BF^c)X(A + BF^c)^* - S^*XS + BZ^{-1}B^* \prec 0. \qquad (6.14)
\end{aligned}$$

In the SDP defined in (6.13), the constraint $\begin{bmatrix} U & I \\ I & P \end{bmatrix} \succ 0$ can be equivalently expressed as

$$\begin{bmatrix} U_T(t,k) & I \\ I & P_T(t,k) \end{bmatrix} \succ \mu I, \quad \begin{bmatrix} U_i^S(t,k) & I \\ I & P_i^S(t,k) \end{bmatrix} \succ \mu I, \quad \begin{bmatrix} U_j^P(t,k) & I \\ I & P_j^P(t,k) \end{bmatrix} \succ \mu I,$$

for some $\mu > 0$ and all $(t,k) \in \mathbb{Z} \times V$, $i = 1, \ldots, d$, and $j = 1, \ldots, m$. The SDP defined in (6.13) introduces the largest computational burden in Algorithm 6.12. In Algorithm 6.6, all the SDPs to be solved are of a comparable size to the SDP defined in (6.14). For eventually time-periodic subsystems interconnected over a finite graph, both SDPs become finite dimensional, and exact expressions for their computational complexity measures can be obtained by formulating the corresponding dual problems [6]. An example-specific comparison between the computational costs of both SDPs is given in Section 6.7. The higher computational cost of the SDP defined in (6.13) may render the coprime factors reduction method detailed in Algorithm 6.12 inapplicable to some systems, whereas the coprime factors reduction methods detailed in Algorithm 6.6 and Section 6.5, respectively, may still be applicable because the involved SDPs are less computationally demanding.

For the sake of completeness, the computational complexity of the SDP defined in (6.13) is also compared to the computational complexity of the feasibility version of the SDP, which is defined as follows:

$$\text{Find } P \in \mathcal{X} \text{ and } W \in \mathcal{F} \text{ subject to (6.6).} \tag{6.15}$$

As will be seen in Section 6.7, the computational cost of the SDP defined in (6.13) is significantly higher than the computational cost of the SDP defined in (6.15). This increase in computational complexity is partly due to the addition of the variable $U \in \mathcal{X}$ needed to render the optimization problem in (6.13) convex. Namely, U is introduced since the constraint (6.6) is in terms of P, and the desired objective function to be minimized is in terms of P^{-1}.

6.5 Second Method for Coprime Factors Reduction using CCFs

In this section, a second method for the coprime factors reduction of distributed NSLPV systems using CCFs is presented that extends the method discussed in [14] for the coprime factors reduction of single uncertain systems using CCFs. This method builds on the coprime factors reduction method of Section 6.2. Namely, after obtaining the RCF $(\mathcal{N}_\delta, \mathcal{M}_\delta)$ of system

$\mathcal{G}_\delta$ as per Theorem 6.5, a scaling factor is solved for and is applied to the aforementioned coprime factorization to ensure that it becomes contractive. The desired scaling factor is shown to always exist. The modifications in Algorithms 6.6 and 6.12 needed to incorporate the discussion in this section are also presented.

Lemma 6.15. *Consider a distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. If system $\mathcal{G}_\delta$ is strongly stable, then there exist $\gamma > 0$ and $X \in \mathcal{X}$ such that (5.7) holds.*

Proof. Since system $\mathcal{G}_\delta$ is strongly stable, there exists $X \in \mathcal{X}$ satisfying (5.10). Then, by the Schur complement formula, X satisfies (5.7) for some $\gamma > 0$ if and only if X satisfies

$$\begin{aligned} & B^*S^*XSB + D^*D \\ & \quad - (A^*S^*XSB + C^*D)^* (A^*S^*XSA + C^*C - X)^{-1} (A^*S^*XSB + C^*D) \prec \gamma^2 I. \end{aligned} \tag{6.16}$$

Clearly, if the left-hand-side of (6.16) is bounded, then it is possible to find a sufficiently large $\gamma > 0$ such that (6.16) holds. Given that the solution $X \in \mathcal{X}$ and the state-space operators are bounded, and since the sum and product of bounded operators are bounded, then it only remains to show that the inverse of $Y = -(A^*S^*XSA + C^*C - X)$ is bounded to prove the boundedness of the left-hand-side of (6.16). Since $Y \succ 0$, then the perturbed version of the inequality, i.e., $Y \succ \epsilon I$, holds for a sufficiently small $\epsilon > 0$. By applying the Schur complement formula twice, it can be seen that the previous inequality is equivalent to $0 \prec Y^{-1} \prec (1/\epsilon)\, I$, which concludes the proof. □

Theorem 6.16. *Consider a strongly stabilizable and strongly detectable distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. Suppose that $F \in \mathcal{F}$ is a strongly stabilizing feedback operator for system $\mathcal{G}_\delta$. Then, there exist $\tilde{T} \in \mathcal{X}$ and $Z^{-1} \succ 0$ such that*

$$\begin{bmatrix} A+BF \\ C+DF \\ F \end{bmatrix} \tilde{T} \begin{bmatrix} (A+BF)^* & (C+DF)^* & F^* \end{bmatrix} - \begin{bmatrix} S^*\tilde{T}S & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} B \\ D \\ I \end{bmatrix} Z^{-1} \begin{bmatrix} B^* & D^* & I \end{bmatrix} \prec 0. \quad (6.17)$$

Furthermore, system $\mathcal{G}_\delta$ admits a strongly stable CCF denoted by $(\mathcal{N}_\delta^c, \mathcal{M}_\delta^c)$, where the realizations of systems $\mathcal{N}_\delta^c$ and $\mathcal{M}_\delta^c$ are given by $(A+BF, BZ^{-1/2}, C+DF, DZ^{-1/2}, \mathbf{\Delta})$ and $(A+BF, BZ^{-1/2}, F, Z^{-1/2}, \mathbf{\Delta})$, respectively.

Proof. Since $F \in \mathcal{F}$ is a strongly stabilizing feedback operator for system $\mathcal{G}_\delta$, the system defined by the realization $\left(A+BF, B, \begin{bmatrix} C+DF \\ F \end{bmatrix}, \begin{bmatrix} D \\ I \end{bmatrix}, \mathbf{\Delta} \right)$ is strongly stable. Then, by Lemma 6.15, there exist $X \in \mathcal{X}$ and some sufficiently large $\gamma > 0$ such that the inequality (5.7) holds when it is expressed for the aforementioned realization. Applying the Schur complement formula twice to (5.7), one retrieves (6.17) with $\tilde{T} = X^{-1} \in \mathcal{X}$ and $Z^{-1} = (1/\gamma^2)\, I \succ 0$. Thus, it has been shown that there exist $\tilde{T} \in \mathcal{X}$ and $Z^{-1} \succ 0$ satisfying (6.17).

By definition, systems $\mathcal{N}_\delta^c$ and $\mathcal{M}_\delta^c$ are strongly stable. An argument similar to the ones in the proofs of Theorems 6.5 and 6.11, respectively, can be used to show that $(\mathcal{N}_\delta^c, \mathcal{M}_\delta^c)$ is an RCF for system $\mathcal{G}_\delta$. This RCF is now proved to be contractive. To do so, define the strongly stable augmented system $\mathcal{H}_\delta^c = \begin{bmatrix} (\mathcal{N}_\delta^c)^* & (\mathcal{M}_\delta^c)^* \end{bmatrix}^*$ that has the following realization:

$$(A_H^c, B_H^c, C_H^c, D_H^c, \mathbf{\Delta}) = \left(A+BF, BZ^{-\frac{1}{2}}, \begin{bmatrix} C+DF \\ F \end{bmatrix}, \begin{bmatrix} DZ^{-\frac{1}{2}} \\ Z^{-\frac{1}{2}} \end{bmatrix}, \mathbf{\Delta} \right).$$

Then, the RCF $(\mathcal{N}_\delta^c, \mathcal{M}_\delta^c)$ is contractive if and only if $\|H_\delta^c\| \leq 1$ for all $\Delta \in \mathbf{\Delta}$. For system $\mathcal{H}_\delta^c$ defined above, it is in fact possible to show that $\|H_\delta^c\| < 1$ (strict inequality) for all $\Delta \in \mathbf{\Delta}$, and hence the RCF can be shown to be strictly contractive. This claim is proved by showing that there exists a solution in $\mathcal{X}$ to (5.7) when $\gamma = 1$ and the inequality is expressed for the realization of system $\mathcal{H}_\delta^c$. By applying the Schur complement formula twice to (6.17), it can be seen that $\tilde{T}^{-1} \in \mathcal{X}$ is one such solution, which concludes the proof. $\square$

Consider a strongly stabilizable and strongly detectable distributed NSLPV system $\mathcal{G}_\delta$ that has a realization denoted by $(A, B, C, D, \mathbf{\Delta})$. The changes in Algorithm 6.12 required to apply the method of this section are now outlined. In steps 1 and 2, solve for $P \in \mathcal{X}$ such that $APA^* - S^*PS - BB^* \prec 0$, and define $F = -\left(B^*S^*P^{-1}SB\right)^{-1} B^*S^*P^{-1}SA \in \mathcal{F}$. Then, compute the scaling factor $Z^{-1/2}$ by solving the following SDP:

$$\text{Find } \tilde{T} \in \mathcal{X} \text{ and } Z^{-1} \succ 0 \text{ subject to (6.17).}$$

Note that inequality (6.17) is linear in Z^{-1}, and the inverse sign is retained for notational consistency. These steps are also similar to the first two steps in Algorithm 6.6, with the additional computation and use of the scaling factor. In step 3, construct the CCF for system $\mathcal{G}_\delta$ according to Theorem 6.16. In step 5, solve for a generalized observability gramian of system $\mathcal{H}_\delta^c$, i.e., find $Y \in \mathcal{X}$ such that $(A_H^c)^* S^*YSA_H^c - Y + (C_H^c)^* C_H^c \prec 0$, and correspondingly, define $\Sigma = TXT^* = (T^{-1})^*YT^{-1}$ in step 6.

This section is concluded by the following remark on a possible heuristic that can be followed to make the CCF constructed in Theorem 6.16 approach normalization.

Remark 6.17. To make the CCF constructed in Theorem 6.16 approach normalization, one should seek $F \in \mathcal{F}$, $\tilde{T} \in \mathcal{X}$, and $Z^{-1} \succ 0$ such that the left-hand-side of (6.17) is as close to zero as possible. As per Theorem 6.2, one typical choice of $F \in \mathcal{F}$ is given by

$F = -(B^*S^*P^{-1}SB)^{-1}B^*S^*P^{-1}SA$, where $P \in \mathcal{X}$ satisfies (6.3). Thus, finding the optimal solutions requires solving non-convex coupled inequalities. To circumvent this difficulty, suboptimal solutions can be sought instead. [14] propose to iterate over the solutions until some prespecified distance from zero is achieved. Two problems are solved per iteration, and at the end of each iteration, the system operators are updated by applying a transformation that can be constructed from $\tilde{T} \in \mathcal{X}$. Such an algorithm needs to be further studied and thoroughly tested; however, this issue is not pursued here.

6.6 Robust Stability Analysis

This section derives a result that allows for interpreting the error bound from the coprime factors redution method in terms of robust stability of the closed-loop system. Specifically, the result provides a robust stability margin on the coprime factors reduction error such that a controller that stabilizes the full order system also stabilizes the obtained reduced order system. Similar discussions for uncertain systems are found in [16, 54].

In the following, it is assumed that the coprime factors reduction method is applied to reduce a strongly stabilizable and strongly detectable distributed NSLPV system $\mathcal{G}_\delta$ that has a strongly stable coprime factorization denoted by $(\mathcal{N}_\delta, \mathcal{M}_\delta)$. The resulting reduced order system is denoted by $\mathcal{G}_{r,\delta}$ and is assumed to be well-posed, and its strongly stable coprime factorization is denoted by $(\mathcal{N}_{r,\delta}, \mathcal{M}_{r,\delta})$. Let ϵ denote the coprime factors reduction error bound, i.e.,

$$\left\| \begin{bmatrix} N_\delta \\ M_\delta \end{bmatrix} - \begin{bmatrix} N_{r,\delta} \\ M_{r,\delta} \end{bmatrix} \right\| = \left\| \begin{bmatrix} \overline{N}_\delta \\ \overline{M}_\delta \end{bmatrix} \right\| < \epsilon \text{ for all } \Delta \in \mathbf{\Delta}, \tag{6.18}$$

where $\overline{N}_\delta = N_\delta - N_{r,\delta}$ and $\overline{M}_\delta = M_\delta - M_{r,\delta}$ for all $\Delta \in \mathbf{\Delta}$. A bound on ϵ is sought such that if a distributed NSLPV controller $\mathcal{K}_\delta$ stabilizes the full order system $\mathcal{G}_\delta$, then $\mathcal{K}_\delta$ also stabilizes the reduced order system $\mathcal{G}_{r,\delta}$.

Consider a distributed NSLPV controller $\mathcal{K}_\delta$ that stabilizes system $\mathcal{G}_\delta$, and inherits the interconnection and uncertainty structures of $\mathcal{G}_\delta$. Such a controller can be designed using the synthesis technique of Chapter 4. Denote the realization of $\mathcal{K}_\delta$ by $\left(A^K, B^K, C^K, D^K, \mathbf{\Delta}^K\right)$.

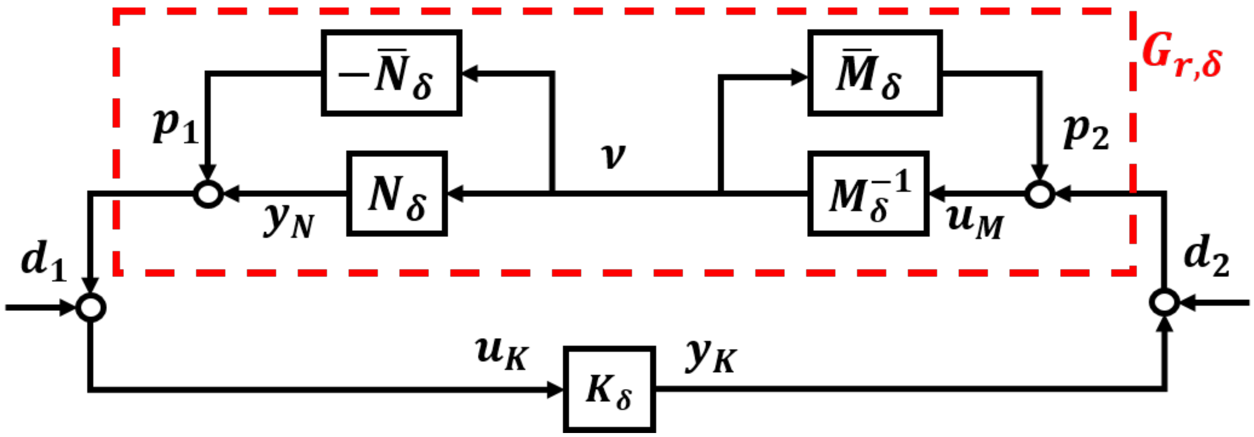

Figure 6.1: This figure shows the standard feedback configuration employed in robust stability analysis. Namely, the figure shows the distributed NSLPV controller $\mathcal{K}_\delta$, and the reduced order system $\mathcal{G}_{r,\delta}$ represented using its RCF $(\mathcal{N}_{r,\delta}, \mathcal{M}_{r,\delta})$, i.e., $G_{r,\delta} = N_{r,\delta} M_{r,\delta}^{-1}$ for all $\Delta_r \in \mathbf{\Delta}_r$. The input-output maps $N_{r,\delta}$ and $M_{r,\delta}$ are expressed as $N_{r,\delta} = N_\delta - \overline{N}_\delta$ and $M_{r,\delta} = M_\delta - \overline{M}_\delta$ for all $\Delta \in \mathbf{\Delta}$.

Figure 6.1 shows the standard feedback interconnection formed by the reduced order system $\mathcal{G}_{r,\delta}$ and the distributed NSLPV controller $\mathcal{K}_\delta$. The reduced order system $\mathcal{G}_{r,\delta}$ is represented using its RCF $(\mathcal{N}_{r,\delta}, \mathcal{M}_{r,\delta})$, i.e., its input-output map $G_{r,\delta}$ is expressed as $G_{r,\delta} = N_{r,\delta} M_{r,\delta}^{-1}$ for all $\Delta_r \in \mathbf{\Delta}_r$. Moreover, $N_{r,\delta}$ and $M_{r,\delta}$ are rewritten as $N_{r,\delta} = N_\delta - \overline{N}_\delta$ and $M_{r,\delta} = M_\delta - \overline{M}_\delta$ for all $\Delta \in \mathbf{\Delta}$, as defined in (6.18). The derivation of the robust stability margin relies on showing that this interconnection is equivalent to the interconnection shown in Figure 6.2, and then applying the small gain theorem.

It is assumed that the interconnection shown in Figure 6.1 is well-defined. Define $d_{in} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

and $p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$, where d_1, d_2 are the exogenous signals and p_1, p_2 are internal signals as shown in Figure 6.1. The output from system $\mathcal{N}_\delta$ is denoted by y_N, the input to system $\mathcal{M}_\delta^{-1}$ is denoted by u_M, and the input to and output from the controller $\mathcal{K}_\delta$ are denoted by u_K and y_K, respectively. Denote the realization of $\mathcal{N}_\delta$ by $(A_N, B_N, C_N, D_N, \mathbf{\Delta})$, and the realization of $\mathcal{M}_\delta$ by $(A_M, B_M, C_M, D_M, \mathbf{\Delta})$. By defining the signal ν as shown in Figure 6.1, it can be seen that the following equations hold for all $\Delta \in \mathbf{\Delta}$:

$$\begin{aligned} p &= \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} -\overline{N}_\delta \\ \overline{M}_\delta \end{bmatrix} \nu, & y_N &= N_\delta\, \nu, \\ \nu &= M_\delta^{-1} u_M, & u_M &= M_\delta\, \nu, \\ \varphi &= \Delta\, \xi, & \xi &= SA_N \varphi + SB_N \nu, \\ y_N &= C_N \varphi + D_N \nu, & u_M &= C_M \varphi + D_M \nu. \end{aligned} \tag{6.19}$$

For each $\Delta \in \mathbf{\Delta}$, the equations for systems $\mathcal{N}_\delta$ and $\mathcal{M}_\delta$ in (6.19) are written similarly to the equations in (5.3), and the simplifications $\varphi_N = \varphi_M = \varphi$ and $\xi_N = \xi_M = \xi$ are applied, since $A_N = A_M$ and $B_N = B_M$ for all the coprime factorizations used in the paper, see Sections 6.2, 6.4, and 6.5. Similarly, for all $\Delta^K \in \mathbf{\Delta}^K$, the controller equations are given by

$$\varphi_K = \Delta^K \xi_K, \qquad \xi_K = SA^K \varphi_K + SB^K u_K, \qquad y_K = C^K \varphi_K + D^K u_K. \tag{6.20}$$

Combining (6.19), (6.20), $u_K = d_1 + y_N + p_1$, and $u_M = p_2 + d_2 + y_K$ yields

$$\begin{bmatrix} \xi \\ \xi_K \end{bmatrix} = \left(\begin{bmatrix} SA_N & 0 \\ SB^K C_N & SA^K \end{bmatrix} + \begin{bmatrix} SB_N \\ SB^K D_N \end{bmatrix} (D_M - D^K D_N)^{-1} \begin{bmatrix} D^K C_N - C_M & C^K \end{bmatrix} \right) \begin{bmatrix} \varphi \\ \varphi_K \end{bmatrix}$$

$$+\left(\begin{bmatrix} 0 & 0 \\ SB^K & 0 \end{bmatrix} + \begin{bmatrix} SB_N \\ SB^K D_N \end{bmatrix} \left(D_M - D^K D_N\right)^{-1} \begin{bmatrix} D^K & I \end{bmatrix}\right) \begin{bmatrix} p_1 + d_1 \\ p_2 + d_2 \end{bmatrix},$$

$$\nu = (D_M - D^K D_N)^{-1} \begin{bmatrix} D^K C_N - C_M & C^K \end{bmatrix} \begin{bmatrix} \varphi \\ \varphi_K \end{bmatrix} + (D_M - D^K D_N)^{-1} \begin{bmatrix} D^K & I \end{bmatrix} (p + d_{in}).$$

That is,

$$\begin{aligned} \begin{bmatrix} \xi \\ \xi_K \end{bmatrix} &= \tilde{S} A_W \begin{bmatrix} \varphi \\ \varphi_K \end{bmatrix} + \tilde{S} B_W (p + d_{in}), \\ \nu &= C_W \begin{bmatrix} \varphi \\ \varphi_K \end{bmatrix} + D_W (p + d_{in}), \end{aligned} \tag{6.21}$$

for all $\Delta \in \mathbf{\Delta}$ and $\Delta^K \in \mathbf{\Delta}^K$, where $\tilde{S} = \mathrm{diag}(S, S)$ and

$$\begin{aligned} A_W &= \begin{bmatrix} A_N & 0 \\ B^K C_N & A^K \end{bmatrix} + \begin{bmatrix} B_N \\ B^K D_N \end{bmatrix} \left(D_M - D^K D_N\right)^{-1} \begin{bmatrix} D^K C_N - C_M & C^K \end{bmatrix}, \\ B_W &= \begin{bmatrix} 0 & 0 \\ B^K & 0 \end{bmatrix} + \begin{bmatrix} B_N \\ B^K D_N \end{bmatrix} \left(D_M - D^K D_N\right)^{-1} \begin{bmatrix} D^K & I \end{bmatrix}, \\ C_W &= \left(D_M - D^K D_N\right)^{-1} \begin{bmatrix} D^K C_N - C_M & C^K \end{bmatrix}, \\ D_W &= \left(D_M - D^K D_N\right)^{-1} \begin{bmatrix} D^K & I \end{bmatrix}. \end{aligned} \tag{6.22}$$

Moreover, $\begin{bmatrix} \varphi \\ \varphi_K \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta^K \end{bmatrix} \begin{bmatrix} \xi \\ \xi_K \end{bmatrix}$ for all $\Delta \in \mathbf{\Delta}$ and $\Delta^K \in \mathbf{\Delta}^K$. The distributed NSLPV system thus constructed that maps $(p + d_{in})$ to ν is denoted by $\mathcal{W}_\delta$, and its input-output map is denoted by W_δ for all $\Delta \in \mathbf{\Delta}$ and $\Delta^K \in \mathbf{\Delta}^K$. The preceding discussion establishes that the interconnections shown in Figures 6.1 and 6.2 are equivalent.

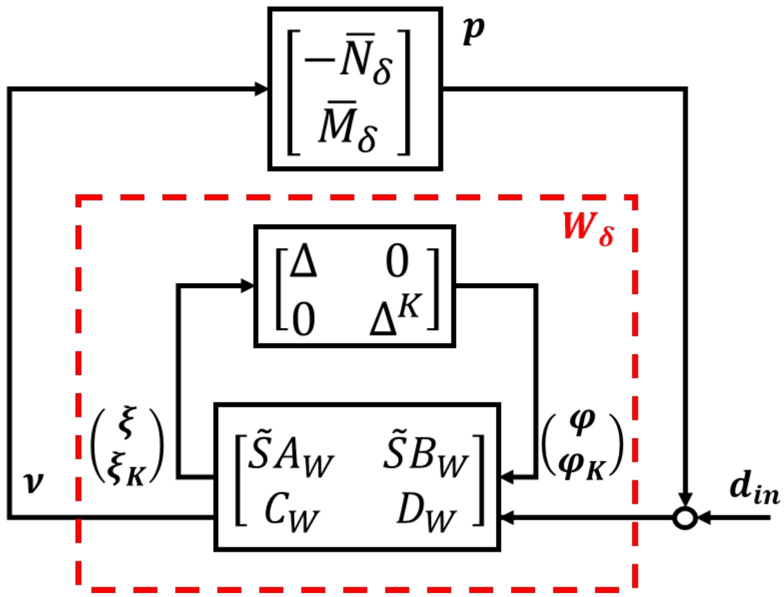

Figure 6.2: This figure shows an interconnection that is equivalent to the interconnection shown in Figure 6.1. The shown interconnection is used in the derivation of the robust stability margin. For each $\Delta \in \mathbf{\Delta}$, $\overline{N}_\delta$ and $\overline{M}_\delta$ are defined in (6.18), and the operators A_W, B_W, C_W, D_W are defined in (6.22).

In order to establish the robust stability theorem, it remains to apply the small gain theorem to the interconnection of systems shown in Figure 6.2. Prior to doing so, it is noted that the equations of system $\mathcal{W}_\delta$ given in (6.21) are not in the standard form shown in (5.2) (or (5.3)), and so they need to be rearranged to become in this form. Specifically, this rearrangement is made to group together the temporal, spatial, and parameter states of system $\mathcal{N}_\delta$ (system $\mathcal{M}_\delta$) with their counterparts in the controller $\mathcal{K}_\delta$, since the uncertainty structures Δ^K and Δ are not independent from each other and neither are the interconnection structures of the plant $\mathcal{G}_\delta$ and the controller $\mathcal{K}_\delta$. Define $\gamma_{\min}$ as the square root of the optimal value of the following optimization problem:

minimize γ^2 subject to $X \in \mathcal{X}$ and inequality (5.7) expressed for the realization of system $\mathcal{W}_\delta$ with the system equations written in standard form. (6.23)

The existence of γ_{min} ensures that system $\mathcal{W}_\delta$ is strongly stable and $\|W_\delta\| < \gamma_{\text{min}}$ for all permissible parameter trajectories. The robustness theorem can now be readily stated as follows.

Theorem 6.18. *Consider a strongly stabilizable and strongly detectable distributed NSLPV system $\mathcal{G}_\delta$ that has a strongly stable coprime factorization denoted by $(\mathcal{N}_\delta, \mathcal{M}_\delta)$. Suppose that one of the coprime factors reduction methods discussed in this chapter is applied to reduce system $\mathcal{G}_\delta$, and denote the resulting reduced order system by $\mathcal{G}_{r,\delta}$ and its strongly stable coprime factorization by $(\mathcal{N}_{r,\delta}, \mathcal{M}_{r,\delta})$. Moreover, let ϵ be the coprime factors reduction error bound defined in (6.18). Suppose that $\mathcal{K}_\delta$ is a distributed NSLPV controller that stabilizes the full order system $\mathcal{G}_\delta$ and further renders the closed-loop system $\mathcal{W}_\delta$ strongly stable, where system $\mathcal{W}_\delta$ is shown in Figure 6.2 and is described by the system equations defined in (6.21). Define γ_{min} as the square root of the optimal value of the optimization problem defined in (6.23). Then, the controller $\mathcal{K}_\delta$ also stabilizes the reduced order system $\mathcal{G}_{r,\delta}$ if $\epsilon \leq \frac{1}{\gamma_{\text{min}}}$.*

This result follows by the application of the small gain theorem [24, 88] to the interconnection of systems shown in Figure 6.2.

6.7 Illustrative Example

This section applies the coprime factors reduction methods discussed in Sections 6.2, 6.4, and 6.5, respectively, to reduce a distributed NSLPV system $\mathcal{G}_\delta$ formed by four subsystems interconnected as shown in Figure 6.3. The purpose of this example is to discuss the differences between the methods in Sections 6.2, 6.4, and 6.5. Special emphasis is placed on discussing the computational complexity of the SDPs involved in the application of each method. As per Remark 6.7, the example also shows the need to impose/verify the well-posedness of the reduced order system that results from applying any of the coprime factors reduction methods to reduce a distributed NSLPV system. For the illustration of other features of the proposed methods such as the time-varying truncation of various types of state variables and the simplification of the interconnection structure, the reader is referred to the examples in Sections 2.8 and 3.5.

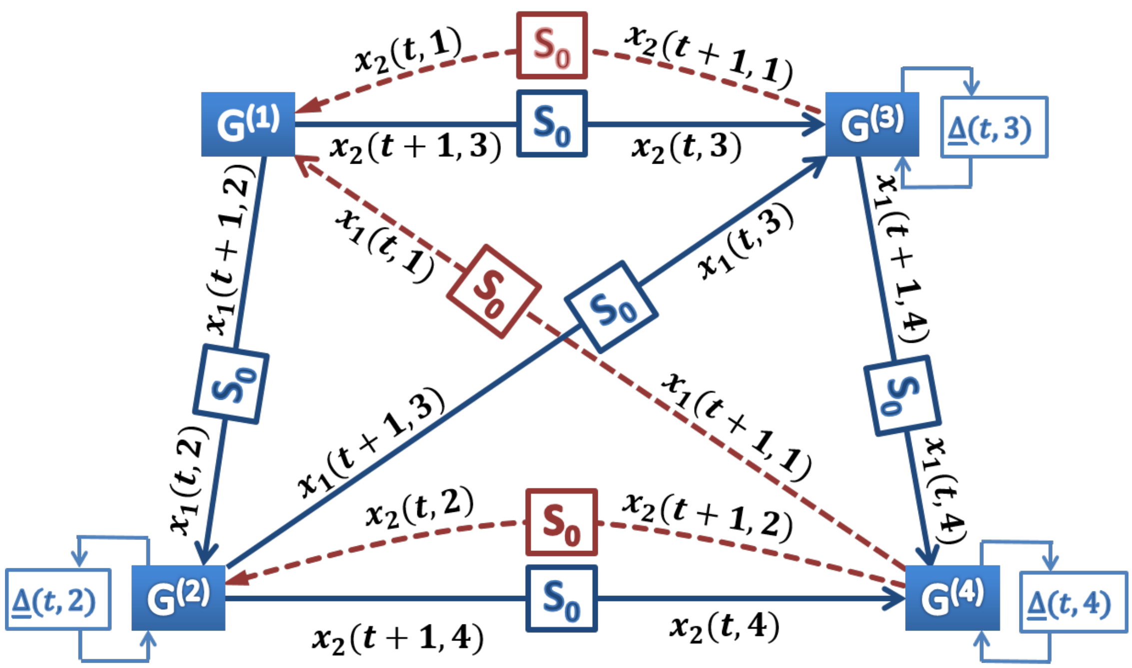

Figure 6.3: Distributed NSLPV system to be reduced by the application of the coprime factors reduction methods of Sections 6.2, 6.4, and 6.5, respectively.

The leader, i.e., subsystem $G^{(1)}$, is described using a $q = 28$ time-periodic discrete-time LTV model, and the followers, i.e., subsystems $G^{(2)}$, $G^{(3)}$, and $G^{(4)}$, are described using discrete-time standard LPV models. There is only one parameter affecting each of the followers, i.e., $m = 1$. Since the leader has an LTV model and the followers have LPV models, then system $\mathcal{G}_\delta$ is a distributed NSLPV system. The realization $(A, B, C, D, \boldsymbol{\Delta})$ of system $\mathcal{G}_\delta$ is defined next.

For the leader, all the state-space matrices except $\overline{A}_{TT}(t,1)$ are constants. The matrix-valued function $\overline{A}_{TT}(t,1)$ is $q = 28$ time-periodic, i.e., $\overline{A}_{TT}(t+q,1) = \overline{A}_{TT}(t,1)$ for all

$t \in \mathbb{N}_0$. Specifically, $\overline{A}_{TT}(t,1) = \mathcal{A}$ for $t = 0,\ldots,6$; $\overline{A}_{TT}(t,1) = \mathcal{Q}\mathcal{A}\mathcal{Q}^*$ for $t = 7,\ldots,13$; $\overline{A}_{TT}(t,1) = \mathcal{Q}^2\mathcal{A}(\mathcal{Q}^*)^2$ for $t = 14,\ldots,20$; $\overline{A}_{TT}(t,1) = \mathcal{Q}^3\mathcal{A}(\mathcal{Q}^*)^3$ for $t = 21,\ldots,27$, where

$$\mathcal{A} = 0.15\begin{bmatrix} 9 & 5 & 1 & 0.3 & -0.2 & 0.2 \\ -2 & 7 & 1 & 0.1 & 0.1 & -0.3 \\ 1 & -1 & -1 & 0.2 & 0.3 & 0.1 \\ 0.1 & -0.2 & -0.3 & 0 & 0 & 0 \\ -0.3 & 0.1 & -0.1 & 0 & 0 & 0 \\ -0.2 & 0.3 & 0.2 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since there are no incoming edges to vertex $k = 1$, and subsystem $G^{(1)}$ is not affected by any parameter, the state-space matrices $A_e^{TS}(t,1)$, $\overline{A}_{TP}(t,1)$, $A_{ie}^{SS}(t,1)$, $A_i^{SP}(t,1)$, $\overline{A}_{PT}(t,1)$, $A_e^{PS}(t,1)$, $\overline{A}_{PP}(t,1)$, $\overline{B}_P(t,1)$, $C_e^S(t,1)$, and $\overline{C}_P(t,1)$ for $i,e = 1,2$ have at least one dimension equal to zero, i.e., are non-existent. The remaining state-space matrices are defined as follows:

$$A_i^{ST}(t,1) = 0.1\begin{bmatrix} 2 & 2 & -1 & 0 & 0 & 0 \\ -2 & 2 & 1 & 0 & 0 & 0 \\ 0.1 & -0.1 & 0.2 & 0 & 0 & 0 \end{bmatrix}, \qquad B_i^S(t,1) = 0.1\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\overline{C}_T(t,1) = \begin{bmatrix} I_2 & 0_{2\times 4} \end{bmatrix}, \qquad \overline{B}_T(t,1) = 0.1\begin{bmatrix} I_2 \\ 0_{4\times 2} \end{bmatrix},$$

for all $t \in \mathbb{N}_0$.

The state-space matrices of the followers are constants for all $t \in \mathbb{N}_0$, $i,e = 1,2$, and

$k = 2, 3, 4$, and are given by

$$\overline{A}_{TT}(t,k) = 0.15 \begin{bmatrix} 7 & 4 & 1 & 0.2 & -0.1 & 0.2 \\ -3 & 5 & 1 & 0.2 & -0.2 & -0.1 \\ 1 & -3 & -2 & 0.1 & 0.1 & 0.2 \\ -0.1 & 0.3 & 0.1 & 0 & 0 & 0 \\ 0.2 & 0.1 & -0.2 & 0 & 0 & 0 \\ 0.1 & -0.2 & 0.1 & 0 & 0 & 0 \end{bmatrix},$$

$$\overline{A}_{TP}(t,k) = 0.01 \begin{bmatrix} 1 & 3 & 2 & 0.1 \\ 2 & 1 & 1 & 0.1 \\ -3 & 4 & -3 & -0.3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_e^{TS}(t,k) = 0.05 \begin{bmatrix} -3 & 2 & 0.1 \\ 4 & 4 & 0.2 \\ 2 & -3 & -0.2 \\ -0.2 & -0.1 & 0 \\ 0.1 & 0.3 & 0 \\ 0.3 & -0.2 & 0 \end{bmatrix},$$

$$\overline{A}_{PT}(t,k) = \begin{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & -0.5 \end{bmatrix} & 0_{2\times 4} \\ \begin{bmatrix} -0.1 & 0 \\ 0 & 0.1 \end{bmatrix} & 0_{2\times 4} \end{bmatrix}, \qquad A_e^{PS}(t,k) = \begin{bmatrix} 0.1 & 0.2 & 0 \\ -0.2 & 0.1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\overline{A}_{PP}(t,k) = 0.1 I_4, \qquad \overline{B}_T(t,k) = 0.1 \begin{bmatrix} I_2 \\ 0_{4\times 2} \end{bmatrix}, \qquad \overline{B}_P(t,k) = 0_{4\times 2},$$

$$\overline{C}_T(t,k) = \begin{bmatrix} I_2 & 0_{2\times 4} \end{bmatrix}, \qquad C_e^S(t,k) = 0_{2\times 3}, \qquad \overline{C}_P(t,k) = 0_{2\times 4},$$

$$A_i^{ST}(t,k) = \mathcal{W}\overline{A}_{TT}(t,k), \qquad A_{ie}^{SS}(t,k) = \mathcal{W}A_e^{TS}(t,k), \qquad A_i^{SP}(t,k) = \mathcal{W}\overline{A}_{TP}(t,k),$$
$$B_i^S(t,k) = \mathcal{W}\overline{B}_T(t,k), \qquad \mathcal{W} = 0.25 \begin{bmatrix} I_3 & 0_3 \end{bmatrix}.$$

Finally, $\overline{D}(t,k) = 0$ for all $(t,k) \in \mathbb{N}_0 \times V = \{1,2,3,4\}$.

The distributed NSLPV system $\mathcal{G}_\delta$ thus formed is not strongly stable, and so it cannot be reduced by the application of the balanced truncation method of Section 5.3. However, system $\mathcal{G}_\delta$ is strongly stabilizable and strongly detectable, and so it can be reduced by the application of any of the coprime factors reduction methods presented in Sections 6.2, 6.4, and 6.5. Since the subsystems are $q = 28$ time-periodic, then the sought solutions to all the considered SDPs are $q = 28$ time-periodic. These problems are modeled using Yalmip [58] and are solved using SDPT3 [79]. The computations are carried out in Matlab 9.4.0.813654 (R2018a) on a ASUSTek laptop with i7-7700HQ Intel Cores, 2.80 GHz processors, and 32 GB of RAM running Windows 10.

6.7.1 Coprime Factors Model Reduction

In this section, system $\mathcal{G}_\delta$ is reduced by the application of the coprime factors reduction method of Section 6.2 and Algorithm 6.6. The resulting reduced order system is denoted by $\mathcal{G}_{r1,\delta}$, and its obtained realization is denoted by $(A_{r1}, B_{r1}, C_{r1}, D, \mathbf{\Delta}_{r1})$. Note that only the final reduced order system and its obtained realization will be denoted distinctively across Sections 6.7.1, 6.7.2, and 6.7.3. Dummy variables in the solved SDPs and intermediate systems may be denoted using the same symbols without ambiguity.

First, the following SDP is solved:

$$\text{Find } P \in \mathcal{X} \text{ such that } APA^* - S^*PS - BB^* \prec 0. \tag{6.24}$$

To verify that system $\mathcal{G}_\delta$ is strongly detectable, the following SDP is also solved:

$$\text{Find } P_0 \in \mathcal{X} \text{ such that } A^*S^*P_0SA - P_0 - C^*C \prec 0.$$

Then, the feedback operator $F = -(B^*S^*P^{-1}SB)^{-1}B^*S^*P^{-1}SA \in \mathcal{F}$ is defined, and the strongly stable augmented system $\mathcal{H}_\delta$ is formed. The realization of system $\mathcal{H}_\delta$ is given by $(A_H, B_H, C_H, D_H, \mathbf{\Delta})$, where $A_H = A + BF$, $B_H = B$, $C_H = \begin{bmatrix}(C+DF)^* & F^*\end{bmatrix}^*$, and $D_H = \begin{bmatrix}D^* & I\end{bmatrix}^*$. Using Lemma 5.2, an upper bound $\gamma = 2.23$ on $\|H_\delta\|$ for all $\Delta \in \mathbf{\Delta}$ is computed. Namely, the following SDP is solved to find γ:

$$\begin{aligned}&\text{minimize } \gamma^2 \text{ subject to: } P \in \mathcal{X} \text{ and}\\ &\qquad \begin{bmatrix} A_H & B_H \\ C_H & D_H \end{bmatrix}^* \begin{bmatrix} S^*PS & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_H & B_H \\ C_H & D_H \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & \gamma^2 I \end{bmatrix} \prec 0. \end{aligned} \tag{6.25}$$

Then, a balanced generalized gramian Σ is found for system $\mathcal{H}_\delta$ as per Algorithm 5.8. Namely, the following SDPs are solved:

$$\begin{aligned}&\text{minimize } \textstyle\sum_{t=0}^{27}\left(\sum_{k=1}^{4}\left(\operatorname{trace} X_T(t,k) + \sum_{i=1}^{2}\operatorname{trace} X_i^S(t,k)\right) + \sum_{k=2}^{4}\operatorname{trace} X_P(t,k)\right)\\ &\text{subject to: } X \in \mathcal{X} \text{ and } A_H X A_H^* - S^*XS + B_H B_H^* \prec 0; \text{ and}\end{aligned} \tag{6.26}$$

$$\begin{aligned}&\text{minimize } \textstyle\sum_{t=0}^{27}\left(\sum_{k=1}^{4}\left(\operatorname{trace} Y_T(t,k) + \sum_{i=1}^{2}\operatorname{trace} Y_i^S(t,k)\right) + \sum_{k=2}^{4}\operatorname{trace} Y_P(t,k)\right)\\ &\text{subject to: } Y \in \mathcal{X} \text{ and } A_H^* S^* Y S A_H - Y + C_H^* C_H \prec 0.\end{aligned} \tag{6.27}$$

Using X and Y, a balanced realization $(A_{H,\text{bal}}, B_{H,\text{bal}}, C_{H,\text{bal}}, D_H, \mathbf{\Delta})$ for system $\mathcal{H}_\delta$ is constructed as in the proof of Lemma 5.4. Finally, the following SDP is solved to find the balanced generalized gramian Σ:

$$\text{minimize } a_1 \times \epsilon + \sum_{t=0}^{27} \left(\sum_{k=1}^{4} \left(\|\text{vect}(\Sigma_T(t,k) - \epsilon I)\|_1 + \sum_{i=1}^{2} \left\|\text{vect}(\Sigma_i^S(t,k) - \epsilon I)\right\|_1 \right) + \sum\nolimits_{k=2}^{4} \|\text{vect}(\Sigma_P(t,k) - \epsilon I)\|_1 \right)$$

$$\begin{aligned} \text{subject to:} \qquad & \Sigma \succeq \epsilon I,\ \epsilon > 0,\ \Sigma \in \mathcal{X}, \\ & [\![\Sigma]\!](t,k) \text{ is a diagonal matrix for all } t = 0,\ldots,27,\ k = 1,\ldots,4, \\ & A_{H,\text{bal}} \Sigma A_{H,\text{bal}}^* - S^* \Sigma S + B_{H,\text{bal}} B_{H,\text{bal}}^* \prec 0, \\ & A_{H,\text{bal}}^* S^* \Sigma S A_{H,\text{bal}} - \Sigma + C_{H,\text{bal}}^* C_{H,\text{bal}} \prec 0. \end{aligned} \tag{6.28}$$

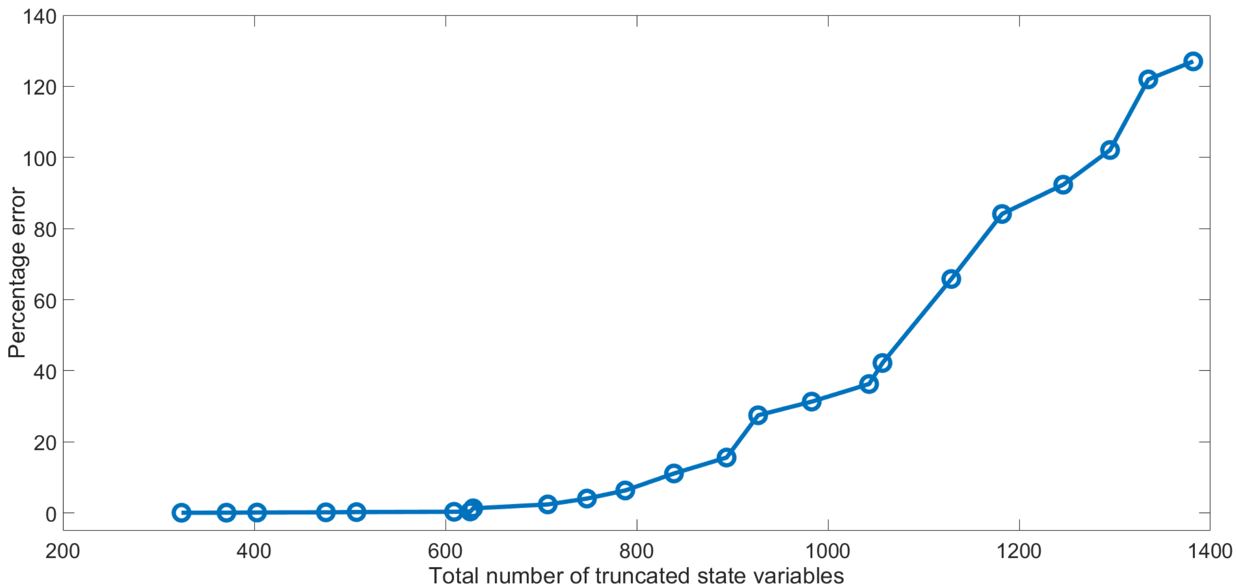

Figure 6.4: This figure shows the trade-off curve between the competing objectives of a small error bound and a large number of truncated state variables. Specifically, the percentage error is computed as $2\epsilon/\gamma \times 100\%$, and the total number of truncated state variables is the sum from $t = 0$ to $t = 27$ of the number of truncated temporal, spatial, and parameter state variables at each time-step t.

The use of the SDPs defined in (6.26), (6.27), and (6.28) for improving on the error bound computation is motivated in the discussion surrounding Algorithm 5.8. In Figure 6.4, the value of the weight $a_1 > 0$ in the SDP defined in (6.28) is varied to trace the trade-off curve between the competing objectives of a small error bound and a large number of truncated state variables. In this example, the value $a_1 = 740$ is chosen, which results in $\epsilon = 0.027$ and $\|(H_\delta - H_{r,\delta})\| < 2.41\%\gamma$. System $\mathcal{H}_\delta$ is reduced by the application of the balanced truncation method of Section 5.3. Namely, each state variable with a corresponding entry in Σ equal to ϵ is truncated. The resulting reduced order system is denoted by $\mathcal{H}_{r,\delta}$ and its balanced realization is denoted by $(A_{H,r}, B_{H,r}, C_{H,r}, D_H, \mathbf{\Delta}_{r1})$.

The realization $(A_{r1}, B_{r1}, C_{r1}, D, \mathbf{\Delta}_{r1})$ of the reduced order system $\mathcal{G}_{r1,\delta}$ is formed from the realization of system $\mathcal{H}_{r,\delta}$ as follows:

$$B_{r1} = B_{H,r}, \qquad \begin{bmatrix} C_{r1} + DF_{r1} \\ F_{r1} \end{bmatrix} = C_{H,r}, \qquad A_{r1} = A_{H,r} - B_{r1}F_{r1}.$$

This example applies the coprime factors reduction method to reduce a distributed NSLPV system, and so the well-posedness of the obtained reduced order system needs to be verified/imposed separately, since this property is not guaranteed as part of the reduction procedure. For the example under consideration, the condition expressed by inequality (6.5) is satisfied. Namely, the inequality $A_{PP}\, X_P\, A_{PP}^* - X_P \prec 0$ holds, where $X \in \mathcal{X}$ is the solution to the SDP defined in (6.26). Thus, by Remark 6.7, the reduced order system $\mathcal{G}_{r1,\delta}$ is well-posed.

The computational complexity and solution time of each SDP invoked when applying the method of this section are summarized in Table 6.1.

Table 6.1: This table shows the computational cost associated with each SDP that appears when applying the coprime factors reduction method of Section 6.2. Time is reported in seconds.

SDP	number of constraints	dimension of SDP variable	number of SDP blocks	dimension of linear variable	Yalmip time	Solver time
(6.24)	4032	2856	448	0	3.0361	4.4869
(6.25)	4033	3080	448	1	4.0325	12.6555
(6.26)	4032	2856	448	0	3.1061	4.3499
(6.27)	4032	2856	448	0	3.4996	4.7804
(6.28)	3193	4284	560	3193	4.4889	5.0661

6.7.2 First Method for Coprime Factors Reduction using Contractive Coprime Factorizations

In this section, system $\mathcal{G}_\delta$ is reduced by the application of the coprime factors reduction method of Section 6.4 and Algorithm 6.12. The resulting reduced order system is denoted by $\mathcal{G}_{r2,\delta}$, and its obtained realization is denoted by $(A_{r2}, B_{r2}, C_{r2}, D, \mathbf{\Delta}_{r2})$.

First, the following SDP is solved:

$$\text{minimize} \;\; \sum_{t=0}^{27}\Big(\sum_{k=1}^{4}\big(\operatorname{trace} U_T(t,k) + \sum_{i=1}^{2}\operatorname{trace} U_i^S(t,k)\big) + \sum_{k=2}^{4}\operatorname{trace} U_P(t,k)\Big)$$

$$\text{subject to: } P \in \mathcal{X},\ W \in \mathcal{F},\ U \in \mathcal{X},\ \begin{bmatrix} U & I \\ I & P \end{bmatrix} \succ 0, \text{ and}$$

$$\begin{bmatrix} -P & (AP+BW)^* & W^* & (CP+DW)^* \\ AP+BW & -S^*PS & 0 & 0 \\ W & 0 & -I & 0 \\ CP+DW & 0 & 0 & -I \end{bmatrix} \prec 0. \tag{6.29}$$

For sake of comparison, the following SDP is also solved:

$$\begin{aligned}&\text{Find}\ \ P_0 \in \mathcal{X} \text{ and } W_0 \in \mathcal{F} \text{ such that}\\ &\begin{bmatrix} -P_0 & (AP_0+BW_0)^* & W_0^* & (CP_0+DW_0)^* \\ AP_0+BW_0 & -S^*P_0S & 0 & 0 \\ W_0 & 0 & -I & 0 \\ CP_0+DW_0 & 0 & 0 & -I \end{bmatrix} \prec 0.\end{aligned} \tag{6.30}$$

Specifically, the SDP defined in (6.30) is the feasibility version of the SDP defined in (6.29). That is, the SDP defined in (6.30) only checks if the inequality constraint is feasible, and there is no objective function to be minimized. This SDP is solved to show how its associated computational cost and solution time compare with those associated with the SDP defined in (6.29), wherein an objective function is minimized and an additional variable $U \in \mathcal{X}$ is introduced.

Then, $Z = I + D^*D + B^*S^*P^{-1}SB$ and $F^c = -Z^{-1}(B^*S^*P^{-1}SA + D^*C) \in \mathcal{F}$ are defined, and the strongly stable augmented system $\mathcal{H}_\delta^c$ is formed. The realization of system $\mathcal{H}_\delta^c$ is given by $(A_H^c, B_H^c, C_H^c, D_H^c, \mathbf{\Delta})$, where $A_H^c = A+BF^c$, $B_H^c = BZ^{-\frac{1}{2}}$, $C_H^c = \begin{bmatrix}(C+DF^c)^* & (F^c)^*\end{bmatrix}^*$, and $D_H^c = \begin{bmatrix}(DZ^{-\frac{1}{2}})^* & Z^{-\frac{1}{2}}\end{bmatrix}^*$. Using Lemma 5.2, an upper bound $\gamma \approx 1$ on $\|H_\delta^c\|$ for all $\Delta \in \mathbf{\Delta}$ is computed. Namely, the following SDP is solved to find γ:

$$\begin{aligned}&\text{minimize } \gamma^2 \text{ subject to:}\ \ Q \in \mathcal{X} \text{ and}\\ &\begin{bmatrix} A_H^c & B_H^c \\ C_H^c & D_H^c \end{bmatrix}^* \begin{bmatrix} S^*QS & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_H^c & B_H^c \\ C_H^c & D_H^c \end{bmatrix} - \begin{bmatrix} Q & 0 \\ 0 & \gamma^2 I \end{bmatrix} \prec 0.\end{aligned} \tag{6.31}$$

Then, a balanced generalized gramian Σ is found for system $\mathcal{H}_\delta^c$ as per Algorithm 5.8. For the method of Section 6.4, the generalized observability gramian is defined as $Y = P^{-1} \in \mathcal{X}$, where $P \in \mathcal{X}$ is obtained by solving the SDP defined in (6.29). The generalized controllability gramian $X \in \mathcal{X}$ is found by solving the following SDP:

$$\begin{aligned}&\text{minimize} \;\; \textstyle\sum_{t=0}^{27}\left(\sum_{k=1}^{4}\left(\operatorname{trace} X_T(t,k) + \sum_{i=1}^{2}\operatorname{trace} X_i^S(t,k)\right) + \sum_{k=2}^{4}\operatorname{trace} X_P(t,k)\right)\\ &\text{subject to:} \;\; X \in \mathcal{X} \text{ and } A_H^c X (A_H^c)^* - S^* X S + B_H^c (B_H^c)^* \prec 0. \end{aligned} \tag{6.32}$$

Using the computed X and Y, a balanced realization for system $\mathcal{H}_\delta^c$ is constructed as in the proof of Lemma 5.4. This realization is denoted by $(A_{H,\text{bal}}^c, B_{H,\text{bal}}^c, C_{H,\text{bal}}^c, D_H^c, \boldsymbol{\Delta})$. Then, the following SDP is solved to find the balanced generalized gramian Σ:

$$\begin{aligned}\text{minimize } a_1 \times \epsilon + \sum_{t=0}^{27}\Bigg(\sum_{k=1}^{4}\Bigg(\|\operatorname{vect}(\Sigma_T(t,k) - \epsilon I)\|_1 + \sum_{i=1}^{2}\left\|\operatorname{vect}(\Sigma_i^S(t,k) - \epsilon I)\right\|_1\Bigg)\\ + \textstyle\sum_{k=2}^{4}\|\operatorname{vect}(\Sigma_P(t,k) - \epsilon I)\|_1\Bigg)\end{aligned}$$

$$\begin{aligned}\text{subject to:} \quad & \Sigma \succeq \epsilon I,\ \epsilon > 0,\ \Sigma \in \mathcal{X},\\ & [\![\Sigma]\!](t,k) \text{ is a diagonal matrix for all } t = 0,\dots,27,\ k = 1,\dots,4,\\ & A_{H,\text{bal}}^c \Sigma \left(A_{H,\text{bal}}^c\right)^* - S^*\Sigma S + B_{H,\text{bal}}^c\left(B_{H,\text{bal}}^c\right)^* \prec 0,\\ & \left(A_{H,\text{bal}}^c\right)^* S^*\Sigma S A_{H,\text{bal}}^c - \Sigma + \left(C_{H,\text{bal}}^c\right)^* C_{H,\text{bal}}^c \prec 0. \end{aligned} \tag{6.33}$$

The value of $a_1 = 720$ is chosen when solving the SDP defined in (6.33), which results in $\epsilon = 0.02$ and $\|(H_\delta^c - H_{r,\delta}^c)\| < 4\%\gamma$ for all $\Delta \in \boldsymbol{\Delta}$. As with the balanced truncation error bound, the coprime factors reduction error bound helps in deciding on how many state variables to truncate. However, since the coprime factors reduction method is a model reduction method in a closed-loop sense, i.e., the coprime factors reduction error bound has closed-loop robust stability interpretations as discussed in Section 6.6, then the coprime factors reduction error bound is not the only factor that determines how far one should proceed with the reduction of a given open-loop system. In general, guidelines still need to be developed for determining what constitutes a good reduced order model when applying any coprime factors reduction method. The example of Section 3.5 also discusses this issue, and some recommendations are given in Algorithm 6.6, namely, check the percentage error as well as the relative and absolute orders of the entries in Σ. Moreover, it appears from this example (and some others, e.g., see the example in [6]) that the error bounds obtained by applying the coprime factors reduction method of Section 6.4 are more conservative than the error bounds obtained when the other methods are applied. That is, for comparable reduction and reduced order system open-loop behaviour, the computed percentage error is larger when the method of Section 6.4 is applied. These observations need further testing and analysis before making final conclusions. Nonetheless, the coprime factors reduction error bound can still be used to quantitatively compare between two reduced order systems obtained by applying the same coprime factors reduction method, and so, this error bound can help in expediting and guiding any comparison between reduced order systems that is based on simulations.

System $\mathcal{H}^c_\delta$ is reduced by the application of the balanced truncation method of Section 5.3. Namely, each state variable with a corresponding entry in Σ equal to ϵ is truncated. The reduced order system is denoted by $\mathcal{H}^c_{r,\delta}$ and its balanced realization is denoted by $(A^c_{H,r}, B^c_{H,r}, C^c_{H,r}, D^c_H, \mathbf{\Delta}_{r2})$.

The realization $(A_{r2}, B_{r2}, C_{r2}, D, \mathbf{\Delta}_{r2})$ of the reduced order system $\mathcal{G}_{r2,\delta}$ is formed from the realization of system $\mathcal{H}^c_{r,\delta}$ as follows:

$$B_{r2} = B^c_{H,r} Z^{\frac{1}{2}}, \qquad \begin{bmatrix} C_{r2} + DF^c_{r2} \\ F^c_{r2} \end{bmatrix} = C^c_{H,r}, \qquad A_{r2} = A^c_{H,r} - B_{r2} F^c_{r2}.$$

System $\mathcal{G}_{r2,\delta}$ is verified to be well-posed by checking that inequality (6.5) holds, where $X \in \mathcal{X}$ is the solution to the SDP defined in (6.32).

The computational complexity and solution time of each SDP invoked when applying the method of this section are summarized in Table 6.2.

Table 6.2: This table shows the computational cost associated with each SDP that appears when applying the coprime factors reduction method of Section 6.4. Time is reported in seconds.

SDP	number of constraints	dimension of SDP variable	number of SDP blocks	dimension of linear variable	Yalmip time	Solver time
(6.29)	10920	6160	448	0	5.2516	304.2304
(6.30)	6888	4732	448	0	2.6470	79.7100
(6.31)	4033	3080	448	1	3.9411	11.3749
(6.32)	4032	2856	448	0	3.1608	4.6312
(6.33)	3193	4284	560	3193	4.5732	5.0358

6.7.3 Second Method for Coprime Factors Reduction using Contractive Coprime Factorizations

In this section, system $\mathcal{G}_\delta$ is reduced by the application of the coprime factors reduction method of Section 6.5. The resulting reduced order system is denoted by $\mathcal{G}_{r3,\delta}$, and its obtained realization is denoted by $(A_{r3}, B_{r3}, C_{r3}, D, \mathbf{\Delta}_{r3})$.

First, the SDP defined in (6.24) is solved, and its solution $P \in \mathcal{X}$ is used to define the operator $F = -(B^*S^*P^{-1}SB)^{-1}B^*S^*P^{-1}SA \in \mathcal{F}$. Then, the scaling factor $Z^{-\frac{1}{2}}$ is determined by solving the following SDP:

Find $\tilde{T} \in \mathcal{X}$ and $Z^{-1} \succ 0$ such that

$$\begin{bmatrix} A+BF \\ C+DF \\ F \end{bmatrix} \tilde{T} \begin{bmatrix} A+BF \\ C+DF \\ F \end{bmatrix}^* - \begin{bmatrix} S^*\tilde{T}S & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} B \\ D \\ I \end{bmatrix} Z^{-1} \begin{bmatrix} B^* & D^* & I \end{bmatrix} \prec 0. \quad (6.34)$$

For the SDP defined in (6.34), the number of constraints is 4368, the dimensions of the SDP variable is 3528, the number of SDP blocks is 560, the dimension of the linear variable is 0, Yalmip time is 6.2454 seconds, and the solver time is 8.1826 seconds.

The strongly stable augmented system $\mathcal{H}_\delta^c$ is then formed, and its realization is defined by $(A_H^c, B_H^c, C_H^c, D_H^c, \mathbf{\Delta})$, where $A_H^c = A + BF$, $B_H^c = BZ^{-\frac{1}{2}}$, $C_H^c = \begin{bmatrix} (C+DF)^* & F^* \end{bmatrix}^*$, and $D_H^c = \begin{bmatrix} (DZ^{-\frac{1}{2}})^* & Z^{-\frac{1}{2}} \end{bmatrix}^*$. An upper bound $\gamma = 0.532$ on $\|H_\delta^c\|$ for all $\Delta \in \mathbf{\Delta}$ is computed by solving the SDP defined in (6.31). Then, a balanced generalized gramian Σ is found for system $\mathcal{H}_\delta^c$ as per Algorithm 5.8. Namely, generalized gramians X and Y in $\mathcal{X}$ are computed by solving SDPs similar to the ones defined in (6.26) and (6.27), respectively. Using these generalized gramians, a procedure similar to the one in the proof of Lemma 5.4 is followed to construct a balanced realization for system $\mathcal{H}_\delta^c$. This realization is denoted by $(A_{H,\mathrm{bal}}^c, B_{H,\mathrm{bal}}^c, C_{H,\mathrm{bal}}^c, D_H^c, \mathbf{\Delta})$. Finally, the balanced generalized gramian Σ is computed by solving the SDP defined in (6.33). For the method of this section, the value of $a_1 = 735$ is chosen when solving the SDP defined in (6.33), which results in $\epsilon = 0.007$ and $\|(H_\delta^c - H_{r,\delta}^c)\| < 2.58\%\gamma$ for all $\Delta \in \mathbf{\Delta}$. System $\mathcal{H}_\delta^c$ is reduced by the application of the balanced truncation method of Section 5.3, and each state variable with a corresponding entry in Σ equal to ϵ is truncated. The reduced order system is denoted by $\mathcal{H}_{r,\delta}^c$ and its balanced realization is denoted by $(A_{H,r}^c, B_{H,r}^c, C_{H,r}^c, D_H^c, \mathbf{\Delta}_{r3})$.

The realization $(A_{r3}, B_{r3}, C_{r3}, D, \mathbf{\Delta}_{r3})$ of the reduced order system $\mathcal{G}_{r3,\delta}$ is formed from the realization of system $\mathcal{H}^c_{r,\delta}$ as follows:

$$B_{r3} = B^c_{H,r} Z^{\frac{1}{2}}, \qquad \begin{bmatrix} C_{r3} + DF_{r3} \\ F_{r3} \end{bmatrix} = C^c_{H,r}, \qquad A_{r3} = A^c_{H,r} - B_{r3} F_{r3}.$$

System $\mathcal{G}_{r3,\delta}$ thus defined is well-posed, since the condition expressed by inequality (6.5) is satisfied.

6.7.4 Summary of the Example

The findings from the example are now summarized. The method of Section 6.2 is the least computationally intensive among the various methods for coprime factors reduction discussed in this chapter. However, this method does not guarantee the contractiveness of the coprime factorizations under consideration. The method of Section 6.4 computes CCFs for the full order and reduced order systems, and as discussed in Remark 6.14, the employed heuristic achieves approximate normalization for the coprime factorization of the full order system. However, this method is the most computationally intensive among the presented methods. For instance, see Table 6.2 for the computational complexity and solution time corresponding to the SDPs defined in (6.29) and (6.30), and compare these measures with the ones for the SDP defined in (6.34) and the other SDPs presented in Tables 6.1 and 6.2. Finally, the method of Section 6.5 guarantees the contractiveness of the coprime factorizations and is computationally less intensive than the method of Section 6.4. However, as it is currently implemented, this method does not seek approximate normalization for the coprime factorization of the full order system. It may be possible to appropriately choose the objective function for this purpose, and as discussed in Remark 6.17, it is possible that an iterative algorithm can be coupled with solving the SDPs defined in (6.24) and (6.34) to achieve approximate normalization. In this algorithm, the computational cost per iteration, i.e., the cost of solving the SDPs defined in (6.24) and (6.34), is less than the computational cost of solving the SDP defined (6.29). The total computational cost of such an algorithm will depend on the convergence properties of the algorithm. However, more work still needs to be done in this direction.

To conclude this example, the full order system $\mathcal{G}_\delta$ and the reduced order systems $\mathcal{G}_{r1,\delta}$, $\mathcal{G}_{r2,\delta}$, and $\mathcal{G}_{r3,\delta}$ are simulated for the sake of comparison. All the systems are subjected to the same set of sinusoidal inputs of various amplitudes and frequencies for the first 100 time-steps and are left to evolve on their own afterwards. The same parameter values that vary randomly between -1 and 1 are used for the various simulated systems. Sample responses of the full order system and the reduced order systems are plotted in Figure 6.5. It can be seen that the responses of the reduced order systems satisfactorily approximate the response of the full order system.

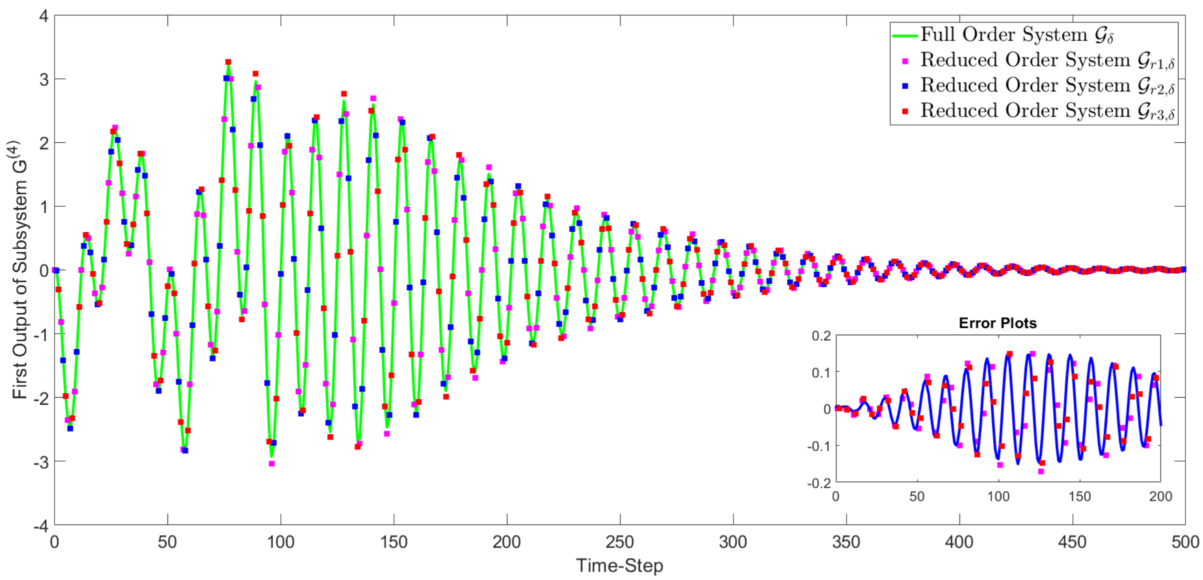

Figure 6.5: This figure plots the first output of subsystem $G^{(4)}$ in the full order system $\mathcal{G}_\delta$ and the reduced order systems $\mathcal{G}_{r1,\delta}$, $\mathcal{G}_{r2,\delta}$, and $\mathcal{G}_{r3,\delta}$, respectively. Plots of the corresponding errors between the said output in the full order system and each of the reduced order systems are also provided.

Chapter 7

Conclusions

This dissertation deals with the structure-preserving model reduction and distributed control problems for systems that are interconnected over arbitrary directed graphs. The subsystems are allowed to be heterogeneous and are assumed to have complex models, namely, LTV, LPV, and NSLPV models. The adopted frameworks model the interconnections between the subsystems using spatial states.

Chapter 2 articulates the structure-preserving balanced truncation method for the class of interconnected LTV systems. The developed balanced truncation method is applicable to strongly stable systems, i.e., systems that possess appropriately structured generalized gramians, ensures the strong stability of the resulting reduced order systems, and comes with a priori error bounds. The example in this chapter shows that the proposed method can be applied to individually truncate each temporal and spatial state, and that truncation can be time-varying even if the dimensions of the full order system are constants. The example also illustrates the use of heuristics for improving on the error bound computations.

Chapter 3 extends the balanced truncation method of Chapter 2 to systems that are not necessarily strongly stable but have strongly stable coprime factorizations. It is shown that strongly stabilizable and strongly detectable systems possess the desired coprime factorizations, and so such systems are reducible using the coprime factors reduction method. The example in this chapter shows how the proposed methods can be used to simplify the interconnection structure of the system. Namely, the balanced truncation and coprime factors reduction methods provide quantitative information on the importance of each interconnection for the overall interconnected system, and the interconnections can be classified as critical or negligible and truncated accordingly.

Chapter 4 develops an operator theoretic framework for compactly representing distributed NSLPV systems, i.e., interconnections of NSLPV subsystems and/or mixes of LTV and standard LPV subsystems. By following a parameter-independent Lyapunov function approach, convex analysis and synthesis conditions are derived for distributed NSLPV systems. The derived results extend standard results from robust control theory to the distributed NSLPV system setting, whereby the standard results gain novel interpretations and characteristics. The synthesized controller is a distributed NSLPV controller that inherits the interconnection structure of the plant. The controller subsystems have NSLPV models, are formulated in an LFT framework, and are scheduled using the same parameters as the corresponding plant subsystems. The proposed synthesis technique is applied to an illustrative example.

Chapter 5 extends the balanced truncation method of Chapter 2 to the class of interconnected NSLPV systems. The results are presented using the operator theoretic framework of Chapter 4. In addition to the novel features and characteristics of the balanced truncation method of Chapter 2, the method of this chapter allows for the truncation of the parameter states, i.e., the signals introduced by formulating the subsystems in an LFT framework. Thus, in addition to simplifying the interconnection structure of the system, the method in this chapter can be used to simplify the uncertainty structure of the system. The example in this chapter discusses the application of the various derived error bound expressions to obtain the most meaningful and least conservative results.

Chapter 6 extends the application of the balanced truncation method of Chapter 5 to strongly stabilizable and strongly detectable distributed NSLPV systems. This chapter contains a discussion on contractive coprime factorizations, which are the natural extension/relaxation of normalized coprime factorizations for systems with inherent interconnection and/or uncertainty structures. Namely, Chapter 6 first extends the coprime factors reduction method of Chapter 3 to distributed NSLPV systems, and then proposes two ways to modify the aforementioned method so as to ensure the contractiveness of the coprime factorizations. Chapter 6 also provides an interpretation of the coprime factors reduction error bound in terms of robust feedback stability. The example in this chapter focuses on the computational complexity of the SDPs that are solved when applying the various proposed methods for coprime factors reduction.

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