# ASYMPTOTIC STOCHASTIC ANALYSIS OF A GRAVITY MODEL FOR INERTIAL NAVIGATION SYSTEMS

by

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# I. Introduction

Mathematical models of the earth's gravity field are required in the operation of inertial navigation systems. Inaccuracies in these models (due to an inability to adequately account for the fine scale structure of the gravity field) are an important source of error [1]. Since the structure of the small scale (short spatial wavelength) variations of the gravity field is "noiselike" and difficult to model adequately, probabilistic descriptions are often used [2] - [6]. The purpose of such stochastic modeling is to provide a framework wherein statistical error estimates can be made and errors minimized using the techniques of estimation theory.

Much of modern estimation theory involves differential equations driven by white noise [1]. The mathematics for such statistical models has been extensively developed. It is not surprising, therefore, that much of the early probabilistic modeling of gravity field errors has involved ad hoc stationary, isotropic white noise assumptions in order to utilize the existing mathematical machinery. Unfortunately, these assumptions are idealizations that do not relate well to physical reality [1]. The objective of this thesis is to develop a probabilistic model for the gravity field that does not require these idealizing assumptions and also proves amenable to asymptotic stochastic analysis.

Three theorems are proven which permit an error analysis of inertial navigation systems; a gravity model is used which is not necessarily stationary or isotropic. The gravity field is assumed to consist of two parts, a large scale (long spatial wavelength) component that is assumed to be deterministic (i.e., known) and a small scale (unknown) probabilistic component. On physical grounds, one would expect that the large scale structure of the earth's gravity field is determined by similarly large (i.e., global) features of the earth while the small scale, short wavelength components arise from local topographic and density variations. Guided by this proposition, we assume that the small scale random component of the gravity field possesses a mixing (asymptotic independence) property. Otherwise, however, this random component need be neither stationary nor isotropic; its statistics can undergo large scale spatial modulation. Although small in amplitude, this rapidly varying random component exerts a nontrivial cumulative effect over long paths (e.g., vehicle trajectories). It enters the analysis as a stochastic perturbation to a deterministic initial value problem incorporating the known large scale variations of the gravity field. The three theorems presented in Chapter III conclude that in an appropriate asymptotic limit, vehicular motion through the gravity field may be approximated by a deterministic centering trajectory (determined by the known large scale variations of the gravity field) plus a small zero mean Gauss-Markov fluctuation process. The covariance matrix of this limiting Gaussian process can be computed in terms of the centering trajectory and the two point statistics of the random gravity field.

This general type of asymptotic analysis of stochastic differential equations was widely developed on a physical level by Stratonovich [7] and later made rigorous by Khas'minskii [8]. White [9] used a similar analysis for stochastic delay-differential equations. The main difference between these prior works and this analysis is our consideration of motion through random fields. Related asymptotic analyses leading to limiting diffusion processes but involving fundamentally different time scales have been developed by Papanicolaou and Keller [10], Papanicolaou and Varadhan [11], Kohler and Papanicolaou [12], Borodin [13], and Kesten and Papanicolaou [14], [15].

Our results are presented in the following manner. The basic problem is delineated in Chapter II. Relevant physical ideas are discussed, the basic scalings and hypotheses are introduced, and the stochastic initial value problem is abstracted. The three asymptotic limit theorems are proven in Chapter III while Chapter IV contains some concluding remarks concerning the results.

# II. Problem Formulation

## A. Inertial Navigation Systems:

Navigation is the determination of the position of moving vehicles relative to the earth. Inertial navigation systems perform this function by mounting instruments such as gyroscopes and accelerometers on the vehicle to measure rotational velocity and acceleration. Vehicle attitude is then computed by integrating the angular velocity; position and velocity are subsequently computed by integrating the acceleration.

Acceleration is measured by comparing the acceleration of a suspended mass to the acceleration of the instrument case. This difference in acceleration is called the specific force. Since the acceleration due to gravity affects both the instrument case and the test mass, it needs to be added to the specific force to obtain the total acceleration of the vehicle. (Gravity, as used here, includes not only the force due to the earth's gravitional field, but also the centrifugal force due to the earth's rotation.) As mentioned before, this total acceleration vector is integrated once to obtain the velocity vector which, in turn, is integrated again to obtain the position vector.

The acceleration due to gravity is stored onboard the vehicle in the form of a computerized mathematical model. Input to this model is position in three-dimensional space while the output is a vector approximating the acceleration due to gravity at the prescribed position. Inertial navigation systems have been developed that are quite accurate, enough so that inaccuracies in the gravity model can be a significant source of error. The errors in the gravity model will not only accumulate with time but will also have a second order effect. Gravity model errors lead in turn to inaccurate knowledge of position. Therefore, the navigation system generally will find itself not only using an imperfect gravity model but also evaluating this model at the wrong position.

There are several types of inertial navigation systems and the equations describing the effect of gravity errors are not the same for every system. For the purpose of discussion, we shall focus upon one particular type of system, the space-stabilized inertial navigation system with altitude damping. However, the analysis used will be general in nature and should be applicable to other types of inertial navigation systems as well.

## B. Review of Geopotential Theory:

The earth's gravitational field is a conservative field, i.e., it can be represented as the gradient of a scalar potential. The force due to the earth's rotation is also a conservative field. Consequently, gravity models used in inertial navigation systems usually are represented as gradients of scalar potentials. The gravity model error is also the gradient of a scalar potential, which can ultimately be characterized as the solution of an exterior third boundary value problem of potential theory. The derivation of this problem, i.e., the differential equation and associated boundary condition, along with the introduction of conventional terminology is presented in this section.

Only a brief summary is given; it is thoroughly discussed in Heiskanen and Moritz [16], which has served as the main source for this discussion.

As previously mentioned, both the earth's gravitational field and the force due to the earth's rotation are conservative (curl-free) vector fields. Thus, the earth's gravity field can be represented as the gradient of a scalar potential. Specifically, this potential W is

$$W = V + \frac{1}{2}\,\omega^2 r^2 \cos^2\phi, \tag{2.1}$$

where V is the gravitational potential, $\omega$ is the earth's rate of rotation, r is the radius, and $\phi$ is geocentric latitude. The gravitational force $\nabla V$ is an inverse square force, similar in nature to the electrostatic force. Therefore, the gravitational potential is given by

$$V(x,y,z) = k\iiint \frac{\rho(x',y',z')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{1}{2}}}\,dv' \tag{2.2}$$

where $\rho$ is the mass density and k is Newton's universal gravitational constant. From (2.1) and (2.2) it readily follows that W satisfies the Poisson equation

$$\nabla^2 W = -4\pi k\rho + 2\omega^2. \tag{2.3}$$

Thus, exterior to the surface of the earth (where there is no mass) the gravitational potential satisfies Laplace's equation, while the gravity potential satisfies (2.3) with $\rho = 0$. The problem is to find a surface that encloses all the attracting mass, specify an appropriate boundary condition on this surface and solve the resulting boundary value problem in the region exterior to this surface.

Before discussing the specific formulation of the boundary value problem, some terms will be defined. Surfaces on which the potential is constant are called equipotential surfaces. In particular, the geoid is a name given to that equipotential surface which has the same potential as mean sea level. Because of the earth's density variations and surface topography, the geoid is an irregular surface and hence not very convenient to use. However, ellipsoids of revolution can be determined which closely approximate the actual equipotential surfaces; these are generally used instead as a basis for a coordinate system and gravity computations. The particular ellipsoid of revolution that is used to approximate the geoid is called the reference ellipsoid. (To be more precise, the reference ellipsoid is an oblate spheroid.)

Assume that P is a point on the geoid. Let Q be the point on the reference ellipsoid which lies on the line that passes through P and is perpendicular to the reference ellipsoid (c.f. Figure 1).

Let G be the actual gravity vector and $\Gamma$ be the gravity vector associated with the reference ellipsoid. These vectors are thus perpendicular to the geoid and reference ellipsoid, respectively. A gravity anomaly vector $\delta G$ is defined as the difference between the actual gravity vector at P, G(P), and the gravity model vector at Q, $\Gamma(Q)$; i.e.,

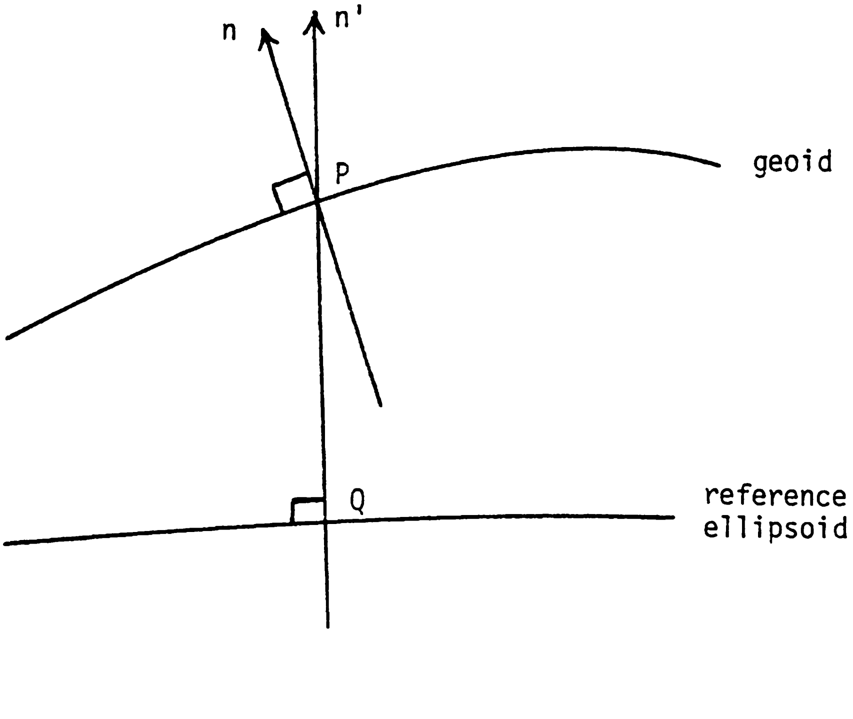
$$\delta G(P) = G(P) - \Gamma(Q). \tag{2.4}$$

The difference in the magnitudes of these gravity vectors is called the gravity anomaly $\Delta g$. Thus

$$\Delta g(P) = g(P) - \gamma(Q) \tag{2.5}$$

where
$$g(P) \equiv |G(P)|$$
$$\gamma(Q) \equiv |\Gamma(Q)|.$$

The difference in direction between the two gravity vectors is the deflection of the vertical. The deflection of the vertical is usually decomposed into a north-south component $\xi$ and an east-west component $\eta$. The distance $|P-Q|$ is called the geoidal undulation N or, sometimes, the geoidal height.




Relationship of Reference Ellipsoid and Geoid

Figure 1

Let W be the actual gravity potential, i.e., $G = \nabla W$, and U be the gravity potential associated with the reference ellipsoid ($\Gamma = \nabla U$). The difference in the two potentials, denoted by T, is called the disturbing potential. Thus

$$T(P) \equiv W(P) - U(P) \tag{2.6}$$

where P denotes a generic point. The geoid is assumed to be a surface enclosing all the mass of the earth and the potential of gravity W is a constant, say $W_0$, on the geoid. The potential U is assumed to have this same constant value $W_0$ on the reference ellipsoid. Therefore, with the additional assumption that the reference ellipsoid encloses all the mass of the earth, the potential of gravity U everywhere exterior to the reference ellipsoid may be determined. Thus U is considered known; the goal is to determine T (and hence W). The gravity anomaly $\Delta g$ can be measured and is thus assumed to be known. The geoid and undulation N are assumed to be unknown. The approach generally used is to derive a third boundary value problem for the disturbing potential T exterior to the geoid. Since the geoid is unknown but closely approximated by the reference ellipsoid, the boundary condition is assumed to hold on the reference ellipsoid. In practice, the boundary value problem for T is solved exterior to this latter surface.

We shall now outline the derivation of this boundary condition. Noting Figure 1, let h denote distance measured along n relative to the intersection point P. Since

$$g = -\frac{\partial W}{\partial n} \quad \text{and} \quad \gamma = -\frac{\partial U}{\partial n'} \cong -\frac{\partial U}{\partial n},$$

we obtain

$$- \frac{\partial T(P)}{\partial h} = g(P) - \gamma(P)$$

or, to first approximation

$$- \frac{\partial T(P)}{\partial h} = g(P) - \gamma(Q) - \frac{\partial \gamma(P)}{\partial h} N(P)$$

$$= \Delta g(P) - \frac{\partial \gamma}{\partial h} N(P) \qquad (2.7)$$

Using a first order approximation for U, we obtain from (2.6)

$$W(P) = U(Q) - \gamma(P)N(P) + T(P).$$

But $W(P) = U(Q)$. Therefore

$$N(P) = \frac{T(P)}{\gamma(P)} \cdot \qquad (2.8)$$

This equation is known as Bruns formula [16]; it cannot be used as a Dirichlet boundary condition for T on the geoid since the geoidal undulation N is generally unknown. However, using Bruns formula in (2.7), we obtain

$$\frac{\partial T(P)}{\partial h} - \frac{1}{\gamma(P)} \frac{\partial \gamma(P)}{\partial h} T(P) + \Delta g(P) = 0. \qquad (2.9)$$

This expression, known as the fundamental equation of physical geodesy, serves as a boundary condition on the surface of the geoid. In practice, as previously mentioned, boundary condition (2.9) is assumed to hold on the reference ellipsoid. Then, assuming that all the earth's attracting mass lies within this ellipsoid, one solves Laplace's equation (the term $2\omega^2$ in (2.3) due to the rotation of the earth applies to both W and U) in the region exterior to the reference ellipsoid, subject to boundary condition (2.9).

For our purposes, the deflections of the vertical will ultimately be of most interest. The type of inertial navigation system being considered uses altitude measurements to regularly correct for errors in the vertical position; consequently, the error in the third component is prevented from accumulating and growing. The deflections of the vertical, however, enter directly into the dynamic equations, and lead to an accumulation of error in transverse position.

The gravity anomaly and the deflections of the vertical are related however. Two important expressions, Stokes formula and the Vening Meinesz equation, relate the undulation of the geoid and the deflections of the vertical, respectively, to the gravity anomaly. These expressions are based upon an additional spherical approximation of the reference ellipsoid (introducing a relative error on the order of 3 x $10^{-3}$, [16]). The coordinate system on the surface of the sphere is shown in Figure 2. $\psi$ is defined to be the distance between reference point P and another point on the spherical surface while $\alpha$ is an angle specifying the direction from P to the point. The equations use the Stokes function

$$S(\psi) \equiv \frac{1}{\sin\left(\frac{\psi}{2}\right)} - 6 \sin\frac{\psi}{2} + 1 - 5\cos\psi - 3\cos\psi \, \ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right). \tag{2.10}$$

In terms of the Stokes function $S(\psi)$ and the angle $\alpha$ defined in Figure 2, Stokes formula gives the geoidal undulation N at point P as

$$N(P) = \frac{R}{4\pi\bar{g}} \int_S \Delta g(\psi,\alpha)\ S(\psi)ds. \tag{2.11}$$

The Vening Meinesz equation, on the other hand, gives the deflection of the vertical at point P as

$$\begin{bmatrix} \xi(P) \\ \eta(P) \end{bmatrix} = \frac{1}{4\pi\bar{g}R^2} \int_S \Delta g(\psi,\alpha)\ \frac{dS(\psi)}{d\psi} \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} ds \tag{2.12}$$

In equations (2.11) and (2.12):

$R \equiv$ mean radius of the geoid,

$S \equiv$ sphere of radius R,

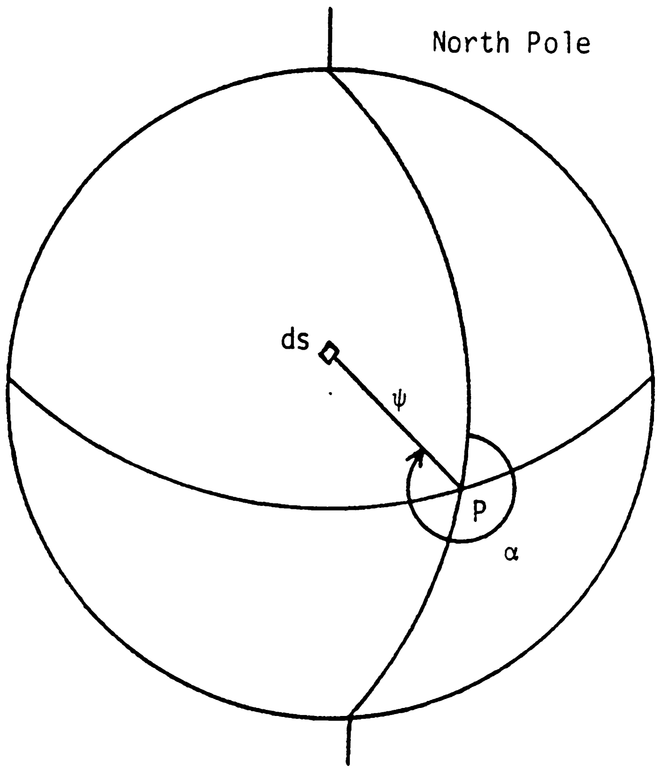
$\bar{g}$ = average magnitude of gravity over S,

$\psi$ = distance on S between P and the integration point,

$\alpha$ = azimuth angle between P and the integration point on S (c.f. Figure 2) and,

$\xi, \eta$ = north-south and east-west components of the deflection of the vertical.

Using these relationships, the information necessary to describe the error in the gravity model may be obtained from a knowledge of the gravity anomaly $\Delta g$ on the bounding surface. The disturbing potential T is, in principle, found by solving the exterior boundary value problem consisting of Laplace's equation and boundary condition (2.9). The undulation of the geoid and the deflection of the vertical are obtained through Stokes formula (2.11) and the Vening Meinesz equation (2.12), respectively.




Coordinates on Surface of Sphere

Figure 2

C. The Role of Stochastic Modeling:

The gravity anomaly must be known on the bounding surface in order to use the fundamental equation of physical geodesy (2.9) as a boundary condition. Likewise, the gravity anomaly must be known to use Stokes formula and the Vening Meinesz equations.

One way of representing the gravity anomaly is by means of a spherical harmonic expansion. Such expansions have in fact been developed with accurate coefficients of order and degree out to approximately fifteen; such a finite expansion accurately models gravity anomalies having wavelengths greater than 3000 km [1]. However, most of the fluctuation in the gravity anomaly lies at shorter wavelengths. Accurate representation of this short wavelength structure of the gravity anomaly by a spherical harmonic expansion would thus require extensive gravity surveys and the accurate computation of many more expansion coefficients. Nash and Jordan [1] estimate that computing coefficients out to order and degree 40 would still accurately model only 20 per cent of the gravity anomaly fluctuation.

One is therefore confronted with an inability to model the fine structure of the gravity field to an extent required for the satisfactory operation of inertial navigation systems. Since this fine structure has an erratic "noiselike" character, a natural approach is the adoption of a probabilistic description. In such a model, the gravity anomaly or the deflections of the vertical (at least their short wavelength portion) is assumed to be a random field.

Random process models have been used in a number of error analyses of inertial navigation systems. Some typical studies are those of Nash [2], [3]; Levine and Gelb [4]; Nash, D'Appolito, and Roy [5]; and Jordan [6]. These analyses generally assumed that the errors in the gravity anomaly and the deflections of the vertical are stationary isotropic white noise processes.

White noise refers to a Gaussian process (usually stationary with mean zero) with constant spectral density. That is, if $f(\sigma)$ is the spectral density and $V(x)$ is the covariance function for the process $\phi(x,\omega)$, then

$$f(\sigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\sigma x}\, V(x)\, dx = c \tag{2.13}$$

for some constant c. However, the only function V satisfying this condition is Dirac's delta function $\delta$. Consequently, white noise is sometimes characterized as being a "delta-correlated" process. Since this function does not exist in the usual sense, in practice, white noise is approximated by ordinary Gaussian processes which have covariance functions which converge in some sense to $\delta$. One example of such a function is

$$V(x) = a\, e^{-b|x|}, \quad a > o, \ b > o. \tag{2.14}$$

For this covariance function,

$$f(\sigma) = \frac{ab}{\pi(b^2+\sigma^2)} \tag{2.15}$$

If a and b increase to infinity in such a manner that $a/b \to 1/2$, then

$$f(\sigma) \to \frac{1}{2\pi} \text{ for all } \sigma, \text{ and } V(x) \to \delta(x).$$

Thus a Gaussian process with this covariance function approximates, in a sense, white noise.

A stationary process is one whose finite-dimensional distributions are invariant under translation. To be specific, the random field $\{X(P,\omega),\ P \in R^3,\ \omega \in \Omega\}$ is stationary if the multivariate distribution of the random variables $X(P_1 + h,\omega)$, $X(P_2 + h,\omega), \ldots, X(P_n + h,\omega)$ is independent of the common translation h for every integer $n < \infty$ and every possible set of sampling points $P_1, P_2, \cdots, P_n$. In particular,

$E\{X(P,\omega)X(P+\Delta P,\omega)\}$ is independent of P, i.e., it is a function only of the relative displacement $\Delta P$.

If the process or random field is, in addition, isotropic, its second order statistics are independent of direction. In this case, $E\{X(P,\omega)X(P+\Delta P,\omega)\}$ is a function of $|\Delta P|$ .

These stationary, isotropic assumptions do not relate well to the physical world where the gross environment can vary significantly, e.g., some regions are smooth (over the ocean) while others are rough (mountains and valleys).

The random field model adopted in this thesis is assumed to have continuous sample functions, to be almost surely bounded, and to satisfy a strong mixing condition. This last assumption will basically assume that the correlation between random variables obtained by sampling the random field at two different points decreases as the distance between the points increases. A precise definition of this property is given in Chapter III.

## D. The Dynamic Equations:

Thus far, we have discussed the gravity model errors and have motivated the need for the stochastic modeling of their fine scale structure. Now, we shall consider the relevant equations of motion and specify how these errors affect the vehicle dynamics; this development follows Britting [17].

As previously mentioned, accelerometers measure the specific force, the difference between the total acceleration of the platform and the acceleration due to gravity. This fact can be expressed mathematically as:

$$f(r) = \ddot{r} - G(r), \tag{2.16}$$

where the dot denotes differentiation with respect to time, $r = r(t)$ is the position vector at time t, $f(r)$ is the specific force vector, $\ddot{r} = \ddot{r}(t)$ is the acceleration vector at time t, and $G(r)$ is the acceleration vector due to gravity at position r.

Since measured and computed quantities are used instead of the true quantities, error terms must be introduced. Let $\tilde{r}(t)$ denote the estimated position vector at time t and $\delta r(t)$ the corresponding position vector error at time t. Then

$$\tilde{r} = r + \delta r, \qquad \ddot{\tilde{r}} = \ddot{r} + \delta\ddot{r}. \tag{2.17}$$

Let $\tilde{f}$ denote the measured specific force vector and $\delta f$ the corresponding specific force vector measurement error. Then

$$\tilde{f} = f + \delta f. \tag{2.18}$$

Lastly, let $\tilde{G}$ denote the gravity vector specified by the onboard gravity model and let $\delta G$ denote the gravity anomaly vector (c.f. (2.4)). Then

$$\tilde{G} = G + \delta G. \tag{2.19}$$

There is another important aspect of the error to be considered. The gravity vector used in onboard calculations is evaluated at the estimated position rather than the true position. Thus, the equation that must actually be used is

$$\ddot{\tilde{r}} = \tilde{f}(r) + \tilde{G}(\tilde{r}) \tag{2.20}$$

or

$$\ddot{r} + \delta\ddot{r} = f(r) + \delta f(r) + \tilde{G}(r+\delta r). \tag{2.21}$$

Using (2.16), we obtain

$$\delta\ddot{r} = \delta f(r) - G(r) + \tilde{G}(r+\delta r). \tag{2.22}$$

Let $\delta\tilde{G}$ denote the error caused by evaluating the gravity model at the estimated position. Then

$$\tilde{G}(r+\delta r) = \tilde{G}(r) + \delta\tilde{G}(r+\delta r). \tag{2.23}$$

$\delta\tilde{G}$ depends on the external altimetry measurements. Let $\tilde{h}$ be the external altitude measurement and let $\delta h$ be the error in the altitude measurement. Then a first order linearized approximation to the error due to using an inaccurate position vector in the gravity model is ([17], p. 92)

$$\delta\tilde{G}(r+\delta r) = \frac{\mu}{|\tilde{r}|^3}\left[\frac{(k\delta h)r}{|\tilde{r}|} + (3-k)\frac{(r^T\delta r)r}{|\tilde{r}|^2} - \delta r\right], \tag{2.24}$$

where k is a weight typically chosen between 2 and 3, $\mu$ is the product of the universal gravitational constant and mass of the earth, T indicates transpose and $r^T\delta r$ is interpreted as a vector product.

Substituting (2.19), (2.23), and (2.24) into (2.22) we obtain

$$\delta\ddot{r} = \frac{\mu}{|\tilde{r}|^3}\left[(3-k)\frac{(r^T\delta r)r}{|\tilde{r}|^2} - \delta r\right] + B(r)\delta\hat{G}(r) + E(r), \tag{2.25}$$

where

$$\delta\hat{G} = \begin{bmatrix} \xi g \\ \eta g \\ \Delta g \end{bmatrix} \qquad (2.26)$$

has components coinciding with the deflections of the vertical and the gravity anomaly. B(r) is a coordinate transformation matrix. E(r) contains the error terms not due to gravity, such as the specific force measurement error $\delta f(r)$ and the external altitude measurement error $\delta h(r)$.

The first term on the right side of this error equation represents the error due to using an inaccurate position in evaluating the gravity model while the second term represents the error inherent in the gravity model. Since only questions of gravity modeling are of interest here, all other terms will be combined into a general term E (the third term on the right side of (2.25)) for this analysis. Some of the sources of error not already mentioned are gyro drift rate error, accelerometer alignment errors, and system alignment errors. These errors, including the measurement errors for specific force and altitude, are neglected in this analysis and only errors arising from the use of a gravity model are considered. Thus we shall set $E \equiv 0$. However, these errors could be incorporated, if desired, into the analysis. The basic assumptions that would be required are that such errors are suitably small (otherwise they would completely overshadow gravity errors) and statistically independent of the gravity model errors.

As mentioned before, the vertical channel is aided by an external altitude measurement which helps the vehicle compensate for altitude errors. Only positional accuracy in the lateral directions suffers the cumulative buildup of small random gravity errors. Consequently, the error term in the vertical channel will ultimately be neglected; we shall eventually assume that there are no altitude errors in the analysis.

E. Formulation and Scaling of the Abstract Initial Value Problem:

The basic error equation (2.25) is

$$\delta\ddot{r} = \frac{\mu}{|\tilde{r}|^3}\left[(3-k)\frac{(r^T\delta r)r}{|\tilde{r}|^2} - \delta r\right] + B(r)\delta\hat{G}(r) + E(r)$$

where

$$\delta\tilde{G}(r) = \begin{bmatrix} \xi(r)g(r) \\ \eta(r)g(r) \\ \Delta g(r) \end{bmatrix}$$

(and, as previously mentioned, we shall ultimately neglect the third component). The basic units for length and time that will be used are the kilometer and the second. Typical values for the quantities appearing in equation (2.25) are (c.f. Heiskanen and Moritz [16]) $6.67 \times 10^{-8}$ cm$^3$/gram·sec$^2$ for the universal gravitational constant and $5.98 \times 10^{27}$ grams for the mass of the earth. Thus $\mu$ is $3.986329 \times 10^{20}$ cm$^3$/sec$^2$ or $3.986329 \times 10^5$ km$^3$/sec$^2$. The radius of the earth $R_e$ is 6371 km. Therefore

$$\frac{\mu}{|\tilde{r}|^3} \cong \frac{\mu}{R_e^{\,3}} \cong \frac{3.986329 \times 10^5}{(6371)^3}\ \text{sec}^{-2} = 1.5 \times 10^{-6}\ \text{sec}^{-2} \qquad (2.27)$$

Also, since $[(3-k)\frac{(r^T\delta r)r}{|\tilde{r}|^2} - \delta r] = O(\delta r)$,

$$\frac{\mu}{|\tilde{r}|^3}\left[(3-k)\frac{(r^T\delta r)r}{|\tilde{r}|^2} - \delta r\right] = 10^{-6}\cdot O(\delta r)\ \sec^{-2} \tag{2.28}$$

The mean value of g is 979.8 $cm/sec^2 \cong 9.8 \times 10^{-3}$ km/sec while typical values for the deflection of the vertical components $\xi$ and $\eta$ are 4-12 arc-seconds (c.f. [1]).

Therefore, $\xi g$, $\eta g$ are typically $0.2 \times 10^{-6} - 0.6 \times 10^{-6}$ $km/sec^2$ while the gravity anomaly $\Delta g$ is 25-85 $cm/sec^2 = 25 \times 10^{-5} - 85 \times 10^{-5}$ $km/sec^2$ [1]. Nominal correlation lengths for $\xi g$ and $\eta g$ are 10-20 n.m. = 18.5-37 km ([2], [4], [6]). Correlation lengths for $\Delta g$ are about twice as long as those for $\xi g$ and $\eta g$.

The vehicle or platform velocity $V_0$ will be assumed to have a nominal magnitude:

$$|V_0| = O(1) \text{ km/sec.} \tag{2.29}$$

(Note that 1 km/sec $\cong$ 2,000 mph.)

Now consider the scaled nondimensionalized version of (2.25). Specifically, we shall introduce a small dimensionless parameter $\varepsilon$ by setting

$$\varepsilon^2 = \frac{\mu}{R_e^3} \times 1\ \sec^2 \cong 10^{-6}. \tag{2.30}$$

Define

$$\phi = \frac{1}{\varepsilon^2}\begin{bmatrix}\xi g\\ \eta g\end{bmatrix} \tag{2.31}$$

Note that the components of $\phi$ are O(1) and that the dimension has been reduced from 3 to 2 in accordance with our decision to neglect any error in the vertical direction. Then the nondimensionalized 2-dimensional version of (2.25) becomes

$$\delta\ddot{r} = \varepsilon^2 A(r(t))\ \delta r + \varepsilon^2 B(r(t))\ \phi(r(t),\omega) \tag{2.32}$$

where $r(t)$ is now the nondimensionalized 2-dimensional actual vehicle position at time t and $\delta r$ the corresponding vehicle trajectory error. In (2.32), A and B are 2 X 2 matrices having $O(1)$ elements.

We shall be interested in the cumulative effect of random errors over vehicle flight times that are $O(\varepsilon^{-1})$ (typically, $10^3$ sec $\cong$ 17 min.). Trajectory lengths associated with this magnitude of flight time are on the order of 1000 km. Accordingly, we shall define a scaled temporal variable $\tau$ by

$$\tau = \varepsilon t. \tag{2.33}$$

We are thus interested in vehicle flight times that are $O(1)$ on the $\tau$ scale. Our asymptotic analysis will be done on this scale. In terms of the slow time $\tau$, (2.32) becomes

$$\frac{d^2\delta r(\tau)}{d\tau^2} = A(r(\tau/\varepsilon))\ \delta r(\tau) + B(r(\tau/\varepsilon))\ \phi(r(\tau/\varepsilon),\omega). \tag{2.34}$$

We shall assume there is no initial error, i.e.,

$$\delta r(0) = 0 \text{ and } \frac{d}{d\tau}\delta r(0) = 0. \tag{2.35}$$

Let us define the 4-dimensional vector $X^\varepsilon(\tau)$ as

$$X^\varepsilon(\tau) = \begin{bmatrix} \delta r(\tau) \\ \text{--------} \\ \frac{d}{d\tau}\delta r(\tau) \end{bmatrix} \tag{2.36}$$

Then (2.34), (2.35) can be rewritten as

$$\frac{dX^\varepsilon(\tau)}{d\tau} = \hat{A}(r(\tau/\varepsilon))X^\varepsilon(\tau) + \hat{B}(r(\tau/\varepsilon))\,\xi(r(\tau/\varepsilon),\omega),\ X^\varepsilon(0) = 0, \tag{2.37}$$

where

$$\hat{A} = \left[\begin{array}{c|c} 0 & I \\ \hline A & 0 \end{array}\right], \tag{2.38}$$

$$\hat{B} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & B \end{array}\right], \tag{2.39}$$

and $$\xi = \left[\begin{array}{c} 0 \\ \hline \phi \end{array}\right]. \tag{2.40}$$

The initial value problem (2.37) will serve as the prototype for the asymptotic analysis of Chapter III.

# III. Asymptotic Limit Theorems

In this chapter we study an abstract version of the stochastic initial value problem developed in Chapter II. Three limit theorems are proved. The first theorem establishes a deterministic correction of the nominal trajectory due to the large scale variations of the gravity field; the result is analogous to a weak law of large numbers. The second and third theorems establish convergence of finite dimensional distributions and weak convergence respectively for a suitably scaled fluctuation process. These results are analogous to a central limit theorem.

The starting point of our current considerations is the initial value problem for the error in acceleration due to the deflection of the vertical (eq. 2.37)

$$\frac{dX^{\varepsilon}(\tau)}{d\tau} = \hat{A}(r(\tau/\varepsilon))\; X^{\varepsilon}(\tau) + \hat{B}(r(\tau/\varepsilon))\xi(r(\tau/\varepsilon),\omega),$$
$$X^{\varepsilon}(0) = 0,$$

where $\tau = \varepsilon t$ is the slow time. This problem will be recast into a slightly more general setting. The problem will be considered in $R^n$, with the n x n matrices $\hat{A}$ and $\hat{B}$ now viewed as nonrandom functions of $X^{\varepsilon}(\tau)$, $\tau$, and $\tau/\varepsilon$. The n-dimensional random vector field $\xi$ will be centered, with the mean value $\hat{B}\; E\{\xi\}$ being combined with the $\hat{A}$ term. Thus, we replace $\hat{A}X^{\varepsilon} + \hat{B}\; E\{\xi\}$ with a known (deterministic) n-dimensional vector $A(X^{\varepsilon},\tau,\tau/\varepsilon)$. The zero mean process $\hat{B}(\xi - E\{\xi\})$ will now, for simplicity, be replaced by $B(X^{\varepsilon},\tau,\tau/\varepsilon)\xi$. B is assumed to be a known matrix-valued function while $\xi$ is now a zero mean random field (arising from the small-scale structure of the deflections of the vertical).

Let $x_0(\tau/\varepsilon)$ be the ideal or "nominal" trajectory defined so that the actual trajectory $\begin{bmatrix} r(\tau/\varepsilon) \\ \hline \frac{d}{d\tau} r(\tau/\varepsilon) \end{bmatrix}$ is the sum of the ideal trajectory and the trajectory error, i.e.,

$$\begin{bmatrix} r(\tau/\varepsilon) \\ \hline \frac{d}{d\tau} r(\tau/\varepsilon) \end{bmatrix} = X^{\varepsilon}(\tau,\omega) + x_0(\tau/\varepsilon) \tag{3.1}$$

(Note that the term trajectory is used here in a generic way. Recall that the original vector $X^{\varepsilon}$ in (2.36) included both position and velocity. For ease of future discussion, however, $X^{\varepsilon}$ and $x_0$ will be called "positions." The results of this chapter will subsequently be related back to the basic problem of interest in Chapter IV.)

In terms of the slight generalizations noted and the new notation introduced, the initial value problem of interest becomes

$$\begin{aligned} \frac{dX^{\varepsilon}(\tau,\omega)}{d\tau} &= A(X^{\varepsilon}(\tau,\omega),\tau,\tau/\varepsilon) + B(X^{\varepsilon}(\tau,\omega),\tau,\tau/\varepsilon)\cdot \\ &\cdot\xi(X^{\varepsilon}(\tau,\omega)+x_0(\tau/\varepsilon),\omega), \qquad 0 \leq \tau \leq \tau_0 < \infty. \\ &X^{\varepsilon}(0,\omega) = 0. \end{aligned} \tag{3.2}$$

Throughout the discussion we use the following notation:

$|a|$ for the absolute value of a scalar a,

$\|x\|$ for the Euclidean norm of an n-dimensional vector, i.e.,

$\|x\| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$, and

$\|B\|$ for the associated operator norm of an n x n matrix B, i.e.,

$\|B\| = \sup_{\|x\|=1} \|Bx\|$ .

For the n-vector A and n x n matrix B in equation (3.2) we adopt the following hypotheses:

(i) $A(x,\tau,t) : R^n \times [0,\tau_0] \times [0,\infty) \to R^n$ and

$B(x,\tau,t) : R^n \times [0,\tau_0] \times [0,\infty) \to R^n \times R^n$

are continuous nonrandom functions that are twice continuously differentiable with respect to x.

(ii) A and B are Lipschitz continuous as follows:

For $(x,\tau,t)$ and $(y,\sigma,t) \in R^n \times [0,\tau_0] \times [0,\infty)$,

$$\begin{aligned} \|A(x,\tau,t) - A(y,\sigma,t)\| &\leq L[\|x-y\| + |\tau-\sigma|] \\ \|B(x,\tau,t) - B(y,\sigma,t)\| &\leq L[\|x-y\| + |\tau-\sigma|]. \end{aligned} \tag{3.3}$$

(iii) $A(x,\tau,t)$ and $B(x,\tau,t)$, together with their first and second derivatives with respect to x, are uniformly bounded for x in any compact subset of $R^n$, $\tau \in [0,\tau_0]$ and $t \in [0,\infty)$.

With respect to the nominal trajectory $x_0$, we assume

(iv) $x_0(t) : [0,\infty) \to R^n$ is a continuous nonrandom function satisfying

$$\|x_0(t) - x_0(s)\| \geq \beta\,|t-s| \tag{3.4}$$

for some positive constant $\beta$ and all $t, s \in [0,\infty)$.

The purpose of this hypothesis is to ensure that the nominal trajectory (which is also the dominant component of the actual trajectory) evolves through the random field as a function of time without doubling back upon itself, stopping for periods of time, etc. This hypothesis ensures that large time intervals are accompanied by correspondingly large spatial displacements; the constant $\beta$ serves as a minimum speed.

For the random field $\xi$, we make the following assumptions:

(v) Let $(\Omega, F, P)$ denote the underlying probability space. Let $\xi(x,\omega) : R^n \times \Omega \to R^n$ be a random field that is jointly measurable relative to $B^n \times F$ where $B^n$ is the $\sigma$-algebra of Borel sets in $R^n$. In addition, assume that

a) $\|\xi(x,\omega)\| \leq \alpha$ almost surely in $\omega$ (3.5)
   for all $x \in R^n$,

b) $E\{\xi(x,\omega)\} = 0, \quad x \in R^n,$ (3.6)

c) $\xi(\cdot,\omega)$ is a continuous function of $x$, almost surely in $\omega$.

In addition to these hypotheses, we also assume that $\xi$ satisfies the following strong mixing condition:

Let $D \in B^n$ and let $F_D \subset F$ denote the minimal $\sigma$-algebra of subsets of $\Omega$ such that $\xi(x,\cdot)$ is measurable for all $x \in D$.

$$\text{Let } N(x,\delta) = \{y \in R^n : \|y-x\| \leq r + \delta\} \tag{3.7}$$

where $r$ is a certain fixed positive constant to be specified in Lemma 1 and $\delta$ is a small nonnegative constant, $0 \leq \delta \leq \delta_0$. Thus $N(x,\delta)$ is the closed ball in $R^n$ having center at $x$ and radius $r + \delta$. In the special case

$$D = \{x \in N(x_0(u),\delta_0) : s \leq u \leq t\} \tag{3.8}$$

we adopt the notation

$$F_D \equiv F_s^t\,(x_0,N(x,\delta_0)) \equiv F_s^t\,, \quad 0 \leq s \leq t < \infty, \tag{3.9}$$

where $\delta_0$ is a fixed, positive constant. In terms of this notation, the strong mixing (asymptotic independence) property assumed for the random field $\xi$ is

$$\sup_{s>0} \sup_{\substack{A \in F_{s+t}^{\infty} \\ B \in F_0^s}} |P(A|B) - P(A)| = \rho(t) \downarrow 0 \text{ as } t \to \infty. \tag{3.10}$$

where the monotonic decrease of $\rho$ as $t \to \infty$ is sufficiently fast that

$$\int_0^{\infty} t^2 \rho^{\frac{1}{2}}(t)dt < \infty. \tag{3.11}$$

Note that in (3.10) and (3.11) time, t, is related to the n-dimensional vector space via $x_0$ which is used to define the family of $\sigma$-algebras. Thus, (3.4) insures that $\sigma$-algebras separated in time are also separated by some minimal distance (directly related to the separation in time) in the n-dimensional vector space.

These hypotheses for the random field $\xi$ seem reasonable on physical grounds. The almost sure bound and sample function continuity are consistent with the measured deflections of the vertical while the zero mean property results from the explicit centering of the process. The strong mixing hypothesis reflects the assumption that the fine structure of the random field is due to local topographic and density fluctuations. The rate at which this asymptotic independence occurs is governed by inequality (3.11). This condition represents a "sufficiently rapid asymptotic independence" assumption. (Note that (3.4) is implicity used in (3.10) and (3.11).)

Thus far, we have assumed quite modest regularity properties for the random field, i.e., continuous sample functions. For technical reasons more regularity will be required. This regularity will be achieved through use of a mollifier, i.e., we shall define a new mollified random field and prove the results for a sequence of mollified problems.

We define a mollifier $\Lambda_\delta(x) : R^n \to R$ in the customary way

$$\Lambda_\delta(x) \equiv \begin{cases} C_\delta \exp\{-(1-\|\delta^{-1}x\|^2)^{-1}\}, & \|x\| < \delta \\ 0, & \|x\| \geq \delta \end{cases} \tag{3.12}$$

with the normalization constant $C_\delta$ chosen so that

$$\int_{R^n} \Lambda_\delta(x)dx = 1. \tag{3.13}$$

Likewise define a mollified random field $\xi_\delta$ by

$$\xi_\delta(x,\omega) = \int_{R^n} \Lambda_\delta(x-y)\xi(y,\omega)dy, \qquad \delta > 0. \tag{3.14}$$

For the mollified field $\xi_\delta$ and the deterministic vector $A(x,\tau,s)$ we assume the following averaged limits:

$$\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T}\int_{t_0}^{t_0+T} E\{[B(x,\tau,s)\xi_\delta(x+x_0(s),\omega)]_i[B(x,\tau,t)\cdot$$

$$\xi_\delta(x+x_0(t),\omega)]_j\}dsdt \equiv E_{ij}^{\delta}(x,\tau) \tag{3.15}$$

and

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} A(x,\tau,s)ds \equiv \bar{A}(x,\tau) \tag{3.16}$$

exist uniformly for x in any compact subset of $R^n$, independently of

$t_0 \in [0,\infty)$. Note that $\bar{A}$ likewise satisfies the Lipschitz condition (ii) and the bounded condition (iii).

We define the "averaged" problem,

$$\frac{d\bar{X}(\tau)}{d\tau} = \bar{A}(\bar{X}(\tau),\tau), \qquad \bar{X}(0) = 0. \tag{3.17}$$

Now assume, for the n-dimensional vector A(x,s,t),

$$\left\| \int_0^\tau [A(\bar{X}(s),s,s/\varepsilon)-\bar{A}(\bar{X}(s),s)]ds \right\| < c\varepsilon\tau, \text{ and} \tag{3.18}$$

$$\left\| \int_0^\tau \left[\frac{\partial A(\bar{X}(s),s,s/\varepsilon)}{\partial x} - \frac{\partial \bar{A}(\bar{X}(s),s)}{\partial x}\right] ds \right\| < c\varepsilon\tau, \tag{3.19}$$

for $\tau \in [0,\tau_0]$. c is some constant, depending on the value of $\tau_0$, whose precise value is unimportant.

Now define the mollified problem,

$$\frac{dX_\delta^\varepsilon(\tau)}{d\tau} = A(X_\delta^\varepsilon(\tau),\tau,\tau/\varepsilon) + B(X_\delta^\varepsilon(\tau),\tau,\tau/\varepsilon)\cdot$$

$$\cdot\xi_\delta(X_\delta^\varepsilon(\tau)+x_0(\tau/\varepsilon),\omega), \qquad X_\delta^\varepsilon(0) = 0. \tag{3.20}$$

The solution $\bar{X}$ corresponds to the change in the nominal trajectory $x_0$ caused by averaged large scale effects. In the absence of the small scale zero mean random fluctuations $\xi$, the Method of Averaging would predict a resulting deterministic trajectory formed by the sum of the original nominal trajectory $x_0$ and the correction $\bar{X}$. In Theorem 1 we prove that the actual change in trajectory (including the small scale zero mean random effects) will still tend asymptotically to $\bar{X}$. (The impact of the random field $\xi$ will become evident when we subsequently consider a suitably scaled fluctuation process.)

Remark: The introduction of mollifiers and subsequent consideration of the mollified problem represents a compromise dictated by technical considerations. On the one hand, the avowed purpose of this discussion is to assess asymptotically the impact of the small amplitude, rapidly varying fluctuations introduced by the random field $\xi$. On the other hand, mollification of $\xi$ (producing $\xi_\delta$) is tantamount to a certain smoothing of this small scale structure; spatial frequencies on the order of $\delta^{-1}$ or greater are damped by this process. Our justification for this approach ultimately lies in the following observations:

i) Recall that the basic unit of length is the kilometer and that $\varepsilon \cong .001$. Thus, if $\delta$ is small, say $\delta \cong \varepsilon$, the smoothing is performed over a quite small patch (i.e., having a radius of one meter).

ii) Some regularity beyond sample function continuity seems essential for the proof of theorems regarding asymptotic behavior (c.f. [14]). The use of mollifiers permits us to achieve this regularity in a very convenient way (i.e., without making any additional assumptions about the underlying initial random field $\xi$).

Theorem 1: Let $\delta(\varepsilon)$ be such that $\varepsilon/\delta(\varepsilon) = o(1)$ as $\varepsilon \to 0$. Then

$$\lim_{\varepsilon\to 0} \sup_{0\leq\tau\leq\tau_0} E\{\|X^{\varepsilon}_{\delta(\varepsilon)}(\tau,\omega) - \bar{X}(\tau)\|\} = 0. \tag{3.21}$$

(Thus the solution of the averaged problem, $\bar{X}(\tau)$, serves as the centering correction for the stochastic problem.) Four lemmas are presented to prove this theorem.

Lemma 1: There exists a positive constant r such that

$$\sup_{\substack{0\leq\tau\leq\tau_0 \\ 0\leq\delta\leq\delta_0 \\ 0\leq\varepsilon\leq\varepsilon_0}} \|X^{\varepsilon}_{\delta}(\tau,\omega)\| \leq r, \text{ almost surely } \omega. \tag{3.22}$$

Proof: From (3.20)

$$X^{\varepsilon}_{\delta}(\tau,\omega) = \int_0^{\tau}[A(X^{\varepsilon}_{\delta}(s,\omega),s,s/\varepsilon) + B(X^{\varepsilon}_{\delta}(s,\omega),s,s/\varepsilon)\cdot$$

$$\cdot\xi_{\delta}(X^{\varepsilon}_{\delta}(s,\omega)+x_0(s/\varepsilon),\omega)]ds.$$

Therefore, noting (3.3), (3.5), and hypothesis (iii)

$$\|X^{\varepsilon}_{\delta}(\tau,\omega)\| \leq \int_0^{\tau}\|A(X^{\varepsilon}_{\delta}(s,\omega),s,s/\varepsilon) - A(0,s,s/\varepsilon)\|ds$$

$$+\int_0^{\tau}\|A(0,s,s/\varepsilon)\|ds$$

$$+\alpha\int_0^{\tau}\|B(X^{\varepsilon}_{\delta}(s,\omega),s,s/\varepsilon) - B(0,s,s/\varepsilon)\|ds$$

$$+\int_0^{\tau}\|B(0,s,s/\varepsilon)\ \xi_{\delta}(X^{\varepsilon}_{\delta}(s,\omega)+x_0(s/\varepsilon),\omega)\|ds$$

$$\leq L(1+\alpha)\int_0^{\tau} \|X^{\varepsilon}_{\delta}(s,\omega)\|ds + M(1+\alpha)\tau,$$

Where L is the Lipschitz constant in (3.3) and M is a generic constant associated with hypothesis (iii). (We have also anticipated the fact that $\|\xi_{\delta}\| \leq \alpha$; c.f. (3.26)).

Applying the Gronwall inequality,

$$\|X_\delta^\varepsilon(\tau,\omega)\| \leq M\,(1+\alpha)\tau + \int_0^\tau LM(1+\alpha)^2 s\; e^{L(1+\alpha)(\tau-s)}\,ds$$

$$\leq \frac{M}{L}\,(e^{L(1+\alpha)\tau_0}-1)$$

$$\equiv r. \tag{3.23}$$

Remark: Several hypotheses involved a restriction of the spatial variable x to a compact subset of $R^n$. Lemma 1 establishes that the mollified process remains almost surely within the ball of radius r. Thus, the hypotheses can be henceforth thought of as pertaining to this compact set. Moreover, the constant r defined by (3.23) is the same as that referred to in (3.7).

Lemma 2: There exists a positive constant c such that

(a) $$\left\| \frac{\partial \xi_\delta(x,\omega)}{\partial x_i} \right\| \leq c\,\delta^{-1} \tag{3.24}$$

(b) $$\left\| \frac{\partial^2 \xi_\delta(x,\omega)}{\partial x_i \partial x_j} \right\| \leq c\,\delta^{-2} \tag{3.25}$$

for $i,j = 1,2,\cdots,n$.

This straightforward result is presented without proof. Note that some additional properties of the mollified process are:

(c) $$\|\xi_\delta(x,\omega)\| \leq \alpha \quad \text{almost surely } \omega \tag{3.26}$$

(We have anticipated this result in the proof of Lemma 1).

(d) $E\{\xi_\delta(x,\omega)\} = 0$ (3.27)

(e) $\xi_\delta(\cdot,\omega) \;\varepsilon\; C^\infty(R^n;R^n)$ (3.28)

where $C^\infty(R^n;R^n)$ is the space of infinitely differentiable continuous functions with range and domain $R^n$.

(f) $\Lambda_\delta(x-y) = 0$ if $\|x-y\| > \delta$. Thus, if $\xi(y,\omega)$ is $F_s^t\,(x_0,N(x_0,\delta))$-measurable then $\xi_\delta(x,\omega)$ is $F_s^t\,(x_0,N(x_0,0))$-measurable.

Remark: In the following arguments the phrase "almost surely $\omega$" will be dropped as it is unnecessarily repetitious and it is clear when it applies.

Lemma 3: This lemma is lemma 1 of reference [12] (p.652). It is stated without proof.

Let $(\Omega,F,P)$ be a probability space and let $F_s^t$, $0 \le s \le t < \infty$, be a family of $\sigma$-algebras contained in $F$ such that

$$F_{s_1}^{t_1} \subset F_{s_2}^{t_2}, \quad 0 \le s_2 \le s_1 \le t_1 \le t_2 < \infty.$$

Assume the $\sigma$-algebras $F_s^t$ are mixing relative to $P$ in the sense,

$$\sup_{s \geq 0} \sup_{\substack{A \,\varepsilon\, F_{s+t}^\infty \\ B \,\varepsilon\, F_0^s}} |\, P(A\,|\,B) - P(A)\,| = \rho(t) \downarrow 0 \text{ as } t \uparrow \infty.$$

Assume $\int_0^\infty \rho^{\frac{1}{2}}(s)ds < \infty$. Let $H(\omega,\omega')$ be a function on $\Omega \times \Omega$ such that, for fixed $\omega'$, $H(\cdot,\omega')$ is $F_{t+s}^{\infty}$- measurable and, for fixed $\omega$, $H(\omega,\cdot)$ is $F_0^s$-measurable.

Let $\| H(\omega,\omega')\| \leq \phi(\omega')$. Define (3.29)

$$\bar{H}(\omega') = E\ \{H(\cdot,\omega')\} = \int_\Omega H(\omega,\omega')P(d\omega). \tag{3.30}$$

Then

$$\| E\{H(\cdot,\omega') \mid F_0^s\} - \bar{H}(\omega')\| \leq 2\ \rho(t)\phi(\omega'). \tag{3.31}$$

Lemma 4: There is a positive constant c such that

$$E\{ \| \frac{1}{\sqrt{\varepsilon}}\int_{\tau_1}^{\tau_2} B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(X_\delta^\varepsilon(s,\omega)+x_0(s/\varepsilon),\omega)ds\|^2 \} \leq c\delta^{-1}| \tau_2-\tau_1|$$

$$\text{for all } \tau_1,\tau_2 \in [0,\tau_0],\ 0 \leq \delta \leq \delta_0. \tag{3.32}$$

Proof: Let $0 \leq \tau_1 \leq \tau_2 \leq \tau_0$ and define

$$I^\varepsilon(\tau_1,\tau_2,\omega) = \frac{1}{\sqrt{\varepsilon}}\int_{\tau_1}^{\tau_2} B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(X_\delta^\varepsilon(s,\omega)+x_0(s/\varepsilon),\omega)ds$$

$$= \sqrt{\varepsilon}\int_{\tau_1/\varepsilon}^{\tau_2/\varepsilon} B(\bar{X}(\varepsilon s),\varepsilon s,s)\xi_\delta(X_\delta^\varepsilon(\varepsilon s,\omega)+x_0(s),\omega)ds.$$

Then

$$\|I^\varepsilon(\tau_1,\tau_2,\omega)\|^2 = \varepsilon\int_{\tau_1/\varepsilon}^{\tau_2/\varepsilon}\int_{\tau_1/\varepsilon}^{\tau_2/\varepsilon} [B(\bar{X}(\varepsilon s_1),\varepsilon s_1,s_1)\xi_\delta(X_\delta^\varepsilon(\varepsilon s_1,\omega)+x_0(s_1),\omega)]^T \cdot$$

$$B(\bar{X}(\varepsilon s_2),\varepsilon s_2,s_2)\xi_\delta(X_\delta^\varepsilon(\varepsilon s_2,\omega)+x_0(s_2),\omega)\ ds_2ds_1$$

$$= 2\varepsilon\int_{\tau_1/\varepsilon}^{\tau_2/\varepsilon}\int_{\tau_1/\varepsilon}^{s_1} [B(\bar{X}(\varepsilon s_1),\varepsilon s_1,s_1)\xi_\delta(X_\delta^\varepsilon(\varepsilon s_1,\omega)+x_0(s_1),\omega)]^T\cdot$$

$$\cdot[B(\bar{X}(\varepsilon s_2),\varepsilon s_2,s_2)\xi_\delta(X_\delta^\varepsilon(\varepsilon s_2,\omega)+x_0(s_2),\omega)]ds_2ds_1.$$

From the fundamental theorem of calculus,

$$\xi_\delta(X_\delta^\varepsilon(\varepsilon s_1,\omega)+x_0(t),\omega) = \xi_\delta(X_\delta^\varepsilon(\varepsilon s_2,\omega)+x_0(t),\omega)$$

$$+\int_{s_2}^{s_1} \frac{d}{d\eta}\xi_\delta(X_\delta^\varepsilon(\varepsilon\eta,\omega)+x_0(t),\omega)d\eta.$$

Noting (3.20)

$$\frac{d}{d\eta}\xi_\delta(X_\delta^\varepsilon(\varepsilon\eta,\omega)+x_0(t),\omega) = \varepsilon\,\nabla\xi_\delta(X_\delta^\varepsilon(\varepsilon\eta,\omega)+x_0(t),\omega)\cdot$$

$$\cdot[A(X_\delta^\varepsilon(\varepsilon\eta,\omega),\varepsilon\eta,\eta)+B(X_\delta^\varepsilon(\varepsilon\eta,\omega),\varepsilon\eta,\eta)\xi_\delta(X_\delta^\varepsilon(\varepsilon\eta,\omega)+x_0(\eta),\omega)].$$

Thus

$$\|I^\varepsilon(\tau_1,\tau_2,\omega)\|^2 = 2\varepsilon\int_{\tau_1/\varepsilon}^{\tau_2/\varepsilon}\int_{\tau_1/\varepsilon}^{s_1} [B(\bar{X}(\varepsilon s_1),\varepsilon s_1,s_1)\cdot$$

$$\cdot\xi_\delta(X_\delta^\varepsilon(\varepsilon s_2,\omega)+x_0(s_1),\omega)]^T[B(\bar{X}(\varepsilon s_2),\varepsilon s_2,s_2)\xi_\delta(X_\delta^\varepsilon(\varepsilon s_2,\omega)+x_0(s_2),\omega)]ds_2ds_1$$

$$+2\varepsilon\int_{\tau_1/\varepsilon}^{\tau_2/\varepsilon}\int_{\tau_1/\varepsilon}^{s_1} [B(\bar{X}(\varepsilon s_1),\varepsilon s_1,s_1)\cdot\varepsilon\int_{s_2}^{s_1} \nabla\xi_\delta(X_\delta^\varepsilon(\varepsilon\eta,\omega)+x_0(s_1),\omega)\cdot$$

$$\cdot[A(X_\delta^\varepsilon(\varepsilon\eta,\omega),\varepsilon\eta,\eta)+B(X_\delta^\varepsilon(\varepsilon\eta,\omega),\varepsilon\eta,\eta)\xi_\delta(X_\delta^\varepsilon(\varepsilon\eta,\omega)+x_0(\eta),\omega)]d\eta]^T\cdot$$

$$\cdot[B(\bar{X}(\varepsilon s_2),\varepsilon s_2,s_2)\xi_\delta(X_\delta^\varepsilon(\varepsilon s_2,\omega)+x_0(s_2),\omega)]ds_2ds_1$$

$$\equiv I_1^{\varepsilon}(\tau_1,\tau_2,\omega) + I_2^{\varepsilon}(\tau_1,\tau_2,\omega) \tag{3.33}$$

First consider $I_1^{\varepsilon}(\tau_1,\tau_2,\omega)$. Let

$$H(\omega,\omega') = [B(\bar{X}(\varepsilon s_1),\varepsilon s_1,s_1)\xi_\delta(X_\delta^{\varepsilon}(\varepsilon s_2,\omega')+x_0(s_1),\omega)]^T\cdot$$

$$\cdot[B(\bar{X}(\varepsilon s_2),\varepsilon s_2,s_2)\xi_\delta(X_\delta^{\varepsilon}(\varepsilon s_2,\omega')+x_0(s_2),\omega')].$$

For fixed $\omega'$, $H(\cdot,\omega')$ is $F_{s_1}^{\infty}$-measurable. For fixed $\omega$, $H(\omega,\cdot)$ is $F_0^{s_2}$- measurable (with $s_2 < s_1$). Applying Lemma 3 (with $\phi(\omega') = \alpha^2M^2$),

$$|E\{H(\cdot,\omega') \mid F_0^{s_2}\} - \bar{H}(\omega')| < 2\,\rho(s_1-s_2)\alpha^2 M^2.$$

However in this case, (3.27) implies that $\bar{H}(\omega') = 0$. Therefore

$$|E\{H(\cdot,\omega') \mid F_0^{s_2}\}| < 2\alpha^2M^2\rho(s_1-s_2).$$

Using the fact that

$$|E\{H\}| = |E\{E\{H \mid F_0^{s_2}\}\}| < E\{|E\{H \mid F_0^{s_2}\}|\},$$

we obtain

$$E\{|I_1^{\varepsilon}(\tau_1,\tau_2,\omega)|\} < 2\,\varepsilon \int_{\tau_1/\varepsilon}^{\tau_2/\varepsilon}\int_{\tau_1/\varepsilon}^{s_1} 2\alpha^2M^2\rho(s_1-s_2)ds_2ds_1$$

$$= 4\alpha^2M^2\varepsilon\int_{\tau_1/\varepsilon}^{\tau_2/\varepsilon}\int_{\tau_1/\varepsilon}^{s}\rho(s_1-s_2)ds_2ds_1$$

$$= 4\alpha^2 M^2 \varepsilon \int_0^{\frac{\tau_2-\tau_1}{\varepsilon}} \int_{\tau_1/\varepsilon}^{\frac{\tau_2}{\varepsilon}-u} \rho(u)dvdu$$

$$\leq 4\alpha^2 M^2 | \tau_2-\tau_1 | \int_0^{\infty} \rho(u)du$$

$$\leq c_1 | \tau_2-\tau_1 |. \qquad (3.34)$$

Now consider $I_2^{\varepsilon}(\tau_1,\tau_2,\omega)$. Let

$$H(\omega,\omega') = B(\bar{X}(\varepsilon s_1,\varepsilon s_1,s_1)\nabla\xi_\delta(X_\delta^{\varepsilon}(\varepsilon\eta,\omega')+x_0(s_1),\omega)\cdot$$

$$\cdot[A(X_\delta^{\varepsilon}(\varepsilon\eta,\omega'),\varepsilon\eta,\eta)+B(X_\delta^{\varepsilon}(\varepsilon\eta,\omega'),\varepsilon\eta,\eta)\xi_\delta(X_\delta^{\varepsilon}(\varepsilon\eta,\omega')+x_0(\eta),\omega')]^T.$$

$$\cdot[B(\bar{X}(\varepsilon s_2),\varepsilon s_2,s_2)\xi_\delta(X_\delta^{\varepsilon}(\varepsilon s_2,\omega')+x_0(s_2),\omega')].$$

Recall that $s_2 \leq \eta \leq s_1$. For fixed $\omega'$, $H(\cdot,\omega')$ is $F_{s_1}^{\infty}$-measurable and for fixed $\omega$, $H(\omega,\cdot)$ is $F_0^{\eta}$-measurable.

Using Lemma 2,

$$\left\| \frac{\partial \xi_\delta(x,\omega)}{\partial x_i} \right\| \leq c\delta^{-1}, \qquad i=1, \cdots, n.$$

Therefore

$$| H(\omega,\omega') | \leq \phi(\omega') = M^2 \alpha c\delta^{-1}(M+\alpha M) \equiv c_2\delta^{-1}.$$

Noting that $E\{\nabla\xi_\delta(x,\omega)\} = 0$, Lemma 3 implies that

$$| E\{H\} | \leq 2\rho(s_1-\eta)\, c_2\delta^{-1}.$$

Therefore

$$E\{\,|\,I_2^\varepsilon(\tau_1,\tau_2,\omega)|\,\} < 4\,\varepsilon^2 c_2\delta^{-1}\int_{\tau_1/\varepsilon}^{\tau_2/\varepsilon}\int_{\tau_1/\varepsilon}^{s_1}\int_{s_2}^{s_1}\rho(s_1-\eta)d\eta ds_2 ds_1$$

$$= 4\varepsilon^2 c_2\delta^{-1}\int_{\tau_1/\varepsilon}^{\tau_2/\varepsilon}\int_{\tau_1/\varepsilon}^{s_1}\int_{\tau_1/\varepsilon}^{\eta}\rho(s_1-\eta)ds_2 d\eta ds_1$$

$$= 4\varepsilon^2 c_2\delta^{-2}\int_{\tau_1/\varepsilon}^{\tau_2/\varepsilon}\int_{\tau_1/\varepsilon}^{s_1}(\eta-\tau_1/\varepsilon)\rho(s_1-\eta)d\eta ds_1$$

$$= 4\varepsilon^2 c_2\delta^{-1}\int_0^{\frac{\tau_2-\tau_1}{\varepsilon}}\int_{\tau_1/\varepsilon}^{\frac{\tau_2}{\varepsilon}-u}(v-\tau_1/\varepsilon)\rho(u)dvdu$$

$$< 3\varepsilon^2 c_2\delta^{-1}\left(\frac{\tau_2-\tau_1}{\varepsilon}\right)^2\int_0^{\frac{\tau_2-\tau_1}{\varepsilon}}\rho(u)du$$

$$< 3c_2\delta^{-1}|\,\tau_2-\tau_1|^2\int_0^\infty \rho(u)du$$

$$< 3c_2\delta^{-1}\tau_0\,|\,\tau_2-\tau_1\,|\int_0^\infty\rho(u)du$$

$$< \ c_3\delta^{-1}|\,\tau_2-\tau_1|. \tag{3.35}$$

Thus, (3.33), (3.34) and (3.35) imply

$$E\{\|I^\varepsilon(\tau_1,\tau_2,\omega)\|^2\} = E\{I_1^\varepsilon\ (\tau_1,\tau_2,\omega)+I_2^\varepsilon(\tau_1,\tau_2,\omega)\}$$

$$< E\{|\,I_1^\varepsilon(\tau_1,\tau_2,\omega)\ |\} + E\{|\,I_2^\varepsilon(\tau_1,\tau_2,\omega)\ |\} < c\ \delta^{-1}|\,\tau_2-\tau_1\ |.$$

Proof of Theorem 1:

$$X_\delta^\varepsilon(\tau,\omega)-\bar{X}(\tau) = \int_0^\tau [A(X_\delta^\varepsilon(s,\omega),s,s/\varepsilon)+B(X_\delta^\varepsilon(s,\omega)s,s/\varepsilon)\cdot$$

$$\cdot\xi_\delta(X_\delta^\varepsilon(s,\omega)+x_0(s/\varepsilon),\omega)-\bar{A}(\bar{X}(s),s)]ds$$

$$=\int_0^\tau [A(X_\delta^\varepsilon(s,\omega),s,s/\varepsilon)-A(\bar{X}(s),s,s/\varepsilon)]ds$$

$$+\int_0^\tau [B(X_\delta^\varepsilon(s,\omega),s,s/\varepsilon)-B(\bar{X}(s),s,s/\varepsilon)]\xi_\delta(X_\delta^\varepsilon(s,\varepsilon)+x_0(s/\varepsilon),\omega)ds$$

$$+\int_0^\tau [A(\bar{X}(s),s,s/\varepsilon)-\bar{A}(\bar{X}(s),s)]ds$$

$$+\int_0^\tau B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(X_\delta^\varepsilon(s,\omega)+x_0(s/\varepsilon),\omega)ds.$$

Noting (3.3) and (3.25),

$$\|X_\delta^\varepsilon(\tau,\omega)-\bar{X}(\tau)\| \leq \int_0^\tau L\|X_\delta^\varepsilon(s,\omega)-\bar{X}(s)\|ds$$

$$+\int_0^\tau L\|X_\delta^\varepsilon(s,\omega)-\bar{X}(s)\|\alpha\, ds$$

$$+\left\|\int_0^\tau [A(\bar{X}(s),s,s/\varepsilon)-\bar{A}(\bar{X}(s),s)]ds\right\|$$

$$+\left\|\int_0^\tau B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(X_\delta^\varepsilon(s,\omega)+x_0(s/\varepsilon),\omega)ds\right\|.$$

From (3.18)

$$\left\|\int_0^\tau [A(\bar{X}(s),s,s/\varepsilon)-\bar{A}(\bar{X}(s),s)]ds\right\| \leq c_1\varepsilon\tau.$$

Thus

$$E\{\|X_\delta^\varepsilon(\tau,\omega)-\bar{X}(\tau)\|\} \leq L(1+\alpha)\int_0^\tau E\{\|X_\delta^\varepsilon(s,\omega)-\bar{X}(s)\|\}ds$$

$$+ c_1\varepsilon\tau + E\{\|\int_0^\tau B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(X_\delta^\varepsilon(s,\omega)+x_0(s/\varepsilon),\omega)ds\|\}.$$

Using the Cauchy-Schwarz inequality and the results of Lemma 4,

$$E\{\|\int_0^\tau B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(X_\delta^\varepsilon(s,\omega)+x_0(s/\varepsilon),\omega)ds\|\}$$

$$\leq \quad [E\{\|\int_0^\tau B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(X_\delta^\varepsilon(s,\varepsilon)+x_0(s/\varepsilon),\omega)ds\|^2\}]^{\frac{1}{2}}$$

$$\leq \quad [c_2\delta^{-1}\varepsilon\tau]^{\frac{1}{2}}.$$

Therefore

$$E\{\|X_\delta^\varepsilon(\tau,\omega)-\bar{X}(\tau)\|\} \leq L(1+\alpha)\int_0^\tau E\{\|X_\delta^\varepsilon(s,\omega)-\bar{X}(s)\|\}ds$$

$$+ c_1\varepsilon\tau \quad + [c_2\delta^{-1}\varepsilon\tau]^{\frac{1}{2}}.$$

Applying the Gronwall inequality,

$$\sup_{0\leq\tau\leq\tau_0} E\{\|X_\delta^\varepsilon(\tau,\omega)-\bar{X}(\tau)\|\} \leq [c_1\varepsilon\tau_0 + (c_2\delta^{-1}\varepsilon\tau_0)^{\frac{1}{2}}] \exp[L(1+\alpha)\tau_0]$$

$$\leq c \left(\frac{\varepsilon}{\delta}\right)^{\frac{1}{2}}$$

and assertion (3.21) follows.

Thus, the random trajectory perturbations $X_{\delta(\varepsilon)}^\varepsilon$, arising as solutions to the sequence of mollified problems, converge as $\varepsilon$ goes to zero to the centering "averaged" trajectory correction $\bar{X}$. In order to study the asymptotic behavior of the difference $X_\delta^\varepsilon - \bar{X}$, an appropriately scaled (magnified) fluctuation process will be considered. We specify this fluctuation process and show in Theorems 2 and 3 that this fluctuation (or error) process converges weakly to a Gauss-Markov process.

The method we use to show weak convergence is outlined in Billingsley [18]. In order to show weak convergence in the space of n-dimensional vector-valued continuous functions on a finite interval, i.e., $C([0,\tau_0];R^n)$ it is sufficient to prove convergence of the finite-dimensional distributions and relative compactness. We initially state both theorems and subsequently present their proofs. The first of these, Theorem 2, establishes convergence of the finite-dimensional distributions.

Theorem 2:

Define the fluctuation process

$$Y_\delta^\varepsilon(\tau,\omega) = \frac{1}{\sqrt{\varepsilon}}[X_\delta^\varepsilon(\tau,\omega)-\bar{X}(\tau)] \tag{3.36}$$

where $X_\delta^\varepsilon(\tau,\omega)$ and $\bar{X}(\tau)$ are solutions of the initial value problems (3.20) and (3.17), respectively. Then there exists a $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that the finite-dimensional distributions of $Y_{\delta(\varepsilon)}^\varepsilon$ converge to the finite-dimensional distributions of the process $Y_0(\tau,\omega)$ defined as the solution of

$$Y_0(\tau,\omega) = W_0(\tau,\omega) + \int_0^\tau \frac{\partial\bar{A}(\bar{X}(s),s)}{\partial x} Y_0(s,\omega)ds \tag{3.37}$$

where $W_0(\tau,\omega)$ is a Gaussian process with independent increments, zero mean and correlation matrix

$$\int_0^{\tau} E^0_{ij}\ (X(\bar{s}),s)ds, \quad i,j, = 1,\cdots, n. \tag{3.38}$$

Note that the matrix element $E^0_{ij}$ is obtained from (3.15) by setting $\delta = 0$; $\bar{A}$ is defined by (3.16).

Prohorov's theorem states that, for a separable and complete metric space, relative compactness and tightness are equivalent [18]. A family $\pi$ of probability measures is tight if for every positive $\varepsilon$ there exists a compact set K such that $P(K) > 1-\varepsilon$ for every P in $\pi$. Alternately, a sequence of probability measures $\{P_n\}$ is tight if for every positive $\varepsilon$ there exists a compact set K such that $P_n(K) > 1 - \varepsilon$ for all n. Since the metric space $C([0,\tau_0];R^n)$ of n-dimensional continuous functions on $[0,\tau_0]$ with uniform topology $(\rho(x,y) \equiv \sup_{0\leq\tau\leq\tau_0} \|x(\tau)-y(\tau)\|)$ is separable and complete, the tightness criterion may be substituted for relative compactness. Theorem 3 will prove that the family of measures associated with the fluctuation process $Y^{\varepsilon}_{\delta}$ (for fixed $\delta$ and index $\varepsilon$) is tight.

Sufficient criteria for tightness are given in the following theorem ([18], theorem 12.3):

Let $\{\phi_m(\tau)\}$ be a sequence of random elements in $C([0,\tau_0];R^n)$. Then $\{\phi_m\}$ is tight if it satisfies

(i) $\{\phi_m(0)\}$ is tight, and (3.39)

(ii) There exist constants $\gamma \geq 0$ and $\alpha > 1$ and a nondecreasing, continuous function F on $[0,\tau_0]$ such that

$$P\{\|\phi_m(\tau_2) - \phi_m(\tau_1)\| > \lambda\} < \frac{1}{\lambda^\gamma} | F(\tau_2)-F(\tau_1) |^\alpha \tag{3.40}$$

holds for all $\tau_1,\tau_2$, and m and all positive $\lambda$.

To obtain a more useful inequality than (3.40), involving moments of the process, we use Chebyshev's inequality, i.e.,

$$P\{\|x\| > c\} < \frac{1}{c^p} E\{\|x\|^p\},\ p > 0,\ c > 0. \tag{3.41}$$

Let $X \equiv \phi_m(\tau_2) - \phi_m(\tau_1)$, $c = \lambda$, and $p = \gamma$. Then (3.41) becomes

$$P\{\|\phi_m(\tau_2)-\phi_m(\tau_1)\| > \lambda\} < \frac{1}{\lambda^\gamma} E\{\|\phi_m(\tau_2)-\phi_m(\tau_1)\|^\gamma\}.$$

Let $F = h^{1/\alpha}\tau$ for some $h > 0$. If the inequality

$$E\{\|\phi_m(\tau_2)-\phi_m(\tau_1)\|^\gamma\} < h\,|\tau_2-\tau_1|^\alpha \tag{3.42}$$

can be shown to hold for some $\gamma > 0$, $\alpha > 1$, $h > 0$, and all $\tau_1,\tau_2 \in [0,\tau_0]$, then the family $\{\phi_m\}$ will be tight and hence relatively compact.

Verifying this condition is the content of Theorem 3.

Theorem 3: Define

$$W_\delta^\varepsilon(\tau,\omega) = \frac{1}{\sqrt{\varepsilon}}\int_0^\tau [A(\bar{X}(s),s,s/\varepsilon) + B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega) - \bar{A}(\bar{X}(s),s)]ds \tag{3.43}$$

and

$$Y_\delta(\tau,\omega) = W_\delta(\tau,\omega) + \int_0^\tau \frac{\partial\bar{A}(\bar{X}(s),s)}{\partial x} Y_\delta(s,\omega)ds \tag{3.44}$$

where $W_\delta$ is a zero mean, independent increment Gaussian process with correlation matrix

$$\int_0^\tau E_{ij}^\delta(\bar{X}(s),s)ds, \qquad i,j=1,\cdots,n, \tag{3.45}$$

and $E_{ij}^\delta$ is defined by (3.15).

Then, for every fixed $\delta > 0$, the family of random vectors $\{Y_\delta^\varepsilon(\tau,\omega)\}$ defined by (3.36) converges weakly to $Y_\delta(\tau,\omega)$ as $\varepsilon \to o$.

Remark: Note that in Theorem 2, the mollifier parameter $\delta$ is coupled to $\varepsilon$ while in Theorem 3 the parameter $\delta$ is fixed. We have been unable to establish weak convergence in terms of a single limit. Instead, weak convergence to the ultimate process of interest is achieved by first letting $\varepsilon \to 0$ and subsequently letting $\delta \to 0$.

Define a process $Z_\delta^\varepsilon$ as the solution of the Volterra equation

$$Z_\delta^\varepsilon(\tau,\omega) \equiv W_\delta^\varepsilon(\tau,\omega) + \int_0^\tau \frac{\partial\bar{A}(\bar{X}(s),s)}{\partial x} Z_\delta^\varepsilon(s,\omega)ds. \tag{3.46}$$

White [9] has basically shown that as $\varepsilon \to 0$ (with $\delta$ fixed) $W_\delta^\varepsilon(\tau,\omega)$ converges weakly to the limiting process $W_\delta(\tau,\omega)$, which is a Gaussian process with independent increments, zero mean and correlation matrix (3.45). In Theorem 2, the parameter $\delta$ is to be a function of $\varepsilon$ that vanishes as $\varepsilon \to 0$. In order to obtain the weak convergence of $W_{\delta(\varepsilon)}^\varepsilon(\tau,\omega)$ to the limiting process $W_0(\tau,\omega)$ as required by Theorem 2, we can use the approach of White by obtaining estimates (on constants and measurability gaps) that are independent of $\delta$. Note, in (3.43), how the measurability of $W_\delta^\varepsilon$ is related to the mollified process $\xi_\delta$. If the original process $\xi$ is $F_s^t(x_0,N(x_0,\delta_0))$- measurable, $\xi_\delta$ is $F_s^t(x_0,N(x_0,0))$- measurable for all $0 < \delta < \delta_0$ ($\delta_0$ is arbitrarily chosen as some maximum permissible value). With this observation, it is a straightforward task to adapt the arguments of White to the problem at hand. We present without proof in Lemma 5 the results of this adaptation, the principal result being the weak convergence of $W^{\varepsilon}_{\delta(\varepsilon)}$ to $W_0$ as $\varepsilon \to 0$. Lemmas 6 and 7, likewise restate results from other sources. Lemma 7 is a restatement of two lemmas from [8]. These lemmas assume a strong mixing hypothesis expressed in a different way than the conditions we use. Papanicolaou and Varadhan [11] have shown the mixing condition we use implies the condition of [8]. This lemma is stated as Lemma 6. We now present these three lemmas.

Lemma 5:

This lemma is a combination of three lemmas from White [9]. The results are recast in our notation and stated without proof. From (3.43),

$$W^{\varepsilon}_{\delta}(\tau,\omega) = \frac{1}{\sqrt{\varepsilon}}\int_0^{\tau} [A(\bar{X}(s),s,s/\varepsilon) + B(\bar{X}(s),s,s/\varepsilon)\cdot$$

$$\cdot\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)-\bar{A}(\bar{X}(s),s)]ds.$$

It follows that:

a) $$\lim_{\varepsilon\to 0} E\{W^{\varepsilon}_{\delta}(\tau)\} = 0 \qquad (3.47)$$

$$\lim_{\varepsilon\to 0} E\{W^{\varepsilon}_{\delta i}(\tau)W^{\varepsilon}_{\delta j}(\tau)\} = \int_0^{\tau} E^{\delta}_{ij}(\bar{X}(s),s)ds \qquad (3.48)$$

for every $\delta > 0$, $i,j,=1\cdots,n$.

where the $E^{\delta}_{ij}$ are defined by (3.15).

b) For every $\delta > 0$ the processes $W_\delta^\varepsilon(\tau,\omega)$ are relatively compact in $C([0,\tau_0];R^n)$, the space of all continuous n-dimensional functions (with uniform topology) over the finite interval $[0,\tau_0]$. This is proven by showing that

$$E\{\|W_\delta^\varepsilon(\tau_2)-W_\delta^\varepsilon(\tau_1)\|^4\} \leq c_\delta |\tau_2 - \tau_1|^2. \tag{3.49}$$

c) For every $\delta > 0$, $(W_\delta^\varepsilon, Z_\delta^\varepsilon)$ converges weakly in $C([0,\tau_0];R^n x R^n)$ to $(W_\delta, Y_\delta)$ where $W_\delta$ is a Gaussian process with zero mean, independent increments, and correlation matrix

$$\int_0^\tau E_{ij}^\delta(\bar{X}(s),s)ds, \tag{3.50}$$

and $Y_\delta$ is the solution of (3.44).

d) Using the arguments of White, one can show that if $\delta = \delta(\varepsilon) = o(1)$ as $\varepsilon \to 0$, then $(W_{\delta(\varepsilon)}^\varepsilon, Z_{\delta(\varepsilon)}^\varepsilon)$ converges weakly to $(W_0, Y_0)$ where $W_0$ is the zero mean Gaussian process having correlation matrix (3.38) and $Y_0$ is the solution of (3.44) with $\delta = 0$.

Lemma 6:

This lemma is a trivial modification of Lemma 1 of [11] and is stated without proof.

Let U be $F_{s+t}^\infty$- measurable and W be $F_0^s$- measurable and let $\alpha_1$ and $\alpha_2$ denote almost sure $\omega$ bounds for $\|U\|$ and $\|W\|$, respectively. Let $\rho(t)$ be defined by

$$\sup_{s>0} \sup_{\substack{A \in F_{t+s}^\infty \\ B \in F_0^s}} | P(A|B) - P(A) | = \rho(t) \downarrow 0 \text{ as } t \to \infty.$$

Then

$$\|E\{UW\} - E\{U\}E\{W\}\| \leq 2\alpha_1\alpha_2\rho(t). \tag{3.51}$$

Lemma 7:

This lemma is a restatement of two lemmas from Khas'minskii [8]. The results are stated without proof.

Let $(\Omega,F,P)$ be a probability space. Suppose that $F_s^t$, $-\infty \leq s \leq t < \infty$ is a family of $\sigma$-algebras of sets in $\Omega$ that satisfy

(1) $F_s^t \subset F$ for all s and t,

(2) $F_s^t \subset F_{s_1}^{t_1}$ if $s_1 \leq s$ and $t \leq t_1$ ,

(3) the family $F_s^t$ satisfies a stong mixing condition

$$\sup_t \sup_{\xi,\eta} \left| E\{\xi\eta\} - E\{\xi\} E\{\eta\} \right| = \alpha(\tau) \downarrow 0 \quad \text{as } \tau \to \infty,$$

where $\sup_{\xi,\eta}$ is taken over all $\xi$ measurable relative to $F_{-\infty}^{t}$ ,

$|\xi| < 1$ and $\eta$ measurable relative to $F_{t+\tau}^{\infty}$, $|\eta| < 1$.

Let $\phi_1(t,\omega),\cdots, \phi_{2k}(t,\omega)$ be stochastic processes in one-dimensional Euclidean space E such that $\{\omega:\phi_i(t,\omega) \in A\} \in F_t^t$ if $A \in B$, the set of all Borel sets on the real line.

a) Suppose $E\{\phi_i(t)\} = 0$, $|\phi_i(t)| < N$ and
$$\int_0^\infty \tau^{m-1}\alpha(\tau)d\tau = A_m < \infty,\ m = 1,2,\cdots,k.$$

Then the estimate

$$\int\cdots\int_{D_{2k}} \left| E\{\phi_1(s_1)\cdots\phi_{2k}(s_{2k})\} \right| ds_1\cdots ds_{2k} < c_{2k}N^{2k}T^k \tag{3.52}$$

holds for all $t_0 > 0$ and $T > 0$, where $D_{2k}$ is the cube $t_0 \leq s_i \leq t_0 + T$, $i=1,\cdots,2k$, and $c_{2k}$ is a constant depending only on $A_1,\cdots,A_k$.

b) Suppose instead of $E\{\phi_i(t)\} = 0$ we have

$$\left|\int_{t_0}^{t_0+T} E\{\phi_i(t,\omega)\}dt\right| \leq B\sqrt{T} \tag{3.53}$$

Then

$$\left| E \int\cdots\int_{D_{2k}} \phi_1(s_1,\omega)\cdots\phi_{2k}(s_{2k},\omega)ds_1\cdots ds_{2k}\right| < c^{(2k)}N^{2k}T^k \tag{3.54}$$

where $c^{(2k)}$ is a constant depending only on $A_1,\cdots,A_k$ and B.

c) The region $D_{2k}$ may be replaced in (3.54) by a region D which is a direct product of k 2-dimensional regions, each of which can be included in a square of side T.

We now present a proof of Theorem 2.

Proof of Theorem 2:

Recall that our goal is to exhibit a $\delta(\varepsilon) = o(1)$ as $\varepsilon \to 0$ such that the finite-dimensional distributions of $Y^{\varepsilon}_{\delta(\varepsilon)}$ defined by (3.36) converge to those of $Y_0$, defined as the solution of Volterra equation (3.37).

The key results in proving Theorem 2 are provided by Lemmas 8 and 9. Since these lemmas are lengthy, they are presented at the end of this chapter. In particular, Lemma 9 shows that there exists a $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$\lim_{\varepsilon\to 0} E\{\| Y^{\varepsilon}_{\delta(\varepsilon)}(\tau,\omega) - Z^{\varepsilon}_{\delta(\varepsilon)}(\tau,\omega)\|\} = 0, \quad \tau \in [0,\tau_0] \tag{3.55}$$

where $Z^{\varepsilon}_{\delta}$ is defined by (3.46).

Let $S \equiv \{\tau_i : 0 < \tau_1 < \tau_2 < \cdots < \tau_N < \tau_0\}$ be an arbitrary but fixed set of partition points. We must show that

$$\lim_{\varepsilon\to 0} P\{\|Y^{\varepsilon}_{\delta(\varepsilon)} - Y_0\| > 0 \text{ for at least one } \tau_i \in S\} = 0.$$

But, for any $c > 0$

$$0 \le P\{\|Y^{\varepsilon}_{\delta(\varepsilon)} - Y_0\| > c\} \le P\{\|Y^{\varepsilon}_{\delta(\varepsilon)} - Z^{\varepsilon}_{\delta(\varepsilon)}\| > c/2\}$$
$$+ P\{\|Z^{\varepsilon}_{\delta(\varepsilon)} - Y_0\| > c/2\}. \tag{3.56}$$

From Lemma 5d, the finite dimensional distributions of $Z^{\varepsilon}_{\delta(\varepsilon)}$ converge to those of $Y_0$. Therefore, for any $c > 0$,

$$\lim_{\varepsilon\to 0} P\{\|Z^{\varepsilon}_{\delta(\varepsilon)} - Y_0\| > c/2 \text{ for at least one } \tau_i \in S\} = 0.$$

From Lemma 9 (i.e. (3.55)) and Chebyshev's inequality

$$\lim_{\varepsilon\to 0} P\{\|Y^{\varepsilon}_{\delta(\varepsilon)} - Z^{\varepsilon}_{\delta(\varepsilon)}\| > c/2 \text{ for at least one } \tau_i \in S\} = 0$$

for any $c > 0$. Therefore, noting (3.56), Theorem 2 is proved.

We now consider the proof of Theorem 3. Recall that in this case, $\delta > 0$ is some fixed constant. The reason for this somewhat undesirable restriction is the fact that our tightness argument for $Y^{\varepsilon}_{\delta}$ fails to be true if $\delta = \delta(\varepsilon) = o(1)$ as $\varepsilon \to 0$.

<u>Proof of Theorem 3</u>:

Convergence of finite-dimensional distributions has been shown in Theorem 2. It remains to establish tightness. We must show that

(i) $\{Y^{\varepsilon}_{\delta}(0)\}$ is tight, and

(ii) $E\{\|Y^{\varepsilon}_{\delta}(\tau_2,\omega) - Y^{\varepsilon}_{\delta}(\tau_1,\omega)\|^4\} \le h\,|\tau_2 - \tau_1|^2$ for some positive constant h and all $\tau_1, \tau_2 \in [0, \tau_0]$. From (3.17), (3.20) and (3.36), $Y^{\varepsilon}_{\delta}(0,\omega) = 0$ for all

$\delta$ and $\varepsilon > 0$. Therefore $\{Y_\delta^\varepsilon(0)\}$ is tight and it remains to consider the fourth moment condition.

Noting (3.17), (3.20), and (3.36)

$$\|Y_\delta^\varepsilon(\tau_2,\omega)-Y_\delta^\varepsilon(\tau_1,\omega)\| = \|\frac{1}{\sqrt{\varepsilon}}[X_\delta^\varepsilon(\tau_2,\omega)-X_\delta^\varepsilon(\tau_1,\omega)-\bar{X}(\tau_2)+\bar{X}(\tau_1)\|$$

$$= \frac{1}{\sqrt{\varepsilon}}\|\int_{\tau_1}^{\tau_2}[A(X_\delta^\varepsilon(s,\omega),s,s/\varepsilon)+B(X_\delta^\varepsilon(s,\omega),s,s/\varepsilon)\xi_\delta(X_\delta^\varepsilon(s,\omega)+x_0(s/\varepsilon),\omega)$$

$$-\bar{A}(\bar{X}(s),s)]ds\|$$

$$= \frac{1}{\sqrt{\varepsilon}}\|\int_{\tau_1}^{\tau_2}\{A(X_\delta^\varepsilon(s,\omega),s,s/\varepsilon)-A(\bar{X}(s),s,s/\varepsilon)$$

$$+ [B(X_\delta^\varepsilon(s,\omega),s,s/\varepsilon)-B(\bar{X}(s),s,s/\varepsilon)]\xi_\delta(X_\delta^\varepsilon(s,\omega)+x_0(s/\varepsilon),\omega)$$

$$+ B(\bar{X}(s),s,s/\varepsilon)[\xi_\delta(X_\delta^\varepsilon(s,\omega)+x_0(s/\varepsilon),\omega)-\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)]$$

$$+ A(\bar{X}(s),s,s/\varepsilon)+B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)-\bar{A}(\bar{X}(s),s)\}ds\|$$

$$< \frac{1}{\sqrt{\varepsilon}}\int_{\tau_1}^{\tau_2}\|A(X_\delta^\varepsilon(s,\omega),s,s/\varepsilon)-A(\bar{X}(s),s,s/\varepsilon)\|ds$$

$$+ \frac{1}{\sqrt{\varepsilon}}\int_{\tau_1}^{\tau_2}\|B(X_\delta^\varepsilon(s,\omega),s,s/\varepsilon)-B(\bar{X}(s),s,s/\varepsilon)\|\cdot\|\xi_\delta(X_\delta^\varepsilon(s,\omega)+x_0(s/\varepsilon),\omega)\|ds$$

$$+ \frac{1}{\sqrt{\varepsilon}}\|\int_{\tau_1}^{\tau_2} B(\bar{X}(s),s,s/\varepsilon)[\xi_\delta(X_\delta^\varepsilon(s,\omega)+x_0(s/\varepsilon),\omega)-\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)]ds\|$$

$$+ \frac{1}{\sqrt{\varepsilon}}\|\int_{\tau_1}^{\tau_2} [A(\bar{X}(s),s,s/\varepsilon)+B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)-\bar{A}(\bar{X}(s),s)]\|.$$

Define

$$I(\tau_1,\tau_2) = \frac{1}{\sqrt{\varepsilon}}\int_{\tau_1}^{\tau_2}B(\bar{X}(s),s,s/\varepsilon)[\xi_\delta(X_\delta^\varepsilon(s,\omega)+x_0(s/\varepsilon),\omega)-$$

$$\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)]ds. \tag{3.57}$$

Then, noting (3.43), hypotheses (ii), and Lemma 2c,

$$\|Y_\delta^\varepsilon(\tau_2,\omega)-Y_\delta^\varepsilon(\tau_1,\omega)\| \le \frac{1}{\sqrt{\varepsilon}}\int_{\tau_1}^{\tau_2} L\|X_\delta^\varepsilon(s,\omega)-\bar{X}(s)\|ds$$

$$+\frac{1}{\sqrt{\varepsilon}}\int_{\tau_1}^{\tau_2} L\alpha\|X_\delta^\varepsilon(s,\omega)-\bar{X}(s)\|ds$$

$$+\|W_\delta^\varepsilon(\tau_2,\omega)-W_\delta^\varepsilon(\tau_1,\omega)\| + \|I(\tau_1,\tau_2)\|$$

$$= L(1+\alpha)\int_{\tau_1}^{\tau_2} \|Y_\delta^\varepsilon(s,\omega)\|ds + \|W_\delta^\varepsilon(\tau_2,\omega)-W_\delta^\varepsilon(\tau_1,\omega)\| + \|I(\tau_1,\tau_2)\|. \quad (3.58)$$

We first digress slightly and establish a bound for $E\{\|Y_\delta^\varepsilon(\tau,\omega)\|^2\}$. For the particular choices $\tau_1 = 0$ and $\tau_2 = \tau$,

$$\|Y_\delta^\varepsilon(\tau,\omega)\| \le L(1+\alpha)\int_0^\tau \|Y_\delta^\varepsilon(s,\omega)\|ds + \|W_\delta^\varepsilon(\tau,\omega)\| + \|I(0,\tau)\|.$$

Therefore

$$E\{\|Y_\delta^\varepsilon(\tau,\omega)\|^2\} \le 3L(1+\alpha)E\{[\int_0^\tau \| Y_\delta^\varepsilon(s,\omega)\|ds]^2\}$$

$$+ 3E\{\|W_\delta^\varepsilon(\tau,\omega)\|^2\} + 3E\{\|I(o,\tau)\|^2\},$$

where we have used the inequality

$$(a+b+c)^2 \le 3(a^2+b^2+c^2).$$

However

$$E\{\|I(0,\tau)\|^2\} \le 2E\{\|\frac{1}{\sqrt{\varepsilon}}\int_0^\tau B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(X_\delta^\varepsilon(s,\omega)+x_0(s/\varepsilon),\omega)ds\|^2\}$$

$$+ 2E\{\|\frac{1}{\sqrt{\varepsilon}}\int_0^\tau B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)ds\|^2\},$$

where we have used $(a+b)^2 \le 2(a^2+b^2)$.

Lemma 4 is used to estimate the first term while Lemmas 6 and 7a can be used to bound the second term. (This is because

$$\left\|\frac{1}{\sqrt{\varepsilon}}\int_0^{\tau} B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)ds\right\|^2$$

$$= \left\|\frac{1}{\sqrt{\varepsilon}}\int_0^{\tau/\varepsilon} B(\bar{X}(\varepsilon s),\varepsilon s,s)\xi_\delta(\bar{X}(\varepsilon s)+x_0(s),\omega)ds\right\|^2$$

$$= \varepsilon c\tau/\varepsilon = c\tau.)$$

We obtain

$$E\{\|I(0,\tau)\|^2\} \leq c_1\delta^{-1}\tau \leq c_1\delta^{-1} \quad \text{for } \tau \in [0,\tau_0].$$

Using the fact that $E\{\|W_\delta^\varepsilon\|^2\} \leq [E\{\|W_\delta^\varepsilon\|^4\}]^{1/2}$ and (3.49), we obtain the inequality $E\{\|W_\delta^\varepsilon(\tau,\omega)\|^2\} \leq c_2$, where $c_2$ is a constant that may be chosen independently of $\delta$ (recall that $0 < \delta \leq \delta_0 < \infty$). Thus, using the Cauchy-Schwarz inequality again,

$$E\{\|Y_\delta^\varepsilon(\tau,\omega)\|^2\} \leq 2L(1+\alpha)\tau\int_0^{\tau} E\{\|Y_\delta^\varepsilon(s,\omega)\|^2\}ds + c_3\delta^{-1}.$$

Applying the Gronwall inequality we obtain

$$E\{\|Y_\delta^\varepsilon(\tau,\omega)\|^2\} \leq c_4\delta^{-1} \text{ for } 0 \leq \tau \leq \tau_0 < \infty. \tag{3.59}$$

Having established bound (3.59), we reconsider inequality (3.58). The Mean Value inequality for vector-valued functions of vector-valued arguments and (3.24) lead to the estimate

$$\|\xi_\delta(X_\delta^\varepsilon(s,\omega)+x_0(s/\varepsilon),\omega)-\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)\| \leq c_5\delta^{-1}\|X_\delta^\varepsilon(s,\omega)-\bar{X}(s)\|.$$

Using this inequality and noting hypothesis (iii),

$$\|I(\tau_1,\tau_2)\| \leq \frac{1}{\sqrt{\varepsilon}}\int_{\tau_1}^{\tau_2}\|B(\bar{X}(s),s,s/\varepsilon)\|\cdot\|\xi_\delta(X_\delta^\varepsilon(s,\omega)+x_0(s/\varepsilon),\omega)$$

$$-\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)\|ds \leq Mc_5\delta^{-1}\int_{\tau_1}^{\tau_2}\|Y_\delta^\varepsilon(s,\omega)\|ds.$$

Thus, from (3.58),

$$\|Y_\delta^\varepsilon(\tau_2,\omega)-Y_\delta^\varepsilon(\tau_1,\omega)\| \leq [L(1+\alpha)+Mc_5\delta^{-1}] \int_{\tau_1}^{\tau_2} \|Y_\delta^\varepsilon(s,\omega)\| ds$$

$$+\|W_\delta^\varepsilon(\tau_2,\omega)-W_\delta^\varepsilon(\tau_1,\omega)\|.$$

Noting that $(a+b)^4 \leq (2[a^2+b^2])^2 \leq 8(a^4+b^4)$,

$$E\{\|Y_\delta^\varepsilon(\tau_2,\omega)-Y_\delta^\varepsilon(\tau_1,\omega)\|^4\} \leq 8c_6\delta^{-1}E\{[\int_{\tau_1}^{\tau_2} \|Y_\delta^\varepsilon(s,\omega)\| ds]^4\}$$

$$+ 8E\{\|W_\delta^\varepsilon(\tau_2,\omega)-W_\delta^\varepsilon(\tau_1,\omega)\|^4\}.$$

Applying the Cauchy-Schwarz inequality and Lemma 5b we obtain

$$E\{\|Y_\delta^\varepsilon(\tau_2,\omega)-Y_\delta^\varepsilon(\tau_1,\omega)\|^4\} \leq 8c_6\delta^{-1}|\tau_2-\tau_1|^3 \int_{\tau_1}^{\tau_2} E\{\|Y_\delta^\varepsilon(s,\omega)\|^4\} ds$$

$$+ c_7|\tau_2-\tau_1|^2. \tag{3.60}$$

Let $\tau_1 = 0$ and $\tau_2 = \tau$. Then, in particular,

$$E\{\|Y_\delta^\varepsilon(\tau,\omega)\|^4\} \leq 4c_6\delta^{-1}\tau^3 \int_0^\tau E\{\|Y_\delta^\varepsilon(s,\omega)\|^4\} ds + c_7\tau^2.$$

Applying the Gronwall inequality, we obtain

$$E\{\|Y_\delta^\varepsilon(\tau,\omega)\|^4\} \leq \exp[c_8\delta^{-1}], \qquad 0 \leq \tau \leq \tau_0. \tag{3.61}$$

Thus, using (3.61) in (3.60) leads to

$$E\{\|Y_\delta^\varepsilon(\tau_2,\omega)-Y_\delta^\varepsilon(\tau_1,\omega)\|^4\} \leq 8c_6\delta^{-1}|\tau_2-\tau_1|^3 \int_{\tau_1}^{\tau_2} \exp(c_8\delta^{-1}) ds + c_7|\tau_2-\tau_1|^2$$

$$\leq \exp(c_9\delta^{-1})|\tau_2-\tau_1|^2 \text{ for } \tau_1,\tau_2 \in [0,\tau_0].$$

We conclude that, for fixed $\delta > 0$ and $\varepsilon \to 0$, $\{Y_\delta^\varepsilon(\tau,\omega)\}$ is tight in $C([o,\tau_0];R^n)$. Since the finite-dimensional distributions of $Y_\delta^\varepsilon(\tau,\omega)$ converge to $Y_\delta^\varepsilon(\tau,\omega)$ (by virtue of Theorem 2), $Y_\delta^\varepsilon(\tau,\omega)$ converges weakly to $Y_\delta(\tau,\omega)$ and Theorem 3 is proven.

We now prove Lemmas 8 and 9 to conclude the arguments.

Lemma 8: Let

$$I_\delta^\varepsilon(\tau,\omega) = \int_0^\tau \{\frac{\partial A(\bar{X}(s),s,s/\varepsilon)}{\partial x} - \frac{\partial \bar{A}(\bar{X}(s),s)}{\partial x}$$

$$- \frac{\partial}{\partial x}[B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)]\}Z_\delta^\varepsilon(s,\omega)ds,$$

where $Z_\delta^\varepsilon(\tau,\omega)$ is defined as the solution of (3.44). Then there exists a positive constant c such that

$$E\{\|I_\delta^\varepsilon(\tau,\omega)\|^2\} \leq c\varepsilon\delta^{-2}. \tag{3.62}$$

Proof:

$$Z_\delta^\varepsilon(\tau,\omega) = W_\delta^\varepsilon(\tau,\omega) + \int_0^\tau \frac{\partial \bar{A}(\bar{X}(s),s)}{\partial x} Z_\delta^\varepsilon(s,\omega)ds. \tag{3.46}$$

Since $\|\frac{\partial \bar{A}(\bar{X}(s),s)}{\partial x}\| \leq M$ where the constant M is independent of $\tau$, there exists a (non-random) differentiable kernel $K(\tau,s)$ defined on $[0,\tau_0]X[0,\tau_0]$ satisfying

$$\|K(\tau,s)\| \leq M\exp[M(\tau-s)] \leq c_1, \quad 0 \leq \tau,s < \tau_0 \tag{3.63}$$

(c.f. Miller [19]) such that

$$\bar{Z}_\delta^\varepsilon(\tau,\omega) = W_\delta^\varepsilon(\tau,\omega) + \int_0^\tau K(\tau,s)W_\delta^\varepsilon(s,\omega)ds.$$

Therefore

$$E\{\|I_\delta^\varepsilon I(\tau,\omega)\|^2\} = E\{\|\int_0^\tau \Big[\frac{\partial A(\bar{X}(s),s,s/\varepsilon)}{\partial x} - \frac{\partial \bar{A}(\bar{X}(s),s)}{\partial x} +$$

$$+ \frac{\partial}{\partial x}\left[B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)\right]\Big] [W_\delta^\varepsilon(s,\omega)$$

$$+ \int_0^s K(s,u)\; W_\delta^\varepsilon(u,\omega)du]ds\|^2\}$$

$$<2E\ \{\|\int_0^\tau \Big[\frac{\partial A(\bar{X}(s),s,s/\varepsilon)}{\partial x} - \frac{\partial \bar{A}(\bar{X}(s),s)}{\partial x}$$

$$+ \frac{\partial}{\partial x}\left[B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)\right]\Big]\ W_\delta^\varepsilon(s,\omega)ds\|^2\}$$

$$+2E\{\|\int_0^\tau \Big[\frac{\partial A(\bar{X}(s),s,s/\varepsilon)}{\partial x} - \frac{\partial \bar{A}(\bar{X}(s),s)}{\partial x}$$

$$+ \frac{\partial}{\partial x}\left[B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)\right]\Big]\int_0^s K(s,u)W_\delta^\varepsilon(u,\omega)duds\|^2\}$$

$$\equiv 2I_{\delta 1}^\varepsilon(\tau,\omega) + 2I_{\delta 2}^\varepsilon(\tau,\omega). \tag{3.64}$$

Let
$$\phi(s,s/\varepsilon) = A(\bar{X}(s),s,s/\varepsilon)+B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega) -\bar{A}(\bar{X}(s),s). \tag{3.65}$$

Let $\phi_k$ be the $k^{th}$ element of $\phi$ and $\phi_{kj} = \frac{\partial}{\partial x_j}\phi_k$. Then

$$W_\delta^\varepsilon(s,\omega) = \frac{1}{\sqrt{\varepsilon}}\int_0^s \phi(u,u/\varepsilon)du$$

and

$$I_{\delta_1}^\varepsilon(\tau,\omega) = E\{\|\frac{1}{\sqrt{\varepsilon}}\int_0^\tau \frac{\partial\phi(s,s/\varepsilon)}{\partial x}\int_0^s \phi(u,u/\varepsilon)du\ ds\|^2\},$$

$$I_{\delta 2}^\varepsilon(\tau,\omega) = E\{\|\frac{1}{\sqrt{\varepsilon}}\int_0^\tau \frac{\partial\phi(s,s/\varepsilon)}{\partial x}\int_0^s K(s,u)\int_0^u \phi(v,v/\varepsilon)dvduds\|^2\}.$$

Consider the $k^{th}$ term of $I_{\delta 1}^\varepsilon$,

$$E\{[\sum_j \frac{1}{\sqrt{\varepsilon}} \int_0^\tau \phi_{kj}(s,s/\varepsilon) \int_0^s \phi_j(u,u/\varepsilon)duds]^2\}$$

$$= E\{[\sum_j \varepsilon^{3/2} \int_0^{\tau/\varepsilon} \phi_{kj}(\varepsilon s,s) \int_0^s \phi_j(\varepsilon u,u)duds]^2\}.$$

Since $E\{\phi_k(\varepsilon s,s)\} = [A(\bar{X}(\varepsilon s),\varepsilon s,s)-\bar{A}(\bar{X}(\varepsilon s),\varepsilon s)]_k$

and $E\{\phi_{kj}(\varepsilon s,s)\} = [\frac{\partial}{\partial x_j}A(\bar{X}(\varepsilon s),\varepsilon s,s) - \frac{\partial}{\partial x_j}\bar{A}(\bar{X}(\varepsilon s),\varepsilon s)]_k$,

from (3.18) and (3.19),

$$\left|\int_0^\tau E\{\phi_k(\varepsilon s,s)\}ds\right| \leq c_2\tau \text{ and} \tag{3.66}$$

$$\left|\int_0^\tau E\{\phi_{kj}(\varepsilon s,s)\}ds\right| \leq c_2\tau. \tag{3.67}$$

Also, note, using Lemma 2,

$$|\phi_k| \leq c_3 \tag{3.68}$$

and $$|\phi_{kj}| \leq c_4\delta^{-1}. \tag{3.69}$$

Thus we can apply (3.54) with $D_2$ replaced by $D$ in Lemma 7c with

$D = \{(u,s) : o \leq u \leq s,\ o \leq s \leq \tau/\varepsilon\} \times \{(u,s) : o \leq u \leq s,\ o \leq s \leq \tau/\varepsilon\}$.

(Note that if $\tau > 1$ choose $B = c_2\sqrt{\tau_0}$ and if $\tau \leq 1$, $c_2\tau \leq c_2\sqrt{\tau}$.)

$$E\{[\sum_j \varepsilon^{3/2}\int_0^{\tau/\varepsilon} \phi_{kj}(\varepsilon s,s) \int_0^s \phi_j(\varepsilon u,u)duds]^2\}$$

$$= \varepsilon^3 \sum_{j,\ell} E\{\int_0^{\tau/\varepsilon}\int_0^{\tau/\varepsilon}\int_0^{s_1}\int_0^{s_2} \phi_{kj}(\varepsilon s_1,s_1)\phi_{k\ell}(\varepsilon s_2,s_2)\cdot$$

$$\phi_j(\varepsilon u_1,u_1)\phi_\ell(\varepsilon u_2,u_2)\}du_2du_1ds_2ds_1 \leq \varepsilon^3 c_5\delta^{-2}(\frac{\tau}{\varepsilon})^2$$

$$= c_6\varepsilon\delta^{-2}, \text{ for } 0 \leq \tau \leq \tau_0. \tag{3.70}$$

Now consider the $k^{th}$ term of $I^{\varepsilon}_{\delta_2}$,

$$E\{[\sum_j \frac{1}{\sqrt{\varepsilon}}\int_0^{\tau}\phi_{kj}(s,s/\varepsilon)\int_0^{s}\sum_{\ell}K_{j\ell}(s,u)\int_0^{u}\phi_{\ell}(v,v/\varepsilon)dvduds]^2\}$$

$$< E\{\varepsilon^3[\sum_j\int_0^{\tau/\varepsilon}\phi_{kj}(\varepsilon s,s)\int_0^{\varepsilon s}\sum_{\ell}K_{j\ell}(\varepsilon s,u)\int_0^{u/\varepsilon}\phi_{\ell}(\varepsilon v,v)dvduds]^2\}$$

$$= \varepsilon^3 E\{[\sum_j\int_0^{\tau}\int_0^{u/\varepsilon}\int_0^{u/\varepsilon}\phi_{kj}(\varepsilon s,s)\sum_{\ell}K_{j\ell}(\varepsilon s,u)\phi_{\ell}(\varepsilon v,v)dvdsdu]^2\}$$

But from (3.19) and the differentiability of K

$$\left|\int_0^{\tau/\varepsilon} E\phi_{kj}(\varepsilon s,s)K_{j\ell}(\varepsilon s,u)ds\right| < c_7\tau. \tag{3.71}$$

Also, using Lemma 2 and (3.63)

$$|\phi_{kj}K_{j\ell}| < c_8\delta^{-1}. \tag{3.72}$$

Noting (3.66) and (3.68), we may apply (3.54) to obtain the bound

$$\varepsilon^3\sum_{j,j_1}\int_0^{\tau}\int_0^{\tau} E\{\int_{u_1/\varepsilon}^{\tau/\varepsilon}\int_{u_1/\varepsilon}^{\tau/\varepsilon}\int_0^{u_1/\varepsilon}\int_0^{u_2/\varepsilon}\phi_{kj}(\varepsilon s_1,s_1)\phi_{kj_1}(\varepsilon s_2,s_2)\cdot$$

$$\sum_{\ell,\ell_1}K_{j\ell}(\varepsilon s_1,u_1)K_{j\ell_1}(\varepsilon s_2,u_2)\phi_{\ell}(\varepsilon v_1,v_1)\phi_{\ell_1}(\varepsilon v_2,v_2)dv_1dv_2ds_1ds_2\}du_1du_2$$

$$< \varepsilon^3\int_0^{\tau}\int_0^{\tau} c_9\delta^{-2}(\tau/\varepsilon)^2du_1du_2$$

$$< c_{10}\varepsilon\delta^{-2}, \quad \tau \in [0,\tau_0]. \tag{3.73}$$

Substituting (3.70) and (3.73) into (3.64),

$$E\{\|I^{\varepsilon}_{\delta}(\tau,\omega)\|^2\{ < c\varepsilon\delta^{-2}.$$

Lemma 9:

From (3.46), $Z^{\varepsilon}_{\delta}(\tau,\omega)$ is defined to be the solution of

$$Z^{\varepsilon}_{\delta}(\tau,\omega) = W^{\varepsilon}_{\delta}(\tau,\omega) + \int_0^{\tau} \frac{\partial\bar{A}(\bar{X}(s),s)}{\partial x} Z^{\varepsilon}_{\delta}(s,\omega)ds.$$

There exists a $\delta=\delta(\varepsilon) = o(1)$ such that

$$\lim_{\varepsilon\to 0} \{ E\|Y^{\varepsilon}_{\delta}(\tau,\omega) - Z^{\varepsilon}_{\delta}(\tau,\omega)\|\} = 0, \quad \tau \in [0,\tau_0]. \tag{3.74}$$

Proof: Define the difference vector

$$U^{\varepsilon}_{\delta}(\tau,\omega) \equiv Y^{\varepsilon}_{\delta}(\tau,\omega) - Z^{\varepsilon}_{\delta}(\tau,\omega). \tag{3.75}$$

Then

$$U^{\varepsilon}_{\delta}(\tau,\omega) = \frac{1}{\sqrt{\varepsilon}}[X^{\varepsilon}_{\delta}(\tau,\omega)-\bar{X}(\tau)]-W^{\varepsilon}_{\delta}(\tau,\omega)-\int_0^{\tau} \frac{\partial\bar{A}(\bar{X}(s),s)}{\partial x} Z^{\varepsilon}_{\delta}(s,\omega)ds$$

$$= \frac{1}{\sqrt{\varepsilon}}\int_0^{\tau}[A(X^{\varepsilon}_{\delta}(s,\omega),s,s/\varepsilon)-A(\bar{X}(s),s,s/\varepsilon)-\sqrt{\varepsilon}\frac{\partial\bar{A}(\bar{X}(s),s)}{\partial x} Z^{\varepsilon}_{\delta}(s,\omega)]ds$$

$$+ \frac{1}{\sqrt{\varepsilon}}\int_0^{\tau} [B(X^{\varepsilon}_{\delta}(s,\omega),s,s/\varepsilon)\xi_{\delta}(X^{\varepsilon}_{\delta}(s,\omega)+x_0(s/\varepsilon),\omega)-B(\bar{X}(s),s,s/\varepsilon)\cdot$$

$$\cdot\xi_{\delta}(\bar{X}(s)+x_0(s/\varepsilon),\omega)]ds$$

$$= \frac{1}{\sqrt{\varepsilon}}\int_0^{\tau}[A(X^{\varepsilon}_{\delta}(s,\omega),s,s/\varepsilon)-A(\bar{X}(s),s,s/\varepsilon)- \frac{\partial A(\bar{X}(s),s,s/\varepsilon)}{\partial x}\sqrt{\varepsilon}Y^{\varepsilon}_{\delta}(s,\omega)]ds +$$

$$+\frac{1}{\sqrt{\varepsilon}}\int_0^\tau\left[\frac{\partial A(\bar{X}(s),s,s/\varepsilon)}{\partial x}\sqrt{\varepsilon}Y_\delta^\varepsilon(s,\omega)-\frac{\partial \bar{A}(\bar{X}(s),s)}{\partial x}\sqrt{\varepsilon}Z_\delta^\varepsilon(s,\omega)\right.$$

$$\left.+\frac{\partial}{\partial x}[B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)]\sqrt{\varepsilon}Z_\delta^\varepsilon(s,\omega)\right]ds$$

$$+\frac{1}{\sqrt{\varepsilon}}\int_0^\tau\left[B(X_\delta^\varepsilon(s,\omega),s,s/\varepsilon)\xi_\delta(X_\delta^\varepsilon(s,\omega)+x_0(s/\varepsilon),\omega)\right.$$

$$-B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)$$

$$\left.-\frac{\partial}{\partial x}[B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)]\sqrt{\varepsilon}Y_\delta^\varepsilon(s,\omega)\right]ds$$

$$+\frac{1}{\sqrt{\varepsilon}}\int_0^\tau\frac{\partial}{\partial x}[B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)]\sqrt{\varepsilon}[Y_\delta^\varepsilon(s,\omega)-Z_\delta^\varepsilon(s,\omega)]ds$$

$$=\frac{1}{\sqrt{\varepsilon}}\int_0^\tau[A(X_\delta^\varepsilon(s,\omega),s,s/\varepsilon)-A(\bar{X}(s),s,s/\varepsilon)-\frac{\partial A(\bar{X}(s),s,s/\varepsilon)}{\partial x}\sqrt{\varepsilon}Y_\delta^\varepsilon(s,\omega)]ds$$

$$+\frac{1}{\sqrt{\varepsilon}}\int_0^\tau[\frac{\partial A(\bar{X}(s),s,s/\varepsilon)}{\partial x}\sqrt{\varepsilon}Y_\delta^\varepsilon(s,\omega)-\frac{\partial A(\bar{X}(s),s,s/\varepsilon)}{\partial x}\sqrt{\varepsilon}Z_\delta^\varepsilon(s,\omega)]ds$$

$$+\frac{1}{\sqrt{\varepsilon}}\int_0^\tau\frac{\partial}{\partial x}[B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)]\sqrt{\varepsilon}U_\delta^\varepsilon(s,\omega)ds$$

$$+\frac{1}{\sqrt{\varepsilon}}\int_0^\tau\left[\frac{\partial A(\bar{X}(s),s,s/\varepsilon)}{\partial x}-\frac{\partial \bar{A}(\bar{X}(s),s)}{\partial x}+\frac{\partial}{\partial x}[B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)\right.$$

$$\left.+x_0(s/\varepsilon),\omega)]\right]\sqrt{\varepsilon}\,Z_\delta^\varepsilon(s,\omega)ds\ +$$

$$+ \frac{1}{\sqrt{\varepsilon}} \int_0^{\tau} \Big[ B(X_\delta^\varepsilon(s,\omega),s,s/\varepsilon)\xi_\delta(X_\delta^\varepsilon(s,\omega)+x_0(s/\varepsilon),\omega)$$

$$- B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)$$

$$- \frac{\partial}{\partial x}[B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)\sqrt{\varepsilon}Y_\delta^\varepsilon(s,\omega)\Big] \, ds$$

Let $I_\delta^\varepsilon(\tau,\omega) \equiv \int_0^{\tau} \Big[ \frac{\partial A(\bar{X}(s),s,s/\varepsilon)}{\partial x} - \frac{\partial \bar{A}(\bar{X}(s),s)}{\partial x}$

$$+ \frac{\partial}{\partial x}[B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)]\Big] \; Z_\delta^\varepsilon(s,\omega)ds.$$

Then

$$\|U_\delta^\varepsilon(\tau,\omega)\| \leq \frac{1}{\sqrt{\varepsilon}}\int_0^{\tau} \|A(X_\delta^\varepsilon(s,\omega),s,s/\varepsilon)-A(\bar{X}(s),s,s/\varepsilon)$$

$$- \frac{\partial A(\bar{X}(s),s,s/\varepsilon)}{\partial x}\sqrt{\varepsilon}Y_\delta^\varepsilon(s,\omega)\| ds$$

$$+ \int_0^{\tau} \Big\{ \Big\|\frac{\partial A(\bar{X}(s),s,s/\varepsilon)}{\partial x}\Big\| + \Big\|\frac{\partial}{\partial x}[B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)]\Big\|\Big\}$$

$$\cdot \|U_\delta^\varepsilon(s,\omega)\| ds + \| I_\delta^\varepsilon(\tau,\omega)\|$$

$$+ \frac{1}{\sqrt{\varepsilon}} \int_0^{\tau}\Big\{\|B(X_\delta^\varepsilon(s,\omega),s,s/\varepsilon)\xi_\delta(X_\delta^\varepsilon(s,\omega)+x_0(s/\varepsilon),\omega)-B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)$$

$$+ x_0(s/\varepsilon),\omega) - \frac{\partial}{\partial x}[B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)]\sqrt{\varepsilon}Y_\delta^\varepsilon(s,\omega)\| \, ds.$$

Noting hypothesis (iii) and Lemma 1,

$$\|A(X_\delta^\varepsilon(s,\omega),s,s/\varepsilon)-A(\bar{X}(s),s,s/\varepsilon)-\frac{\partial A(\bar{X}(s),s,s/\varepsilon)}{\partial x}[X_\delta^\varepsilon(s,\omega)-\bar{X}(s)]\|$$

$$< M\varepsilon\ \|Y_\delta^\varepsilon(s,\omega)\|^2$$

Likewise, noting Lemma 2 as well,

$$\| B(X_\delta^\varepsilon(s,\omega),s,s/\varepsilon)\xi_\delta(X_\delta^\varepsilon(s,\omega)+x_0(s/\varepsilon),\omega)-B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)$$
$$-\frac{\partial}{\partial x}[B(\bar{X}(s),s,s/\varepsilon)\xi_\delta(\bar{X}(s)+x_0(s/\varepsilon),\omega)]\sqrt{\varepsilon}Y_\delta^\varepsilon(s,\omega)\| < M\varepsilon\delta^{-2}\|Y_\delta^\varepsilon(s,\omega)\|^2.$$

Therefore

$$E\{\|U_\delta^\varepsilon(\tau,\omega)\|\} < c_3\delta^{-1}\int_0^\tau E\{\|U_\delta^\varepsilon(s,\omega)\|\}ds+\sqrt{\varepsilon}\ c_4\delta^{-2}\int_0^\tau E\{\|Y_\delta^\varepsilon(s,\omega)\|^2\}ds$$
$$+ E\{\| I_\delta^\varepsilon(\tau,\omega)\|\}.$$

However, we have shown (c.f.(3.59) that

$$E\{\|Y_\delta^\varepsilon(s,\omega)\|^2\} < c_5\delta^{-1}.$$

Also, from Lemma 8,

$$E\{\|I_\delta^\varepsilon(\tau,\omega)\|^2\}< c_6\varepsilon\delta^{-2}.$$

Therefore

$$E\{\|U_\delta^\varepsilon(\tau,\omega)\|\} < c_3\delta^{-1}\int_0^\tau E\{\|U_\delta^\varepsilon(s,\omega)\|\}ds+\sqrt{\varepsilon}\ c_4\delta^{-3}c_5\int_0^\tau ds + c_6\varepsilon\delta^{-2}$$

$$< c_3\delta^{-1}\int_0^\tau E\{\|U_\delta^\varepsilon(s,\omega)\|\}ds + c_7\sqrt{\varepsilon}\delta^{-3}.$$

Applying the Gronwall inequality,

$$E\{\|U_\delta^\varepsilon(\tau,\omega)\|\} < c_8\sqrt{\varepsilon}\delta^{-3}\exp\left(\frac{c\tau_0}{\delta}\right).$$

In particular, therefore, for $\delta = \delta(\varepsilon) = \dfrac{4c\tau_0}{\ln(1/\varepsilon)}$,

$\delta \to 0$ as $\varepsilon \to 0$ and

$$\sqrt{\varepsilon}\delta^{-3}(\varepsilon)\exp\left(\frac{c\tau_0}{\delta(\varepsilon)}\right) = \sqrt{\varepsilon}\frac{[\ln(1/\varepsilon)]^3}{16c^2\tau_0^2}\exp\left(\frac{\ln(1/\varepsilon)}{4}\right)$$

$$= O(\varepsilon^{1/4}[\ln(1/\varepsilon)]^3) = o(1) \text{ as } \varepsilon \to 0.$$

For this choice of $\delta$,

$$\lim_{\varepsilon\to 0} E\{\|U_\delta^\varepsilon(\tau,\omega)\|\} = \lim_{\varepsilon\to 0} E\{\|Y_\delta^\varepsilon(\tau,\omega) - Z_\delta^\varepsilon(\tau,\omega)\|\} = 0.$$

## IV. Concluding Remarks

The analysis of Chapter III leads to the conclusion that the error in the gravity model for the inertial navigation system considered may be asymptotically approximated by the superposition of a deterministic centering trajectory and a suitably scaled Gauss Markov fluctuation process. The use of mollifiers is a technical artifice. The basic processes of physical interest are $W_0$ and $Y_0$ and the basic asymptotic approximation is

$$X^{\varepsilon}(\tau,\omega) \cong \sqrt{\varepsilon}\, Y_0\ (\tau,\omega) + \bar{X}(\tau) \tag{4.1}$$

where $\bar{X}(\tau)$ is the solution to the ordinary differential equation (3.17)

$$\frac{d\bar{X}(\tau)}{d\tau} = \bar{A}(\bar{X}(\tau))\ , \qquad \bar{X}(0) = 0 \tag{4.2}$$

and $Y_0(\tau,\omega)$ is the solution to (3.44) with $\delta = 0$, i.e.,

$$Y_0(\tau,\omega) = W_0(\tau,\omega) + \int_0^{\tau} \frac{\partial \bar{A}(\bar{X}(s),s)}{\partial x}\, Y_0(s,\omega)ds \tag{4.3}$$

where $W_0$ is the zero mean, independent increment Gaussian process having correlation matrix (3.38)

$$\int_0^{\tau} E_{ij}^{0}\ (\bar{X}(s),s)ds \qquad\qquad i,j = 1,\cdots,4$$

and $E_{ij}^{0}$ is defined by (3.15).

The computational complexity of this approximation is relatively simple for two reasons. First, note only the first two moments of the random process model of the deflection of the vertical need be specified.

The mean of the process is included in the A matrix, and hence $\bar{A}$. The covariance is included in the specification of the $W_0$ process.

This approximation, in terms of a canonical fluctuation process, should prove useful in the error analysis and subsequent control of inertial navigation systems. The idealistic assumptions of stationarity and isotropy are not required. The fundamental assumptions of sample function continuity, almost sure boundedness and the strong mixing property seem reasonable on physical grounds.

The second reason is that when the results of Chapter III are reapplied to the basic problem of interest (c.f. (2.31)-(2.40)), some structural simplification results. Let

$A,B,\xi$ correspond to their use in Chapter III,

$x_1, x_2$ be the 2X1 vectors $x_1 \equiv \delta r$ and $x_2 = \frac{d\delta r}{d\tau}$, and

$\mathcal{A}, \mathcal{B}$ correspond to A,B as used in Chapter II.

Then (2.37) becomes

$$\frac{d}{d\tau}\begin{bmatrix} x_1 \\ \hline x_2 \end{bmatrix} = \left[\begin{array}{c|c} 0 & I \\ \hline \mathcal{A} & 0 \end{array}\right] \begin{bmatrix} x_1 \\ \hline x_2 \end{bmatrix} + \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathcal{B} \end{array}\right] \begin{bmatrix} 0 \\ \hline \phi \end{bmatrix} ,$$

$$\begin{bmatrix} x_1(0) \\ \hline x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ \hline 0 \end{bmatrix} \tag{4.4}$$

Let $\bar{\phi} = E\{\phi\}$, $\delta\phi = \phi - \bar{\phi}$. Then

$$A = \begin{bmatrix} x_2 \\ \hline \mathcal{A}x_1 + \mathcal{B}\bar{\phi} \end{bmatrix} = \begin{bmatrix} x_2 \\ \hline A_2(x_1,\tau,t) \end{bmatrix} , \tag{4.5}$$

$$B = \left[\begin{array}{cc} 0 & 0 \\ \hline 0 & \mathcal{B} \end{array}\right] \quad \text{, and} \tag{4.6}$$

$$\xi = \left[\begin{array}{c} 0 \\ \hline \delta\phi \end{array}\right] \quad . \tag{4.7}$$

Therefore $\bar{A}$ has the structure

$$\bar{A} = \left[\begin{array}{c} x_2 \\ \hline \bar{A}_2(x_1,\tau) \end{array}\right] \tag{4.8}$$

and $W_0(\tau,\omega)$ has the structure

$$W_0(\tau,\omega) = \left[\begin{array}{c} 0 \\ \hline w_0(\tau,\omega) \end{array}\right] \quad . \tag{4.9}$$

Also

$$\frac{\partial \bar{A}}{\partial x} = \left[\begin{array}{c|c} 0 & I \\ \hline \frac{\partial \bar{A}_2}{\partial x_1} & 0 \end{array}\right] \tag{4.10}$$

and $Y_0$ satisfies a Volterra equation of the form

$$Y_0(\tau,\omega) = \left[\begin{array}{c} 0 \\ \hline w_0(\tau,\omega) \end{array}\right] + \int_0^\tau \left[\begin{array}{c|c} 0 & I \\ \hline \frac{\partial \bar{A}_2(\bar{X}(s),s)}{\partial x_1} & 0 \end{array}\right] Y_0(s,\omega)ds. \tag{4.11}$$

In summary, it should be emphasized that $\bar{X}(\tau)$ and the statistics of $Y_0(\tau,\omega)$, which are used to asymptotically characterize $X^{\varepsilon}(\tau)$, are explicitly computable. $Y_0(\tau,\omega)$ is a Gaussian process which is amenable to standard signal processing techniques. The procedure for evaluating the approximating process is:

a. Specify the nominal vehicle trajectory $x_0$, which would be the actual trajectory in the absence of any error in the gravity model.

b. Formulate a model for the error in the gravity model consisting of two parts, a large-scale deterministic model and a short wavelength random process model for the deflection of the vertical errors.

c. The dynamic equations, the deterministic error model, and the mean of the random process error model determine A and hence $\bar{A}$. $\bar{X}(\tau)$ is determined as the solution of the initial value problem (4.2). The sum $\bar{X}(\tau)+x_0(\tau/\varepsilon)$ forms a corrected centering trajectory (which accounts for the known large-scale gravity model errors).

d. $W_0(\tau,\omega)$ is a Gaussian process with zero mean, independent increments and correlation matrix computed along the trajectory $x_0 + \bar{X}$:

$$\int_0^{\tau} \lim_{T\to\infty} \frac{1}{T} \int_0^{T} \int_0^{T} E\{[B(\bar{X}(\sigma),\sigma,s)\xi(\bar{X}(\sigma)+x_0(s),\omega)]_i \cdot$$

$$\cdot [B(\bar{X}(\sigma),\sigma,t)\xi(\bar{X}(\sigma)+x_0(t),\omega)]_j\} ds\,dt\,d\sigma. \quad (4.12)$$

e. $Y_0(\tau,\omega)$ is a Gaussian process with zero mean. The correlation matrix (which completely determines the statistics of $Y_0(\tau,\omega)$) may be explicitly computed from the statistics of $W_0(\tau,\omega)$ using (4.3). Let $K(\tau,s)$ be the resolvent kernel for (4.3). Then

$$K(\tau,s) = \Phi(\tau,s) \frac{\partial \bar{A}(\bar{X}(s),s)}{\partial x} \quad (4.13)$$

where $\Phi(\tau,s)$ is the fundamental matrix solution to

$$\frac{d}{d\tau}\Phi(\tau,s) = \frac{\partial \bar{A}(\bar{X}(\tau),\tau)}{\partial x} \Phi(\tau,s), \quad \Phi(s,s) = I. \quad (4.14)$$

Then

$$Y_0(\tau,\omega) = W_0(\tau,\omega) + \int_0^{\tau} K(\tau,s)W_0(s,\omega)ds \quad (4.15)$$

can be used to explicitly compute the correlation matrix for $Y_0(\tau,\omega)$.

f. The asymptotic approximation $X^{\varepsilon}(\tau,\omega)$ is also a Gaussian process and the mean of $X^{\varepsilon}(\tau,\omega)$ is $\bar{X}(\tau)$. The correlation matrix is easily computable from the statistics of $Y_0(\tau,\omega)$ via (4.1).

This asymptotic approximation provides a mathematical framework which enables error analyses of inertial navigation systems to be performed using a more general class of random process models for the deflection of the vertical errors. These models are not restricted by the physically unrealistic conditions that have been heretofore imposed. Note that one ultimately needs only the knowledge of the second order statistics of the deflection of the vertical errors, evaluated along the path $x_0+\bar{X}$, in order to evaluated the asymptotic approximation.

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Asymptotic Stochastic Analysis of a Gravity Model

for Inertial Navigation Systems

by

Mark T. Torgrimson

(ABSTRACT)

Inertial navigation systems require a precise knowledge of gravity to function properly. The inability of models to account for the small amplitude, short wavelength components of the gravity field leads to errors which are frequently viewed as random; these random errors can introduce a significant cumulative impact on system performance.

A model is studied which, in the context of an appropriate scaling, consists of a gravity field having a known deteministic long scale behavior and an unknown random short scale behavior. The short wavelength random fluctuations are assumed to satisfy a strong mixing (asymptotic independence) property; no a priori stationary or isotropy assumptions are made. Results of Khas'minskii (Theory of Probability and Its Applications, Vol. XI, No. 2, 1966, pp 211-228) are extended and applied. In an appropriate asymptotic limit, the vehicle motion is approximated by the sum of a deterministic trajectory and a Gauss-Markov fluctuation process.