FATIGUE LIMIT ANALYSIS INVOLVING BIAXIAL
STRESS COMPONENTS

by

Edgar Gray Munday

Dissertation submitted to the Graduate Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Mechanical Engineering

APPROVED:

L. D. Mitchell, Chairman

H. H. Mabie

K. L. Reifsnider

R. G. Mitchiner

W. W. Stinchcomb

R. A. Comparin

February, 1984

Blacksburg, Virginia

DEDICATION

To my parents, Mr. and Mrs. Ernest Logan Munday.

ACKNOWLEDGEMENTS

The author would like to express his gratitude to his major advisor Dr. Larry D. Mitchell for all his advice, assistance, and guidance throughout this work.

The author also wishes to thank the other members of his committee, Dr. Robert A. Comparin, Dr. Hamilton H. Mabie, Dr. Reginald G. Mitchiner, Dr. Kenneth L. Reifsnider, and Dr. Wayne W. Stinchcomb, for their encouragement and assistance both in and out of the classroom. Thanks are also in order to Ms. Eloise Lafon for her patience in typing this dissertation.

Finally, the author would be especially remiss if he did not acknowledge the help of God.

TABLE OF CONTENTS

TABLE OF CONTENTS (cont.)

LIST OF TABLES

LIST OF FIGURES

LIST OF FIGURES (cont.)

NOMENCLATURE

C	Ratio of reversed bending fatigue limit to reversed torsion fatigue limit squared.
$\overline{C}$	Ratio of reversed normal stress fatigue limit with zero gradient to reversed shear fatigue limit with zero gradient squared.
d	Diameter of fatigue specimen
F_a	Magnitude of sinusoidally varying force
L_1	Half-length of ellipse major axis
L_2	Half-length of ellipse minor axis
M_1	Slope of torsional stress distribution
M_2	Slope of bending stress distribution
N	Number of stress cycles
n_f	Safety factor to compensate for uncertainties in fully corrected fatigue limit, S'_e
n_L	Safety factor to compensate for uncertainty in loading
n_u	Safety factor to compensate for uncertainty in ultimate strength, S_u
n_y	Safety factor to compensate for uncertainty in yield strength, S_y
S	Finite life endurance limit plotted against cycles to failure on a S-N curve
$\overline{S}$	Standard deviation
S_e	Reversed bending fatigue limit
S'_e	Fully corrected reversed bending fatigue limit used for design purposes
$\overline{S}_{se}$	Reversed normal stress fatigue limit with zero gradient
S_{se}	Reversed torsion fatigue limit
$\overline{S}_{se}$	Reversed shear fatigue limit with zero gradient
S_u	Ultimate strength

S_y	Yield strength

Greek Symbols

α	Coefficient of mean stress influence
α_1, α_2	Symbols used to denote angles describing relative orientation of mean and alternating principal stress axes
$\vec{\nabla}$	Symbol for gradient
σ_a	Uniaxial alternating normal stress
σ'_a	Equivalent uniaxial alternating normal stress
σ_a(MASS)	Alternating normal stress component on the plane of maximum alternating shearing stress
$\sigma_{ij}, \sigma'_{ij}$	Two dimensional stress tensor, with $i = 1,2$ and $j = 1,2$. Prime notation used to distinguish between coordinate systems
σ'_m	Equivalent uniaxial mean stress
σ_{xa}, σ_{ya}	Biaxial alternating normal stress components in x and y directions, respectively
$\sigma_{xa}(x,y,z)$	Biaxial alternating normal stress component in x direction shown as a function of position
$\sigma_{ya}(x,y,z)$	Biaxial alternating, normal stress component in the y direction shown as a function of position
σ_{xm}, σ_{ym}	Biaxial mean normal stress components in x and y directions, respectively
σ_{1a}, σ_{2a}	alternating principal stresses
σ_{1m}, σ_{2m}	Mean principal stresses
τ_a(MASS)	Alternating shear stress component of the plane of maximum alternating shear stress
τ_{xya}	Alternating shear stress component
τ_{xym}	Mean shear stress component
$\tau_{xya}(x,y,z)$	Alternating shear stress component shown as a function of position

Special Conventions

- o An alternating stress component followed by mean stress components contained in parentheses indicates that the mean stress components are viewed in the same coordinate system as the alternating stress component. Example: $\sigma_{xa}(\sigma_{xm}, \sigma_{ym}, \tau_{xym})$
- o A "bar" (-) placed over a stress component symbol denotes zero gradient for that component. Example: $\bar{\sigma}_{xa}$
- o A 2x2 array shown in brackets describes a two-dimensional stress tensor

$$\text{Example:} \quad \begin{bmatrix} \sigma_{xa} & \tau_{xya} \\ \tau_{xya} & \sigma_{ya} \end{bmatrix}$$

Superscripts

' " Prime superscripts are used to make a distinction among coordinate systems. A stress component symbol with a prime (or double prime) superscript indicates that the component is viewed in the prime (or double prime) system.

Example: $\bar{\sigma}'_{xa}$, $\bar{\tau}''_{xya}$

Acronyms

MASS Maximum Alternating Shear Stress

1. GENERAL OVERVIEW OF FATIGUE AREA

The design of mechanical components to withstand repeated loading is a many faceted problem. There is no general methodology or unifying theory which embraces every aspect of the problem. It is also safe to say that much of the basic philosophy is still being worked out. Low-cycle fatigue, which is dominated by plastic strain, is usually analyzed by employing a strain-based model. With high-cycle fatigue, a classical stress-based approach may be acceptable. If the part has a large dimension in the direction of crack growth then a fracture mechanics model should be used.

It is not always clear when one method should be selected over another, or how they should be used to complement each other. If a part is being designed for finite life, it seems reasonable to predict the number of cycles to initiate a crack and then predict the number of cycles before unstable crack growth occurs. But even among those who would agree to this approach for certain problems, there may be disagreement as to the meaning of the word "crack" as well as how to predict the crack initiation. It is no doubt advisable in certain cases to assume that a "crack" already exists in the most unfavorable orientation and location, and proceed immediately with the fracture mechanics model, thus disregarding altogether any crack initiation stage.

If a part is being designed for "infinite" or indefinite life, which will be referred to as "fatigue limit analysis", two approaches are viable. Using a fracture mechanics model one could assume some small initial crack and then design so as to obtain a stress intensity factor below the threshold value. This approach is still being refined and although conceptually sound it becomes very difficult to implement for some problems. One example is the shaft subjected to rotating bending and constant torsion. The principal stress axes which result from the superposition of the bending and torsional stresses do not have fixed orientation. It is difficult to determine which orientation for an assumed crack is critical. Difficulty is also encountered in evaluating the stress intensity factor with the presence of complex geometry and mixed mode loading. In this example the geometry is indeed complex since the critical crack orientation will be neither perpendicular nor parallel to the axis of the shaft. Mixed mode loading results since both the bending moment and the torque have components perpendicular and parallel to the plane of the crack face. A serious attempt to evaluate the stress intensity factor in this case would press the limits of present fracture mechanics technology. For problems in which the geometries and loadings are those for which expessions for stress intensity factors have been developed, the fracture mechanics model is a feasible approach.

Another approach to fatigue limit analysis is the classical stress-based concept. There are many suggested methods for implementing the stress-based concept when handling combined stresses. Some of these methods are discussed later. Most of these which pertain to multiaxial stresses have probably been accepted because their various advocates have found them to be conservative. However, the fact that the methods differ indicates that there is room for additional research. For certain fatigue limit analysis problems (e.g., the rotating shaft mentioned above) the classical stress based methods may offer the most reasonable and most economical design approach.

Fatigue limit analysis is applicable primarily to ferrous metals which exhibit a well-defined endurance limit and perform in a non-corrosive environment. Many important engineering materials such as aluminum and plastics and high strength steels do not possess a well-defined endurance limit. When components made from such materials are designed for long life, careful testing may be necessary to guarantee satisfactory performance.

One must be aware of the limitations of the classical stress-based theories when performing finite life calculations. If there are significant geometrical differences between the part being designed and the specimens from which the finite life endurance limit was obtained, the results are not likely to be reliable. For example, one could not reliably estimate the finite life of a plate subjected to alternating tension by using the finite life endurance limit obtained from rotating bending specimens. This is because crack propagation in the two cases is controlled by radically different geometries and stress fields even though the nominal stresses are equal.

There seems to be no limit to the complexity of the fatigue problem with the presence of the many factors which represent departures from ideal test conditions. Some of these factors, only to name a few are, variable amplitude loadings with cummulative damage, multiaxial stresses, variable stress components either out-of-phase with arbitrary phase angles or at different frequencies, environmental effects, material anisotropy and/or inhomogeneity, surface condition and stress concentration, and other geometrical considerations affecting stress gradient. In general, the design/analysis methods are approximate to such an extent that careful prototype testing is required, and in some cases a strict inspection program is required for the product in service.

2. INTRODUCTION

The design of mechanical components to withstand an indefinite number of stress cycles is an important facet of the general fatigue area. It is often necessary for mechanical components to withstand a very large number of alternating stress cycles and, at the same time, not be unduly over-designed. For example, a centrally loaded shaft turning at 2500 RPM would experience more than a billion stress cycles in one year. Yet, many industries require high speed production machinery which must operate twenty-four hours per day, for a period of years, with only a minimum of down time. The machinery must not only be reliable but it must also be efficiently designed to be cost competitive.

If alternating bending stress (S) is plotted against cycles (N) to failure for test specimens of a steel which exhibit a well-defined fatigue limit, the data points will typically be distributed as in Fig. 2.1. If the stress level is low enough, some specimens will be able to endure an indefinite number of cycles without failure. Experience indicates that many specimens will endure indefinitely if they endure 10^6 or 10^7 cycles. The stress level, S_e, which permits an indefinite number of cycles is called the fatigue limit. Usually, S_e values correspond to 50 percent reliabilty, which means that half the parts stressed at that level will survive indefinitely and half will fail. One may achieve any desired reliability by reducing S_e appropriately [2.1]*. In order to use S_e for design purposes, S_e must be further

* Numbers in square brackets refer to references listed at the end of each chapter.

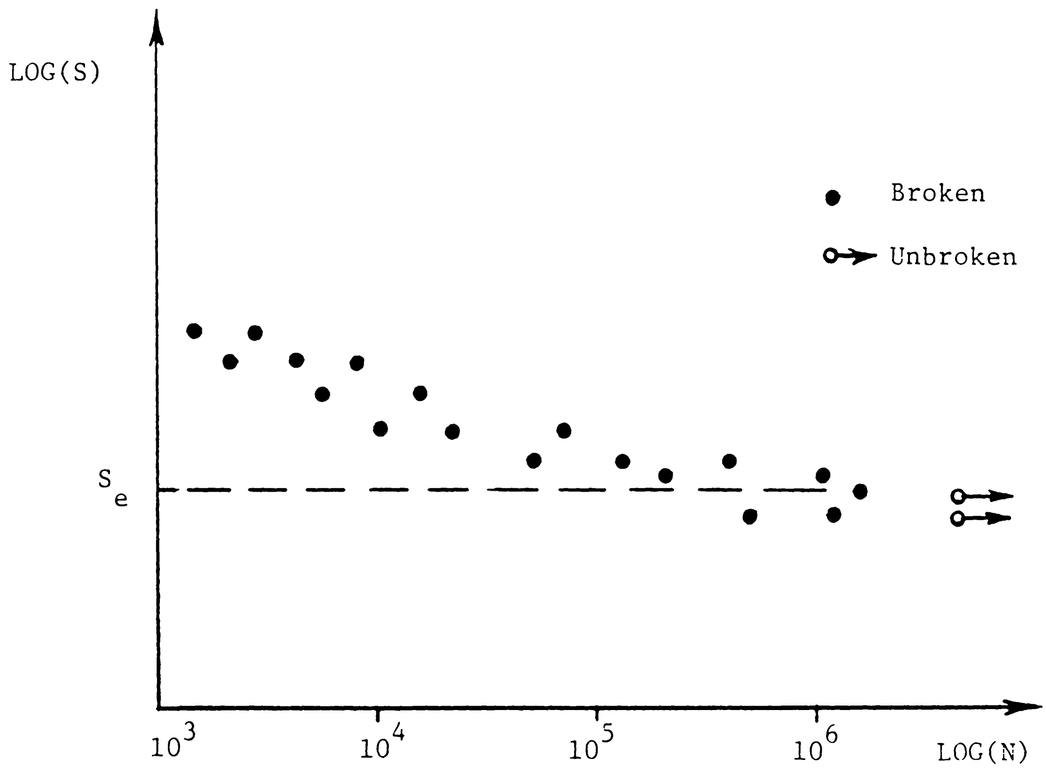


Fig. 2.1 Typical S-N Data for a Steel with a Well-Defined Fatigue Limit.

corrected to account for any departures from ideal test conditions. This is discussed more fully in reference 2.1. If a mechanical component is to be subjected only to uniaxial alternating normal stress with no mean stress, then the design should be such that

$$\sigma_a = \frac{S_e'}{n_f \cdot n_L} \qquad (2.1)^{**}$$

where, σ_a is the nominal stress for the component, S_e' is the fully corrected fatigue limit, n_f is the desired factor of safety to compensate for uncertainties in material properties and n_L is the factor of safety to compensate for uncertainties in loading.

Fatigue limit analysis is more complicated if there is a combination of stresses rather than the simple uniaxial alternating normal (or bending) stress. The combination may consist of purely alternating components or both alternating and mean stress components. The simplest case of combined stresses is uniaxial alternating normal stress, σ_a, with a superimposed mean normal stress, σ_m. Fatigue tests are performed on many specimens in order to determine the alternating stress fatigue limit corresponding to different levels of mean stress. A typical plot of such fatigue limit data is given in Fig. 2.2. By obtaining a curve fit to the data, the typical σ_a - σ_m diagram shown in Fig. 2.3 is obtained. The left side of the diagram pertains to compressive mean stress and the right side pertains to tensile mean stress. In order to perform careful design, it would be necessary to test many

---

** Numbers in parentheses identify mathematical expressions.

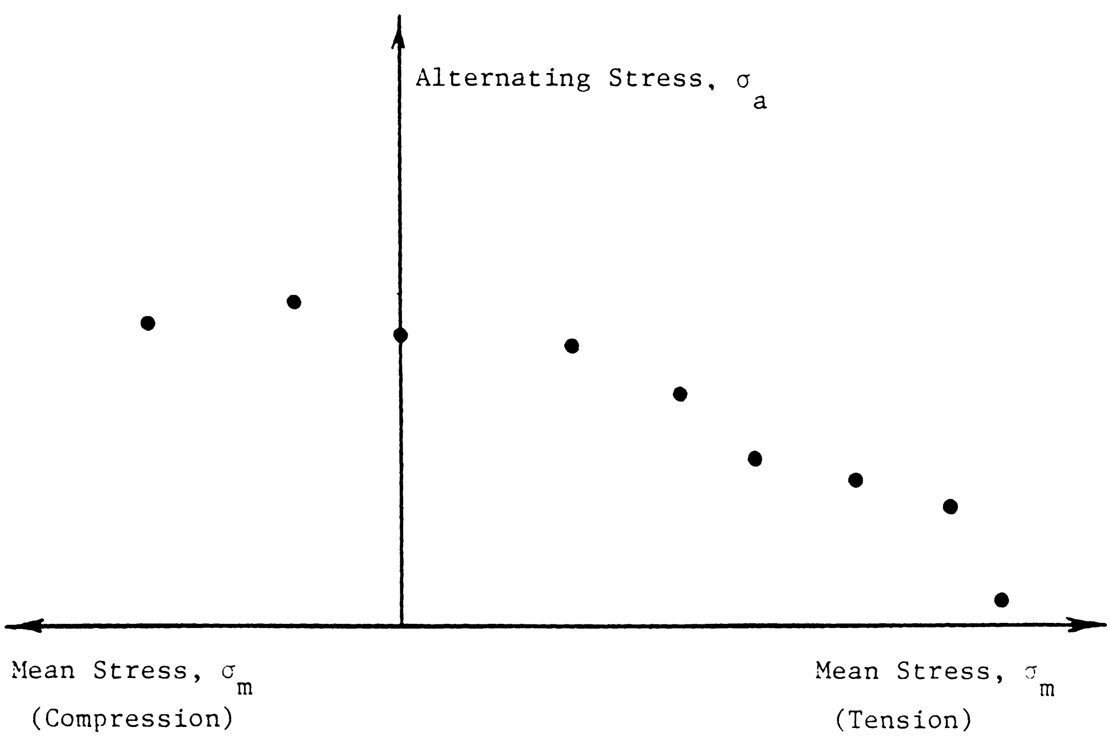


Fig. 2.2 Typical Fatigue Limit Data from Tests Involving Uniaxial Alternating Normal Stress, σ_a, with Superimposed Mean Normal Stress, σ_m.

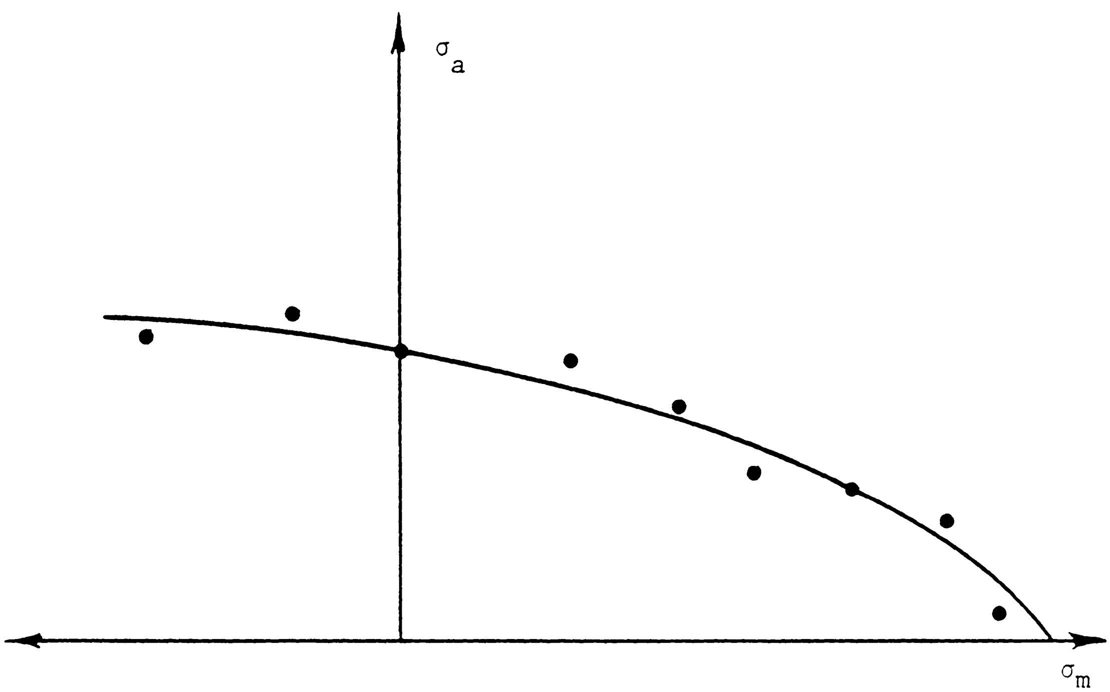


Fig. 2.3 σ_a-σ_m Diagram Obtained by Fitting Curve to Data.

fatigue specimens in order to generate accurate $\sigma_a - \sigma_m$ diagrams. To avoid extensive testing, a number of different methods have been suggested for obtaining approximate and usually conservative $\sigma_a - \sigma_m$ diagrams. Since compressive yield strengths for ductile steels are difficult to obtain, it is often assumed that the yield strength in compression is equal to the yield strength in tension. With this assumption, the compressive side of the $\sigma_a - \sigma_m$ diagram may be approximated by constructing two lines,

$$\sigma_a - \sigma_m = S_y \tag{2.2}$$

and,

$$\sigma_a = S_e' \tag{2.3}$$

where S_y is the static yield strength of the material and S_e' is the fully corrected fatigue limit. Lines representing Eqs. 2.2 and 2.3 are shown in Fig. 2.4. The area below these two lines is the "safe" region on the compressive side.

A variety of equations exist for approximating the tension side of the diagram. Some of the more popular equations are given in reference 2.2. The designer must chose among the alternatives based on his judgement and experience. Most of the proposed equations depend only on the zero mean stress endurance limit S_e and one static material property, such as yield strength S_y or ultimate strength S_u. Figure 2.4 shows a $\sigma_a - \sigma_m$ diagram in which the modified Goodman fracture line is used to approximate the tension side. The equation of the line is also given. Typically, the modified Goodman fracture line tends to be conservative between the end points. Nevertheless, it is usually assumed for

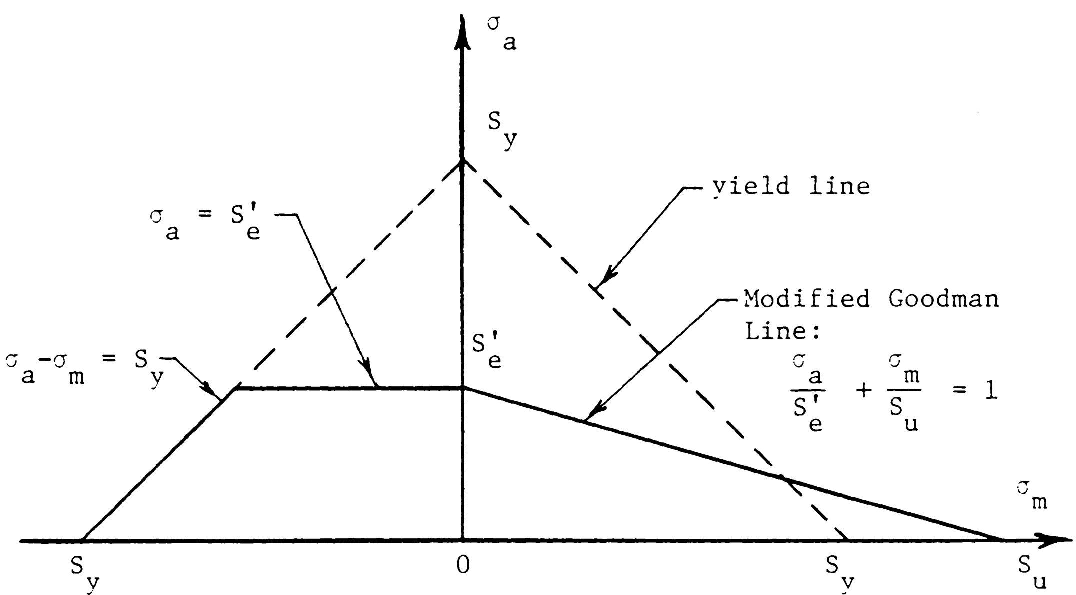


Fig. 2.4 Approximate σ_a-σ_m Diagram Using Modified Goodman Line.

design purposes that the reliability of the Goodman line is everywhere equal to the reliability associated with S_e. This is true not only in the case of the modified Goodman fracture line but with the other lines as well. Shigley and Mitchell [2.1] recommend that safety factors be applied to S_u, S_y and S'_e to compensate for any uncertainties in any of these material properties. Thus, the diagram to be used for design purposes is given in Fig. 2.5. Design safety may be determined by calculating σ_a and σ_m and then locating the point on the σ_a-σ_m diagram. If the point is below the failure line, then the point is expected to have indefinite life with a safety factor against overload greater than 1.0 and a reliability equal to that of the failure line. If the point is above the failure line then the overload safety factor is less than one and ultimate failure is predicted. The value of the overload safety factor depends on how the overload is expected to occur. Determination of safety factor based on overload is discussed in references 2.1, 2.2, and 2.3.

The combined stress problem becomes more difficult if the stress condition is not uniaxial. A number of methods appear in the literature which seek to predict the fatigue behavior under multiaxial loading. The methods are derived largely from biaxial stress data since triaxial data is scarce. Therefore, extension of the methods to the general triaxial stress state may be questionable. Of course, the biaxial stress fatigue criterion is important in its own right since most fatigue failures originate at a free surface where the most general stress state is biaxial.

Certain methods seek to reduce the more general combined stress

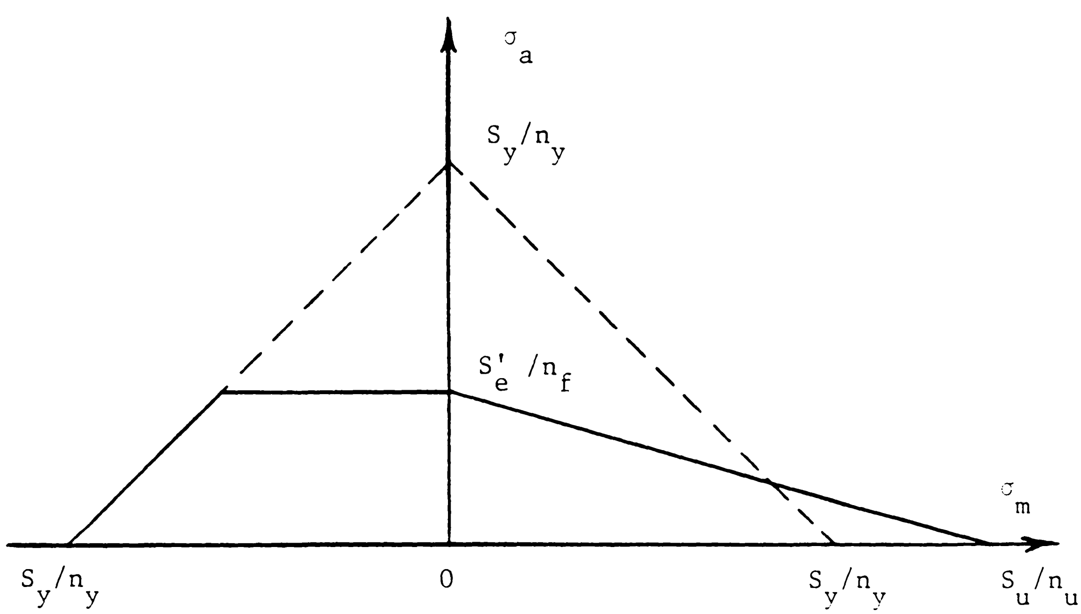


Fig. 2.5 σ_a-σ_m Diagram Showing Factors of Safety for Uncertainties in Material Properties.

problem to one of "equivalent" uniaxial mean and "equivalent" uniaxial alternating stress. A σ_a-σ_m diagram is then used to predict fatigue behavior. This approach has the character of extrapolating available data to some other condition assuming that a correlation exists between the two cases. The correlation exists only if the same fatigue mechanism will be operative in the same way in both cases. The existence of a controlling mechanism, however elusive, must be assumed or else there is no way to proceed toward a useful theory.

The objective of this present work is to examine closely the existing biaxial data in order to determine how much foundation exists for a rational approach to classical biaxial fatigue limit analysis. A partial review is given of the methods presently in vogue and new methods are proposed for obtaining equivalent mean and equivalent alternating stresses. Some ground work is laid for the consideration of stress gradient effects. There are also some observations concerning the Distortional Energy and Tresca Theories. This work assumes that the alternating stress components are proportionally applied. The definition of proportional components is given in Appendix A. It is shown in Appendix B that the alternating principal stresses formed from these components have fixed orientation.

REFERENCES

[2.1] Shigley, J. E., and Mitchell, L. D., Mechanical Engineering Design, 4th Ed., McGraw Hill, New York, NY, 1983.

[2.2] Vaughan, D. T., and L. D. Mitchell, "A General Method for the Fatigue Resistant Design of Mechanical Components. Part II: Analytical," Journal of Engineering for Industry, Transactions of the ASME, 97 (B-3) August 1975, pp. 970-975.

[2.3] Luk, Y. W., and L. D.Mitchell, "Development of an Interactive Computer Program for Fatigue Analysis," ASME Paper No. 79-DE-E-4, 1979.

3. CURRENT METHODS

A number of methods have been suggested in textbooks and other literature for handling the problem of fatigue limit analysis when the alternating stress components and/or mean stress components are biaxial. A method advocated by Shigley and Mitchell [3.1] uses formulas similar in appearance to the von Mises yield equation to obtain equivalent uniaxial mean and equivalent uniaxial alternating stresses. The method is described in the following excerpt [3.1].

> "To apply the theory, construct two stress elements, one for the mean stresses and one for the alternating stresses. Two Mohr's circles are then drawn, one for each element, and the principal mean stresses obtained from one circle and the principal alternating stresses obtained from the other. We can then define mean and alternating von Mises stresses as
>
> $$\sigma'_m = [\sigma_{1m}^2 - \sigma_{1m}\sigma_{2m} + \sigma_{2m}^2]^{1/2} \tag{3.1}$$
>
> $$\sigma'_a = [\sigma_{1a}^2 - \sigma_{1a}\sigma_{2a} + \sigma_{2a}^2]^{1/2} \tag{3.2}$$
>
> for the biaxial state."

More will be said later in several places about this approach.

In recommending the von Mises formula for equivalent alternating stress, Juvinall [3.2] tends to assume that, at most, the alternating components will consist of an alternating normal stress, σ_{xa}, and alternating shearing stress, τ_{xya}. The von Mises formula becomes,

$$\sigma'_a = [\sigma_{xa}^2 + 3\ \tau_{xya}^2]^{1/2} \tag{3.3}$$

Juvinall seems to tentatively recommend Eq. 3.2 for the more general case of biaxial stress. He recommends that the equivalent mean stress be taken as the "maximum principal stress resulting from the superposition of all existing static (mean) stresses [3.2]." But again, he tends to assume that, at most, the mean stress components will consist of a single mean normal stress and a mean shearing stress. For such a case, only one principal stress is positive. Apparently, Juvinall's method for obtaining an equivalent mean stress is based on the assumption that a negative principal mean stress has either no influence on fatigue behavior or else has a beneficial effect.

The method proposed by Sines [3.3] assumes that the allowable alternating octahedral shear stress is affected in a linear fashion by the octahedral normal stress, which is also the hydrostatic stress. All the data which Sines had in view as he developed his criteria corresponded to stress cases in which the principal stresses formed from the alternating stress components had fixed orientation with respect to the critical stress element. For biaxial stress fatigue limit analysis, Sines' criterion is expressed in equational form as,

$$(\sigma_{1a}^2 - \sigma_{1a}\sigma_{2a} + \sigma_{2a}^2)^{\frac{1}{2}} \leq S_e - \alpha\,(\sigma_{1m} + \sigma_{2m}) \tag{3.4}$$

where α is the coefficient of mean stress influence. Sines recommends that α be determined from a test involving a high positive mean stress, but not so high that the maximum stress exceeds the yield strength. In

Eq. 3.3, σ_{1a}, σ_{2a}, σ_{1m}, and σ_{2m}, may be obtained according to Shigley's instructions given previously. Of course, there is no need to compute σ_{1m} and σ_{2m} since, due to invariance, the sum of the two is equal to the sum of the diagonal terms of the mean component stress tensor, regardless of the plane from which it is viewed. Since the mean stress effect depends only on the mean stress tensor diagonal terms, the criterion assumes that any mean shearing stress has no effect. The method presented by Sines as given in reference 3.3 is modified in reference 3.4 for high mean stresses.

The method proposed by Garud [3.5] for the evaluation of fatigue under multiaxial loading relates fatigue life to plastic work during a cycle of loading. This concept might conceivably be applied to fatigue limit analysis if a "threshold" value of plastic work can be determined which permits indefinite life.

The method proposed by Langer uses the maximum alternating shear stress component as a fatigue criterion [3.6], known as the Tresca criterion. Reference 3.6 is concerned with fatigue of pressure vessels which is often a case of low-cycle fatigue. However, Langer's endorsement of the Tresca theory does not seem to be confined to low-cycle fatigue, but (one would think) applies to high cycle fatigue and fatigue limit analysis as well. The Tresca theory may seem reasonable since initial fatigue cracks are observed along planes of maximum alternating shear stress. However, as Little [3.7] points out, one may rightly question the validity of ignoring the other components of stress which exist on the plane of maximum alternating shear stress. In addition to the alternating shear stress, there are in general three other stress components when the stress state is biaxial. These three components are the mean shear stress, alternating normal stress and mean normal stress. Langer recommends the Tresca criterion for "general use", by which he means that it may be used for problems involving stress components which are not proportional. However, this claim is not supported by the non-proportional data of Nishihara and Kawamoto [3.8], as pointed out by Little [3.7].

In his state of the art survey paper, Garud [3.9] discusses multiaxial fatigue criteria presented prior to 1979. Other than the simple application of static yield theories to fatigue analysis, the only stress-based method presented in the research literature before 1970 which has gained popularity as a design tool is that of Sines [3.3], which was discussed above. Other methods such as those by Nishihara and Kawamoto [3.10], Stulen and Cummings [3.11], and Findley [3.12] have been disregarded either because an inordinate amount of data is required to evaluate empirical constants or because they were founded on too few data to inspire confidence.

Methods presented after 1970 were by Langer [3.6], McDiarmid [3.13], Krempl [3.14], Hull [3.15], Grubisic and Simbürger [3.16] and Garud [3.17]. The methods of Langer and Garud were discussed above.

McDiarmid's method [3.13] assumes a particular $\tau_a - \sigma_m$ relationship in order to account for mean stress effects. Krempl [3.14] presents a multiaxial failure criteria which contains the von Mises and Sines criteria as special cases.

The stress-based theory by Hull [3.15] uses a plane of maximum octahedral shear stress as a reference plane. The amplitude of the shear stress along the direction of the maximum octahedral shear stress is determined. Additionally, the mean normal stress on the plane of maximum octahedral shear stress is determined. Hull's theory assumes that failure is indicated by a critical combination of the alternating shear stress component and the mean normal stress component. The theory disregards both the alternating normal stress component and the mean shear stress component on the plane of maximum octahedral shear stress.

Garud [3.9] summarizes the theory by Gubisic and Simbürger [3.16] for ductile materials as follows:

"(1) Consider the variation of shear stress on a plane for the complete cycle of loading.

(2) Obtain the mean and amplitude values of shear stress on this plane in 'all' directions. Plot these on mean-versus-amplitude coordinates. This is called a 'stress diagram' for the plane.

(3) On the same coordinates,for a given life, a 'strength diagram' (that is, a Goodman type of R-M diagram for the material) is assumed to be known.

(4) From these two diagrams choose the most unfavorable combination of mean and amplitude and obtain the 'maximum straining' for this plane.

(5) Repeat Steps 1 through 4 for 'all' other planes.

(6) From these 'maximum straining' values, the 'effective straining' is calculated as some sort of an average. If this value is greater than unity, then failure is indicated."

Garud concludes by saying, "The use of mean shear stress in the 'stress diagram' is not clear to this author. General application of this intriguing theory to design problems appears to be difficult in the present form."

All the methods for fatigue limit analysis which have been discussed above have at least one thing in common. They were developed only on the basis of stress state without any regard to the influence of stress gradients. The methods presented in this current work will be developed in view of the stress gradient influence as well as the influence of the stress state.

REFERENCES

[3.1] Shigley, J. E., and Mitchell, L. D., Mechanical Engineering Design, 4th Ed., McGraw-Hill Book Co., New York, NY, 1983, pp. 332-333.

[3.2] Juvinall, R. C., Fundamentals of Machine Component Design, John Wiley & Sons, New York, NY, 1983, p. 217.

[3.3] Sines, G., "Failure of Metals Under Combined Repeated Stresses with Superimposed Static Stresses," NACA Tech., Note 3495, 1955.

[3.4] Sines, G., and Ohgi,G., "Fatigue Critria Under Combined Stresses or Strains," Journal of Engineering Materials and Technology, Transactions of the ASME, Vol. 103,No. 2, April, 1981, pp.82-90.

[3.5] Garud, Y. S., "A New Approach to the Evaluation of Fatigue Under Multiaxial Loadings," Journal of Engineering Materials and Technology, Transactions of the ASME, Vol. 103, No. 2, April, 1981, pp. 118-125.

[3.6] Langer, B. F., "Design of Pressure Vessels Involving Fatigue," Pressure Vessel Engineering, R. W. Nichols Ed., Elsevier Publishing Co., Amsterdam, 1971.

[3.7] Little, R. E., "A Note on the Shear Stress Criterion for Fatigue Failure Under Combined Stress," Machine Design, V. 38, N 1, Jan. 6, 1966, pp. 145-149.

[3.8] Nishihara, T. and Kawamoto, M., "The Strength of Metals and Combined Alternating Bending and Torsion with Phase Difference," Memoirs, College of Enginering, Kyoto Imperial University, Vol. XI, pp. 85-112, 1945.

[3.9] Garud, Y. S., "Multiaxial Fatigue: A Survey of the State of the Art," Journal of Testing and Evaluation, Vol. 9, No. 3, May 1981, pp. 165-178.

[3.10] Nishihara, T., and Kawamoto, M., "A New Criterion for the Strength of Metals Under Combined Alternating Stresses," Memoirs, College of Engineerng, Kyoto Imperial University, Vol. XI, 1944.

[3.11] Stulen, F. B., and Cummings, H. N., "A Failure Criterion for Multi-Axial Fatigue Stresses," Proceedings, American Society Testing Materials, Vol. 54, 1954, pp. 822-835.

[3.12] Findley, W. N., "A Theory for the Effect of Mean Stress on Fatigue of Metals Under Combined Torsion and Axial Load or Bending," Trans. ASME, Ser. B., Vol. 81, No. 4, pp. 301-306, November, 1959.

[3.13] McDiarmid, M. L., "A General Criterion of Fatigue Failure Under Multi-axial Stress," Proceedings of the 2nd International Conference on Pressure Vessel Technology, ASME, New York, 1973, pp.851-862.

[3.14] Krempl, E., "Multiaxial Fatigue - Present and Future Methods of Correlation," AGARD Conference Proceedings, No. 155, 1974, pp. 5.1-5.12.

[3.15] Hull, W. C., "A Rational Theory of Failure for Combined Stress High-Cycle Fatigue," in Proceedings of the ASME Design Engineering Technology Conference, American Society of Mechanical Engineers, New York, 1977, pp. 179-195.

[3.16] Grubisic, V., and Simbürger, A., "Fatigue Under Combined Out-of-Phase Multiaxial Stresses," in International Conference on Fatigue Testing and Design, Society of Environmental Engineers, London, 1976, pp. 27.1-27.8.

4. REVIEW OF DATA

Unfortunately, there is no great abundance of the different kinds of carefully obtained and well documented test data needed to firmly establish a rational design approach when biaxial alternating stresses are present. This is especially true if mean stresses must be accounted for, which is often the case. For example, only in special cases is data available for uniaxial mean and alternating normal stress which is the simplest case of combined mean and alternating stresses. Moreover, this case may be regarded as biaxial only in a trivial sense. Samples of this type of data are found in references 4.1, 4.2, 4.3 and 4.4. Results from [4.3] and [4.4] are given in [3.3].

Quite a number of investigators have performed tests involving pure alternating torsional stress superimposed on pure mean torsional stress [4.5, 4.6 4.7, 4.8, 4.9. 4.10, 4.11]. Results from references 4.5 through 4.11 are given in reference 3.3. The many tests indicate that mean torsional stress does not influence the allowable alternating torsional stress as long as the maximum torsional stress does not exceed the torsional yield strength [3.3]. This information is especially valuable to the designer of helical springs and other members loaded in torsion. However, the aforementioned tests do not prove that the allowable alternating shear stress would necessarily be independent of mean shearing stress if the stress distribution is other than the special triangular distribution induced by torsion in a member of circular cross section.

Data involving alternating normal stress and mean torsional stress are found in references 4.12, 4.13, and 4.14. The data shows a weak dependence of alternating normal stress on mean torsional stress below the yield point. Results from references 4.13 and 4.14 are given in reference 3.3.

A number of investigators have obtained data from tests involving proportionally applied, reversed bending and torsion [4.15, 4.16, 4.17]. It is shown in Appendix A that proportionally applied reversed bending and torsion always produce alternating principal stresses which are 180° out-of-phase. In addition, Appendix B shows that proportionally applied stresses give principal stress axes which have a fixed orientation with respect to the critical element. For ductile steels the data typically has the distribution shown in Fig. 4.1. Gough and Pollard [4.15] proposed the following equation to describe the data when the material is ductile steel.

$$\left(\frac{\sigma_{xa}}{S_e}\right)^2 + \left(\frac{\tau_{xya}}{S_{se}}\right)^2 = 1 \qquad (4.1)$$

It is worthwhile to determine how well Eq. 4.1 fits not only the data by Gough and Pollard, but also that of other investigators. Eighty one data points from three different investigators have been examined. In every case the material was either carbon or carbon alloy steel. To avoid consideration of sub-surface crack initiation, no tests were included that involved nitriding. Each point represents an endurance limit combination of alternating bending and alternating torsional stresses. Therefore, each point resulted from the fatigue testing of more than one specimen. The percent error between the actual data

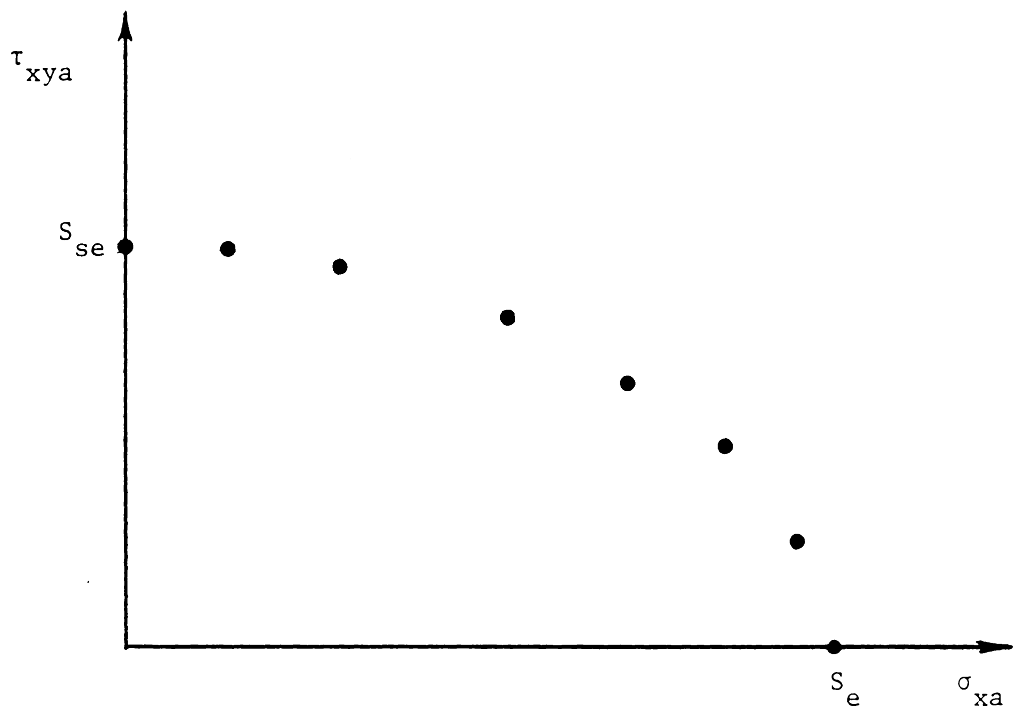


Fig. 4.1 Typical Distribution of Fatigue Limit Combinations of Reversed Bending-Reversed Torsion Stresses.

points and Eq. 4.1 is defined in Fig. 4.2. The percent errors for the eighty one data points are given in Table 4.1. The majority of the errors have a magnitude less than 3.0 percent. Only two data points, entries 20 and 53, have errors greater than five percent. The errors are -7.73 and -6.43 percent, respectively. These errors are not surprisingly large since entry 20 is a very high carbon steel (.9% carbon) and entry 53 is a high-strength alloy steel. Such materials are known to manifest more scatter in fatigue limit data than low carbon steels. There are 35 data points with positive errors and 46 data points with negative errors. The average error is computed as,

$$\mathrm{ERROR}_{ave} = \sum_{i=1}^{81} \frac{(\text{PER CENT ERROR})_i}{81} = .021\% \qquad (4.2)$$

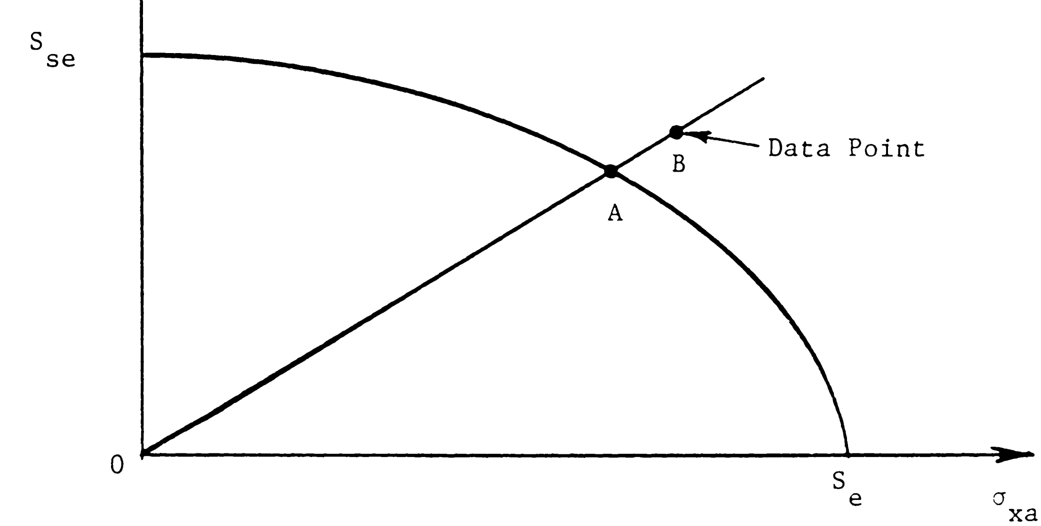
and the standard deviation ($\overline{S}$) of the error is,

$$\overline{S} = \sum_{i=1}^{81} \sqrt{\frac{\left[(\text{PER CENT ERROR})_i - \mathrm{ERROR}_{ave}\right]^2}{80}} = 2.33\% \qquad (4.3)$$

Thus, it is seen that equation (4.1) indicates central tendency with considerable accuracy. The column labelled "C" in Table 4.1 is presently unimportant, but will be referred to later after "C" has been defined in Chapter 5.

Equation 4.1 has been extended by Lee [4.19] to include non-proportionally applied reversed bending and reversed torsional stresses.

$$\left(\frac{\sigma_{xa}}{S_e}\right)^2 + \left(\frac{\tau_{xya}}{S_{se}}\right)^2 = 1 \qquad (4.1)$$



$$\text{Percent Error} = \left[\frac{OB-OA}{OA}\right] \times 100\%$$

Fig . 4.2 Definition of Percent Error for Comparison of Reversed Bending-Reversed Torsion Data and Eq. 4.1.

Table 4.1 Reversed Bending, Reversed Torsional Fatigue Data with Percent Errors Resulting from Comparison with Eq. 4.1

No.	Reference/Material/ Stress Units Used in Reference	Reversed Bending Stress σ_{xa}*	Reversed Torsional Stress τ_{xya}*	Reversed Bending Fatigue Limit S_e*	Reversed Torsion Fatigue Limit S_{se}*	C	Percent Error as Defined in Figure 4.2
1	[4.15]/0.1% C Steel Normalized/tons per sq.in.	17.0* (234)	2.31* (31.7)	7.4* (240)	9.85* (136)	3.12	.45
2	Ditto	15.7 (216)	4.55 (62.7)	Ditto	Ditto	Ditto	1.37
3	"	13.3 (183)	6.7 (92.4)	"	"	"	2.32
4	"	9.5 (132)	8.3 (114)	"	"	"	.41
5	"	5.1 (70.3)	9.5 (132)	"	"	"	1.30
6	[4.15]/0.4% C. steel Normalized/tons per sq.in.	21.3 (294)	2.85 (39.3)	21.5 (296)	13.4 (185)	2.57	1.25

*Numbers not in parentheses have units from the original reference shown in column 2. Number in parentheses are in megapascals.

Table 4.1 (continued)

No.	Reference/Material/ Stress Units Used in Reference	Reversed Bending Stress σ_{xa}*	Reversed Torsional Stress τ_{xya}*	Reversed Bending Fatigue Limit S_e*	Reversed Torsion Fatigue Limit S_{se}*	C	Percent Error as Defined in Figure 4.2
7	[4.15]/0.4% C. Steel Normalized/tons per sq. in.	19.2* (265)	5.55* (76.5)	21.5* (296)	13.4* (185)	2.57	-1.56
8	"	16.4 (226)	8.2 (113)	"	"		-2.21
9	"	12.3 (170)	10.6 (146)	"	"	"	-2.37
10	"	6.6 (91.0)	12.2 (168)	"	"	"	-3.91
11	[4.15]/0.4% C. Steel Spheriodised/tons per sq.in.	17.8 (245)	2.4 (33.1)	17.8 (245)	10.1 (139)	3.11	2.79
12	Ditto	16.1 (222)	4.65 (64.1)	Ditto	Ditto	Ditto	1.27
13	"	13.5 (186)	6.8 (93.8)	"	"	"	1.42

*Numbers not in parentheses have units from the original reference shown in column 2. Numbers in parentheses are in megapascals.

Table 4.1 (continued)

No.	Reference/Material/ Stress Units Used in Reference	Reversed Bending Stress σ_{xa}*	Reversed Torsional Stress τ_{xya}*	Reversed Bending Fatigue Limit S_e*	Reversed Torsion Fatigue Limit S_{se}*	C	Percent Error as Defined in Figure 4.2
14	[4.15]/0.4% C. Steel Spheriodised/tons per sq.in.	10.0* (138)	8.65* (119)	17.8* (245)	10.1* (139)	3.11	2.43
15	Ditto	5.1 (70.3)	9.5 (131)	Ditto	Ditto	Ditto	-1.68
16	[4.15]/0.9% C. steel Pearlitic/tons per sq.in.	22.8 (314)	3.05 (42.1)	22.8 (314)	15.6 (215)	2.14	1.89
17	Ditto	20.8 (287)	6.0 (82.7)	Ditto	Ditto	Ditto	-1.00
18	"	18.2 (251)	9.1 (125)	"	"	"	-1.13
19	"	13.8 (190)	11.9 (164)	"	"	"	-2.62
20	"	7.3 (101)	13.5 (186)	"	"	"	-7.73

*Numbers not in parentheses have units from the original reference shown in column 2. Numbers in parentheses are in megapascals.

Table 4.1 (continued)

No.	Reference/Material/ Stress Units Used in Reference	Reversed Bending Stress σ_{xa}*	Reversed Torsional Stress τ_{xym}*	Reversed Bending Fatigue Limit S_e*	Reversed Torsion Fatigue Limit S_{se}*	C	Percent Error as Defined in Figure 4.2
21	[4.15]/5]3% Ni. steel (30/35 ton) /tons per sq.in.	21.2* (292)	2.85* (39.3)	22.2* (306)	13.3* (183)	2.79	-2.13
22	Ditto	19.7 (272)	5.7 78.6)	Ditto	Ditto	Ditto	-1.45
23	"	17.2 (237)	8.6 (119)	"	"	"	.92
24	"	12.3 (170)	10.7 (148)	"	"	2.79	-2.32
25	"	6.75 (93.1)	12.6 (174)	"	"	"	-0.51
26	[4.15]/3-3 1/2% Ni steel (45/50 ton) /tons per sq.in.	27.2 (375)	2.65 (50.3)	28.8 (397)	17.3 (239)	2.77	-3.23

*Numbers not in parentheses have units from the original reference shown in column 2. Numbers in parentheses are in megapascals.

Table 4.1 (continued)

No.	Reference/Material/ Stress Units Used in Reference	Reversed Bending Stress σ_{xa}*	Reversed Torsional Stress τ_{xya}*	Reversed Bending Fatigue Limit S_e*	Reversed Torsion Fatigue Limit S_{se}*	C	Percent Error as Defined in Figure 4.2
27	[4.15]/3-3 1/2% Ni. Steel (45/50 ton) /tons per sq.in.	25.4* (350)	7.35* (101)	28.8* (397)	17.3* (239)	2.77	-2.11
28	Ditto	21.5 (296)	10.8 (149)	Ditto	Ditto	Ditto	-2.68
29	"	15.9 (219)	13.8 (190)	"	"	"	-3.00
30	"	8.6 (119)	16.0 (221)	"	"	"	-2.82
31	[4.15]/Cr.Va. Steel (45/50 ton)/tons per sq.in.	26.6 (367)	3.6 (49.6)	27.8 (383)	16.7 (230)	2.77	-1.92
32	Ditto	24.6 (339)	7.1 (97.9)	Ditto	Ditto	Ditto	-1.83

*Numbers not in parentheses have units from the original reference shown in column 2. Numbers in parentheses are in megapascals.

Table 4.1 (continued)

No.	Reference/Material/ Stress Units Used in Reference	Reversed Bending Stress σ_{xa}*	Reversed Torsional Stress τ_{xya}*	Reversed Bending Fatigue Limit S_e*	Reversed Torsion Fatigue Limit S_{se}*	C	Percent Error as Defined in Figure 4.2
33	[4.15]/Cr.Va. Steel (45/50 ton) /tons per sq.in.	21.4* (295)	10.7* (148)	27.8* (383)	16.7* (230)	2.77	0.16
34	Ditto	15.9 (219)	13.8 (190)	Ditto	Ditto	Ditto	0.50
35	"	8.6 (119)	15.9 (219)	"	"	"	.12
36	[4.15]/3 1/2% Ni Cr. Steel (Normal Impact) /tons per sq.in.	35.0 (483)	4.7 (64.8)	35.0 (483)	22.8 (314)	2.36	2.10
37	Ditto	31.8 (439)	9.2 (127)	Ditto	Ditto	Ditto	-0.60
38	"	26.5 (365)	13.3 (183)	"	"	"	-4.42
39	"	21.3 (294)	18.4 (254)	"	"	"	1.08

*Numbers not in parentheses have units from the original reference shown in column 2. Numbers in parentheses are in megapascals.

Table 4.1 (continued)

No.	Reference/Material/ Stress Units Used in Reference	Reversed Bending Stress σ_{xa}*	Reversed Torsional Stress τ_{xya}*	Reversed Bending Fatigue Limit S_e*	Reversed Torsion Fatigue Limit S_{se}*	C	Percent Error as Defined in Figure 4.2
40	[4.15]/3 1/2% Ni. Cr. Steel (Normal Impact) /tons per sq.in.	11.6* (160)	21.6* (298)	3.50* (483)	22.8* (314)	2.36	.40
41	[4.15]/3 1/2% Ni.Cr. Steel (Low Impact) /tons per sq.in.	33.4 (461)	4.5 (62.1)	33.0 (455)	21.0 (290)	2.47	3.46
42	Ditto	30.5 (421)	8.8 (121)	Ditto	Ditto	Ditto	1.48
43	"	25.8 (356)	12.9 (178)	"	"	"	-0.57
44	"	20.4 (281)	17.7 (244)	"	"	"	4.53
45	"	10.9 (150)	20.2 (279)	"	"	"	1.70

*Numbers not in parentheses have units from the original reference shown in column 2. Numbers in parentheses are in megapascals.

Table 4.1 (continued)

No.	Reference/Material/ Stress Units Used in Reference	Reversed Bending Stress σ_{xa}*	Reversed Torsional Stress τ_{xya}*	Reversed Bending Fatigue Limit S_e*	Reversed Torsion Fatigue Limit S_{se}*	C	Percent Error as Defined in Figure 4.2
46	[4.16]/Cr-Mo Steel (61.8 ton)/tons per sq.in.	31.74* (438)	4.25* (58.6)	32.97* (455)	19.88* (274)	2.75	-1.39
47	Ditto	31.01 (428)	5.83 (80.4)	Ditto	Ditto	Ditto	-1.49
48	"	29.80 (411)	8.60 (119)	"	"	"	0.21
49	"	25.54 (352)	12.76 (176)	"	"	"	0.60
50	"	18.45 (254)	15.97 (220)	"	"	"	-2.10
51	"	9.86 (136)	18.39 (354)	"	"	"	-2.78

*Numbers not in parentheses have units from the original reference shown in column 2. Number in parentheses are in megapascals.

Table 4.1 (continued)

No.	Reference/Material/ Stress Units Used in Reference	Reversed Bending Stress σ_{xa}*	Reversed Torsional Stress τ_{xya}*	Reversed Bending Fatigue Limit S_e*	Reversed Torsion Fatigue Limit S_{se}*	C	Percent Error as Defined in Figure 4.2
52	[4.16]/Ni-Cr-Mo Steel (S_{ut} = 60.2 tons per sq.in.)Solid Specimen/tons per sq.in.	30.90* (426)	4.14* (57.1)	33.00* (455)	20.00* (276)	2.72	-4.10
53	Ditto	29.50 (407	5.53 (76.3)	Ditto	Ditto	Ditto	-6.43
54	"	29.00 (400)	8.37 (115)	"	"	"	-2.67
55	"	25.45 (351)	12.72 (175)	"	"	"	-0.04
56	"	19.30 (266)	16.70 (230)	"	"	"	1.95
57	"	10.00 (138)	18.70 (258)	33.00 (455)	20.00 (276)	2.72	-1.71
58	Ditto Except Hollow Specimen	30.60 (422)	4.10 (56.5)	32.2 (444)	20.00 (276)	2.59	-2.78

---

*Numbers not in parentheses have units from the original reference shown in column 2. Numbers in parentheses are in megapascals.

Table 4.1 (continued)

No.	Reference/Material/ Stress Units Used Reference	Reversed Bending Stress σ_{xa}*	Reversed Torsional Stress τ_{xya}*	Reversed Bending Fatigue Limit S_e*	Reversed Torsion Fatigue Limit S_{se}*	C	Percent Error as Defined in Figure 4.2
59	[4.16]/Ni-Cr-Mo Steel (S_{ut} = 60.2 tons per sq.in.)Hollow Specimen/tons per sq.in.	30.10* (415)	5.64* (77.8)	32.2* (444)	20.00* (276)	2.59	-2.36
60	Ditto	28.20 (389)	8.20 (113)	Ditto	Ditto	Ditto	-3.34
61	"	25.20 (347)	12.60 (174)	"	"	"	0.47
62	"	18.80 (259)	16.30 (225)	"	"	"	0.26
63	"	9.90 (137)	18.40 (254)	"	"	"	-3.00

*Numbers not in parentheses have units from the original reference shown in column 2. Numbers in parentheses are in megapascals.

Table 4.1 (continued)

No.	Reference/Material/ Stress Units Used Reference	Reversed Bending Stress σ_{xa}*	Reversed Torsional Stress τ_{xya}*	Reversed Bending Fatigue Limit S_e*	Reversed Torsion Fatigue Limit S_{se}*	C	PerCent Error as Defined in Figure 4.2
64	[4.16]/Ni-Cr-V steel (S_{ut} = 90.5 tons per sq.in.)Solid Specimen/tons per sq.in.	41.79* (576)	5.60* (77.2)	43.26* (597)	25.28* (349)	2.93	-0.90
65	Ditto	40.02 (552)	7.52 (104)	Ditto	Ditto	Ditto	-2.83
66	"	38.32 (528)	11.06 (153)	"	"	"	-1.20
67	"	32.35 (446)	16.17 (223)	"	"	"	-1.60
68	"	24.30 (335)	21.03 (290)	"	"	"	0.38
69	"	12.96 (179)	24.17 (333)	"	"	"	0.20
70	Ditto Except Hollow Specimen	41.20 (568)	5.52 (76.1)	42.75 (590)	25.04 (345)	2.91	-1.14

*Numbers not in parentheses have units from the original reference shown in column 2. Numbers in parentheses are in megapascals.

Table 4.1 (continued)

No.	Reference/Material Stress Units Used Reference	Reversed Bending Stress σ_{xa}*	Reversed Torsional Stress $_{xya}$*	Reversed Bending Fatigue Limit S_e*	Reversed Torsion Fatigue Limit S_{se}*	C	Percent Error as Defined in Figure 4.2
71	[4.16]/Ni-Cr-V Steel (S_{ut} - 90.5 tons per sq.in.)Hollow Specimen/tons per sq.in.	39.43* (544)	7.41* (102)	42.75* (590)	25.04* (345)	2.91	-3.14
72	Ditto	37.39 (516)	10.79 (149)	Ditto	Ditto	Ditto	-2.50
73	"	32.68 (451)	16.34 (225)	"	"	"	0.51
74	"	24.12 (333)	20.88 (288)	"	"	"	0.68
75	"	12.73 (176)	23.76 (328)	"	"	"	-0.55
76	[4.17]/Hard Steel/ kg/mm^2	14.08 (138)	17.00 (167)	32.00 (314)	20.00 (196)	2.56	-4.29

*Numbers not in parentheses have units from the original reference shown in column 2. Numbers in parentheses are in megapascals.

Table 4.1 (continued)

No.	Reference/Material/ Stress Units Used Reference	Reversed Bending Stress σ_{xa}*	Reversed Torsional Stress τ_{xya}*	Reversed Bending Fatigue Limit S_e*	Reversed Torsional Fatigue Limit S_{se}*	C	Percent Error as Defined in Figure 4.2
77	[4.17]/Hard Steel/ kg/mm^2	25.00* (245)	12.50* (123)	32.00* (314)	20.00* (196)	2.56	0.05
78	Ditto	30.49 (299)	6.31 (61.9)	Ditto	Ditto	Ditto	0.37
79	[4.17]/Mild steel/ kg/mm^2	10.18 (99.8)	12.29 (121)	24.00 (235)	14.00 (137)	2.94	-2.50
80	Ditto	18.38 (180)	9.19 (90.1)	Ditto	Ditto	Ditto	
81	"	22.73 (223)	4.71 (46.2)	"	"	"	0.51

*Numbers not in parentheses have units from the original reference shown in column 2. Numbers in parentheses are in megapascals.

According to Sines [3.3], Sawert [4.18] may be the only investigator to obtain biaxial alternating stress data for which the alternating principal stresses are in-phase. In order to achieve the desired ratios of maximum alternating principal stress to minimum alternating principal stress, Sawert had to use a variety of specimen geometries.

Rotvel [4.20] performed biaxial fatigue tests using tubular specimens. Although his tests are valuable from a conceptual point of view, he was unable to avoid unwanted mean stress components. For this reason Rotvel's data will not be considered in this current work.

Gough and Clenshaw [4.15] obtained fatigue limit data for solid cylindrical specimens from tests involving proportionally applied alternating bending and torsional stresses with superimposed mean bending and torsional stresses.

REFERENCES

[4.1] Grover, H. J., Gordon, S. A., and Jackson, L. R., Fatigue of Metals and Structures, Government Printing Office, Washington, D.C., 1960.

[4.2] Lessells, J. M., Strength and Resistance of Metals, John Wiley & Sons, Inc., New York, 1954.

[4.3] Ros, M., and Eichinger, A., "Dic Bruchgefahr fester Korper bei wiederhalten Beansprachung-Ermudung," Metalle, Bericht 173, Eidgenossische Materialprufungs-und Versuchsanstalt fur Industrie, Bauwesen und Gewerbe (Zurich), Sept. 1950.

[4.4] Nishihara, Toshio, and Sakurai, Tadakuzu, "Fatigue Strength of Steel for Repeated Tension and Compression," Trans. Soc. Mech. Eng. (Japan), Vol. 5, No. 18, Feb.1939, pp. I-25 - I-29. (Text in Japanese. English summary, pp. S-8 - S-9.)

[4.5] Hankins, G. A., "Torsional Fatigue Tests on Spring Steels." Special Rep. No. 9, Eng. Res., Dept. Sci. and Ind. Res. (Gr. Brit.), 1929.

[4.6] Pomp, A., and Hempel, M., "Das Verhalten von Gusseisen unter Zug-Druck-Wechselbeanspruchung," Stahl und Eisen, Bd. 57, Heft 40, Oct. 1937, pp. 1125-1127.

[4.7] Johnson, J. B., "Fatigue Characteristics of Helical Springs," The Iron Age, Vol. 133, No. 11, Mar. 15, 1934; No. 12, Mar. 22, 1934, pp. 24-26.

[4.8] Moore, H. F., and Jasper, T. M., "An Investigation of the Fatigue of Metals." Bull. No. 142, Eng. Exp. Station, Univ. of Ill., Vol. 21, No. 37, May 26, 1924, p. 60.

[4.9] McAdam, D. J., Jr., "The Endurance Range of Steel," Proc. A.S.T.M., Vol. 24, pt. 2, 1924, pp. 574-600.

[4.10] Ludwik, P., and Krystof, J., "Einfluss der Vorspannung auf die Dauerfestigkeit," Z.V.D.I., Bd. 77, Nr. 24, June 17, 1933, pp. 629-635.

[4.11] Smith, James O., "The Effect of Range of Stress on the Torsional Fatigue Strength of Steel," Bull. No. 316, Eng. Exp. Station, Univ. of Ill., Vol. 33, 1939.

[4.12] Kececioglu, Dimitri, Chester, Louis B., and Dodge, Thomas M., "Combined Bending Torsion Fatigue Reliability of AISI 4340 Steel Shafting with K_t = 2.34," ASME Paper No. 74-WA/DE-12,1974.

[4.13] Lea, F. C., and Budgen, H. P., "Combined Torsional and Repeated Bending Stresses," Engineering, Vol. CXXII, Augs. 20, 1926, pp. 242-245.

[4.14] Ono, Akimasu, "Fatigue of Steels Under Combined Bending and Torsion." Memoirs, College of Eng. Kyushu Imperial Univ., Vol. II, No. 2, 1921, pp. 117-142.

[4.15] Gough, H. J., Pollard, H. V., and Clenshaw, W. S., Some Experiments on the Resistance of Metals to Fatigue Under Combined Stresses, Ministry of Supply, Aeronautical Research Council Reports and Memoranda, His Majesty's Stationery Office, London, 1951.

[4.16] Frith, P. H., "Fatigue Tests on Crankshaft Steels," Journal of the Iron and Steel Institute, Vol. 159, London, 1948.

[4.17] Nishihara, T. and Kawamoto, M., "The Strength of Metals and Combined Alternating Bending and Torsion with Phase Difference," Memoirs, College of Engineering, Kyoto Imperial University, Vol. IX, pp. 85-112, 1945.

[4.18] Sawert, Walter, "Verhalten der Baustahle bei wechseinder meharchsiger Beanspruchung," Z.V.D.I., Bd. 87, Nr. 39/40, Oct. 2, 1943, pp. 609-615.

[4.19] Lee, S. B., "A Criterion for Fully-Reversed Out-of-Phase Torsion and Bending," Proceedings, ASTM, International Symposium of Biaxial/Multiaxial Fatigue, San Francisco, California, 15-17, December, 1982. (to be published)

[4.20] Rotvel, F., "Biaxial Fatigue Tests with Zero Mean Stresses Using Tubular Specimens," Int. J. Mech. Sci., Vol. 12, 1970, pp. 597-613.

5. BIAXIAL ALTERNATING STRESSES WITH NO MEAN STRESSES

It was seen that Eq. 4.1 fits the proportionally applied, reversed bending, reversed torsional stress data of Table 4.1 quite well. It was pointed out that this stress condition gives alternating principal stresses which are 180° out-of-phase, in addition to the fact that for a particular fatigue limit combination of σ_{xa} and τ_{xya}, the alternating principal stress axes have fixed orientation with respect to the critical stress element. It is important to keep these facts in mind since they are assumptions which must continue to be enforced if Eq. 4.1 is used to establish a more general fatigue criterion.

Let Eq. 4.1 be written as,

$$\sigma_{xa}^{2} + C\,\tau_{xya}^{2} = S_{e}^{2} \tag{5.1}$$

where,

$$C = (S_e/S_{se})^2 \tag{5.2}$$

Equation 5.1 can be made to take on the appearance of greater generality by writing it in terms of alternating principal stresses. First, the alternating principal stresses may be written in terms of σ_{xa} and τ_{xya} as,

$$\sigma_{1a} = \frac{\sigma_{xa}}{2} + \sqrt{\left(\frac{\sigma_{xa}}{2}\right)^2 + \tau_{xya}^{2}} \tag{5.3}$$

$$\sigma_{2a} = \frac{\sigma_{xa}}{2} - \sqrt{\left(\frac{\sigma_{xa}}{2}\right)^2 + \tau_{xya}^{2}} \tag{5.4}$$

From Eqs. 5.3 and 5.4, expressions for σ_{xa}^2 and τ_{xya}^2 may be obtained, which are,

$$\sigma_{xa}^2 = (\sigma_{1a} + \sigma_{2a})^2 \tag{5.5}$$

$$\tau_{xya}^2 = -\sigma_{1a}\sigma_{2a} \tag{5.6}$$

Substituting Eqs. 5.5 and 5.6 into Eq. 5.1 gives,

$$\sigma_{1a}^2 + (2-C)\,\sigma_{1a}\sigma_{2a} + \sigma_{2a}^2 = S_e^2 \tag{5.7}$$

Equation 5.7 will be a more general statement than Eq. 5.1 if it can be shown that Eq. 5.7 predicts fatigue limit combinations of σ_{1a} and σ_{2a} which arise not only from the special case of superimposed reversed bending and reversed torsion, but also from a more general biaxial stress state. For the general biaxial stress state the expressions for σ_{1a} and σ_{2a} are,

$$\sigma_{1a} = \frac{\sigma_{xa} + \sigma_{ya}}{2} + \sqrt{\left(\frac{\sigma_{xa} - \sigma_{ya}}{2}\right)^2 + \tau_{xya}^2} \tag{5.8}$$

and,

$$\sigma_{2a} = \frac{\sigma_{xa} + \sigma_{ya}}{2} - \sqrt{\left(\frac{\sigma_{xa} - \sigma_{ya}}{2}\right)^2 + \tau_{xya}^2} \tag{5.9}$$

Where, σ_{xa}, σ_{ya}, and τ_{xya} are alternating stress components viewed from any orientation of the critical stress element. By substituting Eqs. 5.8 and 5.9 into Eq. 5.7 the following result is obtained:

$$\sigma_{xa}^2 + (2-C)\,\sigma_{xa}\sigma_{ya} + \sigma_{ya}^2 + C\,\tau_{xya}^2 = S_e^2 \tag{5.10}$$

The ability of Eq. 5.10 to predict fatigue limit combinations of superimposed reversed bending and torsional stresses is identical to that of Eq. 5.1. This is obvious since Eq. 5.10 reduces to Eq. 5.1 if σ_{ya} is equal to zero. However, it is unknown at this point if Eq. 5.10 will accurately predict fatigue limit combinations of σ_{xa}, σ_{ya}, and τ_{xya} in the general biaxial stress state. Concerning the general biaxial stress state, it should be emphasized that Eqs. 5.7 and 5.10 are equivalent statements since either equation can be transformed into the other. Therefore, in seeking to establish the general validity of the two equations, reference will usually be made to one equation or the other, depending on which is the most convenient for the purpose at hand.

Since Eq. 5.10 is not derived from an energy concept, it is important to recognize that the expression is nevertheless invariant with respect to the orientation of the critical stress element from which the stress components are viewed. This may be obvious from Eq. 5.7, but if not, it is clearly seen by rewriting Eq. 5.10 as,

$$(\sigma_{xa} + \sigma_{ya})^2 - C(\sigma_{xa}\sigma_{ya} - \tau_{xya}^2) = S_e^2 \tag{5.11}$$

The first expression in parentheses is the sum of the alternating stress tensor diagonal terms and the second expression in parentheses is the stress tensor determinant. Since these expressions are well-known stress invariants, Eq. 5.10 is invariant.

It is shown in Appendix A that the requirement for 180° out-of-phase alternating principal stresses is,

$$\sigma_{xa}\sigma_{ya} < \tau_{xya}^{2} \tag{A.10}$$

If it is assumed that only the stress state of the critical element controls fatigue behavior, with no influence from such factors as stress gradients, anisotropy, grain orientation, etc., the following statement can be made. By virtue of Eq. 5.1, Eq. 5.10 is valid for any fatigue limit combination of σ_{xa}, σ_{ya}, and τ_{xya}, provided expression A.10 is satisfied. This is true since the two-dimensional stress tensor,

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xa} & \tau_{xya} \\ \tau_{xya} & \sigma_{ya} \end{bmatrix} \tag{5.12}$$

can be transformed to,

$$\sigma_{ij}' = \begin{bmatrix} \sigma_{xa}' & \tau_{xya}' \\ \tau_{xya}' & 0 \end{bmatrix} \tag{5.13}$$

This is demonstrated graphically in Fig. 5.1.

Expression A.10 is not satisfied if σ_{1a} and σ_{2a} are in-phase. At this point, it is unknown how well Eqs. 5.7 and 5.10 predict central tendency for fatigue limit combinations of alternating stress components when σ_{1a} and σ_{2a} are in phase. Perhaps the only in-phase data available for checking the criterion is that of Sawert [4.18]. Sawert's data may be conveniently compared to Eq. 5.7 because his results are expressed as

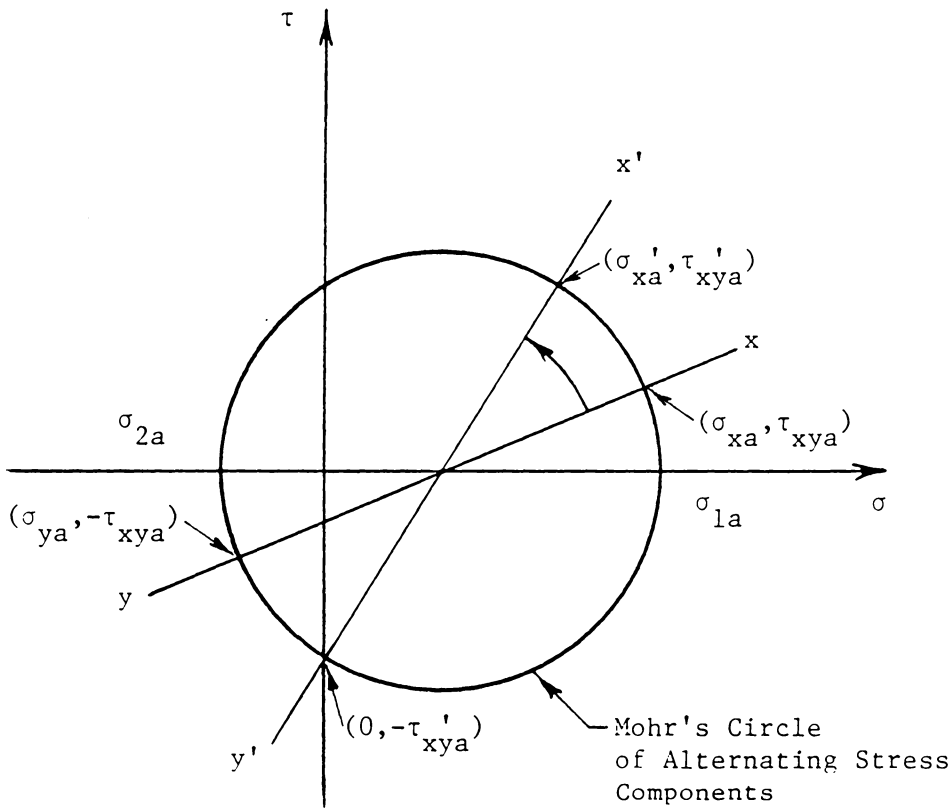


Fig. 5.1 Graphical Demonstration that Eqs. 5.1 and 5.10 are Equivalent Provided Expression A.9 is Satisfied.

fatigue limit combinations of alternating principal stresses. However, before the comparison is made, it should be pointed out that there are some important differences in the nature of Sawert's tests and those from which the data of Table 4.1 were obtained. These differences may exert a significant influence on the accuracy with which the Sawert data fits the ellipse of Eq. 5.7. When the errors were computed for the data of Table 4.1, the bending fatigue limit (S_e) and the torsional fatigue limit (S_{se}) were both obtained from specimens of the same diameter. Furthermore, this same diameter was used throughout a given test series to obtain the various fatigue limit combinations of σ_{xa} and τ_{xya}. Bending induced stresses and torsionally induced stresses in members with circular cross section have distributions of familiar triangular shape. If the diameter is denoted by d, the slope M_1 of the torsional stress distribution is given by

$$M_1 = \frac{2\ \tau_{xya}}{d} \tag{5.14}$$

The slope M_2 of the bending stress distribution is given by

$$M_2 = \frac{2\ \sigma_{xa}}{d} \ . \tag{5.15}$$

Solving for τ_{xya} in Eq. 4.1 gives,

$$\tau_{xya} = \sqrt{\frac{1}{C}\left(S_e^{\,2} - \sigma_{xa}^{\,2}\right)} \ . \tag{5.16}$$

Substituting Eq. 5.16 into Eq. 5.14 gives

$$M_1 = \frac{2\sqrt{\frac{1}{C}\left(S_e^{\ 2} - \sigma_{xa}^{\ 2}\right)}}{d} \tag{5.17}$$

Since d is a constant, Eqs. 5.15 and 5.17 show that the slopes of the bending stress distribution and the torsional stress distributions can be expressed as simple, mathematically continuous functions of the allowable bending stress, σ_{xa}. This fact helps to account for the considerable accuracy with which Eq. 4.1 was able to predict the data of Table 4.1. If the specimens diameters had been allowed to vary within a test series, the different "size effects" would no doubt have caused much larger departures from the curve than those obtained.

The first difficulty to be considered in comparing Sawert's data to Eq. 5.7 is that his reversed bending and reversed torsional endurance limits are obtained from specimens of different diameters. The specimens for pure reversed bending were 16 mm in diameter and the specimens for pure reversed torsion were 14 mm. Perhaps the discrepancy resulting from the difference in diameters will not be very large. However, another consideration which could have much greater influence on error is the diversity of gradient effects which results from Sawert's variety of specimen geometries. For example, all of Sawert's in-phase principal stress data were obtained from axially loaded tension-compression specimens. Each group of specimens had a circumferential groove, the radius of which was chosen so as to provide the desired ratio of σ_{1a} to σ_{2a}. There were four different specimen geometries of this type, two of which were hollow. Except for the reversed torsion data, all of Sawert's out-of-phase principal stress data were also obtained from axially loaded tension-compression specimens. However, the special geometry of these specimens was free from stress concentration in the vicinity of the critical element. Therefore, the alternating principal stress distributions had zero slope at the critical element.

Table 5.1 contains a listing of Sawert's data. The accuracy with which Eq. 5.7 predicts Sawert's data is indicated by the percent error defined in Fig. 5.2. Two pieces of data were omitted because the specimens were produced using a forging operation, for which reason Sawert believed them to have a "mixed fiber direction" and excluded them from his consideration. Also omitted here are all of Sawert's data for nitrided specimens. This was to avoid the complication of subsurface fatigue behavior.

All the rest of Sawert's data is given in Table 5.1. However, not all of it corresponds to in-phase σ_{1a} and σ_{2a}. Both σ_{1a} and σ_{2a} are positive in Table 5.1 for the in-phase condition. It is seen from Table 5.1 that entry 13 has the maximum error of -13.6 percent. The negative sign indicates that the data point falls inside the ellipse in Fig. 5.2. Thus, a negative error is a non-conservative error. This is probably not an unreasonably large error when one considers the comments made previously concerning Sawert's data. Note that entries 4 through 7 and 11 through 14 are the in-phase data, which are presently the main concern. Using all the data in Table 5.1 the average error is computed as -2.55% with a standard deviation of 7%. The computation of these values is similar to expressions 4.2 and 4.3, respectively. The scanty

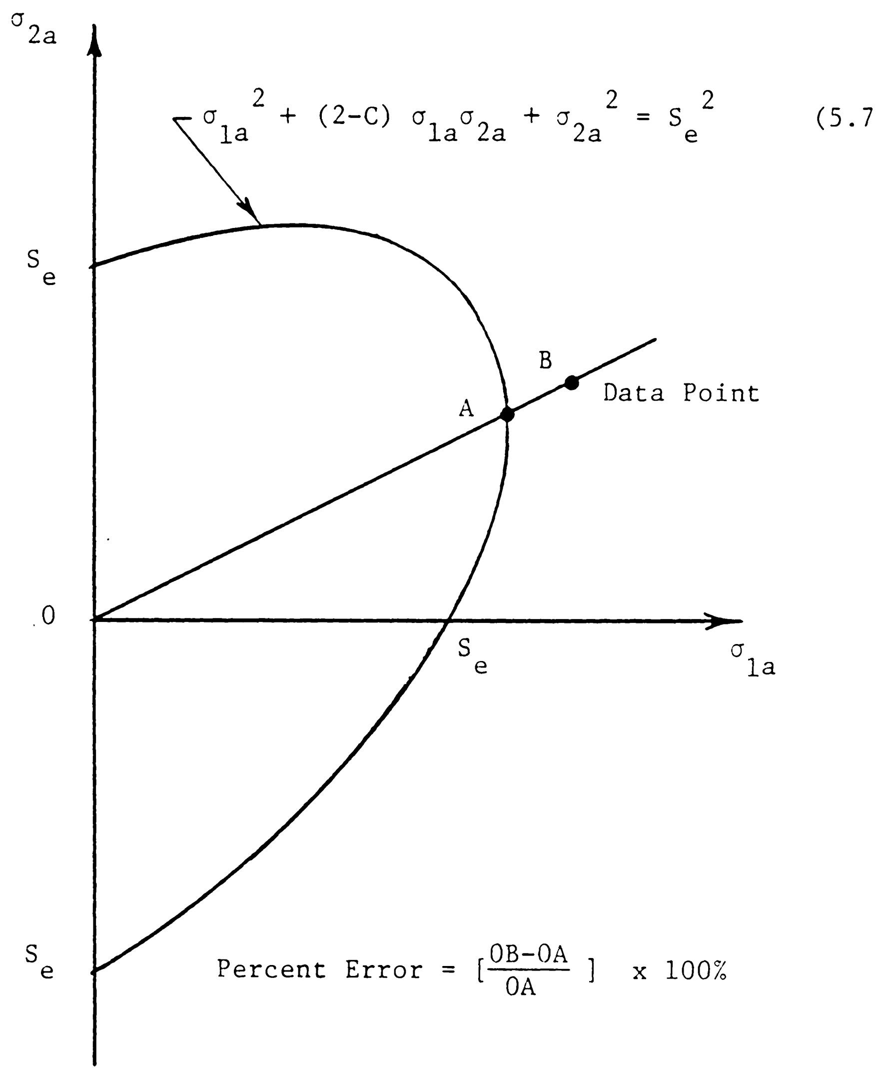


Fig. 5.2 Definition of Percent Error for Comparison of Sawert's Data to Eq. 5.7.

Table 5.1 Sawert's Data [4.18]

No.	Material	Maximum Alternating Principal Stress σ_{1a}*	Minimum Alternating Principal Stress σ_{2a}*	Reversed Bending Fatigue Limit S_e*	Reversed Torsion Fatigue Limit S_{se}*	C	Percent Error as defined in Figure 5.2
1	1.14% C steel Normalized at 910°C	19,200* (132)	-19,200* (-132)	32,000* (221)	19,200* (132)	2.778	0.00
2	Ditto	21,300 (147)	-12,100 (-83.4)	Ditto	Ditto	Ditto	-11.6
3	"	25,900 (178)	-7,400 (-51.0)	"	"	"	-7.6
4	"	34,700 (239)	6,300 (43.1)	"	"	"	2.4
5	"	35,600 (245)	6,800 (47.1)	"	"	"	4.83
6	"	34,300 (236)	9,500 (65.7)	"	"	"	-0.53
7	"	38,100 (263)	10,400 (71.6)	"	"	"	10.56

*The numbers without parentheses have units of lb per sq in. from English translation. Numbers in parentheses are in megapascals.

Table 5.1 (continued)

No.	Material	Maximum Alternating Principal Stress σ_{1a}*	Minimum Alternating Principal Stress σ_{2a}*	Reversed Bending Fatigue Limit S_e*	Reversed Torsion Fatigue Limit S_{se}*	C	Percent Error as defined in Figure 5.1
8	Cr-V steel oil quenched from 860°C	44,800* (309)	-44,800* (-309)	81,800* (564)	44,800* (309)	3.334	0.00
9	drawn 1 hr at 600°C, oil quenched	55,500 (382)	-31,600 (-218)	"	"	"	-2.06
10	Ditto	64,800* (446)	-18,100 (125)	81,800 (564)	44,800 (309)	3.334	-4.59
11	"	87,500 (603)	15,500 (107)	"	"	"	-4.63
12	"	89,000 (613)	16,600 (115)	"	"	"	-3.54
13	"	83,500 (575)	22.200 (218)	"	"	"	-13.63
14	"	90,000 (622)	25,300 (175)	"	"	"	-7.68

*The numbers without parentheses have units of lb per sq in. from English translation. Numbers in parentheses are in megapascals.

data and the unknown gradient effects make it unrealistic to ascribe much significance to the aforementioned statistical parameters.

The influence of stress gradient seems to be especially identifiable with respect to Sawert's out-of-phase data. The out-of-phase entries are 1 through 3 and 8 through 10. If there were equal gradient effects, one might expect Sawert's out-of-phase data to be predicted by Eq. 5.7 almost as well as the data of Table 4.1 was predicted by Eq. 4.1. Entries 1 and 8 are exactly correct since σ_{1a} and σ_{2a} are from the reversed torsion fatigue limit. But entries 2, 3, 9, and 10 have errors that are significantly larger than those typical of Table 4.1. Of the out-of-phase data entries, 2 has the largest error, which is -11.6 percent. It is important to note that for all of Sawert's out-of-phase data, the non-zero errors are negative. This is significant because, except for entries 1 and 8, Sawert's out-of-phase data were obtained from tests in which the alternating principal stress distributions had zero gradient at the critical element. This point will be made clearer in Chapter 6. With zero stress gradients, there is no beneficial gradient effect as there is when the gradients are not zero. In other words, assuming equal stress states, a zero gradient is more detrimental in terms of fatigue damage than a non-zero gradient. This is why the reversed tension-compression fatigue limit is usually lower than reversed bending fatigue limit for the same material. Since Eq. 5.7 is based on data for which there is a beneficial gradient effect, the negative errors in Table 5.1 for Sawert's out-of-phase data are expected. It is not possible to predict, in like fashion, the sign of the errors for Sawert's in-phase principal stress data since the gradients are non-zero. Although the loading is axial tension-compression, the non-zero gradient is induced by a circumferential groove in the specimens. Conceivably, the gradient effect may be more severe or less severe than that anticipated by Eq. 5.7. Thus, it is not surprising that errors for the in-phase data in Table 5.1 would be both positive and negative, as they are seen to be.

Although the results are tolerable, the application of Sawert's data to Eq. 5.7 suggests that the stress gradient influence merits further investigation. The treatment of gradient effects is the subject of the next chapter.

It is emphasized that all the data with which this chapter has reference is based on proportionally applied alternating stress components with no mean stresses present. The assumption of zero mean stress will not be dropped until chapter nine.

6. ALTERNATING STRESS GRADIENTS

In order to take a more careful look at stress gradients, it is first necessary to define a coordinate system with respect to the critical element. This is done in Fig. 6.1. The position of the critical element on the surface of the mechanical component is denoted by A. The origin of the coordinate system is at A with the positive z axis pointing into the component and perpendicular to the surface. The x and y axes have a conveniently chosen but fixed orientation with respect to the mechanical component. It is assumed that the negative z face of the critical element is a free surface, since only the biaxial stress state is being considered. It is also assumed that in the vicinity of the critical element the stress tensor components on planes perpendicular to the x-y-z axes are known in terms of their x,y,z coordinates. This information could be obtained from finite element analysis if no elasticity solution exists, which often is the case. The stresses on the x and y faces of the critical element are shown in Fig. 6.1.

It should be observed that in Fig. 6.1 the stress component notation has been modified. For example, the alternating normal stress component in the x direction is denoted by $\sigma_{xa}(0,0,0)$ instead of just simply σ_{xa}. This is to emphasize that the stress components are functions of position. The coordinate (0,0,0) is the position of the critical stress element. The more elaborate notation will be used only when there would be a loss of clarity without it. For the biaxial stress state, the gradients at the critical element are expressed as,

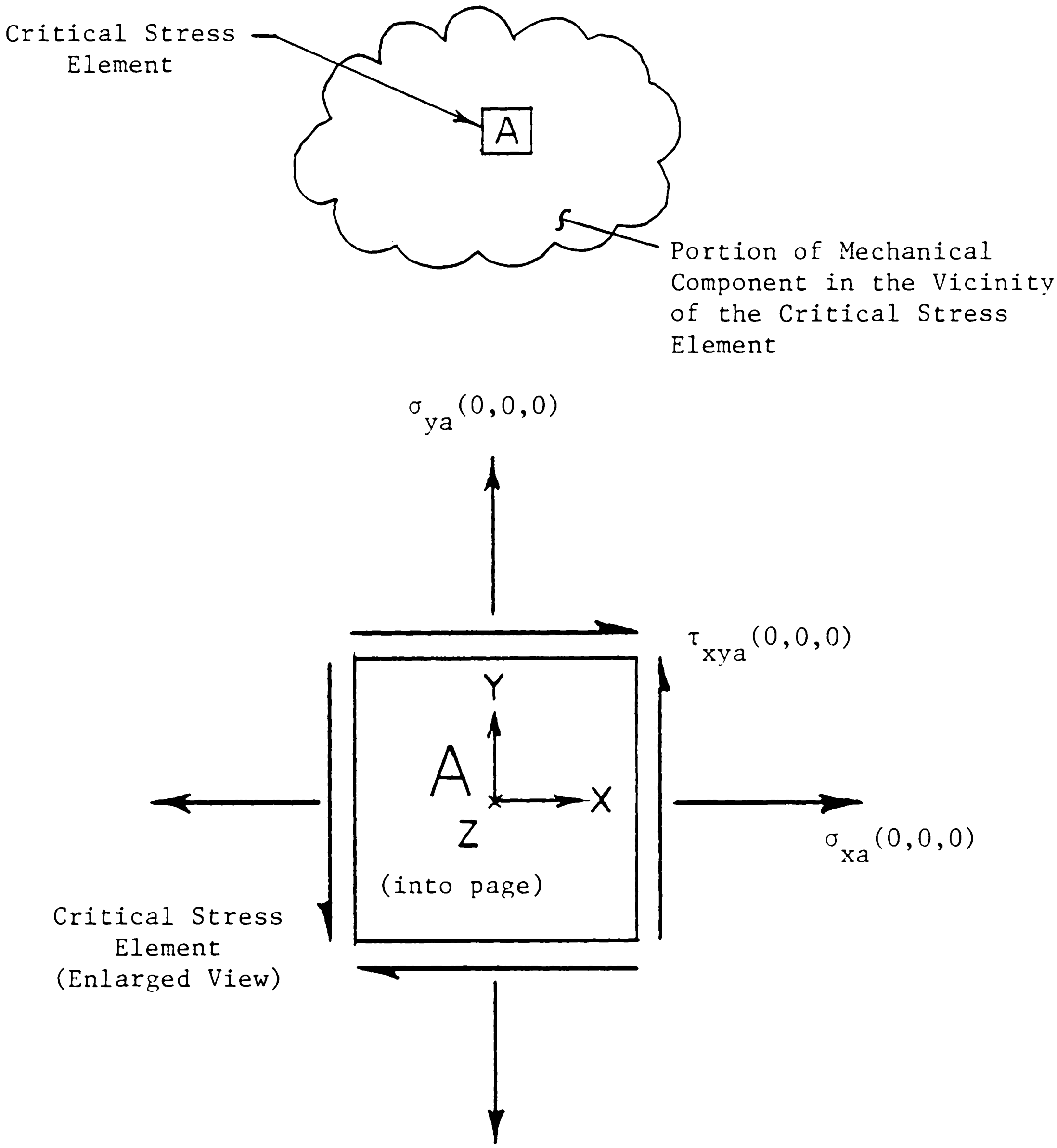


Fig. 6.1 Definition of x-y-z Axes with Respect to the Critical Stress Element.

$$\vec{\nabla}\,\sigma_{xa}(x,y,z)\Big|_{(0,0,0)} = \frac{\partial\,\sigma_{xa}(x,y,z)}{\partial x}\Big|_{(0,0,0)}\hat{i} + \frac{\partial\,\sigma_{xa}(x,y,z)}{\partial y}\Big|\,\hat{j}$$

$$+\frac{\partial\,\sigma_{xa}(x,y,z)}{\partial z}\Big|_{(0,0,0)}\hat{k} \tag{6.1}$$

$$\vec{\nabla}\,\sigma_{ya}(x,y,z)\Big|_{(0,0,0)} = \frac{\partial\,\sigma_{ya}(x,y,z)}{\partial x}\Big|_{(0,0,0)}\hat{i} + \frac{\partial\,\sigma_{ya}(x,y,z)}{\partial y}\Big|_{(0,0,0)}\hat{j}$$

$$+\frac{\partial\,\sigma_{ya}(x,y,z)}{\partial z}\Big|_{(0,0,0)}\hat{k} \tag{6.2}$$

$$\vec{\nabla}\,\tau_{xya}(x,y,z)\Big|_{(0,0,0)} = \frac{\partial\,\tau_{xya}(x,y,z)}{\partial x}\Big|_{(0,0,0)}\hat{i} + \frac{\partial\,\tau_{xya}(x,y,z)}{\partial y}\Big|_{(0,0,0)}\hat{j}$$

$$+\frac{\partial\,\tau_{xya}(x,y,z)}{\partial z}\Big|_{(0,0,0)}\hat{k} \tag{6.3}$$

Where $\vec{\nabla}$ denotes gradient, and the vertical lines with the coordinate (0,0,0) at the lower ends indicate that the quantity is being evaluated at the critical element. Unit vectors in the x,y, and z directions are denoted by $\hat{i}$, $\hat{j}$, and $\hat{k}$, respectively.

In Fig. 6.2a a circular member is shown loaded in alternating bending and alternating torsion. It was mentioned in Chapter 5 that the stress distributions for bending and torsion of a circular member were

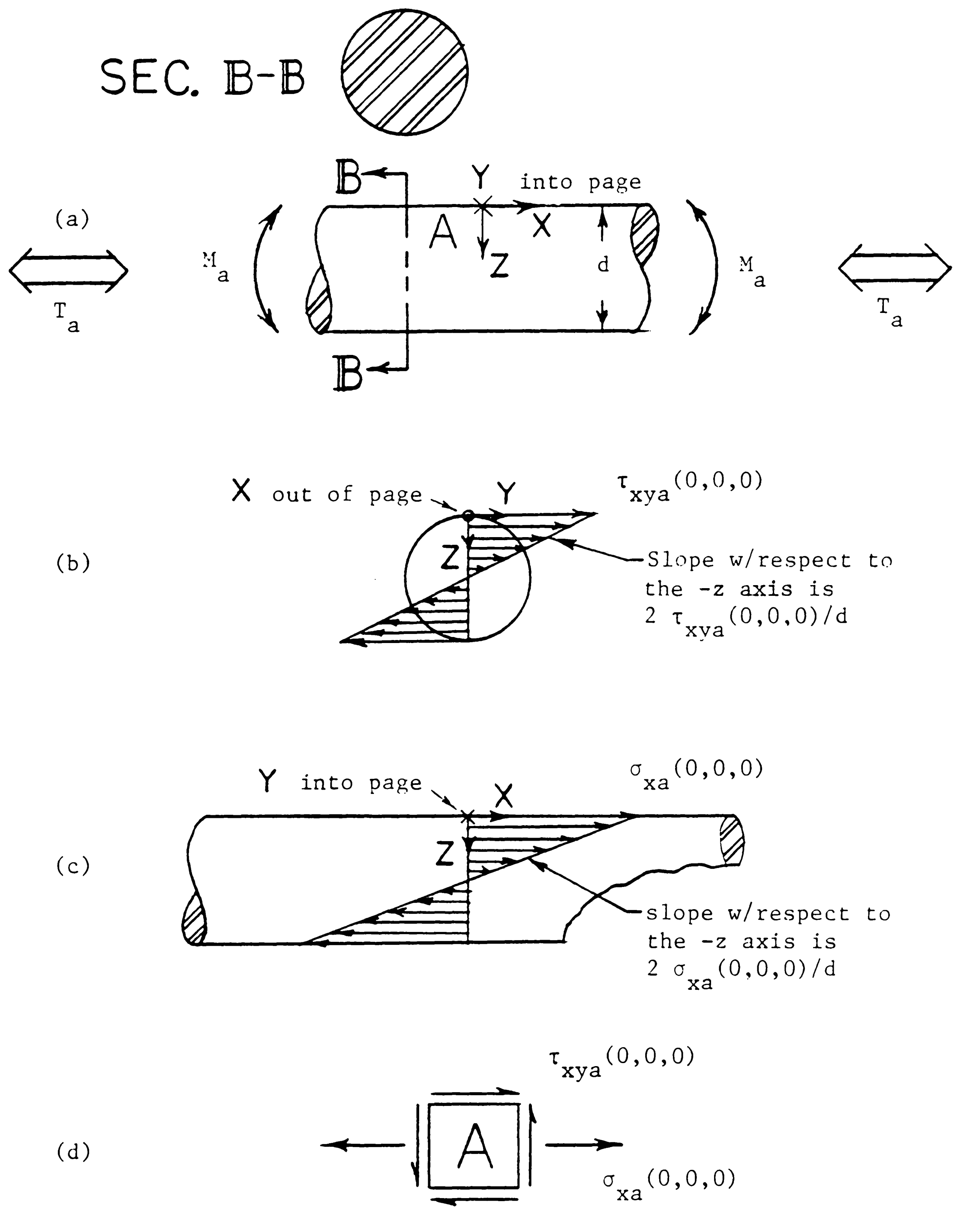


Fig. 6.2 Circular Member Loaded with Alternating Bending and Torsion. (a) Loading Configuration, (b) Torsional Stress Distribution, (c) Bending Stress Distribution, (d) Critical Element

triangularly shaped and the slopes of the distributions were given by Eqs. 5.14 and 5.15. These distributions are shown in Figs. 6.2b and 6.2c with respect to a common set of coordinate axes. Since the bending is about the y axis, the critical element is located on the top (or bottom) of the member and is denoted by point A. Figure 6.2d shows the stresses acting on the critical element. The values of the stress components acting on the critical element are represented by $\tau_{xya}(0,0,0)$ and $\sigma_{xa}(0,0,0)$, replacing the less complete notation used in Eqs. 5.14 and 5.15 and elsewhere. It is now possible to express the gradients for reversed bending and reversed torsion in a way that is complete and unambiguous. In terms of Eqs. 6.1, 6.2, and 6.3, they are,

$$\vec{\nabla}\, \sigma_{xa}(x,y,z) \Big|_{(0,0,0)} = \frac{2\, \sigma_{xa}(0,0,0)}{d}\, \hat{k} \tag{6.4}$$

$$\vec{\nabla}_{\sigma_{ya}}(x,y,z) \Big|_{(0,0,0)} = 0 \tag{6.5}$$

$$\vec{\nabla}\, \tau_{xya}(x,y,z) \Big|_{(0,0,0)} = \frac{2\, \tau_{xya}(0,0,0)}{d}\, \hat{k} \tag{6.6}$$

Equation 5.7 fails to predict the Sawert data with the degree of accuracy seen in Table 4.1, at least partially, and perhaps primarily, because the stress gradients in Sawert's specimens are not equivalent to Eqs. 6.4, 6.5, and 6.6. Furthermore, Eqs. 5.7 is significantly non-conservative for some cases, especially entry number 13 of Table 5.1 for which the error was -13.7 percent. We will now consider how to modify Eq. 5.7 so as to remove any beneficial gradient effect.

Suppose fatigue tests were performed to obtain fatigue limit combinations of reversed normal and reversed shearing stresses such that the gradients as expressed by Eqs. 6.1, 6.2, and 6.3 are all zero. With zero gradients, let $\overline{S}_e$ represent the endurance limit for pure reversed normal stress and let $\overline{S}_{se}$ represent the endurance limit for pure reversed shearing stress. Moreover, a bar placed above any stress component symbol will indicate zero gradient for that component. Would data from such tests mentioned above yield the equation of an ellipse similar to Eq. 4.1? That is, would the following relationship hold true?

$$\left(\frac{\overline{\sigma}_{xa}}{\overline{S}_e}\right)^2 + \left(\frac{\overline{\tau}_{xya}}{\overline{S}_{se}}\right)^2 \overset{?}{=} 1 \tag{6.7}$$

Although there is need for additional testing, it will be shown from Sawert's out-of-phase data that there is reason to believe that the answer is affirmative. It was pointed out in chapter 5 that with the exception of entries 1 and 8 in Table 5.1, all of Sawert's out-of-phase data came from tension-compression tests in which the gradients are zero. Since Sawert's out-of-phase data is now especially important to our consideration it will be examined more closely.

The specimen geometry used by Sawert to obtain the out-of-phase data is given in Fig. 6.3a. Two different specimens were used which were identical in every respect except in one case the D dimension was 25 mm and in the other case the D dimension was 27 mm. The critical

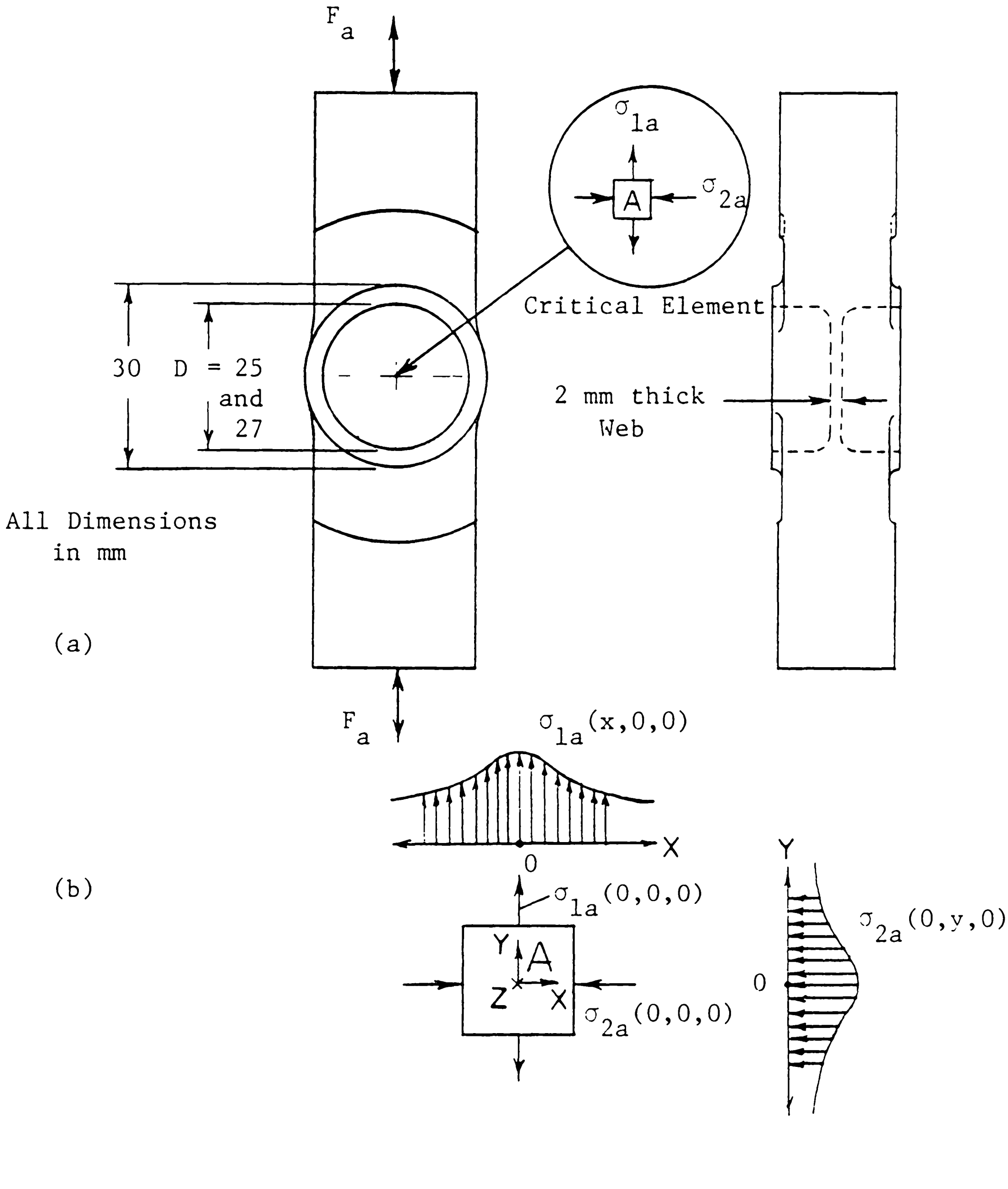


Fig. 6.3 Sawert's Specimen [4.18] for Out-of-Phase Alternating Principal Stresses. (a) Specimen Geometry, (b) Stress Distribution.

element labeled "A" is located in the center of the 2 mm thick web. At the critical element the stresses σ_{1a} and σ_{2a} are uniformly distributed across the 2 mm thickness. In Fig. 6.3b the critical element is shown with an x-y-z coordinate system attached according to the convention of Fig. 6.1. Sawert used a brittle varnish to show that in the vicinity of the critical element the alternating principal stresses are distributed in the x and y directions as shown in Fig. 6.3b. With uniform stress distribution through the 2 mm thickness and the distributions given in Fig. 6.2b it is clear that,

$$\left.\frac{\partial\sigma_{1a}(x,y,z)}{\partial x}\right|_{(0,0,0)} = 0; \quad \left.\frac{\partial\sigma_{1a}(x,y,z)}{\partial y}\right|_{(0,0,0)} = 0; \quad \left.\frac{\partial\sigma_{1a}(x,y,z)}{\partial z}\right|_{(0,0,0)} = 0$$

$$\left.\frac{\partial\sigma_{2a}(x,y,z)}{\partial x}\right|_{(0,0,0)} = 0; \quad \left.\frac{\partial\sigma_{2a}(x,y,z)}{\partial x}\right|_{(0,0,0)} = 0; \quad \left.\frac{\partial\sigma_{2a}(x,y,z)}{\partial z}\right|_{(0,0,0)} = 0 \tag{6.8}$$

Since the σ_{1a} and σ_{2a} are out-of-phase, they may be transformed to σ_{xa} and τ_{xya} by Eqs. 5.5 and 5.6. Using the more complete notation we write,

$$\sigma_{xa}^{2}(x,y,z)\Big|_{(0,0,0)} = \left(\sigma_{1a}(x,y,z)\Big|_{(0,0,0)} + \sigma_{2a}(x,y,z)\Big|_{(0,0,0)}\right)^2 \tag{6.9}$$

$$\tau_{xa}^{2}(x,y,z)\Big|_{(0,0,0)} = -\sigma_{1a}(x,y,z)\Big|_{(0,0,0)} \cdot \sigma_{2a}(z,y,z)\Big|_{(0,0,0)} \tag{6.10}$$

Equations 6.9 and 6.10 may each be partially differentiated with respect to the three coordinates x,y,z, and then solved for the partial derivatives of σ_{xa} and τ_{xya}. However, in the expressions for the partial derivatives of σ_{xa} and τ_{xya}, every term will contain some partial derivative of σ_{1a} or σ_{2a}. But, by Eq. 6.8 these are all zero and thus we have,

$$\frac{\partial\ \sigma_{xa}(x,y,z)}{\partial x}\Bigg|_{(0,0,0)} = 0\ ;\quad \frac{\partial\ \sigma_{xa}(x,y,z)}{\partial y}\Bigg|_{(0,0,0)} = 0\ ;\quad \frac{\partial\ \sigma_{xa}(x,y,z)}{\partial z}\Bigg|_{(0,0,0)} = 0 \tag{6.11}$$

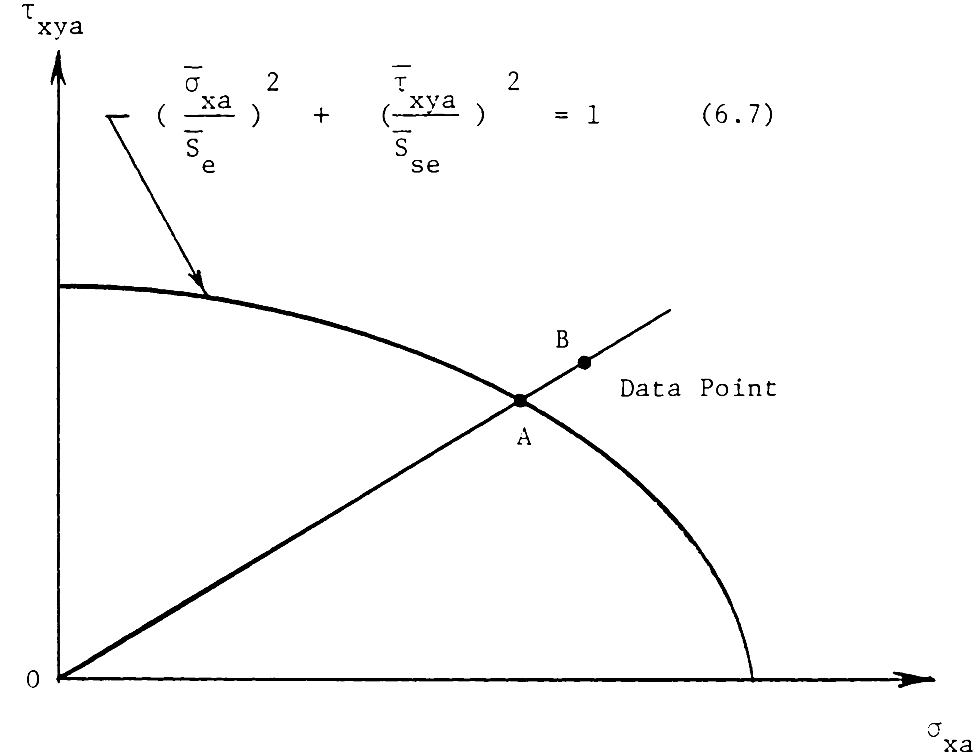
$$\frac{\partial\ \tau_{xya}(x,y,z)}{\partial x}\Bigg|_{(0,0,0)} = 0\ ;\quad \frac{\partial\ \tau_{xya}(x,y,z)}{\partial y}\Bigg|_{(0,0,0)} = 0\ ;\quad \frac{\partial\tau_{xya}(x,y,z)}{\partial z}\Bigg|_{(0,0,0)} = 0$$

By substituting Eqs. 6.11 into Eqs. 6.1 and 6.3 the gradients are,

$$\vec{\nabla}\ \sigma_{xa}(x,y,z)\Big|_{(0,0,0)} = 0 \tag{6.12}$$

$$\vec{\nabla}\ \tau_{xya}(x,y,z)\Big|_{(0,0,0)} = 0 \tag{6.13}$$

Thus, it has been shown that if Sawert's out-of-phase σ_{1a} and σ_{2a} are transformed to σ_{xa} and τ_{xya}, the resulting gradients for σ_{xa} and τ_{xya} are zero. Therefore, with the exception of entries 1 and 8 in Table 5.1, Sawert's out-of-phase data meet the conditions necessary to assess the validity of Eq. 6.7. In addition to the out-of-phase data, Sawert also obtained uniaxial tension-compression fatigue limits for the two materials given in Table 5.1. This data also meets the zero gradient requirement and, therefore, it may also be used to check Eq. 6.7. With the uniaxial tension-compression data included, for each of the two materials represented in Table 5.1 there will be a total of three points with which to check Eq. 6.7. In each case, since $\overline{S}_{se}$ is unknown, the ellipse of Eq. 6.7 was assumed and $\overline{S}_e$ and $\overline{S}_{se}$ were obtained using a least squares fit to the three data points. The least squares fit was accomplished by obtaining $\overline{S}_e$ and $\overline{S}_{se}$ in Eq. 6.7 such that the sum of the squares of the errors were minimized with error defined as in Fig. 6.4. The results are contained in Table 6.1. The maximum error is seen to be 1.78 percent for entry 2. Although the data are lamentably few, the small errors strongly tend to support the validity of Eq. 6.7. The values for $\overline{S}_e$ are 29,700 psi (205 MPa) and 75,400 psi (520 MPa) which compare to the experimentally determined uniaxial tension-compression fatigue limits of 29,500 psi (203.4 MPa) and 75,500 psi (521 MPa), entries 3 and 6, respectively. By allowing $\overline{S}_e$ to vary in the minimization process it is effectively being assumed that there may be some error in the experimental tension-compression fatigue limits. However, due to the relative simplicity of the tension-compression test, the values are expected to be especially reliable. If the minimization process had yielded $\overline{S}_e$ values which departed significantly from the



$$\text{Percent Error} = \left[\frac{OB-OA}{OA}\right] \times 100\%$$

Fig. 6.4 Definition of Percent Error for Comparison of Sawert's 180° Out-of-Phase Alternating Principal Stress Data to Eq. 6.7.

Table 6.1 Sawert's In-Phase Data [4.18] with Zero Stress Gradients

Number	Material	$\bar{\sigma}_{xa}$ *	$\bar{\tau}_{xya}$ *	$\bar{S}_{e}$ *	$\bar{S}_{se}$ *	Percent Error as Defined in Fig. 6.4
1	1.14 per cent C steel Normalized at 910^{o}C	9,200* (63.4)	16,050* (11)	29,700* (205)	17,200* (119)	-1.66
2	Ditto "	18,500 (128)	13,840 (95.4)	29,700 (205)	17,200 (119)	1.78
3	"	29,500 (203)	0.0 (0.0)	29,700 (205)	17,300 (119)	-.67
4	Cr-V Steel Oil Quenched from 860^{o}	23,900 (165)	41,880 (289)	75,400 (520)	44,000 (303)	-.32
5	Ditto	46,700 (372)	34,250 (236)	75,400 (510)	44,000 (302)	-.53
6	"	75,500 (521)	0.0 (0.0)	75,400 (520)	44,00 (303)	.13

* Numbers without parentheses have units of lb per sq in. from English Translation.
Numbers in parentheses are in megapascals.

experimental tension-compression fatigue limit values, the least squares fit would have been repeated with $\overline{S}_e$ fixed at the experimentally determined, tension-compression fatigue limit value, allowing only $\overline{S}_{se}$ to vary in the minimization process.

It is an interesting speculation that Eq. 6.7 may be more universally correct than Eq. 4.1. Gough and Clenshaw [4.15] found that reversed bending-reversed torsion fatigue limit data for cast iron and cast steel did not fit the ellipse quadrant of Eq. 4.1 nearly as well as did the data from unnotched specimens of mild carbon steel. Of course, it may be debated as to whether or not these cast materials have a well-defined fatigue limit. However, the data tended to be consistently below the curve with the largest deviation about midway along the curve. Perhaps the fit of data to Eq. 4.1, for some materials more than others, is adversely influenced by the bending induced and torsionally induced stress gradients. Since all gradient effect is excluded in the data for which Eq. 6.7 applies, a material whose reversed bending-reversed torsion data fails to satisfy Eq. 4.1, might, nevertheless, yield data that is well-described by Eq. 6.7.

Further testing is needed to fully establish Eq. 6.7. However, on the basis of the scanty Sawert data let it now be assumed that Eq. 6.7 is valid. The process by which Eq. 5.1 led to Eq. 5.7 may be repeated in a similar fashion to give,

$$\overline{\sigma}_{1a}{}^2 + (2-\overline{C})\,\overline{\sigma}_{1a}\overline{\sigma}_{2a} + \overline{\sigma}_{2a}{}^2 = \overline{S}_e{}^2 \qquad (6.14)$$

where

$$\overline{C} = \left(\frac{\overline{S}_e}{\overline{S}_{se}}\right)^2 \tag{6.15}$$

By removing the bar notation from $\overline{\sigma}_{1a}$ and $\overline{\sigma}_{2a}$ in Eq. 6.14 we have,

$$\left[\sigma_{1a}{}^2 + (2-\overline{C})\ \sigma_{1a}\sigma_{2a} + \sigma_{2a}{}^2\right]^{1/2} \geq \overline{S}_e \tag{6.16}$$

The equality sign in expression 6.16 is expected to hold for fatigue limit combinations of σ_{1a} and σ_{2a} when σ_{1a} and σ_{2a} have zero gradients. The inequality is expected to hold if σ_{1a} and σ_{2a} do not have zero gradients. Because of the beneficial gradient effect which results from a non-zero gradient, the direction of the inequality is as shown.

It is unfortunate that so few data exist by which the validity of expression 6.16 can be checked, not only for out-of-phase σ_{1a} and σ_{2a}, but for in-phase σ_{1a} and σ_{2a} as well. The best check available is the entire set of Sawert data contained in Table 5.1. If expression 6.16 is valid, then each of Sawert's data should satisfy either the equality or else it should satisfy the inequality in the direction indicated. In order to compare the Sawert data to expression 6.16, a percent error will be defined as,

$$\text{Percent Error} = \left\{\frac{\left[\sigma_{1a}{}^2 + (2-\overline{C})\sigma_{1a}\sigma_{2a} + \sigma_{2a}{}^2\right]^{1/2}\ \overline{S}_e}{\overline{S}_e}\right\} \times 100\% \tag{6.17}$$

It should be noted that a zero percent error means that the equality is satisfied. A positive error means that the inequality is satisfied. Due to unavoidable scatter in fatigue data, it would be possible to obtain a negative error. But, any negative error must be very small (essentially zero) or else the validity of expression 6.16 would be suspect. Sawert's alternating principal stress values from each of the entries in Table 5.1 were substituted into Eq. 6.17. The results of the comparison are shown in Table 6.2. It is seen that the resulting errors satisfy all the expectations which arise from assuming that Eq. 6.16 is valid, not only for 180° out-of-phase principal stresses but in-phase principal stresses as well. Errors for entries 2,3,9, and 10 should be very small because as mentioned before these entries correspond to the specimen geometry of Fig. 6.2 in which the gradients are zero. From Table 6.2 it is seen that they are in fact quite small, with the maximum error of 1.8 percent. Any large error is positive as would be expected since presumably the error is a result of beneficial gradient effect. Entry 2 has the largest negative error of -1.7 percent, which satisfies the requirement of being essentially equal to zero.

It has been shown as far as the few existing data will allow, that Eq. 6.16 is valid. It accurately predicts central tendency for fatigue limit combinations of σ_{1a} and σ_{2a} when σ_{1a} and σ_{2a} have zero gradients. The inequality holds when the gradients are not zero.

An equivalent alternating stress σ_a' may be defined from Eq. 6.16 as,

$$\sigma_a' = \sqrt{\sigma_{1a}^2 + (2-\bar{C})\,\sigma_{1a}\sigma_{2a} + \sigma_{2a}^2} \qquad (6.18)$$

In Eq. 6.18, σ_{1a} and σ_{2a} are design stresses and, therefore, not necessarily fatigue limit values. The goal in design is to achieve stress levels which are sufficiently lower than fatigue limit values so as to

Table 6.2 Comparison of Sawert's Data [4.18] and Eq. 6.16

Entry Number from Table 5.1	$\bar{S}_e$*	$\bar{S}_{se}$*	Percent Error
1	29,700* (205)	17,200* (119)	11.6
2	Ditto	Ditto	-1.7
3	"	"	1.8
4	"	"	8.0
5	"	"	10.4
6	"	"	3.6
7	"	"	15.2
8	74,500 (520)	44,000 (303)	1.8
9	Ditto	Ditto	.32
10	"	"	-.53
11	"	"	8.0
12	"	"	9.5
13	"	"	.39
14	"	"	7.81

*Numbers without parentheses in lb per sq in. Numbers in parentheses are in megapascals.

permit some margin of safety. Let it be assumed that for Eq. 6.18 all the partial derivatives of σ_{1a} and σ_{2a} with respect to the coordinate axes x,y, and z attached to the critical element are known. If the partials for σ_{1a} and σ_{2a} are all zero, then σ_a' computed from Eq. 6.18 must be less than $\bar{S}_e$ to achieve any margin of safety. Of course, in design, the values for both $\bar{S}_e$ and $\bar{C}$ would have to be modified to compensate for any departures from the ideal test conditions under which these values were obtained. Suppose the partial derivatives for σ_{1a} and σ_{2a} are not all zero. We will now discuss a possible means for determining ideally what gradient should be present when the alternating normal stress endurance limit is obtained.

The partial derivatives of σ_a' with respect to the coordinate axes may be expressed as,

$$\left.\frac{\partial\ \sigma_a'(x,y,z)}{\partial x}\right|_{(0,0,0)} = \frac{1}{2}\left[\sigma_{1a}^{\ 2}(0,0,0) + (2-\bar{C})\sigma_{1a}(0,0,0)\sigma_{2a}(0,0,0) + \sigma_{2a}^{\ 2}(0,0,0)\right]^{-\frac{1}{2}}$$

$$\cdot\left[2\ \sigma_{1a}(0,0,0)\left.\frac{\partial\ \sigma_{1a}(x,y,z)}{\partial x}\right|_{(0,0,0)} + (2-\bar{C})\left(\left.\frac{\partial\ \sigma_{1a}(x,y,z)}{\partial x}\right|_{(0,0,0)}\sigma_{2a}(0,0,0) + \sigma_{1a}(0,0,0)\left.\frac{\partial\ \sigma_{2a}(x,y,z)}{\partial x}\right|_{(0,0,0)}\right) + 2\ \sigma_{2a}(0,0,0)\left.\frac{\partial\ \sigma_{2a}(x,y,z)}{\partial x}\right|_{(0,0,0)}\right] \tag{6.19}$$

$$\left.\frac{\partial\ \sigma_a'(x,y,z)}{\partial y}\right|_{(0,0,0)} = \text{SAME AS EQUATION 6.18 EXCEPT } \partial x \text{ is REPLACED BY } \partial y \tag{6.20}$$

$$\left.\frac{\partial\ \sigma_a'(x,y,z)}{\partial z}\right|_{(0,0,0)} = \text{SAME AS EQUATION 6.18 EXCEPT } \partial x \text{ IS REPLACED BY } \partial z. \tag{6.21}$$

The gradient of σ_a' may be written,

$$\vec{\nabla}\ \sigma_a'(x,y,z)\Big|_{(0,0,0)} = \left.\frac{\partial\ \sigma_a'(x,y,z)}{\partial x}\right|_{(0,0,0)} \hat{i} + \left.\frac{\partial\ \sigma_a'(x,y,z)}{\partial y}\right|_{(0,0,0)} \hat{j} \tag{6.22}$$

$$+ \left.\frac{\partial\ \sigma_a(x,y,z)}{\partial z}\right|_{(0,0,0)} \hat{k}$$

Suppose S_e is obtained from a test in which the gradient is equal to $\vec{\nabla}\sigma_a'$. We are inclined to speculate that if this value were substituted for σ_a' in Eq. 6.18 that the new equation would accurately predict fatigue limit combinations of σ_{1a} and σ_{2a} whose partial derivatives are used in Eqs. 6.19, 6.20, and 6.21.

In biaxial fatigue the critical stress element represents a stress state of maximum severity usually at a free surface. Therefore, the partial derivatives of σ_a' with respect to x and y will frequently be zero. For such cases we have

$$\vec{\nabla}\ \sigma_a' = \left.\frac{\partial\ \sigma_a'}{\partial z}(x,y,z)\right|_{(0,0,0)} \hat{k} \tag{6.23}$$

This means that S_e could be obtained from a simple bending endurance limit test in which the gradient expressed by Eq. 6.4 is equal to the partial derivative of σ_a' with respect to z. Expressed in equational form,

$$\frac{2\,S_e}{d}\,\hat{k} = \frac{\partial\,\sigma_a'}{\partial z}(x,y,z)\bigg|_{(0,0,0)}\,\hat{k} \tag{6.24}$$

From Eq. 6.24 it is possible, at least in theory, to determine what size specimen should be used in the rotating bending test so as to create a gradient effect which is identical to that of the actual mechanical component. There is, of course, a considerable practical difficulty since the left-hand side of Eq. 6.24 depends not only on the diameter of the reversed bending specimen, but it depends equally on what the fatigue limit turns out to be. In other words an iterative process would have to be performed according to the following procedure:

a) Estimate S_e

b) Divide S_e by $\frac{\partial\,\sigma_a'}{\partial z}(x,y,z)\bigg|_{(0,0,0)}$ to get $\frac{d}{2}$.

c) Perform rotating bending tests on specimens with the above radius, d/2, to obtain an improved value for S_e.

d) Divide the improved S_e by $\frac{\partial\,\sigma_a'}{\partial z}(x,y,z)\bigg|_{(0,0,0)}$ to get a new value for the specimen radius. If the new radius is not sufficiently close to the old value, the process is repeated to satisfactory convergence.

The above process would yield a value of S_e with which σ_a' could be compared to determine the factor of safety. If the required specimen radius turns out to be too small or too large for a practical test, the conventional reversed bending specimen would need to be replaced by a notched specimen in which the appropriate gradient is achieved.

The above discussion which began on page 74 is offered mainly to provide a logical starting point for the rational inclusion of gradient effects in fatigue limit analysis. Even if it is completely correct it may have only academic value at this time.

In many design problems there may be no objection to the conservative error resulting from beneficial gradient effects. If so, in order to determine safety, the equivalent alternating stress computed from Eq. 6.18 should be compared to the reversed tension compression endurance limit of unnotched specimens, which of course is $\overline{S}_e$.

It is now necessary to discuss the matter of obtaining $\overline{C}$. From Eq. 6.15 both $\overline{S}_e$ and $\overline{S}_{se}$ are needed. From Sawert's data it was possible to obtain $\overline{S}_{se}$ indirectly. By generating test specimens with geometries similar to Sawert's, a similar procedure as before could be used to obtain $\overline{S}_{se}$. However, the test description which follows shows a possible alternate method for obtaining $\overline{S}_{se}$.

The test specimens have circular cross section and are loaded as shown in Fig. 6.4a. Figure 6.4b shows the critical element at A with the reversed shear stress, τ_{xya}. The radius of the groove in Fig. 6.4a should be as large as possible to minimize stress concentration, but small enough to avoid reversed bending failure at some distance away from the specimen midspan. From elementary strength of materials, the shear stress distribution should be approximately parabolic in the y direction with maximum value at y=0. All partial derivatives of the

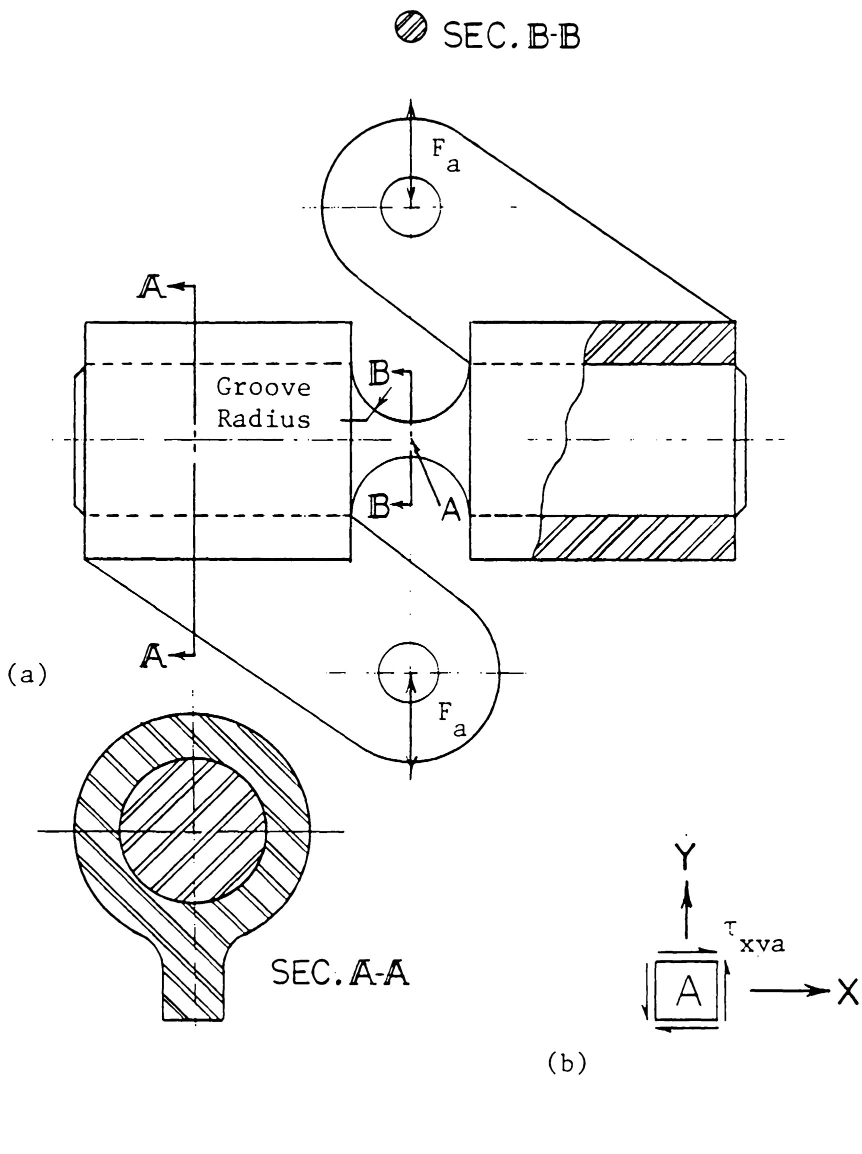


Fig. 6.5 Test Configuration for Obtaining $\overline{S}_{se}$. (a) Specimen Configuration and Loading, (b) Critical Stress Element at Point A Showing Shear Stress Components.

shear stress with respect to the three coordinate axes are probably "close" to zero but, due to the stress concentration, a careful stess analysis would be in order before any tests of this fashion are undertaken.

There is yet another very important observation to be made concerning the determination of $\overline{C}$. The equality from expression 6.16 is an ellipse on σ_{1a}-σ_{2a} coordinates whose major axis has half-length given by,

$$L_1 = \frac{\overline{S}_e}{\sqrt{4-\overline{C}}} \tag{6.25}$$

and minor axis has half-length given by,

$$L_2 = \frac{\overline{S}_e}{\sqrt{\overline{C}}} \tag{6.26}$$

The major axis lies in the in-phase σ_{1a}-σ_{2a} quadrant and the minor axis in the out-of-phase quadrant. With $\overline{C}$ in the vicinity of 3 it is readily verifiable by substitution of trial numbers into Eqs. 6.25 and 6.26 that expression 6.16 can be much more sensitive to errors in $\overline{C}$ if σ_{1a} and σ_{2a} are in-phase than if they are out-of-phase. For this reason, it may be necessary to obtain $\overline{C}$ from in-phase $\overline{\sigma}_{1a}$-$\overline{\sigma}_{2a}$ test data whenever the value is to be used to analyze in-phase alternating principal stresses.

A final remark applies to both $\overline{S}_e$ and $\overline{C}$. These values are likely to be influenced by grain orientation which results from the processes involved in steel production as well as subsequent machining operations. Ideally, the grain orientation of the test specimens should somehow be compatible with the grain orientation of the mechanical component being designed. Due to the non-precise nature of actual fatigue calculations, this problem will not be addressed. A certain amount of error, hopefully small, is likely to result due to material anisotropy.

7. REMARKS CONCERNING THE DISTORTIONAL ENERGY CRITERION

It may be observed that if $\overline{C}$ is equal to 3, then Eq. 6.18 is identical to the von Mises formula which is recommended by Shigley, and others, for obtaining an equivalent alternating stress. During a complete stress cycle involving fully reversed (no mean stress), proportionally applied, stress components, the distortional energy varies from zero to maximum value. It is inherently assumed by the von Mises formula that if the fluctuating distortional energy resulting from fully reversed biaxial stress components has the same maximum value as the maximum distortional energy in a fully reversed uniaxial test, then the fatigue behavior will be the same in both cases. Equation 6.18 is based solely upon data for which the stress components are proportional. It was mentioned earlier and demonstrated in Appendix B, that the alternating principal stress axes have fixed orientation whenever the alternating stress components are proportional. Since the von Mises formula may be viewed as a special case of Eq. 6.18, there is no justification for assuming that the distortional energy concept will be valid if the alternating principal stress axes are not fixed. In fact, Findley, et al. [7.1] describe a case in which the distortional energy is <u>constant</u>, but nevertheless, fatigue failure occurs. The fatigue action somehow seems to be related to the rotation of the alternating principal stress axes.

The distortional energy criterion requires that the ratio of reversed bending fatigue limit to reversed torsion fatigue limit be equal to $\sqrt{3}$. However, experimental values show that the ratio can depart significantly from $\sqrt{3}$. Since S_e/S_{se} is equal to $\sqrt{\overline{C}}$, this is amply demonstrated in Table 4.1 where C values range from 2.14 to 3.12. These values seem to discredit the distortional energy criterion since it requires a C value of 3. It may be possible that the inconsistency is due to the non-zero gradients associated with S_e and S_{se}. It is seen from Table 5.1 that the C values for Sawert's two materials are 2.78 and 3.33. These values do not fully satisfy the expectations of the distortional energy criterion. However, for the same two materials mentioned above, the values for $\overline{C}$ may be computed using the values for $\overline{S}_e$ and $\overline{S}_{se}$ from Table 6.1. The two values are 2.98 and 2.94 which are indeed quite close to 3. Of course, it is unknown how much more nearly equal to 3 the C values for Sawert's materials might have been, if he had not used two different specimen diameters to obtain S_e and S_{se}. The fact that the diameters were different was pointed out in Chapter 5. It is also unknown if $\overline{C}$ values for other materials would be as close to 3 as were the $\overline{C}$ values for the two Sawert materials, thus giving further support to the distortional energy criterion. It would be enlightening to have C and $\overline{C}$ values for a number of different steels.

REFERENCES

[7.1] Findley, W. W. N., Mathur, P. N., Szczepanski E., and Temel, A. A. O., "Energy Versus Stress Theories for Combined Stress - A Fatigue Experiment Using a Rotating Disk," American Society of Mechanical Engineers, Transactions, Vol. 83, Series D., Journal of Basic Engineering, 1961, pp. 10-14.

8. SOME OBSERVATIONS CONCERNING THE MAXIMUM ALTERNATING SHEAR STRESS THEORY, ALSO KNOWN AS THE TRESCA THEORY.

The tensor of alternating stress components may be transformed to the plane of maximum alternating shear stress. The abbreviation MASS will be used to denote "maximum alternating shear stress." For out-of-phase σ_{1a} and σ_{2a}, the alternating normal and shear components on the MASS plane may be expressed in terms of the alternating principal stresses as,

$$\sigma_a(\text{MASS}) = (\sigma_{1a} + \sigma_{2a})/2 \tag{8.1}$$

and

$$\tau_a(\text{MASS}) = (\sigma_{1a} - \sigma_{2a})/2 \tag{8.2}$$

where σ_a(MASS) is the normal component on the MASS plane and τ_a(MASS) is the alternating shear component. Since σ_{1a} and σ_{2a} are alternating components, we may assume with no loss of generality that σ_{1a} is positive. Then, σ_{2a} will be positive or negative depending on whether σ_{1a} and σ_{2a} are in-phase or 180 out-of-phase, respectively. It will also be assumed that the principal stresses are labeled such that, $\sigma_{1a} \geq |\sigma_{2a}|$.

Equations 8.1 and 8.2 are valid only if σ_{1a} and σ_{2a} are 180^o out-of-phase. For in-phase σ_{1a} and σ_{2a} the MASS plane components are given by,

$$\sigma_a(\text{MASS}) = \sigma_{1a}/2 \tag{8.3}$$

$$\tau_a(\text{MASS}) = \sigma_{1a}/2 \tag{8.4}$$

The MASS theory, known to some as the Tresca theory, assumes that ultimate fatigue failure is predicted whenever the maximum alternating shear stress component, $\tau_a(\text{MASS})$, given appropriately by either Eq. 8.2 or Eq. 8.4, reaches a critical value. Consistent with the theory, the critical value may be assumed to be equal to the reversed torsion fatigue limit, S_{se}. It is inherently assumed by the Tresca theory that the normal alternating component, $\sigma_a(\text{MASS})$, may be ignored. We will now consider the implication of ignoring the $\sigma_a(\text{MASS})$ component, as well as the advisability of using $\tau_a(\text{MASS}) < S_{se}$ as a fatigue limit criterion.

Assuming out-of-phase alternating principal stresses, σ_{1a} and σ_{2a} may be obtained from Eq. 8.1 and 8.2. We have,

$$\sigma_{1a} = \sigma_a(\text{MASS}) + \tau_a(\text{MASS}) \tag{8.5}$$

$$\sigma_{2a} = \sigma_a(\text{MASS}) - \tau_a(\text{MASS}) \tag{8.6}$$

If Eqs. 8.5 and 8.6 are substituted into expression 6.16 the following is obtained:

$$(4 - \overline{C})\sigma_a^{\,2}(\text{MASS}) + \overline{C}\tau_a^{\,2}(\text{MASS}) \geq \overline{S}_e^{\,2} \tag{8.7}$$

The equality sign in expression 8.7 is expected to hold for fatigue limit combinations of $\sigma_a(\text{MASS})$ and $\tau_a(\text{MASS})$ when $\sigma_a(\text{MASS})$ and $\tau_a(\text{MASS})$ have zero gradients. The inequality is expected to hold if $\sigma_a(\text{MASS})$ and

τ_a(MASS) do not have zero gradients. As in expression 6.16, due to the beneficial gradient effect which results from a non-zero gradient, the direction of the inequality is as shown. It is clear from expression 8.7 that the alternating normal component, σ_a(MASS), has some influence on fatigue behavior, as well as τ_a(MASS), It will be shown that σ_a(MASS) may be safely ignored only if compensation is made when choosing the critical value for τ_a(MASS).

It is shown in Appendix A that if σ_{1a} and σ_{2a} are out-of-phase, the following condition is satisfied.

$$\sigma_{xa}\sigma_{ya} < \tau_{xya}^2 \tag{A.10}$$

The components on the MASS plane may be written in terms of the components σ_{xa}, σ_{ya}, and τ_{xya} which are viewed from any other element orientation. We have,

$$\sigma_a(\text{MASS}) = \frac{\sigma_{xa} + \sigma_{ya}}{2} \tag{8.8}$$

and

$$\tau_a(\text{MASS}) = \frac{(\sigma_{xa} - \sigma_{ya})^2}{2} + \tau_{xya}^2 \tag{8.9}$$

By squaring both sides of Eqs. 8.8 and 8.9 and subtracting, we have,

$$\sigma_a^2(\text{MASS}) - \tau_a^2(\text{MASS}) = \sigma_{xa}\sigma_{ya} - \tau_{xya}^2 \tag{8.10}$$

Equation A.10 may be written as,

$$\sigma_{xa}\sigma_{ya} - \tau_{xya}^{2} \leq 0. \tag{8.11}$$

Equation 8.11 together with Eq. 8.10 requires for 180° out-of-phase σ_{1a} and σ_{2a} that,

$$\sigma_a(\mathrm{MASS}) \leq \tau_a(\mathrm{MASS}) \tag{8.12}$$

In order to see how the alternating normal stress component, σ_a(MASS), influences the allowable value for τ_a(MASS) when σ_{1a} and σ_{2a} are in-phase, the equality from expression 8.7 is represented as a curve in Fig. 8.1. The shaded area above the curve represents the inequality of expression 8.7. The upper boundary for the shaded area is unknown since it represents fatigue limit combinations of σ_a(MASS) and τ_a(MASS) with infinite gradients. Our remarks from this point will assume, that when the Tresca theory is employed in design, one often does not know the magnitude of the stress gradients, or else, does not know how to account for their effect. In such cases, a designer may assume that the gradients which pertain to design stresses are all zero, in order to be conservative. By assuming zero gradients we may confine our discussion to the curve in Fig. 8.1, thus disregarding the shaded area. Assuming zero stress gradients, design values for τ_a(MASS) and σ_a(MASS) must describe a point below the curve in order to be safe. It was assumed in the derivation of expression 8.7 that σ_{1a} and σ_{2a} were 180° out-of-phase. Therefore, by virtue of expression 8.12, the curve

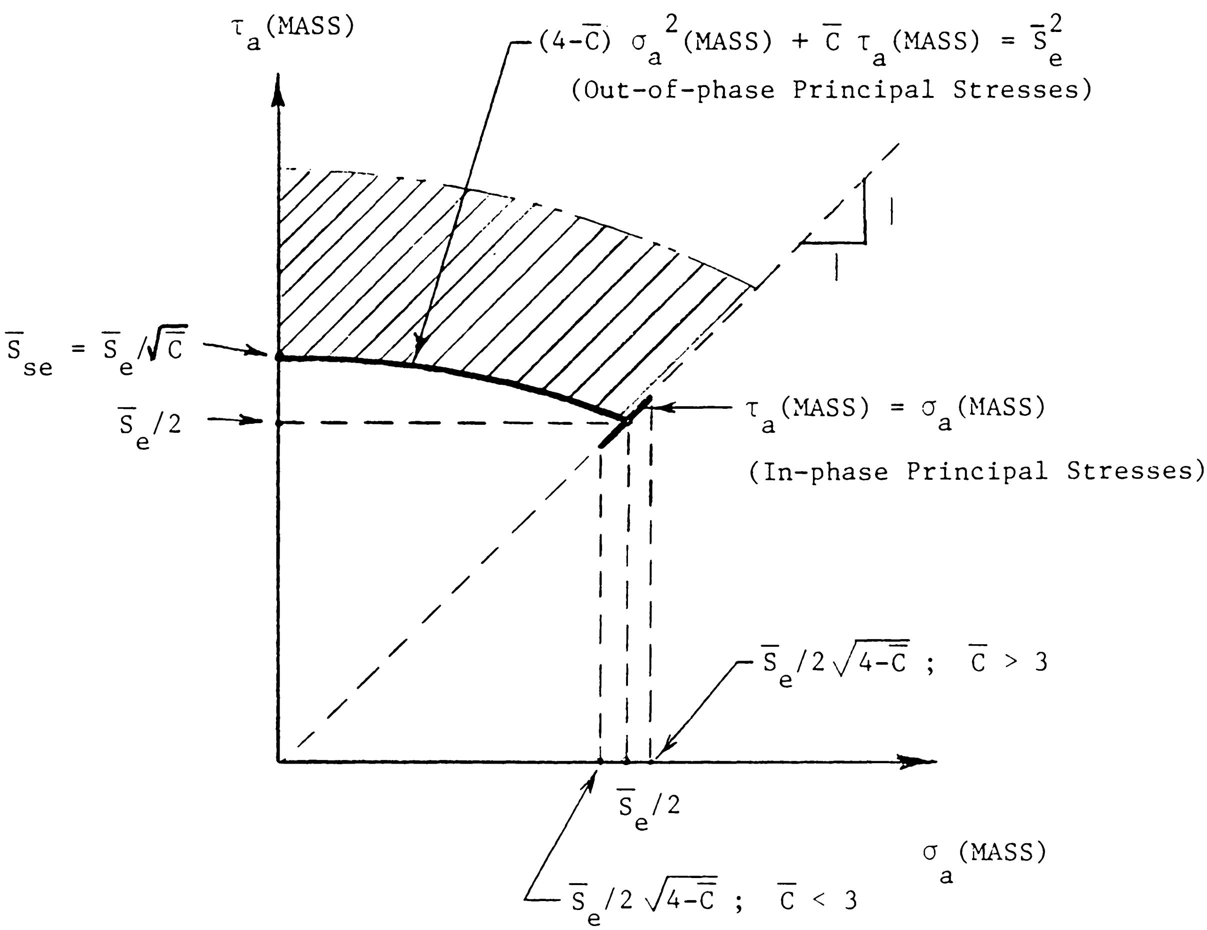


Fig. 8.1 Graphical Depiction of How the Allowable Value for τ_a(MASS) is Influenced by σ_a(MASS).

representing the equality from expression 8.7 extends only to the short line segment along which τ_a(MASS) is equal to σ_a(MASS). The short line segment pertains to in-phase σ_{1a} and σ_{2a} as will soon be explained.

For fully reversed tension-compression, the maximum allowable value for σ_{1a} is $\overline{S}_e$ with σ_{2a} equal to zero. From Eqs. 8.1 and 8.2 we have,

$$\sigma_a(\text{MASS}) = \overline{S}_e/2 \tag{8.13}$$

$$\tau_a(\text{MASS}) = \overline{S}_e/2 \tag{8.14}$$

Since the above values for σ_a(MASS) and τ_a(MASS) satisfy the equality of expression 8.7, they are a point on the curve as indicated in Fig. 8.1. This point is the intersection of the curve which represents the equality from expression 8.7 and the short line segment. It is the point of intersection since reversed tension-compression is the "boundary" between in-phase σ_{1a} and σ_{2a}, and 180° out-of-phase σ_{1a} and σ_{2a}.

Assuming σ_{1a} and σ_{2a} have zero gradients and σ_{1a} is equal to σ_{2a}, it is easy to verify from the equality in expression 6.16 that,

$$\sigma_{1a} = \frac{\overline{S}_e}{\sqrt{4-\overline{C}}} \tag{8.15}$$

But also, from Eqs. 8.3 and 8.4, for in-phase principal stresses we have,

$$\sigma_a(\text{MASS}) = \frac{\overline{S}_e}{2\sqrt{4-\overline{C}}} \tag{8.16}$$

$$\tau_a(\text{MASS}) = \frac{\overline{S}_e}{2\sqrt{4-\overline{C}}} \tag{8.17}$$

Assuming $\overline{C}$ in Eqs. 8.16 and 8.17 to be either greater than 3 or less than 3, the end points for the short line segment in Fig. 8.1 are established. With σ_{1a} and σ_{2a} in-phase, other points along the short line segment are established depending upon the values for $\overline{C}$ and σ_{2a}, with σ_{2a} not equal to σ_{1a}.

Several observations may now be made from Fig. 8.1. If the alternating principal stresses are in-phase, one may achieve safe design while ignoring the σ_a(MASS) component as long as $\tau_a(\text{MASS}) < \overline{S}_e/2$ is the criterion. This criterion would guarantee that the design point is not above the curve, regardless of the value for σ_a(MASS). One would not have a similar guarantee if the criterion were $\tau_a(\text{MASS}) < \overline{S}_{se}$. But it would be especially unadvisable to use the criterion, $\tau_a(\text{MASS}) < S_{se}$, since S_{se} is an even larger value than $\overline{S}_{se}$ due to the beneficial gradient in the reversed torsion specimen. A simple illustration is now given to demonstrate how non-conservative the criterion, $\tau_a(\text{MASS}) < S_{se}$, may be.

Suppose a reversed bending fatigue test for a particular mild carbon steel yields a fatigue limit value of S_e. If similar specimens are tested in reversed tension-compression, the fatigue limit, $\overline{S}_e$, might realistically be in the neighborhood of .85 S_e. If reversed torsion fatigue tests are performed, again on similar specimens, the reversed torsion fatigue limit, S_{se}, might realistically be in the neighborhood of $.58\ S_e$. Now suppose the criterion $\tau_a(\text{MASS}) < S_{se}$ is used to predict the maximum allowable stress for reversed tension-compression in some mechanical component for which the above data is applicable. With reversed tension-compression, the MASS plane is rotated 45^o from the loading direction. On this plane the maximum alternating shear stress is equal to σ_{xa} (allow)/2, where σ_{xa}(allow) is the magnitude of the maximum allowable tension-compression stress. There is also an alternating normal component whose magnitude is also σ_{xa}(allow)/2, but which the criterion disregards. According to the criterion,

$$\sigma_{xa}(\text{allow}) = 2\ S_{se} \tag{8.18}$$

but,

$$S_{se} = .58\ S_e \tag{8.19}$$

Therefore,

$$\sigma_{xa}(\text{allow}) = 1.16\ S_e \tag{8.20}$$

but, the maximum tension-compression value actually allowed is the experimental tension-compression fatigue limit. We compute the error as follows:

$$\text{ERROR} = \left[\frac{.85\ S_e - 1.16\ S_e}{.85\ S_e}\right] \times 100\% = -36\% \tag{8.21}$$

The error is seen to be -36 percent, which by most standards is unacceptably high. The negative sign means that the error is non-conservative. Admittedly, this is an extreme case. But it demonstrates the need for caution when using the MASS criterion.

In recommending the criterion, $\tau_a(\text{MASS}) \leq \overline{S}_e/2$, it was assumed that σ_{1a} and σ_{2a} are out-of-phase. If σ_{1a} and σ_{2a} are in-phase and $\overline{C}$ is unknown, then one cannot be as confident of conservative design using the criterion, $\tau_a(\text{MASS}) \leq \overline{S}_e/2$. This is because the location of the design point on the short line segment in Fig. 8.1 is unknown.

This entire discussion has assumed that the alternating stress components are proportionally applied. There is no reason to expect the Tresca theory to give conservative predictions if the alternating stresses are non-proportional. Nishihara and Kawamoto [8.1] obtained fatigue limit data for non-proportional reversed bending and torsion in combination (zero mean stress). The stress components were non-proportional because they were applied with various phase angles different from 0^o and 180^o. In one instance, when the phase angle was 90^o, the actual fatigue limit value for $\tau_a(\text{MASS})$ was only 13.15 kg/mm^2(129 MPa). The reversed torsion fatigue limit was 20.00 kg/mm^2 (196 MPa). If $\tau_a(\text{MASS}) \leq S_{se}$ is the criteria, the resulting error is given by,

$$\text{ERROR} = \left[\frac{13.15 - 20.00}{13.15}\right] \times 100\% = -52\%.$$

The reversed bending fatigue limit was 32.00 Kg/mm^2 (314 MPa). If the criterion, $\tau_a(\text{MASS}) \leq S_e/2$ is used, the error may be computed as,

$$\text{ERROR} = \left[\frac{13.15 - 16.00}{13.15}\right] \times 100\% = -22\%.$$

These non-conservative errors are unacceptibly large. They point to the non-applicability of the Tresca criterion to cases involving non-proportional stresses.

It has been assumed throughout this discussion that no mean stress components are present. Perhaps it should be stated that there is nothing "special" about the MASS plane which inherently justifies ignoring any mean stress components which exist on that plane. To do so is equivalent to assuming that mean stress has no influence on fatigue behavior.

REFERENCES

[8.1] Nishihara, T. and Kawamoto, M., "The Strength of Metals Under Combined Alternating Bending and Torsion with Phase Difference," Memoirs, College of Engineering, Kyoto Imperial University, Vol. XI, pp. 85-112, 1945.

9. THE COMBINED INFLUENCE OF MEAN AND ALTERNATING BIAXIAL STRESS COMPONENTS

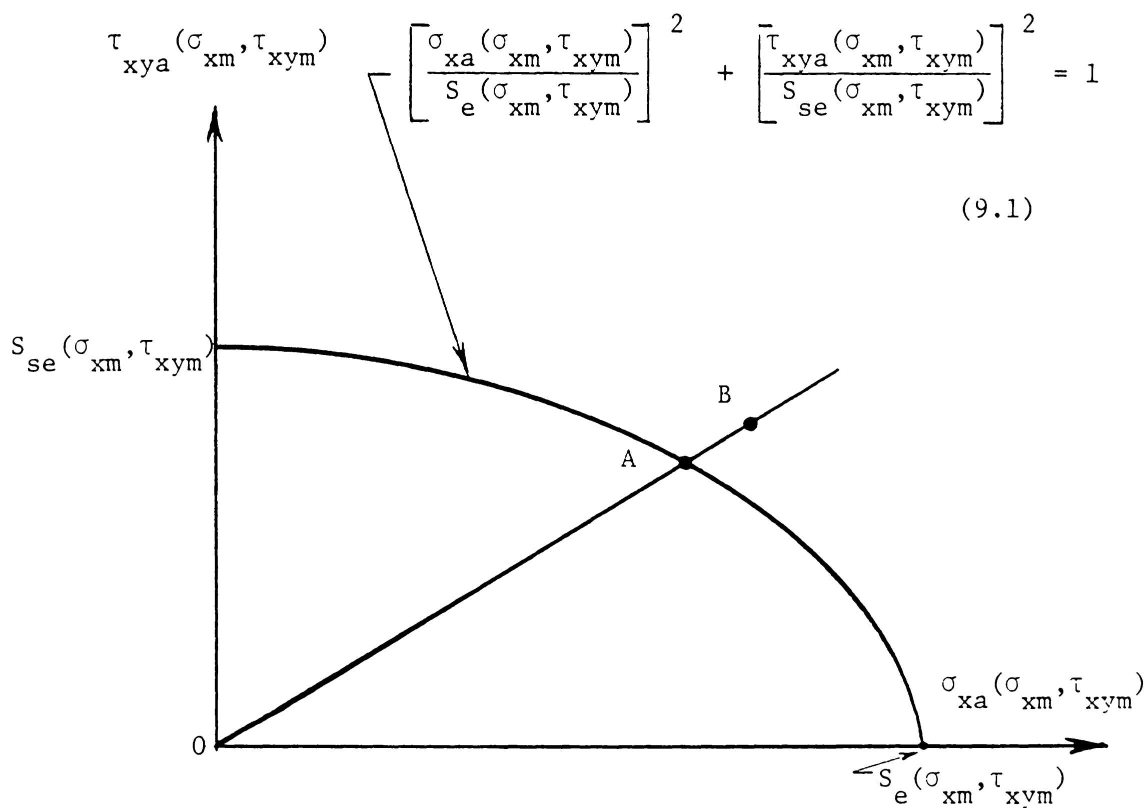
Gough and Clenshaw may be the only investigators to obtain data involving biaxial alternating stresses with superimposed biaxial mean stresses [4.15] such that the mean stress components were held constant while the fatigue limit combinations of alternating components were obtained. The tests they performed involved alternating bending and torsional stresses with superimposed mean bending and torsional stresses. While holding the mean bending stress, σ_{xm}, and the mean torsional stress, τ_{xym}, constant, they found the fatigue limit combinations of σ_{xa} and τ_{xya}. They observed that the data for their solid specimens were described quite well by the equation,

$$\left[\frac{\sigma_{xa}(\sigma_{xm},\tau_{xym})}{S_e(\sigma_{xm},\tau_{xym})}\right]^2 + \left[\frac{\tau_{xya}(\sigma_{xm},\tau_{xym})}{S_{se}(\sigma_{xm},\tau_{xym})}\right]^2 = 1 \qquad (9.1)$$

where $S_e(\sigma_{xm},\tau_{xym})$ and $S_{se}(\sigma_{xm},\tau_{xym})$ are the alternating bending and alternating torsional endurance limits with superimposed σ_{xm} and τ_{xym}. The alternating components σ_{xa} and τ_{xya} in Eq. 9.1 are also followed by the mean stress components shown in parentheses. This is to indicate that σ_{xa} and τ_{xya} are seen with respect to the same coordinate system as σ_{xm} and τ_{xym}. Apparently the superimposed mean stresses introduced some scatter in the fatigue data since Gough and Clenshaw were not able to determine the fatigue limit combinations of σ_{xa} and τ_{xya} as confidently as the data of Table 4.1 for which there were no mean stresses. In some instances they give a range of values instead of a single value.

In order to check the accuracy of Eq. 9.1 relative to the Gough and Clenshaw data, the equation is assumed to be correct in form and $S_e(\sigma_{xm}, \tau_{xym})$ and $S_{se}(\sigma_{xm}, \tau_{xym})$ are obtained using a least squares fit to each of the σ_{xm} and τ_{xym} data for which σ_{xm} and τ_{xym} have constant values. Groups of data for which σ_{xm} and τ_{xym} are constants are shown in Table 9.1 as groups A, B, C, and D. The least squares fit was accomplished by obtaining $S_e(\sigma_{xm}, \tau_{xym})$ and $S_{se}(\sigma_{xm}, \tau_{xym})$ in Eq. 9.1 such that the sum of the squares of the errors were minimized with error defined as in Fig. 9.1. The results are given in Table 9.1. Wherever a range of values are given, the average value is used. It is seen that only entries 7 and 13 have errors greater than 5 percent. The largest error is that of entry 13 which is -6.0 percent. Thus it has been shown that Eq. 9.1 is quite accurate with respect to the Gough and Clenshaw data, especially considering the scatter introduced by the mean stresses.

In Eq. 9.1 the bending and torsional stresses, whether mean or alternating have triangularly shaped distributions. Solely on the basis of our previous experience in Chapter 6 with reversed stress, it will be assumed that if a test were to be performed in which the stress distributions for both mean and alternating stresses have zero gradient, that the form of Eq. 9.1 would be unchanged. The alternating normal stress fatigue limit which results when the alternating stress gradients are zero and the mean stress gradients are zero will be denoted by $\overline{S}_e(\overline{\sigma}_{xm}, \overline{\tau}_{xym})$. The "bar" notation indicates zero gradient as in Chapter 6. The alternating shearing stress fatigue limit with all zero gradients will be denoted by $\overline{S}_{se}(\overline{\sigma}_{xm}, \overline{\tau}_{xym})$. The symbols σ_{xa} and τ_{xya} and continue to represent alternating normal stress and alternating



$$\left[\frac{\sigma_{xa}(\sigma_{xm},\tau_{xym})}{S_e(\sigma_{xm},\tau_{xym})}\right]^2 + \left[\frac{\tau_{xya}(\sigma_{xm},\tau_{xym})}{S_{se}(\sigma_{xm},\tau_{xym})}\right]^2 = 1 \qquad (9.1)$$

$$\text{Percent Error} = \left[\frac{OB-OA}{OA}\right] \times 100\%$$

Fig. 9.1 Definition of Percent Error for Checking Eq. 9.1
Relative to the Gough and Clenshaw Data.

Table 9.1 Gough and Clenshaw's Reversed Bending and Reversed Torsional Data with Superimposed Mean Bending and Torsion [4.15]

Group/ σ_{xm}, τ_{xym}*	Entry Number	$\sigma_{xa}(\sigma_{xm}, \tau_{xym})$*	$\tau_{xya}(\sigma_{xm}, \tau_{xym})$*	$S_e(\sigma_{xm}, \tau_{xym})$* (from least squares fit)	$S_{se}(\sigma_{xm}, \tau_{xym})$* (from least squares fit)	Percent Error (as defined in Figure 9.1)
Group A/	1	0.0* (0.0)	20.2 (279)	36.6 (505)	20.9 (288)	-3.3
σ_{xm} = 17.25 (238)	2	25.0 (345)	16.65 (230)	36.6 (505)	20.9 (288)	4.9
τ_{xym} = 0.0 (0.0)	3	35.8 (494)	0.0 (0.0)	36.6 (505)	20.9 (288)	-2.2
Group B/	4	0.0 (0.0)	21.4-22.5** (295-310)	35.9 (495)	22.2 (306)	-.9
σ_{xm} = 0.0 (0.0)	5	24.85 (343)	16.55 (228)	35.9 (495)	22.2 (306)	1.7
τ_{xym} = 11.0 (152)	6	35.6 (491)	0.0 (0.0)	35.9 (495)	22.2 (306)	-.8

* Numbers not in parentheses are in tons per sq in. from reference 4.15. Numbers shown in parentheses are in megapascals.
** Average value was used.

Table 9.1 (continued)

Group/ σ_{xm}, τ_{xym}*	Entry Number	$\sigma_{xa}(\sigma_{xm}, \tau_{xym})$*	$\tau_{xya}(\sigma_{xm}, \tau_{xym})$*	S_e $\sigma_{xm}, \tau_{xym})$* (from least squares fit)	S_{se} $(\sigma_{xm}, \tau_{xym})$* (from least squares fit)	Percent Error (as defined in Figure 9.1)
Group C/	7	0.0* (0.0)	18.4-21 (254-290)	36.2 (499)	20.8 (287)	-5.3
σ_{xm} = 17.25 (238)	8	10.45 (144)	20.85 (288)	36.2 (499)	20.8 (287)	4.3
τ_{xym} = 11.0 (152)	9	24.25 (334)	16.15 (223)	36.2 (499)	20.8 (287)	2.5
	10	31.3-33.0** (432-455)	8.9-9.4** (123-130)	36.2 (499)	20.8 (287)	-1.2
	11	36.0 (496)	0.0 (0.0)	36.2 (499)	20.8 (287)	-.6

* Numbers not in parentheses are in tons per sq in. from reference 4.15. Numbers shown in parentheses are in megapascals.
** Average value was used.

Table 9.1 (Continued)

Group/ σ_{xm}, τ_{xym}*	Entry Number	$\sigma_{xa}(\sigma_{xm}, \tau_{xym})$*	$\tau_{xya}(\sigma_{xm}, \tau_{xym})$*	$S_e(\sigma_{xm}, \tau_{xym})$* (from least squares fit)	$S_{se}(\sigma_{xm}, \tau_{xym})$* (from least squares fit)	Percent Error (as defined in Figure 9.1)
Group D	12	0.0 (0.0)	18.5 (255-269)	30.6 (422)	18.1 (250)	5.0
σ_{xm} = 34.5 (476)	13	8.2 (113)	16.3 (225)	30.6 (422)	18.1 (250)	-6.0
τ_{xym} = 22.25 (307)	14	20.4 (281)	13.6 (188)	30.6 (422)	18.1 (250)	.45
	15	26-29.5 (359-407)	7.3-8.4 (101-116)	30.6 (422)	18.1 (250)	.67
	16	30.6 (422)	0.0 (0.0)	30.6 (422)	18.1 (250)	0.0

* Numbers not in parentheses are in tons per sq in. from reference 4.15. Numbers shown in parentheses are in megapascals.
** Average value was used.

shearing stress. If σ_{xa} and τ_{xya} have zero gradients, then by assumption,

$$\left[\frac{\overline{\sigma}_{xa}(\overline{\sigma}_{xm},\overline{\tau}_{xym})}{\overline{S}_{e}(\overline{\sigma}_{xm},\overline{\tau}_{xym})}\right]^{2} + \left[\frac{\overline{\tau}_{xya}(\overline{\sigma}_{xm},\overline{\tau}_{xym})}{\overline{S}_{se}(\overline{\sigma}_{xm},\overline{\tau}_{xym})}\right]^{2} = 1 \qquad (9.2)$$

Equation 9.2 can be made to take on the appearance of greater generality by a process similar to that by which Eq. 5.1 led to Eq. 5.7. Because of the presence of mean stresses, the process will be carried out in detail. Equation 9.2 may be written as,

$$\overline{\sigma}_{xa}^{\,2}(\overline{\sigma}_{xm},\overline{\tau}_{xym}) + \overline{C}(\overline{\sigma}_{xm},\overline{\tau}_{xym})\,\overline{\tau}_{xya}^{\,2}(\overline{\sigma}_{xm},\overline{\tau}_{xym})$$

$$= \overline{S}_{e}^{\,2}(\overline{\sigma}_{xm},\overline{\tau}_{xym}) \qquad (9.3)$$

where,

$$\overline{C}(\overline{\sigma}_{xm},\overline{\tau}_{xym}) = \left[\frac{\overline{S}_{e}(\overline{\sigma}_{xm},\overline{\tau}_{xym})}{\overline{S}_{se}(\overline{\sigma}_{xm},\overline{\tau}_{xym})}\right]^{2} \qquad (9.4)$$

As with Eq. 9.1, in Eq. 9.3 the alternating and mean stress components are viewed from the same element orientation. This orientation will be referred to as the unprimed coordinate system. In order to express the alternating principal stresses in terms of $\overline{\sigma}_{xa}(\overline{\sigma}_{xm},\overline{\tau}_{xym})$ and $\overline{\tau}_{xya}(\overline{\sigma}_{xm},\overline{\tau}_{xym})$, the notation must reflect a transformation of the mean stress components to the coordinate system of the alternating principal stresses. Let the coordinate system of the alternating principal stresses be denoted by a single prime. We write,

$$\bar{\sigma}_{1a}'(\bar{\sigma}_{xm}',\bar{\sigma}_{ym}',\bar{\tau}_{xym}') = \frac{\bar{\sigma}_{xa}(\bar{\sigma}_{xm},\bar{\tau}_{xym})}{2} + \left[\frac{\bar{\sigma}_{xm}^{2}(\bar{\sigma}_{xm},\bar{\tau}_{xym})}{2}\right.$$

$$\left. + \tau_{xya}^{2}(\bar{\sigma}_{xm},\bar{\tau}_{xym})\right]^{1/2} \qquad (9.5)$$

$$\bar{\sigma}_{2a}'(\bar{\sigma}_{xm}',\bar{\sigma}_{ym}',\bar{\tau}_{xym}') = \frac{\bar{\sigma}_{xa}(\bar{\sigma}_{xm},\bar{\sigma}_{xym})}{2} - \left[\frac{\bar{\sigma}_{xa}^{2}(\bar{\sigma}_{xm},\bar{\tau}_{xym})}{2}\right.$$

$$\left. + \tau_{xya}^{2}(\sigma_{xm},\tau_{xym})\right]^{1/2} \qquad (9.6)$$

From Eq. 9.5 and 9.6, $\bar{\sigma}_{xa}^{2}(\bar{\sigma}_{xm},\bar{\tau}_{xym})$ and $\bar{\tau}_{xya}^{2}(\bar{\sigma}_{xm},\bar{\tau}_{xym})$ may be obtained. That is,

$$\bar{\sigma}_{xa}^{2}(\bar{\sigma}_{xm},\bar{\tau}_{xym}) = (\bar{\sigma}_{1a}'(\bar{\sigma}_{xm}',\bar{\sigma}_{ym}',\bar{\tau}_{xym}')$$

$$+ \bar{\sigma}_{2a}'(\bar{\sigma}_{xm}',\bar{\sigma}_{ym}',\bar{\tau}_{xym}'))^{2} \qquad (9.7)$$

$$\bar{\tau}_{xya}^{2}(\bar{\sigma}_{xm},\bar{\tau}_{xym}) = -\bar{\sigma}_{1a}'(\bar{\sigma}_{xm}',\bar{\sigma}_{ym}',\bar{\tau}_{xym}')\bar{\sigma}_{2a}'(\bar{\sigma}_{xy}',\bar{\sigma}_{ym}',\bar{\tau}_{xym}') \qquad (9.8)$$

Substitution of Eqs. 9.7 and 9.8 into Eq. 9.3 gives,

$$\bar{\sigma}_{1a}'^{2}(\bar{\sigma}_{xm}',\bar{\sigma}_{ym}',\bar{\tau}_{xym}') + [2-\bar{C}(\bar{\sigma}_{xm},\bar{\tau}_{xym})]\,\bar{\sigma}_{1a}'(\bar{\sigma}_{xm}',\bar{\sigma}_{ym}',\bar{\tau}_{xym}')$$

$$\cdot\,\bar{\sigma}_{2a}'(\bar{\sigma}_{xm}',\bar{\sigma}_{ym}',\bar{\tau}_{xym}') + \bar{\sigma}_{2a}^{2}(\bar{\sigma}_{xm}',\bar{\sigma}_{ym}',\bar{\tau}_{xym}') = \bar{S}_{e}^{2}(\bar{\sigma}_{xm},\bar{\tau}_{xym}) \qquad (9.9)$$

Equation 9.9 is no more general than Eq. 9.2 provided that,

(a) $\bar{\sigma}_{1a}'(\bar{\sigma}_{xm}', \bar{\sigma}_{ym}', \bar{\tau}_{xym}')$ and $\bar{\sigma}_{2a}'(\bar{\sigma}_{xm}', \bar{\sigma}_{ym}', \bar{\tau}_{xym}')$ are 180° out-of-phase.

(b) $\bar{\sigma}_{xm}' \, \bar{\sigma}_{ym}' < \bar{\tau}_{xym}'^{2}$

(c) The coordinate transformation which transforms,

$$\begin{bmatrix} \bar{\sigma}_{xm}' & \bar{\tau}_{xym}' \\ \bar{\tau}_{xym}' & \bar{\sigma}_{ym}' \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} \bar{\sigma}_{xm} & \bar{\tau}_{xym} \\ \bar{\tau}_{xym} & \bar{0} \end{bmatrix}$$

also transforms,

$$\begin{bmatrix} \bar{\sigma}_{1a}' & \bar{0} \\ \bar{0} & \bar{\sigma}_{2a}' \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} \bar{\sigma}_{xa} & \bar{\tau}_{xya} \\ \bar{\tau}_{xya} & \bar{0} \end{bmatrix}$$

Constraints (a) and (b) above do not restrict the generality of Eq. 9.9 nearly as much as constraint (c). Without some means to relax this severe constraint, Eq. 9.9 has no advantage over Eq. 9.2.

We will now make an assumption which will permit more generality than Eq. 9.9 with the three provisions mentioned above. The assumption which will be made seems to be acceptable based on our limited past experience. However, it must be acknowledged that actual test data is needed. We will assume that by introducing a mean normal stress with zero gradient in the y direction in the unprimed system, that the form of Eq. 9.2 will be unchanged. Expressed equationally,

$$\left[\frac{\bar{\sigma}_{xa}(\bar{\sigma}_{xm}, \bar{\sigma}_{ym}, \bar{\tau}_{xym})}{\bar{S}_{e}(\bar{\sigma}_{xm}, \bar{\sigma}_{ym}, \bar{\tau}_{xym})}\right]^{2} + \left[\frac{\bar{\tau}_{xym}(\bar{\sigma}_{xm}, \bar{\sigma}_{ym}, \bar{\tau}_{xym})}{\bar{S}_{se}(\bar{\sigma}_{xm}, \bar{\sigma}_{ym}, \bar{\tau}_{xym})}\right]^{2} = 1 \qquad (9.10)$$

Repeating the process by which Eq. 9.2 is transformed to Eq. 9.9 we have,

$$\bar{\sigma}_{1a}'^{2}(\bar{\sigma}_{xm}',\bar{\sigma}_{ym}',\bar{\tau}_{xym}') + [(2-\bar{C}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})]\ \bar{\sigma}_{1a}'(\bar{\sigma}_{xm}',\bar{\sigma}_{ym}',\bar{\tau}_{xym}')$$

$$\cdot\ \bar{\sigma}_{2a}'(\bar{\sigma}_{xm}',\bar{\sigma}_{ym}',\bar{\tau}_{xym}') + \bar{\sigma}_{2a}'^{2}(\bar{\sigma}_{xm}',\bar{\sigma}_{ym}',\bar{\tau}_{xym}) = \bar{S}_{e}^{2}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) \tag{9.11}$$

where,

$$\bar{C}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) = \left[\frac{\bar{S}_{e}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})}{\bar{S}_{se}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})}\right]^{2} \tag{9.12}$$

As the result of having assumed Eq. 9.10 to be valid, Eq. 9.11 has only two constraints. They are,

(a) The alternating principal stresses are out-of-phase

(b) The coordinate transformation which transforms,

$$\begin{bmatrix} \bar{\sigma}_{xm}' & \bar{\tau}_{xym}' \\ \bar{\tau}_{xym}' & \bar{\sigma}_{ym} \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} \bar{\sigma}_{xm} & \bar{\tau}_{xym} \\ \bar{\tau}_{xym} & \bar{\sigma}_{ym} \end{bmatrix}$$

also transforms

$$\begin{bmatrix} \bar{\sigma}_{1a}' & \bar{0} \\ \bar{0} & \bar{\sigma}_{2a}' \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} \bar{\sigma}_{xa} & \bar{\tau}_{xya} \\ \bar{\tau}_{xya} & \bar{0} \end{bmatrix}$$

An admittedly speculative element has been introduced by assuming that the inclusion of $\bar{\sigma}_{ym}$ in Eq. 9.10 does not alter the equational form. However, the tests required to check the validity of the assumption would be difficult and costly. The test specimens would be thin-walled cylinders with internal pressure. There would be a mean axial load to achieve the desired ratio of $\bar{\sigma}_{xm}$ and $\bar{\sigma}_{ym}$ and a mean torque to generate $\bar{\tau}_{xym}$. To produce the $\bar{\sigma}_{xa}$ component, there would be alternating axial load and to obtain $\bar{\tau}_{xya}$ there would be an alternating torque. The cylinders would need to have a large diameter-to-wall-thickness ratio in order to approximate the zero gradients for $\bar{\tau}_{xym}$ and $\bar{\tau}_{xya}$. Although Eq. 9.11 is clearly an extrapolation, the extrapolation seems to be rather modest compared to others that are frequently encountered in the realm of metal fatigue.

Equation 9.12 involves stress components from both the primed system and the unprimed system. It is possible to develop another equation which contains stress components which are viewed from a single coordinate system. In order to begin, the alternating principal stresses will be expressed in terms of the alternating stress components which are viewed with respect to some other element orientation, which will be referred to as the double prime system. We write,

$$\bar{\sigma}_{1a}'(\bar{\sigma}_{xm}',\bar{\sigma}_{ym}',\bar{\tau}_{xym}') = \frac{\bar{\sigma}_{xa}''(\bar{\sigma}_{xm}'',\bar{\sigma}_{ym}'',\bar{\tau}_{xym}'') + \bar{\sigma}_{ya}''(\bar{\sigma}_{xm}'',\bar{\sigma}_{ym}'',\bar{\tau}_{xym}'')}{2}$$

$$+ \left\{\left[\frac{\bar{\sigma}_{xa}''(\bar{\sigma}_{xm}'',\bar{\sigma}_{ym}'',\bar{\tau}_{xym}'') - (\bar{\sigma}_{ya}''(\bar{\sigma}_{xm}'',\bar{\sigma}_{ym}'',\bar{\tau}_{xym}'')}{2}\right]^2\right.$$

$$\left. + \bar{\tau}_{xya}^{2}{}''(\bar{\sigma}_{xm}'',\bar{\sigma}_{ym}'',\bar{\tau}_{xym}'')\right\}^{1/2} \qquad (9.13)$$

$$\bar{\sigma}_{2a}'(\bar{\sigma}_{xm}',\bar{\sigma}_{ym}',\bar{\tau}_{xym}') = \frac{\bar{\sigma}_{xa}''(\bar{\sigma}_{xm}'',\bar{\sigma}_{ym}'',\bar{\tau}_{xym}'') + \bar{\sigma}_{ya}''(\bar{\sigma}_{xm}'',\bar{\sigma}_{ym}'',\bar{\tau}_{xym}'')}{2}$$

$$- \left\{\left[\frac{\bar{\sigma}_{xa}''(\bar{\sigma}_{xm}'',\bar{\sigma}_{ym}'',\bar{\tau}_{xym}'') - \bar{\sigma}_{ya}''(\bar{\sigma}_{xm}'',\bar{\sigma}_{ym}'',\bar{\tau}_{xym}'')}{2}\right]^2\right.$$

$$+ \quad \bar{\tau}_{xya}''(\bar{\sigma}_{xm}'', \bar{\sigma}_{ym}'', \bar{\tau}_{xym}'')\}^{1/2} \tag{9.14}$$

substituting Eqs. 9.13 and 9.14 into Eq. 9.11 we have,

$$\bar{\sigma}_{xa}''^{2}(\bar{\sigma}_{x,}'', \bar{\sigma}_{ym}'', \bar{\tau}_{xym}'') + \bar{\sigma}_{ya}''^{2}(\bar{\sigma}_{xm}'', \bar{\sigma}_{ym}'', \bar{\tau}_{xym}'') + [2-\bar{C}(\bar{\sigma}_{xm}, \bar{\sigma}_{ym}, \bar{\tau}_{xym})] \cdot$$

$$\bar{\sigma}_{xa}''(\bar{\sigma}_{x,}'', \bar{\sigma}_{ym}'', \bar{\tau}_{xym}'')\ \bar{\sigma}_{ya}''(\bar{\sigma}_{xm}'', \bar{\sigma}_{ym}'', \bar{\tau}_{xym}'') + \bar{C}(\bar{\sigma}_{xm}, \bar{\sigma}_{ym}, \bar{\tau}_{xym}) \cdot$$

$$\bar{\tau}_{xya}^{\ 2}(\bar{\sigma}_{xm}'', \bar{\sigma}_{ym}'', \bar{\tau}_{xym}'') = \bar{S}_{e}^{\ 2}(\bar{\sigma}_{xm}, \bar{\sigma}_{ym}, \bar{\tau}_{xym}) \tag{9.15}$$

Equation 9.15 is subject to the following conditions.

(a) The alternating principal stresses are out-of-phase.

(b) The coordinate transformation which transforms,

$$\begin{bmatrix} \bar{\sigma}_{xm}'' & \bar{\tau}_{xym}'' \\ \bar{\tau}_{xym}'' & \bar{\sigma}_{ym}'' \end{bmatrix} \text{ to } \begin{bmatrix} \bar{\sigma}_{xm} & \bar{\tau}_{xym} \\ \bar{\tau}_{xym} & \bar{\sigma}_{ym} \end{bmatrix}$$

also transforms,

$$\begin{bmatrix} \bar{\sigma}_{xa}'' & \bar{\tau}_{xya}'' \\ \bar{\tau}_{xya}'' & \bar{\sigma}_{ya}'' \end{bmatrix} \text{ to } \begin{bmatrix} \bar{\sigma}_{xa} & \bar{\tau}_{xya} \\ \bar{\tau}_{xya} & \bar{0} \end{bmatrix}$$

Let the double prime system of Eq. 9.15 be the system in which the mean stress components,

$$\begin{bmatrix} \bar{\sigma}_{xm}'' & \bar{\tau}_{xym}'' \\ \bar{\tau}_{xym}'' & \bar{\sigma}_{ym}'' \end{bmatrix} \text{ are equal to } \begin{bmatrix} \bar{\sigma}_{xm} & -\bar{\tau}_{xym} \\ -\bar{\tau}_{xym} & \bar{\sigma}_{ym} \end{bmatrix}$$

The Mohr's circle of mean stress components in Fig. 9.2 shows that such a double-primed system exists. For this case, Eq. 9.15 may be written as,

$$\bar{\sigma}_{xa}''^{2}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},-\bar{\tau}_{xym})'' + \bar{\sigma}_{ya}''^{2}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},-\bar{\tau}_{xym})'' + [2-\bar{C}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})] \cdot$$

$$\bar{\sigma}_{xa}''(\bar{\sigma}_{xm},\bar{\sigma}_{ym},-\bar{\tau}_{xym})'' \ \bar{\sigma}_{ya}''(\bar{\sigma}_{xm},\bar{\sigma}_{ym},-\bar{\tau}_{xym})'' + \bar{C}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) \cdot$$

$$\bar{\tau}_{xya}''^{2}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},-\bar{\tau}_{xym})'' = \bar{S}_{e}^{2}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) \tag{9.16}$$

where the mean stress components with the double primes outside the parentheses are viewed in the double primed system. But,

$$\bar{S}_{e}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) = \bar{S}_{e}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},-\bar{\tau}_{xym})'' \tag{9.17}$$

and

$$\bar{S}_{se}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) = \bar{S}_{se}(\bar{\sigma}_{xm},\bar{\sigma}_{ym}-\bar{\tau}_{xym})'' \tag{9.18}$$

From Eqs. 9.12, 9.17, and 9.18,

$$\bar{C}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) = \bar{C}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},-\bar{\tau}_{xym}) \tag{9.19}$$

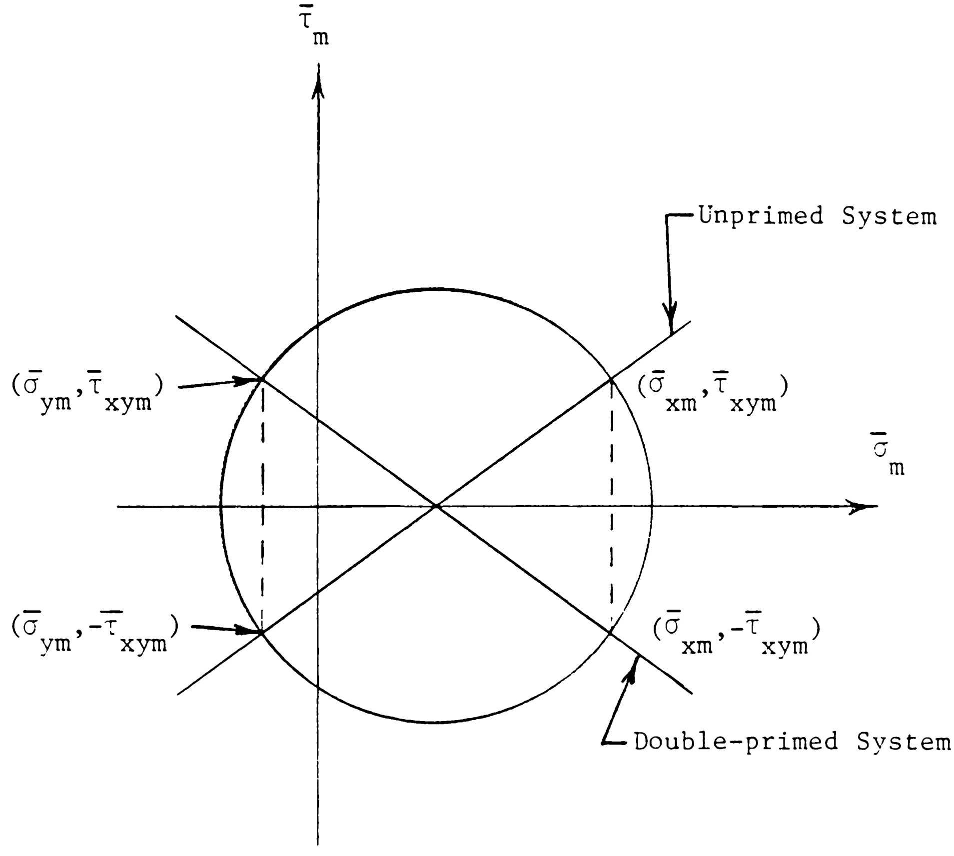


Fig. 9.2 Mohr's Circle of Mean Stress Components.

Since the sign for shear stress is known only convention, by substituting Eqs. 9.17 and 9.19 into Eq. 9.16 we have,

$$\bar{\sigma}_{xa}''^{2}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})'' + \bar{\sigma}_{ya}''^{2}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})'' + [2-\bar{C}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})''] \cdot$$

$$\bar{\sigma}_{xa}''(\bar{\sigma}_{x},,\bar{\sigma}_{ym},\bar{\tau}_{xym})''\ \bar{\sigma}_{ya}''(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})'' + \bar{C}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})'' \cdot$$

$$\bar{\tau}_{xya}''^{2}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})'' = \bar{S}_{e}^{2}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})'' \tag{9.20}$$

Since all stress components are viewed from the double prime system, there is now no need to distinguish between coordinate systems. Dropping the double prime notation we have,

$$\bar{\sigma}_{xa}^{2}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) + \bar{\sigma}_{ya}^{2}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) + [2-\bar{C}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})] \cdot$$

$$\bar{\sigma}_{xa}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})\ \bar{\sigma}_{ya}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) + \bar{C}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) \cdot$$

$$\bar{\tau}_{xya}^{2}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) = \bar{S}_{e}^{2}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) \tag{9.21}$$

Equation 9.21 is subject to the following constraints:

(a) $\bar{\sigma}_{xa}\ \bar{\sigma}_{ya} < \bar{\tau}_{xya}^{2}$

(b) The transformation which takes

$$\begin{bmatrix} \bar{\sigma}_{xa} & \bar{\tau}_{xya} \\ \bar{\tau}_{xya} & \bar{\sigma}_{ya} \end{bmatrix} \text{ to } \begin{bmatrix} \bar{\sigma}_{xa}' & \bar{\tau}_{xya}' \\ \bar{\tau}_{xya}' & \bar{0} \end{bmatrix}$$

also transforms,

$$\begin{bmatrix} \sigma_{xm} & \tau_{xym} \\ \tau_{xym} & \sigma_{ym} \end{bmatrix} \text{ to } \begin{bmatrix} \sigma_{xm} & -\tau_{xym} \\ -\tau_{xym} & \sigma_{ym} \end{bmatrix}$$

If $\bar{\tau}_{xya}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})$ in Eq. 9.21 is zero we have,

$$\bar{\sigma}_{1a}^{\,2}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) + [2-\bar{C}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})]\ \bar{\sigma}_{1a}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) \cdot$$

$$\bar{\sigma}_{2a}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) + \bar{\sigma}_{2a}^{\,2}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) = \bar{S}_e^{\,2}\ (\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) \tag{9.22}$$

Equation 9.22 is subject to constraints similar to those of Eq. 9.21, namely,

(a) $\bar{\sigma}_{1a}$ and $\bar{\sigma}_{2a}$ are 180° out-of-phase

(b) the transformation which takes

$$\begin{bmatrix} \sigma_{1a} & 0 \\ 0 & \sigma_{2a} \end{bmatrix} \text{ to } \begin{bmatrix} \bar{\sigma}_{xa}' & \bar{\tau}_{xya}' \\ \bar{\tau}_{xya}' & \bar{0} \end{bmatrix}$$

also transforms

$$\begin{bmatrix} \sigma_{xm} & \tau_{xym} \\ \tau_{xym} & \sigma_{ym} \end{bmatrix} \text{ to } \begin{bmatrix} \sigma_{xm} & -\tau_{xym} \\ -\tau_{xym} & \sigma_{ym} \end{bmatrix}$$

The (b) constraint for Eq. 9.22 implies that fatigue behavior is influenced by the relative orientation of the alternating principal stress axes and the mean principal stress axes. If it can be assumed that the relative orientation is unimportant, then the (b) constraint may be dropped. It is noted that the methods of Shigley [3.1], Juvinall [3.2], and Sines [3.3], whether unwittingly or otherwise, make this assumption. However, Juvinall [3.2] in particular acknowledges the possible importance of the relative orientation of the alternating principal stress axes and the mean principal stress axes and is therefore more tentative about recommending his method if the mean and alternating principal stress directions are not the same. Figure 9.3 will be used to illustrate why the aforementioned assumption is inherent in each of the three methods mentioned above. Figure 9.3a shows a stress element with both mean and alternating stress components. Also shown are the mean and alternating principal stress directions. Figure 9.3b has the same alternating stress components as Fig. 9.3a but the mean stress components are shown with a prime superscript to denote that they result from some arbitrary transformation of the mean stress components in Fig. 9.3a. In Fig. 9.3b, the mean principal stress directions have changed consistent with the transformation. The method by either Shigley, Juvinall, or Sines will predict the same fatigue behavior for the stress element in Fig. 9.3b as for the one in Fig. 9.3a. Since the angle α_2 in Fig. 9.3b is arbitrarily different from the angle α_1 in Fig. 9.3a, the methods by Shigley, Juvinall, and Sines do not give any importance to the angle. Unfortunately, the validity of this assumption cannot be checked by existing data. The values of mean and alternating stress components from the Gough and Clenshaw data [4.15] specify what the angle will be between the alternating and mean principal stress systems. There is no way to determine if rotation of the mean and

(a)

(b)

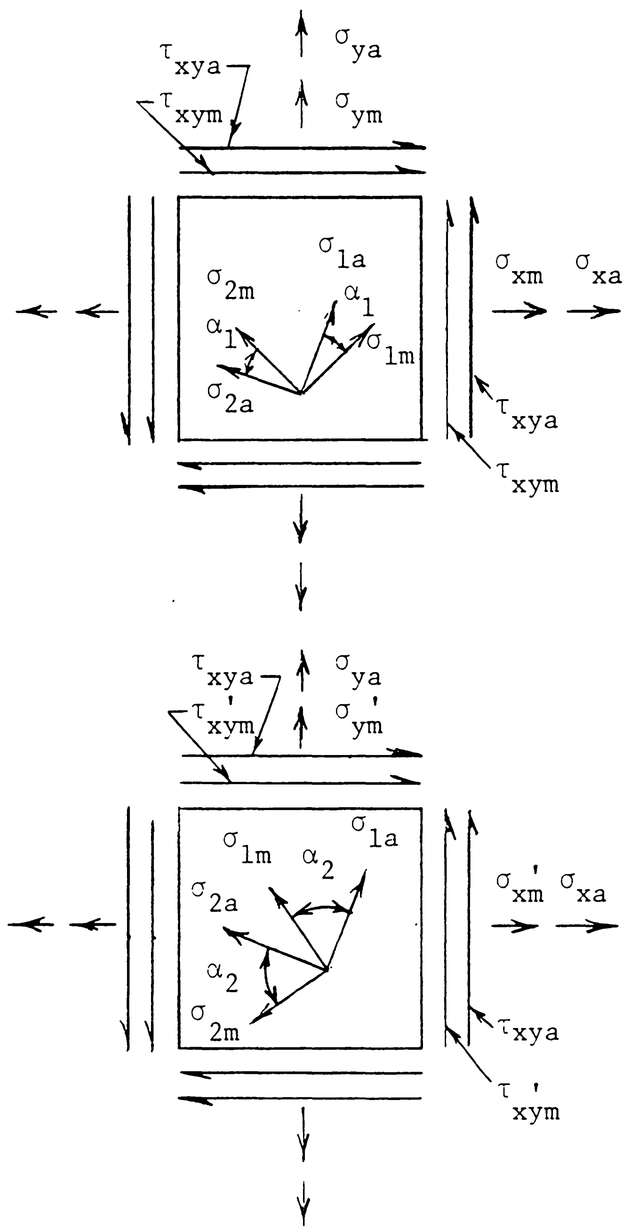


Fig. 9.3 Stress Elements Showing a Rotation of Mean Stress Components. (a) Mean and Alternating Components, (b) Mean Stress Components Transformed to Primed System.

alternating principal stress axes with respect to each other, while keeping the same mean and alternating stress states, will influence the fatigue behavior. This author is unaware of any data that will permit more general conclusions than the data of Gough and Clenshaw.

We are now faced with a dilemma. We may assume, as others have done, that the relative orientation of the mean and alternating principal stress axes is unimportant, or we may deny that there is any rational foundation for handling problems in fatigue limit analysis which are not covered by either Eq. 9.9 or Eq. 9.11 with the stated contraints. Actually, Eqs. 9.9 and 9.11 are not especially practical because values for $\overline{S}_e(\overline{\sigma}_{xm},\overline{\tau}_{xym})$ and $\overline{S}_e(\overline{\sigma}_{xm},\overline{\tau}_{xym})$, or worse, $\overline{S}_e(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$ and $\overline{S}_e(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$, must be determined by experiment.

In order to proceed, we will reluctantly assume that the relative orientation of mean and alternating principal stress axes has negligible importance, thus removing constraint (b). Although it would be extremely difficult to prove the general validity of this assumption, a test will be described below that could provide some evidence. Later on, as we develop simple expressions for equivalent mean and equivalent alternating stresses, we will find it necessary to make conservative assumptions which will help to offset any non-conservative error which may result from ignoring the importance in the relative orientation of the mean and alternating principal stress axes. Since Sawert's data tended to show that the out-of-phase σ_{1a},σ_{2a} requirement would be dropped from Eq. 6.12 which contained no mean stress, we will also assume that Eqs. 9.21 and 9.22 are applicable to in-phase as well as out-of-phase σ_{1a} and σ_{2a}, thus removing constraint (a) as well as (b).

By subjecting a thin-walled cylinder to alternating torsion with superimposed mean torsion, the mean and alternating shearing components, $\bar{\tau}_{xym}$ and $\bar{\tau}_{xya}$, respectively, are obtained. The angle between the mean and alternating principal stress axes will be zero. With $\bar{\tau}_{xym}$ set at fixed value, the fatigue limit value for $\bar{\tau}_{xya}$ may be obtained by testing a group of cylinders. If another set of cylinders which are identical to the others is subjected to an appropriate internal pressure and compressive axial force, it is possible to induce a hoop stress equal to $\bar{\tau}_{xym}$ and a longitudinal stress of $-\bar{\tau}_{xym}$, where $\bar{\tau}_{xym}$ has the same magnitude as in the earlier test. The alternating shear stress, $\bar{\tau}_{xya}$, would again be applied by alternating torque, but now the mean principle stress axes are rotated 45^{o} from the alternating principal stress axes. If both tests yield the same fatigue limit value for $\bar{\tau}_{xya}$, then there is some evidence that fatigue behavior is not influenced by the relative orientation of the mean and alternating principal stress axes. Though helpful, such evidence would still be inconclusive because it may be argued that the case is special.

Equation 9.22 has been assumed to be valid without constraints (a) and (b). From Eq. 9.22 we now write,

$$\sigma_{1a}^{2}(\sigma_{xm},\sigma_{ym},\tau_{xym}) + [2-\bar{C}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})]\ \sigma_{1a}(\sigma_{xm},\sigma_{ym},\tau_{xym}) \cdot$$

$$\sigma_{2a}(\sigma_{xm},\sigma_{ym},\tau_{xym}) + \sigma_{2a}^{2}(\sigma_{xm},\sigma_{ym},\tau_{xym}) \geq \bar{S}_{e}^{2}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) \tag{9.23}$$

It should be emphasized that expression 9.23 only purports to give a mathematical relationship between fatigue limit combinations of σ_{1a} and σ_{2a}, with the presence of mean stresses. The alert designer will recognize that the expression 9.23 must not be used as a design criterion because of the direction of the inequality. Presumably, the equality in expression 9.23 will be satisfied if all gradients associated with both mean and alternating components are zero. The inequality is expected to hold if any gradient is non-zero. The direction of the inequality is as shown since a non-zero gradient has a beneficial effect.

The methods presented earlier from Shigley, Juvinall, and Sines all assume that mean stress effect and alternating stress effect can be separated as two independent effects whose influence can be expressed as simple uniaxial components. However, expression 9.23 suggests a complicated interdependence of the mean and alternating components along with the values, $\overline{S}_e(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$ and $\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$. (A similar statement can be made with reference to Eq. 9.9, which is not as clouded with uncertainty as expression 9.23.) In other words, from a philosophical point of view, it may be impossible to correctly express the influence of mean and alternating components as two independent effects.

In order for Eq. 9.23 to be useful in a practical sense, it is necessary to provide conservative approximations for $\overline{S}_e(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$ and $\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$. We will now seek to provide a conservative evaluation of $\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$.

Unfortunately, no data exists for $\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$. However, Table 9.2 gives values for $C(\sigma_{xm},\tau_{xym})$ determined from Gough's data in Table 9.1. It is seen that values range from 2.47 to 3.52. Reasons will now be given for why $\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$ is not likely to be greater than 4 or less than 2.

To see why $\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$ is probably always less than 4, let the alternating components of Eq. 9.22 be transformed to the MASS plane. Denoting the components on the MASS plane with a single prime we have,

$$[4-\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})]\overline{\sigma}'^{2}_{a}(\text{MASS})(\overline{\sigma}_{xm}',\overline{\sigma}_{ym}',\overline{\tau}_{xym}') + \overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym}) \cdot$$

$$\overline{\tau}'^{2}_{a}(\text{MASS})(\overline{\sigma}_{xm}',\overline{\sigma}_{ym}',\overline{\tau}_{xym}') = \overline{S}_{e}^{2}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym}) \qquad (9.24)$$

Figure 9.4 shows the general shape of the curve representing Eq. 9.24 for $\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$ greater than 4, and $\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$ less than 4. It is seen from Fig. 9.4a that $\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$ greater than 4 implies that the allowable alternating shear stress component, $\overline{\tau}_{a}(\text{MASS})(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$, is increased by an increase in the alternating normal stress component, $\overline{\sigma}_{a}(\text{MASS})(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$. In other words, $\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{xy},\overline{\tau}_{xym})$ greater than 4 would imply that the presence of an alternating normal stress component on the MASS plane is beneficial, which seems unlikely. If no mean stress components are present, Fig. 8.1 shows the influence of the alternating normal stress component $\sigma_{a}(\text{MASS})$ on the allowable alternating shear stress component $\tau_{a}(\text{MASS})$. The $\sigma_{a}(\text{MASS})$ $-\tau_{a}(\text{MASS})$ curve is monotonically decreasing as shown in Fig. 8.1, assuming $\overline{C}$ to be less than 4. Unfortunately, the only values for $\overline{C}$ which are available are the two values derived from the Sawert data [4.18]. However, these values are very nearly equal to 3, as shown in Chapter 6. Although few data exist for $\overline{C}$, Table 4.1 gives C values for

Table 9.2 Values for C (σ_{xm}, τ_{xym}) Determined from Gough and Clenshaw's Data [4.15] in Table 9.1

Group	S_e (σ_{xm}, τ_{xym})*	S_{se} (σ_{xm}, τ_{xym})*	C (σ_{xm} τ_{xym})
A	36.6 (505)	20.9 (288)	3.07
B	35.9 (495)	22.2 (306)	2.62
C	36.2 (499)	20.8 (287)	3.03
D	30.6 (422)	18.1 (250)	2.86
σ_{xm} = 34.5** (476) τ_{xym} = 0.0** (0.0)	34.5** (476)	18.4** (254)	3.52
σ_{xy} = 0.0** (0.0) τ_{xym} = 22.25** (307)	35.0** (483)	22.25** (307)	2.47

* Numbers not in parentheses are in tons per sq in. from reference 4.15. Numbers in parentheses are in megapascals.

** Additional data from [4.15] not included in Table 9.1 because only S_e (σ_{xm}, τ_{xym}) and S_{se} (σ_{xm}, τ_{xym}) were obtained. Numbers not in parentheses are in tons per sq in. from reference 4.15. Numbers in parentheses are in megapascals.

Table 9.2 (continued)

Group	$S_e\ (\sigma_{xm},\tau_{xym})$*	$S_{se}\ (\sigma_{xm},\sigma_{xym})$*	$C\ (\sigma_{xm}\ \tau_{xym})$
σ_{xm} = 17.25** (238) τ_{xym} = 22.25** (307)	35-37*** (483-510)	20.00** (276)	3.24
σ_{xm} = 34.5** (476) τ_{xym} = 11.0** (152)	30.4** (419)	18.2** (251)	2.79
σ_{xm} = 0.0** (0.0) τ_{xym} = 0.0** (0.0)	37.8** (521)	24.0** (331)	2.48

* Numbers not in parentheses are in tons per sq in. from reference 4.15. Numbers in parentheses are in megapascals.

** Additional data from [4.15] not included in Table 9.1 because only $S_e(\sigma_{xm},\tau_{xym})$ and $S_{se}(\sigma_{xm},\sigma_{xym})$ were obtained. Numbers not in parentheses are in tons per sq in. from reference 4.15. Numbers in parentheses are in megapascals.

*** The average value was used.

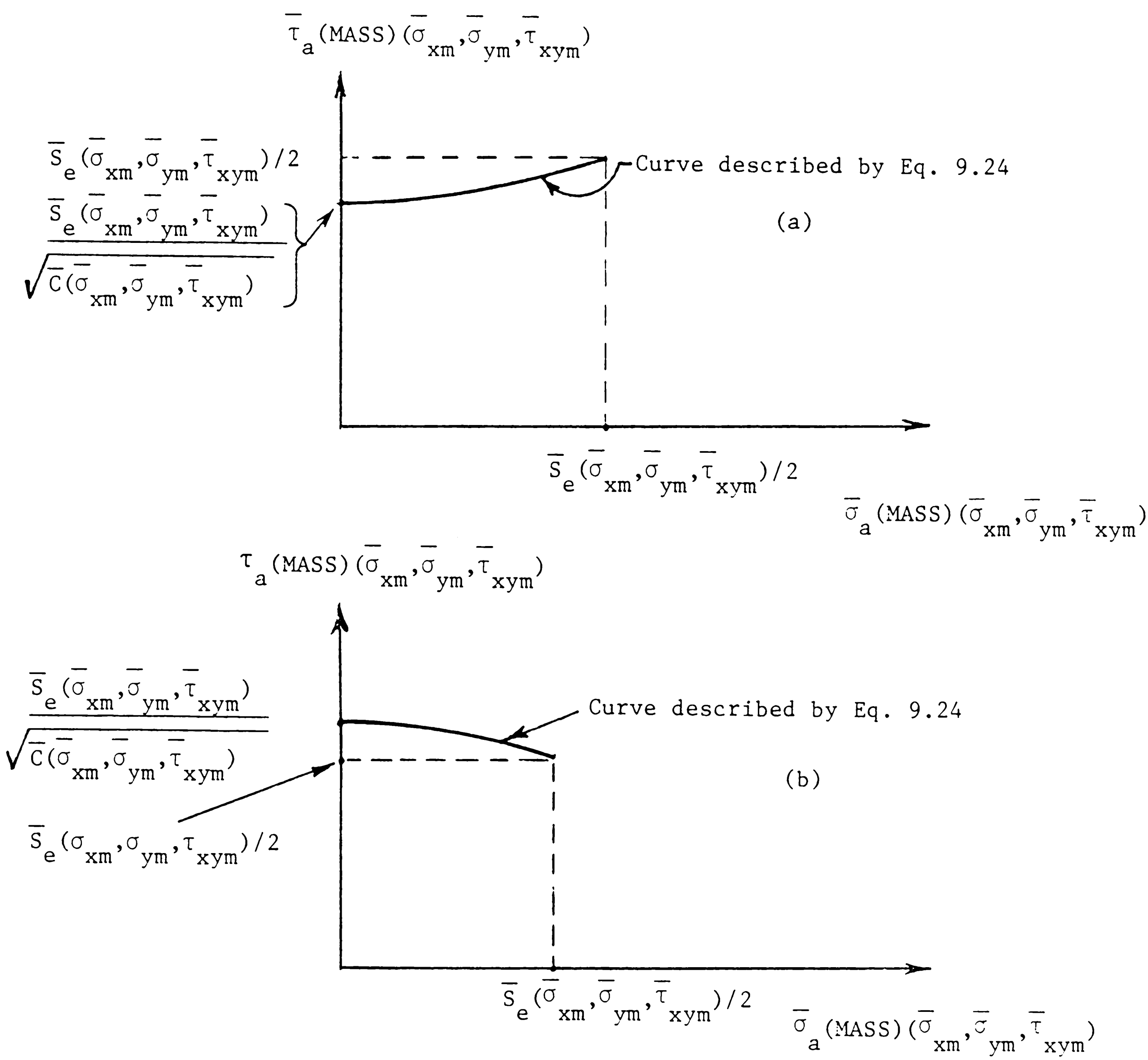


Fig. 9.4 Graphical Depiction of Eq. 9.24.

(a) $\bar{C}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})>4$ (b) $\bar{C}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})<4$

quite a number of steels, with the largest value being 3.12. Assuming that $\overline{C}$ values would not overwhelmingly depart from corresponding C values, it may be said that the monotonic decrease in the curve of Fig. 8.1 is supported. Therefore, with no mean stress components present, the σ_a(MASS) component seems to always reduce the allowable fatigue limit value for τ_a(MASS). We believe that the same will be true if mean stress components are present, since fatigue action results primarily from the alternating stress components. Therefore, $\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$ is probably never greater than 4.

To see why $\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym,})$ is probably never less than 2, let the following be considered. As $\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$ is reduced from 3, the major axis of the Eq. 9.22 ellipse increases in length, and the minor axis decreases in length. In the limit as $\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$ approaches 2, the major axis approaches infinity while the minor axis approaches $\overline{S}_e(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})/\sqrt{2}$. Since we do not expect allowable alternating principal stresses to be infinite, we tend to regard 2 as a lower limit for $\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\sigma}_{xym})$.

We now recommend for conservative approximations that $\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$ be either 2 or 4 depending on which value gives the smallest values in the fatigue limit combinations of $\sigma_{1a}(\sigma_{xm},\sigma_{ym},\tau_{xym})$ and $\sigma_{2a}(\sigma_{xm},\sigma_{ym},\tau_{xym})$ in Eq. 9.23. This means that $\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$ will be 2 if $\sigma_{1a}(\sigma_{xm},\sigma_{ym},\tau_{xym})$ and $\sigma_{2a}(\sigma_{xm},\sigma_{ym},\tau_{xym})$ are in-phase, and $\overline{C}(\sigma_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$ will be 4 if $\sigma_{1a}(\sigma_{xm},\sigma_{ym},\tau_{xym})$ and $\sigma_{2a}(\sigma_{xm},\sigma_{ym},\tau_{xym})$ are 180^o out-of-phase. Note that for $\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$ equal to 2, expression 9.23 becomes,

$$[\sigma_{1a}{}^2(\sigma_{xm},\sigma_{ym},\tau_{xym}) + \sigma_{2a}{}^2(\sigma_{x,},\sigma_{ym},\tau_{xym})]^{1/2} >$$

$$\overline{S}_e(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym}) \qquad (9.25)$$

with C = 4, expression 9.23 becomes

$$|\sigma_{1a}(\sigma_{xm},\sigma_{ym},\tau_{xym}) - \sigma_{2a}(\sigma_{xm},\sigma_{ym},\tau_{xym})| >$$

$$\overline{S}_e(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym}) \qquad (9.26)$$

As with expression 9.23, expressions 9.25 and 9.26 must not be mistakenly used as design criteria.

It has been pointed out that it may be impossible to correctly express the influence of mean and alternating components as two independent effects. However, due to the way in which $\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$ has been approximated, Eq. 9.23 gave rise to expressions 9.25 and 9.26 which permit a convenient separation of the mean and alternating effects. Thus as a practical consideration, we will now present an "equivalent" alternating stress formula and an "equivalent" mean stress formula. These expressions do not claim to represent a "correct" separation of mean and alternating stress effects, but merely conservative approximations which can be used in conjunction with a conventional σ_a-σ_m diagram.

From expressions 9.25 and 9.26 we may define an equivalent alternating stress $\sigma_a{}'$ as:

$$\sigma_a' = \sqrt{\sigma_{1a}^2 + \sigma_{2a}^2} \qquad \text{if } \sigma_{1a} \text{ and } \sigma_{2a} \text{ are in-phase}$$

$$= |\sigma_{1a} - \sigma_{2a}| \qquad \text{if } \sigma_{1a} \text{ and } \sigma_{2a} \text{ are } 180^\circ \text{out-of-phase} \tag{9.27}$$

An equivalent alternating stress value computed using Eq. 9.27 is expected to be conservative with respect to gradient effect only if it is used in conjunction with a σ_a-σ_m diagram based on tension compression fatigue data. We will now suggest a means to obtain an equivalent mean stress.

It has been assumed that the relative orientation of the mean and alternating principal stress axes is unimportant. Consistent with this assumption we may write,

$$\overline{S}_e(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym}) = \overline{S}_e(\overline{\sigma}_{1m},\overline{\sigma}_{2m},\overline{0}) \tag{9.28}$$

where $(\overline{\sigma}_{1m},\overline{\sigma}_{2m},\overline{0})$ is obtained by transforming $(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$. A true equivalent mean stress, by definition, would satisfy the following condition,

$$\overline{S}_e(\overline{\sigma}_m',\overline{0},\overline{0}) = \overline{S}_e(\overline{\sigma}_{1m},\overline{\sigma}_{2m},\overline{0}) \tag{9.29}$$

Figure 9.5a shows a stress element subjected to mean principal stresses $\overline{\sigma}_{1m}$ and $\overline{\sigma}_{2m}$ and superimposed $\overline{\sigma}_{xa}$. The fatigue limit value for $\overline{\sigma}_{xa}$ would, of course, be, $\overline{S}_e(\overline{\sigma}_{1m},\overline{\sigma}_{2m},\overline{0})$. In order to approximate $\overline{\sigma}_m'$, we will determine what single mean stress component, $\overline{\sigma}_m'$, as shown in Fig. 9.5b, will produce mean stress components on a conveniently

(a)

(b)

(c)

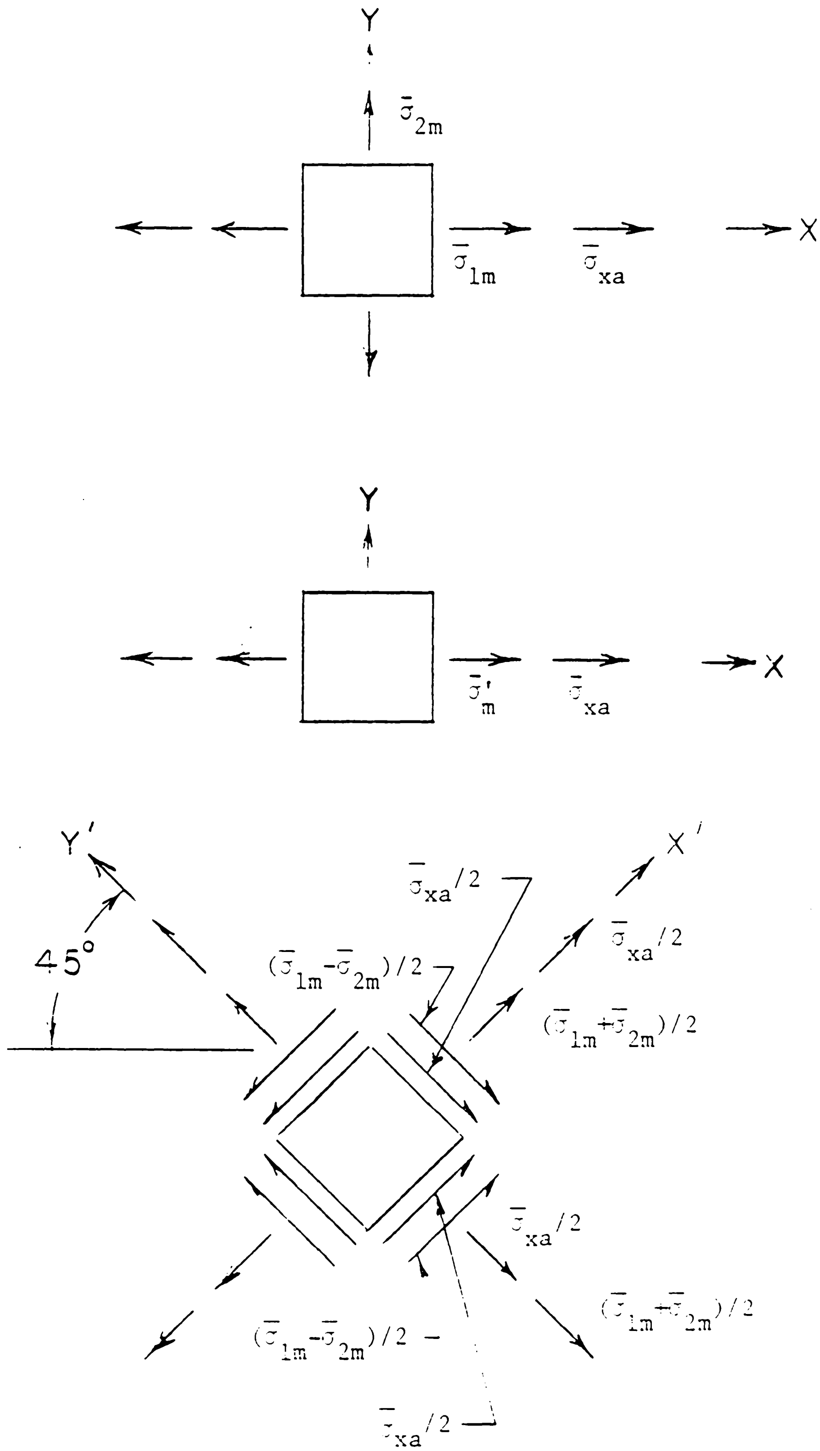


Fig. 9.5 Stress Elements Subjected to Mean Stress Components with Superimposed Alternating Component, $\bar{\sigma}_{xa}$. (a) Unprimed System, (b) Equivalent Mean Stress $\bar{\sigma}'_m$ in Unprimed System, (c) Primed System.

oriented stress element which are at least as severe as those produced by $\bar{\sigma}_{1m}$ and $\bar{\sigma}_{2m}$ on the same element. The most convenient element orientation for this purpose is the one for which the sides of the element are MASS planes. This may be seen by transforming $\bar{\sigma}_{1m}$, $\bar{\sigma}_{2m}$, and $\bar{\sigma}_{xa}$ in Fig. 9.5a to the element rotated 45^o as shown in Fig. 9.5c. Conveniently, on each side of the element in Fig. 9.5c the same mean and alternating stress components exist. Consistent with the usual convention, a normal stress is assumed to be positive if it is tensile. The mean stress $\bar{\sigma}'_m$ in Fig. 9.5b will produce mean stress components on the MASS plane which are at least as severe as those produced by $\bar{\sigma}_{1m}$ and $\bar{\sigma}_{2m}$ in Fig. 9.5c, if $\bar{\sigma}'_m$ is determined by the following rules:

i) If $(\bar{\sigma}_{1m}+\bar{\sigma}_{2m})$ is negative and $|\bar{\sigma}_{1m}+\bar{\sigma}_{2m}|<|\bar{\sigma}_{1m}-\bar{\sigma}_{2m}|$, then $\bar{\sigma}'_m = |\bar{\sigma}_{1m}-\bar{\sigma}_{2m}|$.

<u>Comment</u>: This may be ultra-conservative. However, in this case it is unknown how to improve the estimate for σ'_m and yet be confident of conservative error. It should be clearly observed that σ'_m is positive (tensile) even though $(\sigma_{1m}+\sigma_{2m})$ is negative. This value for $\bar{\sigma}'_m$ will produce shear components and normal components equal to $|\sigma_{1m}-\sigma_{2m}|/2$ on the MASS plane.

ii) If $(\bar{\sigma}_{1m}+\bar{\sigma}_{2m})$ is negative and $|\bar{\sigma}_{1m}+\bar{\sigma}_{2m}|>|\bar{\sigma}_{1m}-\bar{\sigma}_{2m}|$, then $\bar{\sigma}'_m = -|\bar{\sigma}_{1m}-\bar{\sigma}_{2m}|$.

Comment: It is being assumed that a compressive $\bar{\sigma}'_m$ is either beneficial or else has no effect at all on fatigue behavior. This tends to be true if yielding does not occur due to the superposition of both mean and alternating stresses.

iii) If $(\bar{\sigma}_{1m}+\bar{\sigma}_{2m})$ is positive, then $\bar{\sigma}'_m$ is either equal to $(\bar{\sigma}_{1m}+\bar{\sigma}_{2m})$ or $|\bar{\sigma}_{1m}-\bar{\sigma}_{2m}|$, whichever is larger.

An equivalent mean stress whose value is determined according to the above rules, will always give mean stress components in the prime system which are at least as severe as those which result from $\bar{\sigma}_{1m}$ and $\bar{\sigma}_{2m}$.

We may drop the bar notation in the expressions for equivalent mean stress with the understanding that σ'_m computed from mean stress components with non-zero gradients will be conservative with respect to gradient effect if the $\sigma_a - \sigma_m$ diagram employed is based on mean stress values which are induced by axial load. Dropping the bar notation the rules for equivalent mean stresses are,

i) If $(\sigma_{1m}+\sigma_{2m})$ is negative and $|\sigma_{1m}+\sigma_{2m}|<|\sigma_{1m}-\sigma_{2m}|$, then $\sigma'_m = |\sigma_{1m}-\sigma_{2m}|$.

ii) If $(\sigma_{1m}+\sigma_{2m})$ is negative and $|\sigma_{1m}+\sigma_{2m}| >|\sigma_{1m}-\sigma_{2m}|$, then $\sigma'_m = - |\sigma_{1m}-\sigma_{2m}|$.

iii) If $(\sigma_{1m}+\sigma_{2m})$ is positive, then σ'_m is either equal to $(\sigma_{1m}+\sigma_{2m})$ or $|\sigma_{1m}-\sigma_{2m}|$, whichever is larger.

The expressions for σ'_a and σ'_m given above may be used in a manner similar to Shigley's [3.1] approach for fatigue limit analysis. However, it must be born in mind that these expressions purport to be conservative with respect to gradient effects only if the σ_a-σ_m diagram with which they are compared is based on tension-compression loading. It will be the subject of future work to explain in detail how the overload concept of safety factor should be incorporated as well as how the σ_a-σ_m diagram should be made to reflect various design factors which depart from ideal test conditions. It will also be the subject of future work to determine how the tension-compression σ_a-σ_m diagram should be modified to account for beneficial stress gradient effect.

It is freely admitted that the equational development leading to the equivalent alternating and equivalent mean stress formulas in this chapter leave something to be desired. It was necessary to make a number of assumptions which were lacking adequate justification. We probably would not have continued the development beyond Eq. 9.22 if it were not for the following reasons:

- o The development is logical. Although some speculative assumptions are made along the way, the reader is informed of these assumptions, and some thought is given to the matter of checking the assumptions.
- o The equivalent alternating stress formula, Eq. 9.27, presented here is safer than the popular von Mises formula. We believe that the additional safety is warranted because, although there is not enough data to clearly confirm or disqualify the

von Mises formula with the presence of mean stresses, the von Mises formula tends to be disqualified by the $C(\sigma_{xm}, \tau_{xym})$ values in Table 9.2. The values range from 2.47 to 3.52. The von Mises formula would predict a value of 3.

- The development illuminates weaknesses in other methods and shows the need for further research. One very crucial matter which remains to be established is whether or not the relative orientation of the mean and alternating principal stress axes influences fatigue behavior.

Since values for $\overline{C}(\overline{\sigma}_{xm}, \overline{\sigma}_{ym}, \overline{\tau}_{xym})$ in Eq. 9.23 are unknown, conservative assumptions were made which gave rise to the equivalent alternating stress formula given by Eq. 9.27. Conservative error was also introduced by the treatment of mean stress effect as expressed by the three rules. It is unknown if any error, either conservative or non-conservative, was introduced by assuming that the relative orientation of the mean and alternating principal stress axes does not influence fatigue behavior. In order to get some idea of the amount of conservatism in the above method, Eq. 9.27 and the rules for equivalent mean stress will be applied to the Gough and Clenshaw data [4.15] and plotted on a σ_a-σ_m diagram.

Table 9.3 contains the pertinent Gough and Clenshaw data expressed in terms of principal stresses. The last two columns of Table 9.3 contain the equivalent mean stress and equivalent alternating stress values based on Eq. 9.27 and the rules for equivalent mean stress. Each

Table 9.3 Gough and Clenshaw Data [4.15] for Solid Specimens Expressed in Terms of Principal Stress Values with Equivalent Mean and Equivalent Alternating Stress Values Determined from Eq. 9.27 and the Rules for Equivalent Mean Stress

No.	Mean Principal Stress σ_{1m}*	Mean Principal Stress σ_{2m}*	Alternating Principal Stress σ_{1a}*	Alternating Principal Stress σ_{2a}*	Equivalent Mean Stress σ'_m*	Equivalent Alternating Stress σ'_a*
1	0.0* (0.0)	0.0* (0.0)	37.8* (521)	0.0* (0.0)	0.0* (0.0)	37.8* (521)
2	17.25 (238)	0.0 (0.0)	35.8 (494)	0.0 (0.0)	17.25 (238)	35.8 (494)
3	17.25 (238)	0.0 (0.0)	20.2 (279)	-20.2 (-279)	17.25 (238)	40.4 (557)
4	17.25 (238)	0.0 (0.0)	33.32 (459)	- 8.32 (115)	17.25 (238)	41.64 (574)
5	34.5 (476)	0.0 (0.0)	34.5 (476)	0.0 (0.0)	34.5 (476)	34.5 (476)
6	34.5 (476)	0.0 (0.0)	18.4 (254)	-18.4 (-254)	34.5 (476)	36.8 (507)
7	11.0 (152)	-11.0 (-152)	22.0 (303)	-22.0 (-303)	22.0 (303)	44.0 (607)
8	11.0 (152)	-11.0 (-152)	35.6 (491)	0.0 (0.0)	22.0 (303)	35.6 (491)

* Numbers not in parentheses are in tons per sq in. from reference 4.15. Numbers shown in parentheses are in megapascals.

Table 9.3 (continued)

No.	Mean Principal Stress σ_{1m}*	Mean Principal Stress σ_{2m}*	Alternating Principal Stress σ_{1a}*	Alternating Principal Stress σ_{2a}*	Equivalent Mean Stress σ'_m*	Equivalent Alternating Stress σ'_a*
9	11.0* (152)	-11.0* (-152)	33.12* (457)	-8.27* (-114)	22.0* (303)	41.39* (571)
10	22.25 (307)	-22.25 (-307)	22.25 (307)	-22.25 (-307)	44.5 (614)	44.5 (614)
11	22.25 (307)	-22.25 (-307)	35.0 (483)	0.0 (0.0)	44.5 (614)	35.0 (483)
12	22.6 (3.12)	-5.35 (-73.8)	36.0 (496)	0.0 (0.0)	28.0 (386)	36.0 (496)
13	22.6 (312)	-5.35 (-73.8)	19.7 (272)	-19.7 (-272)	28.0 (386)	39.4 (543)
14	22.6 (312)	-5.35 (-73.8)	34.5 (476)	-2.46 (-33.9)	28.0 (386)	37.0 (510)
15	22.6 (312)	-5.35 (-73.8)	32.3 (445)	-8.1 (-112)	28.0 (386)	40.4 (557)
16	22.6 (312)	-5.35 (-73.8)	26.7 (368)	-16.3 (-225)	28.0 (386)	43.0 (593)
17	32.5 (448)	-15.2 (-210)	36.0 (496)	0.0 (0.0)	47.7 (658)	36.0 (496)

* Numbers not in parentheses are in tons per sq in. from reference 4.15. Numbers shown in parentheses are in megapascals.

Table 9.3 (continued)

No.	Mean Principal Sress σ_{1m}*	Mean Principal Stress σ_{2m}*	Alternating Principal Stress σ_{1a}*	Alternating Principal Stress σ_{2a}*	Equivalent Mean Stress σ'_m*	Equivalent Alternating Stress σ'_a*
18	32.5* (448)	-15.2* (-210)	20.00* (276)	-20.00* (-276)	47.7* (658)	40.0* (552)
19	37.7 (520)	-3.21 (-44.3)	30.4 (419)	0.0 (0.0)	40.9 (564)	30.4 (419)
20	37.7 (520)	-3.21 (-44.3)	18.2 (251)	-18.2 (-251)	40.9 (564)	36.4 (502)
21	45.4 (626)	-10.9 (-150)	30.6 (422)	0.0 (0.0)	56.3 (776)	30.6 (422)
22	45.4 (626)	-10.9 (-150)	19.0 (262)	-19.0 (-262)	56.3 (776)	38.0 (524)
23	45.4 (626)	-10.9 (-150)	30.0 (414)	-2.06 (-28.4)	56.3 (776)	32.1 (443)
24	45.4 (626)	-10.9 (-150)	27.2 (375)	-6.8 (93.8)	56.3 (776)	34.0 (496)
25	45.4 (626)	-10.9 (-150)	20.9 (288)	-12.7 (-175)	56.3 (776)	33.6 (463)

* Numbers not in parentheses are in tons per sq in. from reference 4.15. Numbers shown in parentheses are in megapascals.

pair of equivalent mean and equivalent alternating stresses are plotted with respect to the σ_a-σ_m axes in Fig. 9.6. The fatigue limit curve which is also shown in Fig. 9.6 is explained below.

Each pair of equivalent stress values from Table 9.3 would be expected to be conservative with respect to a σ_a-σ_m diagram fatigue limit line only if the line was determined from tension-compresion data. Tension-compression data has been obtained for a variety of materials, many of which may be found in references 9.1 and 4.1. Unfortunately, no axial tension-compression data were obtained by Gough and Clenshaw for the particular material used in their tests. However, they did obtain three data points from which a fatigue limit curve based on bending stresses can be obtained. Entries 1, 2, and 5, in Table 9.3 are these three data points. The equation presented by Kececioglu [9.2] may be used to form the fatigue limit line on the σ_a-σ_m diagram. This equation is expressed as,

$$\left(\frac{\sigma_a}{S_e}\right)^a + \left(\frac{\sigma_m}{S_u}\right)^2 = 1 \qquad (9.30)$$

The value of "a" is determined by transforming Eq. 9.30 into the equation the equation of the straight line

$$y = \frac{1}{a}\,x \qquad (9.31)$$

where,

$$y = \log_e\,(\sigma_a/S_e) \qquad (9.32)$$

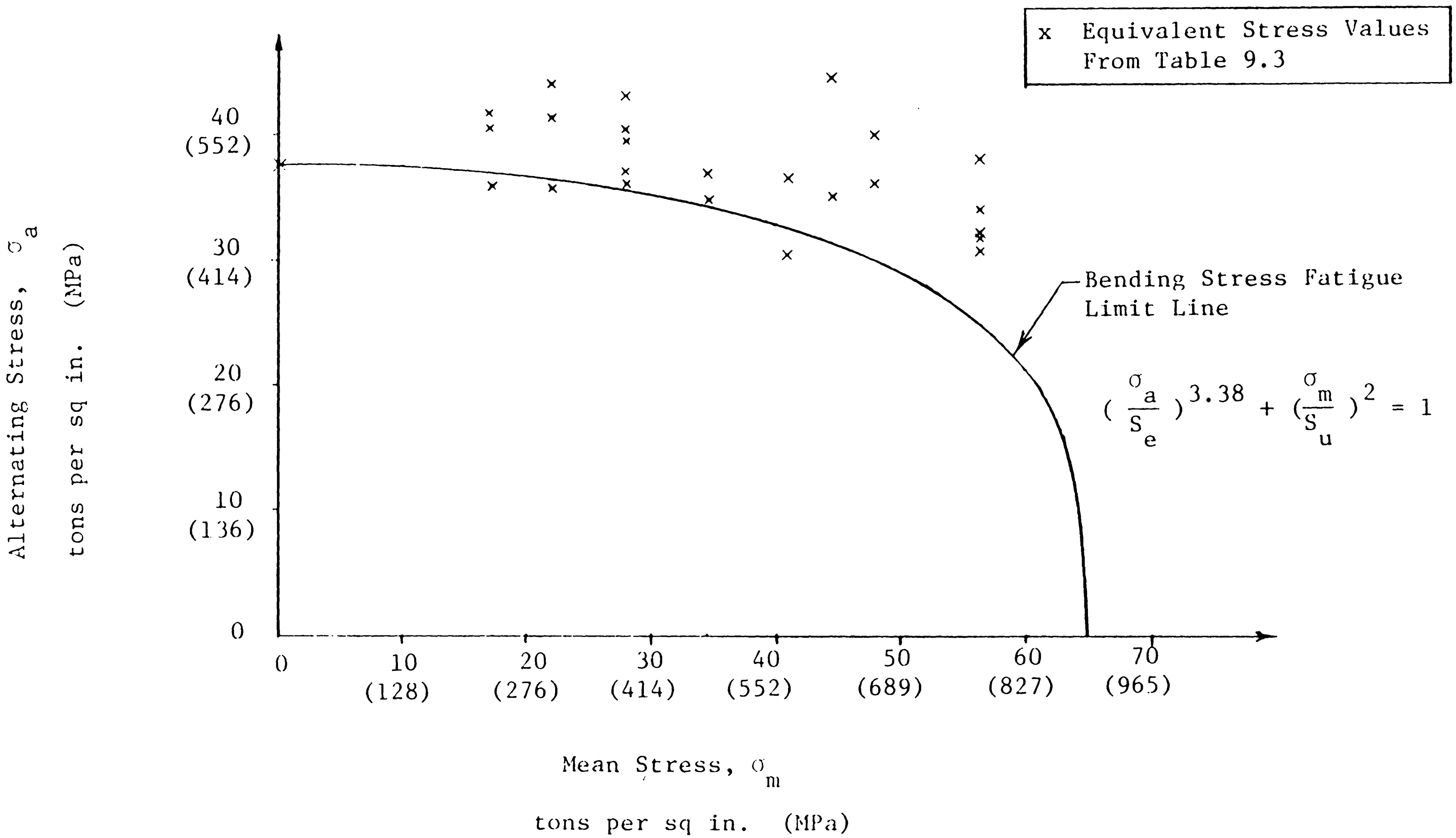


Fig. 9.6 σ_a-σ_m Diagram with Bending Stress Fatigue Limit Line Showing Comparison with Equivalent Stress Values from Table 9.3.

and

$$x = \log_e \left[1 - (\sigma_m/S_u)^2\right] \tag{9.33}$$

The ultimate strength, S_u, from reference 4.15 was 64.8 tons per sq in. (894 MPa). By applying the least-squares method to the three data points mentioned above, the value of "a" is found to be equal to 3.38. The curve shown in Fig. 9.6 represents Eq. 9.30 with "a" equal to 3.38.

In Fig. 9.6, the bending fatigue limit line is seen to be conservative with respect to most of Gough and Clenshaw data to which the rules for equivalent mean stress and the equivalent alternating stress formula have been applied. Although the curve is conservative with respect to most of the data points, it is probably not unduly conservative. A fatigue limit line based on tension-compression data would no doubt be conservative with respect to all of the data points since a curve based on tension-compression data is known to be somewhat below the curve based on bending stresses, for the same material. It should be carefully noted that conservatism is indicated in Fig. 9.6 by the points being above the fatigue limit line. This is because the equivalent mean and equivalent alternating stress values were computed from test data. If allowable values for σ'_a and σ'_m were being selected for design purposes, these values would have to come from below the fatigue limit line in order to provide a conservative design. Moreover, there is no guarantee that such design values will be conservative with respect to gradient effect unless they fall below the fatigue limit line based on tension-compression data as well as the line based on bending data.

It should be pointed out that the comparison which has been discussed above is limited due to the fact that, except where zero values are involved, each pair of mean principal stresses and each pair of alternating principal stresses in Table 9.3 have opposite sign.

REFERENCES

[9.1] Department of Defense, United States of America, Military Standardization Handbook, Metallic Materials and Elements for Aerospace Vehicle Structures, MIL-HDBK-5B, August 15, 1974.

[9.2] Kececioglu, D. B., and L. B. Chester, "Fatigue Reliability Uncer Combined Mean and Alternating Axial Stresses for AISI 1018 and 1038 Steels," ASME Paper No. 75-DET-128.

10. SUMMARY AND CONCLUSIONS

In Chapter 4, Eq. 4.1 by Gough and Pollard [4.15] was used to fit the reversed bending-reversed torsion fatigue limit data from three different investigators. Equation 4.1 is reproduced below

$$\left(\frac{\sigma_{xa}}{S_e}\right)^2 + \left(\frac{\tau_{xya}}{S_{se}}\right)^2 = 1 \tag{4.1}$$

There was a total of 81 data points, all from specimens of either carbon or carbon alloy steel. All specimens represented by the data were without stress raisers, and none of the specimens had been subjected to case hardening heat treatments such as nitriding. It was seen that Eq. 4.1 provided a remarkably good fit to the data.

In Chapter 5, Eq. 5.7 was developed from Eq. 4.1. Equation 5.7, which showed promise of providing a more general fatigue limit criterion than Eq. 4.1, is reproduced below.

$$\sigma_{1a}{}^2 + (2 - C)\ \sigma_{1a}\sigma_{2a} + \sigma_{2a}{}^2 = S_e^2 \tag{5.7}$$

When the Sawert data [4.18] was compared to Eq. 5.7 it appeared that the stress gradients had a predictable influence on fatigue behavior. Therefore, the influence of alternating stress gradients was the subject of Chapter 6. It was assumed in Chapter 5 that no mean stresses were present. It was also assumed that the alternating stresses were proportionally applied. The assumption of no mean stress was continued until Chapter 9. However, the assumption of proportionally applied alternating stress components was continued throughout the entire work.

It was shown in Chapter 6, that there is some support for the validity of Eq. 6.7 given below.

$$\left(\frac{\overline{\sigma}_{xa}}{\overline{S}_e}\right)^2 + \left(\frac{\overline{\tau}_{xya}}{\overline{S}_{se}}\right)^2 = 1 \tag{6.7}$$

The bar notation was used to indicate zero gradients, as explained in Chapter 6. It was suggested that Eq. 6.7 might be more universally correct than Eq. 4.1, since all gradient effect is eliminated from the data for which Eq. 6.7 is applicable.

Equation 6.7 was used to obtain Eq. 6.14, which is,

$$\overline{\sigma}_{1a}^{\ 2} + (2 - \overline{C})\ \overline{\sigma}_{1a}\overline{\sigma}_{2a} + \overline{\sigma}_{2a}^{\ 2} = \overline{S}_e^{\ 2} \tag{6.14}$$

where $\overline{C}$ was defined by

$$\overline{C} = \left(\frac{\overline{S}_e}{\overline{S}_{se}}\right)^2 \tag{6.15}$$

By removing the bar notation from $\overline{\sigma}_{1a}$ and $\overline{\sigma}_{2a}$ in Eq. 6.14, expression 6.16 was obtained.

$$[\sigma_{1a}^{\ 2} + (2-\overline{C})\sigma_{1a}\sigma_{2a} + \sigma_{2a}]^{1/2} \geqslant \overline{S}_e \tag{6.16}$$

Although it would be desirable to have more data, it was seen that the Sawert data [4.18] fully satisfied expression 6.16. If σ_{1a} and σ_{2a} had zero gradients, then the equality in expression 6.16 was satisfied. If σ_{1a} and σ_{2a} did not have zero gradients, the inequality was satisfied in the direction indicated.

From expression 6.16, an equivalent alternating stress formula was defined by Eq. 6.18.

$$\sigma_a' = \sqrt{\sigma_{1a}^{\;2} + (2-\overline{C})\sigma_{1a}\sigma_{2a} + \sigma_{2a}^{\;2}} \tag{6.18}$$

Equation 6.18 is recommended only if no mean stress components are present. A test was described in Chapter 6 for obtaining $\overline{S}_{se}$ which is needed in the computation of $\overline{C}$.

It was observed in Chapter 7 that Eq. 6.18 is equivalent to the von Mises formula if $\overline{C}$ is equal to 3. An example was cited from Findley, et al. [7.1], which shows that the distortional energy criterion is not valid if the alternating principal stress axes are not fixed.

In Chapter 8 it was shown that the alternating normal stress component, σ_a(MASS), on the plane of maximum alternating shearing stress has a definite influnce on fatigue behavior, as does the alternating shearing stress component, τ_a(MASS). It was shown that the σ_a(MASS) component may be safely ignored only if compensation is made by properly choosing the critical value for τ_a(MASS). It was recommended for out-of-phase alternating principal stresses that the Tresca criterion have the form, $\tau_a(\text{MASS}) \leq \overline{S}_e/2$. It was explained why this form of the Tresca criterion cannot be used with as much confidence if the alternating principal stresses are in-phase. Data was presented which shows that the Tresca criterion lacks validity if the alternating stress components are non-proportional.

It was seen in Chapter 9 that Eq. 9.1 by Gough and Clenshaw [4.15] provided a good fit to the data involving reversed bending-reversed torsion with superimposed mean bending and torsion. Since Sawert's data supported the validity of Eq. 6.7, in which the gradients were zero but the form of Eq. 4.1 was retained, it was assumed that the form of Eq. 9.1 would be unchanged if the gradients for both mean and alternating components are zero. The result was Eq. 9.2.

$$\left[\frac{\bar{\sigma}_{xa}(\bar{\sigma}_{xm},\bar{\tau}_{xym})}{\bar{S}_{e}(\bar{\sigma}_{xm},\bar{\tau}_{xym})}\right]^2 + \left[\frac{\bar{\tau}_{xya}(\bar{\sigma}_{xm},\bar{\tau}_{xym})}{\bar{S}_{se}(\bar{\sigma}_{xm},\bar{\tau}_{xym})}\right]^2 = 1 \tag{9.2}$$

It was seen that Eq. 9.2 could not be extended to form a more general fatigue limit criterion without introducing some speculative assumptions. It was first assumed that by introducing a mean stress component with zero gradient in the y direction that the form of Eq. 9.2 would be unchanged. By this assumption, Eq. 9.11 was obtained.

$$\left[\frac{\bar{\sigma}_{xa}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})}{\bar{S}_{e}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})}\right]^2 + \left[\frac{\bar{\tau}_{xym}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})}{\bar{S}_{se}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})}\right]^2 = 1 \tag{9.11}$$

From Eq. 9.11 it was possible to obtain Eq. 9.22.

$$\bar{\sigma}_{1a}^{\,2}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) + \left[2-\bar{C}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})\right]$$

$$\cdot\ \bar{\sigma}_{1a}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) \cdot \bar{\sigma}_{2a}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym})$$

$$+\ \bar{\sigma}_{2a}^{\,2}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) = \bar{S}_{se}^{\,2}(\bar{\sigma}_{xm},\bar{\sigma}_{ym},\bar{\tau}_{xym}) \tag{9.22}$$

Equation 9.22 was subject to two constraints. In order to remove the constraints it was necessary to make two assumptions. It was assumed that the relative orientation of the mean and alternating principal stress axes has no influence on fatigue behavior. It was pointed out in Chapter 9 that although this assumption has been made by others, it remains to be seen how valid the assumption actually is. A test was described in Chapter 9 which could help to establish the validity (or non-validity) of the assumption. It was also assumed that Eq. 9.22 would be valid for in-phase as well as out-of-phase alternating principal stresses. This assumption does not seem to be unreasonable since, earlier, with no mean stresses present, the removal of the out-of-phase constraint from expression 6.16 was supported by the Sawert data.

In order to obtain a practically useful fatigue limit criterion from Eq. 9.22, it was argued that $\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$ is probably never greater than 4 nor less than 2. It was then possible to obtain an equivalent alternating stress formula in the form of Eq. 9.27.

$$\sigma_a' = \begin{cases} \sqrt{\sigma_{1a}^2 + \sigma_{2a}^2} & \text{if } \sigma_{1a} \text{ and } \sigma_{2a} \text{ are in phase} \\ |\sigma_{1a} - \sigma_{2a}| & \text{if } \sigma_{1a} \text{ and } \sigma_{2a} \text{ are } 180^\circ \text{ out-of-phase} \end{cases} \qquad (9.27)$$

Equation 9.27 is recommended when mean stress components are present, whereas Eq. 6.18 is recommended if no mean stress components exist. If $\overline{C}$ is unknown for use in Eq. 6.18, Eq. 9.27 may be used in the place of Eq. 6.18, although in some cases the results may be excessively conservative. Future research may show that $\overline{C}$ values for a variety of steels are consistently close to 3, thus giving considerable justification for the von Mises formula when no mean stresses are present. The equivalent alternating stress formula given by Eq. 9.27 is safer than the popular von Mises formula. By this we mean that Eq. 9.27 always computes a larger value for σ_a' than does the von Mises formula. Equation 9.27 may be too safe in some cases, but with the presence of mean stress components there seems to be no way to improve the value for σ_a' without performing tests to obtain $\overline{C}(\overline{\sigma}_{xm},\overline{\sigma}_{ym},\overline{\tau}_{xym})$.

A method for obtaining an "equivalent" mean stress was given according to the three rules given in Chapter 9.

The tentative nature of the equivalent alternating and equivalent mean stress formulas developed in Chapter 9 was emphasized. It was necessary to make several assumptions whose justification may be questionable. Reasons were given for continuing the development with the presence of such uncertainties.

11. RECOMMENDATIONS FOR FUTURE WORK

It is recommended that more data be taken similar to that of Sawert [4.18] to determine whether or not the validity of Eq. 6.7 is further established. It was suggested in Chapter 6 that Eq. 6.7 might be more universally correct than Eq. 4.1, since all gradient effect is removed from Eq. 6.7. This speculation should be checked by obtaining data for Eq. 6.7 using a material which fails to satisfy Eq. 4.1.

As suggested in Chapter 7, C and $\bar{C}$ values should be obtained for a number of steels. If $\bar{C}$ values are consistently close to 3 with C values showing a much wider deviation, then assuming zero mean stress, support will be given to the distortional energy criterion with discrepancies in C values attributable largely to gradient effects.

In Chapter 9 a test was described which would help to determine whether or not the relative orientation of the mean and alternating principal stress axes influences fatigue behavior. It is recommended that a test of this nature be undertaken. The results would be of great value to the understanding of biaxial fatigue where both mean and alternating components are involved.

It is possible to incorporate the overload concept of safety factor when employing the new formulas for "equivalent" alternating stress and "equivalent" mean stress. This is recommended as an object of future work.

It has been assumed that the new formulas for equivalent mean and equivalent alternating stresses would be used in conjunction with a σ_a-σ_m diagram based on tension-compression data (as opposed to reversed bending with superimposed mean bending). It will be a subject of future work to determine how the $\sigma_a - \sigma_m$ diagram should be modified to take advantage of beneficial gradient effects. It should also be determined how the $\sigma_a - \sigma_m$ diagram should be modified to reflect conditions which depart from the ideal test conditions under which the $\sigma_a - \sigma_m$ data were obtained.

APPENDIX A

A Demonstration that Proportional Reversed Bending and Torsional Stresses, When Combined, Give Alternating Principal Stresses Which are 180 Degrees Out-of-Phase.

The reversed stress components for the biaxial stress state may be expressed as,

$$\sigma_{xa}(t) = \sigma_{xa}\sin\omega t \tag{A.1}$$

$$\sigma_{ya}(t) = \sigma_{ya}\sin\omega t \tag{A.2}$$

$$\tau_{xya}(t) = \tau_{xya}\sin\omega t \tag{A.3}$$

where $\sigma_{xa}(t)$, $\sigma_{ya}(t)$ and $\tau_{xya}(t)$ are the stress components expressed as functions of time. Stress components whose time dependence is described by Eqs. A.1, A.2, and A.3 are called "proportional" because any ratio of component amplitudes is fixed throughout the stress cycle. Clearly, the only phase differences which can exist among $\sigma_{xa}(t)$, $\sigma_{ya}(t)$, and $\tau_{xya}(t)$ are 0^o and 180^o.

The alternating principal stresses expressed as functions of time are,

$$\sigma_{1a}(t) = \left[\frac{\sigma_{xa} + \sigma_{ya}}{2} + \sqrt{\left(\frac{\sigma_{xa} - \sigma_{ya}}{2}\right)^2 + \tau_{xya}{}^2}\right] \sin\omega t \tag{A.4}$$

$$\sigma_{2a}(t) = \left[\frac{\sigma_{xa} + \sigma_{ya}}{2} - \sqrt{\left(\frac{\sigma_{xa} - \sigma_{ya}}{2}\right)^2 + \tau_{xya}^{\ 2}} \right] \sin\omega t \qquad (A.5)$$

or more simply,

$$\sigma_{1a}(t) = \sigma_{1a} \sin\omega t \qquad (A.6)$$

$$\sigma_{2a}(t) = \sigma_{2a} \sin\omega t \qquad (A.7)$$

At some time, t_1, $\sigma_{1a}(t_1)$ will be positive. The principal stresses $\sigma_{1a}(t)$ and $\sigma_{2a}(t)$ will be in-phase if and only if $\sigma_{2a}(t_1)$ is also positive. If σ_{1a} is positive then $\sin\omega t_1$ is positive. For $\sigma_{2a}(t_1)$ to be positive we must have,

$$\frac{\sigma_{xa} + \sigma_{ya}}{2} > \sqrt{\left(\frac{\sigma_{xa} - \sigma_{ya}}{2}\right)^2 + \tau_{xya}^{\ 2}} \qquad (A.8)$$

Squaring both sides and simplifying gives,

$$\sigma_{xa}\ \sigma_{ya} > \tau_{xya}^{\ 2} \qquad (A.9)$$

If σ_{1a} is negative, then at some time t_1, $\sigma_{1a}(t_1)$ will be positive only if $\sin\omega t_1$ is negative. If σ_{1a} is negative, σ_{2a} will of necessity be negative also, thus making $\sigma_{2a}(t_1)$ positive. In order for σ_{1a} to be negative we again have,

$$\sigma_{xa}\sigma_{ya} > \tau_{xya}^{2} \qquad (A.9)$$

Thus, it has been shown that for $\sigma_{1a}(t)$ and $\sigma_{2a}(t)$ to be in-phase that expression A.9 must be satisfied. For the combined reversed bending and torsional stresses σ_{ya} is zero and expression A.9 is not satisfied. Therefore $\sigma_{1a}(t_1)$ and $\sigma_{2a}(t_1)$ have opposite signs, which is equivalent to saying that they are 180^{o} out-of-phase.

In general, for $\sigma_{1a}(t)$ and $\sigma_{2a}(t)$ to be 180^{o} out-of-phase the following requirement must be satisfied:

$$\sigma_{xa}\sigma_{ya} < \tau_{xya}^{2} \qquad (A.10)$$

APPENDIX B

Proof that the Alternating Principal Stress Axes Have Fixed Orientation When Stress Components are Proportional

In Appendix A proportional stresses are defined by Eqs. A.1, A.2, and A.3 A principal stress axis is obtained by rotating the stress element in the proper direction depending on the sign convention for shear stress through either of the following angles,

$$\theta_1 = \frac{1}{2} \tan^{-1} \left[\frac{2 \tau_{xya}(t)}{\sigma_{xa}(t) + \sigma_{ya}(t)}\right] \tag{B.1}$$

$$\theta_2 = \frac{1}{2} \tan^{-1} \left[\frac{2 \tau_{xya}(t)}{\sigma_{xa}(t) + \sigma_{ya}(t)}\right] + \frac{\pi}{2} \tag{B.2}$$

Substituting Eqs. A.1, A.2, and A.3 into Eqs. B.1 and B.2 gives,

$$\theta_1 = \frac{1}{2} \tan^{-1} \left[\frac{2 \tau_{xya}}{\sigma_{xa} + \sigma_{ya}}\right] \tag{B.3}$$

$$\theta_2 = \frac{1}{2} \tan^{-1} \left[\frac{2 \tau_{xya}}{\sigma_{xa} + \sigma_{ya}}\right] + \frac{\pi}{2} \tag{B.4}$$

Since Eqs. B.3 and B.4 are not functions of time, the principal stress axes have fixed orientation.

The vita has been removed from
the scanned document

FATIGUE LIMIT ANALYSIS INVOLVING BIAXIAL
STRESS COMPONENTS

by

Edgar Gray Munday

(ABSTRACT)

Biaxial stress fatigue data is carefully examined in order to determine how much foundation exists for a rational approach to classical stress-based fatigue limit analysis involving biaxial stress components. A review is given of the methods presently in vogue, and new methods are suggested for obtaining equivalent mean and equivalent alternating stresses.

Some ground work is laid for the consideration of stress gradient influence on fatigue behavior. There are also some observations concerning the Distortional Energy and Tresca criteria and how they are incorporated in fatigue design procedures.

The work is restricted to cases in which the alternating principal stress axes have fixed orientation.