ON MONITORING THE

ATTRIBUTES OF A PROCESS

by

Mark O. Marcucci

Dissertation submitted to the Graduate Faculty

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Statistics

APPROVED:

Donald R. Jensen, Chairman

I. J. Good	Marion R. Reynolds, Jr.
Raymond H. Myers	Robert V. Foutz

August, 1982

Blacksburg, Virginia

ACKNOWLEDGEMENTS

I would like to thank the following people, and organizations who have made this undertaking possible.

Donald R. Jensen suggested the topic and provided necessary assistance and guidance throughout the dissertation process.

I. J. Good, Raymond H. Myers, Robert V. Foutz, and Marion R. Reynolds, Jr., served on my committee and made valuable suggestions.

Arnita Perfater and Karen Anderson performed an excellent job in typing the dissertation under less than ideal conditions.

The Department of Statistics and the Department of the Army, through contract No. DAAG-29-78-G-0122, for financial support.

Issac Asimon, who, through the Foundation Triology, planted the seeds of interest in Statistics in a young boy's mind.

Finally, my wife, Dale, who not only painstakingly proofread the manuscript, but also proved to be an abundant source of tea and sympathy.

TABLE OF CONTENTS

LIST OF TABLES

Table

Table

Table

LIST OF FIGURES

Figure

GLOSSARY OF NOTATION AND TERMINOLOGY

$G_r(z;\pi,n)$ or $G_r(z;\pi)$	Krawtchouck polynomial of order r, argument z, and parameters π and n.
$K_r(z;\theta)$	Charlier polynomial of order r with argument z and parameter θ.
$L_r(z;m)$	Laguerre polynomial of order r, argument z, and parameter m.
$b_k(x;n,\underset{\sim}{\pi},\underset{\sim}{\rho})$ or $b_k(n,\underset{\sim}{\pi},\underset{\sim}{\rho})$	k-variate binomial distribution with parameters n, $\underset{\sim}{\pi}$, and $\underset{\sim}{\rho}$.
$P_k(\underset{\sim}{\lambda})$	k-variate Poisson distribution with parameter $\underset{\sim}{\lambda}$.
$G(\theta)$	The geometric distribution with parameter θ.
$N_k(\underset{\sim}{\mu},\underset{\sim}{\Sigma})$	k-variate normal (Gaussian) distribution with mean $\underset{\sim}{\mu}$ and covariance matrix $\underset{\sim}{\Sigma}$.
$W_k(n,\underset{\sim}{\Sigma},\underset{\sim}{\Delta})$	The Wishart distribution of order k, with n degrees of freedom, scale matrix $\underset{\sim}{\Sigma}$ and noncentrality matrix $\underset{\sim}{\Delta}$.
$W_k(n,\underset{\sim}{\Sigma})$	The central Wishart distribution of order k, with n degrees of freedom, and scale matrix $\underset{\sim}{\Sigma}$.
$\chi^2_k(n,\underset{\sim}{\Sigma},\underset{\sim}{\Delta})$	The k-variate chi-squared distribution with n degrees of freedom, scale matrix $\underset{\sim}{\Sigma}$, and noncentrality matrix $\underset{\sim}{\Delta}$.
$\chi^2_k(n,\underset{\sim}{\Sigma})$	$\chi^2_k(n,\underset{\sim}{\Sigma},\underset{\sim}{\Delta})$ when $\underset{\sim}{\Delta} = \underset{\sim}{0}$.
$\Phi(u)$	The cdf of the standard univariate Gaussian distribution.
$L(\underset{\sim}{x})$	The law of the random variable $\underset{\sim}{x}$.
$z^{(j)}$	$z(z-1)\ \ldots\ (z-j+1)$, $j=0,1,\ldots,z$.
$\underset{\sim}{x} \succ \underset{\sim}{y}$	For $\underset{\sim}{x}$ and $\underset{\sim}{y}$ vectors of order n, $x_i \geq y_i$, $i=1,\ldots,n$.
$y = o(x)$	$\lim\limits_{x\to\infty} \frac{y}{x} = 0$.

$y = O(x)$	$\lim_{x\to\infty} \frac{y}{x} = c$, where c is some constant not equal to 0.
$X_n = o_p(1)$	$X_n \xrightarrow{p} 0$, i.e., the sequence of random variables $\{X_n\}$ converges in probability to 0.
$\underset{\sim}{X}_n \xrightarrow{d} \underset{\sim}{X}$	The sequence of random elements $\{\underset{\sim}{X}_n\}$ converges in distribution to the random element $\underset{\sim}{X}$.
$x \sim y$	The random variable x has the distribution y.
$X_1 \vee X_2$	The random variable obtained by generalizing the random variable X_1 with the random variable X_2.
$\underset{\sim}{I}_k$	The identity matrix of order k.
$\underset{\sim}{J}_k$	The k×k matrix all of whose entries are one.
$\mathrm{diag}(\underset{\sim}{A}_1, \underset{\sim}{A}_2, \ldots, \underset{\sim}{A}_k)$	The partitioned matrix with the submatrices $\underset{\sim}{A}_1, \underset{\sim}{A}_2, \ldots, \underset{\sim}{A}_k$ along the diagonal and zeros elsewhere.
$\underset{\sim}{A} \times \underset{\sim}{B}$	The Kronecker product of the matrices $\underset{\sim}{A} = (a_{ij})$ and $\underset{\sim}{B} = (b_{ij})$. If $\underset{\sim}{C} = \underset{\sim}{A} \times \underset{\sim}{B}$, then $\underset{\sim}{C} = (\underset{\sim}{C}_{ij})$, where $\underset{\sim}{C}_{ij} = a_{ij}\underset{\sim}{B}$.
$(L, \gamma_\ell, U, \gamma_u)$	The control chart with the randomized upper and lower control limits L and U, respectively. The randomized portions of the control limits are given by γ_ℓ and γ_u, respectively.
N_C.	The run length of the control chart C.
pmf	probability mass function
cdf	cumulative distribution function
pgf	probability generating function
i.i.d.	independent and identically distributed

ARL	average run length
UMPU	uniformly most powerful unbiased
UMP	uniformly most powerful
R^n	Euclidean n-space

I. INTRODUCTION

1.1 Background

It has been over half a century since Walter A. Shewhart authored the first book devoted to quality control - Economic Control of Quality of Manufactured Product. During the Second World War, military applications gave the use of these techniques an impetus, and the post war era has seen the wide-scale adoption of quality control techniques by industry. More recently the subject has become a small matter of political concern, as some pundits have claimed that one reason for the world prominence of Japanese industry is their mastery of quality control.

This study is devoted to some aspects of statistical quality control. The practice of quality control involves much more than the application of statistical methods (cf. Quality (1980)). While the best statistics in the world will not assure a quality product, statistical techniques, nonetheless, play a crucial role in the entire scheme.

The present chapter gives an introduction and overview of the topics to be covered in the remaining chapters as well as a review of some useful, miscellaneous results from the statistical literature.

1.2 Statistical Quality Control

The methodologies of statistical quality control have applications in far ranging fields, for example in large scale data base management such as in census bureaus, or in the environmental monitoring of pollution levels, but we will motivate the settings and terminologies of the problems to be studied herein from an industrial standpoint. There are two broad topics generally included in the subject matter of statistical quality control - acceptance sampling and process control. Within each topic statistical techniques have been developed for problems dealing with variables, where numerical measures are taken on each item, and attributes, where each item is classified into one of a finite number of categories.

Acceptance sampling provides procedures by which a lot of items may be inspected so as to determine whether or not the lot is of acceptable quality. Existing sampling plans are of varying degrees of sophistication - from simple single sampling plans to various modes of sequential sampling. In this dissertation, no topics in acceptance sampling are specifically covered. However, some of the same arguments and techniques developed for process control can be applied to some areas of acceptance sampling.

In process control, one is faced with the problem of deciding whether or not a particular process is operating within prescribed limits. If so, the process is said to be in control, otherwise the process is said to be out of control. Typically, the statistical device used in process control is the control chart. The maintenance of a control chart may be described as follows. At established intervals over time, the product of the process is sampled and a statistic based on this sample, and possibly some or all of the preceding samples, is graphed. If this statistic falls outside established control limits the process is said to be out of control, causes for this behavior are determined, and the process is adjusted to specification.

Two such approaches to process control are Shewhart-type charts and cumulative sum (CUSUM) charts. The former procedures represent the original statistical methodology used in process control. The samples taken at each monitoring occasion are assumed to be independent. At each occasion a statistic is computed combining the information from the present sample with either a priori knowledge of the process parameters or estimates of these parameters from a period in which the process is in control.

CUSUM charts, on the other hand, are of a more recent vintage (cf. Page (1954) and Barnard (1959)). CUSUM procedures combine information from all monitoring periods by plotting over time the sum, or a function of the sum, of the deviations from the quality standard. For example, if the quality standard is the mean, μ, of the process, a CUSUM procedure would be to plot $\sum_{i=1}^{k} (\bar{X}_i - \mu)$ at the kth sampling occasion. CUSUM procedures have the purported advantage of being able to detect small shifts in the process parameter more quickly than Shewhart-type charts. This characteristic notwithstanding, we will discuss CUSUM charts no further in this study, as they do not readily admit extensions of the types considered here.

Common Shewhart-type control procedures for variables are the $\bar{X}$ and $\bar{R}$ charts. These control charts use the sample mean and range for monitoring the mean and variance, respectively, of a Gaussian process. A discussion of these techniques can be found in any text on quality control (e.g., Duncan (1974)). There has also been some effort at extending Shewhart-type charts to monitor the multivariate normal mean and covariance matrix. For a review of these procedures see Hui (1980) and Alt (1982).

Common Shewhart charts for attributes are the p-chart and c-chart. A p-chart is employed to monitor the proportion defective of a Bernoulli process, while a c-chart is used to monitor the number of defects per unit of product. In the parlance of quality control, a defect is any nonconformance to specification, while an item is defective if it has one or more defects. Monitoring procedures for the proportion defective or the number of defects per unit, generalizations of these procedures, and extensions to multivariate problems will provide the main focus of this dissertation.

The next section provides a review of the literature pertaining to multicharacteristic quality control for attributes.

1.3 A Review of the Multiattribute Quality Control Literature

The statistical literature on multiattribute quality control is relatively sparse. In the work that has been done acceptance sampling problems have received the most attention. There appears to be only one paper specifically dealing with the monitoring of a discrete multivariate process.

Patel (1973) discussed the use of asymptotic chi-squared statistics to monitor multivariate Bernoulli and Poisson processes. A more complete discussion of his suggestion will be reserved for later chapters.

Some authors have proposed techniques for determining acceptance sampling plans for multivariate Bernoulli attributes under the simplifying assumption of independence among the attributes. For example, McCaslin _et al_. (1975) use constrained optimization techniques to construct upper and lower bounds on the operating characteristic (OC) curve for sampling plans under these assumptions. Their plans set lot acceptance limits separately for each characteristic. In a companion paper, Gruska _et al_. (1976) provide simplified techniques for the construction of these lower and upper bounds on the OC curve. Case _et al_. (1975)

discuss the economic design of multiattribute sampling - again under the assumption that the attributes are independent Bernoulli random variables.

Multiple attribute sampling plans for use when dependencies are allowed among the characteristics were considered by Lorenzen (1979). He uses multi-stage single attribute sampling plans to test several dependent attributes simultaneously. A computational algorithm based upon the multinomial distribution is used to develop formulae for determining the sampling plans.

Several authors have discussed acceptance sampling plans for multinomial random variables. Bray *et al*. (1973), for instance, develop sampling plans for lots whose items may be classified as either of good, marginal, or bad quality. Their plans assure that the acceptable number of "bad" items allowable is zero. Clements (1979, 1980) provides additional computational algorithms for deriving three-class attribute sampling plans. He also demonstrates the possible efficiencies of these plans vis-á-vis the standard two-class, i.e., defective-nondefective, sampling plans.

1.4 Some Terminology

In our study of control chart procedures it will be useful to distinguish between two situations concerning the quality standards. In one situation the desired quality standards are assumed to be known - for example, from either managerial fiat or long term experience with the process. This situation will be referred to as *Case I*.

By contrast, it often happens that management has little experience with the process and has no rigid quality standards to offer. In these instances it is necessary to obtain the quality standards through actually operating the process. To this end, suppose the process is observed for a period of time when it can be safely assumed to be in control. This period will be called a base period, and any resulting control chart which uses these observations in its maintenance will be called a Case II monitoring procedure.

An important aspect of this dissertation is the comparison of competing process control procedures. Therefore, a relevant measure by which to compare competing control procedures must be established.

At first glance, it might seem reasonable to borrow the notions of statistical hypothesis testing. Care must be exercised, however. All the control charts that we will study signal eventually with probability one. Therefore, ultimately one ends up deciding that the process is not in control - no matter what is the true state of the process. Hence, the notions of power and level of hypothesis testing cannot be extended in a straightforward manner to the study of control charts.

The Shewhart-type control charts we will study establish control limits at the outset of the monitoring scheme. These limits remain in effect until such time that the process has been determined to have changed. A statistic is computed for the sample information at each monitoring occasion and then plotted on the control chart. If any of these values of the control chart lie outside the control limits the chart is said to signal.

So, we should ask, what characteristics would we desire in a "good" control chart? First, if the process is in control the chart should run a long time before signalling, that is, false alarms should be minimized. On the other hand, if the process is out of control, the chart should signal quickly. The number of monitoring occasions until the control chart signals is called the run length. The two desirable properties of a control chart are embodied in its run-length distribution.

Definition 1.1. Suppose $(\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots)$ represent the values of a control chart, C, and ℓ represents the region of values for which the process is deemed to be in control. The run-length, N_C, of C has the distribution given by

$$P(N_C \leq t) = 1 - P\left(\bigcap_{i=1}^{t} \{\underset{\sim}{x}_i \in \ell\}\right),$$

or, equivalently, $P(N_C > t) = P\left(\bigcap_{i=1}^{t} \{\underset{\sim}{x}_i \in \ell\}\right)$.

Before comparing different control charts we require that they have similar properties when the process is in control. We specifically require that the charts have the same level.

Definition 1.2. Let $(\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots)$ and ℓ be as defined in Definition 1.1. The level of the chart is said to be α if

$$P(\underset{\sim}{x}_i \in \ell) \geq 1-\alpha \qquad i = 1,2,\ldots$$

on each occasion when the process is in control.

The next definition provides the basis for comparing the ability of competing control charts to detect out-of-control behavior.

Definition 1.3. A control chart C is said to dominate a control chart C* if C and C* have the same level, and N_C is stochastically smaller than N_{C*} when the process is not in control.

Definition 1.3 providws a rather strong notion of dominance. A less stringent requirement that is often used is to compare the means of the run-length distributions. Of course, Definition 1.3 implies this weaker version of dominance.

In the theory of statistical tests the idea of unbiasedness plays an important role in the construction of optimal tests. Unbiasedness of a hypothesis test ensures that the power of the test is always at least as large as a trivial test which rejects with probability equal to the size of the Type I error regardless of the sample information. If an α-level control chart is established using a trivial test, then its run-length distribution is the geometric distribution with parameter α. With this observation in mind, the notion of unbiasedness is extended to control charts in the following definition.

Definition 1.4. A control chart C of level α is said to be unbiased if N_C is stochastically smaller than the geometric distribution with parameter α when the process is out of control.

Note that Definition 1.4 assures that the control chart will dominate any trivial control procedure which simply declares the process to be out of control with probability α at each monitoring occasion.

Another weak property which might be required of all process control procedures is admissibility.

Definition 1.5. A control chart is said to be admissible in a class of control chart procedures if there exists no other control chart in the class which dominates it.

The following lemma establishes an important link between the theory of optimal statistical tests and the construction of optimal control charts. This technique was used by Jensen (1982) in discussing optimal control charts for the variance of a Gaussian process.

Lemma 1.1. Suppose $(\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots)$ represent the sample information from successive monitoring periods. Let $(Y_1, Y_2, \ldots)$, where $Y_i = f(\underset{\sim}{x}_i)$, be the values of a control chart for monitoring the process parameters $\underset{\sim}{\theta}$, and suppose the Y_i's are mutually independent. Further, suppose the control limits of the chart are such that each Y_i determines a UMP hypothesis test of level α in a class of tests. Then, the control chart using $\{Y_1, Y_2, \ldots\}$ dominates every other control chart based on test statistics in that class.

Proof. From the independence assumption, the run-length distribution of $(Y_1, Y_2, \ldots)$ is given by:

$$P(N_C > t) = \prod_{i=1}^{t} (1-\beta_i) ,$$

where $\beta_i = P(Y_i \notin \ell)$, and ℓ defines the in-control region of the control chart.

Let N_{C*} be the run length of any other control chart derived from the class of hypothesis tests. So, $P(N_{C*} > t) = \prod_{i=1}^{t} (1-\beta_i^*)$. But from the optimality of Y_i it follows that $\beta_i \geq \beta_i^*$ for all $\underset{\sim}{\theta}$ in the out-of-conrol region of the process. Hence, C dominates C* for any level α.

Lemma 1.1 will be useful in developing optimal control charts directly from existing statistical theory.

In certain applications it will be useful to distinguish between stationary and non-stationary processes. Most discussions of process control techniques assume either that the process is stationary, or that the process is stationary for a period of time and then switches to another stationary state for the rest of the monitoring period. At times, however, stationarity may be an unreasonable assumption. If at each monitoring occasion the quality parameters may be at different values, then this nonstationary process is said to be a drifting process.

1.5 Some Miscellaneous Results

In this section we compile, for the convenience of the reader, some miscellaneous results which will be needed in the sequel. This collection of theorems comes from the study of probability inequalities and asymptotic distribution theory.

A recent book by Tong (1980) provides a comprehensive treatment of probability inequalities. These types of inequalities will be used to give stochastic bounds for the run-length distributions of some control chart procedures. We begin the survey with some well-known results for the multivariate normal distribution.

Theorem 1.1 (Slepian's inequality). Let $\underset{\sim}{X} \sim N_p(\underset{\sim}{0},\underset{\sim}{R})$ and $\underset{\sim}{Y} \sim N_p(\underset{\sim}{0},\underset{\sim}{T})$, where $\underset{\sim}{R} = (\rho_{ij})$ and $\underset{\sim}{T} = (\tau_{ij})$ are positive semidefinite correlation matrices. If $\rho_{ij} \geq \tau_{ij}$ for all (i,j), then

$$P\left(\bigcap_{i=1}^{p} \{X_i \leq u_i\}\right) \geq P\left(\bigcap_{i=1}^{p} \{Y_i \leq u_i\}\right)$$

holds for all $\underset{\sim}{u} = (u_1,\ldots,u_p)$. Furthermore, the inequality is strict if $\rho_{ij} > \tau_{ij}$ for some (i,j).

Proof. Slepian (1962).

Slepian's inequality was partially extended to rectangular regions by Šidák in the following pair of theorems.

Theorem 1.2. Let $\underset{\sim}{X} \sim N_p(\underset{\sim}{0},\underset{\sim}{R}(\underset{\sim}{\lambda}))$, where $\underset{\sim}{R}(\underset{\sim}{\lambda}) = (\rho_{ij}(\underset{\sim}{\lambda}))$ is a correlation matrix depending on $\underset{\sim}{\lambda}$ in the following manner. For $0 \le \lambda_i \le 1$, we have

$$\begin{array}{lll} \rho_{ij} = \lambda_i\lambda_j\tau_{ij} & i \neq j & i,j = 1,\ldots,p \\ \rho_{ii} = 1 & i = 1,\ldots,p & , \end{array}$$

where $\underset{\sim}{T} = (\tau_{ij})$ is a positive semi-definite correlation matrix. Then

$$P[\bigcap_{i=1}^{p} \{|X_i| \le u_i\}] \tag{1.1}$$

is monotonically nondecreasing in each λ_i.

Theorem 1.3 Let $\underset{\sim}{X}$ be as in Theorem 1.2, except that $\rho_{ij} = \lambda\tau_{ij}$, for $\lambda \in [0,1]$. Then (1.1) is strictly increasing in λ if $\underset{\sim}{T} \neq \underset{\sim}{I}$. In particular,

$$P(\bigcap_{i=1}^{p} \{|X_i| \le u_i\}) \ge \prod_{i=1}^{p} P(|X| \le u_i) .$$

Proofs. Šidák (1968).

The following well-known result from Anderson (1955) and its generalization by Mudholkar (1966) prove useful in studying many non-central distributions which arise in hypothesis testing.

Theorem 1.4. Let $f(\underset{\sim}{x})$: $R^k \to [0,\infty)$ be symmetric about the origin and unimodal, i.e. $\{\underset{\sim}{x} | f(\underset{\sim}{x}) \geq u\}$ is convex for all $u \geq 0$. Let $A \subset R^k$ be symmetric about the origin and convex. If $\int_A f(x)dx < \infty$, then

$$\int_A f(\underset{\sim}{x}+\lambda\underset{\sim}{y})d\underset{\sim}{x} \geq \int_A f(\underset{\sim}{x}+\underset{\sim}{y})d\underset{\sim}{x}$$

for all $\underset{\sim}{y} \in R^k$ and $\lambda \in [0,1]$.

Proof. Anderson (1955).

Theorem 1.5. Let $G = \{g\}$ be a group of linear Lebesgue measure-preserving transformations of R^n onto R^n. Let A be a convex, G-invariant region of R^n. Let f be a nonnegative real-valued, G-invariant, and unimodal function on R^n. For any $\underset{\sim}{y} \in R^n$

$$\int_A f(\underset{\sim}{x}+\underset{\sim}{z})d\underset{\sim}{x} \geq \int_A f(\underset{\sim}{x}+\underset{\sim}{y})d\underset{\sim}{x} ,$$

where $\underset{\sim}{z}$ is any point in the convex hull of the G-orbit of $\underset{\sim}{y}$.

Proof. Mudholkar (1966).

Another useful inequality due to Šidák (1973) applies to multivariate distributions constructed in a certain fashion. These types of distributions often arise as the run lengths of Case II monitoring procedures.

Theorem 1.6. Let $\underset{\sim}{Y}_i$ $(i=1,\ldots,k)$ be independent and identically distributed random vectors, and let $\underset{\sim}{U}$ be another random vector which is independent of the $\underset{\sim}{Y}_i$'s. Define, for a Borel set A and a measurable function $f(\cdot,\cdot)$,

$$\beta(r) = P\left(\bigcap_{i=1}^{r} \{f(\underset{\sim}{Y}_i,\underset{\sim}{U}) \in A\}\right) .$$

Then the inequalities:

$$\beta(k) \geq [\beta(k-1)]^{k/(k-1)} \geq \ldots \geq [\beta(2)]^{k/2} \geq [\beta(1)]^{k}$$

hold for $k \geq r \geq 2$.

Proof. Šidák (1973).

The system of Bonferroni inequalities is stated in its most general form by Feller (1968, p. 110). Some useful second-order Bonferroni-type inequalities were given by Kounias (1968).

Theorem 1.7 Let $\underset{\sim}{X}$ be a k-dimensional random variable and $A_1,\ldots,A_k$ be Borel-measurable subsets of the real line. Define $q_i = P(X_i \notin A_i)$, $i=1,\ldots,k$; $Q_1 = \sum_{i=1}^{k} q_i$; $q_{ij} = P(X_i \notin A_i,\ X_j \notin A_j)$; and $Q_2 = \sum_{i>j} q_{ij}$. Then:

(i) $1-Q_1 + \max_{1\leq \ell \leq k} \sum_{i\neq \ell} q_{i\ell} \leq P[\bigcap_{i=1}^{k} \{X_i \varepsilon A_i\}] \leq 1-Q_1+Q_2$;

(ii) for $\underset{\sim}{v} = (q_1,q_2,\ldots,q_k)'$ and $\underset{\sim}{W} = (q_{ij})$

$$P(\bigcap_{i=1}^{k} \{X_i \varepsilon A_i\}) \leq 1-\underset{\sim}{v}'\underset{\sim}{W}^{-}\underset{\sim}{v} ,$$

where $\underset{\sim}{W}\underset{\sim}{W}^{-}\underset{\sim}{W} = \underset{\sim}{W}$.

Proof. Kounias (1968).

We now state a few well-known theorems that will be useful in later chapters when developing the asymptotic behavior of some statistics. The first is a statement of the δ-method.

Theorem 1.8. Let $\underset{\sim}{T}_n$ be a k-dimensional statistic$(T_{1n},\ldots,T_{kn})$ such that the asymptotic distribution of $\sqrt{n}(\underset{\sim}{T}_n-\underset{\sim}{\theta})$ is $N_k(\underset{\sim}{\theta},\underset{\sim}{\Sigma})$. Let $g_1,\ldots,g_q$ be q functions of k variables such that each g_i is totally differentiable.

Then the asymptotic distribution of

$$\{\sqrt{n}(g_i(T_{1n},\ldots,T_{kn}) - g_i(\theta_1,\ldots,\theta_k) \; ; \; i=1,\ldots,q\}$$

is $N_q(\underset{\sim}{0},\underset{\sim}{G}\underset{\sim}{\Sigma}\underset{\sim}{G}')$ where $\underset{\sim}{G}$ is a matrix of order $q\times k$ whose (i,j) element is $(\partial g_i/\partial\theta_j)$, and the rank of the distribution is equal to the rank of $\underset{\sim}{G}\underset{\sim}{\Sigma}\underset{\sim}{G}'$.

Proof. Rao (1973; p. 388).

Theorem 1.9. Let $\{(X_N,Z_N);\ N = 1,2,\ldots\}$ be a sequence in $\mathcal{X}\times Z$ such that X_N converges in distribution to $X\varepsilon\mathcal{X}$ and Z_N converges in probability to $a\varepsilon Z$. Then (X_N,Z_N) converges in distribution to (X,a).

Proof. Billingsley (1968, p. 27).

Theorem 1.10. Let $h\colon \mathcal{X}\to Z$ have the points of discontinuity $D(h)$.

(i) If X_N converges in distribution to X and if $D(h)$, under the limiting measure, has probability zero, then $h(X_N)$ converges in distribution to $h(X)$.

(ii) If X_N converges to a in probability and if h is continuous at a, then $h(X_N)$ converges to $h(a)$ in probability.

Proof. Billingsley (1968, p. 31).

1.6 A Brief Look at Things to Come

Before actually beginning the study of process control procedures for attributes, some necessary groundwork in distribution theory is laid in the next chapter. There, some known results are reviewed, and some new ones are given for the multivariate binomial, Poisson, generalized Poisson, and chi-squared distributions.

Chapter Three considers the standard p-chart and some univariate generalizations of it. Both small- and large-sample, Case I and Case II monitoring procedures are discussed. Emphasis is placed on the distributional properties of the run lengths of the control charts. Some alternatives to the standard p-chart approach are offered.

Chapter Four explores the problem of monitoring a process that can be defective on more than one quality characteristic. First, omnibus procedures for Case I and Case II problems are discussed. These monitoring procedures are strictly large-sample results. Both small- and large-sample procedures are considered for diagnostic control charts of this type.

Finally, Chapter Five discusses process monitoring procedures for the number of defects per unit of output. The same topics covered in the two preceding chapters are extended to the c-chart procedures.

Briefly, we note our convention for referring to other chapters and sections in the text. To refer to a different section within the same chapter we simply say, e.g., Section Three. On the other hand, to refer to a section in another chapter we use, e.g., Section 3.3 to refer to Section 3 of Chapter 3.

II. SOME NECESSARY DISTRIBUTION THEORY

2.1 Introduction

In this chapter some distribution theory that is needed for later developments is reviewed. The multivariate binomial and Poisson distributions will be of interest in the study of techniques for multivariate p- and c-charts, respectively. The generalized Poisson distribution, and its multivariate versions, arise in the problem of monitoring the number of defects of a process. Multivariate chi-squared distributions are relevant as limiting distributions in connection with various univariate and multivariate p-chart problems.

2.2 The Multivariate Binomial Distribution

Some authors refer to the multinominal distribution as the "multivariate binomial." This misnomer is avoided herein. Throughout this study a multivariate binomial population will signify a population whose elements can be in one of two possible states on more than one attribute. For example, samples of wine may be judged as satisfactory or unsatisfactory on the attributes: color, bouquet, and taste.

This section begins with a discussion of the bivariate binomial distribution and some of its properties. The somewhat limited results, now available for the multivariate binomial distribution, are then discussed. Finally, some pertinent asymptotic distribution theory is given.

Aitken and Gonin (1935) considered the bivariate binomial distribution from the starting point of a 2×2 table. The probabilities for the table are

$$\begin{array}{cc|c|c|c}
 & & \multicolumn{2}{c}{\text{Attribute 2}} & \\
 & & B & \bar{B} & \\
\hline
\text{Attribute} & A & \pi_{11} & \pi_{10} & \pi_1 \\
\hline
1 & \bar{A} & \pi_{01} & \pi_{00} & 1-\pi_1 \\
\hline
 & & \pi_2 & 1-\pi_2 &
\end{array} \quad , \qquad (2.1)$$

where A and B represent the occurrence of Attributes one and two, respectively, and $\bar{A}$ and $\bar{B}$ represent the complements of these respective events. A random sample of size n from a population specified by (2.1) can also be represented in a 2×2 table as:

$$\begin{array}{cc|c|c|c}
 & & \multicolumn{2}{c}{\text{Attribute 2}} & \\
 & & B & \bar{B} & \\
\hline
\text{Attribute} & A & X_{11} & X_{10} & X_1 \\
\hline
1 & \bar{A} & X_{01} & X_{00} & n-X_1 \\
\hline
 & & X_2 & n-X_2 & \boxed{n}
\end{array} \quad , \qquad (2.2)$$

where X_{11} is the number of occurrences of the event $A \cap B$, X_{10} is the number of occurrences of the event $A \cap \bar{B}$, et cetera. The joint distribution of (X_1, X_2) is a bivariate binomial distribution.

Aitken and Gonin inverted the factorial moment generating function to obtain the probability mass function for (X_1, X_2) as

$$\begin{aligned}
p(x_1, x_2; \pi_1, \pi_2, \rho, n) &= p(x_1; \pi_1) p(x_2; \pi_2) \\
&\cdot \{1 + \sum_{r=1}^{n} \rho^r G_r(x_1; \pi_1, n) G_r(x_2; \pi_2, n)\} \\
x_1 &= 0, 1, \ldots, n \\
x_2 &= 0, 1, \ldots, n \ ,
\end{aligned} \qquad (2.3)$$

where (i) $\quad G_r(z;\pi,m) \equiv G_r(z;\pi) = (m^{(r)} r!\pi^r (1-\pi)^r)^{-\frac{1}{2}}$

$$\cdot \sum_{i=0}^{r} \binom{r}{i} (-\pi)^i z^{(r-i)} (m-r+i)^{(i)} ,$$

the normalized Krawtchouck polynomial;

(ii) $\quad z^{(j)} = z!/(z-j)!$;

(iii) $\quad p(z;\pi) = \binom{m}{z} \pi^z (1-\pi)^{m-z}$;

and (iv) $$\rho = \frac{\pi_{11} - \pi_1 \pi_2}{[\pi_1 \pi_2 (1-\pi_1)(1-\pi_2)]^{\frac{1}{2}}} .$$

In the terminology of Lancaster (1958), (2.3) is an expansion of the bivariate binomial probability mass function (pmf) in its canonical form, with the canonical correlations, ρ^i, and the canonical variables, $G_r(\cdot,\pi)$. The distribution (2.3) will be denoted by $b_2(n,\underset{\sim}{\pi},\rho)$.

The expression (2.3) can be used to compute probabilities for the bivariate binomial distribution. The key to these computations is an efficient, recursive formula for the Krawtchouck polynomials. The algorithm, along with an accompanying computer program, is explained in Appendix I. Tables of the cumulative distribution function (cdf) for selected examples are also given there.

Analytical properties of the bivariate binomial distribution are scarce. One result, for the special case of identical marginal distributions, was provided by Hamdan and Jensen (1976).

<u>Theorem 2.1</u>. Let (X_1,X_2) have pmf given by (2.3) with $\pi_1 = \pi_2$ and $\rho \geq 0$. If B is any event measurable with respect to $b_1(n,\pi_1)$, then $P(X_1 \varepsilon B, X_2 \varepsilon B)$ is an increasing function of ρ.

Counter examples can be constructed using (2.3) to demonstrate that Theorem 2.1 is not true if either $\rho<0$ or if B is not the same for both X_1 and X_2. Empirical results do suggest, however, that the theorem remains valid if $\pi_1 \neq \pi_2$, but attempts to verify this property analyt-caiily have been unsuccessful.

Krishnamoorthy (1951) extended the Aitken-Gonin expansion to the general multivariate binomial distribution by considering a random sample of size n from a 2^k table. Make the following definitions:

(i) $\pi_{i_1 i_2 \ldots i_k} = P(X_1=i_1, X_2=i_2, \ldots, X_k=i_k) \quad i_j = 0$ or 1;

(ii) $\pi_j = \sum_{\{i_j=1\}} \pi_{i_1 \ldots i_k}$, the probability of occurrence of attribute j , $j=1,\ldots,k$;

(iii) $\pi_{j\ell} = \sum_{\{i_j=1, i_\ell=1\}} \pi_{i_1 \ldots i_k}$, the probability of attributes j and ℓ both occurring;

et cetera ;

(iv) $d_{ij\ldots m} = \pi_{ij\ldots m} - \pi_i \pi_j \ldots \pi_m$

$d'_{ij} = d_{ij}$, $d'_{123} = d_{123} - (\pi_1 d'_{23} + \pi_2 d'_{13} + \pi_3 d'_{12})$,

and in general,

$d'_{ij\ldots m} = d_{ij\ldots m} - (\pi_i d'_{j\ell\ldots m} + \ldots + \pi_m d'_{ij\ldots m-1})$;

(v) $X_j = \sum_{\{i_j=1\}} X_{i_1 i_2 \ldots i_k}$, the number of occurrences of attribute j.

Krishnamoorthy obtained the following symbolic expression for the pmf of the k-variate binomial distribution:

$$p(x_1,\ldots,x_K) = p(x_1;\pi_1)\ \ldots\ p(x_k;\pi_k)$$

$$\{1 + \sum_{i<j} \frac{d'_{ij}G(x_i;\pi_i)G(x_j;\pi_i)}{(\pi_i\pi_j(1-\pi_i)(1-\pi_j))^{\frac{1}{2}}} + \ldots$$

$$+ \frac{d'_{12\ldots k}G(x_1;\pi_1)\ \ldots\ G(x_k;\pi_k)}{(\pi_1\ldots\pi_k(1-\pi_1)\ \ldots\ (1-\pi_k))^{\frac{1}{2}}}\}^n , \tag{2.5}$$

where $\{\cdot\}^n$ is to be expanded formally via the multinomial theorem, and $[G(x_i;\pi_i)]^{m+\ell}$ is to be replaced by $G_{m+\ell}(x_i;\pi_i)$, $(i=1,\ldots,k)$, after collecting terms in the expansion. The notation $b_k(n,\underset{\sim}{\pi},\underset{\sim}{\rho})$ will be used to denote the distribution (2.5). The $2^k-(k+1)$ components of $\underset{\sim}{\rho}$ are functions of the d'_{ij}, and they correspond to the generalized canonical correlations discussed in Lancaster (1969) for the case $k>2$.

It is readily seen that (2.5) is a highly symbolic expression that may be of little practical value. With the available algorithm for generating the Krawtchouck polynomials in (2.5), evaluating the expression for even moderate choices of n and k would be an immense task. Even if this problem could be surmounted, we are still left in the unenviable position of having to specify 2^k-1 parameters.

One way to reduce the number of parameters of the problem, and, at the same time, to ease the severity of the computational task, is to appeal to asymptotic theory. A limiting normal distribution is obtained as a consequence of the multivariate central limit theorem. Let

$$Y_i = \sqrt{n}(\frac{X_i}{n} - \pi_i),\ i=1,\ldots,k\ . \tag{2.6}$$

Then, for $\underset{\sim}{Y}' = (Y_1,\ldots,Y_k)$, as $n\to\infty$ we have

$$\underset{\sim}{Y} \xrightarrow{d} N_k(\underset{\sim}{0},\underset{\sim}{\Sigma}) \ , \tag{2.7}$$

where $\underset{\sim}{\Sigma} = (\sigma_{ij})$, with

$$(\sigma_{ij}) = \begin{cases} \pi_i(1-\pi_i) & \text{if } i=j \\ \pi_{ij} - \pi_i\pi_j & \text{if } i\neq j \end{cases} \ .$$

Note that the normal approximation (2.7) depends only on the marginal binomial probabilities and the usual correlation coefficients. The generalized canonical correlations that appear in (2.5) do not influence the asymptotic behavior. The number of parameters has been reduced from 2^k-1 to $k+\binom{k}{2}$, and well known properties of the normal distribution can be invoked. This particular limiting distribution will be used to advantage in Chapter Four when discussing the monitoring of multivariate Bernoulli processes. A discussion of the accuracy of this approximation in the bivariate case is also provided in that chapter.

The dependence of (2.7) only on the univariate marginals and the correlations suggests an approximation based directly on (2.5), namely,

$$P(x_1,x_2,\ldots,x_k) \doteq p(x_1,\pi_1)\cdots p(x_k,\pi_k) \cdot \{1 + \sum_{i<j} \sum_{r=1}^{n} \rho_{ij}^r G_r(x_i;\pi_i)G_r(x_j;\pi_j)\} \ . \tag{2.8}$$

The expression (2.8) is somewhat reminiscent of (2.4), the bivariate binomial pmf. As such, the algorithm given in Appendix I could be modified to compute (2.8). No attempts have been made here to check the possible accuracy of the approximation (2.8). However, one would expect it to be fairly accurate if, for example, the generalized canonical correlations of the k-variate binomial distribution are negligible.

The multivariate Poisson and generalized Poisson distributions also arise as limiting forms of the multivariate binomial distribution, but a discussion of these properties will be delayed until the former distributions have been reviewed.

2.3 The Multivariate Multinomial Distribution

Multivariate binomial distributions were obtained by sampling from a 2^k table. If, in general, we think of sampling from a $c_1 x c_2 x \ldots x c_k$ table, then a multivariate multinomial distribution is obtained by considering appropriate marginals of the multi-dimensional contingency table.

Results concerning the small-sample properties of the multivariate multinomial distribution are somewhat limited. Wishart (1949) studied the cumulants of the multivariate multinomial distributions, and he gave a set of recurrence relationships. Tracy and Doss (1980) consider multivariate multinomial distributions which are also linear exponential. They show that a k-variate linear exponential distribution of $(\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_k)$ is multinomial if and only if each $\underset{\sim}{X}_i$ is multinomial.

Griffiths (1971) studied the orthogonal polynomials for the multinomial distribution, and he uses these to give a Lancaster-type expansion in two special cases. One case is the instance in which the bivariate multinomial is generated by taking $(\underset{\sim}{X}_1+\underset{\sim}{X}_3, \underset{\sim}{X}_2+\underset{\sim}{X}_3)$ where the $\underset{\sim}{X}_i$'s are independent multinomial random variables. He also provides the canonical correlations for the case when the bivariate multinomial has identically distributed marginals. Unfortunately, Griffiths's expression for the orthogonal polynomials of the multinomial distribution do not appear to be readily computable.

Our attention, therefore, will be restricted to asymptotic distributional properties of the multivariate multinomial distribution. Rather than outline these results here, they will be introduced as needed in the text.

2.4 Multivariate Poisson Distributions

What will be termed here as the multivariate Poisson distribution was first derived in the bivariate case as a limiting form of the bivariate binomial distribution of Aitken and Gonin (1935). Later work provided a different structure in terms of univariate Poisson distributions. Available recursive formulas make the computation of probabilities a considerably easier task for the multivariate Poisson distribution as compared with the multivariate binomial distribution. These and other results concerning the multivariate Poisson distribution are reviewed in this section.

Campbell (1934) first studied the bivariate Poisson distribution as a limiting distribution of certain marginal totals of a 2×2 table. In the notation of (2.1) and (2.2), with the additional assumption that π_{11}, π_1, and π_2 are all $O(n^{-1})$, Campbell showed that

$$P(X_1=x_1,X_2=x_2) = f(x_1;\lambda_1)f(x_2;\lambda_2) \cdot \{1 + \sum_{j=1}^{\infty} \rho^j K_j(x_1;\lambda_1)K_j(x_2;\lambda_2) \quad , \tag{2.9}$$

where (i) $\lambda_1 = \lim_{n\to\infty} n\pi_1$; $\lambda_2 = \lim_{n\to\infty} n\pi_2$;

(ii) $\rho = \dfrac{\lambda_{12}}{\sqrt{\lambda_1\lambda_2}}$, $\lambda_{12} = \lim_{n\to\infty} n\pi_{11}$;

(iii) $f(x;\lambda) = e^{-\lambda}\lambda^x/x!$;

(iv) $K_r(z;\lambda) = (-1)^r\Delta^r\{f(z-r;\lambda)/f(z;\lambda)\}$,

the Charlier polynomials; and

(v) $\Delta g(x) = g(x) - g(x-1)$

$\Delta^r g(x) = \Delta^{r-1}(g(x) - g(x-1))$.

The distribution of (2.9) will be denoted by $P_2(\underset{\sim}{\lambda},\rho)$.

So, one use of the bivariate Poisson distribution may be to approximate bivariate binomial probabilities; other uses will become apparent in later discussions. In Tables 2.1-2.6, some comparisons are made between bivariate binomial cumulative probabilities and their approximating bivariate Poisson probabilities for some selected cases. The tabled values give the differences $P(X_1\leq a, X_2\leq b) - P(Y_1\leq a, Y_2\leq b)$, where $(X_1,X_2)' \sim b_2(n,(\pi,\pi)',\rho)$ and $(Y_1,Y_2) \sim P_2((\lambda,\lambda)',\tau)$, with $\lambda = n\pi$ and $\tau = \pi_{11}/\pi$. The following parameter values are covered: $n=25,35$; $\rho=0.1,0.25,0.40$; and $\pi=0.01(0.005)0.05$.

For these parameter values, the approximation appears to be quite good. As one would expect, smaller values of π or larger values of n generally admit better approximations. It is somewhat surprising, however, that larger values for ρ generally yield a smaller absolute error for the approximation. The bivariate Poisson distribution also seems to assign greater probability to values nearer (0,0). The error of the

TABLE 2.1. $P(X \le A, Y \le B) - P(S \le A, T \le B)$, WHERE (X,Y) HAS A $b_2(N, (\pi,\pi), \rho)$ DISTRIBUTION, AND (S,T) IS ITS APPROXIMATING BIVARIATE POISSON DISTRIBUTION. N IS 25, AND RHO IS 0.1.

	(A,B)					
π	(0,0)	(0,1)	(1,1)	(1,2)	(2,2)	(3,3)
0.010	-0.0028	-0.0008	0.0013	0.0009	0.0004	0.0001
0.015	-0.0050	-0.0019	0.0019	0.0016	0.0011	0.0002
0.020	-0.0070	-0.0035	0.0018	0.0022	0.0021	0.0006
0.025	-0.0087	-0.0053	0.0011	0.0025	0.0033	0.0012
0.030	-0.0099	-0.0071	-0.0004	0.0022	0.0043	0.0021
0.035	-0.0106	-0.0089	-0.0242	0.0014	0.0051	0.0033
0.040	-0.0109	-0.0104	-0.0048	0.0001	0.0054	0.0045
0.045	-0.0109	-0.0116	-0.0074	-0.0016	0.0052	0.0059
0.050	-0.0107	-0.0125	-0.0101	-0.0038	0.0044	0.0072

TABLE 2.2. $P(X \le A, Y \le B) - P(S \le A, T \le B)$, WHERE (X,Y) HAS A $b_2(N, (\pi, \pi), \rho)$ DISTRIBUTION, AND (S,T) IS ITS APPROXIMATING BIVARIATE POISSON DISTRIBUTION. N IS 25, AND RHO IS 0.25.

	(A,B)					
π	(0,0)	(0,1)	(1,1)	(1,2)	(2,2)	(3,3)
0.010	-0.0025	-0.0008	0.0012	0.0009	0.0004	0.0001
0.015	-0.0045	-0.0019	0.0017	0.0016	0.0011	0.0002
0.020	-0.0064	-0.0033	0.0017	0.0021	0.0020	0.0006
0.025	-0.0081	-0.0049	0.0011	0.0023	0.0031	0.0012
0.030	-0.0094	-0.0066	-0.0002	0.0020	0.0041	0.0021
0.035	-0.0102	-0.0082	-0.0020	0.0013	0.0048	0.0031
0.040	-0.0108	-0.0096	-0.0041	0.0001	0.0051	0.0043
0.045	-0.0110	-0.0108	-0.0065	-0.0015	0.0049	0.0056
0.050	-0.0109	-0.0117	-0.0089	-0.0034	0.0042	0.0068

TABLE 2.3. $P(X \le A, Y \le B) - P(S \le A, T \le B)$, WHERE (X,Y) HAS A $b_2(N, (\pi,\pi), \rho)$ DISTRIBUTION, AND (S,T) IS ITS APPROXIMATING BIVARIATE POISSON DISTRIBUTION. N IS 25, AND RHO IS 0.4.

	(A,B)					
π	(0,0)	(0,1)	(1,1)	(1,2)	(2,2)	(3,3)
0.010	-0.0022	-0.0009	0.0011	0.0008	0.0004	0.0001
0.015	-0.0040	-0.0019	0.0016	0.0015	0.0010	0.0002
0.020	-0.0058	-0.0032	0.0017	0.0020	0.0019	0.0006
0.025	-0.0074	-0.0046	0.0011	0.0021	0.0029	0.0012
0.030	-0.0087	-0.0061	0.0000	0.0019	0.0038	0.0020
0.035	-0.0098	-0.0076	-0.0016	0.0012	0.0045	0.0030
0.040	-0.0105	-0.0089	-0.0035	0.0001	0.0048	0.0041
0.045	-0.0108	-0.0100	-0.0056	-0.0013	0.0047	0.0053
0.050	-0.0110	-0.0109	-0.0078	-0.0030	0.0041	0.0064

TABLE 2.4. $P(X \le A,\ Y \le B) - P(S \le A,\ T \le B)$, WHERE (X,Y) HAS A $b_2(N,\ (\pi,\pi),\rho)$ DISTRIBUTION, AND (S,T) IS ITS APPROXIMATING BIVARIATE POISSON DISTRIBUTION. N IS 35, AND RHO IS 0.1.

	(A,B)					
π	(0,0)	(0,1)	(1,1)	(1,2)	(2,2)	(3,3)
0.010	-0.0033	-0.0012	0.0013	0.0011	0.0007	0.0001
0.015	-0.0053	-0.0028	0.0012	0.0016	0.0017	0.0005
0.020	-0.0067	-0.0046	0.0002	0.0017	0.0028	0.0012
0.025	-0.0075	-0.0063	-0.0017	0.0010	0.0036	0.0023
0.030	-0.0079	-0.0078	-0.0042	-0.0004	0.0038	0.0036
0.035	-0.0076	-0.0088	-0.0068	-0.0024	0.0032	0.0049
0.040	-0.0072	-0.0094	-0.0093	-0.0048	0.0018	0.0060
0.045	-0.0065	-0.0095	-0.0114	-0.0074	-0.0004	0.0067
0.050	-0.0058	-0.0093	-0.0131	-0.0099	-0.0031	0.0067

TABLE 2.5. $P(X \le A, Y \le B) - P(S \le A, T \le B)$, WHERE (X,Y) HAS A $b_2(N, (\pi,\pi), \rho)$ DISTRIBUTION, AND (S,T) IS ITS APPROXIMATING BIVARIATE POISSON DISTRIBUTION. N IS 35, AND RHO IS 0.25.

	(A,B)					
π	(0,0)	(0,1)	(1,1)	(1,2)	(2,2)	(3,3)
0.010	-0.0029	-0.0012	0.0012	0.0010	0.0007	0.0001
0.015	-0.0048	-0.0027	0.0012	0.0015	0.0016	0.0005
0.020	-0.0063	-0.0042	0.0002	0.0015	0.0026	0.0012
0.025	-0.0073	-0.0058	-0.0014	0.0009	0.0034	0.0022
0.030	-0.0077	-0.0072	-0.0036	0.0004	0.0036	0.0034
0.035	-0.0078	-0.0082	-0.0060	-0.0022	0.0031	0.0047
0.040	-0.0075	-0.0089	-0.0084	-0.0043	0.0018	0.0057
0.045	-0.0070	-0.0092	-0.0104	-0.0066	-0.0001	0.0063
0.050	-0.0064	-0.0092	-0.0121	-0.0083	-0.0025	0.0063

TABLE 2.6. $P(X \le A, Y \le B) - P(S \le A, T \le B)$, WHERE (X,Y) HAS A $b_2(N, (\pi,\pi),\rho)$ DISTRIBUTION, AND (S,T) IS ITS APPROXIMATING BIVARIATE POISSON DISTRIBUTION. N IS 35, AND RHO IS 0.4.

	(A,B)					
π	(0,0)	(0,1)	(1,1)	(1,2)	(2,2)	(3,3)
0.010	-0.0026	-0.0012	0.0011	0.0010	0.0006	0.0001
0.015	-0.0044	-0.0025	0.0011	0.0014	0.0015	0.0005
0.020	-0.0059	-0.0039	0.0003	0.0014	0.0025	0.0012
0.025	-0.0070	-0.0054	-0.0011	0.0009	0.0032	0.0021
0.030	-0.0076	-0.0067	-0.0031	-0.0003	0.0034	0.0033
0.035	-0.0078	-0.0077	-0.0053	-0.0019	0.0030	0.0044
0.040	-0.0077	-0.0084	-0.0074	-0.0039	0.0019	0.0053
0.045	-0.0074	-0.0088	-0.0094	-0.0059	0.0003	0.0059
0.050	-0.0069	-0.0084	-0.0111	-0.0080	-0.0019	0.0060

approximation, therefore, is negative for values of (a,b) near (0,0), but it becomes positive as a and b increase.

Holgate (1964) gave an equivalent definition for the bivariate Poisson distribution (2.9) which may be more useful for some applications and interpretations. Following Holgate, let Y_1, Y_2 and U be independent Poisson random variables with parameters θ_1,θ_2 and ξ, respectively. Then, the random vector $\underset{\sim}{X} = (X_1,X_2)'$, where $X_i = Y_i+U$, i=1,2, is a bivariate Poisson distribution with parameters $\lambda_1 = \theta_1+\xi$, $\lambda_2 = \theta_2+\xi$, and $\rho = \xi/\sqrt{\lambda_1\lambda_2}$.

Still another model which produces the above bivariate Poisson distribution is given by Hamdan and Tsokos (1971). They showed that if the total number of occurrences of an event follows a Poisson distribution with parameter θ, and if an individual occurrence may be identified as having a bivariate binomial distribution with parameters π_1,π_2, and π_{11}, then a bivariate Poisson distribution is obtained with parameters $\lambda_1 = \theta\pi_1$, $\lambda_2 = \theta\pi_2$, and $\lambda_{11} = \theta\pi_{11}$. In other words, the bivariate Poisson may be obtained by compounding the parameter n of the bivariate binomial distribution with a univariate Poisson distribution.

One of the shortcomings of the particular bivariate Poisson distribution (2.9) is that the value of the correlation coefficient is restricted as follows

$$0 \leq \rho \leq \min\left(\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{2}}, \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1}{2}}\right) . \qquad (2.9a)$$

These bounds on the correlation coefficient may make the bivariate Poisson model inappropriate for some applications. Griffiths et al.

(1979) studied conditions under which bivariate distributions with Poisson marginals may be negatively correlated. Among other results, they show that for every bivariate distribution with fixed Poisson marginals, there is at least one distribution exhibiting a negative correlation.

As mentioned previously, the computation of bivariate Poisson probabilities may be carried out using recursive formulas; these are given by Teicher (1954). If $\underset{\sim}{X} = (X_1,X_2)'$ is $P_2(\underset{\sim}{\lambda},\rho)$ as given at (2.9), then

$$\begin{aligned} P(X_1=a,X_2=b) &= \left(\frac{\lambda_1-\lambda_2}{a}\right)P(X_1=a-1,X_2=b) \\ &\quad + (\lambda_{12/a})P(X_1=a-1,X_2=b-1) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} P(X_1=a,X_2=b) &= \left(\frac{\lambda_2-\lambda_1}{b}\right)P(X_1=a,X_2=b-1) \\ &\quad + (\lambda_{12/b})P(X_1=a-1,X_2=b-1) \ . \end{aligned}$$

If either a-1 or b-1 is negative in (2.10), then that term is zero. Also, $P(X_1=0,X_2=0) = \exp(-\{\lambda_1+\lambda_2-\lambda_{11}\})$.

If the marginals of the bivariate Poisson distribution are identical, then the "random elements in common" characterization of Holgate (1964), together with Theorem 1.6, assures that

$$P(X_1\varepsilon A,X_2\varepsilon A) \geq P(X_1\varepsilon A)P(X_2\varepsilon A) \ .$$

A somewhat stronger result, similar to Theorem 2.1, can be stated.

Theorem 2.2. Let (X_1,X_2) have pmf given by (2.9), with $\lambda_1=\lambda_2$. If B is any event measurable with respect to the Poisson distribution with parameter λ, then $P(X_1\varepsilon B,X_2\varepsilon B)$ is an increasing function of ρ.

<u>Proof</u>. With $\lambda=\lambda_1=\lambda_2$, observe that

$$P(X_1\varepsilon B, X_2\varepsilon B) = [P(X_1\varepsilon B)]^2 + \sum_{j=1}^{\infty} \sum_{x_1\varepsilon B} \rho^j [K_j(x_1;\lambda)f(x_1;\lambda)]^2$$

Upon taking the derivative with respect to ρ of the above probability we obtain,

$$\frac{\partial}{\partial\rho}P(X_1\varepsilon B, X_2\varepsilon B) = \sum_{j=1}^{\infty} \sum_{x_1\varepsilon B} j\rho^{j-1}[K_j(x_1;\lambda)f(x_1;\lambda)]^2 .$$

Since the terms of the sum are all positive, $P(X_1\varepsilon B, X_2\varepsilon B)$ is an increasing function of ρ.

Attempts to relax the assumption of symmetric marginals in the preceding theorem have been as unsuccessful here as they were in the case of the bivariate binomial distribution. However, there is empirical evidence from numerical studies of the cdf to suggest that the theorem, in fact, does hold in the more general case.

We now consider the general k-variate Poisson distribution. Krishnamoorthy (1951) extended the canonical expansion of the bivariate Poisson (2.9) to the multivariate case. Generalizing Campbell's (1934) approach, he considered the multivariate binomial distribution (2.5) as $n\to\infty$ in such a manner that $n\pi_i\to\lambda_i$, $n\pi_{ij}\to\lambda_{ij}$, ..., $n\pi_{12\ldots m}\to\lambda_{12\ldots m}$ $(i=1,\ldots,m, i<j)$. The resulting multivariate Poisson pmf is:

$$f(x_1,\ldots,x_m) = f(x_1,\lambda_1) \ldots f(x_m;\lambda_m)[\sum_{r=0}^{\infty} \frac{1}{r!}\{\sum_{i<j}\lambda_{ij}K(x_i;\lambda_i)K(x_j;\lambda_j) + \ldots + \lambda_{12\ldots m}K(x_1;\lambda_1) \ldots K(x_m;\lambda_m)\}^r] , \qquad (2.11)$$

where $\{\cdot\}^r$ is to be expanded formally using the multinomial theorem and the expression

$$\frac{K(x_i;\lambda_i)^{n_1}}{n_1!} \quad \frac{K(x_i;\lambda_i)^{n_2}}{n_2!}$$

is to be replaced by $K_{n_1+n_2}(x_i;\lambda_i)$ after expanding and collecting terms. The symbolic expression (2.11) shares the same drawback as the corresponding expression (2.5) for the multivariate binomial distribution, in not being amenable to calculation.

This drawback of the pmf is of less concern in this instance, however, as recursive formulas, similar to (2.10), can be obtained by differentiating the probability generating function (pgf) of the multivariate Poisson distribution. These formulas were also given by Teicher (1954) using a less direct approach.

But, even the recursive formulas do not alleviate the need to specify a large number of parameters. One recourse is to adopt an approximation similar to expression (2.8) which relies only on all the bivariate dependencies. Again, if the higher order generalized canonical correlations can be ignored, then such an approximation should not be too inaccurate. No attempt will be made here to check this contention, however.

Moving on to other matters concerning the multivariate Poisson distribution, the characterization of the bivariate Poisson distribution as a "random element in common" model also extends to the multivariate Poisson random variables given by (2.11) (cf. Johnson and Kotz (1968)). Let $Y_1,\ldots,Y_m$ and U be independent Poisson random variables with parameters $\theta_1,\ldots,\theta_m$ and ξ, respectively. Then $\underset{\sim}{X} = (X_1,\ldots,X_m)$, such that

$X_i = Y_i+U$, has the multivariate Poisson distribution (2.11). The correlation between X_i and X_j is given by $\xi/\sqrt{\lambda_i\lambda_j}$. Thus, all the simple correlations of this multivariate Poisson distribution are plagued by the same restrictions, (2.9a), found in the bivariate case.

The bivariate model given by Hamdan and Tsokos (1971) also can be generalized to cover the multivariate Poisson distribution. If the parameter n of an m-variate binomial distribution is compounded by a Poisson distribution with parameter θ, then the m-variate Poisson distribution of (2.11) results. The parameters of the multivariate Poisson distribution are given by: $\{\lambda_i=\theta\pi_i,\ \lambda_{ij}=\theta\pi_{ij},\ldots,\lambda_{12\ldots m} = \theta\pi_{12\ldots m}\}$.

Finally, a limiting normal form of the multivariate Poisson distribution was obtained by Teicher (1954). This result is given in the following theorem and extends the well-known univariate findings.

<u>Theorem 2.3</u>. If $\underset{\sim}{X} = (X_1,\ldots,X_m)'$ has the m-variate Poisson distribution (2.11), and if $\lambda_i\to\infty$, $\lambda_{ij}\to\infty$ $(i=1,\ldots,m;i<j)$ such that $\lambda_{ij}/\sqrt{\lambda_i\lambda_j} \to \rho_{ij}$, then

$$\underset{\sim}{Y} = (Y_1,\ldots,Y_m)' \xrightarrow{d} \underset{\sim}{N}_m(\underset{\sim}{0},\underset{\sim}{R}) ,$$

where $\underset{\sim}{R} = (\rho_{ij})$ and $Y_i = (X_i-\lambda_i)/\sqrt{\lambda_i}$.

2.5 Generalized Poisson Distributions

In this section the generalized Poisson distribution is introduced, and some pertinent properties of both the univariate and the multivariate versions are reviewed. Although the terminology of "generalized Poisson" has been adopted in this discussion, this class of distributions has also been referred to by other names. For instance, Feller

(1968, p.288) uses the terminology "compound Poisson", while Johnson and Kotz (1968, p.112) prefer "stuttering Poisson", and Janossy _et al._ (1950) opt for "composite Poisson".

The standard definition of a generalized random variable is as follows. Let X_1 and X_2 be random variables with pgf's $g_1(z_1)$ and $g_2(z_2)$, respectively. Then, $g_1(g_2(z))$ is the pgf of a generalized X_1 random variable, denoted by $X_1 \vee X_2$. A more general definition uses characteristic functions in place of the pgf's. However, since all of the applications in this study can be formulated in terms of pgf's, we use the former definition. Thus, the generalized Poisson family includes any random variable whose pgf can be expressed as

$$G(t) = \exp[\lambda(g(t)-1)] \ , \tag{2.12}$$

where $g(\cdot)$ is the pgf of a random variable X_2. From the definition of the pgf, (2.12) can be rewritten as:

$$G(t) = \exp[\lambda_1(t-1) + \lambda_2(t^2-1) + \dots] \ , \tag{2.13}$$

where $\lambda_i = \lambda P(X_2=i) \geq 0$ for all i.

Some well known discrete distributions belong to the generalized Poisson family. Clearly, the Poisson distribution, itself, is one, but other members include the negative binomial, Neyman Types A, B, and C, Poisson binomial, Poisson Pascal, and Hermite distributions. See, for example, Gurland (1963), and Kemp and Kemp (1965).

Recursive formulas for the probabilities of any generalized Poisson distribution can be given; these are essentially due to Neyman (1939). Starting with (2.13), we have

$$P(X=j+1) = \frac{1}{(j+1)!} \frac{d^{j+1}}{dt^{j+1}} G(t)\Big|_{t=0}$$

$$= \left(\frac{1}{j+1}\right) \sum_{\ell=0}^{j} (\ell+1)\lambda_{\ell+1} P(X=j-\ell) \qquad (2.14)$$

$$j = 0,\ 1,\ 2,\ \ldots .$$

and $P(X=0) = \exp(-\sum_{i=1}^{\infty} \lambda_i)$. Similar formulas can be derived for the non-central moments of (2.12). In particular, the mean and variance are

$$\mu = \sum_{i=1}^{\infty} i\lambda_i \qquad (2.15)$$

and

$$\sigma^2 = \sum_{i=1}^{\infty} i^2\lambda_i\ , \qquad (2.16)$$

respectively.

An expression for the pmf of a generalized Poisson variate has also been found by Maritz (1952) and Janossy et al. (1950), namely

$$P(X=b) = \exp(-\Sigma\lambda_i) \sum_{\mathcal{K}_b} \frac{\lambda_1^{k_1} \cdots \lambda_b^{k_b}}{k_1! k_2! \cdots k_b!} \quad b = 0,1,2,\ldots\ , \qquad (2.17)$$

where $\mathcal{K}_b = \{k_i | k_1+2k_2+\ldots+bk_b=b\ ,\ k_i$ integers such that $0 \leq k_i \leq b/i\ ,$
$i = 1,\ldots,b\}$.

The property that the Poisson distribution is stochastically increasing in its parameter leads to UMP tests for λ. A weaker property is true for the generalized Poisson distribution. First, we establish the following lemma.

Lemma 2.1. Let X be a generalized Poisson random variate. Then

$$\frac{\partial}{\partial \lambda_i} P(X \leq a) \leq 0$$

for all i.

Proof. From (2.17) we have

$$P(X \leq a) = \exp(-\Sigma\lambda_i)\left(1 + \sum_{b=1}^{a} \sum_{K_b} \prod_{i=1}^{b} \frac{\lambda_i^{k_i}}{k_i!}\right) .$$

Thus, it follows that

$$\frac{\partial}{\partial \lambda_i} P(X \leq a) = \exp(-\Sigma\, \lambda_i)\left[\sum_{b=1}^{a} \sum_{K_b} \frac{1}{k_1! \ldots k_b!} \frac{\partial}{\partial \lambda_i} [\lambda_1^{k_1} \ldots \lambda_b^{k_b}]\right.$$

$$\left. - \left(1 + \sum_{b=1}^{a} \sum_{K_b} \frac{\lambda_1^{k_1} \ldots \lambda_b^{k_b}}{k_1! \ldots k_b!}\right)\right]$$

$$= \exp(-\Sigma\lambda_i)$$

$$\cdot \left\{ \left[\sum_{b=1}^{a} \sum_{K_b \cap \{k_i | k_i \geq 1\}} \frac{\lambda_1^{k_1} \ldots \lambda_i^{k_i - 1} \ldots \lambda_b^{k_b}}{k_1! \ldots (k_i - 1)! \ldots k_b!} \right] \right.$$

$$\left. - \left(1 + \sum_{b=1}^{a} \sum_{K_b} \frac{\lambda_1^{k_1} \ldots \lambda_b^{k_b}}{k_1! \ldots k_b!}\right) \right\}$$

$$\leq 0. \tag{2.18}$$

The inequality in (2.18) follows since each of the nonzero terms in $\frac{\partial}{\partial \lambda_i} \sum_{K_b} [\cdot]$ corresponds to a term in $- \sum_{K_{b-i}} [\cdot]$ if b>i or -1 when b=i.

Lemma 2.1 will now be utilized to prove a stochastic ordering for generalized Poisson distributions induced by an ordering of its parameters.

Theorem 2.4. Let X and Y have generalized Poisson distributions with parameters $\lambda_1,\lambda_2,\ldots,$ and $\nu_1,\nu_2,\ldots,$ respectively. Suppose $\lambda_i \geq \nu_i$ for all i, with strict inequality for at least one i. Then X is stochastically larger than Y.

Proof. Suppose S_1 and S_2 are independent generalized Poisson variables with parameters $\alpha\lambda_1,\alpha\lambda_2,\ldots,$ and $(1-\alpha)\nu_1$, $(1-\alpha)\nu_2,\ldots,$ respectively, for $0\leq\alpha\leq 1$. Let $R = S_1+S_2$. The pgf of R is

$$\begin{aligned} G_R(t) &= \exp\{(\alpha\lambda_1+(1-\alpha)\nu_1)(t-1) \\ &\qquad + (\alpha\lambda_2+(1-\alpha)\nu_2)(t^2-1) + \ldots\} \\ &= \exp(\xi_1(t-1) + \xi_2(t^2-1) + \ldots) , \end{aligned}$$

where $\xi_i = \alpha\lambda_i + (1-\alpha)\nu_i$. Hence, R has a generalized Poisson distribution with parameters $(\xi_1,\xi_2,\ldots)$. Viewing the distribution of R as a function of $(\xi_1,\xi_2,\ldots)$, take the directional derivative of $P(R\leq a)$ with respect to α.

$$\frac{d}{d\alpha} P(R\leq a) = \sum_{i=1}^{\infty} [\frac{\partial}{\partial\xi_i} P(R\leq a)][\frac{d\xi_i}{d\alpha}]$$

By Lemma 2.1, $\frac{\partial}{\partial\xi_i} P(R\leq a) \leq 0$ for all i. Also $\frac{d\xi_i}{d\alpha} = \lambda_i-\nu_i\geq 0$, with strict inequality for some i. Hence, $\frac{d}{d\alpha} P(R\leq a)<0$. Upon taking $\alpha=0$ and $\alpha=1$, we have

$$P(X\leq a) \leq P(Y\leq a) .$$

Some consequences of Theorem 2.4 are given in the following corollaries.

Corollary 2.1. Suppose X_1 and X_2 are $P_1(\theta_1)$ and $P_1(\theta_2)$, respectively, with $\theta_1>\theta_2$. If X_1 and X_2 are generalized by the same discrete distribution Y, then $X_1 \vee Y$ is stochastically larger than $X_2 \vee Y$.

Proof. From (2.13), the parameters of $X_1 \vee Y$ and $X_2 \vee Y$ are $\lambda_{1i} = \theta_1 P(Y=i)$ and $\lambda_{2i} = \theta_2 P(Y=i)$, respectively, for $i=1,2,\ldots$. Hence, $\lambda_{1i} \geq \lambda_{2i}$ for all i, and Theorem 2.3 provides the conclusion of the corollary.

Corollary 2.2. Let Y_1 and Y_2 be two discrete random variables with $P(Y_1=i) \geq P(Y_2=i)$ for $i=1,2,\ldots$. For the Poisson random variable X, $X \vee Y_1$ is stochastically larger than $X \vee Y_2$.

Proof. $\lambda_{1i} = \theta P(Y_1=i) \geq \theta P(Y_2=i) = \lambda_{2i}$ for all i. Hence, $\underset{\sim}{\lambda}_1 > \underset{\sim}{\lambda}_2$ and $P((X \vee Y_1) \geq a) \geq P((X \vee Y_2) \geq a)$. The validity of Corollary 2.2 requires that Y_2 place more probability at 0 than Y_1, but less probability at each succeeding point. Certain binomial distributions, for instance, would satisfy these requirements on Y_1 and Y_2.

We now demonstrate in what sense a connection exists between the generalized Poisson and the multivariate binomial and Poisson distributions. The sum of the components of a multivariate Poisson random vector is a member of the class of generalized Poisson distributions. It then follows that a limiting distribution for the sum of the components of a multivariate binomial random vector is also in the class of generalized Poisson distributions (cf. Maritz (1952)).

Since this last fact will be utilized in later developments, the sense in which the sum of correlated binomial random variables converges to a generalized Poisson random variable is explicity formulated. The distribution of the total number of successes, S, from a k-variate

Bernoulli trial is given by

$$P(S=0) = \pi_{0\ldots0} ,$$
$$P(S=1) = \pi_{10\ldots0} + \ldots + \pi_{0\ldots01} , \ldots ,$$
$$P(S=k) = \pi_{11\ldots1} , \tag{2.19}$$

where $\pi_{i_1 i_2 \ldots i_k}$ is given by (2.4). The characteristic function for the total number of successes in n trials is

$$(1+(e^{it}-1)(\pi_{10\ldots0}+\ldots+\pi_{0\ldots01})+\ldots+(e^{ikt}-1)\pi_{1\ldots1})^n$$
$$= (1 + \frac{(e^{it}-1)n(\pi_{10\ldots0}+\ldots+\pi_{0\ldots01})+\ldots+(e^{ikt}-1)n\pi_{1\ldots1}}{n})^n .$$

Suppose we now let the number of trials, n, increase to infinity such that the following relationships hold

$$\lim_{n\to\infty} n(\pi_{10\ldots0}+\ldots+\pi_{0\ldots01}) = \lambda_1$$
$$\lim_{n\to\infty} n(\pi_{110\ldots0}+\ldots+\pi_{0\ldots011}) = \lambda_2$$
$$\vdots \tag{2.20}$$
$$\lim_{n\to\infty} n\,\pi_{1\ldots1} = \lambda_k .$$

Then the limiting characteristic function of the total number of successes in n trials under the conditions (2.20) is

$$\exp(\lambda_1(e^{it}-1) + \ldots + \lambda_k(e^{ikt}-1)) . \tag{2.21}$$

But (2.21) is simply the characteristic function of a generalized Poisson distribution with $\lambda_j=0$ for $j\geq k+1$.

An extension of the generalized Poisson distributions to more than one dimension will also prove useful in later developments. By analogy to the univariate generalized Poisson law we will define a k-variate generalized Poisson distribution as that corresponding to the pgf obtained by generalizing a k-variate Poisson distribution by another k-variate distribution. For instance, if $h_k(t_1,\ldots,t_k)$ is the pgf of some k-variate distribution, and $h_\ell(t_{i_1},\ldots,t_{i_\ell})$ is the pgf of an ℓ-variate marginal distribution of $h_k(t_1,\ldots,t_k)$, then the pgf of the k-variate generalized Poisson distribution is given by

$$G(t_1,\ldots,t_k) = \exp(-A + \sum_{i=1}^{k} a_i h_1(t_i) + \sum_{i<j} a_{ij} h_2(t_i,t_j) + \ldots + a_{12\ldots k} h_k(t_1,\ldots,t_k)) , \tag{2.22}$$

where $A = \sum_{i=1}^{k} a_i + \sum_{i<j} a_{ij} + \ldots + a_{12\ldots k}$.

In particular, the pgf of the bivariate generalized Poisson distribution can be expressed as

$$G(t_1,t_2) = \exp[\sum_{i=1}^{\infty} \lambda_{1i}(t_1^i-1) + \sum_{i=1}^{\infty} \lambda_{2i}(t_2^i-1) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \xi_{ij}(t_1^i t_2^j-1)] , \tag{2.23}$$

where

$$\lambda_{1i} = a_1 P(X_1=i) + a_{12} P(X_1=i,X_2=0)$$

$$\lambda_{2i} = a_2 P(X_2=i) + a_{12} P(X_1=0,X_2=i)$$

$$\xi_{ij} = a_{12} P(X_1=i,X_2=j) ,$$

$i,j=1,2,\ldots$, and (X_1,X_2) is the generalizing random variable. Some members of the class of bivariate generalized Poisson distributions are: the bivariate Poisson, negative trinomial, bivariate Neyman Type A (type ii and type iii) (cf. Holgate (1965)), and bivariate Poisson-binomial

(Charalambides and Papageorgiou (1981)) distributions.

For the remainder of this discussion our attention will be confined primarily to the bivariate distribution (2.23). There would appear to be no problem in extending the following results to the general k-variate distribution beyond the necessity of tracking more complicated expressions.

First, note that (2.23) yields marginal distributions which are generalized Poisson. For example,

$$G(t_1) = \exp((\lambda_{11}+\xi_{11}+\xi_{12}+\dots)(t_1-1) + (\lambda_{12}+\xi_{21}+\xi_{22}\dots)(t_1^2-1) + \dots) .$$

As in the univariate case, recursive formulas for the bivariate probabilities can be derived from the pgf. For $n_2>0$,

$$P(X_1=n_1,X_2=n_2) = \frac{1}{n_1!}\frac{1}{n_2!}\sum_{\ell=0}^{n_2-1}\sum_{m=0}^{n_1}\binom{n_2-1}{\ell}\binom{n_1}{m} \left[\frac{\partial^{\ell+m}}{\partial t_1^m \partial t_2^\ell}\eta(t_1,t_2)\right]_{t_1=t_2=0} P(X_1=n_1-m,X_2=n_2-\ell-1) \tag{2.24}$$

where

$$\eta(t_1,t_2) = \sum_{i=1}^{\infty} i\lambda_{2i}t_2^{i-1} + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} j\xi_{ij}t_1^i t_2^{j-1} .$$

Therefore, (2.24) yields

$$P(X_1=n_1,X_2=n_2) = \frac{1}{n_2}\sum_{\ell=0}^{n_2-1}(\ell+1)\lambda_{2,\ell+1}P(X_1=n_1,X_2=n_2-\ell-1$$

$$+\frac{1}{n_2}\sum_{m=1}^{n_1}\sum_{\ell=0}^{n_2-1}(\ell+1)\xi_{m,\ell+1}P(X_1=n_1-m,X_2=n_2-\ell-1)$$

for $n_1 = 0,1,2,\ldots \quad n_2 = 1,2,3,\ldots$; (2.25)

$$P(X_1=n_1,X_2=0) = \frac{1}{n_1}\sum_{m=0}^{n_1-1}(m+1)\lambda_{1,m+1}P(X_1=n_1-m-1,X_2=0)$$

for $n_1 = 1,2,\ldots$; and

$$P(X_1=0,X_2=0) = \exp(-(\sum_i\lambda_{1i}+\sum_j\lambda_{2j}+\sum_{ij}\sum\xi_{ij})) .$$

The means and variances of (2.23) can be obtained by identification with the univariate distribution (See (2.15) and (2.16).). For the covariance, referring to (2.23), we have

$$EX_1X_2 = \mu_1\mu_2 + \sum_i\sum_j ij\xi_{ij} ,$$

where $\mu_i = \sum_{j=1}^{\infty} j\lambda_{ij}$. Therefore, $\mathrm{Cov}(X_1,X_2) = \sum_{ij}\sum ij\xi_{ij} \geq 0$. Also, note that $\mathrm{Cov}(X_1,X_2) = 0$ if and only if $\xi_{ij}=0$, for all i,j. That is, X_1 and X_2 are uncorrelated if and only if they are independent.

A means for obtaining the multivariate Poisson distribution of Section Four was the "random component in common" model. A similar technique is sufficient for generating the bivariate generalized Poisson distribution (2.23), but, unlike the bivariate Poisson model, it is not necessary.

In order to demonstrate this last statement, suppose X_1, X_2, and U have independent generalized Poisson distributions with parameters $\underset{\sim}{\lambda}_1$, $\underset{\sim}{\lambda}_2$, and $\underset{\sim}{\xi}$, respectively. Consider the bivariate distribution of

$\underset{\sim}{Y} = (Y_1, Y_2)'$, where $Y_1 = X_1+U$ and $Y_2 = X_2+U$. Then the pgf of $\underset{\sim}{Y}$ is given by

$$Et_1^{Y_1}t_2^{Y_2} = Et_1^{X_1}t_2^{X_2}(t_1t_2)^U = Et_1^{X_1}Et_2^{X_2}E(t_1t_2)^U$$

$$= \exp[\sum_{i=1}^{\infty} \lambda_{1i}(t_1^i-1) + \sum_{i=1}^{\infty} \lambda_{2i}(t_2^i-1)$$

$$+ \sum_{i=1}^{\infty} \xi_i(t_1^it_2^i-1)] \, .$$

But, this pgf is of the type (2.23) with $\xi_{ij} = 0$ for $i \neq j$. This is a rather severe restriction on the type of generalizing random variables that are allowed.

For another property of the bivariate generalized Poisson distribution, recall from the preceding section that Hamdan and Tsokos (1971) produced a bivariate Poisson distribution by compounding a bivariate binomial distribution with a Poisson distribution. A reasonable conjecture is that a bivariate generalized Poisson distribution could be obtained in a similar manner with the univariate generalized Poisson distribution assuming the role previously taken by the Poisson distribution. If the bivariate generalized Poisson distribution is in its most general form, the demonstration of the veracity of this conjecture would entail enormously tedious calculations. Therefore, a special case of the generalized Poisson, the Hermite distribution of Kemp and Kemp (1965), is used to indicate the plausibility of the conjecture.

The pmf of the Hermite distribution is

$$P(X=n) = \exp(-(\lambda_1+\lambda_2)) \sum_{\{k_1+2k_2=n\}} \frac{1}{k_1!}\,\frac{1}{k_2!}\,\lambda_1^{k_1}\lambda_2^{k_2} .$$

So, the pgf of the distribution under consideration is:

$$G(t_1,t_2) = \sum_{n=0}^{\infty} (\pi_{00}+\pi_{10}t_1+\pi_{01}t_2+\pi_{11}t_1t_2)^n$$

$$\cdot \left\{\exp(-(\lambda_1+\lambda_2)) \sum_{\{k_1+2k_2=n\}} \frac{\lambda_1^{k_1}}{k_1!}\,\frac{\lambda_2^{k_2}}{k_2!}\right\}$$

$$= \exp(-(\lambda_1+\lambda_2))\Big\{1 + \sum_{n=1}^{\infty} \frac{((\pi_{00}+\pi_{10}t_1+\pi_{01}t_2+\pi_{11}t_1t_2)\lambda_1)^n}{n!}$$

$$+ \sum_{n=1}^{\infty} \frac{((\pi_{00}+\pi_{10}t_1+\pi_{01}t_2+\pi_{11}t_1t_2)^2\lambda_2)^n}{n!}$$

$$+ \left[\sum_{n=1}^{\infty} \frac{((\pi_{00}+\pi_{10}t_1+\pi_{01}t_2+\pi_{11}t_1t_2)\lambda_1)^n}{n!}\right]$$

$$\cdot \left[\sum_{n=1}^{\infty} \frac{((\pi_{00}+\pi_{10}t_1+\pi_{01}t_2+\pi_{11}t_1t_2)^2\lambda_2)^n}{n!}\right]\Big\}$$

$$= \exp(-(\lambda_1+\lambda_2))[\exp[\lambda_1(\pi_{00}+\pi_{10}t_1+\pi_{01}t_2+\pi_{11}t_1t_2)$$

$$+ \lambda_2(\pi_{00}+\pi_{10}t_1+\pi_{01}t_2+\pi_{11}t_1t_2)^2] . \tag{2.26}$$

After rearrangement of terms, (2.26) can be rewritten as:

$$\begin{aligned}\exp\{&\pi_{10}(\lambda_1+2\lambda_2\pi_{00})(t_1-1) + \pi_{01}(\lambda_1+2\lambda_2\pi_{00})(t_2-1)\\ &+ \pi_{11}(\lambda_1+2\lambda_2\pi_{00}+2\lambda_2\pi_{10}\pi_{01})(t_1t_2-1)\\ &+ \lambda_2\pi_{10}^2(t_1^2-1) + \lambda_2\pi_{01}^2(t_2^2-1) + 2\lambda_2\pi_{01}\pi_{11}(t_1t_2^2-1)\\ &+ 2\lambda_2\pi_{10}\pi_{11}(t_1^2t_2-1) + \lambda_2\pi_{11}^2(t_1^2t_2^2-1)\}\ . \end{aligned} \tag{2.27}$$

It is apparent that the pgf (2.27) is that of a bivariate generalized Poisson distribution whose marginal distributions are Hermite.

A further conjecture, generalizing the corresponding result for the multivariate Poisson distribution, is the following. If the parameter n of a k-variate binomial distribution is compounded by a generalized Poisson distribution, a k-variate generalized Poisson distribution results.

Still another result for the univariate generalized Poisson extends to the multivariate case. To wit, if a multivariate Poisson (binomial) random vector is partitioned into k subvectors of length $\ell_1,\ell_2,\ldots,\ell_k$, and if the components of these subvectors are added together, then the resulting distribution (limiting distribution) is a k-variate generalized Poisson law. Again this result is demonstrated for the bivariate case only, as the more general result is merely a matter of doing some more detailed bookkeeping.

For a random sample of size n, the characteristic function of the above bivariate distribution may be obtained as a limiting function of the following, derived from the multivariate binomial function, namely

$$(1 + \frac{(e^{it_1}-1)n(\pi_{10..0}+..+\pi_{00..01;0..0})+..+(e^{i\ell_1 t_1}-1)n\pi_{11..1;0..0}}{n}$$

$$+ \frac{(e^{it_2}-1)n(\pi_{0..0;10..0}+..+\pi_{0..01})+..+(e^{i\ell_2 t_2}-1)n\pi_{0..0;1..1}}{n}$$

$$+ \frac{(e^{it_1 t_2}-1)n(\pi_{10..0;10..0}+..+\pi_{0..01;0..01})+..+(e^{i(\ell_1 t_1+\ell_2 t_2)}-1)n\pi_{1..1}}{n})^n \tag{2.28}$$

Suppose, as $n\to\infty$, that

(1) $n(\pi_{10...0}+...+\pi_{0...01;0...0}) \to \lambda_{11}$

$n(\pi_{110...0}+...+\pi_{0...011;0...0}) \to \lambda_{12}$

$\vdots$

$n\pi_{1...1;0...0} \to \lambda_{1\ell_1}$

(2) $n(\pi_{0...0;10...0}+...+\pi_{0...01}) \to \lambda_{21}$

$n(\pi_{0...0;110...0}+...+\pi_{0...011}) \to \lambda_{22}$

$\vdots$

$n\pi_{0...0;1...1} \to \lambda_{2\ell_2}$

and (2.29)

(3) $n(\pi_{10...0;10...0}+...+\pi_{0...01;0...01}) \to \xi_{11}$

$n(\pi_{110...0;10...0}+...+\pi_{0...011;0...01}) \to \xi_{21}$

$\vdots$

$n\pi_{11...1} \to \xi_{\ell_1\ell_2}$.

Then (2.28) converges to the characteristic function of the bivariate generalized Poisson distribution (2.23), namely,

$$\exp\Big(\sum_{j=1}^{\ell_1} \lambda_{1j}(e^{it_1 j}-1) + \sum_{j=1}^{\ell_2} \lambda_{2j}(e^{it_2 j}-1)$$
$$+ \sum_{j=1}^{\ell_1} \sum_{k=1}^{\ell_2} \xi_{jk}(e^{it_1 j} e^{it_2 k}-1)\Big) .$$

Another property of the generalized Poisson distribution which will be of interest in the study of c-charts is now given. Once more, we show the result for the bivariate case only, but note that an extension to higher dimensions uses the same approach.

Theorem 2.5. Let (X_1,X_2) have a bivariate generalized Poisson distribution. For w_1 and w_2 positive integers, $w_1X_1+w_2X_2$ has a generalized Poisson distribution.

Proof. The pgf of $w_1X_1+w_2X_2$ is given by

$$\exp\Big[\sum_{i_1=1}^{\infty} \lambda_{1i_1}(t^{w_1 i_1}-1) + \sum_{j_1=1}^{\infty} \lambda_{2j_1}(t^{w_2 j_1}-1)$$
$$+ \sum_{j_2=1}^{\infty} \sum_{i_2=1}^{\infty} \xi_{i_2 j_2}(t^{w_1 i_2+w_2 j_2}-1)\Big]$$

$$= \exp\Big[\sum_{\{w_1 i_1=w_2 j_1=w_1 i_2+w_2 j_2\}} (\lambda_{1i_1}+\lambda_{2_{j_1}}+\xi_{i_2 j_2})(t^{w_1 i_1}-1)$$

$$+ \sum_{\left\{\begin{array}{l} w_1 i_1 \neq w_2 j_1 \\ w_1 i_1 \neq w_1 i_2+w_2 j_2 \end{array}\right\}} \lambda_{1i_j}(t^{w_1 i_1}-1)$$

$$+ \sum_{\left\{\begin{matrix} w_2j_1 \neq w_1i_1 \\ w_2j_1 \neq w_1i_2+w_2j_2 \end{matrix}\right\}} \lambda_{2j_1}(t^{w_2j_1}-1)$$

$$+ \sum_{\left\{\begin{matrix} w_1i_2+w_2j_2 \neq w_1i_1 \\ w_1i_2+w_2j_2 \neq w_2j_1 \end{matrix}\right\}} \xi_{i_2j_2}(t^{w_1i_2+w_2j_2}-1)$$

$$+ \sum_{\{w_1i_1=w_2j_2 \neq w_1i_1+w_2j_2\}} (\lambda_{1i_1}+\lambda_{2j_1})(t^{w_1i_1}-1)$$

$$+ \sum_{\{w_1i_1=w_1i_2+w_2j_2 \neq w_2j_1\}} (\lambda_{1i_1}+\xi_{i_2j_2})(t^{w_1i_1}-1)$$

$$+ \sum_{\{w_2j_1=w_1i_2+w_2j_2 \neq w_1i_1\}} (\lambda_{2j_1}+\xi_{i_2j_2})(t^{w_2j_1}-1)] \qquad (2.30)$$

But (2.30) is merely the pgf of a generalized Poisson distribution with some of its parameters set to zero.

The mean of $w_1X_1+w_2X_2$ from Theorem 2.4 is:

$$\sum_{i_1=1}^{\infty} w_1i_1\lambda_{1i_1} + \sum_{j_1=1}^{\infty} w_2j_1\lambda_{2j_1} + \sum_{i_2}\sum_{j_2} (w_1i_2+w_2j_2)\xi_{i_2j_2} .$$

The variance is given by:

$$\sum_{i_1} (w_1i_1)^2\lambda_{1i_1} + \sum_{j_1} (w_2j_1)^2\lambda_{2j_1} + \sum_{i_2}\sum_{j_2} (w_1i_2+w_2j_2)^2\xi_{i_2j_2} . \qquad (2.31)$$

An analogue of Theorem 2.3, concerning the asymptotic normality of the multivariate Poisson distribution, is available for the multivariate generalized Poisson distribution. For the k-variate generalized Poisson distribution of (2.22), the characteristic function can be written as:

$$\exp\Big(\sum_{j=1}^{k}\sum_{\ell=1}^{\infty}\lambda_{j,\ell}(e^{i\ell t_j}-1)$$

$$+\sum_{j<\ell}\sum_{m_j,m_\ell}\lambda_{j\ell,m_jm_\ell}(e^{i(m_jt_j+m_\ell t_\ell)}-1)+\ldots$$

$$+\sum_{m_1,\ldots,m_k}\lambda_{12\ldots k,m_1\ldots m_k}(e^{i(m_1t_1+\ldots+m_kt_k)}-1)\Big)\ , \qquad (2.32)$$

$m_i = 1,\ldots,\infty$, $i = 1,\ldots,k$, and m_i in $\lambda_{\ell_1\ldots\ell_n,m_1\ldots m_n}$ corresponds to ℓ_i. For example, $\lambda_{123,456}$ is the parameter coefficient of $(\exp[i(4t_1+5t_2+6t_3)]-1)$ in the expansion of the characteristic function.

Let $\overset{(a)}{\underset{\ell}{\Sigma}} X_{\ell_1\ldots\ell_n;m_1\ldots m_n}$ denote the sum over all $X_{\ell_1\ldots\ell_n;m_1\ldots m_n}$ containing the subscript a among the ℓ_j's. Also, let $\overset{[m_a=j]}{\underset{m}{\Sigma^a}}$ denote the sum over all $X_{\ell_1\ldots\ell_n;m_1\ldots m_n}$ such that $m_a=j$, where m_a corresponds to $\ell_i=a$. As an example we could have,

$$\overset{(1)}{\underset{\ell}{\Sigma}}\ \overset{[m_1=2]}{\underset{m}{\Sigma^1}}\ \lambda_{\ell_1\ell_2,m_1m_2} = \sum_{b=2}^{k}\sum_{c=1}^{\infty}\lambda_{1b,2c}\ .$$

The means, variances, and covariances of (2.32) can now be expressed as

$$\mu_i = \sum_{j=1}^{\infty} j(\lambda_{i,j} + \sum_{\ell}^{(i)} \sum_{m}^{[m_i=j]} \lambda_{\ell_1\ell_2,m_1m_2} + \ldots$$

$$+ \sum_{m}^{[m_i=j]} \lambda_{12\ldots k,m_1\ldots m_k}), \; i = 1,\ldots,k \; ;$$

$$\sigma_i^2 = \sum_{j=1}^{\infty} j^2(\lambda_{i,j} + \sum_{\ell}^{(i)} \sum_{m}^{[m_i=j]} \lambda_{\ell_1\ell_2,m_1m_2} + \ldots$$

$$+ \sum_{m}^{[m_i=j]} \lambda_{12\ldots k,m_1\ldots m_k}) \; , \; i = 1,\ldots,k \; ; \tag{2.33}$$

$$\sigma_{ij} = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} rs(\lambda_{ij,rs} + \sum_{\ell}^{(i,j)} \sum_{m}^{[m_i=r,m_j=s]} \lambda_{\ell_1\ell_2\ell_3,m_1m_2m_3}$$

$$+ \ldots + \sum_{m}^{[m_i=r,m_j=s]} \lambda_{12\ldots k,m_1m_2\ldots m_k}) \; , \; i<j \; , \; i,j = 1,\ldots,k \; .$$

Theorem 2.6. Let $(X_1,\ldots,X_k)$ have the k-variate generalized Poisson distribution with characteristic function (2.32), and consider $Y_j = (X_j-\mu_j)/\sigma_j$, $j=1,\ldots,k$. Suppose $\sigma_{ij},\sigma_j^2 \to \infty$ $(i<j,\ j=1,\ldots,k)$ such that $(\sigma_{ij}/\sigma_i\sigma_j) \to \rho_{ij}$. Then the limiting distribution of $\underset{\sim}{Y} = (Y_1,\ldots,Y_k)'$ is $N_k(\underset{\sim}{0},\underset{\sim}{R})$, where $\underset{\sim}{R} = (\rho_{ij})$ with $\rho_{ii} = 1$ and $\rho_{ij}(i\neq j)$ as given.

Proof. The characteristic function of $\underset{\sim}{Y}$ is:

$$\exp(-i[t_1(\frac{\mu_1}{\sigma_1})+\ldots+t_k(\frac{\mu_k}{\sigma_k})])\{\exp[\sum_{j=1}^{k} \sum_{\ell=1}^{\infty} \lambda_{j,\ell}(e^{i(t_j\ell/\sigma_i)}-1)$$

$$+ \sum_{j<\ell} \sum_{m_j,m_\ell} \lambda_{j\ell,m_jm_\ell}(e^{i(t_jm_j/\sigma_j+t_\ell m_\ell/\sigma_\ell)}-1) + \ldots$$

$$+ \sum_{m_1\ldots m_k} \lambda_{12\ldots k,m_1\ldots m_k}(e^{i(t_1m_1/\sigma_1 +\ldots+ t_km_k/\sigma_k)}-1)\}$$

$$= \exp(-i(\sum_{j=1}^{k} t_j(\frac{\mu_j}{\sigma_j}))[\exp(\sum_{j=1}^{k}\sum_{\ell=1}^{\infty}\lambda_{j,\ell}(\frac{it_j\ell}{\sigma_j} - \frac{t_j^2\ell^2}{2\sigma_j^2} + o(\sigma_j^{-2})))$$

$$+ \sum_{j<\ell}\sum_{m_j,m_\ell}\lambda_{j\ell,m_j m_\ell}(i(\frac{t_j m_j}{\sigma_j} + \frac{t_\ell m_\ell}{\sigma_\ell}) - \tfrac{1}{2}(\frac{t_j m_j}{\sigma_j} + \frac{t_\ell m_\ell}{\sigma_\ell})^2 + o(\sigma_j^{-2})) + \ldots$$

$$+ \sum_{m_1\ldots m_k}\lambda_{12\ldots k,m_1\ldots m_k}(i(\frac{t_1 m_1}{\sigma_1} + \ldots + \frac{t_k m_k}{\sigma_k}) - \tfrac{1}{2}(\frac{t_1 m_1}{\sigma_1} + \ldots + \frac{t_k m_k}{\sigma_k})^2$$

$$+ 0(\sigma_j^{-2}))]$$

$$= \exp(-i(\sum_{j=1}^{k} t_j\frac{\mu_j}{\sigma_j}))[\exp\{(i\sum_{j=1}^{k}\frac{t_j}{\sigma_j}\mu_j)$$

$$- \tfrac{1}{2}\sum_{j=1}^{k}\frac{t_j^2}{\sigma_j^2}\sigma_j^2 - \sum_{j<h}\frac{t_j t_h}{\sigma_j\sigma_h}\sigma_{jh} + 0(\sigma_j^{-2})\ ]$$

$$= \exp[-\tfrac{1}{2}(\sum_{j=1}^{k} t_j^2 + 2\sum_{j<h}\frac{\sigma_{jh}}{\sigma_j\sigma_h}\ t_j t_h) + 0(\sigma_j^{-2})]$$

$$\to \exp[-\tfrac{1}{2}(\sum_{j=1}^{k} t_j^2 + 2\sum_{j<h}\rho_{ih}t_j t_h)] \tag{2.34}$$

as σ_j^2 and $\sigma_{ij}\to\infty$ under the conditions of the theorem. But (2.34) is the characteristic function of a $N_k(\underset{\sim}{0},\underset{\sim}{R})$ random vector.

The results of this section will play a major role in Chapter Five, where control charts for monitoring the number of defects of a process are discussed.

2.6 Multivariate χ^2 Distributions

The chi-squared (χ^2) distribution plays a central role as a limiting distribution for certain statistics associated with the multinomial distribution. Multivariate χ^2 distributions arise, for example, as limiting distributions in multiple testing problems involving multinomial distributions. A distinction has to be drawn, however, between the random vector which in our terminology will be said to have a multivariate χ^2 distribution, and more general multivariate distributions having χ^2 marginals. Both types of distributions will be applicable to different problems to be studied in later chapters.

First, let us define the multivariate χ^2 distribution and examine some of its properties. Suppose $\underset{\sim}{W} = (w_{ij})$, a $k \times k$ random matrix, has a Wishart distribution with n degrees of freedom, the scale matrix $\underset{\sim}{\Sigma} = (\sigma_{ij})$, and the noncentrality matrix $\underset{\sim}{\Delta} = (\delta_{ij})$. That is, $L(\underset{\sim}{W}) = \mathcal{W}(n, \underset{\sim}{\Sigma}, \underset{\sim}{\Delta})$. Let $\underset{\sim}{X} = (\frac{w_{11}}{\sigma_{11}}, \frac{w_{22}}{\sigma_{22}}, \ldots, \frac{w_{kk}}{\sigma_{kk}})'$. Then $\underset{\sim}{X}$ is said to have a noncentral multivariate χ^2 distribution with n degrees of freedom, the scale matrix $\underset{\sim}{\Sigma}$, and the noncentrality matrix $\underset{\sim}{\Delta}$, denoted by $\chi_k^2(n, \underset{\sim}{\Sigma}, \underset{\sim}{\Delta})$. If $\underset{\sim}{\Delta} = \underset{\sim}{0}$, the adjective "noncentral" will be dropped, and $\underset{\sim}{X}$ will be denoted by $\chi_k^2(n, \underset{\sim}{\Sigma})$.

The central bivariate chi-squared distribution has been the most widely studied of the multivariate χ^2 distributions. Alternative infinite series expansions for the pdf have been given in terms of the Laguerre polynomials (Kibble (1941)) and univariate χ^2 distributions (cf. Siotani (1959) and Johnson (1962), among others). If $\underset{\sim}{U} = (U_1, U_2)' \sim \chi_2^2(n, \underset{\sim}{R})$, for $\underset{\sim}{R}$ a correlation matrix, then the expansion in Laguerre polynomials is given in terms of $X = (X_1,X_2)' = (U_1/2,U_2/2)'$ as:

$$f(x_1,x_2) = g(x_1)g(x_2) \sum_{r=0}^{\infty} \frac{\rho^{2r} r!\Gamma(n/2)}{\Gamma(\frac{n}{2}+r)} L_r(x_1;n/2)\ L_r(x_2;n/2) \qquad (2.35)$$

where

(i) $\quad g(z) = \dfrac{z^{n/2-1}e^{-z}}{\Gamma(n/2)}$;

(ii) $\quad L_a(y;n) = \sum_{k=0}^{a} (-1)^k \binom{a+n-1}{a-k} \dfrac{y^k}{k!}$,

i.e., the Laguerre polynomial of degree a. Jensen and Howe (1968) used (2.35) to evaluate selected points of the cdf of the bivariate χ^2 distribution.

In terms of its univariate χ^2 marginals, the bivariate χ^2 pdf, with $(Y_1,Y_2)' = U_1/(1-\rho^2),U_2/(1-\rho^2)'$, is given by

$$f(y_1,y_2) = \sum_{j=0}^{\infty} \frac{(1-\rho^2)^{n/2}\Gamma(\frac{n}{2}+j)\rho^{2j}}{\Gamma(n/2)j!} g_{n+2j}(y_1)\ g_{n+2j}(y_2) \qquad (2.36)$$

where $g_m(z) = \dfrac{z^{m/2-1}e^{-z/2}}{2^{m/2}\Gamma(m/2)}$ (cf. Siotani (1959)).

Unsurprisingly, complications arise for expressions of the general multivariate χ^2 pdf. Krishnamoorthy and Parthasarathy (1951) gave such an expression for $\underset{\sim}{U} \sim \chi_k^2(n,\underset{\sim}{R})$, where $\underset{\sim}{R} = (\rho_{ij})$ is a correlation matrix. Let $X_i = U_i/2$, $i=1,2,\ldots,k$; then the pdf of the k-variate χ^2 distribution is

$$f(x_1,\ldots,x_k) = g(x_1)\ldots g(x_k) \sum_{r=0}^{\infty} \frac{(\frac{n}{2})^{(r)}}{r!}$$

$$\cdot \{ \sum_{i<j} c_{ij} \frac{L(x_i;n/2)}{n/2} \frac{L(x_j;n/2)}{n/2} + \ldots$$

$$+ c_{12\ldots k} \cdot \frac{L(x_1;n/2)}{n/2} \cdots \frac{L(x_k;n/2)}{n/2} \}^r , \tag{2.37}$$

where $\{\cdot\}^r$ is to be expanded by the multinomial theorem and $[L(x_i;n/2)/(n/2)]^a$ is to be replaced by $L_a(x_i;n/2)/(n/2)^{(a)}$ after the expansion. Here, $c_{ij\ldots\ell} = (-1)^{p+1}|\underset{\sim}{I}_p - \underset{\sim}{\Omega}_{ij\ldots\ell}|$, where $\underset{\sim}{\Omega}_{ij\ldots\ell}$ is the correlation matrix for the p-variate marginal distribution of $(X_i, X_j, \ldots, X_\ell)'$.

It is clear that (2.37), while in a fairly compact mathematical form, is not amenable to computation. As a result, there have been several suggestions advanced for approximating the multivariate χ^2 distribution using series expansions. Tan and Wong (1978), for instance, advocate using a finite series expansion involving the Laguerre polynomials. Their approximation matches the mixed moments up to some specified order with those of the distribution. For the equicorrelated multivariate χ^2 distribution, Johnson (1962) suggested an approximation whose basis is the bivariate χ^2 pdf (2.36). Explicitly, for the $\chi_k^2(n,\underset{\sim}{R})$ cdf, with $\underset{\sim}{R} = (1-\rho)\underset{\sim}{I}_k + \rho\underset{\sim}{J}_k$, $0 \leq \rho \leq 1$, where $\underset{\sim}{J}_k$ is a k×k matrix all of whose elements are one, his approximation uses

$$P(Y_1 \le y_1, \ldots, Y_k \le y_k) = \sum_{j=0}^{\infty} \frac{(1-\rho^2)^{n/2}\Gamma(\frac{n}{2}+j)\rho^{2j}}{\Gamma(\frac{n}{2})j!} \cdot \prod_{i=1}^{k} P(S_j \le y_i) \ , \qquad (2.38)$$

where $S_j \sim \chi_1^2(n+2j,1)$.

The equicorrelated multivariate χ^2 distribution will be of interest in later chapters. Some effort was made, therefore, to evaluate multivariate χ^2 probabilities when $u_1=u_2=\ldots=u_k=u$ by truncating the infinite series in (2.38). This method of approximation appears to work reasonably well. A more detailed discussion, together with some tables of these types of percentiles, can be found in Appendix II.

We now focus our attention on other aspects of the multivariate chi-squared distribution which will be useful in later endeavors. The next result extends to the noncentral case a lemma of Krishnaiah (1968) for representing the multivariate χ^2 distribution in terms of normal random variables.

<u>Theorem 2.7</u>. Let $\underset{\sim}{Y} = (\underset{\sim}{y}_1,\ldots,\underset{\sim}{y}_k)'$, with $\underset{\sim}{y}_i$ of order $p\times 1$, have the distribution $N_{k,p}(\underset{\sim}{M},\underset{\sim}{\Omega})$, where $\underset{\sim}{M} = (\underset{\sim}{\mu}_1,\underset{\sim}{\mu}_2,\ldots,\underset{\sim}{\mu}_k)'$ and $\underset{\sim}{\Omega} = \underset{\sim}{B}\times\underset{\sim}{\Sigma}$. $\underset{\sim}{B}$ is a $k\times k$ positive definite matrix normalized to have ones along its diagonal, and $\underset{\sim}{\Sigma}$ is a $p\times p$ positive definite matrix. The joint distribution of X_1^2, $X_2^2,\ldots,X_k^2$, where $X_i^2 = \underset{\sim}{y}_i'\underset{\sim}{\Sigma}^{-1}\underset{\sim}{y}_i$ $(i=1,2,\ldots,k)$, is $\chi_k^2(p,\underset{\sim}{B},\underset{\sim}{M}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{M}')$.

<u>Proof</u>. Let $\underset{\sim}{Z} = \underset{\sim}{Y}\underset{\sim}{\Gamma}$ where $\underset{\sim}{\Gamma}'\underset{\sim}{\Sigma}\underset{\sim}{\Gamma} = \underset{\sim}{I}_p$. Then, $\underset{\sim}{Z} \sim N_{k,p}(\underset{\sim}{M}\underset{\sim}{\Gamma},\underset{\sim}{B}\times\underset{\sim}{I}_p)$. Therefore, we have

$$\underset{\sim}{Z}\underset{\sim}{Z}' = \underset{\sim}{W} \sim \mathcal{W}_K(p,\underset{\sim}{B},\underset{\sim}{\Delta}) \ ,$$

where $\underset{\sim}{\Delta} = \underset{\sim}{M}\underset{\sim}{\Gamma}\underset{\sim}{\Gamma}'\underset{\sim}{M}' = \underset{\sim}{M}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{M}'$. Now, note that the diagonal elements of $\underset{\sim}{W}$ are $(X_1^2,X_2^2,\ldots,X_k^2)$. The conclusion of the theorem follows from the definition of the multivariate χ^2 distribution.

A result from Das Gupta et al. (1972, p. 256) yields, directly, a monotonicity property for certain multivariate χ^2 cdf's.

Theorem 2.8. Let $\underset{\sim}{X} = (\underset{\sim}{X}_1', \underset{\sim}{X}_2', \ldots, \underset{\sim}{X}_k') \sim N_{kp}(\underset{\sim}{0}, \underset{\sim}{\Theta}_\lambda \times \underset{\sim}{I}_p)$. Additionally $\underset{\sim}{\Theta}_\lambda$ is a k×k positive definite matrix satisfying the following properties:

(i) $\underset{\sim}{\Theta}_\lambda = (\theta_{ij}(\underset{\sim}{\lambda}))$;

(ii) $\underset{\sim}{\Theta} = (\theta_{ij})$ is a k×k positive definite matrix;

(iii) $\theta_{ii}(\underset{\sim}{\lambda}) = \theta_{ii}$ and $\theta_{ij}(\underset{\sim}{\lambda}) = \lambda_i \lambda_j \theta_{ij}$ $(i \neq j)$;

and

(iv) $\underset{\sim}{\lambda} = (\lambda_1, \ldots, \lambda k)'$, $0 \le \lambda_j \le 1$ $j=1,\ldots,k$.

Then, $P(\underset{\sim}{X}_1' \underset{\sim}{X}_1 \le a_1, \ldots, \underset{\sim}{X}_k' \underset{\sim}{X}_k \le a_k)$ is nondecreasing in each λ_j. That is, the cdf of $\chi_k^2(p, \underset{\sim}{\Theta}_\lambda)$ is nondecreasing in each λ_j.

Šidák (1973) applied Theorem 1.6 to the equicorrelated multivariate chi-squared distribution to obtain another probability inequality. If $\underset{\sim}{X} \sim \chi_k^2(n, \underset{\sim}{R})$, where $\underset{\sim}{R} = (1-\rho)\underset{\sim}{I}_k + \rho \underset{\sim}{J}_k$, $0 \le \rho \le 1$, then we have for the set A,

$$P\left(\bigcap_{i=1}^{k} \{X_i \varepsilon A\}\right) \ge \left[P\left(\bigcap_{i=1}^{k} \{X_i \varepsilon A\}\right)\right]^{k/k-1} \ge \ldots \ge \prod_{i=1}^{k} P(X_i \varepsilon A). \tag{2.39}$$

Some limited results for the noncentral multivariate χ^2 distribution will now be explored. The following lemma leads to one such property.

Lemma 2.2 Let $\underset{\sim}{X} \sim N_{nk}(\underset{\sim}{0}, \underset{\sim}{R} \times \underset{\sim}{I}_n)$, where $\underset{\sim}{X} = (\underset{\sim}{X}_1', \ldots, \underset{\sim}{X}_k')'$ and $\underset{\sim}{X}_i$ is of order n, i=1,...,k, and $f(\underset{\sim}{x})$ is the pdf of $\underset{\sim}{X}$. Define the set A as follows:

$$A = \{\underset{\sim}{X} | \underset{\sim}{X}_1'\underset{\sim}{X}_1 \leq u_1, \ldots, \underset{\sim}{X}_k'\underset{\sim}{X}_k \leq u_k\}.$$

Then for $\underset{\sim}{\mu}_2 = (\underset{\sim}{\mu}_{21}', \ldots, \underset{\sim}{\mu}_{2k}')' \varepsilon R^{nk}$, partitioned conformably with $\underset{\sim}{X}$,

$$\int_A f(\underset{\sim}{x} + \underset{\sim}{\mu}_1) d\underset{\sim}{x} \geq \int_A f(\underset{\sim}{x} + \underset{\sim}{\mu}_2) d\underset{\sim}{x} , \tag{2.40}$$

for any $\underset{\sim}{\mu}_1 \varepsilon R^{nk}$ in the convex hull of the set

$$\{\underset{\sim}{y} | \underset{\sim}{y} = (\underset{\sim}{Q}\underset{\sim}{\mu}_{21}, \underset{\sim}{Q}\underset{\sim}{\mu}_{22}, \ldots, \underset{\sim}{Q}\underset{\sim}{\mu}_{2k})', \underset{\sim}{Q}'\underset{\sim}{Q} = \underset{\sim}{I}_n\} . \tag{2.41}$$

<u>Proof</u>. The proof requires an application of Theorem 1.5. First, therefore, we show that the set $M = \{\underset{\sim}{M}: \underset{\sim}{M} = \text{diag}(\underset{\sim}{Q}, \ldots, \underset{\sim}{Q}) = \underset{\sim}{I}_k \times \underset{\sim}{Q}, \underset{\sim}{Q}'\underset{\sim}{Q} = \underset{\sim}{I}_n\}$ forms a subgroup of the orthogonal group. Clearly, the set is a non-empty subset of the set of orthogonal matrices. Let $\underset{\sim}{M}_1, \underset{\sim}{M}_2 \varepsilon M$. Then $\underset{\sim}{M}_1\underset{\sim}{M}_2^{-1} = \underset{\sim}{M}_1\underset{\sim}{M}_2' = (\underset{\sim}{I}_k \times \underset{\sim}{Q}_1)(\underset{\sim}{I}_k \times \underset{\sim}{Q}_2') = \underset{\sim}{I}_k \times \underset{\sim}{Q}_1\underset{\sim}{Q}_2'$. But $(\underset{\sim}{Q}_1\underset{\sim}{Q}_2')(\underset{\sim}{Q}_2\underset{\sim}{Q}_1') = \underset{\sim}{I}_n$, therefore $\underset{\sim}{M}_1\underset{\sim}{M}_2^{-1} \varepsilon M$. That is, M is a subgroup of the orthogonal group. So, for $\underset{\sim}{M}_1 \varepsilon M$,

$$\begin{aligned} L(\underset{\sim}{M}_1\underset{\sim}{X}) &= nk(\underset{\sim}{0}, \underset{\sim}{M}_1(\underset{\sim}{R} \times \underset{\sim}{I}_n)\underset{\sim}{M}_1') \\ &= nk(\underset{\sim}{0}, (\underset{\sim}{I}_k \times \underset{\sim}{Q}_1)(\underset{\sim}{R} \times \underset{\sim}{I}_n)(\underset{\sim}{I}_k \times \underset{\sim}{Q}_1')) \\ &= nk(\underset{\sim}{0}, \underset{\sim}{R} \times \underset{\sim}{I}_n) , \end{aligned}$$

i.e., $L(\underset{\sim}{X})$ is M-invariant. Also, note that the set A is convex and M-invariant, since $\underset{\sim}{M}_1 A = \{\underset{\sim}{X} | \underset{\sim}{X}_1'\underset{\sim}{Q}_1'\underset{\sim}{Q}_1\underset{\sim}{X}_1 \leq u_1, \ldots, \underset{\sim}{X}_k'\underset{\sim}{Q}_1'\underset{\sim}{Q}_1\underset{\sim}{X}_k \leq u_k\} = A$.

Since (2.41) describes the M-orbit of $\underset{\sim}{\mu}_2$, the conditions of Theorem 1.5 are met, implying the validity of (2.40).

Expression (2.40) implies a stochastic ordering between two non-central multivariate χ^2 distributions. It remains, however, to interpret the exact meaning of this stochastic ordering. This interpretation is contained in the following theorem.

Theorem 2.9 Let $\underset{\sim}{X}$ have the distribution $\chi_k^2(n,\underset{\sim}{R},\underset{\sim}{\Delta})$ and $\underset{\sim}{Y}$ have the distribution $\chi_k^2(n,\underset{\sim}{R},\underset{\sim}{\Theta})$, such that $\underset{\sim}{\Theta} = \lambda\underset{\sim}{\Delta}$, $0\leq\lambda\leq 1$. Then $\underset{\sim}{X}$ is stochastically larger than $\underset{\sim}{Y}$.

Proof. For $\underset{\sim}{\mu}_1$ and $\underset{\sim}{\mu}_2$ as in Lemma 2.2, take $\underset{\sim}{\mu}_1 = \lambda_1\underset{\sim}{\mu}_2 = (\lambda_1\underset{\sim}{\mu}_{21}', \ldots, \lambda_1\underset{\sim}{\mu}_{2k}')'$, $\lambda_1 \epsilon [0,1]$. The choice of $\underset{\sim}{Q} = -\underset{\sim}{I}_n$ in (2.41) shows $\underset{\sim}{\mu}_1$ to be in the convex hull of the set (2.41). Now, let $\underset{\sim}{\mu}_2^* = [(\underset{\sim}{Q}\underset{\sim}{\mu}_{21})', \ldots, (\underset{\sim}{Q}\underset{\sim}{\mu}_{2k})']'$ for $\underset{\sim}{Q}$ orthogonal. $\underset{\sim}{\mu}_2^*$ is then a point on the $\mathcal{M}$-orbit of $\underset{\sim}{\mu}_2$. By the same argument as above $\underset{\sim}{\mu}_1^* = \lambda_1\underset{\sim}{\mu}_2^*$ is also in the convex hull of the set (2.41). Furthermore, all the points $\lambda_1\underset{\sim}{\mu}_2$ generated in this fashion by varying $\underset{\sim}{Q}$ and λ_1 are in the convex hull of (2.41). That is, any point which can be expressed as

$$\underset{\sim}{\mu}_1 = (\lambda_1(\underset{\sim}{Q}\underset{\sim}{\mu}_{21})', \ldots, \lambda_1(\underset{\sim}{Q}\underset{\sim}{\mu}_{2k})')' \tag{2.42}$$

is in the convex hull of (2.41).

From Lemma 2.2, $\underset{\sim}{X}$ is stochastically larger than $\underset{\sim}{Y}$ for all $\underset{\sim}{\mu}_1$ given by (2.42). This restriction implies the noncentrality matrix of $\underset{\sim}{Y}$ is related to the noncentrality matrix of $\underset{\sim}{X}$ in the following fashion

$$\begin{aligned} \underset{\sim}{\Theta} &= \begin{bmatrix} \underset{\sim}{\mu}_{11}' \\ \vdots \\ \underset{\sim}{\mu}_{1k}' \end{bmatrix} [\underset{\sim}{\mu}_{11}, \ldots, \underset{\sim}{\mu}_{1k}] \\ &= \lambda_1^2 \begin{bmatrix} \underset{\sim}{\mu}_{21}'\underset{\sim}{Q}' \\ \vdots \\ \underset{\sim}{\mu}_{2k}'\underset{\sim}{Q}' \end{bmatrix} [\underset{\sim}{Q}\underset{\sim}{\mu}_{21}, \ldots, \underset{\sim}{Q}\underset{\sim}{\mu}_{2k}] \end{aligned}$$

$$= \lambda_1^2 \underset{\sim}{\Delta}.$$

An important special case of the preceding theorem is stated in the following corollary. Its proof is immediate.

Corollary 2.3 Let $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ be as stated in Theorem 2.9 with

$$\underset{\sim}{\Delta} = \begin{bmatrix} \underset{\sim}{0}_{k_1} \\ \underset{\sim}{\mu}_1 \\ \vdots \\ \underset{\sim}{\mu}_1' \end{bmatrix} [\underset{\sim}{0}_{k_1} \underset{\sim}{\mu}_1 \cdots \underset{\sim}{\mu}_1] = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{\mu}_1'\underset{\sim}{\mu}_1 \underset{\sim}{J}_{k-k_1} \end{bmatrix}$$

$$\underset{\sim}{\Theta} = \begin{bmatrix} \underset{\sim}{0}_{k_1} \\ \underset{\sim}{\mu}_2' \\ \vdots \\ \underset{\sim}{\mu}_2' \end{bmatrix} [\underset{\sim}{0}_{k_1} \underset{\sim}{\mu}_2 \cdots \underset{\sim}{\mu}_2] = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{\mu}_2'\underset{\sim}{\mu}_2 \underset{\sim}{J}_{k-k_1} \end{bmatrix} .$$

If $\underset{\sim}{\mu}_2'\underset{\sim}{\mu}_2 = \lambda \underset{\sim}{\mu}_1'\underset{\sim}{\mu}_1$, $0\le\lambda\le1$, then $\underset{\sim}{X}$ is stochastically larger than $\underset{\sim}{Y}$.

Heretofore our attention has been restricted to multivariate χ^2-type distributions with symmetric marginals. A more general class of multivariate χ^2 distributions is obtained if the joint distribution is formed in a way now to be described. Let $\underset{\sim}{X} = (\underset{\sim}{X}_1', \underset{\sim}{X}_2', \ldots, \underset{\sim}{X}_k')' \sim N_{k(\sum_i n_i)}(\underset{\sim}{0}, \underset{\sim}{\Sigma})$ where $\underset{\sim}{X}_i$ is a $(n_i \times 1)$ vector and

$$\underset{\approx}{\Sigma} = \begin{bmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} & \cdots & \underset{\sim}{\Sigma}_{1k} \\ \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22} & \cdots & \underset{\sim}{\Sigma}_{2k} \\ \vdots & & \ddots & \\ \underset{\sim}{\Sigma}_{k1} & \cdots & & \underset{\sim}{\Sigma}_{kk} \end{bmatrix} ,$$

with $\underset{\sim}{\Sigma}_{ij}$ a matrix of order $(n_i \times n_j)$. Then

$$(\underset{\sim}{X}_1' \underset{\sim}{\Sigma}_{11}^{-1} \underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_k' \underset{\sim}{\Sigma}_{kk}^{-1} \underset{\sim}{X}_k) \tag{2.43}$$

is a multivariate distribution with χ^2 marginals, but not necessarily of the type considered previously. Note that the marginals in (2.43) have possibly different degrees of freedom.

Jensen (1970a) studied the bivariate χ^2 distribution of the type (2.43). He provided an expansion of the pdf in the Laguerre polynomials analogous to (2.35), although the canonical correlations are complicated by an additional weighting function. In another paper, Jensen (1970b) considered the general case and gave an expression for the pdf analagous to (2.37). In both instances the greater generality increases the complexity of any computational procedure.

The following theorem and corollary, due to Dykstra (1980), will be useful in dealing with distributions of the type (2.43). These results provide bounds on the probabilities associated with (2.43), and they can be used to devise approximate procedures in multicharacteristic quality control.

<u>Theorem 2.10</u> Suppose $\underset{\sim}{Y} = (\underset{\sim}{Y}_1', \underset{\sim}{Y}_2')'$ is $N_m(\underset{\sim}{0}, \underset{\sim}{\Sigma})$ where

$$\underset{\sim}{\Sigma} = \begin{bmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22} \end{bmatrix}$$

is partitioned conformably with $\underset{\sim}{Y}$. Let C be an arbitrary convex set symmetric about $\underset{\sim}{0}$. If $\underset{\sim}{A}$ is idempotent, and c is a positive constant,

$$P(\underset{\sim}{Y}_1 \varepsilon C,\ \underset{\sim}{Y}_2'\underset{\sim}{A}\underset{\sim}{Y}_2 \leq c) \geq P(\underset{\sim}{Y}_1 \varepsilon C)P(\underset{\sim}{Y}_2'\underset{\sim}{A}\underset{\sim}{Y}_2 \leq c). \tag{2.44}$$

Proof. Dykstra (1980).

Corollary 2.4. Let $\underset{\sim}{X} = (\underset{\sim}{X}_1', \underset{\sim}{X}_2', \ldots, \underset{\sim}{X}_k') \sim N_m(\underset{\sim}{0}, \underset{\sim}{\Sigma})$ where $\underset{\sim}{\Sigma} = (\underset{\sim}{\Sigma}_{ij})$, $\underset{\sim}{\Sigma}_{ii} = \underset{\sim}{I}_{n_i}$ $i,j=1,\ldots,k$. Then

$$P(\underset{\sim}{X}_i'\underset{\sim}{A}_i\underset{\sim}{X}_i \leq c_i,\ i=1,\ldots,k) \geq \prod_{i=1}^{k} P(\underset{\sim}{X}_i'\underset{\sim}{A}_i\underset{\sim}{X}_i \leq c_i) \tag{2.45}$$

for all positive constants $c_1, \ldots, c_k$ and idempotent matrices $\underset{\sim}{A}_i$.

Proof. Dykstra (1980).

Probability inequalities for the noncentral distributions similar to those given in Theorem 2.9 are not available for the more general correlation structure of (2.43) using the same approach. The reflection group arises as the appropriate group of invariant transformations, but this is simply a restatement of Anderson's theorem, Theorem 1.4.

III. MONITORING UNIVARIATE BERNOULLI AND MULTINOMIAL PROPORTIONS

3.1 Introduction

The most commonly encountered situation in the process control of attributes is the monitoring of the proportion defective of a Bernoulli process. The widely accepted solution to this problem invokes central limit theory for independent and identically distributed (i.i.d.) Bernoulli random variables. An extension from the Bernoulli process to a multinomial process, where a single attribute may be classified into more than two states, apparently has not been considered in the quality control setting of process control.

Throughout this chapter the problem studied is assumed to arise in the following manner. Independent random samples from a Bernoulli or multinomial process are taken at specified intervals over time. That is, we observe $(\underset{\sim}{X}_1,\underset{\sim}{X}_2,\ldots)$ where $\underset{\sim}{X}_i = (X_{i1},\ldots,X_{in_i})'$ with

$$X_{ik} = \begin{cases} 1 & \text{with probability } \pi_i \\ 0 & \text{with probability } 1-\pi_i \end{cases}$$

for Bernoulli processes, or $\underset{\sim}{X}_i = (\underset{\sim}{X}'_{i1},\ldots,\underset{\sim}{X}'_{in_i})'$ with $\underset{\sim}{X}_{ik} = (X_{ijk})$ and

$$X_{ijk} = \begin{cases} 1 & \text{with probability } \pi_{ij} \\ 0 & \text{with probability } 1-\pi_{ij} \end{cases} \qquad j=1,\ldots,c$$

for multinomial processes. If $X_{ijk} = 1$, then $X_{ij'k} = 0$, for $j\neq j'$. For drifting Bernoulli processes, $\pi_i\neq\pi_{i'}$ for some $i\neq i'$, otherwise the process is stationary. Similarly $\underset{\sim}{\pi}_i\neq\underset{\sim}{\pi}_{i'}$ for multinomial processes. The objective is to determine as quickly as possible whenever the parameter π_i, or $\underset{\sim}{\pi}_i$ in the case of multinomial processes, shifts to an out-of-control state.

Whenever possible, control charts based upon exact sampling distributions are introduced and their properties examined. Also, the more traditional large-sample approach is studied. In Section Two, the monitoring of a Bernoulli process is considered under Case I, i.e., the quality standard is specified, and some optimality properties of the standard p-chart are given. Next, still under Case I, a control chart based on the Pearson X^2 statistic is introduced for the purpose of monitoring multinomial processes.

In Section Four we turn to the problem of monitoring a Bernoulli process under Case II, that is, when the quality standard is estimated. Some alternatives to the standard process control techniques are examined for both small and large samples. The chapter is closed with a somewhat detailed examination of the properties of a multinomial process monitoring scheme for Case II. This procedure is based upon the chi-squared test for homogeneity of multinomial proportions.

3.2 Monitoring Bernoulli Processes - Case I: The p-Chart

In this section our attention is confined to monitoring Bernoulli processes when the in-control proportion defective is specified. If π_0 denotes the target value for the percent defective, the standard upper and lower control limits against which the sample percent defective is monitored are given by

$$\pi_0 \pm K(\pi_0(1-\pi_0)/n)^{\frac{1}{2}} , \qquad (3.1)$$

where n is the sample size in the ith monitoring period, assumed to be the same for all periods. In (3.1) K is some constant; the usual spirit of quality control invokes the choice of three for this constant. In the terminology of quality control this control chart is referred to as the p-chart; for a thorough exposition of the pertinent methodology see, for instance, Duncan (1974).

The rationale used in obtaining the limits (3.1) is the observation that, as $n \to \infty$, $p_i = \sum_{j=1}^{n} X_{ij}/n$ has the limiting distribution $N_1(\pi_0, \pi_0(1-\pi_0)/n)$ when the process is in control. Therefore, the individual hypothesis tests with acceptance region defined by (3.1) are asymptotically uniformly most powerful unbiased (UMPU) tests at some level. Likewise, if only positive departures from the in-control state are of interest, then the upper control limit in (3.1) defines the critical value of a uniformly most powerful (UMP) test. By invoking Lemma 1.1, the control chart based upon (3.1) will be asymptotically dominant among all unbiased control charts of the Shewhart type. For the one-sided analogue a stronger statement may be made, as the adjective "unbiased" may be omitted from the preceding sentence.

While asymptotic results are often of interest - many times being the statistician's only recourse - whenever feasible, small-sample properties of a procedure deserve greater attention. When monitoring a Bernoulli process under Case I, optimal control charts can be given for any sample size. As before, these statements are based upon well-known results in statistical inference.

To obtain a specified level in testing for the parameter of a discrete random variable often requires a randomized test. Likewise, if a specific level is needed for monitoring the attributes of a process, the control limits will have a randomized portion. For example, if upper and lower randomized control limits are established, then they will be denoted by (L,γ_L,U,γ_U). The interpretation of these limits is as follows. If $T(\underset{\sim}{X})$ represents the control chart value, then the chart signals if $T(\underset{\sim}{X}) > U$ or $T(\underset{\sim}{X}) < L$. Also the charts signals with probability γ_U if $T(\underset{\sim}{X}) = U$ or with probability γ_L if $T(\underset{\sim}{X}) = L$.

With this background, we first consider optimal p-charts for detecting an increase in the process proportion defective.

<u>Theorem 3.1</u>. Let $(\underset{\sim}{X}_1,\underset{\sim}{X}_2,\ldots)$ represent successive samples from the monitoring period. In the class of control charts based upon the statistics $(Y_1,Y_2,\ldots)$, where $Y_i = f(\underset{\sim}{X}_i)$, consider the p-chart using the percent defective together with control limits $(U/n,\gamma)$ $(0\le\gamma\le1)$ such that

$$P(b_1(n,\pi_0)>U) + \gamma P(b_1(n,\pi_0)=U) = \alpha. \tag{3.2}$$

For detecting increases in π_0, this p-chart dominates any other control chart of level α in the class of charts using $(Y_1,Y_2,\ldots)$.

<u>Proof</u>. The monotone likelihood ratio property of the binomial family guarantees that the choice of (U,γ) as in (3.2) yields a UMP test for $\pi\le\pi_0$ versus $\pi>\pi_0$ at each sampling occasion. See, e.g., Lehmann (1959). Combining this fact with the independence of the test statistics among sampling occasions allows an application of Lemma 1.1, which completes the proof.

In the next theorem control limits are given for a control chart optimal for detecting any type of departure from a specified level of proportion defective.

Theorem 3.2. Let $(\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots)$ and $(Y_1, Y_2, \ldots)$ be as in Theorem 3.1. Consider the p-chart with control limits $(L/n, \gamma_\ell, U/n, \gamma_u)$ given by

$$P(L<b_1(n,\pi_0)<U) + (1-\gamma_\ell)P(b_1(n,\pi_0)=L) + (1-\gamma_u)P(b_1(n,\pi_0)=U) = 1-\alpha, \quad (3.3)$$

and

$$P(L\leq b_1(n-1,\pi_0)\leq U-2) + (1-\gamma_\ell)P(b_1(n,\pi_0)=L-1) + (1-\gamma_u)P(b_1(n-1,\pi_0)=U-1) = 1-\alpha ,$$

where $0\leq\gamma_i<1$, $i=\ell,u$. Then this p-chart dominates any other control chart of level α, in the class of unbiased control charts satisfying $Y_i = f(\underset{\sim}{X}_i)$, for detecting shifts in the parameter of a Bernoulli process.

Proof. At each sampling occasion the control limits defined at (3.3) determine a UMPU test for $\pi=\pi_0$ versus $\pi\neq\pi_0$ (Lehmann (1959), p.128). Therefore, by Lemma 1.1, the control chart based on these limits is dominant in the class stated in the theorem.

The most striking distinction between the optimal charts as given by Theorems 3.1 and 3.2 and the standard p-charts is the randomized portion (i.e., the γ's) of the former. Indeed, if randomized charts are excluded from consideration, then the standard p-charts, both the one- and two-sided versions, are the optimal charts at some level for the classes in Theorems 3.1 and 3.2. The wisdom of excluding randomized control limits is discussed in the following paragraph.

The usual arguments for neglecting randomized tests rest on the resulting ease of computation of the acceptance regions, the practitioner's unfamiliarity with these types of tests, and the statistician's stated distaste in imposing a further, seemingly unrelated, randomization on the problem. However, in dismissing the use of randomized tests, one should be cognizant of the effect on the operating characteristics of the control chart. If it is desired to achieve a certain run-length distribution for given process conditions, then it is necessary to employ randomized tests in monitoring the process. Table 3.1 illustrates the actual level, α, of the p-chart under various values for π_0 when the asymptotic limits (3.1) with K = 3 are used. This choice of K gives a nominal value of α of 0.0027. If this value of α truly expresses the characteristics desired to be achieved by use of the chart, Table 3.1 indicates that, for the cases considered, in general, the actual in-control run-length distribution is stochastically smaller than the corresponding nominal run-length distribution.

The empirical evidence in Table 3.1 suggests that the disparity between the actual and nominal levels of the asymptotic p-chart is generally worse for smaller values of π_0. It also appears that the convergence to the true level is slower at these parameter values. Typically, one expects the target values of the proportion defective to be in the range of (0.0,0.05), where the asymptotic test appears to be at its worst. These observations supply credence to the argument that

TABLE 3.1. ACTUAL LEVELS FOR THE P-CHART BASED UPON (3.1) WITH $K = 3$, FOR VARIOUS VALUES OF π_0 AND N.

	N				
π_0	10	15	25	40	50
0.005	0.0489	0.0124	0.0069	0.0172	0.0261
0.010	0.0043	0.0096	0.0258	0.0075	0.0138
0.015	0.0093	0.0207	0.0061	0.0221	0.0067
0.020	0.0162	0.0353	0.0132	0.0082	0.0178
0.025	0.0246	0.0057	0.0235	0.0174	0.0362
0.030	0.0345	0.0094	0.0062	0.0067	0.0037
0.035	0.0043	0.0142	0.0105	0.0125	0.0078
0.040	0.0062	0.0203	0.0165	0.0049	0.0036
0.045	0.0086	0.0037	0.0046	0.0085	0.0068
0.050	0.0115	0.0055	0.0072	0.0034	0.0032
0.060	0.0138	0.0104	0.0031	0.0136	0.0027
0.070	0.0036	0.0028	0.0065	0.0058	0.0073
0.080	0.0053	0.0050	0.0028	0.0087	0.0056
0.090	0.0088	0.0082	0.0053	0.0024	0.0043
0.100	0.0128	0.0127	0.0023	0.0051	0.0032

greater consideration needs to be given to the use of control charts based on (3.2) or (3.3) rather than (3.1), whether or not randomization is used.

A possible compromise could be proffered in order to overcome the last two objections to randomized tests given above. If satisfactory operating characteristics cannot be obtained with non-randomized limits, but there is still reluctance to adopt a randomized procedure, one might take the control limits to be warning limits, and stop the process when the number of times the warning limit is reached times the randomization parameter (γ_i) is greater than or equal to one. This compromise will not, of course, give the average run length (ARL) designed by the limits (3.2) or (3.3), but it will be an improvement over the approximation of the ARL obtained by setting $\gamma_i=0$. This method is discussed no further here.

The situation when the proportion defective is allowed to be in some range, $\pi_\ell \leq \pi \leq \pi_u$, say, might also be of interest in process monitoring. This problem has not been addressed specifically in the quality control literature. The usual approach is to estimate the process parameter when the process is in control and operating within the specified limits. The process is then monitored using (3.1) as control limits with π_0 replaced by the estimated parameter. This procedure is more fully discussed below in Section Five. Instead of estimating the process proportion defective, control limits may be established directly from the requirement that $\pi_\ell \leq \pi \leq \pi_u$.

Theorem 3.3. Consider the same class of control charts given in Theorem 3.2. For monitoring shifts of the proportion defective outside of the range $[\pi_\ell, \pi_u]$, consider the p-chart with control limits $(L/n, \gamma_\ell, U/n, \gamma_u)$ such that

$$\begin{aligned} &P(b_1(n,\pi_i)<L) + \gamma_\ell P(b_1(n,\pi_i)=L) + \\ &P(b_1(n,\pi_i)\geq U) + \gamma_u P(b_1(n,\pi_i)=U) = \alpha \end{aligned} \tag{3.4}$$

for $i = \ell, u$, where $0 \leq \gamma_\ell, \gamma_u < 1$. Then this chart dominates any other control chart of level α in the class.

Proof. At each sampling occasion (3.4) defines a UMPU test (Lehmann (1959), p. 126). As before, Lemma 1.1 completes the proof.

The above discussion of randomized control limits also applies to Theorem 3.3. Letting $\gamma_\ell = \gamma_u = 0$ yields a nonrandomized conservative control chart, with limits chosen such that

$$P(b_1(n,\pi_\ell)<L) + P(b_1(n,\pi_\ell)>U) \leq \alpha$$
$$P(b_1(n,\pi_u)<L) + P(b_1(n,\pi_u)>U) \leq \alpha.$$

This concludes our discussion of Bernoulli process monitoring under Case I. We will return to the problem of monitoring a Bernoulli process in Section Four.

3.3 Monitoring a Multinomial Process - Case I: The Generalized p-Chart

We now consider an extension of the problem of monitoring the proportion defective that does not appear to have been studied in a quality control framework. Suppose an industrial process which produces items of varying levels of quality is to be monitored. For instance, the units of production may be classified as of premium , good, poor or unsatisfactory quality. In this section a control chart for monitoring such multinomial processes is proposed when the individual proportions are specified a priori. The resulting control chart will be referred to as the generalized p-chart.

For hypotheses tests about the parameters of the multinomial distribution, the standard test statistic is the well known Pearson χ^2 statistic,

$$Y_i^2 = \sum_{j=1}^{c} \frac{(X_{ij.} - n_i \pi_{0j})^2}{n_i \pi_{0j}} , \qquad (3.5)$$

where $X_{ij.} = \sum_{k=1}^{n_i} X_{ijk}$, and π_{0j} is the target value for the multinomial probability π_{ij}, $j=1,2,\ldots,c$. A solution to the problem of monitoring a multinomial process, then, would be to plot on a control chart the values found using (3.5) with control limits established based on the asymptotic distribution of Y_i^2, namely the $\chi_1^2(c-1,1)$ distribution.

As there are no optimal small-sample tests for multinomial proportions, as there are for binomial proportions, we shall have to be content with this asymptotic chart. Even among large-sample tests the chi-squared test does not possess universally optimum properties similar to those discussed in the preceding section.

Knowledge of the small-sample properties of the Pearson test for goodness of fit is very limited. Cohen and Sackrowitz (1975), however, have demonstrated for any n that (3.5) gives an unbiased test of the hypothesis $\pi_{0i} = \frac{1}{c}$, i=1,...,c. Morever, they give a counterexample to show that this unbiasedness property does not extend to more general hypotheses. Since it would be a rare occasion in quality control when the hypothesis $\pi_{0i} = \frac{1}{c}$ for all i would be of interest, we cannot, in general, even be assured of the unbiasedness of the generalized p-chart.

An asymptotic property of the generalized p-chart is outlined in the following theorem.

Theorem 3.4. The α-level generalized p-chart based on $(Y_1^2, Y_2^2, \ldots)$ with the control limit u such that $P(U \le u) = 1-\alpha$, where $U \sim \chi_1^2(c-1,1)$, has the following asymptotic property. For out-of-control parameters, π_{ij}, of the form

$$\pi_{ij} - \pi_{0j} = \frac{d_{ij}}{\sqrt{n}} \quad j=1,\ldots,c;\ i=1,2,\ldots, \tag{3.6}$$

the run length of the chart is stochastically decreasing in each of its noncentrality parameters,

$$\delta_i = \sum_{j=1}^{c} \frac{d_{ij}^2}{\pi_{0j}}, \quad i=1,2,\ldots .$$

Proof. Under the alternative parameter sequence (3.6), the test statistic (3.5) is asymptotically $\chi_1^2(c-1,1,\delta_i)$ (cf, Cochran (1952)). Recall, the noncentral chi-squared distribution is stochastically increasing in its noncentrality parameter. Also, from familiar arguments the run-length distribution is given by

$$P(N_{GP}(\underset{\sim}{\delta}) > t) = \prod_{i=1}^{t} P(\chi_1^2(c-1,1,\delta_i) \le u) , \tag{3.7}$$

where $N_{GP}(\underset{\sim}{\delta})$ is the run length of the control chart for $\underset{\sim}{\delta} = (\delta_1, \delta_2, \ldots, \delta_t)'$. For $\underset{\sim}{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_t)'$ with $\gamma_i \geq \delta_i$ for all i, from (3.7) we have,

$$P(N_{GP}(\underset{\sim}{\delta})>t) \geq P(N_{GP}(\underset{\sim}{\gamma})>t) \ .$$

In particular, letting $\underset{\sim}{\delta}=\underset{\sim}{0}$, we see that the generalized p-chart is asymptotically unbiased. Also, if the process is out of control and stationary, Theorem 3.4 assures that, asymptotically, the generalized p-chart will signal more quickly in probability if the process is more out of control. On the other hand, if the process is drifting it is clear from the proof of Theorem 3.4 that the greater the process drifts, as determined by $\gamma_i \geq \delta_i$ for all i, denoted by $\underset{\sim}{\gamma} \succ \underset{\sim}{\delta}$, the faster, asymptotically, the control chart will signal in probability. If upper and lower bounds can be placed on the possible drift of δ_i, then it is clear from the work of Jensen and Hui (1980) that corresponding upper and lower stochastic bounds may be placed on the run-length distribution of the generalized p-chart.

3.4 Control Charts for Monitoring Binomial Proportions - Case II

In the present section, control charts for monitoring binomial proportions are considered for use when the in-control proportions are not specified but, rather, are estimated. Relatively little attention has been paid to the inherent statistical difficulties imposed by using these estimates as parameter values - the typical approach in quality control. Herein these resulting complexities are not ignored, and the effects on the properties of the control charts discussed in Section

Two are explored.

The standard procedure for the p-chart in Case II is to define a base period when the process is assumed to be in control; to estimate the parameter π from this period; and then to substitute this estimate, p_0, say, for π_0 in the expression (3.1) for the control limits of the p-chart. The process is then said to be out of control if at any sampling occasion the proportion defective is outside the bounds

$$p_0 \pm K((1-p_0)p_0/n)^{\frac{1}{2}} . \tag{3.8}$$

When properties of this methodology are discussed no heed generally is paid to the fact that p_0 is an estimate and not a parameter. As a consequence p-charts based upon (3.8) are considered to be no different in their operating characteristics than those based upon (3.1). We will postpone a discussion of the actual asymptotic properties of this p-chart until the next section, since it will be more convenient to view it as a special case of multinomial process monitoring under Case II.

First, though, a possible small-sample procedure for monitoring the fraction defective against a base-period estimate is suggested. Instead of proceeding in the heuristic fashion outlined in the preceding paragraph, it might be beneficial to employ the UMPU test for comparing two binomial populations. This UMPU test is given by Lehmann (1959, p.143); it is a conditional test based upon the hypergeometric distribution. If a control chart is established based upon this test, then the values of the chart will be dependent. This precludes an application of Lemma 1.1, so, unfortunately, the strong optimality properties of the p-chart found in Section Two cannot be duplicated here.

Nonetheless, certain desirable properties can be ascribed to this control chart in the class of control charts which base their decision procedures on unbiased test statistics of the form $\{T(\underset{\sim}{X}_1,\underset{\sim}{X}_0),T(\underset{\sim}{X}_2,\underset{\sim}{X}_0),\ldots\}$, where $\underset{\sim}{X}_0$ represents the sample information in the base period. Denote this class by C_2.

The actual control limits depend upon the nature of the alternatives, but they are analogous to those limits set forth in Theorems 3.1 and 3.2. For example, the set $(L/n,\gamma_\ell,U/n,\gamma_u)$ of control limits for a two-sided p-chart monitoring against $\pi_i \neq \pi_0$ is given by

$$\sum_{y=L+1}^{U-1} \frac{\binom{m}{s-y}\binom{n}{y}}{\binom{m+n}{s}} + (1-\gamma_\ell)\frac{\binom{m}{s-L}\binom{n}{L}}{\binom{m+n}{s}} + (1-\gamma_u)\frac{\binom{m}{s-U}\binom{n}{U}}{\binom{m+n}{s}} = 1-\alpha$$

$$\sum_{y=L+1}^{U-1} \frac{\binom{m}{s-y}\binom{n-1}{y-1}}{\binom{m+n-1}{s-1}} + (1-\gamma_\ell)\frac{\binom{m}{s-1-(L-1)}\binom{n-1}{L-1}}{\binom{m+n-1}{s-1}} + (1-\gamma_u)\frac{\binom{m}{s-1-(U-1)}\binom{n-1}{U-1}}{\binom{m+n-1}{s-1}} = 1-\alpha \;. \tag{3.9}$$

In (3.9), m is the sample size in the base period, n is the sample size in each monitoring period, and s is the sum of the defects observed in the base period and monitoring period i. The following theorem gives some weak optimality properties that can be ascribed to a Case II p-chart based on the UMPU test for comparing two binomial proportions.

Theorem 3.5. For detecting changes in a Bernoulli process under Case II, consider the control chart based upon the conditional distribution

$$P(Y_i=y_i \,|\, Y_i+Y_0=s) = \frac{\binom{m}{s-y_i}\binom{n}{y_i}}{\binom{m+n}{s}} , \tag{3.10}$$

where

(i) $Y_i = \sum_{j=1}^{n_i} X_{ij}$, i=0,1,2,... , the number of defectives observed in the ith monitoring period;

(ii) n_0 = m, the sample size in the base period;

and

(iii) n_i = n, i=1,2.,,, , the sample size on each of the other monitoring occasions.

For stationary processes, when the control limits are chosen as outlined, for example, in (3.9), then

(i) the approximating geometric run-length distribution for (3.10) dominates the approximating run-length distribution of any other member of C_2;

(ii) the control chart is admissible in the class of control charts in C_2 which are based on unbiased tests.

Proof. (i) By Theorem 1.6, letting N_H denote the run length of the control chart based on (3.9) and (3.10), for the acceptance region A, it follows that

$$P(N_H>t) = P\left(\bigcap_{i=1}^{t}\{T_H(Y_i,Y_0)\varepsilon A\}\right)$$

$$\geq [P(T_H(Y_i,Y_0)\varepsilon A)]^t = (1-\beta_H)^t . \qquad (3.11)$$

If $N_{H'}$ is the run length of any other control chart in C_2 with the same level and with acceptance region A',

$$P(N_{H'}>t) \geq [P(T_{H'}(Y_i,Y_0)\varepsilon A')]^t = (1-\beta_{H'})^t . \qquad (3.12)$$

But, $(1-\beta_{H'}) \geq (1-\beta_H)$, since β_H is the power associated with the UMPU test. Now the right hand sides of (3.11) and (3.12) imply that $G(\beta_H)$ and $G(\beta_{H'})$ are upper stochastic bounds for the distributions of the run lengths N_H and $N_{H'}$, respectively. But $G(\beta_H)$ is stochastically smaller than $G(\beta_{H'})$, hence the geometric bound for N_H dominates the geometric stochastic bound for any other procedure in C_2.

(ii) Since the hypothesis test based on (3.9) and (3.10) is UMPU,

$$P(N_H>1) \leq P(N_{H*}>1) ,$$

where N_{H*} is the run length of any control chart in the class under consideration. Hence the control chart in Theorem 3.5 is admissible in the class.

Once more, the test for this discrete problem, in general, involves a randomized part. For convenience our attention will be confined to nonrandomized tests, although the inherent shortcomings of such an approach, as outlined in Section Two, should be kept in mind.

Evaluating the run-length properties of the control chart, beyond those stated in Theorem 3.5, is much more difficult when the process is out of control. Harkness and Katz (1964) and Bennett and Hsu (1960) have studied the power properties of the UMPU test for the comparison of binomial proportions. Their results do not extend to our general situation of control chart run lengths due to the dependencies among the values of the chart. What their results can be used for is to provide a means for calculating the approximating geometric run-length distribution when the process is stationary, but out of control.

Two asymptotic procedures for the problem of this section are now considered. The first procedure is somewhat heuristic and is in keeping with the traditional methods of quality control To wit, estimate π_0 consistently in the base period by p_0; in the monitoring period compare the proportion defective against control limits found as in Section Two using $b_1(n,p_0)$ as the target population. The estimated probabilities converge in probability to the true probabilities. Therefore, this procedure, asymptotically on the size of the base period, has all the optimal properties attributed to the p-charts studied in Section Two.

The monitoring procedure of Theorem 3.5 is asymptotically equivalent to the procedure outlined in the preceding paragraph. To see this fact, consider

$$\operatorname*{plim}_{m\to\infty} \frac{\binom{m}{s-y_i}\binom{n}{y_i}}{\binom{m+n}{s}} = \binom{n}{y_i} \operatorname*{plim}_{m\to\infty} \frac{(y_0+y_i)\ldots(y_0+1)(m+n-y_i-y_0)\ldots(m-y_0+1)}{(m+n)(m+n-1)\ldots(m+1)}$$

$$= \binom{n}{y_i} \operatorname*{plim}_{m\to\infty} \frac{(\frac{y_0}{m}+\frac{y_i}{m})\ldots(\frac{y_0}{m}+\frac{1}{m})(1-\frac{y_0}{m}+\frac{n-y_i}{m})\ldots(1-\frac{y_0}{m}+\frac{1}{m})}{(1+\frac{n}{m})\ldots(1+\frac{1}{m})}$$

$$= \binom{n}{y_i}\pi_0^{y_i}(1-\pi_0)^{n-y_i} \; . \tag{3.13}$$

So, when the process is in control, the probabilities (3.10) converge in probability to the corresponding binomial probability as $m\to\infty$.

That the distribution (3.10) in the out-of-control situation also converges to the correct binomial distribution under the above limiting process follows from the arguments below. The general probability mass function for the conditional distribution of Y_i given Y_i+Y_0 is given by Harkness (1965) as

$$\frac{g(y_i)t^{y_i}}{P(t)} \; , \tag{3.13a}$$

where $g(x) = \dfrac{\binom{n}{x}\binom{m}{s-x}}{\binom{n}{s}}$, $t = \dfrac{\pi_i(1-\pi_0)}{\pi_0(1-\pi_i)}$, $P(\cdot)$ is the probability generating function of $g(x)$, and π_i is the out-of-control parameter value.

We have already seen that $g(y_i)\to\binom{n}{y_i}\pi_0^{y_i}(1-\pi_0)^{n-y_i}$. Under the limiting process $P(t)$ converges to the pgf of $b_1(n,\pi_0)$, i.e., $((1-\pi_0)+\pi_0 t)^n$. Therefore, as $m\to\infty$, (3.13a) becomes

$$\binom{n}{y_i}\pi_0^{y_i}(1-\pi_0)^{n-y_i}\left(\frac{\pi_i(1-\pi_0)}{\pi_0(1-\pi_i)}\right)^{y_i}\left((1-\pi_0)+\frac{\pi_i(1-\pi_0)}{(1-\pi_i)}\right)^{-n}$$

$$= \binom{n}{y_i}\pi_i^{y_i}(1-\pi_i)^{n-y_i} \; ,$$

the corresponding binomial distribution for the out-of-control p-chart of Section Two.

Therefore, the monitoring procedure of Theorem 3.5 also has, asymptotically as the size of the base period increases, the optimal properties ascribed to the p-charts of Section Two. Given the meager small-sample results of Theorem 3.5, it is somewhat reassuring to know that the procedure behaves in such a nice fashion when large base-period samples are available.

Given the aforementioned difficulties with the small-sample characteristics of the run length when the process is out of control, the asymptotic behavior offers some hope for, at least, approximating the run length. Note that under the above limiting process, the observed values of the control chart are asymptotically independent, so the run-length computations are not complicated by a dependency structure. This behavior is true for both stationary and drifting processes.

The second asymptotic procedure is based on a normal approximation for the UMPU test (3.9) given by Hannan and Harkness (1963). Their approximation allows both m and n to increase to infinity, thereby maintaining the dependencies between the control chart values. Under these limiting conditions we have, when the process is in control,

$$P(\ell \leq Y_i \leq u \mid Y_i + Y_0 = s) \doteq \Phi(Z_{u+\frac{1}{2}}) - \Phi(Z_{\ell-\frac{1}{2}})$$

and (3.13b)

$$P(0 \leq Y_i \leq u \mid Y_i + Y_0 = s) \doteq \Phi(Z_{u+\frac{1}{2}}) \ ,$$

where $\Phi(\cdot)$ denotes the cdf of the $N_1(0,1)$ distribution and

$$Z_a = [\frac{(n+m)^3}{nms(n+m-s)}]^{\frac{1}{2}} (a - \frac{ns}{n+m}) . \qquad (3.13c)$$

Asymptotic control limits, (ℓ,u), could then be determined for the Case II p-chart from the percentiles of the standard normal distribution and expression (3.13b).

Now that four different procedures have been introduced as contenders for monitoring a Bernoulli process under Case II, the question arises as to which one, if any, is to be preferred to the rest on the basis of power and level. Of course, (3.10), being a small-sample procedure, gives an exact level. In Table 3.2 we give a comparison of upper control limits using a level of 0.05 found with the procedures (3.10), the binomial approximation, and the normal approximations (3.13b) and (3.8) with K=2.

From this comparison it is seen that the binomial approximation gives control limits closest to those of the p-chart based on Theorem 3.5, while the normal approximation (3.13b) appears to do the worst in approximating these control limits. All three approximations lead to control charts which will signal more often when the process is in control than the p-chart based on Theorem 3.5.

Again, a comparison of the small-sample properties of these procedures when the process is out of control is complicated by the dependencies affecting their individual run-length distributions.

TABLE 3.2. CONTROL LIMITS VIA SEVERAL METHODS FOR CASE II P-CHARTS FOR VARIOUS N, M, AND Y_0.

		METHODS											
		N = 15				N = 25				N = 50			
	M	A	B	C	D	A	B	C	D	A	B	C	D
$Y_0 = 2$	5N	3	2	1	1	3	2	1	1	3	2	1	1
	6N	3	2	0	1	3	1	0	1	3	1	0	1
	7N	3	2	0	1	3	1	0	1	3	1	0	1
	8N	3	2	0	1	3	1	0	1	3	1	0	1
	9N	2	1	0	1	2	1	0	1	2	1	0	1
	10N	2	1	0	1	2	1	0	1	2	1	0	1
$Y_0 = 3$	5N	3	3	1	2	4	2	1	2	4	2	1	2
	6N	3	2	1	1	3	2	1	1	3	2	1	1
	7N	3	2	1	1	3	2	1	1	3	2	1	1
	8N	3	2	0	1	3	2	0	1	3	2	0	1
	9N	3	2	0	1	3	1	0	1	3	1	0	1
	10N	3	2	0	1	3	1	0	1	3	1	0	1
$Y_0 = 4$	5N	4	3	1	2	4	2	1	2	4	2	1	2
	6N	3	3	1	2	4	2	1	2	4	2	1	2
	7N	3	3	1	2	3	2	1	2	3	2	1	2
	8N	3	2	1	1	3	2	1	1	3	2	1	1
	9N	3	2	1	1	3	2	1	1	3	2	1	1
	10N	3	2	0	1	3	2	1	1	3	2	1	1
$Y_0 = 5$	5N	4	4	2	2	4	3	2	2	4	3	2	2
	6N	4	3	1	2	4	2	1	2	4	3	2	2
	7N	4	3	1	2	4	2	1	2	4	2	1	2
	8N	3	3	1	2	3	2	1	2	3	2	1	2
	9N	3	2	1	2	3	2	1	2	3	2	1	2
	10N	3	2	1	1	3	2	1	1	3	2	1	1

METHOD A: UMPU TEST (3.10)
B: BINOMIAL APPROXIMATION (3.13)
C: NORMAL APPROXIMATION (3.13b)
D: NORMAL APPROXIMATION (3.8), with K = 3

3.5 Monitoring Multinomial Processes - Case II

In this section asymptotic control chart procedures are given for monitoring multinomial processes under Case II. As a special case, for Bernoulli processes an asymptotic procedure is obtained which is slightly different from the standard p-chart with limits given by (3.8).

As in the preceding section the problem is to compare the process, here, a multinomial process, at each sampling occasion to the baseline process. A standard statistical test for determining the homogeneity of two multinomial populations is the Pearson X^2 statistic, as given by

$$X_i^2 = \sum_{j=1}^{c} \left\{ \frac{n_i\left(\frac{X_{ij.}}{n_i} - \frac{X_{.j.}}{N}\right)^2}{\hat{\pi}_j} + \frac{n_0\left(\frac{X_{0j.}}{n_0} - \frac{X_{.j.}}{N}\right)^2}{\hat{\pi}_j} \right\}, \tag{3.14}$$

where

(i) $X_{ij.} = \sum_{k=1}^{n_i} X_{ijk}$ and $X_{.j.} = X_{0j.} + X_{ij.}$ and the X_{ijk} are as in Section One;

(ii) $N_i = n_i + n_0$;

and

(iii) $\hat{\pi}_j$ is a consistent estimator of π_{0j}.

Expression (3.14) converges in distribution to $\chi_1^2(c-1,1)$ under the homogeneity hypothesis.

If the statistics $(X_1^2, X_2^2, \ldots)$ are used to construct a control chart, the points on the chart will be dependent. The following theorem gives the joint distribution of any finite number of these statistics. With this distribution the asymptotic behavior of the run length of this chart can be described.

In the following discussion let $n_i=n$, $N_i=N$ for $i=1,2,\ldots,k$ and $n_0=m$.

Theorem 3.6. Consider monitoring a multinomial process under Case II using the statistic (3.14) at each monitoring occasion. As n and $m\to\infty$ such that $m/n\to\eta$, the joint distribution of $(X_1^2,X_2^2,\ldots,X_k^2)$, i.e., k successive values of the control chart, is $\chi_k^2(c-1,\underset{\sim}{R})$ when the process is in control. Here, we have $\underset{\sim}{R} = \frac{\eta}{1+\eta}\underset{\sim}{I}_k + \frac{1}{1+\eta}\underset{\sim}{J}_k$.

Proof. Let $\underset{\sim}{p}_i=(p_{ij})$ be the $(c-1)\times 1$ vector whose elements are defined by

$$p_{ij} = \frac{X_{ij.}}{n} \quad j=1,\ldots,c-1 ,$$

for $i=1,\ldots,k$; and

$$p_{0j} = \frac{X_{0j.}}{m} .$$

Define the vectors

$$\underset{\sim}{Y}_i = \sqrt{n}(\underset{\sim}{p}_i-\underset{\sim}{\pi}_i) \quad i=0,1,\ldots,k.$$

By the multivariate central limit theorem it follows that

$$\underset{\sim}{Y}_i \xrightarrow{d} N_{c-1}(\underset{\sim}{0},\underset{\sim}{\Sigma}_i) \quad i=1,\ldots,k ,$$

and

$$\underset{\sim}{Y}_0 \xrightarrow{d} N_{c-1}(\underset{\sim}{0},\eta^{-1}\underset{\sim}{\Sigma}_0) .$$

The covariance matrix $\underset{\sim}{\Sigma}_i = (\sigma_{j\ell})$ $i=0,1,\ldots,k$, is given by

$$\sigma_{j\ell} = \begin{cases} \pi_{ij}(1-\pi_{ij}) & \text{if } j=\ell \\ -\pi_{ij}\pi_{i\ell} & \text{if } j\neq\ell \end{cases} \tag{3.15}$$

for $j,\ell = 1,2,\ldots,c-1$.

Let $\underset{\sim}{Z}_i = \underset{\sim}{Y}_i - \underset{\sim}{Y}_0 = \sqrt{n}(\underset{\sim}{p}_i - \underset{\sim}{p}_0 - (\underset{\sim}{\pi}_i - \underset{\sim}{\pi}_0))$ $i=1,\ldots,k$, and consider the limiting distribution of $\underset{\sim}{Z} = (\underset{\sim}{Z}_1', \underset{\sim}{Z}_2', \ldots, \underset{\sim}{Z}_k')'$. By Theorem 1.8, this asymptotic distribution is $N_{k(c-1)}(\underset{\sim}{0}, \underset{\sim}{G}\underset{\sim}{\Omega}\underset{\sim}{G}')$, where

$$\underset{\sim}{G} = \begin{bmatrix} -\underset{\sim}{I}_{c-1} & \underset{\sim}{I}_{c-1} & & \\ \vdots & & \ddots & \underset{\sim}{0} \\ \vdots & \underset{\sim}{0} & & \ddots \\ -\underset{\sim}{I}_{c-1} & & & \underset{\sim}{I}_{c-1} \end{bmatrix}$$

and $\underset{\sim}{\Omega} = \mathrm{diag}(\underset{\sim}{\Sigma}_0, \underset{\sim}{\Sigma}_1, \ldots, \underset{\sim}{\Sigma}_k)$. Therefore, the covariance matrix $\underset{\sim}{G}\underset{\sim}{\Omega}\underset{\sim}{G}'$ can be written as

$$(\underset{\sim}{J}_k \times \eta^{-1} \underset{\sim}{\Sigma}_0) + \mathrm{diag}(\underset{\sim}{\Sigma}_1, \ldots, \underset{\sim}{\Sigma}_k) \ . \tag{3.16}$$

When the process is in control $\underset{\sim}{\pi}_i = \underset{\sim}{\pi}_0$, for all i. Using this fact and making an appropriate change of scale, we have

$$\underset{\sim}{Z}^* = \left(\frac{\eta}{1+\eta}\right)^{\frac{1}{2}} \underset{\sim}{Z} \xrightarrow{d} N_{k(c-1)}(\underset{\sim}{0}, \underset{\sim}{R} \times \underset{\sim}{\Sigma}_0) \ , \tag{3.17}$$

where $\underset{\sim}{R} = (r_{ij})$ is a $k \times k$ correlation matrix with

$$r_{ij} = \frac{1}{1+\eta} \quad \text{for } i \neq j \ .$$

Or, we could write $\underset{\sim}{R}$ as

$$\frac{1}{1+\eta} \underset{\sim}{J}_k + \frac{\eta}{1+\eta} \underset{\sim}{I}_k \ .$$

Consider the quadratic forms $\{q_1,\ldots,q_k\}$ defined by

$$q_i = n(\frac{\eta}{1+\eta})(\underset{\sim}{p}_i-\underset{\sim}{p}_0)'\underset{\sim}{\Sigma}_0^{-1}(\underset{\sim}{p}_i-\underset{\sim}{p}_0) \quad , \; i=1,\ldots,k \; . \tag{3.18}$$

An application of Theorem 2.5 in conjunction with Theorem 1.10 gives the asymptotic distribution of $(q_1,q_2,\ldots,q_k)'$ as $\chi_k^2(c-1,\underset{\sim}{R})$.

Since $\underset{\sim}{\Sigma}_0$ is unknown, the expressions (3.18), as they stand, cannot be used for the computation of the control chart values. An alternative is to replace the parameters π_{0j}, $j=1,\ldots,c-1$, in (3.15) with consistent estimators, and proceed to compute (3.18) accordingly. We now show that this substitution leaves the asymptotic distribution theory intact.

First, note that $\underset{\sim}{\Sigma}_0$ can be written as $\text{diag}(\pi_{01},\ldots,\pi_{0,c-1})-\underset{\sim}{\pi}_0\underset{\sim}{\pi}_0'$, so (cf. Rao (1973, p.33))

$$\underset{\sim}{\Sigma}_0^{-1} = \text{diag}(\frac{1}{\pi_{01}},\ldots,\frac{1}{\pi_{0,c-1}}) + \frac{\underset{\sim}{1}\,\underset{\sim}{1}'}{1 - \sum_{j=1}^{c-1}\pi_{0j}}$$

$$= \text{diag}(\frac{1}{\pi_{01}},\ldots,\frac{1}{\pi_{0,c-1}}) + \frac{1}{\pi_{0c}}\underset{\sim}{J}_{c-1} \; ,$$

where $\underset{\sim}{1}' = (1,\ldots,1)$ is of order $(c-1)$.

Expression (3.18) can now be rewritten as

$$n(\frac{\eta}{1+\eta})(\underset{\sim}{p}_i-\underset{\sim}{p}_0)'[\text{diag}(\frac{1}{\pi_{01}},\ldots,\frac{1}{\pi_{0,c-1}}) + \frac{1}{\pi_{0c}}\underset{\sim}{J}_{c-1}](\underset{\sim}{p}_i-\underset{\sim}{p}_0)$$

$$= n(\frac{\eta}{1+\eta})[\sum_{j=1}^{c-1}\frac{1}{\pi_{0j}}(p_{ij}-p_{0j})^2 + \frac{1}{\pi_{0c}}[\sum_{j=1}^{c-1}(p_{ij}-p_{0j})]^2]$$

$$= n(\frac{\eta}{1+\eta})[\sum_{j=1}^{c-1}\frac{1}{\pi_{0j}}(p_{ij}-p_{0j})^2 + \frac{1}{\pi_{0c}}[(1-p_{ic})-(1-p_{0c})]^2)$$

$$= n(\frac{n}{1+n}) \sum_{j=1}^{c} \frac{1}{\pi_{0j}}(p_{ij}-p_{0j})^2 \ . \tag{3.19}$$

Next, consider any consistent estimator, $\hat{\pi}_{0j}$, for π_{0j}. Then (3.19) can be expressed as

$$n(\frac{n}{1+n})[\sum_{j=1}^{c} \frac{(p_{ij}-p_{0j})^2}{\hat{\pi}_{0j}} (\frac{\hat{\pi}_{0j}}{\pi_{0j}})]$$

$$= n(\frac{n}{1+n})[\sum_{j=1}^{c} \frac{(p_{ij}-p_{0j})^2}{\hat{\pi}_{0j}}] + o_p(1) \ , \tag{3.20}$$

where $o_p(1)$ designates terms that go to 0 in probability. Therefore, replacing each π_{0j} in (3.19) by a consistent estimator does not alter the asymptotic distribution theory.

The theorem will be complete with the demonstration that (3.20), omitting the $o_p(1)$ term, is actually the Pearson X^2 statistic given at (3.14). Consider the following expression equivalent to (3.20), namely,

$$n(\frac{n}{1+n})[\sum_{j=1}^{c} \frac{[(p_{ij}-\frac{X_{.j.}}{N}) - (p_{0j}-\frac{X_{.j.}}{N})]^2}{\hat{\pi}_j}] \ . \tag{3.21}$$

Expand (3.21) and make the following substitutions in the cross product terms

$$p_{0j} - \frac{X_{.j.}}{N} = -\frac{n}{m}(p_{ij} - \frac{X_{.j.}}{N})$$

and

$$p_{ij} - \frac{X_{.j.}}{N} = -\frac{m}{n}(p_{0j} - \frac{X_{.j.}}{N}).$$

These substitutions yield expression (3.14), which completes the proof.

The folklore of quality control would suggest a procedure alternative to the one outlined in Theorem 3.6. This folklore tempts one to estimate the π_{0i}'s of the χ^2 goodness-of-fit statistic, (3.5), in a base period, take these as the true parameter values, and then proceed merrily along as in Section Three. The pitfalls of such a naive approach can be seen by comparing the folklore statistic, namely,

$$X_{Fi}^2 = \sum_{i=1}^{c} \frac{n(p_{ij}-p_{0j})^2}{p_{0j}} \tag{3.22}$$

with the statistic (3.19) of Theorem 3.6.

Since p_{0j} is a consistent estimator of π_{0j} $(j=1,2,\ldots,c)$, $(\frac{m}{m+n})X_{Fi}^2$ is asymptotically equivalent to the Pearson X^2 test for homogeneity. Therefore, the folklore statistic (3.22) will be numerically greater than the X^2 homogeneity statistic. Moreover, following the tradition also would lead one to use the χ_{c-1}^2 distribution in establishing control limits for (3.22). Consequently, the control chart would signal more frequently than the chosen limits would indicate.

A further point needs to be addressed in order to ensure the utility of Theorem 3.6. The theorem allows only a finite number of control chart values. For this condition to be true it is essential to demonstrate that the chart eventually signals with probability one. First, suppose that the process is stationary in the sampling period, and that a constant control limit, u, say, is used. Then, denoting the run length by N_{GP} and the cdf of $\underset{\sim}{p}_0$ by $F(\underset{\sim}{p}_0)$, we have

$$\begin{aligned} P(N_{GP}>t) &= P(X_1^2\le u,\ldots,X_t^2\le u) \\ &= \int_{P_0} P\left(\bigcap_{i=1}^{t}\{X_i^2\le u\}\,\middle|\,\underset{\sim}{p}_0\right)dF(\underset{\sim}{p}_0) \\ &= \int_{P_0}[P(X_1^2\le u\,|\,\underset{\sim}{p}_0)]^t\,dF(\underset{\sim}{p}_0) \\ &\le \int_{P_0} dF(p_0) = 1 , \end{aligned}$$

where $P_0 = \{\underset{\sim}{p}_0 | 0\le p_{0j}\le 1,\ \sum_{j=1}^{c} p_{0j}=1\}$. Furthermore, since $P(X_1^2\le u|\underset{\sim}{p}_0)<1$, the dominated convergence theorem implies

$$\lim_{t\to\infty} P(N_{GP}>t) = \int_{P_0} \lim_{t\to\infty} [P(X_1^2\le u|p_0)]^t dF(p_0) = 0 .$$

Therefore, the generalized p-chart under Case II signals with probability one.

An immediate consequence for the run-length distribution of this generalized p-chart follows from Theorem 1.6.

Theorem 3.7. Suppose a multinomial process is stationary in its monitoring period and a control chart with a constant rejection region is established using (3.14). Let N_{GP} represent the run length of this procedure. Then, for $t_1\le t$,

$$P(N_{GP}>t) \ge \left[P\left(\bigcap_{i=1}^{t_1}(X_i^2\le u)\right)\right]^{t/t_1} \ge P(N_G>t) , \tag{3.23}$$

where N_G has the geometric distribution with parameter $\theta = P(X_i^2>u)$. The inequalities (3.23) are valid no matter what the sample size is.

Proof. The monitoring procedure is of the form

$$P\left(\bigcap_{i=1}^{t} \{T(\underset{\sim}{Y}_i, \underset{\sim}{Y}_0) \varepsilon A\}\right) .$$

Therefore, Theorem 1.6 gives (3.23) for any sample size.

When the process is in control, or out of control, but stationary, Theorem 3.7 ensures that the run length of the generalized p-chart is stochastically larger than the geometric distribution with parameter $\theta = P(X_i^2 > u)$. The generalized p-chart signals less frequently than the nominal level of the chart under independence would suggest. This property is satisfactory for in-control situations, but may be less desirable when the process is out of control.

When the process is in control the asymptotic stochastic lower bound based on the geometric distribution can be improved upon. Letting t_1 in (3.23) be equal to two, for instance, allows use of the known expression for the bivariate χ^2 cdf (see section 2.6) to arrive at tighter bounds. If Johnson's approximation for the equicorrelated multivariate chi-squared distribution is as reliable as the preliminary findings in Appendix II would indicate, then t_1 can be taken even larger. Some examples of these improvements in the bounds are given in Table 3.3 using Johnson's approximation for $t_1 > 2$. For the cases investigated, the improved bounds are the same as the geometric bounds to two decimal places. Either there is little to be gained via these approximations, or the geometric bound, by itself, is a decent approximation to the asymptotic run-length distribution.

TABLE 3.3. APPROXIMATIONS TO THE ASYMPTOTIC CDF OF THE RUN LENGTH OF A CASE II GENERALIZED P-CHART USING (3.23). DEGREES OF FREEDOM = 1; RHO = 0.038; LEVEL = 0.01.

	t_1					
t	1	2	4	6	8	10
1	0.01003	0.01003	0.01003	0.01003	0.01003	0.01003
5	0.04914	0.04913	0.04911	0.04911	0.04911	0.04911
10	0.09587	0.09585	0.09582	0.09579	0.09576	0.09573
15	0.14030	0.14027	0.14022	0.14018	0.14014	0.14010
20	0.18255	0.18251	0.18245	0.18240	0.18234	0.18230
25	0.22272	0.22268	0.22260	0.22254	0.22248	0.22242
30	0.26092	0.26087	0.26079	0.26071	0.26064	0.26058
35	0.29724	0.29719	0.29709	0.29701	0.29693	0.29686
40	0.33177	0.33172	0.33161	0.33152	0.33144	0.33137
45	0.36461	0.36455	0.36444	0.36434	0.36426	0.36417
50	0.39583	0.39577	0.39566	0.39555	0.39546	0.39537
60	0.45375	0.45369	0.45356	0.45345	0.45335	0.45326
70	0.50612	0.50605	0.50592	0.50580	0.50569	0.50560
80	0.55347	0.55340	0.55326	0.55314	0.55303	0.55293
90	0.59628	0.59620	0.59607	0.59594	0.59583	0.59573
100	0.63498	0.63491	0.63477	0.63464	0.63453	0.63443
110	0.66998	0.66990	0.66976	0.66964	0.66953	0.66943
120	0.70162	0.70154	0.70141	0.70128	0.70117	0.70107
130	0.73022	0.73015	0.73002	0.72990	0.72979	0.72969
140	0.75609	0.75602	0.75589	0.75577	0.75566	0.75557
150	0.77947	0.77940	0.77928	0.77916	0.77906	0.77897

TABLE 3.3. APPROXIMATIONS TO THE ASYMPTOTIC CDF OF THE RUN LENGTH OF A CASE II GENERALIZED P-CHART USING (3.23). DEGREES OF FREEDOM = 1; RHO = 0.091; LEVEL = 0.01.

	t_1					
t	1	2	4	6	8	10
1	0.01003	0.01003	0.01003	0.01003	0.01003	0.01003
5	0.04914	0.04909	0.04898	0.04893	0.04893	0.04893
10	0.09587	0.09576	0.09556	0.09539	0.09523	0.09508
15	0.14030	0.14015	0.13987	0.13961	0.13939	0.13918
20	0.18255	0.18235	0.18200	0.18168	0.18139	0.18113
25	0.22272	0.22249	0.22206	0.22168	0.22134	0.22103
30	0.26092	0.26065	0.26017	0.25973	0.25934	0.25899
35	0.29724	0.29694	0.29641	0.29593	0.29549	0.29510
40	0.33177	0.33145	0.33087	0.33035	0.32988	0.32945
45	0.36461	0.36427	0.36365	0.36309	0.36258	0.36212
50	0.39583	0.39548	0.39482	0.39422	0.39369	0.39321
60	0.45375	0.45337	0.45265	0.45201	0.45143	0.45090
70	0.50612	0.50571	0.50496	0.50428	0.50367	0.50312
80	0.55347	0.55305	0.55227	0.55156	0.55093	0.55036
90	0.59628	0.59585	0.59505	0.59434	0.59370	0.59312
100	0.63498	0.63455	0.63375	0.63303	0.63239	0.63180
110	0.66998	0.66955	0.66875	0.66804	0.66740	0.66681
120	0.70162	0.70119	0.70041	0.69970	0.69907	0.69849
130	0.73022	0.72981	0.72904	0.72835	0.72773	0.72716
140	0.75609	0.75568	0.75493	0.75426	0.75365	0.75311
150	0.77947	0.77908	0.77835	0.77770	0.77711	0.77658

TABLE 3. 3. APPROXIMATIONS TO THE ASYMPTOTIC CDF OF THE RUN LENGTH OF A CASE II GENERALIZED P-CHART USING (3.23). DEGREES OF FREEDOM = 2; RHO = 0.038; LEVEL = 0.01.

	t_1					
t	1	2	4	6	8	10
1	0.01000	0.01000	0.01000	0.01000	0.01000	0.01000
5	0.04902	0.04901	0.04900	0.04899	0.04899	0.04899
10	0.09563	0.09562	0.09559	0.09557	0.09554	0.09552
15	0.13996	0.13994	0.13990	0.13987	0.13983	0.13980
20	0.18212	0.18210	0.18205	0.18200	0.18196	0.18192
25	0.22221	0.22218	0.22212	0.22207	0.22202	0.22197
30	0.26034	0.26030	0.26024	0.26017	0.26012	0.26006
35	0.29659	0.29656	0.29648	0.29641	0.29635	0.29629
40	0.33107	0.33103	0.33095	0.33088	0.33081	0.33074
45	0.36386	0.36382	0.36373	0.36365	0.36358	0.36351
50	0.39505	0.39500	0.39491	0.39482	0.39474	0.39467
60	0.45290	0.45285	0.45275	0.45266	0.45257	0.45249
70	0.50522	0.50517	0.50506	0.50497	0.50487	0.50479
80	0.55254	0.55248	0.55237	0.55227	0.55218	0.55209
90	0.59533	0.59527	0.59516	0.59506	0.59497	0.59488
100	0.63403	0.63397	0.63386	0.63376	0.63366	0.63357
110	0.66903	0.66897	0.66886	0.66876	0.66867	0.66858
120	0.70068	0.70062	0.70052	0.70042	0.70032	0.70023
130	0.72931	0.72925	0.72915	0.72905	0.72895	0.72887
140	0.75519	0.75514	0.75504	0.75494	0.75485	0.75477
150	0.77860	0.77855	0.77845	0.77836	0.77827	0.77819

TABLE 3.3. APPROXIMATIONS TO THE ASYMPTOTIC CDF OF THE RUN LENGTH OF A CASE II GENERALIZED P-CHART USING (3.23). DEGREES OF FREEDOM = 2; RHO = 0.091; LEVEL = 0.01.

	t_1					
t	1	2	4	6	8	10
1	0.01000	0.01000	0.01000	0.01000	0.01000	0.01000
5	0.04902	0.04897	0.04889	0.04885	0.04885	0.04885
10	0.09563	0.09555	0.09540	0.09525	0.09511	0.09499
15	0.13996	0.13985	0.13962	0.13942	0.13922	0.13904
20	0.18212	0.18197	0.18169	0.18143	0.18118	0.18095
25	0.22221	0.22204	0.22170	0.22139	0.22110	0.22082
30	0.26034	0.26014	0.25975	0.25940	0.25906	0.25875
35	0.29659	0.29637	0.29595	0.29555	0.29518	0.29484
40	0.33107	0.33083	0.33037	0.32994	0.32954	0.32916
45	0.36386	0.36360	0.36311	0.36265	0.36222	0.36182
50	0.39505	0.39477	0.39425	0.39376	0.39331	0.39288
60	0.45290	0.45260	0.45204	0.45151	0.45101	0.45055
70	0.50522	0.50491	0.50431	0.50375	0.50323	0.50274
80	0.55254	0.55221	0.55160	0.55102	0.55048	0.54998
90	0.59533	0.59500	0.59437	0.59379	0.59324	0.59272
100	0.63403	0.63370	0.63307	0.63248	0.63193	0.63141
110	0.66903	0.66870	0.66807	0.66748	0.66694	0.66642
120	0.70068	0.70035	0.69973	0.69916	0.69862	0.69811
130	0.72931	0.72899	0.72838	0.72781	0.72728	0.72678
140	0.75519	0.75488	0.75429	0.75374	0.75322	0.75274
150	0.77860	0.77830	0.77773	0.77719	0.77669	0.77622

TABLE 3.3 . APPROXIMATIONS TO THE ASYMPTOTIC CDF OF THE RUN LENGTH OF A CASE II GENERALIZED P-CHART USING (3.23). DEGREES OF FREEDOM = 3; RHO = 0.038; LEVEL = 0.01.

t_1

t	1	2	4	6	8	10
1	0.01021	0.01021	0.01021	0.01021	0.01021	0.01021
5	0.05002	0.05001	0.05000	0.04999	0.04999	0.04999
10	0.09753	0.09752	0.09749	0.09747	0.09745	0.09743
15	0.14267	0.14265	0.14261	0.14258	0.14255	0.14252
20	0.18555	0.18552	0.18548	0.18544	0.18540	0.18536
25	0.22628	0.22626	0.22620	0.22615	0.22611	0.22606
30	0.26498	0.26495	0.26489	0.26483	0.26478	0.26473
35	0.30174	0.30171	0.30164	0.30158	0.30152	0.30146
40	0.33667	0.33663	0.33656	0.33649	0.33643	0.33636
45	0.36984	0.36980	0.36973	0.36966	0.36959	0.36952
50	0.40136	0.40132	0.40124	0.40116	0.40109	0.40102
60	0.45975	0.45970	0.45961	0.45953	0.45945	0.45938
70	0.51244	0.51239	0.51230	0.51221	0.51213	0.51205
80	0.55999	0.55994	0.55985	0.55975	0.55967	0.55959
90	0.60290	0.60285	0.60276	0.60267	0.60258	0.60249
100	0.64163	0.64158	0.64149	0.64139	0.64131	0.64122
110	0.67658	0.67653	0.67644	0.67635	0.67626	0.67618
120	0.70813	0.70808	0.70798	0.70789	0.70781	0.70773
130	0.73659	0.73654	0.73645	0.73636	0.73628	0.73620
140	0.76228	0.76224	0.76215	0.76206	0.76198	0.76190
150	0.78547	0.78542	0.78534	0.78525	0.78517	0.78510

TABLE 3.3. APPROXIMATIONS TO THE ASYMPTOTIC CDF OF THE RUN LENGTH OF A CASE II GENERALIZED P-CHART USING (3.23). DEGREES OF FREEDOM = 3; RHO = 0.091; LEVEL = 0.01.

	t_1					
t	1	2	4	6	8	10
1	0.01021	0.01021	0.01021	0.01021	0.01021	0.01021
5	0.05002	0.04998	0.04990	0.04987	0.04987	0.04987
10	0.09753	0.09746	0.09732	0.09718	0.09706	0.09694
15	0.14267	0.14256	0.14236	0.14217	0.14199	0.14182
20	0.18555	0.18541	0.18516	0.18492	0.18469	0.18448
25	0.22628	0.22613	0.22582	0.22554	0.22527	0.22501
30	0.26498	0.26480	0.26446	0.26413	0.26382	0.26353
35	0.30174	0.30154	0.30116	0.30080	0.30046	0.30014
40	0.33667	0.33645	0.33604	0.33565	0.33527	0.33492
45	0.36984	0.36961	0.36917	0.36875	0.36836	0.36798
50	0.40136	0.40112	0.40065	0.40021	0.39979	0.39939
60	0.45975	0.45948	0.45898	0.45850	0.45804	0.45761
70	0.51244	0.51216	0.51163	0.51112	0.51064	0.51019
80	0.55999	0.55970	0.55915	0.55863	0.55814	0.55767
90	0.60290	0.60261	0.60205	0.60153	0.60103	0.60055
100	0.64163	0.64134	0.64078	0.64025	0.63975	0.63927
110	0.67658	0.67629	0.67574	0.67521	0.67471	0.67424
120	0.70813	0.70784	0.70729	0.70678	0.70628	0.70582
130	0.73659	0.73631	0.73578	0.73527	0.73479	0.73433
140	0.76228	0.76201	0.76149	0.76100	0.76053	0.76009
150	0.78547	0.78521	0.78470	0.78423	0.78377	0.78334

An important special case of Theorem 3.6 occurs when c=2, giving the standard p-chart. The widely accepted p-chart procedure for Case II leads one down the same primrose path as the statistic (3.22). In terms of the normal distribution, the correct asymptotic bounds are not (3.8), but

$$p_0 \pm K[(\frac{m+n}{mn})\ p_0(1-p_0)]^{\frac{1}{2}}\ . \tag{3.24}$$

This last expression follows readily from (3.18) upon noting that in this instance $\underset{\sim}{\Sigma}_0$ is merely the scalar $\pi_0(1-\pi_0)$-which is consistently estimated by $p_0(1-p_0)$. As (3.8) is always of smaller width than (3.24), the usual p-chart under Case II signals more frequently, in the limit, than its level indicates.

The use of (3.24) has the possible drawback of being a two-sided procedure. While some type of ordered alternatives, as discussed in Bergman (1981), for example, might be reasonable for the generalized p-chart, these will not be pursued here. A brief discussion will be provided, however, of a one-sided asymptotic chart for the binomial parameter. An appropriate control limit for detecting $\pi_i>\pi_0$ would compare p_i against

$$p_0 + K(\frac{m+n}{mn}\ p_0(1-p_0))^{\frac{1}{2}}\ . \tag{3.25}$$

If (3.25) is expressed in the form: declare the process in control when

$$(\frac{mn}{m+n})^{\frac{1}{2}}\ \frac{p_i-p_0}{[p_0(1-p_0)]^{\frac{1}{2}}} \leq K\ , \tag{3.26}$$

then the joint distribution of the p-chart values follows from (3.17). Under the limiting behavior of m and n contained in Theorem 3.6, the p-chart statistics have an equicorrelated normal distribution with correlation parameter $1/1+\eta$. Since Theorem 1.6 is also valid for the p-chart of (3.26), a result analogous to Theorem 3.7 is immediate.

Another adjustment to the generalized p-chart of Theorem 3.6 is required if it is not possible to ensure that equal sample sizes are taken in each monitoring period. Comparatively little adjustment needs to be made in the monitoring scheme - the upper control limit is unchanged but the statistic (3.21) becomes

$$X^2_{n_i} = \left(\frac{n_i m}{n_i+m}\right) \sum_{j=1}^{c} \frac{(p_{ij}-p_{0j})^2}{\hat{\pi}_{0j}} . \tag{3.27}$$

There are more substantial changes, however, in the properties of the chart.

First, the basic result of Theorem 3.6 is unaltered, but a new correlation structure is required for the multivariate χ^2 distribution. The asymptotic distribution of $(X^2_{n_i},\ldots,X^2_{n_k})$ is $\chi^2_k(c-1,\underset{\sim}{R})$, where $\frac{m}{n_i}\to\eta_i$ as n_i, $m\to\infty$ and the elements of $\underset{\sim}{R}=(r_{ij})$ are given by

$$r_{ij} = \begin{cases} 1 & i=j \\ \left(\dfrac{1}{(1+\eta_i)(1+\eta_j)}\right)^{\frac{1}{2}} & i\neq j \end{cases} \qquad i,j=1,2,\ldots,c-1 . \tag{3.28}$$

The proof is identical to the one given for Theorem 3.6 with some additional attention paid to keeping track of the various n_i's.

Since the $X^2_{n_i}$'s are no longer identically distributed we cannot make direct use of Theorem 1.6 to obtain stochastic bounds on the run-length distribution of the control chart. A similar result, however, is available when the process is in control.

Theorem 3.8. Suppose in monitoring a multinomial process different size samples are taken at each observation period. Let N_M be the run length of the control chart using $(X^2_{n_1}, X^2_{n_2}, \ldots, X^2_{n_t})$, where $X^2_{n_i}$ is given by (3.27). Let $N_{M'}$ be the run length of the control chart which uses $n = \min(n_1, n_2, \ldots, n_t)$ in (3.21). Then, when the process is in control,

$$P(N_M>t) \geq P(N_{M'}>t) \geq P[\bigcap_{i=1}^{t_1}\{X_i^2\leq u\}]^{t/t_1}$$

$$\geq [P(X_n^2\leq u)]^t = P(N_G>t)$$

as each $n_i\to\infty$ such that $m/n_i\to\eta_i$, where N_G has the geometric distribution with parameter $\theta = P(X_n^2>u)$, with X_n^2 given by (3.27) with $n_i=n$.

Proof. Define $\lambda_i = 1/1+\eta_i$ for $i=1,\ldots,t$, and let

$$\theta_{ij} = \begin{cases} 1 & \text{if } i=j \\ \lambda_i\lambda_j & \text{if } i\neq j \end{cases}$$

The correlation matrix defined by (3.28) and the quadratic forms $(X^2_{n_1}, \ldots, X^2_{n_t})$ satisfy the conditions of Theorem 2.6. Therefore, $P(X^2_{n_1}\leq u, \ldots, X^2_{n_t}\leq u)$ is nondecreasing in each λ_i, or it is nondecreasing in the correlation parameters. Hence,

$$P(X^2_{n_1} \le u, \ldots, X^2_{n_t} \le u) \ge P(X^2_1 \le u, \ldots, X^2_t \le u) .$$

That is, $P(N_M > t) \ge P(N_{M'} > t)$. The remainder of the proof follows by noting that $N_{M'}$ satisfies the conditions of Theorem 3.7.

Reviewing the procedure for the case of unequal sample sizes in the monitoring period, note first that the control limits, based on the $\chi^2_1(c-1,1)$ distribution, are the same as when all the sample sizes are equal. The run-length distributions differ for the two cases, however. When the process is in control the procedure for varying sample sizes is stochastically larger, asymptotically, than the generalized p-chart with the sample size for each monitoring period equal to n of Theorem 3.8.

To this point in the present section we have dealt mainly with monitoring procedures when the multinomial process is in control. Now, the asymptotic properties of the monitoring procedure and the run-length distribution when the process is out of control and possibly drifting out of control are explored. Pitman-type alternative parameter sequences are assumed for each sampling period. That is, as $n \to \infty$, $\sqrt{n}(\underset{\sim}{\pi}_i - \underset{\sim}{\pi}_0) = \underset{\sim}{d}_i$, or

$$\pi_{ij} = \pi_{0j} + \frac{d_{ij}}{\sqrt{n}} \quad \begin{matrix} j=1,\ldots,c \\ i=1,\ldots,k \end{matrix} \quad . \tag{3.29}$$

The initial step is to find the asymptotic joint distribution of the statistics used in the generalized p-chart when the process is not in control.

Theorem 3.9. For X_i^2 as defined by (3.14), consider the distribution of $(X_1^2,\ldots,X_k^2)$ as $m,n\to\infty$ such that $m/n\to\eta$, and suppose the sequence of parameters is of the type (3.29). Then the joint asymptotic distribution of $(X_1^2,\ldots,X_k^2)$ is $\chi_k^2(c-1,\underset{\sim}{R},\underset{\sim}{\Delta})$, where $\underset{\sim}{R} = \frac{\eta}{1+\eta}\,\underset{\sim}{I}_k + \frac{1}{1+\eta}\,\underset{\sim}{J}_k$, $\underset{\sim}{\Delta} = (\frac{\eta}{1+\eta})\underset{\sim}{M}\underset{\sim}{\Sigma}_0^{-1}\underset{\sim}{M}'$, $\underset{\sim}{M}' = [\underset{\sim}{d}_1,\ldots,\underset{\sim}{d}_k]$, and $\underset{\sim}{\Sigma}_0$ is defined at (3.15).

Proof. Consider,

$$\begin{aligned}\sqrt{n}(\underset{\sim}{p}_i-\underset{\sim}{p}_0) &= \sqrt{n}((\underset{\sim}{p}_i-\underset{\sim}{p}_0) - (\underset{\sim}{\pi}_i-\underset{\sim}{\pi}_0)) + \sqrt{n}(\underset{\sim}{\pi}_i-\underset{\sim}{\pi}_0)\\ &= \underset{\sim}{Z}_i + \sqrt{n}(\underset{\sim}{\pi}_i-\underset{\sim}{\pi}_0)\ ,\end{aligned}$$

where $\underset{\sim}{Z}_i$ is as defined at (3.16). As in the proof of Theorem 3.6, we have

$$\underset{\sim}{Z} = (\underset{\sim}{Z}_1',\ldots,\underset{\sim}{Z}_k')' \xrightarrow{d} N_{k(c-1)}(\underset{\sim}{0},\underset{\sim}{\Omega})\ ,$$

where $\underset{\sim}{\Omega} = (\underset{\sim}{J}_k\times\eta^{-1}\underset{\sim}{\Sigma}_0) + \mathrm{diag}(\underset{\sim}{\Sigma}_0,\ldots,\underset{\sim}{\Sigma}_0)$. This expression for $\underset{\sim}{\Omega}$ follows, since under the assumption (3.29) $\underset{\sim}{\Sigma}_i \xrightarrow{p} \underset{\sim}{\Sigma}_0$.

Hence,

$$\sqrt{n}\begin{bmatrix}\underset{\sim}{p}_1-\underset{\sim}{p}_0\\ \vdots\\ \underset{\sim}{p}_k-\underset{\sim}{p}_0\end{bmatrix} \xrightarrow{d} N_{k(c-1)}(\underset{\sim}{\mu},\underset{\sim}{\Omega})$$

where $\underset{\sim}{\mu} = (\underset{\sim}{d}_1',\ldots,\underset{\sim}{d}_k')'$.

The joint asymptotic distribution of the quadratic forms $(q_1,\ldots,q_k)$, where q_i is defined at (3.18), is, by Theorem 2.5, $\chi_k^2(c-1,\underset{\sim}{R},\underset{\sim}{\Delta})$.

We are now in a position to apply some of the results of Chapter Two concerning the noncentral multivariate chi-squared distribution to multinomial process monitoring when the process is out of control.

Theorem 3.10. Suppose a stationary multinomial process is monitored using (3.14) when the parameter sequence is of the type (3.29). Let $\underset{\sim}{Y}_1 = (Y_{11},\ldots,Y_{1k})'$ and $\underset{\sim}{Y}_2 = (Y_{21},\ldots,Y_{2k})'$ represent two such stationary processes with noncentrality parameters δ_1 and δ_2, respectively. If $\delta_1 > \delta_2$, then the control chart based on $\underset{\sim}{Y}_1$ asymptotically dominates the control chart based on $\underset{\sim}{Y}_2$.

Proof. By Corollary 2.3, $\underset{\sim}{Y}_1$ is stochastically larger than $\underset{\sim}{Y}_2$ in the limit. That is,

$$P(\bigcap_{i=1}^{t} \{Y_{1i} \le u\}) \le P(\bigcap_{i=1}^{t} \{Y_{2i} \le u\}) .$$

So, if u represents the control limit of the chart using the statistic (3.14),

$$P(N_1 > t) \le P(N_2 > t) .$$

Hence, the control chart based on $\underset{\sim}{Y}_1$ asymptotically dominates the control chart based on $\underset{\sim}{Y}_2$.

Letting $\delta_2 = 0$ implies that the procedure (3.14) is asymptotically unbiased. In words, Theorem 3.10 says that if a stationary process is more out-of-control, in the sense that its noncentrality parameter is of greater magnitude, then, in the limit, it signals more frequently in probability.

We next consider a particular type of nonstationary process which can be dealt with using the results of Section 2.6. Suppose a multinomial process is in control until some time t_1, when it switches to an out-of-control, but stationary, state. The noncentrality matrix of such a process is given by

$$M = \begin{bmatrix} \underset{\sim}{0}_{t_1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \delta \underset{\sim}{J}_{t_2} \end{bmatrix} \tag{3.30}$$

for the first $t = t_1 + t_2$ time periods. Denote such a class of drifting processes by S_{t_1}.

Theorem 3.11. Suppose a multinomial process in the class S_{t_1} is monitored under Case II using (3.14). Let $\underset{\sim}{Y}_1 = (Y_{11}, \ldots, Y_{1t})'$ and $\underset{\sim}{Y}_2 = (Y_{21}, \ldots, Y_{2t})'$ represent the control chart values of two such processes with δ in (3.30) equal to δ_1 and δ_2, respectively. If $\delta_1 > \delta_2$, then the $\underset{\sim}{Y}_1$ chart asymptotically dominates the $\underset{\sim}{Y}_2$ chart.

Proof. By the results of Theorem 2.8, $\underset{\sim}{Y}_1$ is stochastically larger than $\underset{\sim}{Y}_2$. The remainder of the proof follows as in the proof of Theorem 3.10.

3.6 Summary

In this chapter we have described various techniques for monitoring Bernoulli and multinomial processes. Results from the theory of statistical hypothesis testing were employed in a process control setting.

First, for Bernoulli processes under Case I monitoring, standard p-charts were shown to have some optimal run-length properties. Generalized p-charts, for monitoring multinomial processes, were also introduced. At each monitoring occasion a Pearson statistic for goodness of fit is used to obtain an asymptotic χ^2 control chart.

For the Case II monitoring of a Bernoulli process we considered a control chart based on the UMPU hypothesis test for comparing two binomial populations. Some weak optimality properties resulted, but no general small-sample run-length properties could be offered for such a control chart. The large-sample properties of this procedure, however, were favorable.

In the last section of the chapter an asymptotic procedure was given for the Case II monitoring problem for multinomial processes. It was shown that the procedure which uses a Pearson statistic for testing the homogeneity of two multinomial populations at each monitoring period gives a run-length distribution whose asymptotic properties depend upon a multivariate chi-squared distribution. This characterization of the run-length distribution holds for both in-control or out-of-control processes. Other asymptotic properties of this procedure were also given.

IV. MONITORING MULTIVARIATE BERNOULLI AND MULTINOMIAL PROCESSES

4.1 Introduction

In the preceding chapter we discussed various aspects of and approaches to monitoring the attributes of a process when there was only one characteristic of interest. It may be the case that each realization of a process must be judged on each of several, possibly dependent, characteristics. Under these circumstances it is desirable to be able to determine whether or not the process is in control based on its behavior over all its characteristics.

Two general approaches to this problem, each with its own merits, will be considered in the present chapter. One approach may be described as omnibus procedures; all attributes are considered concurrently, and a decision is made as to whether the process is in or out of control based on a single statistic. No attempt is made to identify which characteristics cause the chart to signal. Omnibus procedures may be particularly attractive when the assignment of cause is superflous - for instance, when the entire process must be adjusted with respect to all characteristics to return it to an in-control state.

The second methodology, called diagnostic procedures, gives foremost attention to identifying which of the characteristics is causing the chart to signal that the process is out of control. In contradistinction to the situations where omnibus procedures are suitable, diagnostics are to be preferred as more informative when the process may be adjusted separately for each, or several, of the characteristics.

In the absence of otherwise suitable techniques for monitoring multivariate attribute data, quality control practitioners might adopt the ad hoc procedures of monitoring each attribute separately. Since such a procedure is precisely a diagnostic control chart, studying the properties of these charts acquires an added importance.

A more complete statistical description of the general problem considered in this chapter follows. Independent random samples of size n_i are taken from a multivariate Bernoulli or multinomial process at specified intervals over time. Denote these samples by $(\underset{\sim}{X}_0, \underset{\sim}{X}_1, \ldots)$. At times, a more detailed description of the $\underset{\sim}{X}_i$'s will be necessary. The following, admittedly cumbersome, conventions are adopted:

(i) $X_{ijk\ell}$ - refers to the scalar which is the ℓth observation ($\ell = 1,2,\ldots,n_i$) of the kth classification ($k = 1,2,\ldots,c_j$) of the jth attribute ($j = 1,2,\ldots,r$) on the ith sampling occasion ($i = 1,2,\ldots$).

(ii) π_{ijk} - the probability of occurrence of the kth classification of the jth attribute on the ith sampling occasion.

(iii) $X^{(a,b)}_{ijk\ell}$ - the ℓth observation of the ath classification of attribute j and the bth classification of attribute k on the ith sampling occasion; $X^{(a,b)}_{ijk\ell} = 1$ if both occur on the ℓth observation, 0 otherwise.

(iv) $\pi^{(a,b)}_{ijk}$ - the probability of the ath classification of attribute j occurring with the bth classification of attribute k on the ith sampling occasion.

The outer subscripts may be deleted to yield appropriate vectors; e.g., $\underset{\sim}{X}_{ij}$ denotes all the observations on the jth attribute for the ith sampling occasion.

The definitions of the statistical terminology for the omnibus procedures are identical to the univariate procedures discussed in Chapter Three. For the in-control region A, the level of the chart is given by α, where

$$P(T(\underset{\sim}{X}_i) \in A|\text{in control}) = 1 - \alpha$$

$$P(T(X_i,X_0) \in A|\text{in control}) = 1 - \alpha, \; i = 1,2,\ldots$$

for Cases I and II, respectively. The run-length distribution is found for Case I from $P(N > t) = P(\bigcap_{i=1}^{t} \{T(\underset{\sim}{X}_i) \in A\})$.

Diagnostic procedures, on the other hand, require a slight reformulation of these definitions. For in-control limits $A_1,\ldots,A_r$, the level of the chart is now given by α, where

$$P(\bigcap_{j=1}^{r} \{T_j(\underset{\sim}{X}_i) \in A_j\}|\text{process in control}) = 1 - \alpha$$

$$P(\bigcap_{j=1}^{r} \{T_j(\underset{\sim}{X}_i,\underset{\sim}{X}_0) \in A_j\}|\text{process in control}) = 1 - \alpha$$

$(i = 1,2,\ldots)$,

for Cases I and II, respectively. The diagnostic charts will be said to signal out of control if at any sampling occasion the value of at least one of the r individual control charts is outside its control limits. Therefore, the run-length distribution is defined by

$$P(N_D > t) = P(\bigcap_{i=1}^{t} \bigcap_{j=1}^{r} \{T_j(\underset{\sim}{X}_i) \in A_j\})$$

$$= P(N_1 > t, N_2 > t, \ldots, N_r > t)$$

$$= P(\min_j N_j > t) \ ,$$

where N_j is the run length of the control chart for the jth attribute, and N_D is the run length of the diagnostic procedure. In general, the r run lengths, $N_1, \ldots, N_r$, will be dependent.

It is appropriate to relate the notions of omnibus and diagnostic control charts to the theory of statistical hypothesis testing. With omnibus procedures we are testing the hypotheses H: $\underset{\sim}{\pi}_i = \underset{\sim}{\pi}_0$ versus A: $\underset{\sim}{\pi}_i \neq \underset{\sim}{\pi}_0$ at sampling occasion i. Here, $\underset{\sim}{\pi}_0$ represents either a known quality standard, or the standard as determined from a base period.

Diagnostic charts, on the other hand, test the r hypotheses

$$H: \quad \bigcap_{j=1}^{r} \{\underset{\sim}{\pi}_{ij} = \underset{\sim}{\pi}_{0j}\}$$

$$\text{versus} \qquad A: \quad \bigcup_{j=1}^{r} \{\underset{\sim}{\pi}_{ij} \neq \underset{\sim}{\pi}_{0j}\}$$

simultaneously at each monitoring period.

In Sections Two and Three of this chapter omnibus procedures and some of their properties are discussed for multivariate Bernoulli and multinomial processes. In Sections Four and Five diagnostic methods are considered for the same problems.

4.2 Omnibus Procedures for Multivariate Bernoulli and Multinomial Processes - Case I

New problems of practicality immediately arise for multivariate Bernoulli or multinomial process monitoring under Case I. As seen in Chapter Two, the additional dimensions of the problem induce a somewhat exponential growth in the number of population parameters. For even moderate dimensioned binomial random vectors the number of parameters quickly becomes unmanageable. For example, in four dimensions there are fifteen parameters, for five dimensions, thirty-one. It is unreasonable to expect that all these parameters can be specified *a priori*.

A partial remedy to this preponderance of parameters is to use procedures based upon the asymptotic normality of the multivariate distributions in question. In this section we will concentrate mainly on asymptotic procedures for the multivariate binomial distribution. With this approach, for instance, the number of parameters of an r-variate binomial distribution that need to be specified is reduced from 2^r-1 to $r + \binom{r}{2}$. We need only specify the marginal binomial parameters $\pi_1,\ldots,\pi_r$ and the pairwise correlation coefficients $\rho_{12},\ldots,\rho_{r-1,r}$. There still may be times, however, when knowing even this reduced number of parameters is an unreasonable assumption. This problem will be more fully addressed in Sections Three and Five.

Patel (1973) proposes a monitoring procedure for the multivariate binomial distribution along the lines of the preceding paragraph. In the tradition of quality control he proposes using the same test statistic both for specified levels of quality and for base-period estimates of quality levels. His α-level control chart uses

$$n(\underset{\sim}{p}_i - \underset{\sim}{\pi}_0)'\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{p}_i - \underset{\sim}{\pi}_0) \tag{4.1}$$

for the control chart values with a control limit established at the 100(1-α)th percentile of the $\chi_1^2(r,1)$ distribution. In (4.1), $\underset{\sim}{p}_i$ is of order r×1 with components $p_{ij} = \frac{1}{n}\sum_{\ell=1}^{n} X_{ij1\ell}$, j = 1,...,r, and

$$\sqrt{n}\,\underset{\sim}{p}_i \xrightarrow{d} N_r(\sqrt{n}\,\underset{\sim}{\pi}_0, \underset{\sim}{\Sigma}) ,$$

where $\underset{\sim}{\Sigma}$ is as given at (2.7), and $\underset{\sim}{\pi}_0$ is the specified quality standard for the binomial proportions. When $\underset{\sim}{\pi}_0$ and $\underset{\sim}{\Sigma}$ are unknown, Patel suggests estimating them in a base period by the quantities

$$\hat{\underset{\sim}{\pi}}_0 = (\hat{\pi}_{0i}) \ , \ \hat{\pi}_{0i} = \sum_{\ell=1}^{m} \frac{X_{0i1\ell}}{m} \quad i = 1,\ldots,r$$

and

$$\underset{\sim}{S} = (s_{ij}) \ , \ s_{ij} = \sum_{\ell=1}^{m} \frac{(X_{0i1\ell} - \hat{\pi}_{0i})(X_{0j1\ell} - \hat{\pi}_{0j})}{n - 1}$$

$$i,j = 1,\ldots,r , \tag{4.2}$$

respectively, and replacing $\underset{\sim}{\pi}_0$ and $\underset{\sim}{\Sigma}$ in (4.1) by these estimates. Any further discussion of a Case II omnibus procedure will be postponed until the next section. It will be seen there that Patel's procedure suffers from the same shortcomings as the traditional univariate _ad hoc_ procedures for using estimates.

Some amplifying remarks are necessary, however, concerning the use of (4.1) when the target parameters are known. First, for the bivariate case, the small-sample properties of the asymptotic procedure

(4.1) can be explored. A procedure for finding the acceptance regions based on (4.1) in terms of the number defective is given in the following paragraph. The probability contents of these regions can then be evaluated using the bivariate binomial distribution.

The acceptance region for an α-level asymptotic chart based on (4.1) when r = 2 can be expressed as

$$a_i^2 + b_i^2 - 2\rho a_i b_i - \frac{(1-\rho^2)}{n} u \leq 0, \tag{4.3}$$

where $$a_i = a(\underset{\sim}{X}_i) = \frac{(\sum_{\ell=1}^{n} X_{i11\ell}/n) - \pi_{01}}{\sqrt{\pi_{01}(1-\pi_{01})}},$$

$$b_i = b(\underset{\sim}{X}_i) = \frac{(\sum_{\ell=1}^{n} X_{i21\ell}/n) - \pi_{02}}{\sqrt{\pi_{02}(1-\pi_{02})}},$$

and $P(\chi_1^2(2,1) \leq u) = 1 - \alpha$.

Ignoring the inequality sign in (4.3) for the moment, for given values of $b_i = b$ the roots of (4.3) considered as a function of a_i are

$$t_1 = \rho b + \sqrt{(1-\rho^2)(u/n - b^2)}$$

$$\text{and} \quad t_2 = \rho b - \sqrt{(1-\rho^2)(u/n - b^2)} \quad . \tag{4.4}$$

If the roots are complex, i.e. $|b| > \sqrt{u/n}$, the quadratic form (4.3) is always greater than zero. It is only necessary, then, to compute (4.4) for those values of $\sum_{\ell=1}^{n} X_{i11\ell} \varepsilon \{0,1,2,\ldots,n\}$ which give real roots. The values of $(\sum_{\ell=1}^{n} X_{i11\ell}, \sum_{\ell=1}^{n} X_{i21\ell})$ for which the process signals out of control are those $\sum_{\ell=1}^{n} X_{i11\ell}$ for given $\sum_{\ell=1}^{n} X_{i21\ell}$ such that

$$\sum_{\ell=1}^{n} X_{i11\ell} > [t_1(\pi_{01}(1-\pi_{01}))^{\frac{1}{2}} + \pi_{01}]$$

$$\text{or,} \quad \sum_{\ell=1}^{n} X_{i11\ell} < [t_2(\pi_{01}(1-\pi_{01}))^{\frac{1}{2}} + \pi_{01}] \ ,$$

where $[\cdot]$ denotes the largest integer function.

For a nominal level of 0.01 the actual level of (4.1) was computed using the bivariate binomial cdf for various n, ρ, and $\underset{\sim}{\pi}$. These results are presented in Tables 4.1-4.3. A striking feature of these tables is the underestimation of the true level of the control chart by the nominal level. At least for these ranges of the parameter values of the bivariate binomial distribution, the control chart based upon (4.1) will signal more frequently when the process is in control than the nominal level of the procedure would indicate. So, care must be exercised in selecting an asymptotic level which reflects the desired in-control small-sample run-length properties. Recall that the standard univariate p-chart (3.1) also exhibited these tendencies.

When the process is not in control, the quadratic form (4.1) converges in law to the noncentral χ^2 distribution if the alternative parameter sequence is of the type (3.29). Therefore, for two run lengths N_1 and N_2 with associated noncentrality parameters $\underset{\sim}{\delta}_1$ and $\underset{\sim}{\delta}_2$,

TABLE 4.1. ACTUAL LEVELS OF THE OMNIBUS PROCEDURE (4.1) FOR A BIVARIATE BERNOULLI PROCESS FOR VARIOUS VALUES OF N AND $\underset{\sim}{\pi}_0$ WHEN ρ IS 0.10. THE NOMINAL LEVEL IS 0.01.

$\pi_{20} = 0.010$

π_{10}	N = 15	N = 25	N = 35
0.010	0.0189	0.0498	0.0153
0.015	0.0297	0.0314	0.0259
0.020	0.0440	0.0381	0.0198
0.025	0.0248	0.0481	0.0324
0.030	0.0312	0.0315	0.0401

$\pi_{20} = 0.0125$

0.015	0.0346	0.0442	0.0245
0.020	0.0487	0.0508	0.0173
0.025	0.0312	0.0606	0.0246
0.030	0.0237	0.0443	0.0235

$\pi_{20} = 0.0150$

0.015	0.0399	0.0159	0.0303
0.020	0.0543	0.0247	0.0238
0.025	0.0261	0.0369	0.0260
0.030	0.0296	0.0148	0.0289

$\pi_{20} = 0.0175$

0.020	0.0607	0.0291	0.0321
0.025	0.0329	0.0325	0.0335
0.030	0.0363	0.0185	0.0361

$\pi_{20} = 0.020$

0.020	0.0679	0.0260	0.0211
0.025	0.0405	0.0363	0.0266
0.030	0.0439	0.0230	0.0317

TABLE 4.2. ACTUAL LEVELS OF THE OMNIBUS PROCEDURE (4.1) FOR A BIVARIATE BERNOULLI PROCESS FOR VARIOUS VALUES OF N AND $\underset{\sim}{\pi}_0$ WHEN ρ IS 0.25. THE NOMINAL LEVEL IS 0.01.

$\pi_{02} = 0.010$

π_{01}	N = 15	N = 25	N = 35
0.010	0.0182	0.0479	0.0526
0.015	0.0287	0.0307	0.0422
0.020	0.0426	0.0370	0.0357
0.025	0.0202	0.0465	0.0284
0.030	0.0182	0.0309	0.0306

$\pi_{02} = 0.0125$

π_{01}	N = 15	N = 25	N = 35
0.015	0.0333	0.0434	0.0239
0.020	0.0470	0.0494	0.0293
0.025	0.0196	0.0586	0.0197
0.030	0.0230	0.0436	0.0220

$\pi_{02} = 0.0150$

π_{01}	N = 15	N = 25	N = 35
0.015	0.0388	0.0156	0.0294
0.020	0.0522	0.0278	0.0345
0.025	0.0255	0.0418	0.0254
0.030	0.0288	0.0269	0.0275

$\pi_{02} = 0.0175$

π_{01}	N = 15	N = 25	N = 35
0.020	0.0583	0.0216	0.0352
0.025	0.0322	0.0317	0.0327
0.030	0.0354	0.0202	0.0347

$\pi_{02} = 0.020$

π_{01}	N = 15	N = 25	N = 35
0.020	0.0651	0.0253	0.0259
0.025	0.0397	0.0353	0.0243
0.030	0.0428	0.0248	0.0268

TABLE 4.3. ACTUAL LEVELS OF THE OMNIBUS PROCEDURE (4.1) FOR A BIVARIATE BERNOULLI PROCESS FOR VARIOUS VALUES OF N AND $\underset{\sim}{\pi}_0$ WHEN ρ IS 0.40. THE NOMINAL LEVEL IS 0.01.

$\pi_{02} = 0.010$

	N		
π_{01}	15	25	35
0.010	0.0172	0.0452	0.0470
0.015	0.0273	0.0488	0.0269
0.020	0.0408	0.0356	0.0288
0.025	0.0390	0.0444	0.0262
0.030	0.0175	0.0435	0.0282

$\pi_{02} = 0.0125$

0.015	0.0315	0.0584	0.0230
0.020	0.0447	0.0477	0.0366
0.025	0.0412	0.0561	0.0250
0.030	0.0221	0.0542	0.0247

$\pi_{02} = 0.0150$

0.015	0.0366	0.0490	0.0282
0.020	0.0494	0.0277	0.0251
0.025	0.0447	0.0350	0.0245
0.030	0.0278	0.0331	0.0264

$\pi_{02} = 0.0175$

0.020	0.0550	0.0207	0.0314
0.025	0.0494	0.0413	0.0316
0.030	0.0343	0.0308	0.0286

$\pi_{02} = 0.020$

0.020	0.0615	0.0243	0.0213
0.025	0.0551	0.0339	0.0232
0.030	0.0416	0.0256	0.0269

if $\underset{\sim}{\delta}_1 > \underset{\sim}{\delta}_2$ then, asymptotically, N_1 is stochastically smaller than N_2. The independence of these test statistics implies that this result holds for processes drifting in $\underset{\sim}{\delta}$ as well as for stationary processes.

The methodology of this section also may be extended to multivariate multinomial processes. Since the number of parameters that would have to be specified increases dramatically over the number in the multivariate binomial distribution, this Case I procedure may be of little practical value. This topic will be pursued no further here, since Case II procedures, which will be discussed next, seem more appropriate.

4.3 Omnibus Procedures for Monitoring Multivariate Multinomial Processes - Case II: Generalized p-charts for Several Characteristics

In Section Two, the severe limitations inherent in requiring the knowledge of all the marginal probabilities plus the first order correlations were discussed. Case II procedures, therefore, are of paramount interest. In this section an extension of the univariate procedure of Section 3.5 is given, and some of its properties are determined.

Recall that in the case of a single attribute we proposed a control chart which was, at least asymptotically, equivalent to performing multiple tests of homogeneity for 2×c contingency tables. Unfortunately, no such nice interpretation is available for the multivariate multinomial monitoring procedure to be proposed. This shortcoming aside, we can mostly parallel the developments of Section 3.5 in this more general setting.

Make the following definition:

$$\underset{\sim}{Y}_i = \begin{bmatrix} \underset{\sim}{Y}_{i1} \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{Y}_{ir} \end{bmatrix} = n^{\frac{1}{2}} \begin{bmatrix} \underset{\sim}{p}_{i1} - \underset{\sim}{\pi}_{i1} \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{p}_{ir} - \underset{\sim}{\pi}_{ir} \end{bmatrix}, \quad i=0,1,\ldots,k \qquad (4.5)$$

$$\text{where } \underset{\sim}{Y}_{ij} = \left(\frac{\sum_{\ell=1}^{n_i} X_{ij1\ell}}{n_i}, \ldots, \frac{\sum_{\ell=1}^{n_i} X_{ijc_j-1,\ell}}{n_i} \right)' \quad \begin{matrix} i=0,\ldots,k \\ j=1,\ldots,r \end{matrix},$$

$$\text{and} \quad n_i = \begin{cases} m & \text{if } i = 0 \\ n & \text{otherwise} \end{cases}.$$

Theorem 4.1. Suppose an r-variate multinomial process is monitored under Case II. Let $\underset{\sim}{Y}_0$ of (4.5) represent the base period, and $\{\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_k\}$ the k monitoring periods; and suppose the $\underset{\sim}{Y}_i$'s are independent. As m and n of (4.5) tend to ∞ such that $m/n \to \eta$, consider the statistics

$$Z_i = \frac{n\eta}{(1+\eta)} \begin{bmatrix} \underset{\sim}{Y}_{i1} - \underset{\sim}{Y}_{01} \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{Y}_{ir} - \underset{\sim}{Y}_{0r} \end{bmatrix}' \hat{\underset{\sim}{\Sigma}}_0^{-1} \begin{bmatrix} \underset{\sim}{Y}_{i1} - \underset{\sim}{Y}_{01} \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{Y}_{ir} - \underset{\sim}{Y}_{0r} \end{bmatrix}, \qquad (4.6)$$

$i=1,\ldots,k,$

where $\hat{\underset{\sim}{\Sigma}}_0$ is a consistent estimator for the covariance matrix defined below at (4.7). Suppose an asymptotic control chart is established which uses $(Z_1,Z_2,\ldots,Z_k)$ for monitoring the process through the first k time periods. When the process is in control the joint asymptotic distribution of these control chart values is $\chi^2_k(\sum_{i=1}^{r} c_i - r, \frac{\eta}{1+\eta} \underset{\sim}{T}_k + \frac{1}{1+\eta} \underset{\sim}{J}_k)$.

Proof. In order to prevent the notation from becoming too unwieldy the proof will be given for the bivariate case (r=2) only. The extension to higher dimensions is straightforward, simply requiring changes in notation and additional bookkeeping.

By the multivariate central limit theorem, it follows that

$$\underset{\sim}{Y}_i \xrightarrow{d} N_{c_1+c_2-2}(\underset{\sim}{0}, \underset{\sim}{\Sigma}_i) \quad i=1,\ldots,k \ ,$$

and

$$\underset{\sim}{Y}_0 \xrightarrow{d} N_{c_1+c_2-2}(\underset{\sim}{0}, n^{-1}\underset{\sim}{\Sigma}_0) \ .$$

Partition $\underset{\sim}{\Sigma}_0$ conformably with $\underset{\sim}{Y}_{01}$ and $\underset{\sim}{Y}_{02}$ as:

$$\underset{\sim}{\Sigma}_0 = \begin{bmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22} \end{bmatrix}, \tag{4.7}$$

where $\underset{\sim}{\Sigma}_{ii} = (\sigma_{ij\ell})$ with

$$\sigma_{ij\ell} = \begin{cases} \pi_{0ij}\,(1 - \pi_{0ij}) & \text{if } j=\ell \quad j=1,\ldots,c_i-1 \\ -\pi_{0ij}\,\pi_{0i\ell} & \text{if } j\neq\ell \end{cases},$$

$$\underset{\sim}{\Sigma}_{12} = (\sigma_{ij}), \text{ with}$$

$$\sigma_{ij} = \pi_{012}^{(i,j)} - \pi_{01i}\,\pi_{02j} \qquad \begin{array}{l} i=1,\ldots,c_1-1 \\ j=1,\ldots,c_2-1 \end{array} ,$$

and $\underset{\sim}{\Sigma}_{21} = \underset{\sim}{\Sigma}'_{12}$.

Under the in-control hypothesis, by Theorem 1.8 the joint distribution of the appropriate differences satisfies

$$\begin{bmatrix} \underset{\sim}{Q}_1 \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{Q}_k \end{bmatrix} = \begin{bmatrix} \underset{\sim}{Y}_{11} - \underset{\sim}{Y}_{01} \\ \underset{\sim}{Y}_{12} - \underset{\sim}{Y}_{02} \\ \hline \cdot \\ \cdot \\ \cdot \\ \hline \underset{\sim}{Y}_{k1} - \underset{\sim}{Y}_{01} \\ \underset{\sim}{Y}_{k2} - \underset{\sim}{Y}_{02} \end{bmatrix} \xrightarrow{d} N_{k(c_1+c_2-2)}(\underset{\sim}{0}, \underset{\sim}{G}\underset{\sim}{\Omega}\underset{\sim}{G}'), \qquad (4.8)$$

where $\underset{\sim}{\Omega} = \mathrm{diag}(\eta^{-1}\underset{\sim}{\Sigma}_0, \underset{\sim}{\Sigma}_0, \ldots, \underset{\sim}{\Sigma}_0)$,

and

$$\underset{\sim}{G} = \begin{bmatrix} -\underset{\sim}{I}_{c_1-1} & \underset{\sim}{0} & \underset{\sim}{I}_{c_1-1} & & & \\ \underset{\sim}{0} & -\underset{\sim}{I}_{c_2-1} & & \underset{\sim}{I}_{c_2-1} & & \\ \vdots & & & & \ddots & \underset{\sim}{0} \\ & & & \underset{\sim}{0} & & \\ -\underset{\sim}{I}_{c_1-1} & \underset{\sim}{0} & & & \underset{\sim}{I}_{c_1-1} & \\ \underset{\sim}{0} & -\underset{\sim}{I}_{c_2-1} & & & & \underset{\sim}{I}_{c_2-1} \end{bmatrix} .$$

Therefore, $\underset{\sim}{G}\underset{\sim}{\Omega}\underset{\sim}{G}' = (\underset{\sim}{I}_k + \eta\underset{\sim}{J}_k)\times\underset{\sim}{\Sigma}_0$.

Theorems 2.7 and 1.10 imply that the joint distribution of the quadratic forms $(q_1,\ldots,q_k)$, where

$$q_i = n(\frac{\eta}{1+\eta})\ \underset{\sim}{Q}_i'\ \underset{\sim}{\Sigma}_0^{-1}\ \underset{\sim}{Q}_i\ ,\ i=1,\ldots,k\ , \tag{4.9}$$

is $\chi_k^2(c_1+c_2-2,\ \frac{\eta}{1+\eta}\ \underset{\sim}{I}_k + \frac{1}{1+\eta}\ \underset{\sim}{J}_k)$.

The q_i are in terms of the unknown covariance matrix, $\underset{\sim}{\Sigma}_0$. As in Theorem 3.6, if $\underset{\sim}{\Sigma}_0$ is estimated consistently, the asymptotic distribution theory is unaltered. The veracity of this statement follows from an application of Theorem 1.9.

Even with the results of Theorem 4.1, some pertinent outstanding problems still remain concerning the monitoring procedure. The first question to be settled is: What estimator should be used for $\underset{\sim}{\Sigma}_0$ in (4.9)? We stated earlier that Patel (1973) opted for employing (4.2) - the method-of-moments estimator for the covariance matrix. While this approach certainly would have appeal if there were access only to the sample marginal proportions, i.e. that the $X_{ijk\ell}^{(a,b)}$ were not available, it is likely to suffer in efficiency.

A more suitable estimator can be obtained from the work of Hamdan and Martinson (1971). They gave maximum likelihood estimators for the parameters of a bivariate binomial distribution. The estimate for each covariance term in $\underset{\sim}{\Sigma}_0$ could be computed following their suggestion. Hence, using the partition (4.7), we would have

$$\hat{\Sigma}_{ii} = (\hat{\sigma}_{i\ell k}) \;,\; \hat{\sigma}_{i\ell k} = \begin{cases} p_{0i\ell}(1-p_{0i\ell}) & i=1,2;\ \ell=1,\ldots,c_i-1 \\ -p_{0i\ell}\, p_{0ik} & i=1,2;\ \ell < k \end{cases}$$

$$\hat{\Sigma}_{12} = (\hat{\sigma}_{ij}) \;,\; \hat{\sigma}_{ij} = p_{012}^{(i,j)} - p_{01i}\, p_{02j} \qquad \begin{array}{l} i=1,2,\ldots,c_1-1 \\ j=1,2,\ldots,c_2-1 \end{array}$$

where the p_{0ij} are the components of the vector p_{0i} of (4.5) and

$$p_{012}^{(i,j)} = \sum_{\ell=1}^{m} \frac{1}{m} (X_{012\ell}^{(i,j)}) \qquad \begin{array}{l} i=1,2,\ldots,c_1-1 \\ j=1,2,\ldots,c_2-1 \end{array} .$$

For the multivariate binomial case, the test criterion (4.1), which is an *ad hoc* procedure that substitutes base-period estimates for parameter values, also overstates the value of the appropriate asymptotic test statistic by a factor of $(1+\eta^{-1})$. The same drawback that existed with these types of *ad hoc* charts in the univariate Bernoulli monitoring problem is present once more in the multivariate problem. A control chart based on (4.1) will signal more often, in the limit, than the established asymptotic level leads one to believe.

Results parallel to those given in connection with Theorem 3.6 are valid also for the multivariate monitoring procedure of Theorem 4.1. First, the Shewhart-type chart based upon (4.6) signals with probability one. The necessary arguments are exactly those given in Section 3.5 and will not be repeated here. This fact, then, justifies using the finite-dimensional distribution theory of Theorem 4.1 to describe some properties of the control chart.

Theorem 3.7 extends naturally and requires no new proof. This extension is given below.

<u>Theorem 4.2</u>. Suppose a stationary multivariate multinomial process is monitored using (4.6). If N_Z represents the run length of this procedure, then, for $t_1 = 1,2,\ldots,t$,

$$P(N_Z > t) \geq [\Pr(\bigcap_{i=1}^{t_1} Z_i \leq a)]^{t/t_1} . \tag{4.10}$$

Further, these inequalities hold whatever the sample size.

Clearly, the remarks following Theorem 3.7 could be repeated here. Suffice it to say that the chart will take longer, in probability, to signal than it would if the parameters were specified rather than taken from a base period.

Since the covariance structure of the multivariate χ^2 distribution in Theorem 4.1 matches that of Theorem 3.8, similar conclusions can be drawn concerning the effect of varying the sample size in the separate monitoring periods. The in-control run-length distribution for unequal sample sizes will be stochastically larger than for the monitoring procedure which uses the minimum of the unequal sample sizes at each sampling occasion.

Next, the asymptotic properties of the chart using (4.6) when the process is out of control need to be considered. Define the alternative parameter sequence as $\underset{\sim}{e}_i = \sqrt{n}(\underset{\sim}{\pi}_i - \underset{\sim}{\pi}_0)$ as $n\to\infty$, or, component-wise, as

$$\begin{aligned} \pi_{ij\ell} &= \pi_{0j\ell} + \frac{e_{ij\ell}}{\sqrt{n}} \\ \pi_{ij\ell}^{(a,b)} &= \pi_{0j\ell}^{(a,b)} + \frac{e_{ij\ell}^{(a,b)}}{\sqrt{n}} \end{aligned} \qquad \begin{aligned} i&=1,\ldots,k \\ j&=1,\ldots,r \\ \ell&=1,\ldots,c_j-1 \\ a&=1,\ldots,c_j-1 \\ b&=1,\ldots,c_j-1 \end{aligned} \tag{4.11}$$

The following theorem is, again, a generalization of a theorem from Chapter Three.

<u>Theorem 4.3</u>. Suppose a multivariate multinomial process is operating out of control such that its parameters follow (4.11). Consider the control chart established using (4.6). Under the limiting conditions of Theorem 4.1, the values $(Z_1,\ldots,Z_k)$ of the control chart are asymptotically distributed as $\chi_k^2(\sum_{i=1}^{r} c_i - r,\ \underset{\sim}{R},\ \underset{\sim}{\Delta})$, where

$$\underset{\sim}{R} = \frac{\eta}{1+\eta}\ \underset{\sim}{I}_k + \frac{1}{1+\eta}\ \underset{\sim}{J}_k$$

and

$$\underset{\sim}{\Delta} = \left(\frac{\eta}{1+\eta}\right) \begin{bmatrix} \underset{\sim}{e}_1' \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{e}_k' \end{bmatrix} \underset{\sim}{\Sigma}_0^{-1} [\underset{\sim}{e}_1 \cdots \underset{\sim}{e}_k] \ .$$

<u>Proof</u>. Note that $\underset{\sim}{\Sigma}_i \xrightarrow{p} \underset{\sim}{\Sigma}_0$ under the alternative parameter sequence (4.11), since

$$\pi_{ij\ell}^{(a,b)} - \pi_{ija}\,\pi_{i\ell b} = \pi_{0j\ell}^{(a,b)} - \pi_{0ja}\,\pi_{0\ell b} + \pi_{ija}\frac{e_{i\ell b}}{\sqrt{n}} + \pi_{i\ell b}\frac{e_{ija}}{\sqrt{n}} + \frac{e_{i\ell b}\,e_{ija}}{n} + \frac{e_{ij\ell}^{(a,b)}}{\sqrt{n}}$$

$$\xrightarrow{p} \pi_{0j\ell}^{(a,b)} - \pi_{0ja}\,\pi_{0\ell b} \quad \text{for all } i,j,\ell,a,b.$$

Likewise, $\pi_{ij\ell}(1-\pi_{ij\ell}) \xrightarrow{P} \pi_{0j\ell}(1-\pi_{0j\ell})$ and

$$-\pi_{ij\ell}\,\pi_{ijk} \xrightarrow{P} -\pi_{0j\ell}\,\pi_{0jk} \quad \text{for all } i,j,\ell,k.$$

For $\underset{\sim}{Q}_i$ defined in (4.8), we have

$$\sqrt{n}\begin{bmatrix} \underset{\sim}{p}_{i1} - \underset{\sim}{p}_{01} \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{p}_{ir} - \underset{\sim}{p}_{0r} \end{bmatrix} = \underset{\sim}{Q}_i + \underset{\sim}{e}_i \ .$$

The remainder of the proof follows as in Theorem 3.9.

The properties of the noncentral multivariate χ^2 distribution can now be used to establish certain asymptotic properties for the monitoring scheme (4.6) when the multinomial process is out of control, but stationary. These properties are the same as those in Section 3.5, and they will only be summarized here.

Consider two stationary multivariate multinomial processes with noncentrality parameters δ and γ for processes one and two, respectively. If $\delta > \gamma$, then, asymptotically, the control chart of Theorem 4.1 signals more quickly in probability for process one than the same chart for process two. Likewise, if the two processes are in control until some time t and then switch to an out-of-control, but stationary, state with noncentrality parameters δ and γ, process one will signal faster, in probability, than process two.

4.4 Diagnostic Procedures for Monitoring Multivariate Attributes - Case I

We now move from omnibus tests to diagnostic procedures for multivariate Bernoulli and multinomial processes. For small-sample procedures the initial emphasis will be placed upon problems which can be handled using the bivariate binomial distribution. After discussing some small-sample diagnostic charts for Bernoulli processes, diagnostics based upon asymptotic properties of the multivariate multinomial distribution will be proposed and studied.

Starting with the easiest case first, suppose it is necessary to monitor both of the marginal proportions defective of a bivariate Bernoulli process. The only omnibus procedures found for this problem were asymptotic and for two-sided alternatives only. If the parameters of the bivariate process can be specified, then diagnostic control charts can be devised for both small samples and one-sided alternatives. As indicated in Chapter Three, one-sided alternatives are often more suitable for monitoring the percent defective, while the large-sample properties are sometimes quite crude approximations of the actual properties for moderate sized samples.

From heuristic foundations, we will choose the optimal univariate test procedures as the basis for the individual diagnostic charts. First, consider the situation where it is desired to detect if the parameters π_1 or π_2 exceed the specified standards π_{01}, π_{02}. Using limits of the form (3.2) for the two diagnostic charts, and for the moment eschewing randomized tests, the goal is to find rectangular acceptance regions, $(0,U_1)$ and $(0,U_2)$, say, of level α. That is, we need to find (U_1,U_2) such that

$$P(X_1 \leq U_1,\ X_2 \leq U_2) \geq (1-\alpha) \tag{4.12}$$

where $(X_1,X_2)' \sim b_2(n,(\pi_{01},\pi_{02}),\rho)$.

An iterative method is proposed to arrive at values for U_1 and U_2. For a control chart of level α, begin by selecting L_1 and L_2 such that:

$$\begin{aligned} &P(X_1 > L_1) \leq \alpha/k \\ \text{and} \qquad &P(X_2 > L_2) \leq (1-1/k)^{\alpha}. \end{aligned} \tag{4.13}$$

Generally k is chosen to be two, but different weights can be more appropriate. Then, evaluate (4.12), using U_1=L_1 and U_2=L_2. Next decrement L_1 and L_2 by one, and evaluate $P(X_1 \leq L_1-1,\ X_2 \leq L_2-1)$. When limits, U_1^* and U_2^*, say, are found such that the inequality in (4.12) holds for (U_1^*,U_2^*), but not (U_1^*-1,U_2^*-1), decrement one limit at a time and evaluate the joint probabilities. Finally, choose the control limits as those which yield the minimum probability that is greater than or equal to $1-\alpha$. A more detailed explanation of this procedure is provided via a flowchart in Figure 4.1.

In order to fix ideas on the method of obtaining the control limits for the diagnostic chart based on the bivariate binomial distribution, an illustration is presented. Suppose the specified quality level for the percent defective for two characteristics is $(\pi_{01},\pi_{02})'$ = (0.0125, 0.0150)'. In addition, a random sample of size twenty-five

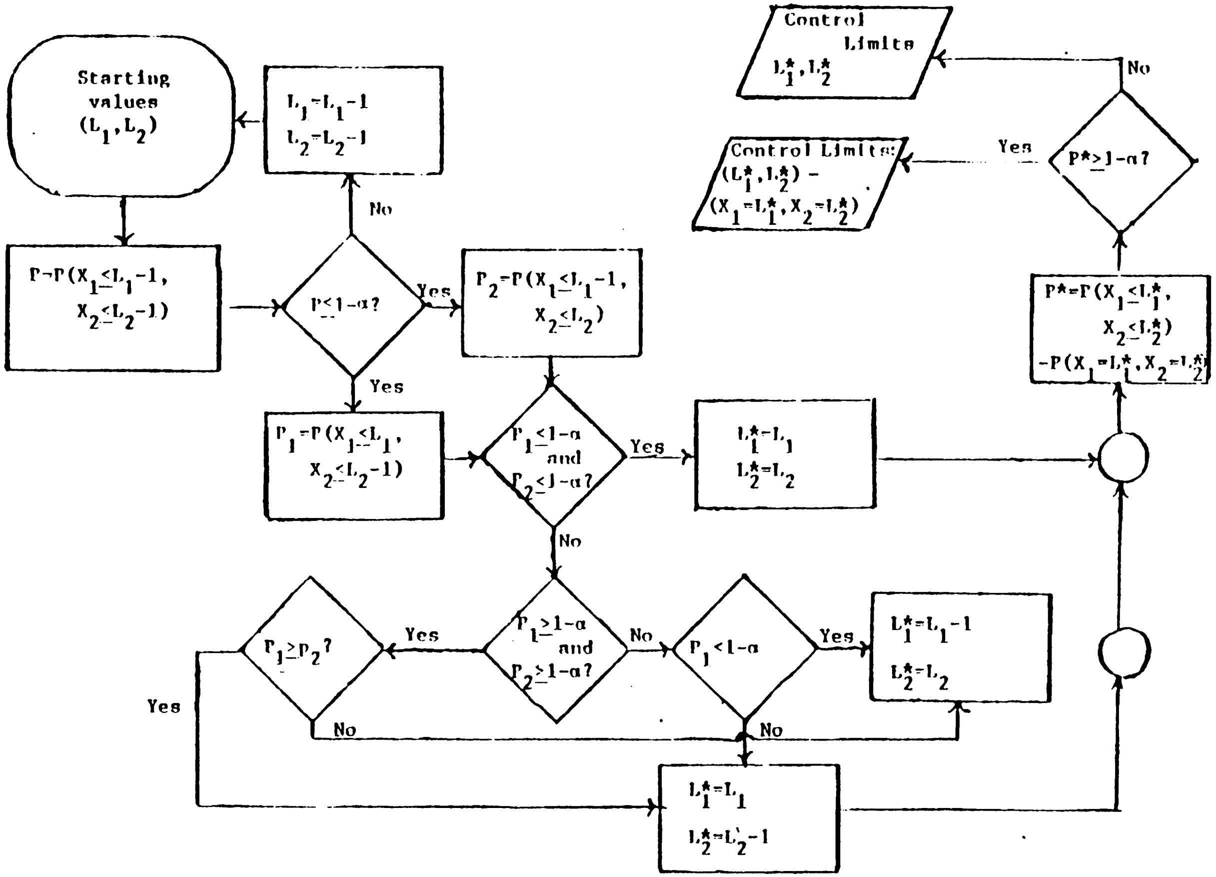

Figure 4.1 Flowchart for the establishment of a bivariate binomial diagnostic chart

is to be taken at each sampling occasion in the monitoring period. The level of the chart is chosen to be 0.01.

For the computation of the starting values from the binomial marginals, take k=2 in (4.13). Then determine that

$P(b_1(25, 0.0125) \leq 1) = 0.9612$, $P(b_1(25, 0.0125) \leq 2) = 0.9963$,
$P(b_1(25, 0.015) \leq 2) = 0.9939$, and $P(b_1(25, 0.015) \leq 3) = 0.99502$.

Hence the starting values are (2,3) for the control limits. In Table 4.4 the necessary additional steps are given for computing the control limits for four different values of ρ.

As with the nonrandomized univariate p-charts it generally will be impossible to guarantee a specific level for the bivariate diagnostic p-charts obtained in the manner described above. In fact, establishing other general properties of the run-length distribution is also difficult due to the paucity of analytical results for the bivariate binomial distribution. See Chapter Two.

What can be done is to give evidence, in the form of investigations of particular cases via the bivariate binomial cdf, that the diagnostic chart is operating as one would like. At the very least, it is desirable that the procedure should be unbiased. If we take the space of alternative parameters to be the set $P_a = \{(\pi_1,\pi_2,\rho): \pi_1 \geq \pi_{01}, \pi_2 \geq \pi_{02}, \rho = \rho_0\}$, then all indications are that the run-length distribution is stochastically decreasing for $\underset{\sim}{\pi}^* \succ \underset{\sim}{\pi}$ - at least for $\pi_{01},\pi_{02} \in (0.005, 0.05)$. The results of some of these investigations are given in Figures 4.2-4.7.

TABLE 4.4. ILLUSTRATION OF THE METHOD FOR OBTAINING CONTROL LIMITS FOR A BIVARIATE BINOMIAL DIAGNOSTIC CHART FOR N = 25, π_0 = (0.0125, 0.0150), AND VARIOUS ρ.

STEPS		ρ			
		0.10	0.25	0.50	0.60
1 .	$P(X_1 \leq 2, X_2 \leq 3)$	0.99586	0.99589	0.99600	0.99607
2 .	$P(X_1 \leq 1, X_2 \leq 2)$	0.95581	0.95660	0.95825	0.95896
3A.	$P(X_1 \leq 2, X_2 \leq 2)$	0.99037	0.99057	0.99123	0.99165
3B.	$P(X_1 \leq 1, X_2 \leq 3)$	0.96081	0.96091	0.96108	0.96114
4 .	$P(X_1 \leq 2, X_2 \leq 2)$	0.00309	0.00561	0.01177	0.01535
5 .	CONTROL LIMITS	(2,2)	(2,2)	(2,2)	(2,2)

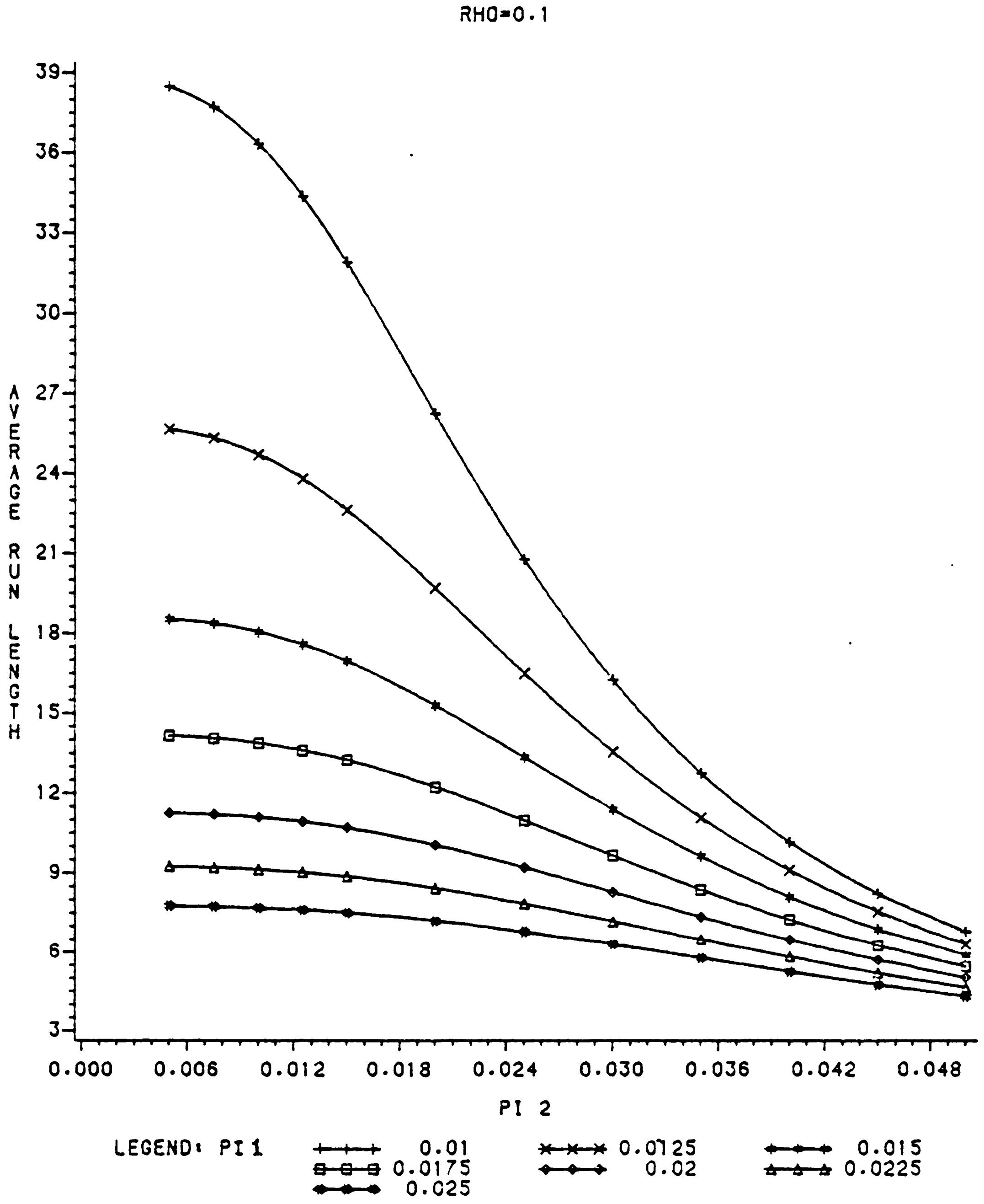

FIGURE 4.2 . AVERAGE RUN LENGTH OF A BIVARIATE BINOMIAL CHART FOR VARIOUS PI 1 AND PI 2 WITH N = 25. THE CONTROL LIMIT FOR THE FIRST VARIABLE IS ONE AND THE SECOND VARIABLE IS TWO.

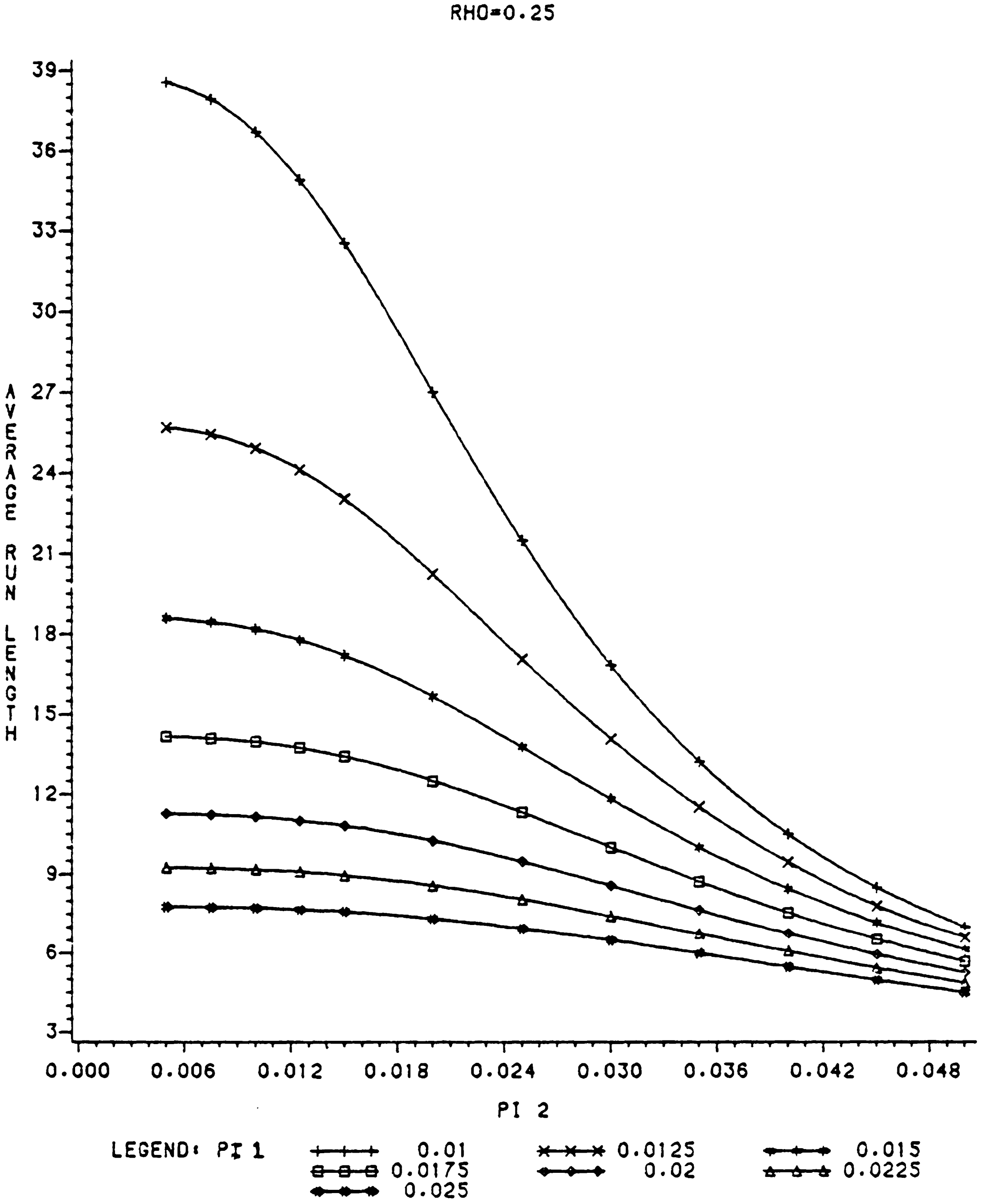

FIGURE 4.3 . AVERAGE RUN LENGTH OF A BIVARIATE BINOMIAL CHART FOR VARIOUS PI 1 AND PI 2 WITH N = 25. THE CONTROL LIMIT FOR THE FIRST VARIABLE IS ONE AND THE SECOND VARIABLE IS TWO.

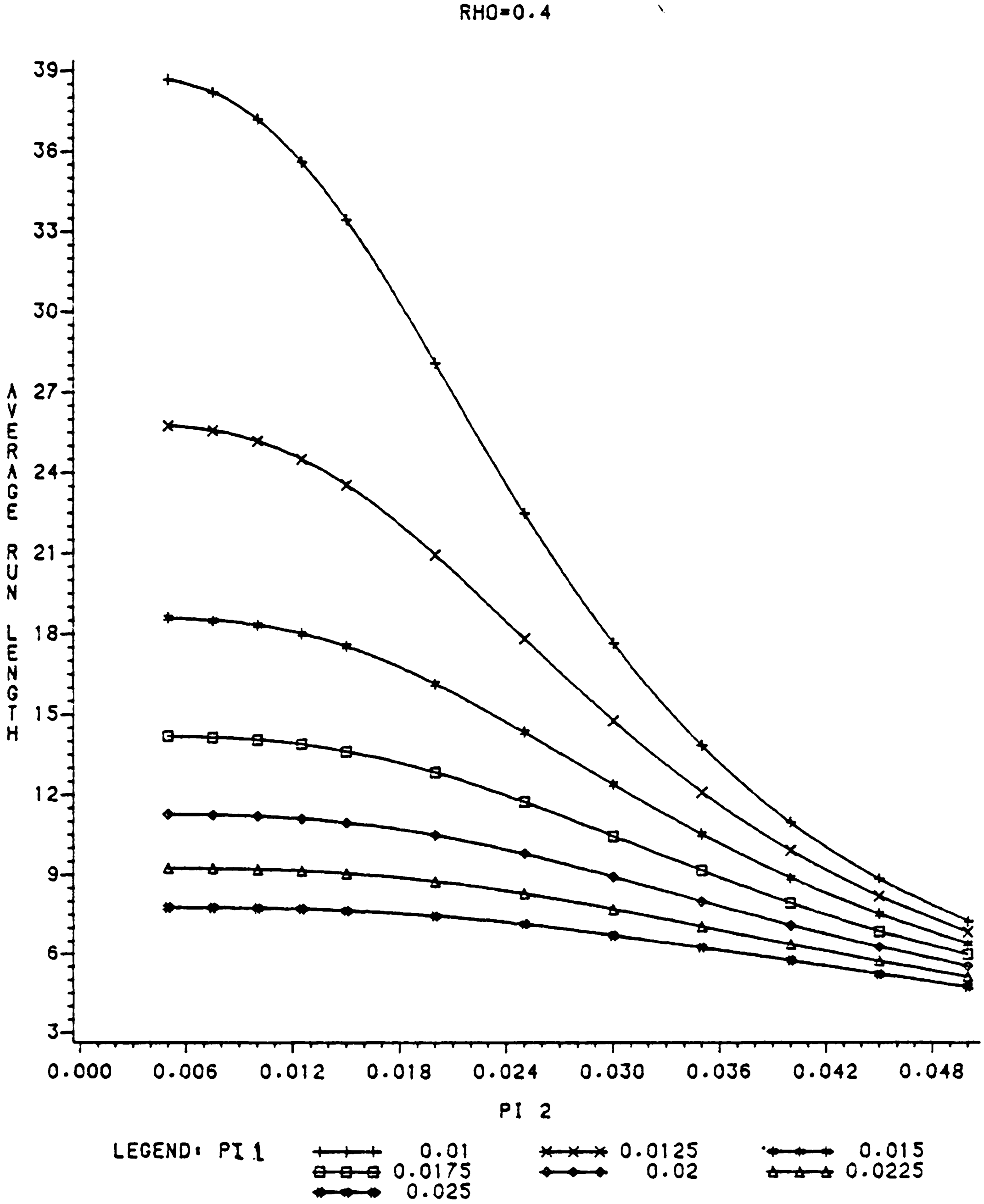

FIGURE 4.4 . AVERAGE RUN LENGTH OF A BIVARIATE BINOMIAL CHART FOR VARIOUS PI 1 AND PI 2 WITH N = 25. THE CONTROL LIMIT FOR THE FIRST VARIABLE IS ONE AND THE SECOND VARIABLE IS TWO.

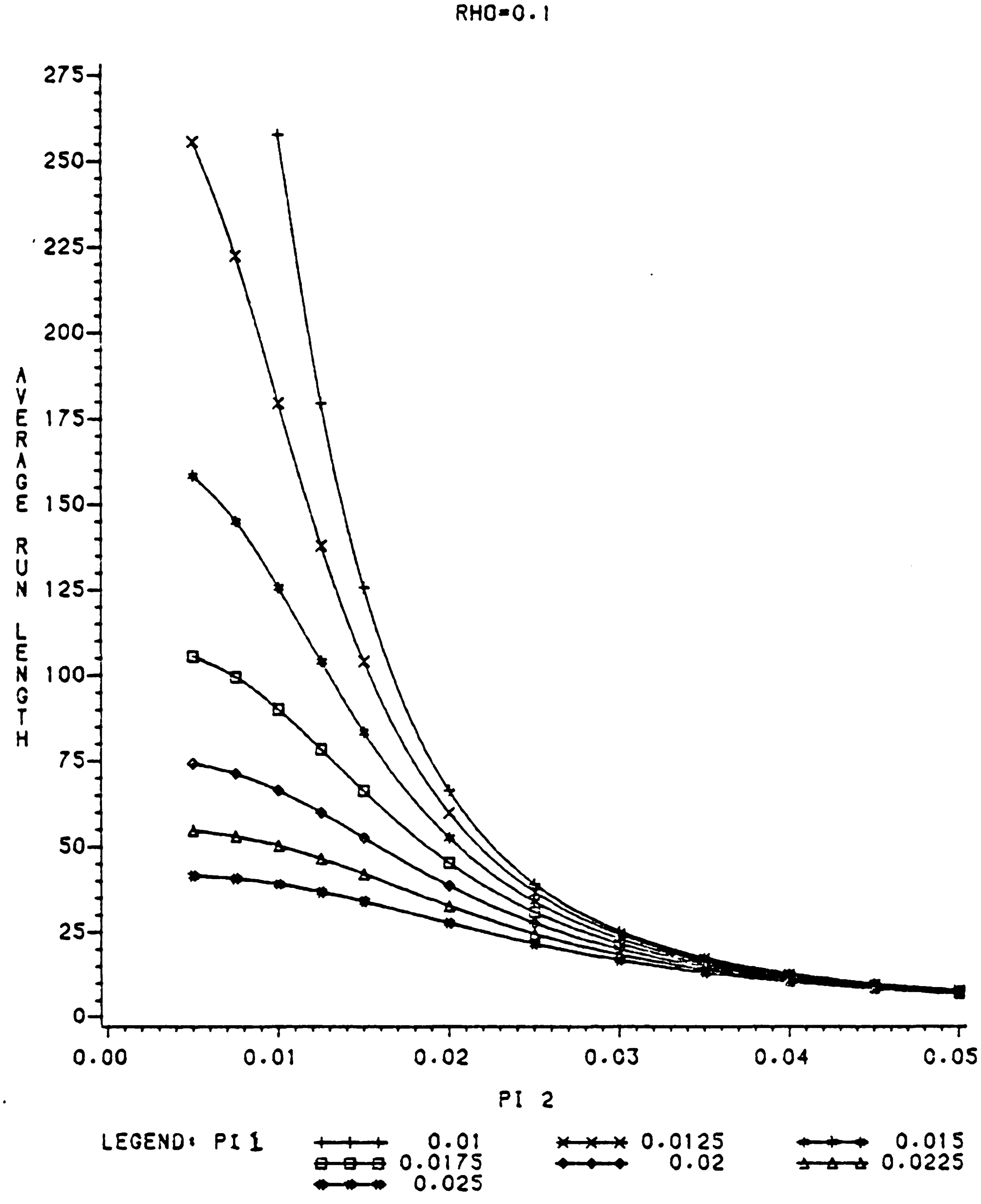

FIGURE 4.5 . AVERAGE RUN LENGTH OF A BIVARIATE BINOMIAL CHART FOR VARIOUS PI 1 AND PI 2 WITH N = 25. THE CONTROL LIMIT FOR BOTH VARIABLES IS TWO.

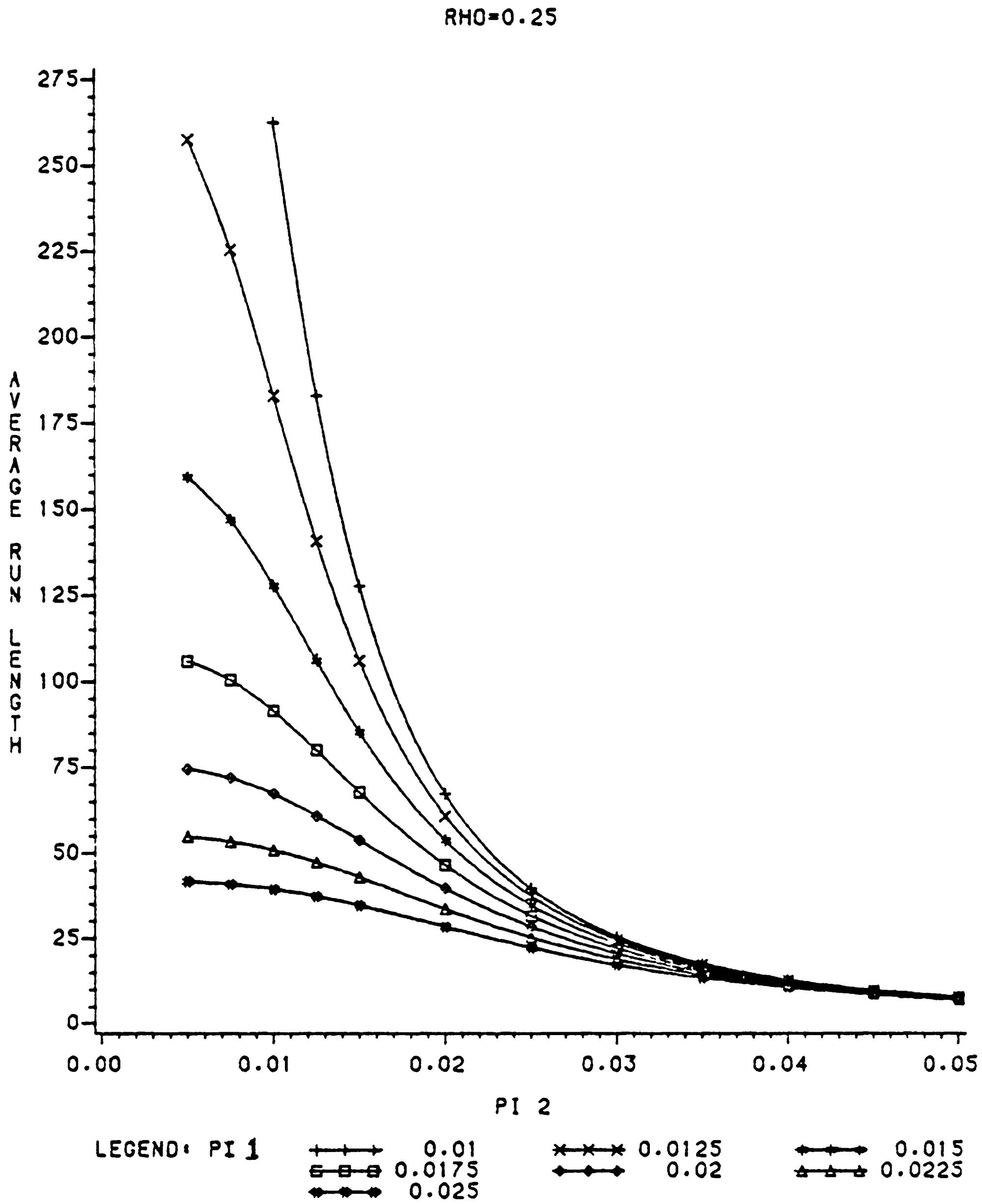

FIGURE 4.6 . AVERAGE RUN LENGTH OF A BIVARIATE BINOMIAL CHART FOR VARIOUS PI 1 AND PI 2 WITH N = 25. THE CONTROL LIMIT FOR BOTH VARIABLES IS TWO.

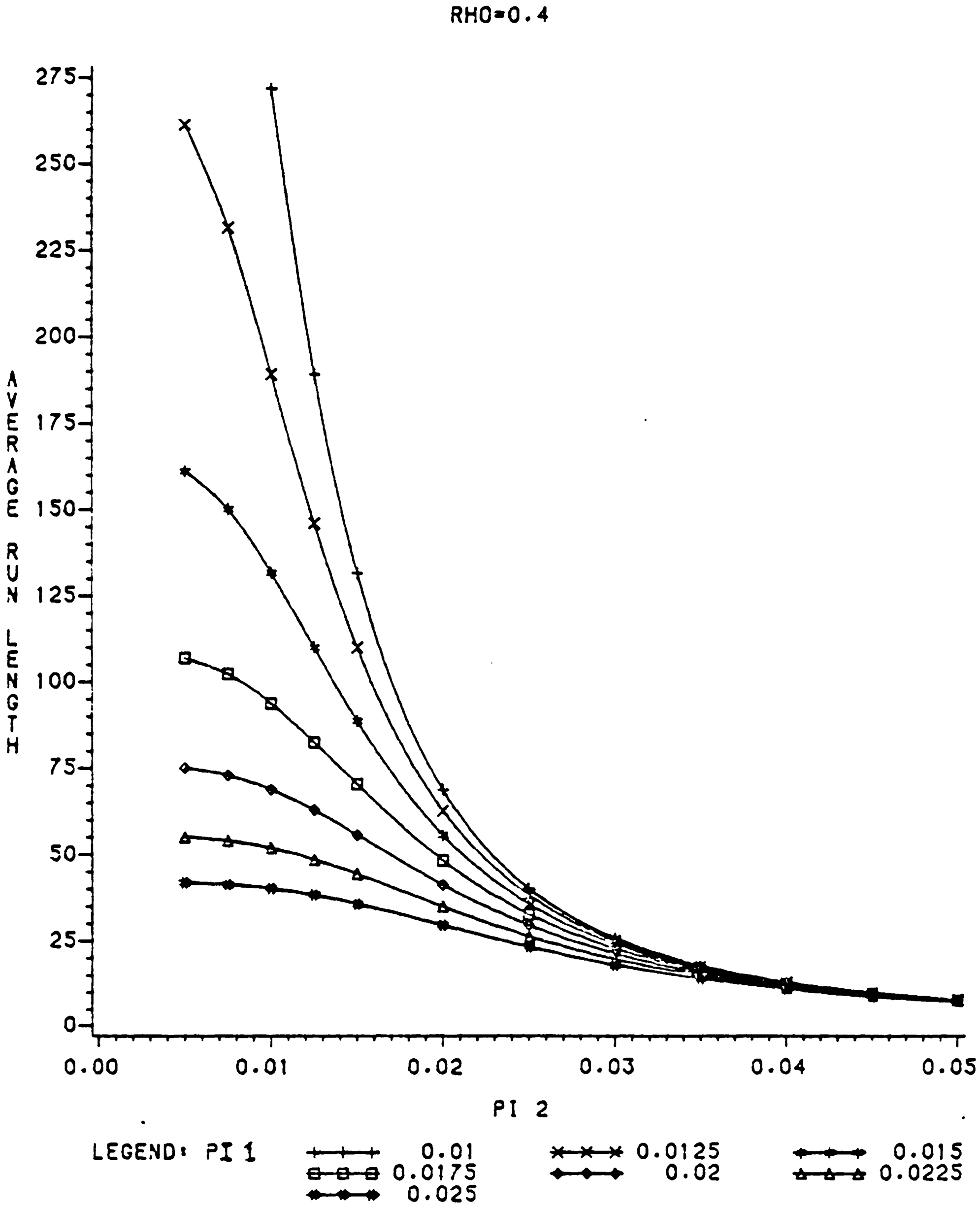

FIGURE 4.7 . AVERAGE RUN LENGTH OF A BIVARIATE BINOMIAL CHART FOR VARIOUS PI 1 AND PI 2 WITH N = 25. THE CONTROL LIMIT FOR BOTH VARIABLES IS TWO.

Since it may be unreasonable to expect that the correlation parameter remains constant when the parameters π_1 and π_2 change, one might hope that there is a similar monotonic behavior for ρ. Some possibilities of such behavior were discussed in Chapter Two. The implication of that discussion here is that the behavior of the run-length distribution might not be so charitable if the correlation parameter is allowed to wander with $\underset{\sim}{\pi}$, especially when the acceptance regions are not square and the change in ρ is relatively large.

A further consideration of the case of symmetrical marginals is necessary. When monitoring for increases in the proportions defective in this situation, it is quite reasonable to take the control limits to be identical, that is $U_1 = U_2$ in (4.12), although these types of limits are not guaranteed if the algorithm outlined above is used. It is then possible, if ρ is unknown but positive, to obtain through the use of Theorem 2.1, an envelope for the run-length distribution based on an appropriate minimum and maximum value for the correlation parameter. The run-length distribution based on the minimum correlation will be a stochastic lower bound, while the maximum correlation yields a stochastic upper bound for the true run-length distribution. So, for monitoring bivariate Bernoulli processes with identical marginals using the same control limit, we have

$$P(N_{\rho_{max}} \leq t) \leq P(N_\rho \leq t) \leq P(N_{\rho_{min}} \leq t), \qquad (4.14)$$

where $N_{\rho_{max}}$ is the run length associated with the maximum correlation; $N_{\rho_{min}}$ is the run length associated with the minimum correlation; and

N_ρ is the actual run length of the process. Expression (4.14) also provides bounds for processes in which the correlation parameter drifts between ρ_{min} and ρ_{max} while the marginal probabilities remain the same.

In earlier developments some progress was made in studying the properties of control charts when the quality parameters were drifting from their target values. There appear to be no analytical results along these lines for the bivariate binomial distribution except for the special case of $\rho = 0$. At least for the cases studied numerically, however, it appears for $\underset{\sim}{X} \sim b_2(n, \underset{\sim}{\pi}_1, \rho)$ and $\underset{\sim}{Y} \sim b_2(n, \underset{\sim}{\pi}_2, \rho)$ such that $\underset{\sim}{\pi}_1 \succ \underset{\sim}{\pi}_2$ that the inequality

$$P(X_1 \leq U_1,\ X_2 \leq U_2) \leq P(Y_1 \leq U_1,\ Y_2 \leq U_2)$$

holds. If this empirical result holds generally, or otherwise in some range of $\underset{\sim}{\pi}$ of interest, upper and lower stochastic bounds could be constructed for the run-length distribution.

Heretofore, only the diagnostic chart for two one-sided alternatives has been discussed. Two-sided diagnostic p-charts could also be constructed using Theorems 3.2 and 3.3 as starting points. The control limits could then be adjusted using the bivariate binomial distribution in much the same manner as for the one-sided control charts. One reason for not pursuing such procedures in more detail is that for nonrandomized charts with $\pi_i \in (0.0, 0.1)$, $i = 1,2$, $n \leq 35$, and standard choices for the level, the lower control limit will not be greater than zero. Since most quality control problems fall within the above realm of parameters, any study of nonrandomized two-sided charts is rendered virtually superfluous.

Before considering the general multivariate Bernoulli process monitoring problem, an example is given of a bivariate diagnostic p-chart. Layman (1967) discusses the following bivariate process monitoring problem. A delay cartridge in a pilot ejection system must meet certain performance specifications on two characteristics, delay time and output. Suppose the cartridge is allowed the nonconformance to specifications rates of one percent for delay time and one and one-half percent for output. Further, suppose the correlation for the process is known to be 0.25.

The testing of product quality was done at high and low temperature extremes using samples of size three and two, respectively. These samples were then combined to obtain a sample of size five for each monitoring occasion. Thirty-one sets of observations were available.

Under the conditions outlined above, a 0.05 level diagnostic p-chart was established with the following control limits. The process is declared out of control if the number of nonconforming items exceeds one on either characteristic, or equals one on both characteristics. The monitoring charts are given in Figure 4.8. The process was deemed out of control at five of the thirty-one monitoring times.

A χ^2 chart based on the omnibus procedure (4.1) is a possible alternative to the bivariate binomial chart used above. If, for this example, this alternative is implemented, then the control chart will signal for all samples except those having no defectives among the five observations. Thus, the actual level of the omnibus procedure

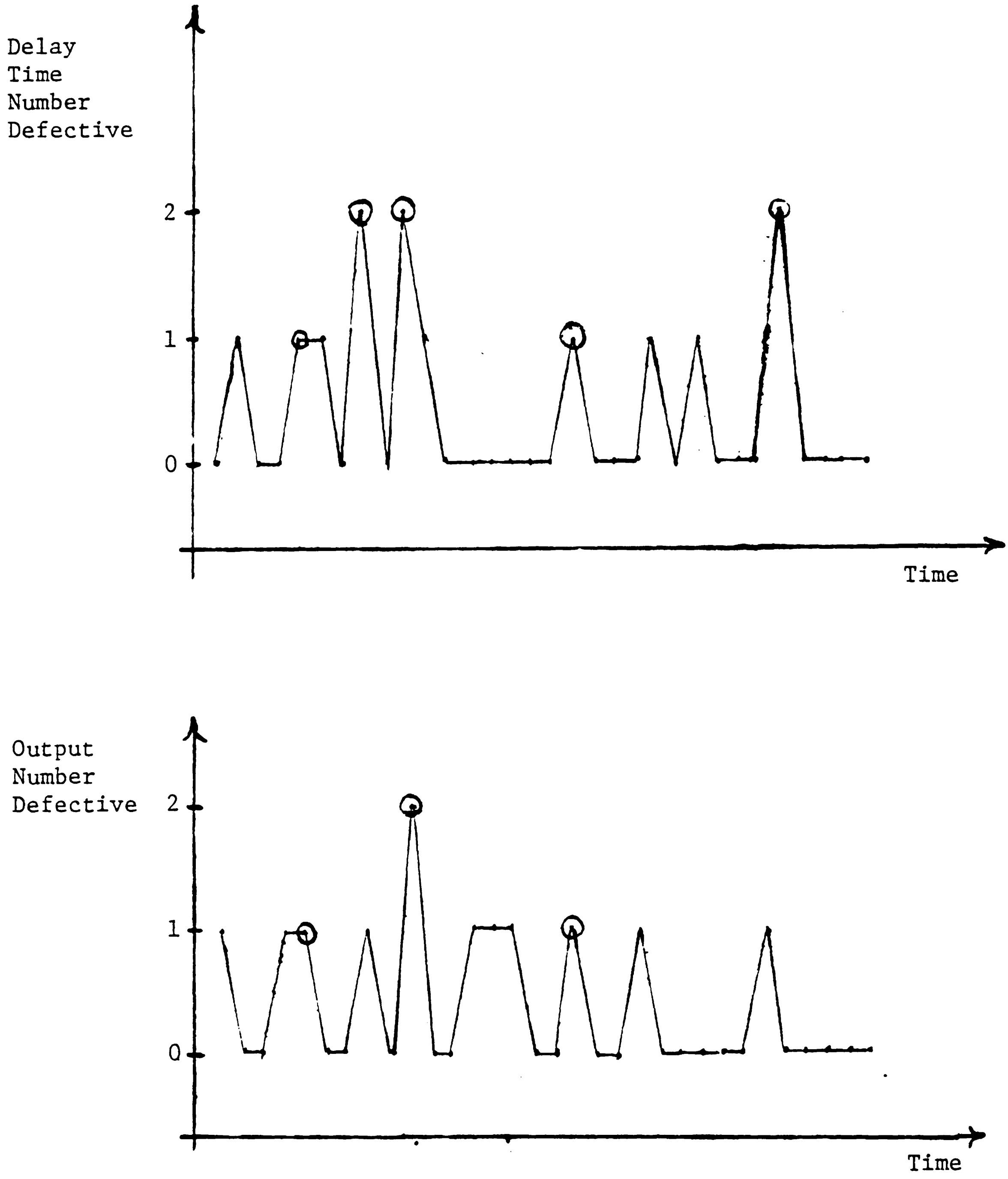

Figure 4.8. An example of a diagnostic bivariate p-chart.

(4.1) will be 0.1045. The observed nonconservative nature of this asymptotic procedure, as discussed in Section Two, is reflected once more in this particular example.

Let us now drop the restriction of only two dichotomous quality attributes, and consider a generalization to r Bernoulli attributes. The appropriate distribution for a diagnostic control chart is the r-variate binomial distribution. Per our observations in Section 2.2, Krishnamoorthy's (1951) series representation of the multivariate binomial distribution is not readily amenable to computation. The proliferation of parameters with the increase in dimensions also lobbies against the direct utilization of this representation. Therefore, instead of an exact small-sample procedure we will settle for conservative control limits based upon second order Bonferroni-type inequalities.

Suppose the goal is to construct a diagnostic control chart for detecting increases in any of the r marginal proportions defective, when the value of the in-control proportions is known. Further, suppose the values of the $\binom{r}{2}$ correlation coefficients are also known, for example from past experience with the process. The following inequality is a special case of Theorem 1.7

$$P(X_1 \leq U_1, \ldots, X_r \leq U_r) \geq \max_{1 \leq \ell \leq r} \{ \sum_{i \neq \ell} P(X_i \leq U_i, X_\ell \leq U_\ell) - (r-2)\, P(X_\ell \leq U_\ell) \} . \qquad (4.15)$$

The requisite control limits of the multivariate binomial chart can be taken to be the U_i's such that the right hand side of (4.15) is greater than or equal to $1-\alpha$. The control limits will therefore be conservative and based solely on the univariate and bivariate binomial distributions.

The following procedure is a possible means for selecting the appropriately restricted U_i's in (4.15) to establish an α-level diagnostic chart. First, use the univariate binomial distribution to find starting values of the U_i's; i.e., choose U_i's such that

$$P(X_i > U_i) = \alpha_i \quad i=1,\ldots,r, \tag{4.16}$$

where $0 \leq \alpha_i \leq \alpha$ for all i and $\sum_{i=1}^{r} \alpha_i = \alpha$. Next, subtract one from each U_i and compute the right hand side of (4.15). If the result is greater than or equal to $(1-\alpha)$ continue subtracting one from each limit until the inequality is violated. Then increment one U_i at a time and compute the resulting r right hand sides of (4.15). If the minimum of these r values is greater than or equal to $(1-\alpha)$ then these U_i's determine the desired control limits. Otherwise, continue the search, this time incrementing two U_i's at a time. Continue in the fashion described until satisfactory control limits have been found.

An illustration of this technique of obtaining control limits for a multivariate Bernoulli process is now given. Suppose we are monitoring a four-variate Bernoulli process, taking samples of size 25, with the following specified parameters: $\underset{\sim}{\pi} = (0.01, 0.015, 0.01, 0.02)'$; $\rho_{12} = 0.25$, $\rho_{13} = 0.15$, $\rho_{14} = 0.10$, $\rho_{23} = 0.40$, $\rho_{24} = 0.20$, and $\rho_{34} = 0.05$. Use $\alpha = 0.05$.

Step 1. Find the starting values from the univariate binomial marginals.

(a) $P(X_1 \leq 1) = 0.97425$ $\quad P(X_1 \leq 2) = 0.99805$

(b) $P(X_2 \leq 1) = 0.94621$ $\quad P(X_2 \leq 2) = 0.99394$

(c) $P(X_3 \leq 1) = 0.97425$ $\quad P(X_3 \leq 2) = 0.99805$

(d) $P(X_4 \leq 2) = 0.98676$ $\quad P(X_4 \leq 3) = 0.99855$

Therefore, the starting values are $(U_1, U_2, U_3, U_4) = (2, 2, 2, 3)$. From (4.15), compute $P(X_1 \leq 2, X_2 \leq 2, X_3 \leq 2, X_4 \leq 3) \geq 0.98738$.

Step 2. Decrement each limit by one; then compute (4.15).

(a) $\ell=1$: 0.89057

(b) $\ell=2$: 0.89979

(c) $\ell=3$: 0.89403

(d) $\ell=4$: 0.88558

Therefore, $P(X_1 \leq 1, X_2 \leq 1, X_3 \leq 1, X_4 \leq 2) \geq 0.89978 \not\geq 0.95$.

Step 3. Increment one limit at a time; then compute (4.15).

(a) $P(X_1 \leq 2, X_2 \leq 1, X_3 \leq 1, X_4 \leq 2) \geq 0.91837 \not\geq 0.95$

(b) $P(X_1 \leq 1, X_2 \leq 2, X_3 \leq 1, X_4 \leq 2) \geq 0.93396 \not\geq 0.95$

(c) $P(X_1 \leq 1, X_2 \leq 1, X_3 \leq 2, X_4 \leq 2) \geq 0.91505 \not\geq 0.95$

(d) $P(X_1 \leq 1, X_2 \leq 1, X_3 \leq 1, X_4 \leq 3) \geq 0.90937 \not\geq 0.95$

Step 4. Increment two limits at a time; then compute (4.15).

(a) $P(X_1 \leq 2, X_2 \leq 2, X_3 \leq 1, X_4 \leq 2) \geq 0.95577$

(b) $P(X_1 \leq 2, X_2 \leq 1, X_3 \leq 2, X_4 \leq 2) \geq 0.93363$

(c) $P(X_1 \leq 2, X_2 \leq 1, X_3 \leq 1, X_4 \leq 3) \geq 0.92795$

(d) $P(X_1 \leq 1, X_2 \leq 2, X_3 \leq 2, X_4 \leq 2) \geq 0.95524$

(e) $P(X_1 \leq 1, X_2 \leq 2, X_3 \leq 1, X_4 \leq 3) \geq 0.94524$

(f) $P(X_1 \leq 1, X_2 \leq 1, X_3 \leq 2, X_4 \leq 3) \geq 0.92340$

Therefore, possible control limits are given by (2, 2, 1, 2) or (1, 2, 2, 2). The latter limit gives a slightly lower level, closer to 0.05.

The preceding method for finding control limits merely gives an upper bound for the level of the control chart. Lower bounds can also be provided for the level of the chart using the minimum of the inequalities in Theorems 1.7. Specializing to the case at hand, we have

$$P(X_1 \leq U_1,\dots,X_r \leq U_r) \leq \frac{(r-1)(r-2)}{2} - (r-2)\sum_{i=1}^{r} P(X_i \leq U_i) + \sum_{i=2}^{r}\sum_{j=1}^{i-1} P(X_i \leq U_i,\ X_j \leq U_j) \tag{4.17a}$$

and $P(X_1 \leq U_1,\dots,X_r \leq U_r) \leq 1 - \underset{\sim}{v}'\underset{\sim}{W}^{-1}\underset{\sim}{v}$ (4.17b)

where $\underset{\sim}{v} = (1 - P(X_1 \leq U_1),\dots,1 - P(X_r \leq U_r))$ and $\underset{\sim}{W} = (w_{ij})$ with

$$w_{ij} = \begin{cases} 1 - P(X_i \leq U_i) & \text{if } i=j \\ 1 - P(\{X_i \leq U_i\} \cup \{X_j \leq U_j\}) & i \neq j\ . \end{cases}$$

The bounds (4.15) and the minimum of (4.17a), (4.17b) can be combined to yield lower and upper stochastic bounds, respectively, for the run-length distribution of the multivariate binomial diagnostic chart.

For the hypothetical situation used to illustrate the establishment of the diagnostic chart, we provide in Table 4.5 the bounds obtained on the parameter of the run-length distribution using (4.15) and (4.17) when the process is in control and for selected cases when the process is out of control. The resulting bounds on the ARL of the chart are also given in Table 4.5. These bounds are quite tight, so there appears to be some promise that (4.15) and (4.17) can be used to determine operating characteristics of competing charts that can be utilized to choose the most appropriate control limits.

This concludes the discussion of small-sample diagnostic charts for the multivariate binomial distribution under Case I. Diagnostic charts based on the large-sample properties of the multivariate binomial distribution are the next topic of study. The main advantages of these types of procedures is the relative ease of computation involved in establishing and maintaining the charts. We can also expand our attention to the multivariate multinomial processes - which have not been successfully studied in small samples. First, though, consider the multivariate Bernoulli process.

As with the small-sample charts, the initial step will be to consider diagnostic charts designed to detect increases in the proportion defective of any of the r different attributes. From previous discussions it is clear that in the notation of (4.5),

$$\underset{\sim}{g}_i = (g_{ij}) \xrightarrow{d} N_r(\underset{\sim}{0},\underset{\sim}{R}) \quad i=1,2,\ldots, \tag{4.18}$$

TABLE 4.5. SOME BOUNDS ON THE PARAMETER OF THE GEOMETRIC RUN LENGTH DISTRIBUTION AND THE CORRESPONDING BOUNDS ON THE ARL FOR THE ILLUSTRATION OF A FOUR-VARIATE BINOMIAL CHART USED IN THE TEXT. THE IN-CONTROL PARAMETER VECTOR IS $\pi_0' = (0.010, 0.015, 0.010, 0.020)$.

CONTROL LIMITS = (2, 2, 1, 2)

π'	LOWER BOUND (4.15)	UPPER BOUND (4.17)	BOUNDS ON THE ARL LOWER	BOUNDS ON THE ARL UPPER
(0.010, 0.015, 0.010, 0.020)	0.9558	0.9566	22.61	23.03
(0.010, 0.020, 0.010, 0.020)	0.9505	0.9514	20.18	20.57
(0.010, 0.015, 0.015, 0.020)	0.9293	0.9301	14.15	14.31
(0.015, 0.015, 0.010, 0.020)	0.9520	0.9532	20.85	21.38
(0.010, 0.015, 0.010, 0.025)	0.9456	0.9467	18.33	18.77
(0.020, 0.015, 0.020, 0.020)	0.8870	0.8887	8.85	8.98
(0.010, 0.025, 0.010, 0.025)	0.9324	0.9338	14.79	15.11
(0.010, 0.025, 0.020, 0.020)	0.8849	0.8869	8.69	8.84
(0.015, 0.025, 0.020, 0.020)	0.8818	0.8844	8.46	8.65
(0.015, 0.025, 0.010, 0.030)	0.9156	0.9134	11.85	12.25

CONTROL LIMITS = (1, 2, 2, 2)

π'	LOWER BOUND (4.15)	UPPER BOUND (4.17)	BOUNDS ON THE ARL LOWER	BOUNDS ON THE ARL UPPER
(0.010, 0.020, 0.010, 0.020)	0.9494	0.9506	19.76	20.24
(0.010, 0.015, 0.015, 0.020)	0.9515	0.9530	20.62	21.30
(0.015, 0.015, 0.010, 0.020)	0.9287	0.9297	14.02	14.22
(0.010, 0.015, 0.010, 0.025)	0.9452	0.9465	18.25	18.69
(0.020, 0.015, 0.020, 0.020)	0.8863	0.8885	8.80	8.97
(0.010, 0.025, 0.010, 0.025)	0.9308	0.9326	14.44	14.83
(0.010, 0.025, 0.020, 0.020)	0.9316	0.9341	14.62	15.17
(0.015, 0.025, 0.020, 0.020)	0.9055	0.9095	10.58	11.05
(0.015, 0.025, 0.010, 0.030)	0.8918	0.8959	9.24	9.60

where $$g_{ij} = \sqrt{n}\,\frac{(p_{ij}-\pi_{ij})}{\sqrt{\pi_{ij}(1-\pi_{ij})}} \quad j=1,2,\ldots,r;$$

and $\underset{\sim}{R} = (\rho_{\ell k})$ with

$$\rho_{\ell k} = \begin{cases} 1 & \text{if } \ell=k \\ \dfrac{\pi_{i11}^{(\ell,k)} - \pi_{i\ell}\pi_{ik}}{\sqrt{\pi_{i\ell}\,\pi_{ik}\,(1-\pi_{i\ell})(1-\pi_{ik})}} & \text{if } \ell \neq k \end{cases}$$

$\ell,k = 1,2,\ldots,r.$

An asymptotic monitoring procedure is to find upper control limits $\{U_1,\ldots,U_r\}$ such that

$$P(g_{i1} \leq U_1,\ldots,g_{ir} \leq U_r) = 1 - \alpha. \qquad (4.19)$$

Obtaining the U_i's in (4.19) might become an unwieldy task, unless further constraints can be placed on the problem. For instance, we might require that $U_i = U$ for all i, or take $\rho_{\ell k} = \rho$ for all $\ell \neq k$. Justifications for adopting these constraints are given below.

<u>Theorem 4.4</u>. Let N_D be the run length of the diagnostic control chart (4.19). Let N_{D*} be the run length of the diagnostic chart (4.19) for which $\rho_{\ell k} = \rho = \min_{(\ell,k)} \rho_{\ell k}$ for all (ℓ,k). Then, asymptotically,

$$P(N_D > t) \geq P(N_{D*} > t) .$$

Proof. From Slepian's inequality (Theorem 1.1) it follows that, asymptotically,

$$P_{\underset{\sim}{R}}(g_{i1} \le U_1, \ldots, g_{ir} \le U_r) \ge P_{\underset{\sim}{R}*}(g_{i1} \le U_1, \ldots, g_{ir} \le U_r), \qquad (4.20)$$

where $\underset{\sim}{R}* = (1-\rho)\underset{\sim}{I}_r + \rho\underset{\sim}{J}_r$. Determining the U_i such that

$$P_{R*}(g_{i1} \le U_1, \ldots, g_{ir} \le U_r) = 1 - \alpha$$

yields the required asymptotic inequality on the run-length distributions.

Suppose it is reasonable to take the r control limits in (4.19) to be equal, i.e., $U_1 = U_2 = \ldots = U_r = U$. From the preceding theorem, a monitoring procedure using $\rho_{\ell k} = \rho = \min_{(\ell,k)} \rho_{\ell k}$ for all (ℓ,k) will provide an asymptotic control chart which signals less often, in probability, than a control chart utilizing the actual correlations.

With the above simplifications, tables of the maximum of equicorrelated standard normal variables given by Gupta et al. (1973) can be used to establish the upper control limit, U. These tables are available for r = 1(1)10(2)50; α = 0.01, 0.025, 0.05, 0.10, 0.25; and for fifteen values of ρ between 0.10 and 0.90. An added benefit of the utilization of this particular procedure is that it is only necessary to maintain one, not r, control charts. At each sampling occasion it is necessary only to maintain a control chart for the maximum of $(g_{i1}, \ldots, g_{ir})$. The process is said to be out of control if this maximum is greater than U.

Now the behavior of the diagnostic control charts of the type (4.19) is considered when the process is out of control. Suppose the out-of-control sequence of parameters is given by (4.11). Then, we have

$$(g_{i1},\ldots,g_{ir})' \xrightarrow{d} N_r(\underset{\sim}{\mu}_i,\underset{\sim}{R}),\ i=1,2,\ldots,$$

where $\underset{\sim}{R}$ is as given at (4.18), and

$$\mu_{ij} = \sqrt{n}\,\frac{(\pi_{ij} - \pi_{0j})}{\sqrt{\pi_{0j}(1 - \pi_{0j})}} \geq 0 \quad j=1,2,\ldots,r\ .$$

Theorem 4.5. Consider the monitoring of two multivariate Bernoulli processes which are out of control in the manner described above. Suppose for k monitoring periods processes one and two have the parameters $(\underset{\sim}{\mu}_1, \underset{\sim}{\mu}_2,\ldots,\underset{\sim}{\mu}_k)$ and $(\underset{\sim}{\mu}_1^*,\ldots,\underset{\sim}{\mu}_k^*)$, respectively, with $\underset{\sim}{\mu}_i \succ \underset{\sim}{\mu}_i^*$, $i = 1,2,\ldots,k$. For the control chart based on (4.19), the run-length distribution of process one is, asymptotically, stochastically smaller than the run-length distribution of process two.

Proof. For process two, asymptotically we have, for $i = 1,2,\ldots,k$

$$P(g_{i1}\leq U_1,\ldots,g_{ir}\leq U_r) = P(g_{i1}-\mu_{i1}^*\leq U_1-\mu_{i1}^*,\ldots,g_{ir}-\mu_{ir}^*\leq U_r-\mu_{ir}^*).$$

Likewise, for process one we have

$$P(g_{i1}\leq U_1,\ldots,g_{ir}\leq U_r) = P(g_{i1}-\mu_{i1}\leq U_1-\mu_{i1},\ldots,g_{ir}-\mu_{ir}\leq U_r-\mu_{ir}).$$

Note that $U_j-\mu_{ij}\leq U_j-\mu_{ij}^*$ for $j = 1,\ldots,r$ and $(g_{i1}-\mu_{i1},\ldots,g_{ir}-\mu_{ir})$ has the same asymptotic distribution as $(g_{i1}-\mu_{i1}^*,\ldots,g_{ir}-\mu_{ir}^*)$. Thus,

$$P(g_{i1}-\mu^*_{i1} \leq U_1-\mu^*_{i1},\ldots,g_{ir}-\mu^*_{ir} \leq U_r-\mu^*_{ir})$$

$$\geq P(g_{i1}-\mu_{i1} \leq U_1-\mu_{i1},\ldots,g_{ir}-\mu_{ir} \leq U_r-\mu_{ir})$$

for i = 1,2,...,k. Hence, we have the conclusion of the theorem.

Before ending the discussion of diagnostic p-charts we offer some remarks concerning the use of these charts with upper and lower control limits. Using Theorem 1.3, one can show that, in the limit, the two-sided p-charts will run longer than r separate p-charts which are independent. Thus, conservative large-sample control limits can be found using the $N_1(0,1)$ cdf. If the process is out of control with a parameter sequence of the type (4.11), then Theorem 1.4 can be used to show that the monitoring procedure is asymptotically unbiased. The details of these statements are omitted.

We now consider the more general case of using diagnostic control charts to monitor multivariate multinomial processes under Case I. Following the route used for the other diagnostic charts, the r multinomial attributes will be monitored using r separate, but dependent, generalized p-charts. Asymptotically, these r generalized p-charts will give dependent chi-squared statistics at each sampling occasion. The relevant limiting joint distribution is the multivariate chi-squared distribution given at (2.43).

This limiting distribution arises in the following manner. In the notation of (4.5), note that

$$\underset{\sim}{Y}_i = (\underset{\sim}{Y}'_{i1}, \ldots, \underset{\sim}{Y}'_{ir})' \xrightarrow{d} N_{r[\sum_i(c_i-1)]}(\underset{\sim}{0}, \underset{\sim}{\Sigma}), \qquad (4.21)$$

where

$$\underset{\sim}{\Sigma} = \begin{bmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} & \cdots & \underset{\sim}{\Sigma}_{1r} \\ \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22} & & \\ \vdots & & \ddots & \\ \underset{\sim}{\Sigma}_{r1} & & & \underset{\sim}{\Sigma}_{rr} \end{bmatrix},$$

and $\underset{\sim}{\Sigma}_{ij}$ is a matrix of order $(c_i-1) \times (c_j-1)$. For convenience, rewrite (4.21) in a canonical form as

$$\underset{\sim}{Y}^*_i = ((\underset{\sim}{\Sigma}_{11}^{-\frac{1}{2}}\underset{\sim}{Y}_{i1})', \ldots, (\underset{\sim}{\Sigma}_{rr}^{-\frac{1}{2}}\underset{\sim}{Y}_{ir})')' \xrightarrow{d} N_{r(\sum_i(c_i-1))}(\underset{\sim}{0}, \underset{\sim}{\Sigma}^*), \qquad (4.22)$$

where

$$\underset{\sim}{\Sigma}^* = \begin{bmatrix} \underset{\sim}{I}_{c_1-1} & \underset{\sim}{\Sigma}^*_{12} & \cdots & \underset{\sim}{\Sigma}^*_{1r} \\ \underset{\sim}{\Sigma}^*_{21} & \underset{\sim}{I}_{c_2-1} & & \\ \vdots & & \ddots & \\ \underset{\sim}{\Sigma}^*_{r1} & & & \underset{\sim}{I}_{c_r-1} \end{bmatrix},$$

with $\underset{\sim}{\Sigma}^*_{ij} = \underset{\sim}{\Sigma}_{ii}^{-\frac{1}{2}} \underset{\sim}{\Sigma}_{ij} \underset{\sim}{\Sigma}_{jj}^{-\frac{1}{2}}$. It is apparent from (4.22) that the joint distribution of $(\underset{\sim}{Y}'_{i1} \underset{\sim}{\Sigma}_{11}^{-1} \underset{\sim}{Y}_{i1}, \ldots, \underset{\sim}{Y}'_{ir} \underset{\sim}{\Sigma}_{rr}^{-1} \underset{\sim}{Y}_{ir})$ is of the type (2.43). Over the monitoring periods, the diagnostic generalized p-charts will be independent realizations of this multivariate distribution.

Although it is not now possible to give exact asymptotic control limits for this diagnostic chart, conservative bounds on the level can be obtained using Theorem 2.10.

Theorem 4.6. Suppose the r separate control charts based on the statistics $(\underset{\sim}{Y}'_{i1} \underset{\sim}{\Sigma}^{-1}_{11} \underset{\sim}{Y}_{i1}, \ldots, \underset{\sim}{Y}'_{ir} \underset{\sim}{\Sigma}^{-1}_{rr} \underset{\sim}{Y}_{ir})$ with upper control limits $(U_1, \ldots, U_r)$ are used to monitor the specified values of the multinomial proportions $(\underset{\sim}{\pi}_{01}, \ldots, \underset{\sim}{\pi}_{0r})$. When the process is in control the asymptotic run-length distribution of the control chart is stochastically larger than the geometric distribution with parameter $1 - \prod_{j=1}^{r} P(W_j \leq U_j)$, where $W_j \sim \chi_1^2(c_j-1,1)$.

Proof. From Theorem 2.10 we obtain, for the run-length N_D,

$$P(N_D > t) = [P(\bigcap_{j=1}^{r} \{\underset{\sim}{Y}'_{1j} \underset{\sim}{\Sigma}^{-1}_{jj} \underset{\sim}{Y}_{1j} \leq U_j\})]^t$$

$$\geq [\prod_{j=1}^{r} P(\underset{\sim}{Y}'_{1j} \underset{\sim}{\Sigma}^{-1}_{jj} \underset{\sim}{Y}_{1j} \leq U_j)]^t$$

$$= [\prod_{j=1}^{r} P(W_j \leq U_j)]^t .$$

If the level α is chosen such that

$$\prod_{j=1}^{r} P(W_j \leq U_j) = 1 - \alpha,$$

then the diagnostic generalized p-chart will run longer than the geometric distribution with parameter α when the process is in control.

Recall that no results are available at present for non-central distributions of the type (2.43). However, we can assert the asymptotic unbiasedness of the control chart of Theorem 4.6 based on Theorem 1.4.

4.5 Diagnostic Procedures for Monitoring Multivariate Attributes - Case II

We now consider a few methods for monitoring multivariate Bernoulli or multinomial attributes when the target quality standards, or nuisance parameters, or both, are estimated in a base period. For the most part, our attention will have to be restricted to asymptotic procedures. A small-sample diagnostic chart that is simply an extension to the bivariate Bernoulli process of the monitoring procedure of Section 3.4 will be considered. The cumbersomeness of this tactic leads to a reconsideration of the small-sample monitoring procedures of the preceding section, appropriately modified for Case II. Lastly, a start will be presented for monitoring multivariate multinomial populations with diagnostics under Case II.

In the notation of this chapter, let $\underset{\sim}{X}_0$ represent the m observations obtained from monitoring the process in a base period. For the purposes of the immediate discussion let r=2 and c_1=c_2=2 - so that a bivariate Bernoulli process is to be monitored. Suppose two separate charts are established based on the UMPU test, as discussed in Section 3.4, for each marginal. Marginally, the test statistics are the conditional random variables

$$(X_{i1}|X_{i1} + X_{01} = S_1) \text{ and } (X_{i2}|X_{i2} + X_{02} = S_2).$$

The relevant joint distribution of interest for the two diagnostic charts is

$$P(X_{i1} = x_1, X_{i2} = x_2 | X_{i1} + X_{01} = S_1, X_{i2} + X_{02} = S_2) \tag{4.23}$$

$$= \frac{P(X_{i1} = x_1, X_{i2} = x_2)\ P(X_{01} = S_1 - x_1, X_{02} = S_2 - x_2)}{P(X_{i1} + X_{01} = S_1, X_{i2} + X_{02} = S_2)}$$

The moment generating function of the denominator of (4.23) is

$$g(t_1,t_2) = (1+\pi_{i1}(e^{t_1}-1) + \pi_{i2}(e^{t_2}-1) + \pi_{i11}^{(1,2)}(e^{t_1+t_2}-e^{t_1}-e^{t_2}-1))^n$$
$$\cdot (1+\pi_{01}(e^{t_1}-1) + \pi_{02}(e^{t_2}-1) + \pi_{011}^{(1,2)}(e^{t_1+t_2}-e^{t_1}-e^{t_2}-1))^m . \tag{4.24}$$

When the process is in control, $\pi_{i1} = \pi_{01}$, $\pi_{i2} = \pi_{02}$, and $\pi_{i11}^{(1,2)} = \pi_{011}^{(1,2)}$. The mgf (4.24) then, is that of a $b_2(m+n, (\pi_{01},\pi_{02}), \rho_0)$. Since the numerator of (4.23) is the product of independent bivariate binomial probabilities, (4.23) can be expressed in the following manner when the process is in control:

$$\frac{b_1(x_1;\ n,\ \pi_{01})\ b_1(x_2;\ n,\ \pi_{02})\ b_1(S_1 - x_1;\ m,\ \pi_{01})\ b_1(S_2 - x_2;\ m,\ \pi_{02})}{b_1(S_1;\ m+n,\ \pi_{01})\ b_1(S_2;\ m+n,\ \pi_{02})}$$
$$\frac{\{\sum_{j=0}^{n} \rho_0^{\,j} G_j(x_1;\pi_{01},n) G_j(x_2;\pi_{02},n)\}\{\sum_{j=0}^{m} \rho_0^{\,j} G_j(S_1-x_1;\pi_{01},m) G_j(S_2-x_2;\pi_{02},m)\}}{\sum_{j=0}^{m+n} \rho_0^{\,j} G_j(S_1;\ \pi_{01},\ m+n)\ G_j(S_2;\ \pi_{02},\ m+n)} \tag{4.25}$$

where $G_j(Z;\ \pi,K)$ is the Krawtchouck polynomial.

Note that the first term of (4.25) corresponds to the univariate conditional test for each marginal, i.e. to

$$\frac{\binom{n}{x_1}\binom{m}{S_1-x_1}}{\binom{m+n}{S_1}} \quad \frac{\binom{n}{x_2}\binom{m}{S_2-x_2}}{\binom{m+n}{S_2}} .$$

The remainder of (4.25), however, is not independent of the nuisance parameters π_{01}, π_{02}, and ρ_0. Furthermore, no simplification seems possible. So, while (4.25) is a bivariate distribution with the required marginals, it is of no use in evaluating probabilities for the purpose of implementing control procedures. Given the seeming intractability of the small-sample procedures considered above, a realistic alternative approach may be to modify the Case I procedures of the preceding section.

As an initial step, suppose the correlation parameter of a bivariate Bernoulli process is unknown, but estimated in a base period. As mentioned in the course of the discussion of the diagnostic monitoring of a multivariate Bernoulli process under Case I, knowledge of the nuisance parameter ρ might be an unreasonable assumption. Suppose the maximum likelihood estimator (mle), given by Hamdan and Martinson (1971) as

$$\hat{\rho} = \frac{\sum_{\ell=1}^{m} \frac{X_{011\ell}^{(1,2)}}{m} - \left(\sum_{\ell=1}^{m} \frac{X_{01\ell}}{m}\right)\left(\sum_{\ell=1}^{m} \frac{X_{02\ell}}{m}\right)}{\left[\frac{1}{m}\sum_{\ell=1}^{m} X_{01\ell}\left(1 - \frac{1}{m}\sum_{\ell=1}^{m} X_{01\ell}\right) \frac{1}{m}\left(\sum_{\ell} X_{02\ell}\right)\left(1 - \frac{1}{m}\sum_{\ell} X_{02\ell}\right)\right]^{\frac{1}{2}}} , \quad (4.26)$$

is used to obtain a consistent estimator of ρ in the bivariate binomial distribution. In turn, consistent estimates of the bivariate binomial probabilities for the in-control process are obtained for each (x_1,x_2) when $P(X_1 = x_1, X_2 = x_2)$ is computed with $\hat{\rho}$ replacing ρ in the pmf (2.3). That is,

$$\hat{P}(X_1=x_1,\ X_2=x_2) = b_1(x_1;\ n,\ \pi_1)\ b_1(x_2;\ n,\ \pi_2) \sum_{r=0}^{n} \hat{\rho}^r G_r(x_1,n) G_r(x_2,n). \quad (4.27)$$

The consistency of the estimated probabilities (4.27) can be verified as follows. Substitute $\rho(1 + o_p(1))$ for $\hat{\rho}$ in (4.27). Then as $m\to\infty$, (4.27) can be expressed as the pmf (2.3) plus terms that go to zero in probability. Therefore, if (4.27) is used to compute the probability (4.12), the procedure of Section Four is asymptotically correct for monitoring against increases in either proportion defective. The same conclusion applies for monitoring departures in either direction from specified proportions defective.

In addition to the correlation parameter, suppose (π_1,π_2) is also estimated. Proceeding as in (4.27), estimate the bivariate binomial probabilities with π_1 and π_2 also replaced by their mle's - which are, again, consistent. Once more the estimated probabilities are consistent estimates of the true probabilities. Replacing π_1 and π_2 in (4.27) by p_1 and p_2, respectively, yields $P(X_1 = x_1, X_2 = x_2)$ plus terms that go to zero in probability as $m\to\infty$.

The run-length properties of control charts constructed with the bivariate binomial parameters estimated in a base period are asymptotically equivalent to the run-length properties for the situation when the parameters are known a priori. The small-sample run-length properties of these Case II procedures are doubly complicated by the dependence of the control chart values and the actual bivariate distribution. The values of the diagnostic control charts for different time periods are, however, asymptotically independent. In small samples, the source of the dependence is the maximum likelihood estimators common to each control chart value. In the limit, however, these estimators converge to a constant, eliminating the dependencies. The asymptotic run-length distribution, therefore, is the geometric distribution with parameter $1 - P(X_1 \leq U_1, X_2 \leq U_2)$.

If, in contrast to the bivariate monitoring problem, we are concerned with the general r-binomial attribute monitoring, then the relevant parameters might also be estimated in an appropriate base period. The procedure for multivariate Bernoulli process monitoring set forth in Section Four depends only on the univariate and bivariate marginals. Both the univariate and bivariate probabilities can be estimated consistently if the correlation parameters or the marginal proportions defective are not specified a priori. The properties of these procedures as discussed in Section Four are now asymptotically valid for these Case II procedures.

The next topic to be discussed is the monitoring of multivariate multinomial processes in Case II with diagnostic charts. It is proposed to monitor each multinomial marginal using the Case II procedure of Section 3.5. To avoid unnecessarily complicating the notation, only the bivariate multinomial problem is explicitly studied, but the extension to more than two dimensions is mainly a bookkeeping chore.

Theorem 4.7. In monitoring a bivariate multinomial process when the marginal multinomial parameters and their covariance matrices are estimated in a base period, consider the statistics and control limits given by:

$$q_{i1} = \left(\frac{\eta}{1+\eta}\right) (\underset{\sim}{Y}_{i1} - \underset{\sim}{Y}_{01})' \, \hat{\underset{\sim}{\Sigma}}_{11}^{-1} \, (\underset{\sim}{Y}_{i1} - \underset{\sim}{Y}_{01}) \le U_1$$

and (4.28)

$$q_{i2} = \left(\frac{\eta}{1+\eta}\right) (\underset{\sim}{Y}_{i2} - \underset{\sim}{Y}_{02})' \, \hat{\underset{\sim}{\Sigma}}_{22}^{-1} \, (\underset{\sim}{Y}_{i2} - \underset{\sim}{Y}_{02}) \le U_2 ,$$

$$i = 1,2,\ldots$$

where $\ell im_{m,n\to\infty} \frac{m}{n} = \eta$, and the $\underset{\sim}{Y}_{ij}$'s and $\underset{\sim}{\Sigma}_{jj}$'s, $j = 1,2$, are defined at (4.5) and (4.7). Here $\hat{\underset{\sim}{\Sigma}}_{jj}$ is a consistent estimator of $\underset{\sim}{\Sigma}_{jj}$. When the process is in control the following probability inequalities are asymptotically correct, as $m,n\to\infty$ such that $\frac{m}{n}\to\eta$, namely,

(i) $$P(q_{11} \le U_1,\ldots,q_{K1} \le U_1,\; q_{12} \le U_2,\ldots,q_{K2} \le U_2)$$

$$\ge \prod_{i=1}^{K} P(q_{i1} \le U_1)\, P(q_{i2} \le U_2) \qquad (4.29)$$

(ii) $$P(q_{11} \le U_1,\ldots,q_{K1} \le U_1,\; q_{12} \le U_2,\ldots,q_{K2} \le U_2)$$

$$\ge P(q_{11} \le U_1,\ldots,q_{K1} \le U_1) \prod_{i=1}^{K} P(q_{i2} \le U_2) \qquad (4.30)$$

where $(q_{11},\ldots,q_{K1}) \xrightarrow{d} \chi_K^2(c_1-1,\; \frac{\eta}{1+\eta}\underset{\sim}{I}_K + \frac{1}{1+\eta}\underset{\sim}{J}_K)$ and similarly for $(q_{12},\ldots,q_{K2})$.

Proof. From the proof of Theorem 4.1 and expression (4.8) we have

$$\underset{\sim}{Y} = \begin{bmatrix} \underset{\sim}{Y}_{i1} - \underset{\sim}{Y}_{01} \\ \underset{\sim}{Y}_{i2} - \underset{\sim}{Y}_{02} \\ \vdots \\ \underset{\sim}{Y}_{K1} - \underset{\sim}{Y}_{01} \\ \underset{\sim}{Y}_{K2} - \underset{\sim}{Y}_{02} \end{bmatrix} \quad \overset{d}{\to} \quad N_{2K(c_1+c_2-2)}(\underset{\sim}{0}, (\underset{\sim}{I}_k + \eta^{-1}\underset{\sim}{J}_k) \times \underset{\sim}{\Sigma}_0),$$

where

$$\underset{\sim}{\Sigma}_0 = \begin{bmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22} \end{bmatrix}$$

is defined by (4.7). Take $\underset{\sim}{Y}$ into a canonical form via the transformation $\underset{\sim}{Y} \to ((\frac{\eta}{1+\eta})^{\frac{1}{2}} \underset{\sim}{I}_K \times \mathrm{diag}(\underset{\sim}{\Sigma}_{11}^{-\frac{1}{2}}, \underset{\sim}{\Sigma}_{22}^{-\frac{1}{2}}))\ \underset{\sim}{Y} = \underset{\sim}{Y}^*$. It follows that

$$\underset{\sim}{Y}^* \overset{d}{\to} N_{2K(c_1+c_2-2)}(\underset{\sim}{0}, (\frac{\eta}{1+\eta} \underset{\sim}{I}_K + \frac{1}{1+\eta} \underset{\sim}{J}_K) \times \underset{\sim}{\Sigma}_0^*),$$

where

$$\underset{\sim}{\Sigma}_0^* = \begin{bmatrix} \underset{\sim}{I}_{c_1-1} & \underset{\sim}{\Sigma}_{11}^{-\frac{1}{2}} \underset{\sim}{\Sigma}_{12} \underset{\sim}{\Sigma}_{22}^{-\frac{1}{2}} \\ \underset{\sim}{\Sigma}_{22}^{-\frac{1}{2}} \underset{\sim}{\Sigma}_{21} \underset{\sim}{\Sigma}_{11}^{-\frac{1}{2}} & \underset{\sim}{I}_{c_2-1} \end{bmatrix}.$$

Note that rearranging $\underset{\sim}{Y}^*$ into the more convenient form

$$\underset{\sim}{Y}^* = [(\tfrac{\eta}{1+\eta})^{\frac{1}{2}}\underset{\sim}{\Sigma}_{11}^{-\frac{1}{2}}(\underset{\sim}{Y}_{11} - \underset{\sim}{Y}_{01})', \ldots, (\tfrac{\eta}{1+\eta})^{\frac{1}{2}}\underset{\sim}{\Sigma}_{11}^{-\frac{1}{2}}(\underset{\sim}{Y}_{K1} - \underset{\sim}{Y}_{01})',$$

$$(\tfrac{\eta}{1+\eta})^{\frac{1}{2}}\underset{\sim}{\Sigma}_{22}^{-\frac{1}{2}}(\underset{\sim}{Y}_{21} - \underset{\sim}{Y}_{02}), \ldots, (\tfrac{\eta}{1+\eta})^{\frac{1}{2}}\underset{\sim}{\Sigma}_{22}^{-\frac{1}{2}}(\underset{\sim}{Y}_{K2} - \underset{\sim}{Y}_{02})']'$$

does not affect the limiting distribution.

Let $\underset{\sim}{Y}_1^*$ represent the first $K(c_1-1)$ elements of $\underset{\sim}{Y}^*$, and denote by C the set $\{(\underset{\sim}{Z}_1,\ldots,\underset{\sim}{Z}_K)\colon \underset{\sim}{Z}_1' \underset{\sim}{Z}_1 \leq U_1,\ldots,\underset{\sim}{Z}_K' \underset{\sim}{Z}_K \leq U_1\}$. Note that C is convex and symmetric about $\underset{\sim}{0}$. Hence, by Theorem 2.10, it follows that

$$P(\underset{\sim}{Y}_1^* \in C,\ q_{12} \leq U_2,\ldots,q_{K2} \leq U_2) \geq P(\underset{\sim}{Y}^* \in C) \prod_{i=1}^{K} P(q_{i2} \leq U_2).$$

But $P(\underset{\sim}{Y}_1^* \in C) = P(q_{11} \leq U_1,\ldots,q_{K1} \leq U_1)$, where $(q_{11},\ldots,q_{K1})$ has the same joint distribution as the control chart studied in Theorem 3.6. Hence, (4.30) is proven. Expression (4.29) follows similarly from Corollary 2.4.

Theorem 4.7 gives asymptotic methods for establishing conservative control limits as well as for investigating the distribution of the run length, N_D, of the diagnostic control chart. From familiar arguments the following stochastic upper bounds on the asymptotic run-length distribution are obtained when the process is in control

$$P(N_D \leq t) \leq 1 - P(q_{11} \leq U_1,\ldots,q_{t1} \leq U_1) \prod_{i=1}^{t} P(q_{i2} \leq U_2), \qquad (4.31)$$

and

$$P(N_D \leq t) \leq 1 - \prod_{i=1}^{t} P(q_{i1} \leq U_1)\, P(q_{i2} \leq U_2). \qquad (4.32)$$

Expression (4.32) states that the run-length distribution, in the limit, is stochastically larger than the geometric distribution with parameter $1 - P(Z_1 \leq U_1)\, P(Z_2 \leq U_2)$, where $Z_i \sim \chi_1^2(c_i-1,1)$, $i = 1,2$.

We repeat the caveat that, aside from the asymptotic unbiasedness, which follows from Theorem 1.4, nothing can be said at present concerning the out-of-control properties of the procedure in Theorem 4.7.

4.6 Summary

In this chapter we have considered various techniques for monitoring multivariate Bernoulli and multinomial processes.

Omnibus control chart procedures are particularily relevant when the entire process must be adjusted whatever attributes may be out of control. Asymptotic χ^2 charts were proposed for use in monitoring multivariate Bernoulli and multinomial processes under both Cases I and II. It was shown that for processes which are out of control but stationary, the proposed Case II procedures will signal faster, asymptotically in probability, the more the process is out of control as indicated by the magnitude of its noncentrality parameter. Other stochastic bounds were given also for the run-length distributions of these control charts.

Diagnostic control chart procedures are more relevant to monitoring situations where it is advantageous to determine which quality characteristics are causing the charts to signal. A rather extensive study of Case I diagnostic procedures was undertaken. Both small- and large-sample procedures were given for multivariate Bernoulli process monitoring. Empirical results suggest that the small-sample procedures operate in a suitable fashion, but analytical results for these procedures are not yet available. The large-sample diagnostic charts are based upon the asymptotic normality of the multivariate Bernoulli and multinomial distributions. Various probability inequalities introduced in Chapters One and Two were used to develop approaches for obtaining asymptotic levels of these diagnostic charts.

For Case II diagnostic monitoring we were unable to obtain any small-sample control charts. It was shown, however, that the small-sample Case I diagnostic procedures were asymptotically valid when the parameters were estimated consistently in a base period. Further, some diagnostic control charts for multivariate multinomial processes were considered. These charts used the univariate generalized p-chart for each dimension. They were shown to have properties dependent upon the multivariate chi-squared distributions of Section 2.6.

V. MONITORING THE NUMBER OF DEFECTS OF A PROCESS - C-CHARTS AND EXTENSIONS

5.1 Introduction

In this chapter procedures for monitoring the number of defects per unit output of a process are discussed. The customary control chart for this purpose is based upon the Poisson distribution, and it is referred to as the c-chart. Throughout this chapter the c-chart and some generalizations of it are examined in situations of monitoring univariate or multivariate attributes.

Later developments will demonstrate that the c-chart is valid when each unit has only one possible type of defect. For the more general situation where different types of defects are grouped together as a "single defect", a proper limiting distribution when the defects are dependent is not the Poisson, but the generalized Poisson. Both of these monitoring procedures may be considered to be univariate.

Multivariate procedures become relevant, however, when different types of defects are classified into more than one grouping - for example, defects may be categorized as critical, major, or minor. Such circumstances require results for the multivariate versions of the Poisson and generalized Poisson distributions discussed in Chapter Two in order to implement monitoring procedures. Both omnibus and diagnostic charts are considerd herein for these multivariate problems.

The statistical model for the procedures of this chapter may arise in two ways. First, the process being monitored may genuinely be a Poisson or generalized Poisson process. Alternatively, the Poisson or generalized Poisson distribution may arise as a limiting distribution of a univariate or multivariate Bernoulli process. These models will be elaborated upon in due course.

The first task for this chapter is to review the standard c-chart and provide some optimality properties. Then generalized c-charts are introduced and some of their properties are given. Finally, the multivariate procedures for monitoring defects are elucidated.

5.2 Monitoring One Type of Defect: The c-Chart

The everyday methodology of quality control makes no distinction among defects of different types when monitoring the number of defects per unit of a process. Such an approach may be fallacious, as we will demonstrate in the following section. In the present section, however, only processes with one type of defect will be considered. In these instances, the Poisson distribution is appropriate, either as a limiting distribution of a Bernoulli process or as a realization of a Poisson process.

A thorough description of the c-chart methodology is contained in Duncan (1974). As with p-charts, the suggested procedure is to establish control limits at $\pm K$ standard deviations from the target value for the defects per unit. For example, if the expected number of defects is given by λ_0, then the control limits of the procedure are taken as $\lambda_0 - K\sqrt{\lambda_0}$ and $\lambda_0 + K\sqrt{\lambda_0}$.

Rather than accepting uncritically this methodological approach, results closely paralleling those already expounded for the p-chart in Section 3.2 can be advanced. Optimal procedures for various hypothesis tests of Poisson parameters are well known. Suppose the specified parameter value of the Poisson distribution is λ_0; the alternative out-of-control states will be taken as (i) $\lambda > \lambda_0$ or (ii) $\lambda \neq \lambda_0$. For the one-sided alternative, the following theorem is the analogue of Theorem 3.1.

Theorem 5.1. Suppose the number of defects of a process can be effectively modeled using a Poisson process. Let $(\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots)$ represent independent realizations of this process from a monitoring period. In the class of control charts based on $(Y_1, Y_2, \ldots)$, where $Y_i = f(\underset{\sim}{X}_i)$, consider the c-chart with the control limits (U, γ) $(0 \leq \gamma \leq 1)$ such that

$$P(P_1(\lambda_0) > U) + \gamma P(P_1(\lambda_0) = U) = \alpha \,, \qquad (5.1)$$

where $P_1(\lambda_0)$ is the univariate Poisson distribution with parameter λ_0. Then this c-chart dominates any other control chart of level α for detecting increases in the expected number of defects above λ_0 in the class of charts based on $(Y_1, Y_2, \ldots)$.

Proof. Control limits chosen using (5.1) define a UMP test at each monitoring occasion. The independence of the test statistics and Lemma 1.1 complete the proof.

For two-sided alternatives the next theorem describes an optimal c-chart.

Theorem 5.2. Let $(\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots)$ and $(Y_1, Y_2, \ldots)$ be as in Theorem 5.1. Consider the c-chart with control limits $(L, \gamma_L, U, \gamma_U)$ obtained by replacing $b_1(n, \pi_0)$ and $b_1(n-1, \pi_0)$ in (3.3) with $P_1(\lambda_0)$. Then this c-chart dominates any other control chart of level α for detecting shifts in the expected number of defects, λ_0, in the class of unbiased control charts satisfying $Y_i = f(\underset{\sim}{X}_i)$.

Proof. The control limits define the UMPU test of level α for H_0: $\lambda=\lambda_0$ versus H_1: $\lambda\neq\lambda_0$; cf. Lehmann (1959), p. 128ff. The proof proceeds as before using Lemma 1.1.

The optimal control charts of Theorems 5.1 and 5.2 have randomized control limits. A discussion of the advantages and disadvantages of randomized control limits can be found in Section 3.2. These same arguments are relevant here, and, once more, γ_L and γ_U will, for convenience, be taken to be zero.

The more common methodology of quality control uses an initial estimate of the expected number of defects found in a base period, rather than a level established by fiat. The consequences of pursuing such a methodology when monitoring a single type of defect are similar to those found for the p-chart in Section 3.4. These problems are now considered.

If the in-control parameter value is estimated from a base period by c_0, the standard control limits are chosen to be $c_0 \pm K\sqrt{c_0}$. In studying the properties of the procedure the fact that c_0 is an estimate and not the actual parameter value is usually ignored. This oversight suggests that a reexamination of c-charts is needed for use in Case II.

A UMPU test exists for the comparison of two independent Poisson populations (Lehmann (1959), p.141). This test is a conditional test based upon the binomial distribution.

Recall the definition of the class C_2 from Section 3.4, i.e. control charts based on test statistics of the form $\{T(\underset{\sim}{X}_1,\underset{\sim}{X}_0),T(\underset{\sim}{X}_2,\underset{\sim}{X}_0),\ldots\}$

where $T(\underset{\sim}{X}_i, \underset{\sim}{X}_0)$ (i=1,2,...) yields an unbiased statistical test. The next theorem is the analogue of Theorem 3.5, applicable to c-charts. Again, the results of this theorem give some fairly weak optimality statements.

Theorem 5.3. Under Case II, suppose a Poisson distribution can be used to model the number of defects of a process. Consider the control chart based upon the conditional distribution

$$P(S_i=s_i \mid S_i+S_0=t_i) = \binom{t_i}{s_i}(\tfrac{1}{2})^{t_i} \quad , \tag{5.2}$$

where S_i is the number of defects in monitoring period i. Let the control limits be established based upon the UMP (or UMPU) test for the binomial distribution with $\pi_0 = \frac{1}{2}$. Then for stationary processes

(i) the approximating geometric run-length distribution using (5.2) dominates the approximating geometric run-length distribution of any other member of C_2;

(ii) the control chart is admissible in the class of control charts in C_2 which are based on unbiased tests.

Proof. The proof of Theorem 5.3 mimics the proof of Theorem 3.5.

The c-chart of Theorem 5.3 is based upon a conditional test, but the run-length distribution must be expressed in terms of the unconditional distribution. The run-length distribution of the c-chart based upon (5.2) is given by

$$P(N>k) = \sum_{t_1=s_1}^{\infty} \cdots \sum_{t_k=s_k}^{\infty} \sum_{s_i \in \ell} \binom{t_1}{s_1} \cdots \binom{t_k}{s_k}$$

$$\cdot \left(\frac{\lambda_1}{\lambda_1+\lambda_0}\right)^{s_1} \cdots \left(\frac{\lambda_k}{\lambda_k+\lambda_0}\right)^{s_k} \left(\frac{\lambda_0}{\lambda_1+\lambda_0}\right)^{t_1-s_1} \cdots \left(\frac{\lambda_0}{\lambda_k+\lambda_0}\right)^{t_k-s_k}$$

$$\cdot P(T_1=t_1,\ldots,T_k=t_k) \ , \qquad (5.3)$$

where $T_i = S_i+S_0$ and ℓ represents the in-control region of the control chart.

Note that $(T_1,\ldots,T_k)'$ in (5.3) has a multivariate Poisson distribution. The run-length probabilities for individual cases can be computed, but, otherwise, it is not clear how to obtain general run-length properties from (5.3). We are faced with the same dilemma encountered in trying to analyze the Bernoulli process Case II monitoring procedures. Though there is, in (5.3), more structure available for this problem, we are still left with rather incomplete results regarding the run-length properties of the Case II procedure.

5.3 Monitoring the Sum of More Than One Type of Defect - The Generalized c-Chart

In the preceding section the Poisson distribution was used, with good results, for monitoring a process with only one type of defect. The restriction to one type of defect is weakened in this section. Assume, at first, that there is more than one type of defect that can occur on each unit produced by a process. The joint distribution of the numbers of different types of defects is given by the multivariate binomial distribution. Recall, from Section 2.5, that the generalized Poisson distribution arises as a limiting distribution for the sum of the components of a multivariate binomial distribution.

The parameter function that is of interest in the c-chart monitoring procedure is the expected number of defects. For the standard c-chart this hypothesis was simply $n\pi_0 \to \lambda_0$ - the mean of the Poisson distribution. When there are k types of defects, the expected number of defects of all types in a sample of n units is $n(\pi_1+\pi_2+\ldots+\pi_k)$, where the π_i's are the marginal binomial parameters. In the notation of (2.4) and (2.20) we have that

$$\begin{aligned} n(\sum_i \pi_i) &= n(\pi_{10\ldots0}+\ldots+\pi_{0\ldots01}) \\ &\quad + 2n(\pi_{110\ldots0}+\ldots+\pi_{0\ldots011}) \\ &\quad +\ldots+kn\pi_{1\ldots1} \\ &\to \lambda_1 + 2\lambda_2 +\ldots+ k\lambda_k = c_0 \quad , \end{aligned} \tag{5.4}$$

as $n\to\infty$ as in (2.20). From (5.4), we see that the c-chart is monitoring, at least asymptotically, the mean of a generalized Poisson distribution. Such an application of the generalized Poisson distribution was suggested in a remark by Maritz (1952), but neither he nor anyone else seems to have pursued this approach.

Despite the inadequacy of the Poisson model when dependent defects can occur in each unit, it is, nonetheless, widely advocated for use in monitoring the expected number of defects (5.4). The control limits of the chart are taken as $c_0 \pm K\sqrt{c_0}$. These are the familar $K\sigma$ limits, but the standard deviation used is the one applicable to the Poisson distribution. From (2.16), the standard deviation for the generalized Poisson distribution is $(c_0+2\lambda_2+6\lambda_3+\ldots+k(k-1)\lambda_k)^{1/2}>\sqrt{c_0}$. When used for monitoring defects the generalized Poisson distribution is, by this measure, "more spread out" than the Poisson distribution. This observation suggests that the use of the Poisson σ-limits will lead to a chart which signals more often than the level indicates when the process is in control.

A somewhat more precise statement of this behavior is available through Chebyshev's inequality. If $S_{\underset{\sim}{\lambda}}$, with $\underset{\sim}{\lambda} = (\lambda_1,\lambda_2,\ldots,\lambda_k)'$, denotes the generalized Poisson random variable used in monitoring the number of defects, then

$$P(|S_{\underset{\sim}{\lambda}}-c_0|\geq K\sqrt{c_0}) \leq \frac{1}{K^2} + \frac{2\lambda_2+6\lambda_3+\ldots+k(k-1)\lambda_k}{K^2 c_0} \quad . \tag{5.5}$$

The use of the limits $c_0\pm K\sqrt{c_0}$ gives a higher upper bound on the level of the chart than the correct σ-limits. The difference is given by the second term of the right hand side of (5.5).

While tradition may dictate the choice of σ-limits for the control chart, it is reasonable to ask whether these limits have any statistical justification when the generalized Poisson distribution is employed to monitor the expected number of defects. A partial solution is forthcoming from Theorem 2.4 - $S_{\underset{\sim}{\lambda}^*}$ is stochastically larger than $S_{\underset{\sim}{\lambda}}$ if $\underset{\sim}{\lambda}^*\succ\underset{\sim}{\lambda}$. This fact supports using an upper limit u such that $P(S_{\underset{\sim}{\lambda}}\leq u)=\alpha$, for α-level control charts monitoring against increases in the expected number of defects. Of course $c_0^*>c_0$ does not imply that $\underset{\sim}{\lambda}^*\succ\underset{\sim}{\lambda}$. In fact, if c_0

increases in such a way that $\sum_i \lambda_i$ decreases, then $P(S_{\underset{\sim}{\lambda}*}=0) \geq P(S_{\underset{\sim}{\lambda}}=0)$, hence $S_{\underset{\sim}{\lambda}*}$ does not stochastically order $S_{\underset{\sim}{\lambda}}$ in that instance.

If the generalized c-chart arises as a consequence of the multivariate binomial limiting process (2.20), an interpretation can be given to the restriction that $\underset{\sim}{\lambda}* \succ \underset{\sim}{\lambda}$. In this instance λ_i represents the expected number of items having j defects in the limit, j=1,2,...,k. So, if we are comparing a collection of k-variate Bernoulli processes for which each λ_j is nondecreasing, then the stochastic ordering result can be applied to this class of processes, in the limit. Denote such a class by $\mathcal{P}_I$, if the processes are stationary. Let $\mathcal{P}_{\underset{\sim}{\lambda}_0}$ represent the class of k-variate Bernoulli processes for which $\underset{\sim}{\lambda} \succ \underset{\sim}{\lambda}_0$, at each monitoring occasion.

Theorem 5.4. Consider monitoring against increases in the expected number of defects of a k-variate Bernoulli process under Case I. Suppose the process is declared to be out of control if $S_{\underset{\sim}{\lambda}_0} > U$, where $\underset{\sim}{\lambda}_0$ is the vector of in-control parameters.

(i) In the class $\mathcal{P}_{\underset{\sim}{\lambda}_0}$, this generalized c-chart is asymptotically unbiased.

(ii) In any class of the type $\mathcal{P}_I$, the more out of control the process is, i.e., the greater the expected number of defects, the more quickly, asymptotically, this generalized c-chart signals, in probability.

Proof.

(i) From Theorem 2.4, $P(S_{\underset{\sim}{\lambda}} > U) \geq P(S_{\underset{\sim}{\lambda}_0} > U)$, asymptotically, at each monitoring occasion. From the independence of the monitoring statistics, the run-length distribution for $S_{\underset{\sim}{\lambda}}$ is stochastically smaller, in the limit, than for $S_{\underset{\sim}{\lambda}_0}$.

(ii) Theorem 2.4 can be applied between any two pairs of processes in P_I. As in part (i) the run-length distributions are stochastically ordered.

A necessary condition for $\underset{\sim}{\lambda} \succ \underset{\sim}{\lambda}_0$ for multivariate Bernoulli processes is that, in the limit, the expected number of items having no defects in $\underset{\sim}{\lambda}$ decreases relative to $\underset{\sim}{\lambda}_0$. But, in order to use Theorem 5.4, this decrease in the expected number of no defects must be accompanied by increases, or nondecreases, in the expected number of items having j defects, for $j=1,2,\ldots,k$. Certainly, one goal of a process monitoring scheme for the number of defects is to detect when the expected number of items having no defects decreases.

Up to now we have proposed using a generalized c-chart only as a limiting procedure for monitoring the total number of defects of a multivariate Bernoulli process. However, a generalized c-chart is also appropriate for monitoring stochastic processes arising in a fashion described by Janossy _et al_. (1950). They demonstrated that the probability of T events in a time interval of a Markov process which is homogeneous in time is given by the generalized Poisson distribution. That is, a generalized Poisson process requires that the number of events occurring in the time intervals (t_1,t_2) and (t_3,t_4), for $t_1<t_2\leq t_3<t_4$, are independent, and the probability of T events occurring in any time interval depends only on the length of the interval. Note that the Poisson process requires the additional assumption that the probability

of two or more events in a short time interval is arbitrarily small.

The general results discussed above for the generalized c-chart are also applicable to the monitoring of generalized Poisson processes. The interpretation for generalized Poisson processes is as follows. If the mean number of time periods having one defect, the mean number of time periods having two defects, et cetera, all are nondecreasing, for process one, as compared to process two, then process one will signal faster, in probability, than process two.

To this point we have assumed that the parameters of the generalized Poisson distribution when the process is in control could be specified. Even when the units of the process can be defective on just a few attributes such an assumption is questionable. This difficulty is magnified as the number of possible defects increases. Therefore, we now consider approaches to monitoring the process under Case II.

Unlike the Poisson or the binomial distributions, there are no known UMPU procedures for comparing the parameters of two generalized Poisson distributions. So, a reasonable first step is to examine asymptotic procedures which use consistent estimators in the stead of the generalized Poisson parameters. For the monitoring of generalized Poisson processes, such procedures will be asymptotically correct. In what follows we also show that the asymptotic distribution theory is left unchanged for the generalized c-chart if consistent estimates are used for the parameters.

If $S_{\tilde{\lambda}}$ represents the generalized Poisson random variable used in a monitoring procedure, then the pmf of $S_{\tilde{\lambda}}$ is given by (2.17). Suppose that instead of the parameter values $\underset{\sim}{\lambda}$ we have a consistent estimate $\hat{\underset{\sim}{\lambda}}$, and this $\hat{\underset{\sim}{\lambda}}$ is used to replace $\underset{\sim}{\lambda}$ in (2.17). Then

$$P(S_{\hat{\underset{\sim}{\lambda}}}=b) = \exp(-\Sigma\hat{\lambda}_i)\sum_{K_b} \frac{\hat{\lambda}_1^{k_1}\cdots\hat{\lambda}_b^{k_b}}{k_1!k_2!\cdots k_b!}$$

$$= \exp(-\Sigma(\lambda_i+o_p(1)))\left\{\sum_{K_b} \frac{(\lambda_1+o_p(1))^{k_i}\cdots(\lambda_b+o_p(1))^{k_b}}{k_1!k_2!\cdots k_b!}\right\}$$

$$= \exp(-\Sigma\lambda_i)\exp(-(o_p(1)))\left\{\sum_{K_b} \frac{\prod_{i=1}^{b}\sum_{j=1}^{k_i}\binom{k_i}{j}\lambda_i^j[o_p(1)]^{k_i-j}}{k_1!k_2!\cdots k_b!}\right\}$$

$$= \exp(-\Sigma\lambda_i)\exp(-o_p(1))\left\{\sum_{K_b}\left[\prod_{i=1}^{b}\frac{\lambda_i^{k_i}}{k_i!}+o_p(1)\right]\right\}$$

$$\overset{P}{\rightarrow} \exp(-\Sigma\lambda_i)\sum_{K_b}\left\{\prod_{i=1}^{b}\frac{\lambda_i^{k_i}}{k_i!}\right\} = P(S_{\underset{\sim}{\lambda}}=b) \quad b=0,1,2,\ldots \tag{5.6}$$

Therefore, from (5.6), if $\hat{\underset{\sim}{\lambda}}$ is any consistent estimator of $\underset{\sim}{\lambda}$ the estimated pmf converges in probability to the true pmf. If $S_{\underset{\sim}{\lambda}}$ is obtained via the limiting argument of (2.20), then the use of $S_{\hat{\underset{\sim}{\lambda}}}$ does not alter the asymptotic distribution theory. All of the properties of the generalized c-chart under Case I are now true asymptotically, for Case II.

What remains to be answered, though, is what consistent estimators to use for the generalized Poisson parameters. Several methods of estimation have been proposed by Gurland (1965) and Hinz and Gurland

(1967). Patel (1976) considered some important special cases of the generalized Poisson distribution and their estimation.

Patel (1976) discusses estimation of generalized Poisson distributions, when k in (5.4) is three or four, via maximum likelihood, the method of moments, and a method using sample frequencies. The latter two estimating techniques were found to have rather poor asymptotic efficiencies, but the maximum likelihood estimators faired poorly in small samples and were shackled with a rather large bias term. The mle's are also more burdensome computationally.

Gurland (1965) and Hinz and Gurland (1967) discussed general methods for estimating generalized Poisson distributions. These authors proposed minimum chi-squared estimators using the sample factorial cumulants, sample frequencies, and a combination of sample frequencies and factorial cumulants. The minimum chi-squared estimators are similar in form to weighted least squares estimators, so they have an advantage over the maximum likelihood estimators of being relatively easy to compute. The minimum chi-squared techniques also yield asymptotically efficient estimates. The authors illustrated these estimation procedures with the negative binomial, Neyman Type A, Poisson-binomial, and Poisson-Pascal distributions, but the same methodology is valid for any generalized Poisson distribution (2.13) with a finite number of parameters.

Patel's techniques may be extended to generalized Poisson distributions for any finite k. However, for k>3 the moment estimators are no longer linear. The factorial cumulants do not suffer from this drawback, and, hence, estimators based on the sample factorial cumulants are relatively easy to compute. For instance, if the generalized Poisson distribution has s parameters, the s factorial cumulants, $\kappa_{(1)},\ldots,\kappa_{(s)}$, can be expressed in terms of the parameters $\lambda_1,\ldots,\lambda_s$, as

$$\begin{bmatrix} \kappa_{(1)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \kappa_{(s)} \end{bmatrix} = \begin{bmatrix} 1 & 2 & \cdot & \cdot & \cdot & \cdot & s \\ 0 & 2 & \cdot & \cdot & \cdot & \cdot & s(s-1) \\ & & 6 & \cdot & \cdot & \cdot & s(s-1)(s-2) \\ \vdots & & & \cdot & & & \\ & & & & \cdot & & \\ 0 & & & & & \cdot & s! \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \lambda_s \end{bmatrix} = \underset{\sim}{W}\underset{\sim}{\lambda} \quad .$$

Therefore, a consistent estimator for $\underset{\sim}{\lambda}$ is given by,

$$\hat{\underset{\sim}{\lambda}} = \underset{\sim}{W}^{-1}\underset{\sim}{X} , \tag{5.7}$$

where $\underset{\sim}{X} = [k_{(1)},\ldots,k_{(s)}]'$, with $k_{(i)}$ the ith sample factorial cumulant.

The estimation techniques utilizing sample frequencies can be combined with (5.7) to possibly improve the efficiencies of these estimators. A possible drawback to this approach for quality control problems is the necessity of having observations of zero defects.

5.4 Omnibus Procedures for Monitoring More Than One Type of Defect

Next, we study the problem of monitoring a process when a number of types of defects per unit are considered separately. Omnibus procedures are examined in this section, while diagnostic procedures are reserved for the next section.

In the preceding section it was shown that the generalized Poisson distribution plays a central role for monitoring the total number of more than one type of defect. In a sense, then, that asymptotic procedure could be construed to be an omnibus chart for a multivariate

Bernoulli process. The distinguishing feature of the techniques to be considered in this section is that an effort is made to track the defects of different types. This effort may entail keeping tabs on each defect, or grouping the defects into major subcategories such as critical, major, and minor defects, for example.

First, consider the use of the multivariate Poisson distribution. This distribution is appropriate when each defect classification of the unit arises from only one type of defect. When the multivariate Poisson distribution is used as a limiting distribution of the multivariate binomial distribution, the problem is simply the one encountered in Section 4.4 and 4.5 with the Poisson replacing the binomial distribution.

Another possible model for a multivariate Poisson process uses the "random component in common" model reviewed in Section 2.4. For example, suppose three different components are manufactured independently, and the number of defects in each component is given by a Poisson process. Further, suppose one of the three components is combined with each of the other two in a subassembly. If the number of defects is counted for each subassembly, then the joint distribution of the number of defects is given by a bivariate Poisson distribution. The model can be extended to k dimensions if one component is combined with k independent components in each of k subassemblies.

Recall from Section 2.5 that some multivariate generalized Poisson distributions may also be characterized by a "random component in common" model. Therefore, if in the preceding paragraph the number of defects for each subassembly is a univariate generalized Poisson process, then the joint distribution of the number of defects is a multivariate generalized Poisson distribution.

First, suppose the multivariate Poisson distribution adequately describes the underlying process. Patel (1973) suggested a procedure based on asymptotic normality through the multivariate central limit theorem. For our purposes, though, it seems a somewhat artificial construct to impose central limit theory on a Poisson process. Perhaps a more realistic justification for using an asymptotic normal procedure can be found in Theorem 2.3.

Denote by X_i the observed number of defects and by λ_i the in-control expected number of defects of type i, where i=1,...,m. In the notation of Theorem 2.3, we have

$$\underset{\sim}{Y}'\underset{\sim}{R}^{-1}\underset{\sim}{Y} \xrightarrow{d} \chi_1^2(m,1) \ , \tag{5.7}$$

under the appropriate limiting operations on the λ_i's and λ_{ij}'s.

It would be helpful to see how well a monitoring procedure based on the asymptotic χ^2 statistic (5.7) might approximate the small-sample properties. To this end, the actual probability that a bivariate Poisson process signals, when it is in control, was computed for a control chart based on (5.7) with a nominal level of 0.01. These results are given in Tables 5.1-5.3 for various values of λ_1, λ_2, and ρ. The method used in arriving at the tabled values was essentially the same as the one described in Section 4.2 for the bivariate binomial distribution.

TABLE 5.1. ACTUAL LEVELS OF THE OMNIBUS PROCEDURE (5.7) FOR A BIVARIATE POISSON PROCESS FOR VARIOUS VALUES OF $\underset{\sim}{\lambda}$ WHEN RHO IS 0.10 . THE NOMINAL LEVEL IS 0.01.

λ_1 \ λ_2	0.5	1.0	2.0	3.0	4.0	5.0
0.5	0.02822	0.02724	0.02021	0.02301	0.02272	0.02501
1.0	0.02724	0.02617	0.02043	0.01930	0.02049	0.02012
1.5	0.02039	0.02423	0.01847	0.01514	0.01652	0.01724
2.0	0.02021	0.02043	0.01587	0.01542	0.01506	0.01472
2.5	0.01968	0.02321	0.01586	0.01437	0.01661	0.01429
3.0	0.02301	0.01930	0.01542	0.01383	0.01341	0.01517
3.5	0.02353	0.02152	0.01424	0.01588	0.01431	0.01407
4.0	0.02272	0.02049	0.01506	0.01341	0.01534	0.01381
4.5	0.02652	0.02168	0.01580	0.01427	0.01572	0.01483
5.0	0.02501	0.02012	0.01472	0.01517	0.01381	0.01382

TABLE 5.2. ACTUAL LEVELS OF THE OMNIBUS PROCEDURE (5.7) FOR A BIVARIATE POISSON PROCESS FOR VARIOUS VALUES OF $\underset{\sim}{\lambda}$ WHEN RHO IS 0.25 . THE NOMINAL LEVEL IS 0.01.

λ_1 \ λ_2	0.5	1.0	2.0	3.0	4.0	5.0
0.5	0.02743	0.02581	0.02498	0.02296	0.01996	0.02430
1.0	0.02581	0.02726	0.02305	0.01972	0.01947	0.02072
1.5	0.02599	0.02303	0.02099	0.01789	0.01671	0.01606
2.0	0.02498	0.02305	0.01618	0.01552	0.01573	0.01699
2.5	0.02520	0.01918	0.01514	0.01587	0.01799	0.01574
3.0	0.02296	0.01972	0.01552	0.01444	0.01464	0.01445
3.5	0.02127	0.02350	0.01577	0.01561	0.01643	0.01472
4.0	0.01996	0.01947	0.01573	0.01464	0.01370	0.01439
4.5	0.02131	0.02184	0.01490	0.01645	0.01510	0.01628
5.0	0.02430	0.02072	0.01699	0.01445	0.01439	0.01396

TABLE 5.3. ACTUAL LEVELS OF THE OMNIBUS PROCEDURE (5.7) FOR A BIVARIATE POISSON PROCESS FOR VARIOUS VALUES OF $\underset{\sim}{\lambda}$ WHEN RHO IS 0.40 . THE NOMINAL LEVEL IS 0.01.

	λ_2					
λ_1	0.5	1.0	2.0	3.0	4.0	5.0
0.5	0.02629	0.03059	0.02151	0.02173	*	*
1.0	0.03059	0.02510	0.02399	0.02114	0.02141	0.02129
1.5	0.02493	0.02276	0.02046	0.01739	0.01542	0.01715
2.0	0.02151	0.02399	0.01750	0.01774	0.01713	0.01532
2.5	0.02286	0.01916	0.01768	0.01489	0.01405	0.01557
3.0	0.02173	0.02114	0.01774	0.01546	0.01466	0.01483
3.5	*	0.02145	0.01456	0.01372	0.01510	0.01218
4.0	*	0.02141	0.01713	0.01466	0.01336	0.01515
4.5	*	0.02261	0.01576	0.01727	0.01666	0.01625
5.0	*	0.02129	0.01532	0.01483	0.01515	0.01367

* $\rho > \min\left((\lambda_1/\lambda_2)^{\frac{1}{2}}, (\lambda_2,\lambda_1)^{\frac{1}{2}}\right)$

At least for the range of parameters used to construct these tables, the asymptotic χ^2 procedure is non-conservative when the bivariate Poisson process is in control. As would be expected, the error is greatest for small values of λ_1 or λ_2. But, if a moderate number of defects per unit are expected, the error of the approximate procedure may not be too bad.

The same drawbacks encountered for other multivariate Case I monitoring schemes also limit the usefulness of a χ^2 chart based on (5.7). A modification, therefore, may be in order. As before, we consider monitoring the process in a base period which reflects the in-control operation of the process. Let X_{ijp} represent the number of defects of type j on the pth observation in monitoring period i. Consider $\underset{\sim}{Y}_i = (Y_{i1},\ldots,Y_{i\ell})'$, i=0,1,...,k, where

$$Y_{0j} = \frac{\sum_{p=1}^{m} \frac{X_{0jp}}{m} - \lambda_{0j}}{\sqrt{\lambda_{0j}}}$$

and (5.8)

$$Y_{ij} = \frac{\sum_{p=1}^{n} \frac{X_{ijp}}{n} - \lambda_{ij}}{\sqrt{\lambda_{ij}}} \qquad \begin{matrix} i=1,\ldots,k \\ j=1,\ldots,\ell \end{matrix} \quad .$$

In (5.8) m and n represent the number of observations of the process in the base period and in each monitoring period, respectively, while λ_{ij} is the expected number of defects of type j in monitoring period i.

Now, under the limiting operations of Theorem 2.3 the statistic given by

$$c_i = \left(\frac{mn}{m+n}\right)(\underset{\sim}{Y}_i-\underset{\sim}{Y}_0)'\hat{\underset{\sim}{R}}^{-1}(\underset{\sim}{Y}_i-\underset{\sim}{Y}_0) \ , \quad i=1,\ldots,k \ , \tag{5.9}$$

where $\hat{\underset{\sim}{R}}$ is a consistent estimator of $\underset{\sim}{R}$ defined in Theorem 2.3, is asymptotically distributed as $\chi_1^2(\ell,1)$ when the process is in control. If (5.9) is used for establishing an omnibus control chart for the vector of expected number of defects, the following, familiar looking, theorem results.

<u>Theorem 5.5</u>. Consider the c-chart established by using (5.9) at each monitoring occasion i. Suppose the limiting conditions of Theorem 2.3 hold for each $\underset{\sim}{Y}_i$, i=0,1,2,...,k. Then the distribution of $(c_1,c_2,\ldots,c_k)'$ is asymptotically $\chi_k^2(\ell,\frac{m}{m+n}\underset{\sim}{I}_k + \frac{n}{m+n}\underset{\sim}{J}_k)$, when the process is in control.

<u>Proof</u>. The proof closely parallels the proof of Theorem 3.6. A sketch follows. From Theorems 2.3 and 1.8, we have

$$\begin{bmatrix} \underset{\sim}{Y}_i-\underset{\sim}{Y}_0 \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{Y}_k-\underset{\sim}{Y}_0 \end{bmatrix} \overset{d}{\to} N_{k\ell}(\underset{\sim}{0},\underset{\sim}{G}\ \mathrm{diag}(\tfrac{1}{m}\underset{\sim}{R}_0,\tfrac{1}{n}\underset{\sim}{R}_1,\ldots,\tfrac{1}{n}\underset{\sim}{R}_k)\underset{\sim}{G}') \ ,$$

where

$$\underset{\sim}{G} = \begin{bmatrix} \underset{\sim}{I}_\ell & -\underset{\sim}{I}_\ell & & \underset{\sim}{0} \\ \vdots & & \ddots & \\ \underset{\sim}{I}_\ell & \underset{\sim}{0} & & -\underset{\sim}{I}_\ell \end{bmatrix} .$$

Therefore, $\underset{\sim}{G}\ \mathrm{diag}(\frac{1}{m}\underset{\sim}{R}_0, \frac{1}{n}\underset{\sim}{R}_1, \ldots, \frac{1}{n}\underset{\sim}{R}_k)\ \underset{\sim}{G}'$ is

$$\begin{bmatrix} \frac{1}{m}\underset{\sim}{R}_0 + \frac{1}{n}\underset{\sim}{R}_1 & \frac{1}{m}\underset{\sim}{R}_0 & \cdots & \frac{1}{m}\underset{\sim}{R}_0 \\ \frac{1}{m}\underset{\sim}{R}_0 & \frac{1}{m}\underset{\sim}{R}_0 + \frac{1}{n}\underset{\sim}{R}_2 & & \\ \vdots & & \ddots & \\ \frac{1}{m}\underset{\sim}{R}_0 & \cdots & \cdots & \frac{1}{m}\underset{\sim}{R}_0 + \frac{1}{n}\underset{\sim}{R}_k \end{bmatrix} . \tag{5.10}$$

Under the in-control hypothesis, (5.10) becomes $(\frac{1}{n}\underset{\sim}{I}_k + \frac{1}{m}\underset{\sim}{J}_k) \times \underset{\sim}{R}_0$. By Theorem 2.7, $(Q_1,\ldots,Q_k)'$ where

$$Q_i = (\frac{mn}{m+n})(\underset{\sim}{Y}_i - \underset{\sim}{Y}_0)'\underset{\sim}{R}_0^{-1}(\underset{\sim}{Y}_i - \underset{\sim}{Y}_0) , \tag{5.11}$$

has the multivariate chi-squared distribution of the theorem.

Suppose $\hat{\underset{\sim}{R}}_0$ is a consistent estimator of $\underset{\sim}{R}_0$. Then Theorems 1.9 and 1.10 assure us that the quadratic forms (5.9) and (5.11) have the same asymptotic distribution. The proof is complete.

The properties of the control chart of Theorem 5.5 are very similar to those of the generalized p-chart under Case II. Without repeating the details again, some of the highlights of these properties are given. The run-length distribution of a stationary process can be bounded using Theorem 1.6. If $\sqrt{n}(\underset{\sim}{\lambda}_i - \underset{\sim}{\lambda}_0) = \underset{\sim}{d}_i$, then the asymptotic distribution of (5.9) is a noncentral multivariate χ^2-distribution. For stationary processes, the more out of control the process, as measured by $\underset{\sim}{d}_i'\underset{\sim}{d}_i$, the faster the chart signals, asymptotically.

It is sometimes the case that defects are classified as to their seriousness. The discussion of Section 2.5 shows that the multivariate generalized Poisson distribution arises as a limiting form of the joint distribution of the classification of defects.

Dodge (1928) and Dodge and Torrey (1956) proposed a different scheme for monitoring defects classified as very serious, serious, moderately serious, and not serious. Let X_1, X_2, X_3 and X_4, respectively, denote the number observed in these classifications. Then they use the statistic

$$w_1X_1 + w_2X_2 + w_3X_3 + w_4X_4 \ , \tag{5.12}$$

where w_1, w_2, w_3 and w_4 are positive weights given to each classification, to establish a control chart. The values for the w_i's suggested by Dodge and Torrey (1956) are 100, 50, 10, and 1, respectively. They assume that the X_i are independent Poisson random variables, and establish 3σ limits for (5.12) based on this assumption.

Suppose the assumption that X_1, X_2, X_3, and X_4 in (5.12) are independent Poisson random variables is unwarranted. Then, Theorem 2.5 demonstrates that, in general, the distribution of (5.12) is in the family of generalized Poisson distributions. The 3σ limits proposed by Dodge and Torrey use $w_1^2 \mathrm{var} X_1 + w_2^2 \mathrm{var} X_2 + \ldots + w_4^2 \mathrm{var} X_4$ as the variance term of the control chart value. These limits are understated since they ignore the covariance terms - which are always positive. Correct $K\sigma$ control limits for (5.12) could be found from the variance of the generalized Poisson distribution for (5.12), that is, (2.31).

Appeal can be made to our earlier developments concerning generalized c-charts in order to justify the use of the generalized Poisson distribution for monitoring the statistic (5.12). First, regardless of the choice for the weights in (5.12), Theorem 5.4 is still valid when the underlying process is the multivariate Bernoulli process. The control chart for (5.12) will also signal faster if the intensity of defects of any type in the multivariate generalized Poisson process increases, and none of the intensities decrease. For Case II procedures and properties of (5.12), the reader is referred to the corresponding discussion in Section Three.

Another approach to monitoring a multivariate generalized Poisson process is to use the asymptotic normality property of Theorem 2.6. If the means, variances, and covariances of (2.33) can be specified, then an asymptotically χ^2 quadratic form, as in (5.7), can be proposed to monitor the process. If the knowledge of these parameters is too strong an assumption to make, a Case II monitoring procedure exactly parallel to (5.9) is valid. The run-length properties of such a procedure also follow, without modification, from Theorem 5.5 when the process is in control.

As an example of this last procedure we consider an exercise given by Burr (1953, p. 256). In this problem the numbers of several different types of defects on aircraft assemblies are being monitored. The defects are classified as missing rivets, foreign matter, or other defects. The first twenty-five serial numbers were used as a base period for the multivariate generalized Poisson process. The method-of-moments estimators were used to compute the variances and covariances, and, thus, $\hat{\underset{\sim}{R}}_0$. The base period estimates of the correlation matrix and marginal sample means are given below:

$\overline{X}_1 = 14.04$ (rivets) ; $\overline{X}_2 = 14.24$ (foreign matter) ;

$\overline{X}_3 = 818.92$ (other defects) ;

and

$$\hat{\underset{\sim}{R}}_0 = \begin{bmatrix} 1 & 0 & 0.27843 \\ 0 & 1 & 0.06155 \\ 0.27843 & 0.06155 & 1 \end{bmatrix}$$

In this example, m and n of (5.9) are 25 and 1, respectively. The asymptotic distribution of (5.9) is $\chi_1^2(3,1)$, when the process is in control. So, if an asymptotic level of 0.01 is desired, the control limit of the χ^2 chart is taken to be 11.35.

The quadratic forms used to establish the control chart based on the above base period are

$$c_i = \left(\frac{25}{26}\right) \begin{bmatrix} \dfrac{X_{i1}-14.04}{5.9124} \\[2ex] \dfrac{X_{i2}-14.24}{4.2356} \\[2ex] \dfrac{X_{i3}-818.92}{70.3917} \end{bmatrix}' \hat{\underset{\sim}{R}}_0^{-1} \begin{bmatrix} \dfrac{X_{i1}-14.04}{5.9124} \\[2ex] \dfrac{X_{i2}-14.24}{4.2356} \\[2ex] \dfrac{X_{i3}-818.92}{70.3917} \end{bmatrix}, \tag{5.13}$$

with $i=1,2,\ldots,26$, where $(X_{i1},X_{i2},X_{i3})'$ is the observed number of defects for rivets, foreign matter, and other defects on the ith monitoring occasion. The control chart for (5.13) is given in Figure 5.1.

If the process is in control, the values of the control chart have the distribution $\chi_{26}^2(3,\frac{25}{26}I_3+\frac{I}{26}J_3)$, asymptotically. From the control chart in Figure 5.1, the longest run of in-control values is ten. But, from (2.39), we know that $P(N>10) \geq (0.99)^{10} = 0.9044$, asymptotically,

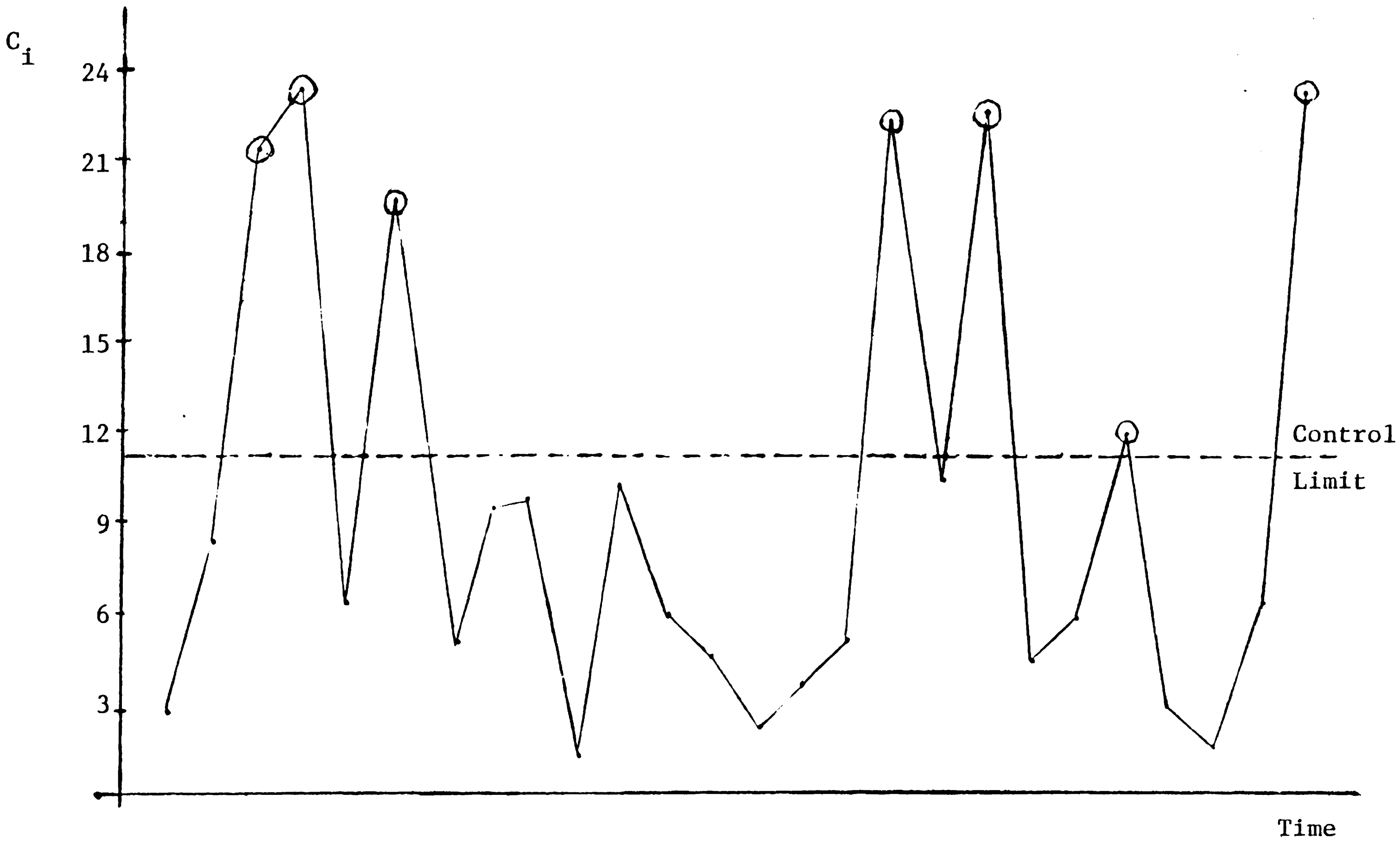

Figure 5.1. A control chart for a generalized Poisson process. The asymptotic level is 0.01. Example from Burr (1953).

where N is the run-length of the chart. Therefore, it is quite unlikely that the process is actually in control. Further consideration of this example using some diagnostic procedures is given in the next section.

5.5 Diagnostic Procedures for Monitoring More Than One Type of Defect

In this section we continue to consider the multivariate problem introduced in the preceding section. Now, however, diagnostic procedures under Cases I and II for monitoring the defect classifications of interest will be studied. The multivariate Poisson and generalized Poisson distributions play the central roles in the discussion.

A reasonable first step towards developing diagnostic procedures for defects is to modify the procedures in Section 4.4 for multivariate Bernoulli processes. Let us begin with the bivariate Poisson - either as an approximation to the bivariate binomial distribution or as a process itself. Figure 4.1 could be used to describe the algorithm for finding the two diagnostic control limits. The recursive formulas (2.10) provide an efficient means to compute bivariate Poisson probabilities. An illustration of this procedure is given in Table 5.4.

It seems quite natural to proceed in the fashion outlined in the preceding paragraph, using the optimal small-sample test for each marginal distribution. However, there are no analytical results that assure that the run-length distribution of the resulting diagnostic chart has suitable properties. At the very least, we would expect that if the in-control parameters are given by $(\lambda_{10},\lambda_{20},\rho_0)$, then the run-length distribution of the bivariate Poisson control chart should be

TABLE 5.4. ILLUSTRATION OF THE METHOD FOR OBTAINING CONTROL LIMITS FOR A BIVARIATE POISSON DIAGNOSTIC CHART FOR $\underset{\sim}{\lambda}_0$ = (0.25, 0.50) AND VARIOUS ρ . THE LEVEL IS 0.01.

		ρ		
STEPS	0.1126	0.2604	0.5066	0.6051
1 . $P(X_1 \leq 2, X_2 \leq 3)$	0.99611	0.99617	0.99638	0.99651
2 . $P(X_1 \leq 1, X_2 \leq 2)$	0.96019	0.96140	0.96407	0.96531
3A. $P(X_1 \leq 1, X_2 \leq 3)$	0.97192	0.97215	0.97261	0.97280
3B. $P(X_1 \leq 2, X_2 \leq 2)$	0.98359	0.98383	0.98456	0.98501
4 . $P(X_1 \leq 2, X_2 \leq 3)$	0.00079	0.00159	0.00328	0.00402
5 . CONTROL LIMITS*	$(X_1 \leq 2, X_2 \leq 3) - (X_1 = 2, X_2 = 3)$	$(X_1 \leq 2, X_2 \leq 3) - (X_1 = 2, X_2 = 3)$	$(X_1 \leq 2, X_2 \leq 3) - (X_1 = 2, X_2 = 3)$	$(X_1 \leq 2, X_2 \leq 3) - (X_1 = 2, X_2 = 3)$

*The minus sign signifies a set difference operation.

stochastically smaller for alternative process parameters in the set $\Lambda_1 = \{(\lambda_1,\lambda_2,\rho): \lambda_1 \geq \lambda_{10}, \lambda_2 \geq \lambda_{20}, \rho=\rho_0\}$. There is some empirical evidence which suggests that such an ordering is appropriate, thus lending credence to the use of this diagnostic procedure.

Figures 5.2-5.4 provide some examples of these empirical results. In all of these figures, it can be seen that increasing both λ_1 and λ_2 (L1 and LAMBDA 2 in the chart) decreases the average run length of the control chart. Other empirical investigations, not given here, suggest that the behavior exhibited in these figures holds more broadly.

A valid objection to the class of alternatives Λ_1 is the seemingly unreasonable requirement that the correlation coefficient be unaltered by changes in the other parameters. Theorem 2.2 can be used, however, to give bounds for the run-length distribution in terms of the maximum and minimum correlation coefficient when the marginals are symmetric. For nonsymmetric marginals, there is empirical evidence to suggest that the same approach may be taken.

Suppose, now, that a bivariate generalized Poisson process is monitored, and that the numbers of different types of defects are tabulated in two categories - major and minor defects, say. A diagnostic control chart could be established using the algorithm in Figure 4.1 with the recursive formula (2.25) used to compute the required probabilities. These developments exactly parallel previous efforts, so the details are omitted.

Next, we comment on diagnostic charts for monitoring general multivariate Poisson or generalized Poisson processes. While there exist

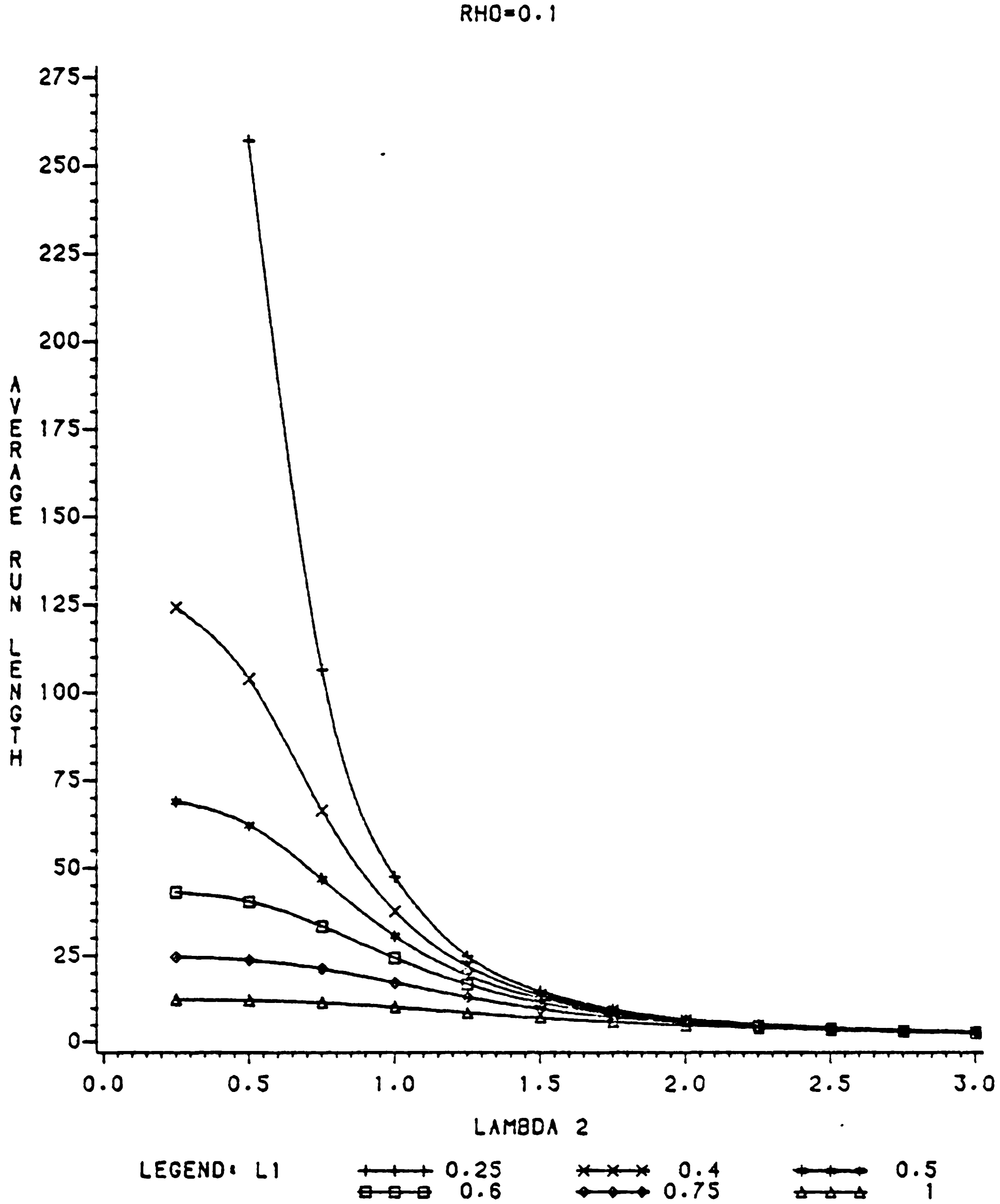

FIGURE 5.2. AVERAGE RUN LENGTH OF A BIVARIATE POISSON CHART. THE CONTROL LIMIT FOR VARIABLE ONE IS TWO AND FOR VARIABLE TWO IS THREE.

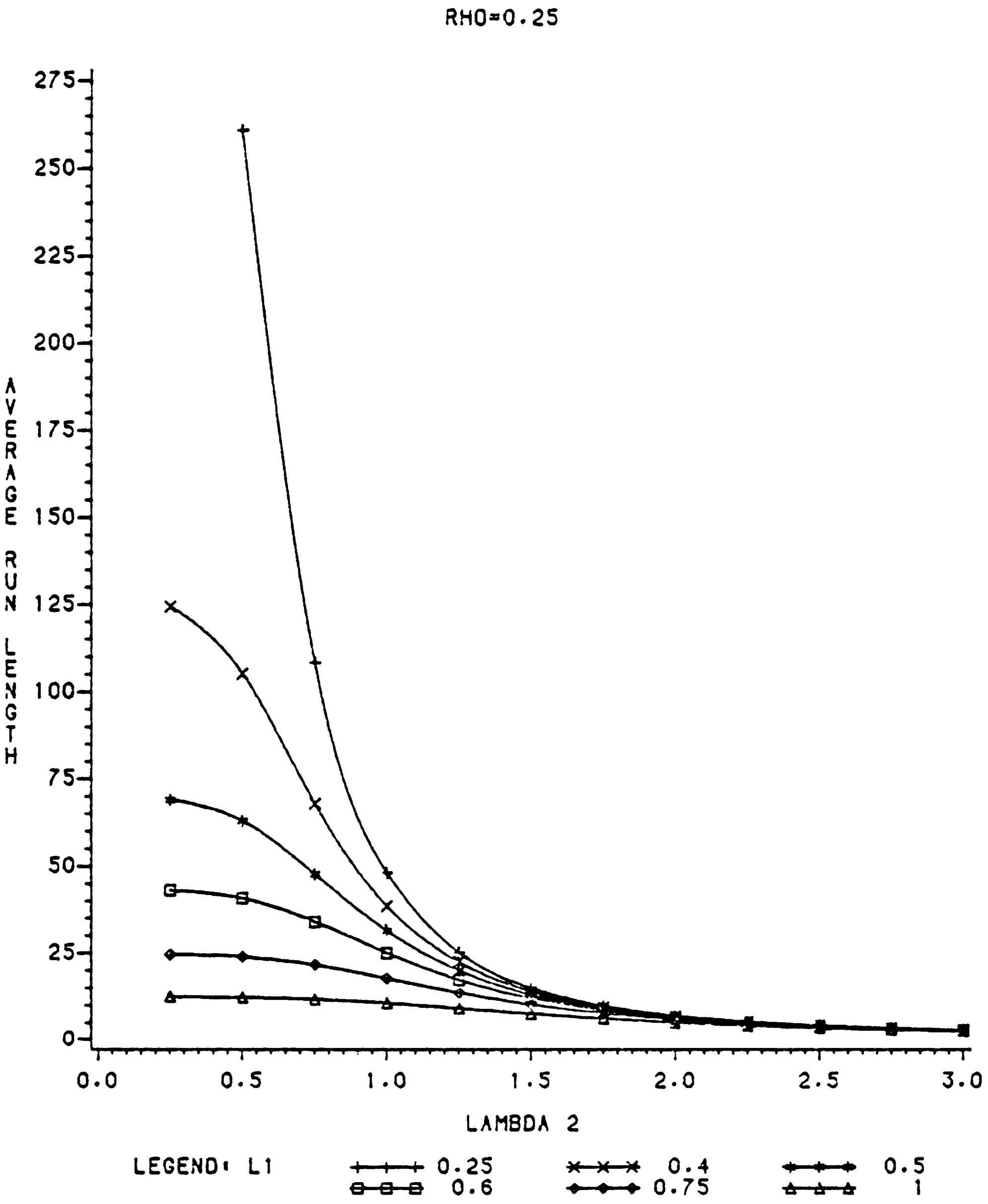

FIGURE 5.3. AVERAGE RUN LENGTH OF A BIVARIATE POISSON CHART. THE CONTROL LIMIT FOR VARIABLE ONE IS TWO AND FOR VARIABLE TWO IS THREE.

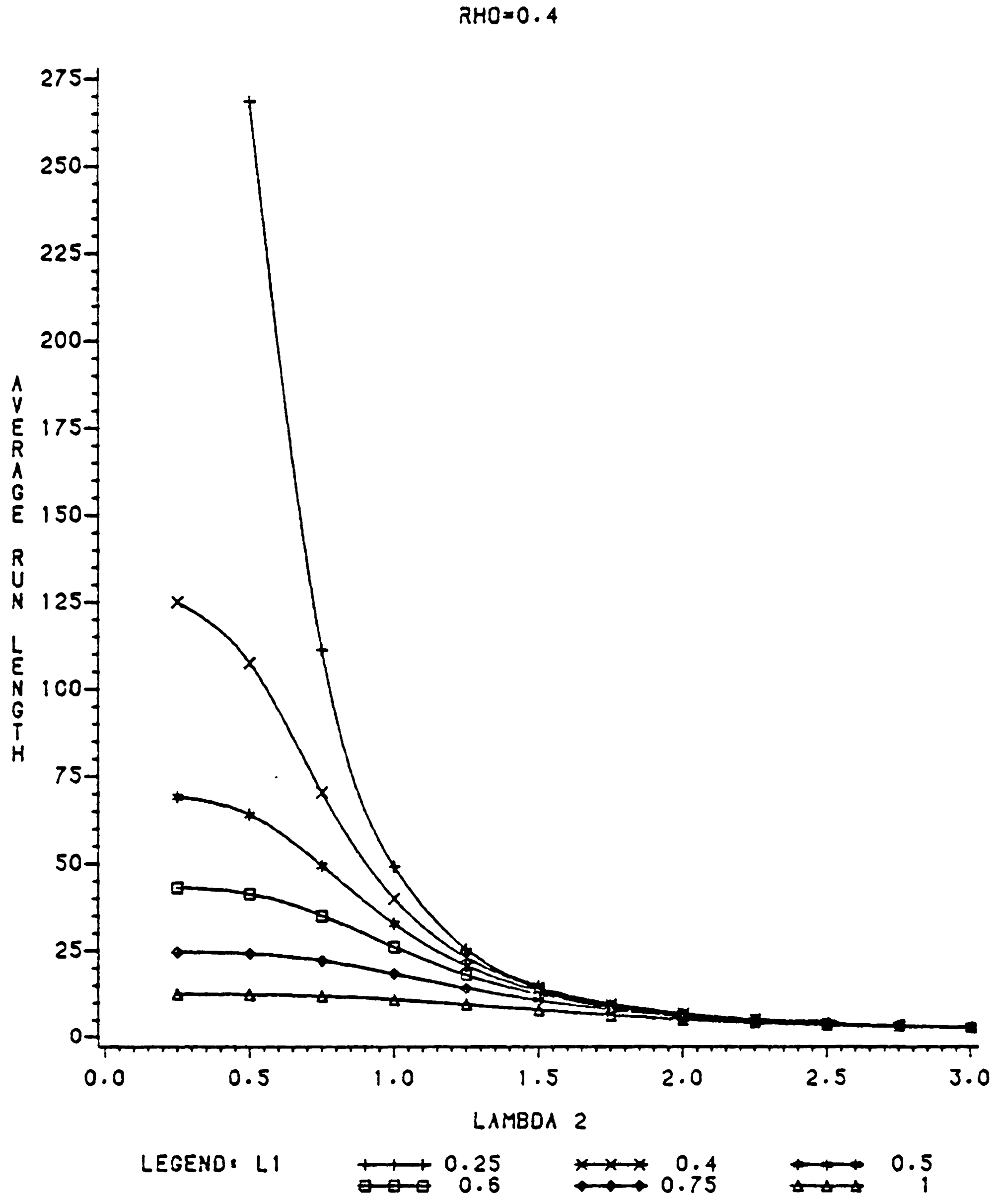

FIGURE 5.4. AVERAGE RUN LENGTH OF A BIVARIATE POISSON CHART. THE CONTROL LIMIT FOR VARIABLE ONE IS TWO AND FOR VARIABLE TWO IS THREE.

general recursive formulas for the probabilities of both of these multivariate families, we are still faced with the prospect of specifying an inordinate number of parameter values. So, instead of hoping for exact α-level procedures, we adopt the compromise settled upon in Section 4.4 for multivariate Bernoulli processes. The inequality (4.15) and the procedure following it can be used to obtain conservative control limits when $\underset{\sim}{X} = (X_1,\ldots,X_r)'$ has either a multivariate Poisson or multivariate generalized Poisson distribution.

An alternative approach for obtaining diagnostic procedures for monitoring r-variate Poisson or generalized Poisson processes relies on asymptotic normality under the limiting conditions of Theorems 2.3 and 2.6, respectively. Each marginal defect would be monitored using

$$y_i = \frac{X_i-\mu_i}{\sigma_i} \quad , \ i=1,\ldots,r \ , \tag{5.13}$$

where X_i is the observed number of defects of type i and μ_i and σ_i^2 are given by (2.33) for the generalized Poisson distribution, while $\mu_i=\lambda_i$ and $\sigma_i = \sqrt{\lambda_i}$ for the Poisson distribution. Then, an asymptotically α-level monitoring procedure uses limits $u_1,\ldots,u_r$ or $\ell_1,\ldots,\ell_r$ such that

$$P(y_1 \le u_1,\ldots,y_r \le u_r) = 1-\alpha \ , \tag{5.14}$$

or

$$P(|y_1| \le \ell_1,\ldots,|y_r| \le \ell_r) = 1-\alpha \quad , \tag{5.15}$$

where y_i is given by (5.13), and the joint asymptotic distribution of $(y_1,\ldots,y_r)'$ is as given in Theorems 2.3 or 2.6. Either expression (5.14) or (5.15) is used depending upon whether an upper or two-sided control limit is desired.

First, consider the monitoring situation represented by (5.14). A conservative solution along the lines of Theorem 4.4 can be found by applying Slepian's inequality. So, if a lower bound, ρ_0 say, can be placed on the correlations of $y_1,\ldots,y_r$, then the equicorrelated normal distribution having $\rho_{ij}=\rho_0$, for all (i,j), can be used in (5.14). Since $\rho_{ij}\geq 0$ for both distributions, we also have

$$P(y_1\leq u_1,\ldots,y_r\leq u_r) \geq \prod_{i=1}^{r} P(y_i\leq u_i) \ .$$

If it is reasonable to take $u_1=\ldots=u_r=u$, then the tables given by Gupta et al. (1973) can be combined with the results in the preceding paragraph to yield conservative control limits for (5.14). As in Chapter Four, we note that an added benefit of this approach is the administrative simplification of having to maintain a control chart only for $\max_i y_i$.

Now consider the use of a two-sided control limit for each defect in the diagnostic monitoring problem. If $\underset{\sim}{X}$ is a realization from a multivariate Poisson or generalized Poisson process, the following theorem may be employed to construct conservative control limits based on asymptotics.

Theorem 5.6. For monitoring the number of defects of a multivariate Poisson or generalized Poisson process, consider the diagnostic procedure using $\{y_i;\ i=1,\ldots,r\}$ given at (5.13). Suppose the in-control regions of the r separate charts are taken as $|y_i| \leq \ell_i$, $i=1,\ldots,r$. Then, when the process is in control, the asymptotic probability of signalling at each monitoring occasion, given by (5.15), is nondecreasing in ξ/λ_i. Here ξ is the expected number of defects of the process common to each of the r defects, as in Section 2.4 or 2.5.

Proof. The correlation between y_i and y_j is $\rho_{ij} = \xi/\sqrt{\lambda_i\lambda_j} = (\xi/\lambda_i)^{\frac{1}{2}}(\xi/\lambda_j)^{\frac{1}{2}}$. Therefore, by Theorem 1.2, the probability statement (5.15) is asymptotically nondecreasing in each ξ/λ_i, $i=1,\ldots,r$.

As one consequence of Theorem 5.6 take $\rho = \xi/\max_i \lambda_i$ and let $\underset{\sim}{Z} \sim N_r(\underset{\sim}{0},\underset{\sim}{S})$, where $\underset{\sim}{S}=(s_{ij})$, with

$$s_{ij} = \begin{cases} 1 & \text{if } i=j \\ \rho & \text{if } i\neq j \end{cases},\quad i=1,\ldots,r\ .$$

Theorem 5.6 then gives the following inequality:

$$P\left(\bigcap_{i=1}^{r}\{|y_i|\leq\ell_i\}\right) \geq P\left(\bigcap_{i=1}^{r}\{|Z_i|\leq\ell_i\}\right)\ . \tag{5.16}$$

If, in addition, we have $\ell_1=\ell_2=\ldots=\ell_r=\ell$ in (5.16), then tables given by Odeh (1982) can be used to determine ℓ. This restriction on ℓ also allows us to maintain only one control chart for the maximum of the $|y_i|$.

To this point of this section only diagnostic procedures for Case I monitoring have been considered. Under Case II for Poisson process monitoring, a first inclination might be to establish diagnostic charts using Theorem 5.3. The same problems that arose with this approach on bivariate Bernoulli processes also occur here. For a bivariate diagnostic chart with marginals as in Theorem 5.3, the joint distribution can be shown to be

$$\binom{t_1}{y_1}\left(\frac{\lambda_{10}}{\lambda_{10}+\lambda_1}\right)^{y_1}\left(1-\frac{\lambda_{10}}{\lambda_{10}+\lambda_1}\right)^{t_1-y_1}\binom{t_2}{y_2}\left(\frac{\lambda_{20}}{\lambda_{20}+\lambda_2}\right)^{y_2}\left(1-\frac{\lambda_{20}}{\lambda_{20}+\lambda_2}\right)^{t_2-y_2}$$
$$\left[\frac{\{\sum_{j=0}^{\infty}\rho_0^j K_j(s_1-y_1;\lambda_{10})K_j(s_2-y_2;\lambda_{20})\}\{\sum_{j=0}^{\infty}\rho_1^j K_j(y_1;\lambda_1)K_j(y_2;\lambda_2)\}}{\{\sum_{j=0}^{\infty}\rho_2^j K_j(t_1;\lambda_{10}+\lambda_1)K_j(t_2;\lambda_{20}+\lambda_2)\}}\right], \quad (5.17)$$

using the notation of (2.9). When the process is in control $\lambda_{10}=\lambda_1$, $\lambda_{20}=\lambda_2$, and $\rho_0=\rho_1$ in (5.17), but this simplification does not eliminate the nuisance parameters in the series expansions of (5.17).

A second approach, which has been relied upon extensively throughout this study, is to estimate the parameters consistently in a base period and to use these estimates in the Case I procedures. This fact again produces valid asymptotic procedures when applied to the small-sample Case I procedures of this section. The details are similar to those given for the multivariate binomial procedures in Section 4.5 and the generalized Poisson procedures in Section Three.

The work that has been done on estimation for multivariate Poisson and generalized Poisson distributions appears to be mainly restricted to particular bivariate distributions. For instance, Holgate (1964) considers estimation of bivariate Poisson parameters by maximum likelihood, method-of-moments, and moment estimators together with sample frequencies. He finds that on the basis of asymptotic efficiency, the maximum likelihood estimator is strongly preferred.

Following this suggestion, a Case II procedure for monitoring a bivariate Poisson process would begin thus. First, compute $\bar{X}_1$, $\bar{X}_2$, and $\hat{\lambda}_{11} = \frac{1}{n}\sum_{i=1}^{n}(X_{1i}-\bar{X}_1)(X_{2i}-\bar{X}_2)$. Form the estimates $\hat{\lambda}_1 = \bar{X}_1+\hat{\lambda}_{11}$, $\hat{\lambda}_2 = \bar{X}_2+\hat{\lambda}_{11}$, and $\hat{\rho} = \hat{\lambda}_{11}/\sqrt{\bar{\hat{\lambda}}_1\bar{\hat{\lambda}}_2}$. Compute

$$\sum_{X_1=1}^{\infty}\sum_{X_2=1}^{\infty}\frac{\hat{P}(X_1-1,X_2-1)}{\hat{P}(X_1,X_2)}\ , \tag{5.18}$$

where $\hat{P}(X_1,X_2)$ is the bivariate Poisson pmf based on the estimates $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\rho}$. Iterate on $\hat{\lambda}_{11}$ until (5.18) equals n.

Estimation for some particular members of the class of bivariate generalized Poisson distributions also has been discussed. See Holgate (1966) and Gillings (1974) for bivariate Neyman Type A distributions; Charlambides and Papageorgiou (1981) for bivariate Poisson-binomial distributions; and Cacoullos and Papageorgiou (1981) for some more general bivariate distributions. Some of the techniques used were method-of-moments, maximum likelihood, and marginal means in conjunction with zero sample frequencies. The only general conclusion from these efforts is the inferiority of the moment estimators.

For general multivariate distributions of the type (2.22), Khatri (1971) discusses maximum likelihood estimation for a class of multivariate contagious distributions which contains some of the class (2.22). For a bivariate contagious distribution not in our class of generalized

Poisson distributions, he concludes that maximum likelihood estimation is more satisfactory than the method of moments.

The use of maximum likelihood or moment estimators entails some computational difficulties if no further structure is placed on the multivariate generalized Poisson distributions. The factorial cumulants, however, are linear functions of the parameters of this distribution. If there are only a finite number of parameters an estimation procedure similar to (5.7) might be useful.

Specifically, we outline such a procedure for the bivariate generalized Poisson distribution with pgf given by (2.23). The factorial cumulants for this distribution, when there are s_1 parameters for the first marginal and s_2 parameters for the second marginal, are

$$\kappa_{(k_1 0)} = \sum_{i=k_1}^{s_1} \frac{i!\lambda_{1i}}{(i-k_1)!} + \sum_{i=k_1}^{s_1} \sum_{j=1}^{s_2} \frac{i!}{(i-k_1)!} \xi_{ij} \quad k_1=1,\dots,s_1$$

$$\kappa_{(0\, k_2)} = \sum_{j=k_2}^{s_2} \frac{j!\lambda_{2j}}{(j-k_2)!} + \sum_{i=1}^{s_1} \sum_{j=k_2}^{s_2} \frac{j!}{(j-k_2)!} \xi_{ij} \quad k_2=1,\dots,s_2$$

$$\kappa_{(k_1\, k_2)} = \sum_{i=k_1}^{s_1} \sum_{j=k_2}^{s_2} \frac{i!j!}{(i-k_1)!(j-k_2)!} \xi_{ij} \quad . \tag{5.19}$$

Now, define the parameter vector by $\underset{\sim}{\lambda} = (\lambda_{11},\dots,\lambda_{1s_1},\lambda_{21},\dots,\ _{2s_2},$ $\xi_{11},\dots,\xi_{s_1 s_2})'$ and take $\underset{\sim}{W}$ to be the matrix whose rows correspond to the coefficients in (5.19). Then, an estimator of $\underset{\sim}{\lambda}$ is the expression

$$\hat{\underset{\sim}{\lambda}} = \underset{\sim}{W}^{-1}\underset{\sim}{K} , \tag{5.20}$$

where $\underset{\sim}{K}$ is a vector of sample factorial cumulants, which themselves are linear functions of the sample moments.

The efficiencies of the estimators (5.20) might be expected to be quite poor for even moderate values of s_1 and s_2. Recall that in the univariate case Hinz and Gurland (1967) were able to improve the efficiencies of these estimators by combining sample frequencies with the sample cumulants in minimum chi-squared estimators. For this same method to be successful in the multivariate case requires observations of zero frequencies for all marginals - a situation which may not exist in quality control applications.

Another possible methodology for the monitoring schemes is to base the Case II diagnostic procedures on asymptotic normal theory. From the proof of Theorem 5.5, under the in-control hypothesis, we have for k monitoring periods that,

$$\begin{bmatrix} Y_{11}-Y_{01} \\ \vdots \\ Y_{1\ell}-Y_{0\ell} \\ \cdots\cdots \\ \vdots \\ \cdots\cdots \\ Y_{k1}-Y_{01} \\ \vdots \\ Y_{k\ell}-Y_{0\ell} \end{bmatrix} \xrightarrow{d} N_{k\ell}(\underset{\sim}{0}, (\frac{1}{n}\underset{\sim}{I}_k + \frac{1}{m}\underset{\sim}{J}_k)\times \underset{\sim}{R}_0) \quad , \tag{5.21}$$

where Y_{ij} is given at (5.8). In practice $\hat{\underset{\sim}{R}}_0$, a consistent estimator of the correlation matrix $\underset{\sim}{R}$ given in Theorem 2.6, is substituted in (5.21). For the ith monitoring period (i=1,2,...,k) the statistics

$$\frac{\frac{1}{n}\sum_{p=1}^{n} X_{ijp} - \frac{1}{m}\sum_{p=1}^{m} X_{0jp}}{[(\frac{m+n}{mn})\,\hat{\sigma}_{ij}]^{1/2}} \qquad j=1,\ldots,r \ , \tag{5.22}$$

could be used to establish an asymptotically normal diagnostic chart. Theorems 1.1 or 1.3 can be used to establish a conservative asymptotic level of the diagnostic chart based on the statistics (5.22). In turn, the same theorems can be applied to (5.21) to yield geometric lower stochastic bounds on the run-length distribution when the process is in control.

As an example of this multivariate normal diagnostic procedure we again consider the exercise given by Burr (1953) used in the preceding section. For this example, r=3 and the three control chart statistics corresponding to (5.22) are:

$$\sqrt{\frac{25}{26}}\left(\frac{X_{i1}-14.04}{5.9124}\right) \quad , \quad \text{for rivets ;}$$

$$\sqrt{\frac{25}{26}}\left(\frac{X_{i2}-14.24}{4.2356}\right) \quad , \quad \text{for foreign matter;}$$

and

$$\sqrt{\frac{25}{26}}\left(\frac{X_{i3}-818.92}{70.3917}\right) \quad , \quad \text{for other defects.}$$

Suppose the diagnostic charts are established to detect increases in the numbers of defects for the three attributes. Further, equal weight is given to the detection of increases in each type of defect. If an asymptotic level of 0.01 is used, Theorem 1.1 gives a control limit of 2.715 for each chart. This conservative control limit is based on the product of the three marginal $N_1(0,1)$ random variables. The control charts are given in Figure 5.5.

It can be seen from Figure 5.5 that only the defect "foreign matter" causes the chart to signal that the process is out of control. It does so at five points. Four of these five points match up with times when the process was said to be out of control using the omnibus procedure of the preceding section. Recall that the omnibus procedure signalled on seven occasions, but it protects against both increases and decreases in the expected numbers of defects. A possible advantage of the diagnostic chart is that it can be devised to ignore, as in this example, decreases in the numbers of defects.

5.6 Summary

In this chapter the univariate c-chart has been reexamined, and some new methods have been proposed for the monitoring of different numbers of defects.

A Poisson process may be an adequate model if only one type of defect may occur per unit of production. The Poisson distribution also may arise as a limiting form of a Bernoulli process. For these situations some optimal control charts were offered under Case I monitoring. Under Case II monitoring, the UMPU hypothesis test for comparing two Poisson populations leads to a control chart with some weak optimality properties.

The next control charts for defects we considered were those based on the generalized Poisson distribution. This generalized c-chart may arise either as a limiting distribution for the total number of defects of a multivariate Bernoulli or Poisson process, or as a result of

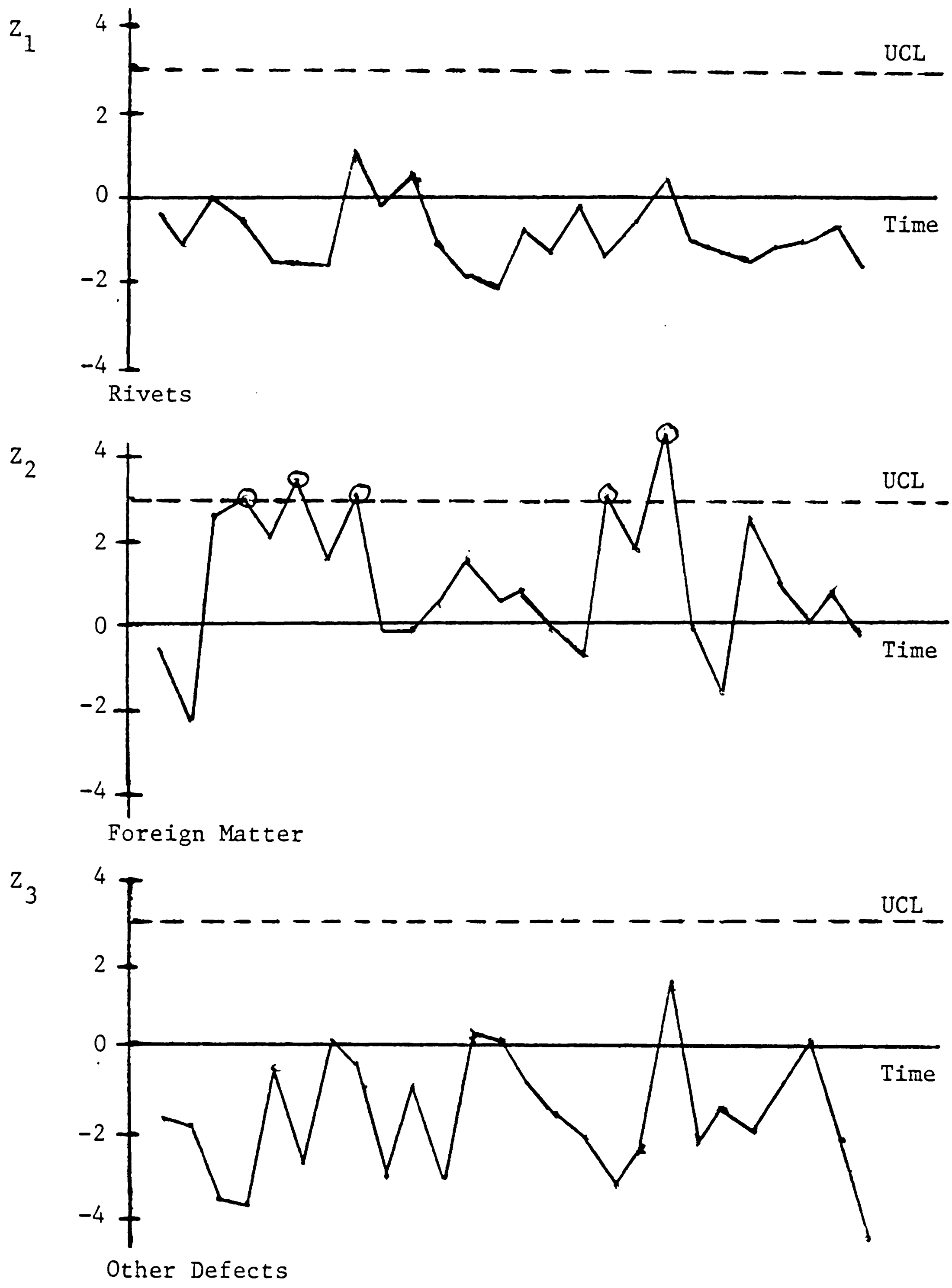

Figure 5.5. An asymptotic diagnostic control chart for a multivariate generalized Poison distribution.

monitoring the number of defects of a generalized Poisson process. For certain classes of generalized Poisson distributions the generalized c-chart was shown to signal faster the more the process was out of control. Some Case II procedures were also discussed. However, the efficient estimation techniques used for these Case II procedures require observations of units with zero defects in the base period.

As was the case with multivariate Bernoulli processes, all the omnibus procedures for multivariate defects are based on the asymptotic normality of the multivariate Poisson or generalized Poisson distributions. For both Case I and Case II monitoring, asymptotic χ^2 charts were proposed for monitoring the numbers of defects. For the Case II procedures presented, the run lengths of these charts were shown to be characterized asymptotically in terms of the multivariate χ^2 distribution.

The chapter ended with an examination of some diagnostic monitoring procedures for the numbers of defects. Using the methods of Chapter Four, some small-sample procedures were proposed for Case I problems. Very few analytical results are available for judging the appropriateness of these approaches. No effective small-sample procedures could be found for Case II problems. Some asymptotic procedures based on the Case I diagnostic charts for defects were given, but these control charts suffer from a lack of availability of good estimation techniques for multivariate Poisson and generalized Poisson parameters. Finally, some asymptotically normal diagnostic procedures were given for Case II monitoring.

LIST OF REFERENCES

Aitken, A. C. and Gonin, H. T. (1935). "On fourfold sampling with and without replacement." *Proc. Royal. Soc. Edin.* 55:114-125.

Alt, Frank B. (1982). "Multivariate quality control: State of the art." Working Paper 82-001. Department of Management Science/ Statistics, University of Maryland.

Anderson, T. W. (1955). "The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities." *Proc. Amer. Math. Soc.* 6:170-176.

Barnard, G. A. (1959). "Control charts and stochastic processes." *J. Roy. Statist. Soc. B.* 21:239-271.

Bennett, B. M. and Hsu, P. (1960). "On the power function of the exact test for the 2 x 2 contingency table." *Biometrika* 47:393-398.

Bergmann, Bo (1981). "Tests against certain ordered multinomial alternatives." *Scan. J. Statist.* 8:218-226.

Beus, Gary B. (1968). *Testing Hypotheses Involving Categorical Data.* Ph.D. Dissertation, Virginia Polytechnic Institute.

Billingsley, Patrick. (1968). *Convergence of Probability Measures.* John Wiley & Sons , New York.

Bray, D. F., Lyon, D. A., and Burr, I. W. (1973). "Three class attributes plans in acceptance sampling." *Technometrics* 15:575-585.

Burr, Irving W. (1953). *Engineering Statistics and Quality Control.* McGraw-Hill, New York.

Cacoullos, Theophilus and Papageorgiou, H. (1981). "On bivariate discrete distributions generated by compounding" in *Statistical Distributions in Scientific Work* - Volume 4 (C. Tallie, G. P. Patil, and B. A. Baldessari, editors), 197-212. D. Reidel, Boston.

Case, K. E., Schmidt, J. W., and Bennett, G. K. (1975). "A discrete economic multiattribute acceptance sampling." *AIIE Trans.* 7:363-369.

Campbell, J. T. (1934). "The Poisson correlation function." *Proc. Edin. Math. Soc. Ser. 2* 4:18-26.

Charalambides, Ch. A. and Papageorgiou, H. (1981). "Bivariate Poisson-binomial distributions." *Biom. J.* 23:437-450.

Clements, John A. (1980). "Three-class attribute sampling plans-continued." *1980 ASQC Tech. Conf. Trans.* 475-481.

Clements, John A. (1979). "Three-class attribute sampling plans." *1979 ASQC Tech. Conf. Trans.* 264-269.

Cochran, W. G. (1952). "The χ^2 goodness of fit test." *Ann. Math. Statist.* 23:315-345.

Cohen, Arthur and Sackrowitz, H. B. (1975). "Unbiasedness of the chi-square, LR and other gof tests." *Ann. Statist.* 3:959-964.

Das Gupta, S., Eaton, M. L., Olkin, I., Perlman, M., Savage, L. J., and Subel, M. (1972). "Inequalities on the probability content of convex regions for elliptically contoured distributions." *Sixth Berkely Symposium on Mathematical Statistics and Probability*, Vol. II, 241-265.

Dodge, H. F. (1928). "A method of rating manufactured product." *Bell System Tech. J.* 8:350-358.

Dodge, H. F. and Torrey, H. N. (1956). "A check inspection and demerit rating plan." *Ind. Qual. Cont.* 13:5-12.

Duncan, A. J. (1974). *Quality Control and Industrial Statistics*, 4th edition, Richard D. Irwin, Homewood, Illinois.

Dutt, J. E. and Soms, A. P. (1976). "An integral representation technique for calculating general multivariate probabilities with an application to multivariate χ^2." *Comm. Statist.- Theor. Meth.* A5:377-388.

Dykstra, R. L. (1980). "Product inequalities involving the multivariate normal distribution." *J. Amer. Statist. Assoc.* 75:646-650.

Feller, William. (1968). *An Introduction to Probability Theory and Its Application*, 3rd edition. John Wiley and Sons, New York.

Gillings, Dennis B. (1974). "Some further results for bivariate generalizations of the Neyman Type A distribution." *Biometrics* 30:619-628.

Griffiths, R. C. (1971). "Orthogonal polynomials on the multinomial distribution." *Austral. J. Statist.* 13:27-35.

Griffiths, R. C., Milne, R. K., and Wood, R. (1979). "Aspects of correlation in bivariate Poisson distributions and processes." *Austral. J. Statist.* 21:238-255.

Gruska, G. F., McClasin, J. A., and Lazur, J. G. (1976). "Multiple characteristic sentencing." *1976 ASQC Tech. Conf. Trans.* 276-284.

Gupta, S. S., Nagel, Klaus, and Panchapakesan, S. (1973). "On the order statistics from equally correlated normal random variables." *Biometrika* 60:403-413.

Gurland, John. (1965). "A method for estimation for some generalized Poisson distributions." in *Classical and Contagious Discrete Distributions*, (Ed., G. P. Patil). Statistical Publishing Society, Calcutta.

Hamdan, M. A. and Jensen, D. R. (1976). "A bivariate binomial distribution and some applications." *Austral. J. of Statist.* 18:163-169.

Hamdan, M. A. and Martinson, E. O. (1971). "Maximum likelihood estimation in the bivariate binomial (0,1) distribution: application to 2 x 2 tables." *Austral. J. Statist.* 13:154-158.

Hamdan, M. A. and Tsokos, C. P. (1971). "A model for physical and biological problems: the bivariate-compound Poisson distribution." *Rev. Int. Statist. Insti.* 39:60-63.

Hannan, J. and Harkness, W. (1963). "Normal approximation to the distribution of two independent binomials, conditional on a fixed sum." *Ann. Math. Statist.* 34:1593-1595.

Harkness, W. L. (1965). "Properties of the extended hypergeometric distribution." *Ann. Math. Statist.* 36:938-945.

Harkness, W. L. and Katz, L. (1964). "Comparison of the power functions for the test of independence in 2 x 2 contingency tables." *Ann. Math. Statist.* 35:1115-1127.

Hinz, Paul and Gurland, John. (1967). "Simplified techniques for estimating parameters of some generalized Poisson distributions." *Biometrika* 54:555-566.

Holgate, P. (1966). "Bivariate generalizations of Neyman's Type A distribution." *Biometrika* 53:241-245.

Holgate, P. (1964). "Estimation for the bivariate Poisson distribution." *Biometrika* 51:241-245.

Hui, Yer Van. (1980). *Topics in Statistical Process Control.* Ph.D. Dissertation. Virginia Polytechnic Institute and State University.

Janossy, L., Renyi, A., and Aczel, J. (1950). "On composite Poisson distributions, I." *Acta Math.* 1:209-224.

Jensen, D. R. (1982). "An optimal property of S^2 charts." *J. Statist. Comp. Sim.* To appear.

Jensen, D. R. (1970a). "The joint distribution of quadratic forms and related distributions." *Austral. J. Statist.* 12:13-22.

Jensen, D. R. (1970b). "The joint distribution of traces of Wishart matrices and some applications." *Ann. Math. Statist.* 41:133-145.

Jensen, D. R. and Beus, G. B. (1968). "Upper percentage points of the maximum of a number of independent chi-square variates." Technical Report No. 7, Department of Statistics, Virginia Polytechnic Institute.

Jensen, D. R. and Howe, R. B. (1968). "Probability content of the bivariate chi-square distribution over rectangular regions." *Virginia J. Science* 19:233-239.

Jensen, D. R. and Hui, Y. U. (1980). "The use of R and S^2 charts under nonstandard conditions." Research Report Number Q-3, Department of Statistics, Virginia Polytechnic Institute and State University.

Johnson, N. L. (1962). "Some notes on the investigation of heterogeneity in interactions." *Trab. de Estudistica* 13:183-199.

Johnson, N. L. and Kotz, S. (1968). *Discrete Distributions.* John Wiley and Sons, New York.

Kemp, C. D. and Kemp, A. W. (1965). "Some properties of the Hermite distribution." *Biometrika* 52:381-394.

Kibble, W. F. (1941). "A two-variate gamma type distribution." *Sankhyā* 5:137-150.

Khatri, C. G. (1971). "On multivariate contagious distributions." *Sankhyā B.* 33:197-216.

Kounias, E. G. (1968). "Bounds for the probability of a union, with applications." *Ann. Math. Statist.* 39:2154-2158.

Krishnaiah, P. R. (1969). "Simultaneous test procedures under general MANOVA models, " in *Multivariate Analysis II* (P.R. Krishnaiah, editor). Academic Press, New York.

Krishnaiah, P. R. and Armitage, J. V. (1965). "Tables for the distribution of the maximum of correlated chi-square variates with one degree of freedom." ARL 65-136, Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio.

Krishnamoorthy, A. S. (1951). "Multivariate binomial and Poisson distributions." *Sankhyā* 11:117-124.

Krishnamoorthy, A. S. and Parthasarathy, M. (1951). "A multivariate gamma-type distribution." *Ann. Math. Statist.* 22:549-557.

Lancaster, H. O. (1969). *The Chi-Squared Distribution.* John Wiley and Sons, New York.

Lancaster, H. O. (1958). "The structure of bivariate distributions." *Ann. Math. Statist.* 29:719-736.

Layman, Bobby C. (1967). *Application of Control Charts to Small Lot Acceptance.* M.S. Thesis, Virginia Polytechnic Institute.

Lehmann, E. L. (1959). *Testing Statistical Hypotheses.* John Wiley and Sons, New York.

Lorenzen, Thomas J. (1979). "The computational aspects of multiple attribute sampling procedures." *Comm. Statist. - Simul. Comput.* B8:291-309.

Maritz, J. S. (1952). "Note on a certain family of discrete distributions." *Biometrika* 39:196-198.

McCaslin, James A., Gruska, G. F., Lazur, J. G. (1975). "OC curves for multiple characteristics." *1975 ASQC Tech. Conf. Trans.* 498-507.

Mudholkar, Govind S. (1966). "The integral of an invariant unimodal function over an invariant convex set - an inequality and applications." *Proc. Amer. Math. Soc.* 17:1327-1333.

Odeh, Robert E. (1982) "Tables of percentage points of the distribution of the maximum absolute value of equally correlated normal random variables." *Comm. Statist. - Simul. Comput.* B11:65-87.

Page, E. S. (1954). "Continuous inspection schemes." *Biometrika* 41: 100-115.

Patel, H. I. (1973). "Quality control methods for multivariate binomial and Poisson distributions." *Technometrics* 15:103-112.

Patel, Y. C. (1976). "Estimation of the parameters of the triple and quadruple stuttering Poisson." *Technometrics* 18:67-73.

Quality. (1980). "Dr. W. Edwards Deming - the statistical control of quality." Feb. 38-41 and March 34-36.

Rao, C. R. (1973). *Linear Statistical Inference and Its Applications*, 2nd ed. John Wiley and Sons, New York.

Šidák, Z. (1973). "A chain of inequalities for some types of multivariate distributions, with nine special cases." *Apl. Mat.* 18:110-118.

Šidák, Z. (1968). "On multivariate normal probabilities of rectangles: Their dependence on correlations." *Ann. Math. Statist.* 39:1425-1434.

Siotani, M. (1959). "The extreme value of the generalized distances of the individual points in the multivariate normal sample." *Ann. Instit. Statist. Math.* 11:183-208.

Slepian, D. (1962). "The one-sided barrier problem for Gaussian noise." *Bell System Tech. J.* 41:463-501.

Tan, W. Y. and Wong, S. P. (1978). "On approximating the central and noncentral multivariate gamma distribution." *Comm. Statist. - Simul. Comput.* B7:227-242.

Teicher, H. (1954). "On the multivariate Poisson distribution." *Skand. Aktuar.* 37:1-9.

Tong, Y. L. (1980). *Probability Inequalities in Multivariate Distributions*. Academic Press, New York.

Tracey, Derrick S. and Doss, Deva C. (1980). "A characterization of multivector multinomial and negative multinomial distributions." in *Multivariate Statistical Analysis* (R.P. Gupta, ed.). North-Holland, Amsterdam.

Wishart, John. (1949). "Cumulants of multivariate multinomial distributions." *Biometrika* 36:47-58.

APPENDIX I

The Computation of Bivariate Binomial Probabilities

In principle there is no problem in finding bivariate binomial probabilities directly from expression (2.3). Such an approach, however, requires tedious and time-consuming computations of the Krawtchouck polynomials. In this appendix a recursive formula for the Krawtchouck polynomials is provided which greatly expedites the computation of bivariate binomial probabilities. A sample FORTRAN program and a few representative values of bivariate binomial cdf's obtained using the program are also given.

Aitken and Gonin (1935) give the following recursive formula for the nonnormalized Krawtchouck polynomials

$$G_r(z+1;\pi,n) = G_r(z;\pi,n) + rG_{r-1}(z;\pi;n-1) \ . \qquad \text{(A.1)}$$

This relationship, by itself, is not useful in generating the polynomials, since it would be necessary to generate all the polynomials with parameters less than n. A recursive formula expressed just in terms of one value of n is needed to simplify the computations.

To achieve that goal, the relationship in the following lemma is helpful.

Lemma A.1. The following recursive relationship holds for the nonnormalized Krawtchouck polynomials.

$$G_r(z;\pi,n) = zG_{r-1}(z-1;\pi,n-1) - \pi(n-r+1)G_{r-1}(z;\pi,n) \ . \qquad \text{(A.2)}$$

Proof. Recall that the standard form of the Krawtchouck polynomials is

$$G_r(z;\pi,n) \equiv G_r(z;n) = \sum_{j=0}^{r} (-1)^j \binom{r}{j} \pi^j (n-r+j)^{(j)} z^{(r-j)} \quad . \qquad \text{(A.3)}$$

We also have

$$zG_{r-1}(z-1;n-1) = \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \pi^j (n-r+j)^{(j)} z^{(r-j)} \ ,$$

and

$$\pi(n-r+1)G_{r-1}(z,n) = \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} \pi^{j+1} (n-r+j+1)^{(j+1)} z^{(r-1-j)} \ .$$

Combining these last two results yields

$$\begin{aligned}
&zG_{r-1}(z-1;n-1) - \pi(n-r+1)G_{r-1}(z,n) \\
&= z^{(r)} + \sum_{j=1}^{r-1} (-1)^j \binom{r-1}{j} \pi^j (n-r+j)^{(j)} z^{(r-j)} \\
&+ \sum_{j=0}^{r-2} (-1)^{j+1} \binom{r-1}{j} \pi^{j+1} (n-r+j+1)^{(j+1)} z^{(r-(j+1))} \\
&\quad + (-1)^r \binom{r-1}{r-1} \pi^r n^{(r)} \\
&= z^{(r)} + \sum_{k=1}^{r-1} (-1)^k \pi^k z^{(r-k)} (n-r+k)^{(k)} \left[\binom{r-1}{k} + \binom{r-1}{k-1}\right] \\
&\quad + (-1)^r \pi^r n^{(r)} \\
&= z^{(r)} + \sum_{k=1}^{r-1} (-1)^k \binom{r}{k} \pi^k z^{(r-k)} (n-r+k)^{(k)} + (-1)^r \pi^r n^{(r)} \\
&= G_r(z;n) \ .
\end{aligned}$$

Now combine (A.1) and (A.2) to arrive at

$$G_r(z-1,n) + r\, G_{r-1}(z-1,n-1)$$

$$= z\, G_{r-1}(z-1,n-1) - \pi(n-r+1)G_{r-1}(z,n).$$

Or, for $z \neq r$ and $r \geq 1$, we have

$$G_{r-1}(z-1,n-1) = (\frac{1}{z-r})G_r(z-1,n) + \pi(n-r+1)G_{r-1}(z,n) \ .$$

Therefore, for $z \neq r$, $r=1,\ldots,n$, $z=0,\ldots,n$

$$G_r(z,n) = z[(\frac{1}{z-r})G_r(z-1,n) + \pi(n-r+1)G_{r-1}(z,n)]$$

$$- \pi(n-r+1)G_{r-1}(z,n)$$

$$= (\frac{z}{z-r})G_r(z-1,n) + (\frac{r}{z-r})\pi(n-r+1)G_{r-1}(z,n) \ . \tag{A.4}$$

An efficient algorithm for computing the Krawtchouck polynomials for given values of n and π can now be outlined.

1. First, compute $G_r(0,n)$ for $r=1,\ldots,n$ directly from the expression:

   $$G_r(0,n) = (-\pi)^r n^{(r)} \quad .$$

2. Compute $G_1(z,n) = z-\pi n$, $z=1,\ldots,n$.
3. For $z \neq r$, use (A.4) to compute $G_r(z,n)$ starting at $z=1$, $r=2$, then incrementing r to n. Then take $z=2$, $r=1$ and again increment r. Continue to increment z to n.
4. For $z=r$, compute $G_r(z,n)$ using (A.3).

The above algorithm was incorporated into a FORTRAN program to compute the cdf of a bivariate binomial random variable. For specified values of n, π_1 and π_2, this program calculates the cdf for up to ten different values of ρ. In the computations the nonnormalized Krawtchouck polynomials (A.3) were used rather than the orthonormal polynomials of expression (2.3), since this procedure led to increased precision. The program was able to handle values of n up to thirty-five. Larger values of n resulted in unresolved computing problems, including underflows and overflows.

The FORTRAN program is given below. Some representative output of the program is also provided as an illustration.

```
      CALL ERRSET(208,256,-1,0,0,208)
C*********************************************************************
C     THIS DOUBLE PRECISION PROGRAM COMPUTES BIVARIATE BINOMIAL
C     PROBABILITIES USING THE EXPANSION IN THE NONNORMALIZED KRAWTCHOUCK
C     POLYNOMIALS FOUND IN AITKEN & GONIN (1935).
C     P(1) IS THE BINOMIAL PROBABILITY OF X, P(2) THE PROBABILITY OF Y.
C     N IS THE VALUE OF THE OTHER BINOMIAL PARAMETER FOR EACH MARGINAL.
C     RHO(I) (I = 1,...,10) IS THE VALUE OF THE CORRELATION PARAMETER OF
C     THE BIVARIATE BINOMIAL.
C*********************************************************************
      INTEGER X, Y, I, R, N
      INTEGER XP, YP
      REAL*8 P(2),C(2),GX(50,50),GY(50,50),NF(50),TSUM(50,10),SUM(50,10)
     1,DI,DP,RHO(10),C(10),PR1(50,50),PR2(50,50),PR3(50,50),PR4(50,50),
     2PR5(50,50),PR6(50,50),PR7(50,50),PR8(50,50),PR9(50,50),PR10(50,50)
     3,CMPRX(50,10),CMPRY(50,10),CMPR1(50,10),CMPR2(50,10),CMPR3(50,10),
     4CMPR4(50,10),CMPR5(50,10),CMPR6(50,10),CMPR7(50,10),CMPR8(50,10),
     5CMPR9(50,10),CMPR10(50,10),SUBPR1(50,10),SUBPR2(50,10),SUBPR3(50,1
     60),SUBPR4(50,10),SUBPR5(50,10),SUBPR6(50,10),SUBPR7(50,10),SUBPR8(
     750,10),SUBPR9(50,10),SUBPRT(50,10)
      REAL*8 PRX(50),PRY(50)
      N = 25
      M = N + 1
      RHO(1) = 0.05D0
      RHO(2) = 0.10D0
      RHO(3) = 0.15D0
      RHO(4) = 0.20D0
      RHO(5) = 0.25D0
      RHO(6) = 0.30D0
      RHO(7) = 0.40D0
      RHO(8) = 0.50D0
      RHO(9) = 0.60D0
```

Figure A1.1. Fortran Program for Computing Bivariate Binomial Cumulative Distribution Functions

```
      RHO(10) = 0.80D0
      NF(1) = 1.D0
      DO 25 I = 2,N
      DI = DFLOAT(I)
      NF(I) = NF(I-1) * DI
   25 CONTINUE
  200 READ (5,300) P1,P2
  300 FORMAT (F6.4,5X,F6.4)
      IF (P1.EQ.0.0) GO TO 400
      P(1) = P1 * 1.D0
      P(2) = P2 * 1.D0
      DO 50 I = 1,2
      Q(I) = 1.D0 - P(I)
   50 CONTINUE
      DO 20 J = 1,10
      C(J) = RHO(J) / (DSQRT(P(1) * Q(1) * P(2) * Q(2)))
   20 CONTINUE
      CALL KRPLY(P(1), NF, GX, N)
      CALL KRPLY(P(2), NF, GY, N)
      CALL BINPR(N,PRX,P(1),Q(1))
      CALL BINPR(N,PRY,P(2),Q(2))
      DO 125 X = 1,10
      DO 150 Y = 1,M
      DO 165 J = 1,10
      TSUM(1,J) = 1.D0
      DO 175 I = 1,N
      IF (I.EQ.N) GO TO 225
      SUM(I,J) = ((C(J)**I)*NF(N-I)) * (GX(X,I)/NF(I)) * (GY(Y,I) / NF(N
     1))
      GO TO 75
  225 SUM(I,J) = (C(J)**I) *(GX(X,I)/NF(I)) * (GY(Y,I)/NF(N))
   75 TSUM(I+1,J) = TSUM(I,J) + SUM(I,J)
```

Figure A1.1. Continued.

```
  175 CONTINUE
  165 CONTINUE
C***********************************************************************
C     PRI(J,K) IS THE BIVARIATE BINOMIAL PROBABILITY FOR RHO(I)
C     (I=1,...,10) AND J = X - 1, K = Y - 1.
C***********************************************************************
      DP = PRX(X) * PRY(Y)
   10 PR1(X,Y) = TSUM(N+1,1) * DP
      CMPRY(Y+1,1) = CMPRY(Y,1) + PR1(X,Y)
   11 PR2(X,Y) = TSUM(N+1,2) * DP
      CMPRY(Y+1,2) = CMPRY(Y,2) + PR2(X,Y)
   12 PR3(X,Y) = TSUM(N+1,3) * DP
      CMPRY(Y+1,3) = CMPRY(Y,3) + PR3(X,Y)
   13 PR4(X,Y) = TSUM(N+1,4) * DP
      CMPRY(Y+1,4) = CMPRY(Y,4) + PR4(X,Y)
   14 PR5(X,Y) = TSUM(N+1,5) * DP
      CMPRY(Y+1,5) = CMPRY(Y,5) + PR5(X,Y)
   15 PR6(X,Y) = TSUM(N+1,6) * DP
      CMPRY(Y+1,6) = CMPRY(Y,6) + PR6(X,Y)
   16 PR7(X,Y) = TSUM(N+1,7) * DP
      CMPRY(Y+1,7) = CMPRY(Y,7) + PR7(X,Y)
   17 PR8(X,Y) = TSUM(N+1,8) * DP
      CMPRY(Y+1,8) = CMPRY(Y,8) + PR8(X,Y)
   18 PR9(X,Y) = TSUM(N+1,9) * DP
      CMPRY(Y+1,9) = CMPRY(Y,9) + PR9(X,Y)
   19 PR10(X,Y) = TSUM(N+1,10) * DP
      CMPRY(Y+1,10) = CMPRY(Y,10) + PR10(X,Y)
  150 CONTINUE
  125 CONTINUE
C***********************************************************************
C     CMPRI(X,Y) IS THE CDF OF THE BIVARIATE BINOMIAL FOR RHO(I)
C     EVALUATED AT (X-1,Y-1).
```

Figure A1.1. Continued.

```
C***********************************************************************
  190 CMPR1(1,1) = PR1(1,1)
      CMPR2 (1,1) = PR2 (1,1)
      CMPR3 (1,1) = PR3 (1,1)
      CMPR4 (1,1) = PR4 (1,1)
      CMPR5 (1,1) = PR5 (1,1)
      CMPR6 (1,1) = PR6 (1,1)
      CMPR7 (1,1) = PR7 (1,1)
      CMPR8 (1,1) = PR8 (1,1)
      CMPR9 (1,1) = PR9 (1,1)
      CMPR10(1,1) = PR10(1,1)
      DO 1 I = 2,10
      CMPR1(I,1) = CMPR1 (I-1,1) + PR1 (I,1)
      CMPR2(I,1) = CMPR2 (I-1,1) + PR2 (I,1)
      CMPR3(I,1) = CMPR3 (I-1,1) + PR3 (I,1)
      CMPR4(I,1) = CMPR4 (I-1,1) + PR4 (I,1)
      CMPR5(I,1) = CMPR5 (I-1,1) + PR5 (I,1)
      CMPR6(I,1) = CMPR6 (I-1,1) + PR6 (I,1)
      CMPR7(I,1) = CMPR7 (I-1,1) + PR7 (I,1)
      CMPR8(I,1) = CMPR8 (I-1,1) + PR8 (I,1)
      CMPR9(I,1) = CMPR9 (I-1,1) + PR9 (I,1)
      CMPR10(I,1)= CMPR10(I-1,1) + PR10(I,1)
    1 CONTINUE
      DO 2 I = 1,10
      SUBPR1(1,I) = PR1(1,I)
      SUBPR2(1,I) = PR2(1,I)
      SUBPR3(1,I) = PR3(1,I)
      SUBPR4(1,I) = PR4(1,I)
      SUBPR5(1,I) = PR5(1,I)
      SUBPR6(1,I) = PR6(1,I)
      SUBPR7(1,I) = PR7(1,I)
      SUBPR8(1,I) = PR8(1,I)
```

Figure A1.1. Continued.

```
      SUBPR9(1,I) = PR9(1,I)
      SUBPRT(1,I) = PR10(1,I)
    2 CONTINUE
      DO 3 I = 1,10
      DO 4 J = 2,10
      SUBPR1 (J,I) = SUBPR1 (J-1,I) + PR1 (J,I)
      SUBPR2 (J,I) = SUBPR2 (J-1,I) + PR2 (J,I)
      SUBPR3 (J,I) = SUBPR3 (J-1,I) + PR3 (J,I)
      SUBPR4 (J,I) = SUBPR4 (J-1,I) + PR4 (J,I)
      SUBPR5 (J,I) = SUBPR5 (J-1,I) + PR5 (J,I)
      SUBPR6 (J,I) = SUBPR6 (J-1,I) + PR6 (J,I)
      SUBPR7 (J,I) = SUBPR7 (J-1,I) + PR7 (J,I)
      SUBPR8 (J,I) = SUBPR8 (J-1,I) + PR8 (J,I)
      SUBPR9 (J,I) = SUBPR9 (J-1,I) + PR9 (J,I)
      SUBPRT(J,I) = SUBPRT(J-1,I) + PR10(J,I)
    4 CONTINUE
    3 CONTINUE
      DO 5 I = 1,10
      DO 6 J = 2,10
      CMPR1 (I,J) = SUBPR1 (I,J) + CMPR1 (I,J-1)
      CMPR2 (I,J) = SUBPR2 (I,J) + CMPR2 (I,J-1)
      CMPR3 (I,J) = SUBPR3 (I,J) + CMPR3 (I,J-1)
      CMPR4 (I,J) = SUBPR4 (I,J) + CMPR4 (I,J-1)
      CMPR5 (I,J) = SUBPR5 (I,J) + CMPR5 (I,J-1)
      CMPR6 (I,J) = SUBPR6 (I,J) + CMPR6 (I,J-1)
      CMPR7 (I,J) = SUBPR7 (I,J) + CMPR7 (I,J-1)
      CMPR8 (I,J) = SUBPR8 (I,J) + CMPR8 (I,J-1)
      CMPR9 (I,J) = SUBPR9 (I,J) + CMPR9 (I,J-1)
      CMPR10(I,J) = SUBPRT(I,J) + CMPR10(I,J-1)
    6 CONTINUE
    5 CONTINUE
C***********************************************************************
```

Figure A1.1. Continued.

```
C         THE REMAINDER OF THE MAIN PROGRAM MERELY FORMATS AND PRINTS
C         THE BIVARIATE BINOMIAL CDF IN TABLULAR FORM FOR X AND Y LESS
C         THAN NINE.
C**********************************************************************
      DO 380 K = 1,10
      WRITE(6,375)
  375 FORMAT('1',13X,'BIVARIATE BINOMIAL CDF', 5X,'(PR(X L.E. A, Y L.E.
     1B))')
      WRITE(6,376) N,P1,P2,RHO(K)
  376 FORMAT('0',13X,'N = ',I2,3X,'PR X = ',F6.4,3X,'PR Y = ',F6.4,3X,
     1 'RHO = ' , F4.2)
      WRITE(6,377)
  377 FORMAT('0',50X, 'B')
      WRITE(6,378)
  378 FORMAT('0',5X,'A',9X,'0',9X,'1',9X,'2',9X,'3',9X,'4',9X,'5')
  379 FORMAT ('0',5X,I1,5X,6(F8.6,2X))
      GO TO (401,402,403,404,405,406,407,408,409,410),K
  401 DO 411 I = 1,10
      IM = I-1
      WRITE(6,379) IM,(CMPR1 (I,J),J=1,6)
  411 CONTINUE
      GO TO 380
  402 DO 412 I = 1,10
      IM = I-1
      WRITE(6,379) IM,(CMPR2 (I,J),J=1,6)
  412 CONTINUE
      GO TO 380
  403 DO 413 I = 1,10
      IM = I-1
      WRITE(6,379) IM,(CMPR3 (I,J),J=1,6)
  413 CONTINUE
      GO TO 380
```

Figure A1.1. Continued.

```
404 DO 414 I = 1,10
    IM = I-1
    WRITE(6,379) IM,(CMPR4 (I,J),J=1,6)
414 CONTINUE
    GO TO 380
405 DO 415 I = 1,10
    IM = I-1
    WRITE(6,379) IM,(CMPR5 (I,J),J=1,6)
415 CONTINUE
    GO TO 380
406 DO 416 I = 1,10
    IM = I-1
    WRITE(6,379) IM,(CMPR6 (I,J),J=1,6)
416 CONTINUE
    GO TO 380
407 DO 417 I = 1,10
    IM = I-1
    WRITE(6,379) IM,(CMPR7 (I,J),J=1,6)
417 CONTINUE
    GO TO 380
408 DO 418 I = 1,10
    IM = I-1
    WRITE(6,379) IM,(CMPR8 (I,J),J=1,6)
418 CONTINUE
    GO TO 380
409 DO 419 I = 1,10
    IM = I-1
    WRITE(6,379) IM,(CMPR9 (I,J),J=1,6)
419 CONTINUE
    GO TO 380
410 DO 420 I = 1,10
    IM = I-1
```

Figure A1.1. Continued.

```
      WRITE(6,379) IM,(CMPR10(I,J),J=1,6)
  420 CONTINUE
  380 CONTINUE
      GO TO 200
  400 RETURN
      END
C******************&****************************************************
C         DOUBLE PRECISION SUBROUTINE TO COMPUTE BINOMIAL PROBABILITIES
C         FOR GIVEN N AND P. PR(I) IS B(I-1; N, P).
C***********************************************************************
      SUBROUTINE BINPR(N,PR,P,Q)
      REAL*8 PR(50), P , Q
      PR(1) = Q**N
      DO 100 I = 1,N
      PR(I+1) = (DFLOAT(N-I+1)/ DFLOAT(I)) * (P/Q) * PR(I)
  100 CONTINUE
      RETURN
      END
C***********************************************************************
C         SUBROUTINE TO COMPUTE NON-NORMALIZED KRAWTCHOUCK POLYNOMIALS.
C         G(K,R) IS THE POLYNOMIAL OF DEGREE R EVALUATED AT X = K - 1,
C         FOR THE SPECIFIED PARAMETERS P (THE BINOMIAL PROBABILITY) AND
C         N.  SEE TEXT FOR THE EXPLANATION OF THE RECURSIVE ALGORITHM
C         USED.
C***********************************************************************
      SUBROUTINE KRPLY(P,NF,G,N)
      REAL*8 G(50,50),NF(50),TGSUM(50),GSUM(50),P
      INTEGER R, K
      M = N + 1
      G(1,1) = (-P) * DFLOAT(N)
      DO 100 R = 2,N
      G(1,R) = (-P) * DFLOAT(N-R+1) * G(1,R-1)
```

Figure A1.1. Continued.

```
100 CONTINUE
    DO 300 K = 2,M
    G(K,1) = DFLOAT(K-1) - ( P * DFLOAT(N))
    DO 200 R = 2,N
    IF (R.EQ.K-1) GO TO 225
    G(K,R) = DFLOAT(K-1) * G(K-1,R) / DFLOAT(K-1-R)  + P*DFLOAT(R) *
   1DFLOAT(N-R+1)/ DFLOAT(K-1-R) * G(K,R-1)
    GO TO 200
225 GSUM(1) = (-P) * DFLOAT(R) * DFLOAT(N-R+1)
    TGSUM(1) = 1.D0 + GSUM(1)
    DO 275 I = 2,R
    GSUM(I) = -1.D0 * P * DFLOAT(R-I+1) * DFLOAT(N-R+I) / DFLOAT(I*I)
   1 * GSUM(I-1)
    TGSUM(I) = TGSUM(I-1) + GSUM(I)
275 CONTINUE
    G(K,R) = TGSUM(R) * NF(R)
200 CONTINUE
300 CONTINUE
    RETURN
    END
```

TABLE A1.1.

BIVARIATE BINOMIAL CDF (PR(X ≤ A, Y ≤ B))

N = 25 PR X = 0.0100 PR Y = 0.0100* RHO = 0.10

	B					
A	0	1	2	3	4	5
0	0.620471	0.761344	0.776697	0.777766	0.777819	0.777821
1	0.761344	0.950157	0.972473	0.974147	0.974237	0.974241
2	0.776697	0.972473	0.996123	0.997945	0.998045	0.998049
3	0.777766	0.974147	0.997945	0.999786	0.999889	0.999893
4	0.777819	0.974237	0.998045	0.999889	0.999991	0.999995
5	0.777821	0.974241	0.998049	0.999893	0.999995	1.000000
6	0.777821	0.974241	0.998049	0.999893	0.999995	1.000000
7	0.777821	0.974241	0.998049	0.999893	0.999996	1.000000
8	0.777821	0.974241	0.998049	0.999893	0.999996	1.000000
9	0.777821	0.974241	0.998049	0.999893	0.999996	1.000000

* The value of π_1 is given by PR X, and the value of π_2 is given by PR Y.

TABLE A1.1.

BIVARIATE BINOMIAL CDF (PR(X ≤ A, Y ≤ B))

N = 25 PR X = 0.0100 PR Y = 0.0100 RHO = 0.25

	B					
A	0	1	2	3	4	5
0	0.644381	0.766115	0.777154	0.777794	0.777820	0.777821
1	0.766115	0.952145	0.972759	0.974171	0.974238	0.974241
2	0.777154	0.972759	0.996191	0.997953	0.998046	0.998049
3	0.777794	0.974171	0.997953	0.999788	0.999889	0.999893
4	0.777820	0.974238	0.998046	0.999889	0.999991	0.999995
5	0.777821	0.974241	0.998049	0.999893	0.999995	1.000000
6	0.777821	0.974241	0.998049	0.999893	0.999996	1.000000
7	0.777821	0.974241	0.998049	0.999893	0.999996	1.000000
8	0.777821	0.974241	0.998049	0.999893	0.999996	1.000000
9	0.777821	0.974241	0.998049	0.999893	0.999996	1.000000

TABLE A1.1.

BIVARIATE BINOMIAL CDF (PR(X ≤ A, Y ≤ B))

N = 25 PR X = 0.0100 PR Y = 0.0100 RHO = 0.40

	B					
A	0	1	2	3	4	5
0	0.669175	0.770157	0.777471	0.777810	0.777821	0.777821
1	0.770157	0.954789	0.973119	0.974196	0.974240	0.974241
2	0.777471	0.973119	0.996321	0.997968	0.998047	0.998049
3	0.777810	0.974196	0.997968	0.999792	0.999889	0.999893
4	0.777821	0.974240	0.998047	0.999889	0.999991	0.999995
5	0.777821	0.974241	0.998049	0.999893	0.999995	1.000000
6	0.777821	0.974241	0.998049	0.999893	0.999996	1.000000
7	0.777821	0.974241	0.998049	0.999893	0.999996	1.000000
8	0.777821	0.974241	0.998049	0.999893	0.999996	1.000000
9	0.777821	0.974241	0.998049	0.999893	0.999996	1.000000

TABLE A1.1.

BIVARIATE BINOMIAL CDF (PR(X ≤ A, Y ≤ B))

N = 25 PR X = 0.0100 PR Y = 0.0100 RHO = 0.60

	B					
A	0	1	2	3	4	5
0	0.703661	0.774309	0.777714	0.777819	0.777821	0.777821
1	0.774309	0.959525	0.973630	0.974224	0.974241	0.974241
2	0.777714	0.973630	0.996630	0.997998	0.998048	0.998049
3	0.777819	0.974224	0.997998	0.999804	0.999890	0.999893
4	0.777821	0.974241	0.998048	0.999890	0.999991	0.999995
5	0.777821	0.974241	0.998049	0.999893	0.999995	1.000000
6	0.777821	0.974241	0.998049	0.999893	0.999996	1.000000
7	0.777821	0.974241	0.998049	0.999893	0.999996	1.000000
8	0.777821	0.974241	0.998049	0.999893	0.999996	1.000000
9	0.777821	0.974241	0.998049	0.999893	0.999996	1.000000

TABLE A1.1.

BIVARIATE BINOMIAL CDF (PR(X ≤ A, Y ≤ B))

N = 25 PR X = 0.0100 PR Y = 0.0100 RHO = 0.80

			B			
A	0	1	2	3	4	5
0	0.739849	0.776916	0.777807	0.777821	0.777821	0.777821
1	0.776916	0.965904	0.974058	0.974238	0.974241	0.974241
2	0.777807	0.974058	0.997169	0.998032	0.998049	0.998049
3	0.777821	0.974238	0.998032	0.999834	0.999892	0.999893
4	0.777821	0.974241	0.998049	0.999892	0.999993	0.999995
5	0.777821	0.974241	0.998049	0.999893	0.999995	1.000000
6	0.777821	0.974241	0.998049	0.999893	0.999996	1.000000
7	0.777821	0.974241	0.998049	0.999893	0.999996	1.000000
8	0.777821	0.974241	0.998049	0.999893	0.999996	1.000000
9	0.777821	0.974241	0.998049	0.999893	0.999996	1.000000

TABLE A1.1.

BIVARIATE BINOMIAL CDF (PR(X ≤ A, Y ≤ B))

N = 25 PR X = 0.0100 PR Y = 0.0150 RHO = 0.10

	B					
A	0	1	2	3	4	5
0	0.549849	0.741895	0.774091	0.777540	0.777805	0.777821
1	0.671500	0.923475	0.968643	0.973794	0.974214	0.974240
2	0.684419	0.944610	0.992042	0.997559	0.998019	0.998048
3	0.685295	0.946170	0.993833	0.999396	0.999862	0.999892
4	0.685338	0.946252	0.993931	0.999498	0.999964	0.999994
5	0.685339	0.946255	0.993936	0.999502	0.999968	0.999998
6	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998
7	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998
8	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998
9	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998

TABLE A1.1.

BIVARIATE BINOMIAL CDF (PR(X ≤ A, Y ≤ B))

N = 25 PR X = 0.0100 PR Y = 0.0150 RHO = 0.25

	B					
A	0	1	2	3	4	5
0	0.575968	0.750058	0.775316	0.777657	0.777813	0.777821
1	0.676456	0.926422	0.969289	0.973875	0.974220	0.974240
2	0.684871	0.945002	0.992175	0.997582	0.998021	0.998048
3	0.685322	0.946200	0.993847	0.999400	0.999862	0.999892
4	0.685339	0.946254	0.993932	0.999498	0.999964	0.999994
5	0.685339	0.946255	0.993936	0.999502	0.999968	0.999998
6	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998
7	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998
8	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998
9	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998

TABLE A1.1.

BIVARIATE BINOMIAL CDF (PR(X ≤ A, Y ≤ B))

N = 25 PR X = 0.0100 PR Y = 0.0150 RHO = 0.40

	B					
A	0	1	2	3	4	5
0	0.603276	0.757363	0.776255	0.777734	0.777818	0.777821
1	0.680414	0.930177	0.970112	0.973970	0.974228	0.974240
2	0.685149	0.945446	0.992409	0.997626	0.998026	0.998048
3	0.685334	0.946228	0.993871	0.999409	0.999864	0.999892
4	0.685339	0.946255	0.993934	0.999499	0.999964	0.999994
5	0.685339	0.946256	0.993936	0.999502	0.999968	0.999998
6	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998
7	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998
8	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998
9	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998

TABLE A1.1.

BIVARIATE BINOMIAL CDF (PR(X ≤ A, Y ≤ B))

N = 25 PR X = 0.0100 PR Y = 0.0150 RHO = 0.60

	B					
A	0	1	2	3	4	5
0	0.641626	0.765610	0.777110	0.777791	0.777820	0.777821
1	0.683971	0.936744	0.971376	0.974092	0.974235	0.974241
2	0.685312	0.945986	0.992942	0.997719	0.998034	0.998049
3	0.685339	0.946251	0.993910	0.999436	0.999868	0.999892
4	0.685339	0.946255	0.993935	0.999501	0.999965	0.999994
5	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998
6	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998
7	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998
8	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998
9	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998

TABLE A1.1.

BIVARIATE BINOMIAL CDF (PR(X ≤ A, Y ≤ B))

N = 25 PR X = 0.0100 PR Y = 0.0150 RHO = 0.80

	B					
A	0	1	2	3	4	5
0	0.682310	0.771936	0.777587	0.777815	0.777821	0.777821
1	0.685333	0.945523	0.972658	0.974183	0.974239	0.974241
2	0.685339	0.946254	0.993851	0.997843	0.998042	0.998049
3	0.685339	0.946256	0.993935	0.999496	0.999876	0.999893
4	0.685339	0.946256	0.993936	0.999502	0.999968	0.999994
5	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998
6	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998
7	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998
8	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998
9	0.685339	0.946256	0.993936	0.999502	0.999969	0.999998

TABLE A1.1.

BIVARIATE BINOMIAL CDF (PR(X ≤ A, Y ≤ B))

N = 25 PR X = 0.0100 PR Y = 0.0200 RHO = 0.10

	B					
A	0	1	2	3	4	5
0	0.486530	0.716964	0.769352	0.776961	0.777754	0.777817
1	0.591776	0.890007	0.961888	0.972921	0.974132	0.974234
2	0.602704	0.909838	0.984909	0.996620	0.997930	0.998041
3	0.603429	0.911278	0.986658	0.998449	0.999772	0.999885
4	0.603463	0.911352	0.986753	0.998550	0.999874	0.999987
5	0.603465	0.911355	0.986756	0.998554	0.999878	0.999992
6	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992
7	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992
8	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992
9	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992

TABLE A1.1.

BIVARIATE BINOMIAL CDF (PR(X ≤ A, Y ≤ B))

N = 25 PR X = 0.0100 PR Y = 0.0200 RHO = 0.25

	B					
A	0	1	2	3	4	5
0	0.513374	0.728459	0.771713	0.777271	0.777783	0.777819
1	0.596647	0.893765	0.962997	0.973110	0.974154	0.974236
2	0.603130	0.910306	0.985115	0.996668	0.997937	0.998042
3	0.603453	0.911311	0.986678	0.998455	0.999773	0.999885
4	0.603464	0.911354	0.986754	0.998550	0.999874	0.999987
5	0.603465	0.911355	0.986756	0.998554	0.999878	0.999992
6	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992
7	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992
8	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992
9	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992

TABLE A1.1.

BIVARIATE BINOMIAL CDF (PR(X ≤ A, Y ≤ B))

N = 25 PR X = 0.0100 PR Y = 0.0200 RHO = 0.40

A	B = 0	1	2	3	4	5
0	0.541637	0.739081	0.773630	0.777492	0.777801	0.777820
1	0.600310	0.898386	0.964407	0.973338	0.974178	0.974237
2	0.603361	0.910789	0.985454	0.996756	0.997950	0.998043
3	0.603462	0.911338	0.986708	0.998470	0.999776	0.999886
4	0.603465	0.911355	0.986755	0.998551	0.999874	0.999987
5	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992
6	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992
7	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992
8	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992
9	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992

TABLE A1.1.

BIVARIATE BINOMIAL CDF (PR(X ≤ A, Y ≤ B))

N = 25 PR X = 0.0100 PR Y = 0.0200 RHO = 0.60

	B					
A	0	1	2	3	4	5
0	0.581652	0.751681	0.775538	0.777677	0.777814	0.777821
1	0.603081	0.906294	0.966647	0.973654	0.974207	0.974239
2	0.603460	0.911272	0.986194	0.996947	0.997975	0.998045
3	0.603465	0.911354	0.986748	0.998514	0.999787	0.999887
4	0.603465	0.911355	0.986756	0.998554	0.999876	0.999988
5	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992
6	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992
7	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992
8	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992
9	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992

TABLE A1.1.

BIVARIATE BINOMIAL CDF (PR(X ≤ A, Y ≤ B))

N = 25 PR X = 0.0100 PR Y = 0.0200 RHO = 0.80

	B					
A	0	1	2	3	4	5
0	0.624497	0.762209	0.776786	0.777772	0.777820	0.777821
1	0.603117	0.916776	0.969098	0.973930	0.974227	0.974240
2	0.603468	0.911267	0.987427	0.997221	0.998004	0.998047
3	0.603465	0.911356	0.986746	0.998607	0.999806	0.999889
4	0.603465	0.911355	0.986757	0.998553	0.999881	0.999989
5	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992
6	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992
7	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992
8	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992
9	0.603465	0.911355	0.986757	0.998554	0.999878	0.999992

APPENDIX II

The Approximation of Multivariate Chi-Squared Probabilities: The Equicorrelated Case

In Section 2.6 some series expansions for the multivariate chi-squared distribution were discussed. There it was noted that bivariate chi-squared probabilities could be obtained by truncating the series expansion (2.35) or (2.36). Beyond the bivariate case the expansion (2.37) quickly becomes too complicated for easy manipulation.

In this appendix we are interested in finding values c_α such that

$$P(X_1 \le c_\alpha, \ldots, X_k \le c_\alpha) = 1-\alpha \tag{A.5}$$

where $(X_1, \ldots, X_k)' \sim \chi^2_k(\nu, \underset{\sim}{R})$ and $\underset{\sim}{R} = (1-\rho)\underset{\sim}{I}_k + \rho \underset{\sim}{J}_k$. For several choices of ρ the c_α's of (A.5) are first found for the case k=2. Then the approximation suggested by Johnson (1962) (see(2.38)) is employed to obtain the c_α's for k=3,4,...,10. Various yardsticks are used to judge the accuracy of the approximation.

There have been some prior attempts to find c_α of (A.5) for some special situations. Jensen and Beus (1968) give extensive tables for ρ=0.0, while Krishnaiah and Armitage (1965) perform a thorough examination of the case ν=1. Dutt and Soms (1976) have given a very limited treatment of the case k=3. Their technique, however, is different in that they utilize an integral representation. All the preceding efforts will provide convenient reference points for the approximations below.

Jensen and Howe (1968) used the expansion in Laguerre polynomials (2.35) to examine probabilities not only of the type (A.5), but also two-sided square regions and one- and two-sided rectangular regions for the bivariate chi-squared distribution. Their technique for evaluating bivariate χ^2 probabilities was implemented in conjunction with an interpolation scheme to arrive at the probabilities reported in Tables A2.1 and A2.2. Expression (2.35) was truncated to evaluate (A.5) for ten different values of c_α for k=2. The range of these different values contained the c_α corresponding to the percentile of interest. The ten values of c_α and their percentiles were then used to compute a cubic spline. The spline was then evaluated at the appropriate percentile to give the points provided in the tables. The tables are for $|\rho|\varepsilon\{0.1, 0.2, 0.4, 0.5, 0.6, 0.8\}$, $\nu=1(1)30, 30(5)55$, and $\alpha=0.05, 0.01$.

The only approximation involved for the bivariate χ^2 cdf is the truncation of the series expansion (2.35). For the values of ρ considered herein, it is not anticipated that much accuracy is lost with the truncation procedure. The bivariate χ^2 distribution will also play a role in evaluating the approximations for the equicorrelated multivariate χ^2 cdf.

The same general procedure was used to compute the probabilities (A.5). Again, ten values of c_α were chosen and the probability (A.5) was evaluated for these values using the approximation (2.38). A cubic spline was computed and evaluated to give the values found in Tables A2.3 - A2.18. These tables are for $k=3(1)10$, $\rho\varepsilon\{0.1, 0.2, 0.4, 0.5, 0.6, 0.8\}$, and $\nu=1(1)15$.

For $\nu=1$, the approximate values can be compared to those given by Krishnaiah and Armitage (1965). These comparisons are summarized in Tables A2.19 and A2.20. In Table A2.21 the values reported by Dutt and Soms (1976) are contrasted with those found using Johnson's approximation. Unfortunately, the values given by Dutt and Soms for fifteen degrees of freedom appear to be in error. Therefore, those values are not included in the comparison.

All the comparisons made in the preceding paragraph indicate that Johnson's approximation is appealing. It would be even more reassuring if the approximations could be checked over a wider range of parameter values. A partial remedy can be offered with the aid of Theorem 1.7. Upper and lower bounds for the values in Tables A2.3 - A2.18 were computed using these second order Bonferroni bounds. For (A.5) the lower bound of Theorem 1.7 becomes:

$$(k-1)P(X_1\le c_\alpha, X_2\le c_\alpha) - (k-2)P(X_1\le c_\alpha)\ , \tag{A.6}$$

where $(X_1,X_2)' \sim \chi_2^2(\nu,\underset{\sim}{R})$, for $\underset{\sim}{R}$ a correlation matrix. The upper bound for (A.5) from Theorem 1.7 is:

$$\min\{\frac{(k-1)(k-2)}{2} - k(k-2)P(X_1\le c_\alpha) + \frac{k(k-1)}{2}\, P(X_1\le c_\alpha, X_2\le c_\alpha)\ ,$$

$$1 - \frac{k(1-P(X_1\le c_\alpha))^2}{(2k-1)[1-P(X_1\le c_\alpha)-(k-1)(1-P(X_1\le c_\alpha, X_2\le c_\alpha))]}\} \tag{A.7}$$

where $(X_1,X_2)'$ are as in (A.6).

Available algorithms for the univariate and bivariate χ^2 cdf's were used to compute the stochastic bounds (A.6) and (A.7) for each value of c_α reported in Tables A2.3 - A2.18. These upper and lower stochastic bounds are given in Tables A2.22 - A2.37. As one would expect the bounds become larger with ρ or k. As the degrees of freedom increase, however, the bounds, in general, become smaller. Another striking feature of these tables is the remarkable tightness of the bounds for $\alpha=0.01$ over all values of ρ, ν, and k. Overall, the results for the bounds (A.6) and (A.7) reinforce an impression that Johnson's approximation is adequate, especially for tail area probabilities with moderate values of ρ or k.

The expansion in Laguerre polynomials (2.35) could be used in place of the expansion using chi-squared marginals for the purpose of obtaining an approximation. The use of the Laguerre polynomials would have the advantage of employing a pdf to approximate a pdf, whereas the approximation (2.38) does not possess this property. With this alternative approach, then, at least certain anomalies would be avoided.

We have dealt explicity only with an equicorrelated structure for the multivariate χ^2 distribution. For certain correlation structures Theorem 2.8 can be used to obtain stochastic bounds in terms of equicorrelated distributions. In turn, approximate bounds obtained by approximating the cdf of these equicorrelated distributions might prove useful.

TABLE A2.1. UPPER 95TH PERCENTILES FOR THE BIVARIATE CHI-SQUARED DISTRIBUTION WITH N DEGREES OF FREEDOM AND CORRELATION COEFFICIENT ρ.

	ρ					
N	0.1	0.2	0.4	0.5	0.6	0.8
1	4.9981	4.9865	4.9362	4.8935	4.8344	4.6330
2	7.3487	7.3377	7.2878	7.2434	7.1798	6.9522
3	9.3170	9.3061	9.2556	9.2096	9.1424	8.8959
4	11.1098	11.0989	11.0476	11.0001	10.9299	10.6677
5	12.7972	12.7863	12.7340	12.6851	12.6121	12.3361
6	14.4124	14.4014	14.3402	14.2979	14.2224	13.9341
7	15.9743	15.9631	15.9090	15.8574	15.7796	15.4799
8	17.4946	17.4834	17.4283	17.3755	17.2955	16.9851
9	18.9815	18.9701	18.9141	18.8601	18.7781	18.4577
10	20.4406	20.4291	20.3722	20.3171	20.2330	19.9033
11	21.8762	21.8646	21.8068	21.7507	21.6647	21.3260
12	23.2917	23.2799	23.2213	23.1640	23.0764	22.7290
13	24.6895	24.6776	24.6181	24.5599	24.4704	24.1149
14	26.0717	26.0597	25.9994	25.9402	25.8491	25.4856
15	27.4401	27.4279	27.3669	27.3067	27.2140	26.8429
16	28.7961	28.7838	28.7218	28.6608	28.5665	28.1880
17	30.1408	30.1283	30.0656	30.0037	29.9078	29.5223
18	31.4752	31.4626	31.3991	31.3364	31.2390	30.8465
19	32.8002	32.7875	32.7233	32.6596	32.5608	32.1615
20	34.1165	34.1037	34.0387	33.9743	33.8741	33.4682
21	35.4249	35.4120	35.3463	35.2811	35.1794	34.7671
22	36.7260	36.7129	36.6464	36.5804	36.4774	36.0588
23	38.0200	38.0068	37.9397	37.8729	37.7686	37.3438
24	39.3077	39.2943	39.2265	39.1589	39.0533	38.6226
25	40.5892	40.5758	40.5073	40.4389	40.3321	39.8954
26	41.8651	41.8515	41.7824	41.7133	41.6052	41.1628
27	43.1357	43.1220	43.0522	42.9824	42.8730	42.4250
28	44.4011	44.3874	44.3170	44.2465	44.1359	43.6822
29	45.6619	45.6480	45.5770	45.5057	45.3940	44.9348
30	46.9181	46.9041	46.8325	46.7605	46.6476	46.1831
35	53.1387	53.1241	53.0494	52.9742	52.8557	52.3656
40	59.2737	59.2587	59.1811	59.1029	58.9792	58.4652
45	65.3392	65.3235	65.2433	65.1620	65.0335	64.4971
50	71.3463	71.3302	71.2475	71.1634	71.0302	70.4725
55	77.3037	77.2872	77.2019	77.1154	76.9776	76.3999

TABLE A2.2. UPPER 99TH PERCENTILES FOR THE BIVARIATE CHI-SQUARED DISTRIBUTION WITH N DEGREES OF FREEDOM AND CORRELATION COEFFICIENT ρ.

	ρ					
N	0.1	0.2	0.4	0.5	0.6	0.8
1	7.8731	7.8675	7.8378	7.8080	7.7615	7.5744
2	10.5902	10.5854	10.5585	10.5298	10.4837	10.2861
3	12.8314	12.8269	12.8013	12.7730	12.7263	12.5200
4	14.8532	14.8491	14.8238	14.7956	14.7481	14.5344
5	16.7423	16.7381	16.7133	16.6850	16.6368	16.4163
6	18.5402	18.5360	18.5112	18.4827	18.4340	18.2069
7	20.2700	20.2661	20.2413	20.2126	20.1631	19.9301
8	21.9471	21.9429	21.9184	21.8895	21.8392	21.6007
9	23.5814	23.5772	23.5525	23.5233	23.4721	23.2286
10	25.1800	25.1758	25.1508	25.1214	25.0700	24.8210
11	26.7485	26.7445	26.7194	26.6897	26.6373	26.3837
12	28.2909	28.2867	28.2618	28.2318	28.1788	27.9206
13	29.8106	29.8067	29.7814	29.7511	29.6977	29.4346
14	31.3103	31.3062	31.2809	31.2503	31.1962	30.9291
15	32.7922	32.7883	32.7625	32.7319	32.6770	32.4054
16	34.2580	34.2539	34.2281	34.1971	34.1418	33.8661
17	35.7090	35.7049	35.6792	35.6477	35.5916	35.3124
18	37.1470	37.1427	37.1166	37.0850	37.0288	36.7451
19	38.5727	38.5683	38.5422	38.5103	38.4534	38.1660
20	39.9869	39.9827	39.9565	39.9245	39.8667	39.5761
21	41.3910	41.3868	41.3605	41.3282	41.2699	40.9753
22	42.7855	42.7814	42.7546	42.7221	42.6634	42.3652
23	44.1711	44.1668	44.1400	44.1071	44.0478	43.7462
24	45.5481	45.5440	45.5167	45.4837	45.4241	45.1191
25	46.9174	46.9129	46.8859	46.8526	46.7922	46.4843
26	48.2793	48.2748	48.2475	48.2139	48.1533	47.8419
27	49.6343	49.6296	49.6023	49.5688	49.5073	49.1926
28	50.9826	50.9781	50.9504	50.9165	50.8546	50.5368
29	52.3246	52.3204	52.2923	52.2585	52.1961	51.8752
30	53.6608	53.6564	53.6286	53.5940	53.5312	53.2076
35	60.2629	60.2586	60.2300	60.1941	60.1292	59.7907
40	66.7539	66.7491	66.7196	66.6831	66.6157	66.2636
45	73.1532	73.1486	73.1186	73.0809	73.0116	72.6467
50	79.4766	79.4719	79.4409	79.4023	79.3314	78.9542
55	85.7352	85.7305	85.6985	85.6595	85.5861	85.1974

TABLE A2. 3 . APPROXIMATE UPPER 95 TH PERCENTILES FOR THE THREE-DIMENSIONAL EQUI-CORRELATED CHI-SQUARED DISTRIBUTION COMPUTED FROM JOHNSON'S APPROXIMATION. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE COMMON CORRELATION COEFFICIENT BY ρ .

	ρ					
N	0.1	0.2	0.4	0.5	0.6	0.8
1	5.6951	5.6761	5.5928	5.5217	5.4223	5.0882
2	8.1490	8.1313	8.0499	7.9766	7.8712	7.4965
3	10.1930	10.1758	10.0941	10.0186	9.9080	9.5039
4	12.0489	12.0318	11.9493	11.8717	11.7565	11.3279
5	13.7915	13.7745	13.6909	13.6112	13.4918	13.0419
6	15.4566	15.4395	15.3547	15.2730	15.1499	14.6805
7	17.0643	17.0471	16.9610	16.8775	16.7507	16.2636
8	18.6273	18.6099	18.5226	18.4372	18.3072	17.8034
9	20.1542	20.1367	20.0481	19.9611	19.8278	19.3084
10	21.6512	21.6335	21.5437	21.4550	21.3187	20.7845
11	23.1229	23.1050	23.0139	22.9237	22.7844	22.2362
12	24.5728	24.5548	24.4625	24.3706	24.2286	23.6670
13	26.0036	25.9854	25.8919	25.7986	25.6539	25.0794
14	27.4176	27.3993	27.3046	27.2099	27.0626	26.4756
15	28.8168	28.7982	28.7024	28.6062	28.4564	27.8574

TABLE A2.4. APPROXIMATE UPPER 95TH PERCENTILES FOR THE FOUR-DIMENSIONAL EQUI-CORRELATED CHI-SQUARED DISTRIBUTION COMPUTED FROM JOHNSON'S APPROXIMATION. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE COMMON CORRELATION COEFFICIENT BY ρ.

	ρ					
N	0.1	0.2	0.4	0.5	0.6	0.8
1	6.1967	6.1721	6.0639	5.9710	5.8418	5.4068
2	8.7186	8.6961	8.5911	8.4961	8.3591	7.8745
3	10.8129	10.7912	10.6864	10.5889	10.4455	9.9244
4	12.7109	12.6894	12.5840	12.4839	12.3350	11.7833
5	14.4905	14.4691	14.3626	14.2601	14.1060	13.5276
6	16.1890	16.1676	16.0598	15.9550	15.7962	15.1935
7	17.8274	17.8060	17.6968	17.5896	17.4265	16.8015
8	19.4190	19.3975	19.2869	19.1775	19.0102	18.3645
9	20.9729	20.9512	20.8391	20.7277	20.5564	19.8912
10	22.4954	22.4734	22.3600	22.2466	22.0716	21.3877
11	23.9913	23.9692	23.8544	23.7390	23.5603	22.8589
12	25.4644	25.4420	25.3258	25.2086	25.0265	24.3081
13	26.9174	26.8950	26.7773	26.6582	26.4728	25.7383
14	28.3529	28.3302	28.2112	28.0903	27.9017	27.1516
15	29.7727	29.7498	29.6294	29.5068	29.3151	28.5499

TABLE A2.5. APPROXIMATE UPPER 95TH PERCENTILES FOR THE FIVE-DIMENSIONAL EQUI-CORRELATED CHI-SQUARED DISTRIBUTION COMPUTED FROM JOHNSON'S APPROXIMATION. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE COMMON CORRELATION COEFFICIENT BY ρ.

	ρ					
N	0.1	0.2	0.4	0.5	0.6	0.8
1	6.5891	6.5601	6.4317	6.3211	6.1672	5.6513
2	9.1611	9.1348	9.0109	8.8982	8.7357	8.1630
3	11.2927	11.2674	11.1443	11.0289	10.8591	10.2445
4	13.2220	13.1971	13.0734	12.9552	12.7791	12.1294
5	15.0292	15.0044	14.8797	14.7588	14.5768	13.8963
6	16.7526	16.7279	16.6019	16.4783	16.2910	15.5824
7	18.4139	18.3892	18.2618	18.1356	17.9433	17.2090
8	20.0269	20.0021	19.8732	19.7446	19.5475	18.7892
9	21.6009	21.5759	21.4454	21.3145	21.1128	20.3319
10	23.1423	23.1173	22.9852	22.8520	22.6460	21.8437
11	24.6564	24.6311	24.4975	24.3621	24.1520	23.3293
12	26.1467	26.1213	25.9862	25.8486	25.6345	24.7924
13	27.6165	27.5908	27.4542	27.3146	27.0967	26.2358
14	29.0680	29.0421	28.9040	28.7624	28.5407	27.6618
15	30.5032	30.4772	30.3375	30.1940	29.9687	29.0724

TABLE A2. 6. APPROXIMATE UPPER 95TH PERCENTILES FOR THE SIX-DIMENSIONAL EQUI-CORRELATED CHI-SQUARED DISTRIBUTION COMPUTED FROM JOHNSON'S APPROXIMATION. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE COMMON CORRELATION COEFFICIENT BY ρ.

	ρ					
N	0.1	0.2	0.4	0.5	0.6	0.8
1	6.9117	6.8790	6.7335	6.6079	6.4331	5.8493
2	9.5231	9.4936	9.3538	9.2262	9.0421	8.3957
3	11.6841	11.6559	11.5172	11.3868	11.1948	10.5022
4	13.6381	13.6104	13.4714	13.3379	13.1390	12.4076
5	15.4671	15.4397	15.2997	15.1633	14.9579	14.1924
6	17.2102	17.1829	17.0417	16.9024	16.6911	15.8946
7	18.8898	18.8624	18.7198	18.5777	18.3609	17.5358
8	20.5197	20.4923	20.3481	20.2034	19.9812	19.1295
9	22.1096	22.0821	21.9362	21.7890	21.5618	20.6851
10	23.6662	23.6385	23.4910	23.3413	23.1094	22.2090
11	25.1947	25.1668	25.0177	24.8655	24.6291	23.7060
12	26.6988	26.6707	26.5200	26.3656	26.1247	25.1800
13	28.1818	28.1535	28.0012	27.8445	27.5993	26.6338
14	29.6460	29.6176	29.4636	29.3047	29.0554	28.0699
15	31.0935	31.0649	30.9093	30.7483	30.4951	29.4902

TABLE A2. 7. APPROXIMATE UPPER 95TH PERCENTILES FOR THE SEVEN-DIMENSIONAL EQUI-CORRELATED CHI-SQUARED DISTRIBUTION COMPUTED FROM JOHNSON'S APPROXIMATION. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE COMMON CORRELATION COEFFICIENT BY ρ.

	ρ					
N	0.1	0.2	0.4	0.5	0.6	0.8
1	7.1858	7.1498	6.9895	6.8509	6.6579	6.0155
2	9.8293	9.7971	9.6435	9.5030	9.3001	8.5904
3	12.0146	11.9838	11.8318	11.6882	11.4770	10.7174
4	13.9889	13.9588	13.8067	13.6600	13.4412	12.6398
5	15.8359	15.8061	15.6531	15.5032	15.2776	14.4392
6	17.5953	17.5657	17.4114	17.2585	17.0266	16.1546
7	19.2899	19.2603	19.1045	18.9487	18.7107	17.8079
8	20.9339	20.9042	20.7468	20.5882	20.3445	19.4129
9	22.5369	22.5071	22.3481	22.1868	21.9377	20.9790
10	24.1060	24.0761	23.9155	23.7514	23.4972	22.5127
11	25.6463	25.6163	25.4539	25.2873	25.0281	24.0192
12	27.1619	27.1317	26.9676	26.7985	26.5345	25.5022
13	28.6558	28.6254	28.4596	28.2881	28.0195	26.9647
14	30.1305	30.0999	29.9325	29.7586	29.4856	28.4091
15	31.5882	31.5574	31.3883	31.2121	30.9348	29.8374

TABLE A2. 8 . APPROXIMATE UPPER 95TH PERCENTILES FOR THE EIGHT-DIMENSIONAL EQUI-CORRELATED CHI-SQUARED DISTRIBUTION COMPUTED FROM JOHNSON'S APPROXIMATION. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE COMMON CORRELATION COEFFICIENT BY ρ .

	ρ					
N	0.1	0.2	0.4	0.5	0.6	0.8
1	7.4240	7.3852	7.2118	7.0616	6.8525	6.1585
2	10.0947	10.0601	9.8944	9.7423	9.5229	8.7576
3	12.3004	12.2675	12.1037	11.9486	11.7203	10.9020
4	14.2920	14.2598	14.0961	13.9377	13.7015	12.8386
5	16.1543	16.1225	15.9579	15.7962	15.5528	14.6505
6	17.9275	17.8960	17.7301	17.5653	17.3150	16.3772
7	19.6349	19.6033	19.4360	19.2681	19.0115	18.0407
8	21.2908	21.2592	21.0902	20.9194	20.6567	19.6552
9	22.9050	22.8734	22.7028	22.5289	22.2606	21.2302
10	24.4847	24.4530	24.2806	24.1040	23.8301	22.7723
11	26.0351	26.0032	25.8292	25.6499	25.3707	24.2869
12	27.5604	27.5283	27.3525	27.1705	26.8863	25.7775
13	29.0636	29.0313	28.8537	28.6692	28.3801	27.2472
14	30.5472	30.5148	30.3355	30.1485	29.8546	28.6987
15	32.0136	31.9810	31.8000	31.6104	31.3120	30.1337

TABLE A2. 9 . APPROXIMATE UPPER 95TH PERCENTILES FOR THE NINE-DIMENSIONAL EQUI-CORRELATED CHI-SQUARED DISTRIBUTION COMPUTED FROM JOHNSON'S APPROXIMATION. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE COMMON CORRELATION COEFFICIENT BY ρ .

	ρ					
N	0.1	0.2	0.4	0.5	0.6	0.8
1	7.6348	7.5934	7.4082	7.2476	7.0241	6.2839
2	10.3289	10.2921	10.1155	9.9531	9.7189	8.9040
3	12.5524	12.5174	12.3430	12.1775	11.9341	11.0634
4	14.5589	14.5248	14.3506	14.1818	13.9299	13.0124
5	16.4343	16.4007	16.2258	16.0535	15.7941	14.8351
6	18.2196	18.1862	18.0101	17.8345	17.5680	16.5714
7	19.9380	19.9047	19.7271	19.5483	19.2750	18.2438
8	21.6042	21.5709	21.3916	21.2098	20.9301	19.8665
9	23.2281	23.1948	23.0139	22.8289	22.5432	21.4492
10	24.8171	24.7836	24.6009	24.4131	24.1216	22.9987
11	26.3763	26.3427	26.1583	25.9675	25.6705	24.5201
12	27.9099	27.8762	27.6899	27.4964	27.1941	26.0173
13	29.4212	29.3873	29.1992	29.0030	28.6955	27.4935
14	30.9127	30.8785	30.6887	30.4899	30.1774	28.9510
15	32.3865	32.3522	32.1606	31.9592	31.6418	30.3919

TABLE A2.10. APPROXIMATE UPPER 95TH PERCENTILES FOR THE TEN-DIMENSIONAL EQUI-CORRELATED CHI-SQUARED DISTRIBUTION COMPUTED FROM JOHNSON'S APPROXIMATION. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE COMMON CORRELATION COEFFICIENT BY ρ.

	ρ					
N	0.1	0.2	0.4	0.5	0.6	0.8
1	7.8238	7.7801	7.5841	7.4141	7.1775	6.3956
2	10.5385	10.4998	10.3132	10.1414	9.8937	9.0340
3	12.7775	12.7407	12.5567	12.3818	12.1245	11.2067
4	14.7972	14.7614	14.5778	14.3994	14.1334	13.1666
5	16.6842	16.6490	16.4647	16.2828	16.0089	14.9988
6	18.4801	18.4452	18.2597	18.0744	17.7929	16.7436
7	20.2082	20.1734	19.9865	19.7978	19.5094	18.4238
8	21.8835	21.8487	21.6601	21.4682	21.1732	20.0537
9	23.5161	23.4812	23.2909	23.0959	22.7945	21.6432
10	25.1131	25.0782	24.8861	24.6880	24.3806	23.1991
11	26.6801	26.6450	26.4512	26.2501	25.9369	24.7267
12	28.2211	28.1859	27.9902	27.7862	27.4675	26.2296
13	29.7395	29.7041	29.5066	29.2998	28.9756	27.7113
14	31.2379	31.2022	31.0029	30.7934	30.4640	29.1742
15	32.7183	32.6826	32.4814	32.2692	31.9347	30.6203

TABLE A2.11. APPROXIMATE UPPER 99TH PERCENTILES FOR THE THREE-DIMENSIONAL EQUI-CORRELATED CHI-SQUARED DISTRIBUTION COMPUTED FROM JOHNSON'S APPROXIMATION. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE COMMON CORRELATION COEFFICIENT BY ρ.

	ρ					
N	0.1	0.2	0.4	0.5	0.6	0.8
1	8.6066	8.5973	8.5487	8.4989	8.4211	8.1100
2	11.3986	11.3910	11.3474	11.3000	11.2226	10.8950
3	13.6972	13.6902	13.6486	13.6020	13.5240	13.1827
4	15.7675	15.7609	15.7206	15.6740	15.5951	15.2417
5	17.6998	17.6933	17.6536	17.6069	17.5271	17.1627
6	19.5368	19.5303	19.4909	19.4442	19.3633	18.9887
7	21.3028	21.2966	21.2572	21.2101	21.1282	20.7444
8	23.0137	23.0074	22.9681	22.9209	22.8378	22.4450
9	24.6797	24.6735	24.6341	24.5866	24.5024	24.1011
10	26.3086	26.3025	26.2632	26.2149	26.1300	25.7205
11	27.9059	27.8997	27.8603	27.8119	27.7256	27.3084
12	29.4759	29.4697	29.4303	29.3814	29.2944	28.8695
13	31.0221	31.0160	30.9764	30.9273	30.8391	30.4072
14	32.5476	32.5412	32.5015	32.4518	32.3628	31.9239
15	34.0540	34.0478	34.0078	33.9579	33.8678	33.4221

TABLE A2.12. APPROXIMATE UPPER 99 TH PERCENTILES FOR THE FOUR-DIMENSIONAL EQUI-CORRELATED CHI-SQUARED DISTRIBUTION COMPUTED FROM JOHNSON'S APPROXIMATION. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE COMMON CORRELATION COEFFICIENT BY ρ.

	ρ					
N	0.1	0.2	0.4	0.5	0.6	0.8
1	9.1302	9.1187	9.0556	8.9904	8.8882	8.4830
2	11.9726	11.9630	11.9069	11.8449	11.7436	11.3175
3	14.3098	14.3012	14.2478	14.1867	14.0850	13.6418
4	16.4131	16.4049	16.3530	16.2924	16.1895	15.7309
5	18.3744	18.3666	18.3157	18.2552	18.1510	17.6785
6	20.2380	20.2302	20.1798	20.1192	20.0137	19.5283
7	22.0287	22.0211	21.9710	21.9100	21.8033	21.3060
8	23.7626	23.7550	23.7049	23.6438	23.5356	23.0270
9	25.4503	25.4427	25.3928	25.3313	25.2219	24.7022
10	27.0997	27.0922	27.0422	26.9803	26.8697	26.3397
11	28.7168	28.7091	28.6592	28.5967	28.4849	27.9450
12	30.3055	30.2981	30.2480	30.1850	30.0720	29.5226
13	31.8700	31.8624	31.8122	31.7489	31.6345	31.0759
14	33.4127	33.4051	33.3548	33.2910	33.1753	32.6078
15	34.9361	34.9286	34.8778	34.8137	34.6968	34.1209

TABLE A2.13. APPROXIMATE UPPER 99TH PERCENTILES FOR THE FIVE-DIMENSIONAL EQUI-CORRELATED CHI-SQUARED DISTRIBUTION COMPUTED FROM JOHNSON'S APPROXIMATION. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE COMMON CORRELATION COEFFICIENT BY ρ.

	ρ					
N	0.1	0.2	0.4	0.5	0.6	0.8
1	9.5382	9.5246	9.4498	9.3719	9.2498	8.7684
2	12.4180	12.4069	12.3404	12.2665	12.1454	11.6401
3	14.7841	14.7741	14.7111	14.6385	14.5171	13.9916
4	16.9120	16.9027	16.8416	16.7695	16.6469	16.1036
5	18.8955	18.8862	18.8264	18.7545	18.6304	18.0709
6	20.7790	20.7700	20.7109	20.6388	20.5133	19.9386
7	22.5882	22.5792	22.5205	22.4484	22.3214	21.7327
8	24.3392	24.3305	24.2721	24.1996	24.0711	23.4690
9	26.0433	26.0346	25.9765	25.9035	25.7733	25.1586
10	27.7084	27.6997	27.6414	27.5681	27.4367	26.8097
11	29.3400	29.3314	29.2731	29.1994	29.0664	28.4280
12	30.9431	30.9346	30.8761	30.8018	30.6676	30.0178
13	32.5212	32.5126	32.4541	32.3793	32.2436	31.5831
14	34.0772	34.0687	34.0097	33.9346	33.7972	33.1264
15	35.6130	35.6045	35.5456	35.4697	35.3313	34.6503

TABLE A2.14. APPROXIMATE UPPER 99TH PERCENTILES FOR THE SIX-DIMENSIONAL EQUI-CORRELATED CHI-SQUARED DISTRIBUTION COMPUTED FROM JOHNSON'S APPROXIMATION. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE COMMON CORRELATION COEFFICIENT BY ρ.

	ρ					
N	0.1	0.2	0.4	0.5	0.6	0.8
1	9.8724	9.8572	9.7724	9.6837	9.5446	8.9990
2	12.7820	12.7696	12.6945	12.6102	12.4724	11.9002
3	15.1711	15.1600	15.0889	15.0064	14.8679	14.2732
4	17.3187	17.3082	17.2393	17.1576	17.0182	16.4033
5	19.3195	19.3095	19.2419	19.1604	19.0195	18.3865
6	21.2190	21.2090	21.1426	21.0609	20.9183	20.2685
7	23.0430	23.0333	22.9673	22.8854	22.7412	22.0757
8	24.8079	24.7984	24.7326	24.6505	24.5047	23.8242
9	26.5251	26.5156	26.4503	26.3675	26.2200	25.5254
10	28.2024	28.1930	28.1277	28.0447	27.8953	27.1873
11	29.8461	29.8367	29.7714	29.6877	29.5368	28.8158
12	31.4606	31.4511	31.3857	31.3016	31.1492	30.4155
13	33.0494	33.0400	32.9744	32.8900	32.7361	31.9903
14	34.6157	34.6065	34.5407	34.4555	34.3002	33.5427
15	36.1621	36.1527	36.0868	36.0011	35.8441	35.0753

TABLE A2.15. APPROXIMATE UPPER 99TH PERCENTILES FOR THE SEVEN-DIMENSIONAL EQUI-CORRELATED CHI-SQUARED DISTRIBUTION COMPUTED FROM JOHNSON'S APPROXIMATION. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE COMMON CORRELATION COEFFICIENT BY ρ .

	ρ					
N	0.1	0.2	0.4	0.5	0.6	0.8
1	10.1557	10.1391	10.0456	9.9473	9.7933	9.1923
2	13.0898	13.0764	12.9935	12.9004	12.7480	12.1179
3	15.4979	15.4857	15.4076	15.3165	15.1634	14.5088
4	17.6618	17.6504	17.5748	17.4845	17.3302	16.6537
5	19.6772	19.6662	19.5922	19.5019	19.3463	18.6501
6	21.5897	21.5793	21.5063	21.4161	21.2585	20.5440
7	23.4260	23.4157	23.3434	23.2530	23.0937	22.3619
8	25.2026	25.1923	25.1203	25.0295	24.8686	24.1206
9	26.9305	26.9203	26.8487	26.7576	26.5947	25.8311
10	28.6181	28.6081	28.5365	28.4450	28.2803	27.5020
11	30.2717	30.2617	30.1902	30.0980	29.9316	29.1389
12	31.8954	31.8854	31.8140	31.7212	31.5532	30.7470
13	33.4935	33.4835	33.4119	33.3188	33.1488	32.3294
14	35.0685	35.0585	34.9869	34.8930	34.7215	33.8895
15	36.6233	36.6133	36.5415	36.4471	36.2740	35.4295

TABLE A2.16. APPROXIMATE UPPER 99TH PERCENTILES FOR THE EIGHT-DIMENSIONAL EQUI-CORRELATED CHI-SQUARED DISTRIBUTION COMPUTED FROM JOHNSON'S APPROXIMATION. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE COMMON CORRELATION COEFFICIENT BY ρ.

	ρ					
N	0.1	0.2	0.4	0.5	0.6	0.8
1	10.4017	10.3838	10.2825	10.1756	10.0084	9.3584
2	13.3564	13.3421	13.2523	13.1512	12.9859	12.3046
3	15.7806	15.7676	15.6832	15.5843	15.4183	14.7109
4	17.9585	17.9462	17.8647	17.7666	17.5992	16.8685
5	19.9861	19.9745	19.8945	19.7970	19.6278	18.8759
6	21.9100	21.8987	21.8200	21.7222	21.5515	20.7798
7	23.7568	23.7457	23.6677	23.5700	23.3971	22.6070
8	25.5430	25.5320	25.4548	25.3566	25.1819	24.3741
9	27.2803	27.2695	27.1925	27.0938	26.9173	26.0929
10	28.9768	28.9660	28.8892	28.7900	28.6116	27.7714
11	30.6386	30.6281	30.5511	30.4514	30.2711	29.4158
12	32.2706	32.2600	32.1831	32.0829	31.9007	31.0304
13	33.8763	33.8657	33.7888	33.6880	33.5039	32.6196
14	35.4588	35.4482	35.3714	35.2697	35.0840	34.1862
15	37.0206	37.0103	36.9330	36.8311	36.6434	35.7321

TABLE A2. 17. APPROXIMATE UPPER 99 TH PERCENTILES FOR THE NINE-DIMENSIONAL EQUI-CORRELATED CHI-SQUARED DISTRIBUTION COMPUTED FROM JOHNSON'S APPROXIMATION. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE COMMON CORRELATION COEFFICIENT BY ρ.

	ρ					
N	0.1	0.2	0.4	0.5	0.6	0.8
1	10.6189	10.6000	10.4915	10.3770	10.1976	9.5039
2	13.5916	13.5763	13.4806	13.3723	13.1949	12.4680
3	16.0300	16.0163	15.9261	15.8200	15.6423	14.8876
4	18.2198	18.2069	18.1197	18.0149	17.8356	17.0563
5	20.2581	20.2458	20.1607	20.0562	19.8752	19.0734
6	22.1919	22.1800	22.0963	21.9915	21.8087	20.9860
7	24.0478	24.0362	23.9533	23.8485	23.6636	22.8211
8	25.8425	25.8310	25.7487	25.6437	25.4568	24.5957
9	27.5879	27.5766	27.4946	27.3889	27.2001	26.3214
10	29.2919	29.2808	29.1989	29.0928	28.9020	28.0066
11	30.9611	30.9500	30.8684	30.7617	30.5689	29.6572
12	32.6000	32.5889	32.5072	32.4000	32.2053	31.2779
13	34.2126	34.2014	34.1198	34.0120	33.8151	32.8729
14	35.8015	35.7906	35.7089	35.6003	35.4017	34.4449
15	37.3697	37.3588	37.2769	37.1679	36.9675	35.9965

TABLE A2.18. APPROXIMATE UPPER 99TH PERCENTILES FOR THE TEN-DIMENSIONAL EQUI-CORRELATED CHI-SQUARED DISTRIBUTION COMPUTED FROM JOHNSON'S APPROXIMATION. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE COMMON CORRELATION COEFFICIENT BY ρ.

	ρ					
N	0.1	0.2	0.4	0.5	0.6	0.8
1	10.8133	10.7936	10.6787	10.5571	10.3669	9.6334
2	13.8020	13.7860	13.6845	13.5695	13.3817	12.6132
3	16.2528	16.2383	16.1430	16.0304	15.8419	15.0444
4	18.4531	18.4398	18.3477	18.2362	18.0464	17.2229
5	20.5012	20.4881	20.3983	20.2875	20.0956	19.2485
6	22.4436	22.4312	22.3426	22.2317	22.0378	21.1689
7	24.3075	24.2953	24.2076	24.0966	23.9008	23.0110
8	26.1097	26.0977	26.0108	25.8995	25.7014	24.7922
9	27.8621	27.8504	27.7639	27.6521	27.4520	26.5240
10	29.5730	29.5613	29.4752	29.3628	29.1607	28.2151
11	31.2486	31.2488	31.1508	31.0381	30.8338	29.8711
12	32.8936	32.8823	32.7962	32.6828	32.4764	31.4973
13	34.5123	34.5006	34.4145	34.3006	34.0923	33.0975
14	36.1071	36.0956	36.0095	35.8949	35.6843	34.6745
15	37.6809	37.6694	37.5833	37.4677	37.2556	36.2306

TABLE A2.19. A COMPARISON OF THE 95TH PERCENTILES FOR THE MAXIMUM OF THE EQUI-CORRELATED K-DIMENSIONAL CHI-SQUARED RANDOM VARIABLES WITH ONE DEGREE OF FREEDOM FOR VARIOUS ρ, THE COMMON CORRELATION COEFFICIENT. THE TOP NUMBER IN EACH CELL IS THE VALUE GIVEN BY KRISHNAIAH AND ARMITAGE (1965), AND THE BOTTOM NUMBER IS FOUND USING JOHNSON'S APPROXIMATION.

	ρ					
K	0.1	0.2	0.4	0.5	0.6	0.8
3	5.695	5.675	5.590	5.518	5.418	5.082
	5.695	5.676	5.593	5.522	5.423	5.088
4	6.196	6.171	6.058	5.962	5.830	5.394
	6.197	6.172	6.064	5.971	5.842	5.407
5	6.589	6.558	6.422	6.307	6.149	5.632
	6.589	6.560	6.432	6.321	6.167	5.651
6	6.911	6.876	6.721	6.590	6.409	5.823
	6.912	6.879	6.734	6.608	6.433	5.849
7	7.185	7.146	6.974	6.828	6.628	5.984
	7.186	7.150	6.990	6.851	6.658	6.016
8	7.423	7.381	7.193	7.034	6.817	6.121
	7.424	7.385	7.212	7.062	6.853	6.159
9	7.633	7.588	7.387	7.216	6.983	6.242
	7.635	7.593	7.408	7.248	7.024	6.284
10	7.822	7.774	7.560	7.378	7.132	6.349
	7.824	7.780	7.584	7.414	7.178	6.396

TABLE A2.20. A COMPARISON OF THE 99TH PERCENTILES FOR THE MAXIMUM OF THE EQUI-CORRELATED K-DIMENSIONAL CHI-SQUARED RANDOM VARIABLES WITH ONE DEGREE OF FREEDOM FOR VARIOUS ρ, THE COMMON CORRELATION COEFFICIENT. THE TOP NUMBER IN EACH CELL IS THE VALUE GIVEN BY KRISHNAIAH AND ARMITAGE (1965), AND THE BOTTOM NUMBER IS FOUND USING JOHNSON'S APPROXIMATION.

	ρ					
K	0.1	0.2	0.4	0.5	0.6	0.8
3	8.607 8.607	8.597 8.597	8.548 8.549	8.497 8.498	8.418 8.421	8.106 8.110
4	9.130 9.130	9.118 9.119	9.054 9.056	8.987 8.990	8.882 8.888	8.474 8.483
5	9.538 9.538	9.524 9.525	9.446 9.450	9.366 9.372	9.240 9.250	8.754 8.768
6	9.873 9.872	9.857 9.857	9.768 9.772	9.675 9.684	9.531 9.545	8.980 8.999
7	10.156 10.156	10.139 10.139	10.040 10.046	9.937 9.947	9.777 9.793	9.169 9.192
8	10.402 10.402	10.383 10.384	10.275 10.283	10.163 10.176	9.988 10.008	9.331 9.358
9	10.619 10.619	10.599 10.600	10.483 10.492	10.362 10.377	10.175 10.198	9.473 9.504
10	10.814 10.813	10.793 10.794	10.669 10.679	10.540 10.557	10.341 10.367	9.599 9.633

TABLE A2.21. APPROXIMATE PROBABILITIES THAT THE MAXIMUM OF THE EQUI-CORRELATED TRIVARIATE CHI-SQUARED DISTRIBUTION IS LESS THAN FIFTEEN, FOR VARIOUS CORRELATIONS, ρ , AND DEGREES OF FREEDOM, N. THE TOP NUMBER IN EACH CELL IS THE VALUE GIVEN BY DUTT AND SOMS (1976), AND THE BOTTOM NUMBER IS FOUND USING JOHNSON'S APPROXIMATION.

N	ρ = 0.2	ρ = 0.4	ρ = 0.6
10	0.6600 0.6598	0.6780 0.6775	0.7081 0.7075
20	0.0132 0.0130	0.0205 0.0199	0.0381 0.0372

TABLE A2. 22. LOWER AND UPPER BOUNDS FOR THE PROBABILITIES CORRESPONDING TO THE APPROXIMATE PERCENTILES REPORTED IN TABLE A2. 3 . THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE CORRELATION COEFFICIENT BY ρ .

LOWER BOUNDS

	ρ					
N	0.1	0.2	0.4	0.5	0.6	0.8
1	0.949665	0.949506	0.948873	0.948399	0.947819	0.946450
2	0.949673	0.949539	0.948977	0.948526	0.947956	0.946524
3	0.949678	0.949558	0.949029	0.948593	0.948031	0.946566
4	0.949682	0.949568	0.949063	0.948639	0.948081	0.946595
5	0.949682	0.949576	0.949088	0.948670	0.948115	0.946618
6	0.949684	0.949581	0.949106	0.948694	0.948144	0.946634
7	0.949686	0.949586	0.949120	0.948714	0.948165	0.946649
8	0.949687	0.949589	0.949133	0.948729	0.948184	0.946660
9	0.949688	0.949593	0.949143	0.948745	0.948200	0.946671
10	0.949688	0.949594	0.949152	0.948756	0.948214	0.946679
11	0.949689	0.949596	0.949158	0.948767	0.948225	0.946687
12	0.949690	0.949600	0.949166	0.948775	0.948236	0.946695
13	0.949690	0.949600	0.949171	0.948784	0.948246	0.946701
14	0.949689	0.949602	0.949177	0.948792	0.948255	0.946707
15	0.949691	0.949604	0.949182	0.948798	0.948262	0.946711

UPPER BOUNDS

1	0.950015	0.950056	0.950362	0.950771	0.951541	0.955809
2	0.950010	0.950036	0.950261	0.950585	0.951234	0.955124
3	0.950010	0.950030	0.950218	0.950503	0.951093	0.954783
4	0.950010	0.950026	0.950193	0.950455	0.951006	0.954566
5	0.950008	0.950024	0.950176	0.950421	0.950944	0.954411
6	0.950008	0.950022	0.950164	0.950395	0.950899	0.954290
7	0.950008	0.950022	0.950154	0.950376	0.950861	0.954195
8	0.950008	0.950020	0.950147	0.950359	0.950833	0.954115
9	0.950008	0.950020	0.950141	0.950348	0.950808	0.954048
10	0.950008	0.950018	0.950136	0.950337	0.950787	0.953990
11	0.950008	0.950018	0.950130	0.950328	0.950768	0.953939
12	0.950008	0.950019	0.950128	0.950319	0.950753	0.953895
13	0.950007	0.950017	0.950124	0.950311	0.950738	0.953855
14	0.950006	0.950017	0.950121	0.950306	0.950726	0.953818
15	0.950008	0.950017	0.950119	0.950299	0.950715	0.953785

TABLE A2.23. LOWER AND UPPER BOUNDS FOR THE PROBABILITIES CORRESPONDING TO THE APPROXIMATE PERCENTILES REPORTED IN TABLE A2.4 . THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE CORRELATION COEFFICIENT BY ρ .

LOWER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.949418	0.949102	0.947819	0.946824	0.945579	0.942457
2	0.949436	0.949174	0.948037	0.947104	0.945887	0.942649
3	0.949445	0.949211	0.948148	0.947249	0.946050	0.942755
4	0.949452	0.949231	0.948220	0.947343	0.946159	0.942826
5	0.949455	0.949246	0.948270	0.947410	0.946235	0.942881
6	0.949458	0.949257	0.948308	0.947463	0.946296	0.942923
7	0.949460	0.949267	0.948340	0.947503	0.946345	0.942958
8	0.949462	0.949275	0.948365	0.947537	0.946385	0.942989
9	0.949464	0.949281	0.948386	0.947568	0.946419	0.943015
10	0.949466	0.949285	0.948405	0.947593	0.946451	0.943035
11	0.949467	0.949290	0.948422	0.947616	0.946476	0.943055
12	0.949469	0.949293	0.948435	0.947636	0.946501	0.943072
13	0.949468	0.949298	0.948447	0.947652	0.946521	0.943088
14	0.949469	0.949300	0.948458	0.947667	0.946540	0.943102
15	0.949471	0.949303	0.948468	0.947682	0.946558	0.943115

UPPER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.950030	0.950116	0.950826	0.951819	0.953757	0.962353
2	0.950021	0.950076	0.950590	0.951376	0.952996	0.962117
3	0.950018	0.950062	0.950489	0.951178	0.952640	0.961970
4	0.950017	0.950053	0.950431	0.951058	0.952423	0.961803
5	0.950015	0.950046	0.950391	0.950976	0.952269	0.961384
6	0.950014	0.950043	0.950363	0.950916	0.952155	0.961062
7	0.950013	0.950041	0.950341	0.950867	0.952065	0.960805
8	0.950013	0.950040	0.950324	0.950829	0.951990	0.960592
9	0.950014	0.950038	0.950309	0.950799	0.951929	0.960412
10	0.950014	0.950035	0.950298	0.950772	0.951878	0.960256
11	0.950014	0.950035	0.950289	0.950750	0.951831	0.960121
12	0.950014	0.950033	0.950279	0.950730	0.951793	0.960000
13	0.950013	0.950034	0.950271	0.950711	0.951757	0.959893
14	0.950013	0.950032	0.950264	0.950695	0.951725	0.959796
15	0.950014	0.950032	0.950258	0.950681	0.951697	0.959708

TABLE A2.24. LOWER AND UPPER BOUNDS FOR THE PROBABILITIES CORRESPONDING TO THE APPROXIMATE PERCENTILES REPORTED IN TABLE A2.5. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE CORRELATION COEFFICIENT BY ρ.

LOWER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.949235	0.948783	0.946903	0.945412	0.943507	0.938508
2	0.949264	0.948891	0.947232	0.945839	0.943989	0.938837
3	0.949278	0.948942	0.947401	0.946062	0.944241	0.939019
4	0.949288	0.948975	0.947506	0.946202	0.944407	0.939142
5	0.949294	0.948997	0.947581	0.946305	0.944526	0.939233
6	0.949298	0.949014	0.947640	0.946383	0.944621	0.939303
7	0.949300	0.949027	0.947686	0.946446	0.944697	0.939363
8	0.949303	0.949038	0.947725	0.946499	0.944760	0.939413
9	0.949307	0.949047	0.947756	0.946544	0.944812	0.939454
10	0.949307	0.949056	0.947783	0.946582	0.944858	0.939491
11	0.949310	0.949061	0.947806	0.946615	0.944899	0.939523
12	0.949310	0.949067	0.947828	0.946644	0.944935	0.939553
13	0.949312	0.949071	0.947846	0.946671	0.944969	0.939580
14	0.949313	0.949076	0.947863	0.946696	0.944997	0.939603
15	0.949314	0.949081	0.947877	0.946718	0.945024	0.939626

UPPER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.950042	0.950175	0.951332	0.953011	0.956373	0.964757
2	0.950028	0.950113	0.950945	0.952266	0.955061	0.964489
3	0.950024	0.950089	0.950782	0.951933	0.954448	0.964321
4	0.950023	0.950077	0.950685	0.951731	0.954072	0.964198
5	0.950022	0.950068	0.950620	0.951594	0.953807	0.964102
6	0.950020	0.950063	0.950574	0.951492	0.953611	0.964023
7	0.950018	0.950059	0.950539	0.951412	0.953455	0.963958
8	0.950018	0.950056	0.950511	0.951349	0.953329	0.963901
9	0.950019	0.950053	0.950486	0.951296	0.953222	0.963850
10	0.950017	0.950052	0.950467	0.951251	0.953132	0.963806
11	0.950018	0.950049	0.950450	0.951213	0.953054	0.963767
12	0.950016	0.950048	0.950436	0.951179	0.952985	0.963731
13	0.950017	0.950046	0.950423	0.951150	0.952926	0.963699
14	0.950017	0.950045	0.950412	0.951124	0.952871	0.963669
15	0.950017	0.950045	0.950401	0.951101	0.952822	0.963642

TABLE A2.25. LOWER AND UPPER BOUNDS FOR THE PROBABILITIES CORRESPONDING TO THE APPROXIMATE PERCENTILES REPORTED IN TABLE A2.6 . THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE CORRELATION COEFFICIENT BY ρ .

LOWER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.949098	0.948525	0.946105	0.944152	0.941614	0.934708
2	0.949136	0.948665	0.946540	0.944721	0.942265	0.935184
3	0.949153	0.948733	0.946757	0.945013	0.942604	0.935447
4	0.949165	0.948774	0.946896	0.945198	0.942826	0.935623
5	0.949173	0.948804	0.946995	0.945334	0.942986	0.935755
6	0.949177	0.948824	0.947072	0.945438	0.943112	0.935858
7	0.949183	0.948840	0.947132	0.945521	0.943214	0.935941
8	0.949185	0.948854	0.947181	0.945591	0.943297	0.936010
9	0.949188	0.948867	0.947221	0.945650	0.943368	0.936072
10	0.949191	0.948875	0.947256	0.945699	0.943431	0.936126
11	0.949194	0.948884	0.947288	0.945743	0.943486	0.936171
12	0.949195	0.948890	0.947314	0.945784	0.943534	0.936213
13	0.949197	0.948897	0.947340	0.945819	0.943578	0.936250
14	0.949198	0.948904	0.947361	0.945850	0.943617	0.936284
15	0.949199	0.948909	0.947380	0.945879	0.943654	0.936315

UPPER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.950053	0.950233	0.951858	0.954288	0.959257	0.966727
2	0.950037	0.950149	0.951313	0.953215	0.957327	0.966442
3	0.950030	0.950117	0.951080	0.952734	0.956426	0.966261
4	0.950028	0.950099	0.950944	0.952443	0.955871	0.966127
5	0.950026	0.950090	0.950854	0.952246	0.955484	0.966022
6	0.950023	0.950081	0.950789	0.952099	0.955194	0.965936
7	0.950023	0.950075	0.950739	0.951985	0.954966	0.965863
8	0.950022	0.950071	0.950698	0.951894	0.954778	0.965799
9	0.950022	0.950068	0.950664	0.951818	0.954623	0.965744
10	0.950021	0.950064	0.950635	0.951752	0.954491	0.965696
11	0.950021	0.950062	0.950613	0.951697	0.954377	0.965651
12	0.950020	0.950059	0.950592	0.951650	0.954276	0.965611
13	0.950021	0.950058	0.950575	0.951608	0.954187	0.965575
14	0.950020	0.950058	0.950559	0.951570	0.954108	0.965541
15	0.950020	0.950056	0.950544	0.951536	0.954037	0.965511

TABLE A2.26. LOWER AND UPPER BOUNDS FOR THE PROBABILITIES CORRESPONDING TO THE APPROXIMATE PERCENTILES REPORTED IN TABLE A2. 7. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE CORRELATION COEFFICIENT BY ρ.

LOWER BOUNDS

N	ρ = 0.1	0.2	0.4	0.5	0.6	0.8
1	0.948992	0.948310	0.945404	0.943022	0.939881	0.931075
2	0.949035	0.948479	0.945932	0.943721	0.940690	0.931703
3	0.949058	0.948561	0.946199	0.944076	0.941113	0.932048
4	0.949071	0.948610	0.946368	0.944308	0.941388	0.932282
5	0.949080	0.948645	0.946488	0.944471	0.941589	0.932452
6	0.949086	0.948671	0.946579	0.944599	0.941747	0.932587
7	0.949092	0.948691	0.946651	0.944702	0.941870	0.932697
8	0.949096	0.948707	0.946710	0.944787	0.941975	0.932789
9	0.949099	0.948719	0.946760	0.944859	0.942065	0.932869
10	0.949102	0.948731	0.946805	0.944921	0.942142	0.932936
11	0.949103	0.948742	0.946842	0.944974	0.942209	0.932998
12	0.949107	0.948751	0.946875	0.945023	0.942269	0.933052
13	0.949109	0.948759	0.946905	0.945066	0.942323	0.933102
14	0.949110	0.948765	0.946931	0.945105	0.942372	0.933146
15	0.949111	0.948771	0.946955	0.945139	0.942417	0.933188

UPPER BOUNDS

N	0.1	0.2	0.4	0.5	0.6	0.8
1	0.950065	0.950287	0.952393	0.955621	0.962338	0.968382
2	0.950042	0.950181	0.951684	0.954200	0.959736	0.968088
3	0.950036	0.950142	0.951382	0.953562	0.958523	0.967898
4	0.950032	0.950120	0.951207	0.953181	0.957777	0.967758
5	0.950030	0.950106	0.951089	0.952917	0.957257	0.967647
6	0.950027	0.950097	0.951003	0.952724	0.956867	0.967555
7	0.950027	0.950090	0.950936	0.952574	0.956559	0.967477
8	0.950026	0.950085	0.950884	0.952452	0.956308	0.967410
9	0.950025	0.950079	0.950841	0.952352	0.956100	0.967351
10	0.950024	0.950076	0.950807	0.952267	0.955922	0.967298
11	0.950023	0.950074	0.950775	0.952194	0.955768	0.967251
12	0.950024	0.950072	0.950749	0.952131	0.955633	0.967208
13	0.950024	0.950070	0.950726	0.952075	0.955513	0.967169
14	0.950022	0.950068	0.950705	0.952025	0.955407	0.967133
15	0.950022	0.950066	0.950687	0.951980	0.955311	0.967100

TABLE A2.27. LOWER AND UPPER BOUNDS FOR THE PROBABILITIES CORRESPONDING TO THE APPROXIMATE PERCENTILES REPORTED IN TABLE A2.8. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE CORRELATION COEFFICIENT BY ρ.

LOWER BOUNDS

	ρ					
N	0.1	0.2	0.4	0.5	0.6	0.8
1	0.948903	0.948127	0.944778	0.941995	0.938282	0.927600
2	0.948954	0.948322	0.945395	0.942815	0.939245	0.928386
3	0.948978	0.948417	0.945704	0.943235	0.939746	0.928816
4	0.948994	0.948472	0.945899	0.943504	0.940073	0.929102
5	0.949006	0.948514	0.946038	0.943697	0.940312	0.929315
6	0.949012	0.948544	0.946144	0.943847	0.940495	0.929484
7	0.949019	0.948565	0.946229	0.943967	0.940646	0.929619
8	0.949025	0.948586	0.946297	0.944068	0.940769	0.929733
9	0.949028	0.948601	0.946358	0.944150	0.940876	0.929831
10	0.949030	0.948615	0.946406	0.944223	0.940965	0.929915
11	0.949032	0.948624	0.946451	0.944287	0.941046	0.929993
12	0.949036	0.948635	0.946489	0.944343	0.941118	0.930060
13	0.949039	0.948644	0.946523	0.944393	0.941182	0.930118
14	0.949039	0.948652	0.946554	0.944439	0.941239	0.930175
15	0.949042	0.948660	0.946583	0.944479	0.941292	0.930224

UPPER BOUNDS

1	0.950074	0.950339	0.952933	0.956990	0.965075	0.969797
2	0.950047	0.950212	0.952058	0.955208	0.962256	0.969500
3	0.950038	0.950165	0.951684	0.954411	0.960713	0.969306
4	0.950034	0.950138	0.951467	0.953932	0.959764	0.969161
5	0.950033	0.950123	0.951321	0.953603	0.959103	0.969046
6	0.950030	0.950113	0.951215	0.953361	0.958605	0.968951
7	0.950030	0.950103	0.951134	0.953172	0.958216	0.968870
8	0.950029	0.950098	0.951069	0.953021	0.957897	0.968800
9	0.950027	0.950093	0.951018	0.952894	0.957632	0.968738
10	0.950026	0.950089	0.950972	0.952788	0.957405	0.968682
11	0.950025	0.950084	0.950935	0.952698	0.957210	0.968634
12	0.950026	0.950082	0.950902	0.952618	0.957039	0.968589
13	0.950026	0.950079	0.950873	0.952548	0.956887	0.968547
14	0.950024	0.950077	0.950848	0.952486	0.956751	0.968510
15	0.950024	0.950076	0.950827	0.952429	0.956629	0.968475

TABLE A2.28. LOWER AND UPPER BOUNDS FOR THE PROBABILITIES CORRESPONDING TO THE APPROXIMATE PERCENTILES REPORTED IN TABLE A2. 9. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE CORRELATION COEFFICIENT BY ρ.

LOWER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.948831	0.947968	0.944212	0.941055	0.936801	0.924271
2	0.948887	0.948186	0.944910	0.941990	0.937909	0.925218
3	0.948917	0.948292	0.945258	0.942466	0.938486	0.925731
4	0.948933	0.948356	0.945479	0.942774	0.938859	0.926074
5	0.948944	0.948400	0.945637	0.942993	0.939134	0.926328
6	0.948953	0.948433	0.945757	0.943163	0.939347	0.926527
7	0.948960	0.948460	0.945852	0.943300	0.939516	0.926690
8	0.948966	0.948482	0.945930	0.943414	0.939659	0.926825
9	0.948968	0.948498	0.945997	0.943508	0.939779	0.926942
10	0.948973	0.948513	0.946053	0.943592	0.939884	0.927044
11	0.948976	0.948526	0.946103	0.943663	0.939976	0.927133
12	0.948978	0.948538	0.946146	0.943727	0.940059	0.927213
13	0.948982	0.948548	0.946185	0.943784	0.940132	0.927287
14	0.948984	0.948555	0.946220	0.943836	0.940198	0.927352
15	0.948985	0.948564	0.946251	0.943882	0.940257	0.927411

UPPER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.950083	0.950388	0.953472	0.958386	0.966199	0.971025
2	0.950052	0.950240	0.952429	0.956234	0.964865	0.970731
3	0.950044	0.950186	0.951984	0.955271	0.962976	0.970534
4	0.950038	0.950156	0.951725	0.954693	0.961813	0.970387
5	0.950034	0.950137	0.951552	0.954296	0.961004	0.970269
6	0.950033	0.950125	0.951426	0.954004	0.960397	0.970172
7	0.950032	0.950116	0.951329	0.953777	0.959918	0.970088
8	0.950031	0.950109	0.951252	0.953594	0.959530	0.970016
9	0.950029	0.950103	0.951190	0.953441	0.959204	0.969952
10	0.950029	0.950098	0.951137	0.953314	0.958928	0.969896
11	0.950028	0.950094	0.951093	0.953203	0.958689	0.969844
12	0.950027	0.950091	0.951053	0.953108	0.958481	0.969798
13	0.950028	0.950089	0.951020	0.953023	0.958295	0.969756
14	0.950027	0.950085	0.950990	0.952949	0.958129	0.969717
15	0.950026	0.950083	0.950963	0.952881	0.957979	0.969680

TABLE A2.29. LOWER AND UPPER BOUNDS FOR THE PROBABILITIES CORRESPONDING TO THE APPROXIMATE PERCENTILES REPORTED IN TABLE A2.10. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE CORRELATION COEFFICIENT BY ρ.

LOWER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.948768	0.947827	0.943694	0.940189	0.935417	0.921078
2	0.948832	0.948070	0.944470	0.941230	0.936663	0.922181
3	0.948863	0.948183	0.944854	0.941761	0.937310	0.922781
4	0.948882	0.948254	0.945101	0.942103	0.937733	0.923180
5	0.948893	0.948303	0.945274	0.942348	0.938040	0.923475
6	0.948903	0.948342	0.945408	0.942539	0.938276	0.923706
7	0.948910	0.948369	0.945513	0.942690	0.938470	0.923893
8	0.948916	0.948392	0.945599	0.942815	0.938629	0.924049
9	0.948922	0.948411	0.945672	0.942923	0.938765	0.924185
10	0.948925	0.948428	0.945735	0.943014	0.938882	0.924303
11	0.948929	0.948441	0.945791	0.943095	0.938987	0.924410
12	0.948931	0.948454	0.945838	0.943164	0.939079	0.924499
13	0.948934	0.948464	0.945882	0.943229	0.939160	0.924583
14	0.948937	0.948473	0.945919	0.943286	0.939235	0.924659
15	0.948938	0.948483	0.945955	0.943338	0.939301	0.924729

UPPER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.950091	0.950436	0.954010	0.959801	0.967215	0.972106
2	0.950058	0.950270	0.952800	0.957272	0.965944	0.971814
3	0.950047	0.950206	0.952282	0.956139	0.965226	0.971618
4	0.950042	0.950173	0.951983	0.955461	0.963913	0.971469
5	0.950037	0.950152	0.951781	0.954995	0.962950	0.971350
6	0.950036	0.950140	0.951635	0.954653	0.962227	0.971251
7	0.950034	0.950128	0.951524	0.954385	0.961660	0.971167
8	0.950033	0.950120	0.951433	0.954169	0.961198	0.971092
9	0.950032	0.950113	0.951360	0.953992	0.960812	0.971028
10	0.950030	0.950108	0.951300	0.953841	0.960483	0.970970
11	0.950030	0.950103	0.951249	0.953713	0.960199	0.970918
12	0.950029	0.950100	0.951203	0.953599	0.959951	0.970869
13	0.950029	0.950096	0.951165	0.953501	0.959730	0.970826
14	0.950030	0.950093	0.951129	0.953413	0.959533	0.970786
15	0.950028	0.950092	0.951099	0.953335	0.959355	0.970749

TABLE A2.30. LOWER AND UPPER BOUNDS FOR THE PROBABILITIES CORRESPONDING TO THE APPROXIMATE PERCENTILES REPORTED IN TABLE A2.11. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE CORRELATION COEFFICIENT BY ρ.

LOWER BOUNDS

N	ρ = 0.1	0.2	0.4	0.5	0.6	0.8
1	0.989984	0.989968	0.989889	0.989817	0.989715	0.989408
2	0.989985	0.989973	0.989907	0.989843	0.989747	0.989437
3	0.989986	0.989975	0.989916	0.989855	0.989763	0.989452
4	0.989986	0.989976	0.989922	0.989863	0.989773	0.989462
5	0.989986	0.989977	0.989926	0.989869	0.989781	0.989470
6	0.989987	0.989978	0.989928	0.989874	0.989787	0.989476
7	0.989986	0.989979	0.989930	0.989876	0.989791	0.989482
8	0.989987	0.989979	0.989932	0.989880	0.989796	0.989486
9	0.989987	0.989979	0.989933	0.989882	0.989799	0.989490
10	0.989987	0.989980	0.989936	0.989884	0.989802	0.989494
11	0.989987	0.989980	0.989936	0.989886	0.989804	0.989496
12	0.989987	0.989980	0.989938	0.989888	0.989807	0.989499
13	0.989986	0.989980	0.989938	0.989890	0.989809	0.989501
14	0.989987	0.989980	0.989939	0.989890	0.989811	0.989503
15	0.989987	0.989980	0.989940	0.989892	0.989812	0.989505

UPPER BOUNDS

N	0.1	0.2	0.4	0.5	0.6	0.8
1	0.990001	0.990001	0.990021	0.990056	0.990136	0.990715
2	0.990000	0.990001	0.990014	0.990040	0.990104	0.990615
3	0.990000	0.990001	0.990011	0.990033	0.990089	0.990564
4	0.990000	0.990000	0.990010	0.990029	0.990080	0.990531
5	0.990000	0.990000	0.990009	0.990026	0.990074	0.990507
6	0.990000	0.990000	0.990008	0.990024	0.990069	0.990489
7	0.990000	0.990000	0.990007	0.990022	0.990065	0.990474
8	0.990000	0.990000	0.990006	0.990021	0.990062	0.990462
9	0.990000	0.990000	0.990005	0.990020	0.990059	0.990451
10	0.990000	0.990001	0.990006	0.990018	0.990057	0.990443
11	0.990000	0.990000	0.990006	0.990018	0.990054	0.990434
12	0.990000	0.990000	0.990005	0.990017	0.990054	0.990427
13	0.990000	0.990000	0.990005	0.990017	0.990052	0.990421
14	0.990000	0.990000	0.990005	0.990016	0.990050	0.990416
15	0.990000	0.990000	0.990005	0.990016	0.990049	0.990410

TABLE A2.31. LOWER AND UPPER BOUNDS FOR THE PROBABILITIES CORRESPONDING TO THE APPROXIMATE PERCENTILES REPORTED IN TABLE A2.12. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE CORRELATION COEFFICIENT BY ρ .

LOWER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.989972	0.989942	0.989787	0.989639	0.989426	0.988756
2	0.989974	0.989951	0.989822	0.989690	0.989491	0.988819
3	0.989975	0.989956	0.989840	0.989715	0.989525	0.988855
4	0.989976	0.989958	0.989850	0.989732	0.989546	0.988878
5	0.989976	0.989960	0.989858	0.989745	0.989563	0.988896
6	0.989977	0.989961	0.989863	0.989753	0.989575	0.988910
7	0.989977	0.989962	0.989868	0.989760	0.989585	0.988922
8	0.989977	0.989963	0.989871	0.989766	0.989593	0.988932
9	0.989977	0.989963	0.989875	0.989772	0.989600	0.988940
10	0.989977	0.989964	0.989877	0.989776	0.989607	0.988948
11	0.989978	0.989964	0.989880	0.989779	0.989612	0.988954
12	0.989977	0.989965	0.989882	0.989782	0.989617	0.988960
13	0.989978	0.989965	0.989883	0.989786	0.989621	0.988965
14	0.989978	0.989965	0.989885	0.989788	0.989625	0.988969
15	0.989978	0.989966	0.989886	0.989791	0.989628	0.988975

UPPER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.990001	0.990004	0.990049	0.990134	0.990333	0.991850
2	0.990000	0.990002	0.990032	0.990095	0.990254	0.991586
3	0.990000	0.990002	0.990025	0.990077	0.990217	0.991453
4	0.990000	0.990001	0.990021	0.990067	0.990193	0.991365
5	0.990000	0.990001	0.990019	0.990061	0.990178	0.991303
6	0.990000	0.990001	0.990017	0.990056	0.990165	0.991254
7	0.990000	0.990001	0.990016	0.990052	0.990156	0.991216
8	0.990000	0.990001	0.990014	0.990049	0.990148	0.991183
9	0.990000	0.990001	0.990013	0.990046	0.990142	0.991155
10	0.990000	0.990001	0.990012	0.990044	0.990137	0.991132
11	0.990001	0.990000	0.990012	0.990042	0.990132	0.991111
12	0.990000	0.990001	0.990011	0.990040	0.990128	0.991093
13	0.990000	0.990000	0.990011	0.990039	0.990124	0.991076
14	0.990000	0.990000	0.990011	0.990038	0.990121	0.991061
15	0.990001	0.990001	0.990010	0.990036	0.990118	0.991048

TABLE A2.32. LOWER AND UPPER BOUNDS FOR THE PROBABILITIES CORRESPONDING TO THE APPROXIMATE PERCENTILES REPORTED IN TABLE A2.13. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE CORRELATION COEFFICIENT BY ρ.

LOWER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.989963	0.989920	0.989697	0.989479	0.989161	0.988120
2	0.989967	0.989934	0.989748	0.989555	0.989257	0.988222
3	0.989968	0.989940	0.989773	0.989592	0.989308	0.988278
4	0.989968	0.989944	0.989789	0.989616	0.989341	0.988315
5	0.989970	0.989946	0.989800	0.989634	0.989364	0.988343
6	0.989970	0.989948	0.989809	0.989647	0.989383	0.988365
7	0.989970	0.989949	0.989814	0.989658	0.989398	0.988384
8	0.989970	0.989950	0.989820	0.989667	0.989411	0.988399
9	0.989970	0.989951	0.989825	0.989674	0.989421	0.988412
10	0.989971	0.989952	0.989828	0.989681	0.989431	0.988424
11	0.989971	0.989952	0.989832	0.989686	0.989438	0.988434
12	0.989971	0.989953	0.989834	0.989691	0.989446	0.988443
13	0.989971	0.989954	0.989837	0.989696	0.989452	0.988451
14	0.989972	0.989955	0.989839	0.989700	0.989457	0.988459
15	0.989971	0.989954	0.989841	0.989702	0.989463	0.988465

UPPER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.990001	0.990006	0.990079	0.990223	0.990566	0.992785
2	0.990001	0.990003	0.990052	0.990158	0.990429	0.992694
3	0.990000	0.990002	0.990040	0.990129	0.990366	0.992569
4	0.990000	0.990002	0.990034	0.990111	0.990327	0.992413
5	0.990001	0.990001	0.990030	0.990100	0.990299	0.992300
6	0.990001	0.990001	0.990027	0.990091	0.990279	0.992212
7	0.990000	0.990001	0.990024	0.990085	0.990263	0.992142
8	0.990000	0.990001	0.990022	0.990080	0.990250	0.992084
9	0.990000	0.990000	0.990021	0.990075	0.990238	0.992034
10	0.990001	0.990001	0.990020	0.990072	0.990230	0.991992
11	0.990000	0.990000	0.990019	0.990069	0.990221	0.991955
12	0.990000	0.990001	0.990018	0.990066	0.990215	0.991921
13	0.990000	0.990001	0.990017	0.990063	0.990209	0.991892
14	0.990001	0.990001	0.990016	0.990062	0.990202	0.991865
15	0.990000	0.990001	0.990016	0.990059	0.990198	0.991840

TABLE A2.33. LOWER AND UPPER BOUNDS FOR THE PROBABILITIES CORRESPONDING TO THE APPROXIMATE PERCENTILES REPORTED IN TABLE A2.14. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE CORRELATION COEFFICIENT BY ρ.

LOWER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.989956	0.989902	0.989618	0.989336	0.988919	0.987516
2	0.989961	0.989920	0.989685	0.989434	0.989045	0.987655
3	0.989962	0.989928	0.989717	0.989483	0.989111	0.987731
4	0.989963	0.989932	0.989736	0.989514	0.989155	0.987782
5	0.989964	0.989936	0.989750	0.989537	0.989187	0.987821
6	0.989965	0.989937	0.989761	0.989554	0.989210	0.987851
7	0.989965	0.989939	0.989770	0.989568	0.989230	0.987877
8	0.989965	0.989941	0.989776	0.989579	0.989247	0.987897
9	0.989965	0.989942	0.989783	0.989589	0.989260	0.987916
10	0.989965	0.989942	0.989787	0.989598	0.989272	0.987931
11	0.989966	0.989944	0.989792	0.989604	0.989282	0.987945
12	0.989967	0.989944	0.989795	0.989611	0.989292	0.987957
13	0.989966	0.989945	0.989798	0.989617	0.989301	0.987969
14	0.989966	0.989946	0.989801	0.989621	0.989308	0.987979
15	0.989967	0.989946	0.989804	0.989626	0.989315	0.987988

UPPER BOUNDS

N	0.1	0.2	0.4	0.5	0.6	0.8
1	0.990001	0.990007	0.990111	0.990318	0.990822	0.993185
2	0.990001	0.990004	0.990073	0.990225	0.990623	0.993089
3	0.990001	0.990003	0.990057	0.990184	0.990529	0.993028
4	0.990000	0.990002	0.990047	0.990159	0.990473	0.992984
5	0.990000	0.990002	0.990041	0.990142	0.990433	0.992949
6	0.990000	0.990001	0.990037	0.990130	0.990402	0.992920
7	0.990000	0.990001	0.990034	0.990120	0.990379	0.992896
8	0.990000	0.990001	0.990030	0.990113	0.990360	0.992875
9	0.990000	0.990001	0.990029	0.990106	0.990344	0.992857
10	0.990000	0.990001	0.990027	0.990102	0.990330	0.992840
11	0.990000	0.990001	0.990026	0.990096	0.990318	0.992825
12	0.990001	0.990001	0.990025	0.990093	0.990309	0.992811
13	0.990000	0.990001	0.990023	0.990089	0.990300	0.992800
14	0.990000	0.990001	0.990022	0.990086	0.990292	0.992788
15	0.990000	0.990001	0.990021	0.990083	0.990284	0.992752

TABLE A2.34. LOWER AND UPPER BOUNDS FOR THE PROBABILITIES CORRESPONDING TO THE APPROXIMATE PERCENTILES REPORTED IN TABLE A2.15. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE CORRELATION COEFFICIENT BY ρ.

LOWER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.989950	0.989888	0.989548	0.989208	0.988698	0.986944
2	0.989956	0.989909	0.989628	0.989326	0.988853	0.987121
3	0.989958	0.989917	0.989666	0.989386	0.988933	0.987216
4	0.989959	0.989923	0.989690	0.989424	0.988986	0.987281
5	0.989960	0.989927	0.989707	0.989450	0.989024	0.987330
6	0.989960	0.989930	0.989720	0.989471	0.989054	0.987369
7	0.989961	0.989931	0.989730	0.989488	0.989078	0.987400
8	0.989962	0.989933	0.989738	0.989501	0.989098	0.987427
9	0.989962	0.989934	0.989745	0.989513	0.989115	0.987449
10	0.989962	0.989935	0.989751	0.989523	0.989129	0.987469
11	0.989963	0.989937	0.989757	0.989532	0.989142	0.987486
12	0.989962	0.989937	0.989761	0.989539	0.989154	0.987502
13	0.989963	0.989938	0.989765	0.989547	0.989164	0.987516
14	0.989962	0.989938	0.989768	0.989552	0.989173	0.987529
15	0.989963	0.989939	0.989772	0.989558	0.989182	0.987541

UPPER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.990001	0.990009	0.990144	0.990418	0.991095	0.993526
2	0.990001	0.990005	0.990094	0.990296	0.990829	0.993427
3	0.990001	0.990003	0.990073	0.990241	0.990704	0.993363
4	0.990001	0.990003	0.990061	0.990208	0.990628	0.993316
5	0.990001	0.990003	0.990053	0.990185	0.990574	0.993279
6	0.990000	0.990003	0.990048	0.990170	0.990534	0.993249
7	0.990000	0.990002	0.990043	0.990157	0.990502	0.993223
8	0.990001	0.990002	0.990039	0.990146	0.990477	0.993201
9	0.990000	0.990001	0.990037	0.990139	0.990456	0.993181
10	0.990000	0.990002	0.990034	0.990132	0.990438	0.993163
11	0.990001	0.990002	0.990033	0.990126	0.990422	0.993147
12	0.990000	0.990001	0.990031	0.990120	0.990408	0.993133
13	0.990001	0.990001	0.990030	0.990116	0.990396	0.993120
14	0.990000	0.990001	0.990029	0.990112	0.990385	0.993107
15	0.990000	0.990001	0.990028	0.990108	0.990376	0.993096

TABLE A2.35. LOWER AND UPPER BOUNDS FOR THE PROBABILITIES CORRESPONDING TO THE APPROXIMATE PERCENTILES REPORTED IN TABLE A2.16. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE CORRELATION COEFFICIENT BY ρ .

LOWER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.989947	0.989875	0.989487	0.989092	0.988496	0.986401
2	0.989952	0.989899	0.989577	0.989228	0.988677	0.986614
3	0.989954	0.989908	0.989622	0.989297	0.988770	0.986730
4	0.989956	0.989915	0.989650	0.989341	0.988831	0.986808
5	0.989957	0.989919	0.989668	0.989373	0.988876	0.986866
6	0.989957	0.989922	0.989683	0.989395	0.988910	0.986912
7	0.989958	0.989924	0.989694	0.989416	0.988938	0.986950
8	0.989958	0.989926	0.989705	0.989432	0.988962	0.986982
9	0.989959	0.989928	0.989713	0.989445	0.988982	0.987009
10	0.989959	0.989929	0.989719	0.989457	0.988999	0.987033
11	0.989959	0.989931	0.989725	0.989466	0.989014	0.987055
12	0.989960	0.989932	0.989730	0.989476	0.989027	0.987073
13	0.989960	0.989932	0.989735	0.989484	0.989039	0.987090
14	0.989960	0.989932	0.989739	0.989490	0.989050	0.987107
15	0.989959	0.989934	0.989742	0.989497	0.989059	0.987119

UPPER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.990002	0.990011	0.990178	0.990521	0.991382	0.993820
2	0.990001	0.990006	0.990115	0.990368	0.991045	0.993719
3	0.990000	0.990004	0.990089	0.990299	0.990887	0.993654
4	0.990001	0.990003	0.990075	0.990258	0.990789	0.993606
5	0.990000	0.990003	0.990064	0.990231	0.990721	0.993567
6	0.990000	0.990002	0.990058	0.990209	0.990670	0.993536
7	0.990000	0.990002	0.990052	0.990195	0.990630	0.993509
8	0.990000	0.990001	0.990048	0.990182	0.990598	0.993485
9	0.990000	0.990002	0.990045	0.990172	0.990572	0.993464
10	0.990000	0.990001	0.990042	0.990163	0.990549	0.993446
11	0.990000	0.990002	0.990040	0.990155	0.990529	0.993430
12	0.990001	0.990002	0.990038	0.990149	0.990511	0.993414
13	0.990001	0.990002	0.990036	0.990143	0.990496	0.993400
14	0.990000	0.990001	0.990035	0.990138	0.990483	0.993388
15	0.990000	0.990002	0.990033	0.990134	0.990471	0.993374

TABLE A2.36. LOWER AND UPPER BOUNDS FOR THE PROBABILITIES CORRESPONDING TO THE APPROXIMATE PERCENTILES REPORTED IN TABLE A2.17. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE CORRELATION COEFFICIENT BY ρ.

LOWER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.989943	0.989865	0.989430	0.988986	0.988307	0.985885
2	0.989949	0.989890	0.989532	0.989140	0.988513	0.986133
3	0.989952	0.989902	0.989582	0.989216	0.988620	0.986268
4	0.989953	0.989908	0.989612	0.989266	0.988689	0.986360
5	0.989954	0.989913	0.989634	0.989301	0.988739	0.986428
6	0.989954	0.989916	0.989651	0.989328	0.988778	0.986482
7	0.989955	0.989919	0.989664	0.989350	0.988811	0.986525
8	0.989955	0.989920	0.989674	0.989368	0.988837	0.986562
9	0.989956	0.989923	0.989683	0.989382	0.988859	0.986594
10	0.989956	0.989924	0.989691	0.989395	0.988879	0.986622
11	0.989957	0.989925	0.989698	0.989407	0.988896	0.986647
12	0.989957	0.989926	0.989703	0.989417	0.988911	0.986663
13	0.989958	0.989927	0.989708	0.989426	0.988924	0.986688
14	0.989957	0.989928	0.989713	0.989434	0.988936	0.986706
15	0.989957	0.989929	0.989717	0.989442	0.988948	0.986722

UPPER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.990002	0.990013	0.990211	0.990627	0.991679	0.994077
2	0.990001	0.990006	0.990137	0.990442	0.991268	0.993976
3	0.990001	0.990005	0.990106	0.990359	0.991075	0.993910
4	0.990001	0.990004	0.990088	0.990310	0.990956	0.993861
5	0.990000	0.990003	0.990076	0.990276	0.990873	0.993822
6	0.990000	0.990002	0.990069	0.990251	0.990811	0.993789
7	0.990000	0.990003	0.990063	0.990233	0.990763	0.993761
8	0.990000	0.990002	0.990057	0.990218	0.990724	0.993737
9	0.990000	0.990002	0.990054	0.990204	0.990691	0.993715
10	0.990000	0.990002	0.990050	0.990194	0.990663	0.993697
11	0.990000	0.990002	0.990048	0.990185	0.990639	0.993679
12	0.990000	0.990002	0.990045	0.990178	0.990617	0.993663
13	0.990001	0.990002	0.990043	0.990171	0.990599	0.993649
14	0.990000	0.990002	0.990041	0.990165	0.990583	0.993636
15	0.990000	0.990002	0.990039	0.990159	0.990569	0.993622

TABLE A2.37. LOWER AND UPPER BOUNDS FOR THE PROBABILITIES CORRESPONDING TO THE APPROXIMATE PERCENTILES REPORTED IN TABLE A2.18. THE DEGREES OF FREEDOM ARE GIVEN BY N AND THE CORRELATION COEFFICIENT BY ρ .

LOWER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.989939	0.989855	0.989379	0.988888	0.988134	0.985393
2	0.989946	0.989883	0.989490	0.989056	0.988362	0.985675
3	0.989949	0.989895	0.989545	0.989141	0.988479	0.985829
4	0.989951	0.989903	0.989579	0.989196	0.988557	0.985933
5	0.989952	0.989907	0.989603	0.989236	0.988613	0.986011
6	0.989953	0.989911	0.989621	0.989265	0.988656	0.986072
7	0.989953	0.989914	0.989635	0.989289	0.988692	0.986122
8	0.989953	0.989916	0.989647	0.989309	0.988720	0.986164
9	0.989954	0.989918	0.989657	0.989326	0.988746	0.986199
10	0.989954	0.989919	0.989665	0.989340	0.988768	0.986232
11	0.989954	0.989955	0.989672	0.989353	0.988787	0.986259
12	0.989954	0.989922	0.989679	0.989364	0.988803	0.986284
13	0.989955	0.989922	0.989684	0.989374	0.988819	0.986307
14	0.989955	0.989924	0.989689	0.989383	0.988832	0.986328
15	0.989956	0.989924	0.989694	0.989390	0.988845	0.986345

UPPER BOUNDS

N \ ρ	0.1	0.2	0.4	0.5	0.6	0.8
1	0.990001	0.990015	0.990245	0.990734	0.991985	0.994305
2	0.990001	0.990008	0.990158	0.990516	0.991497	0.994203
3	0.990001	0.990005	0.990123	0.990419	0.991268	0.994138
4	0.990000	0.990005	0.990102	0.990361	0.991127	0.994088
5	0.990001	0.990003	0.990088	0.990323	0.991030	0.994048
6	0.990001	0.990003	0.990079	0.990294	0.990956	0.994015
7	0.990001	0.990003	0.990071	0.990271	0.990898	0.993987
8	0.990000	0.990002	0.990066	0.990254	0.990851	0.993962
9	0.990000	0.990002	0.990061	0.990238	0.990813	0.993940
10	0.990000	0.990002	0.990057	0.990226	0.990780	0.993921
11	0.990000	0.990001	0.990054	0.990216	0.990752	0.993903
12	0.990000	0.990002	0.990051	0.990207	0.990727	0.993887
13	0.990001	0.990001	0.990048	0.990199	0.990704	0.993872
14	0.990000	0.990002	0.990047	0.990192	0.990685	0.993859
15	0.990000	0.990002	0.990045	0.990186	0.990668	0.993844

The vita has been removed from the scanned document

ON MONITORING THE

ATTRIBUTES OF A PROCESS

by

Mark O. Marcucci

(ABSTRACT)

Two prominent monitoring procedures in statistical quality control are the p-chart for the proportion of items defective, and the c-chart, for the number of defects per item. These procedures are reconsidered, and some extensions are examined for monitoring processes with multiple attributes.

Some relevant distribution theory is reviewed, and some new results are given. The distributions considered are multivariate versions of the binomial, Poisson, and chi-squared distributions, plus univariate and multivariate generalized Poisson distributions. All of these distributions prove useful in the discussion of attribute control charts.

When quality standards are known, p-charts and c-charts are shown to have certain optimal properties. Generalized p-charts, for monitoring multinomial processes, and generalized c-charts are introduced. Their properties are shown to depend upon multivariate chi-squared and generalized Poisson distributions, respectively.

Various techniques are considered for monitoring multivariate Bernoulli, Poisson, multinomial, and generalized Poisson processes. Omnibus procedures are given, and some of their asymptotic properties are derived. Also examined are diagnostic procedures based upon both small- and large-sample.