COMBINED INTRA- AND INTER-BLOCK ANALYSIS FOR FACTORIALS IN INCOMPLETE BLOCK DESIGNS

by

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I. INTRODUCTION

1.1 Historical Review

Incomplete block and quasi-factorial designs, as developed by Yates, Bose, Nair, Harshbarger, and many others, are applied in situations where the number of treatments exceeds the number of homogeneous experimental units in each block. These designs were first introduced in 1936 by Yates [24, 25], in order to obtain a gain in precision due to the use of smaller blocks, at the expense of loss of information on those varietal comparisons which are confounded with blocks.

In 1939 Bose and Nair [3] published a paper on partially balanced incomplete block designs. The same year, Bose [1] discussed the construction of balanced incomplete block designs. During the past few years, Bose and his co-workers have investigated the particular field of partially balanced incomplete block designs with two associate classes. Some of these designs were constructed and classified by Bose and Shimamoto [4] in 1952. This work was extended by Bose, Clatworthy, and Shrikhande [2], the extensions resulting in the production of a set of tables of all known partially balanced incomplete block designs, with two-associate classes, for which $r \leq 10$, $3 \leq k \leq 10$, where r is the number of replications and k is the number of plots per block and when the efficiency factors, E_1 and E_2, are not too different.

In 1947 Harshbarger [8] developed the rectangular lattice design where the number of varieties is expressible as the product of two consecutive integers. During the next several years Harshbarger [9, 10, 11] further extended the development of rectangular lattices.

In the original papers by Yates on incomplete block designs, attention was directed to methods for obtaining intra-block estimators of varietal effects by considering comparisons arising within blocks. A second estimator may be obtained by taking into account comparisons arising among block totals. Denoting these two estimators by t_1 and t_2, Yates [21, 22] showed that the "best combined estimator" having minimum variance is

$$t_c = \frac{t_1(\text{var } t_2) + t_2(\text{var } t_1)}{\text{var } t_1 + \text{var } t_2} \quad . \tag{1.1.1}$$

It can also be shown [6] that the best combined linear estimators of the varietal effects can be obtained by the minimization of a weighted sum of squares consisting of two parts. The first part by itself yields the usual intra-block estimators while the second part yields the inter-block estimators. In practice, the weights are not known but can be estimated fairly accurately in large experiments from the mean squares in the analysis of variance table. In a recent paper by Sprott [20], it has been shown that the two methods for obtaining the combined estimators are not in general equal; therefore, the first method does not of itself yield the best combined estimate. Rao [19] illustrates a method where the combined estimators and their variances and covariances can be obtained from the intra-block formulae by making suitable changes. The required changes are seen to follow easily from a comparison of the two sets of normal equations.

The use of incomplete block designs in the past has been restricted to essentially varietal trials. In recent years, however, the utility of such designs has been greatly increased by incorporating factorial treatment combinations in them. The first use of a factorial in a partially balanced incomplete block design is given by Cornish [7] in 1938. In 1954 Harshbarger [11] considered a 2^3 factorial in a latinized rectangular lattice design. In the past year, Kramer and Bradley [13, 14, 15] and Zelen [26] have placed factorials in several classes of incomplete block designs making possible a study of several factors at a time together with their interactions. Kramer and Bradley considered only the intra-block analysis and Zelen obtained, in addition, the inter-block analysis for certain specific designs.

1.2 Objectives of this Dissertation

The main objective of this dissertation is to extend the work of Kramer and Bradley and place factorials in the several suitable classes of two-associate class, partially balanced incomplete block designs that have not at present been considered in the literature. Throughout the work on factorials by Kramer and Bradley no attempt was made to utilize the recovery of inter-block information, and a major part of this dissertation will be the consideration of this very important aspect of the analysis of experimental designs.

The recovery of inter-block information is used to gain more information on the estimation of treatment differences in situations where there may be assumed to exist a random variability between blocks. Additional assumptions in the mathematical model are made that the block effects are normally and independently distributed with zero means and equal finite variances.

A study will be made of the variances of estimators of treatment differences, and then of factorial treatment contrasts, along with the efficiencies of the contrasts in various designs relative to the corresponding contrasts in completely randomized designs. Tests of significance for factorial effects will be derived and single degree-of-freedom comparisons will be determined in order to investigate trends and special contrasts among the factorial effects.

For the balanced incomplete block and the group divisible, partially balanced incomplete block designs, only the combined intra- and inter-block analysis will be considered. However, in the case of the Latin Square sub-type L_3, partially balanced incomplete block designs, both the intra-block analysis and the combined intra- and inter-block analysis will be presented.

1.3 Review of Partially Balanced Designs with Two Associate Classes

An incomplete block design is said to be partially balanced with two associate classes if it satisfies the following requirements.

(i) The experimental material is divided into b blocks of k units each, different treatments being applied to the units in the same block.

(ii) There are v ($>k$) treatments each of which occurs in r blocks.

(iii) There can be established a relation of association between any two treatments satisfying the following requirements:

(a) Two treatments are either first associates or second associates.

(b) Each treatment has exactly n_i i^{th} associates ($i = 1, 2$).

(c) Given any two treatments which are i^{th} associates, the number of treatments common to the j^{th} associate of the first and the k^{th} associate of the second is p^i_{jk} and is independent of the pair of treatments we start with. Also $p^i_{jk} = p^i_{kj}$ ($i, j, k = 1, 2$).

(iv) Two treatments which are i^{th} associates occur together in exactly λ_i blocks ($i = 1, 2$).

For a proper partially balanced incomplete block design $\lambda_1 \neq \lambda_2$. If $\lambda_1 = \lambda_2$, the design becomes a balanced incomplete block design.

The numbers v, r, k, b, n_1, n_2, λ_1, and λ_2 are called the parameters of the first kind, whereas the numbers p^i_{jk} ($i, j, k = 1, 2$) are called the parameters of the second kind.

The following relations between the parameters are known to hold:

(1.3.1) $$vr = bk ,$$

(1.3.2) $$n_1 + n_2 = v - 1,$$

(1.3.3) $$n_1\lambda_1 + n_2\lambda_2 = r(k-1),$$

(1.3.4) $$p_{11}^1 + p_{12}^1 = n_1 - 1,$$

(1.3.5) $$p_{21}^1 + p_{22}^1 = n_2,$$

(1.3.6) $$p_{11}^2 + p_{12}^2 = n_1,$$

(1.3.7) $$p_{21}^2 + p_{22}^2 = n_2 - 1,$$

(1.3.8) $$n_1 p_{12}^1 = n_2 p_{11}^2,$$

(1.3.9) $$n_1 p_{22}^1 = n_2 p_{12}^2.$$

The parameters p_{jk}^i of the second kind can be exhibited as the elements of the two symmetric matrices

$$P_1 = \begin{bmatrix} p_{11}^1 & p_{12}^1 \\ p_{21}^1 & p_{22}^1 \end{bmatrix} ,$$

and

$$P_2 = \begin{bmatrix} p_{11}^2 & p_{12}^2 \\ p_{21}^2 & p_{22}^2 \end{bmatrix} .$$

Bose, Clatworthy, and Shrikhande define the constants Δ, H, c_1, c_2 by the relations

(1.3.10) $$k^2\Delta = (a+\lambda_1)(a+\lambda_2) + (\lambda_1-\lambda_2)\left[a(f-g)+f\lambda_2-g\lambda_1\right],$$

(1.3.11) $$kH = (2a+\lambda_1+\lambda_2) + (f-g)(\lambda_1-\lambda_2),$$

(1.3.12) $$k\Delta c_1 = \lambda_1(a+\lambda_2) + (\lambda_1-\lambda_2)(f\lambda_2-g\lambda_1),$$

(1.3.13) $$k\Delta c_2 = \lambda_2(a+\lambda_1) + (\lambda_1-\lambda_2)(f\lambda_2-g\lambda_1),$$

where

(1.3.14) $$a = r(k-1),$$

(1.3.15) $$f = p^1_{12},$$

(1.3.16) $$g = p^2_{12}.$$

Partially balanced incomplete block designs with two associate classes are classified into the following types depending upon their association schemes:

(1) Group divisible,

(2) Simple,

(3) Triangular,

(4) Latin Square type, or

(5) Cyclic.

The only designs that will be considered in this dissertation are those for which the number of treatments is non-prime. This restriction is necessary when we consider factorial treatments in incomplete block designs.

II. BALANCED INCOMPLETE BLOCK DESIGNS

2.1 Combined Intra- and Inter-block Estimators for Factorial Treatments

Suppose we have an incomplete block design consisting of $v = mn$ treatments arranged in b blocks containing k plots each, with each treatment replicated r times. If each pair of treatments occurs together in exactly λ blocks, the design will be balanced. We then have the following two relations:

(2.1.1) $$bk = vr, \text{ and}$$

(2.1.2) $$(v-1)\lambda = (k-1)r.$$

To obtain the intra-block estimators, we use the model

(2.1.3) $$y_{ijs} = \mu + \tau_{ij} + \beta_s + \epsilon_{ijs},$$

$i = 1,\ldots,m;\ j = 1,\ldots,n;\ s = 1,\ldots,b$, where y_{ijs} is the observation on the $(ij)^{th}$ treatment in block s if that treatment occurs in block s, μ is the grand mean, τ_{ij} is the $(ij)^{th}$ treatment effect, β_s is the effect of block s, and the ϵ_{ijs}'s are independent normal variates with zero means and homogeneous variances, σ^2.

For the inter-block estimators we use the model

(2.1.4) $$\sum_{ij}\sum \delta^s_{ij} y_{ijs} = k\mu + \sum_{ij}\sum \delta^s_{ij}\tau_{ij} + k\beta_s + \sum_{ij}\sum \delta^s_{ij}\epsilon_{ijs},$$

where $\delta^s_{ij} = 1$ if the $(ij)^{th}$ treatment occurs in block s and is zero otherwise. Equation (2.1.4) may also be written

(2.1.5) $$B_s = k\mu + \tau_{..s} + k\beta_s + \epsilon_{..s},$$

where B_s is the total of the observations in block s, $\tau_{..s}$ is the sum of the τ_{ij}'s in block s, and $\epsilon_{..s}$ is the sum of the ϵ_{ijs}'s in block s. The β_s's are now additionally assumed to be normally and independently distributed around a mean of zero with equal variances, σ_b^2. The β_s's are also assumed to be uncorrelated with the ϵ_{ijs}'s.

The estimators for the effects represented by equation (2.1.3) are obtained from the method of least squares by minimizing the error sum of squares. Therefore, we minimize

$$(2.1.6) \qquad \sum_{sij}\sum\sum \delta_{ij}^{s}\epsilon_{ijs}^{2} = \sum_{sij}\sum\sum \delta_{ij}^{s}(y_{ijs}-\mu-\tau_{ij}-\beta_s)^2$$

and obtain the normal equations, yielding the intra-block estimators, in the form

$$(2.1.7) \qquad \frac{r(k-1)}{k} t_{ij} - \frac{\lambda}{k} \sum_{\substack{i' \\ i'\neq i}} \sum_{\substack{j' \\ j'\neq j}} t_{i'j'} = Q_{ij}$$

where i, i' = 1,...,m; j, j' = 1,...,n,

$$(2.1.8) \qquad Q_{ij} = T_{ij} - B_{ij.}/k,$$

T_{ij} is the total of the observations for the $(ij)^{th}$ treatment, and $B_{ij.}$ is the total of block totals for blocks containing the $(ij)^{th}$ treatment. To obtain determinate solutions of equations (2.1.7), as derived by Yates [25], we impose the condition that $\sum_{ij}\sum t_{ij} = 0$. Then

$$(2.1.9) \qquad t_{ij} = \frac{k}{(\lambda+rk-r)} Q_{ij} = \frac{k}{v\lambda} Q_{ij},$$

where i = 1,...,m and j = 1,...,n.

The combined intra- and inter-block estimators of the τ_{ij}'s may be obtained from equations (2.1.9) by making certain substitutions for λ, r, and Q_{ij}. The required substitutions follow from a comparison of the normal equations for the combined intra- and inter-block estimators with the intra-block equations (2.1.7). If we define $W = 1/\sigma^2$ and $W' = 1/(\sigma^2+k\sigma_\beta^2)$, and assume that they are known without error, Sprott [20] shows that the normal equations for the combined intra- and inter-block estimators of the τ_{ij}'s can be obtained by minimizing

$$(2.1.10) \qquad W\sum_{s}\sum_{i}\sum_{j} \delta_{ij}^{s}\left(y_{ijs} - \frac{B_s}{k} - \tau_{ij} + \frac{\tau_{..s}}{k}\right)^2 + \frac{W'}{k}\sum_{s}(B_s - k\mu - \tau_{..s})^2.$$

Minimizing (2.1.10) subject to the condition that $\sum_{i}\sum_{j} \tau_{ij} = 0$, we obtain the normal equations

$$(2.1.11) \qquad r\left[W+W'/(k-1)\right]\left(\frac{k-1}{k}\right)t'_{ij} - \frac{\lambda(W-W')}{k}\sum_{\substack{i' \\ i'\neq i}}\sum_{\substack{j' \\ j'\neq j}} \delta_{i'j'}^{s}t'_{i'j'} = WQ_{ij} + W'Q'_{ij}$$

$i = 1,\ldots,m$; $j = 1,\ldots,n$, where

$$(2.1.12) \qquad Q'_{ij} = \frac{B_{ij.}}{k} - \frac{rG}{bk};$$

t'_{ij} is the combined inter- and intra-block estimator of τ_{ij}.

It is now obvious that solutions for equations (2.1.11) are the same as for those from equations (2.1.7) if we replace

(2.1.13) $\qquad Q_{ij}$ by $WQ_{ij} + W'Q'_{ij}$,

(2.1.14) $\qquad r$ by $r\left[W+W'/(k-1)\right]$,

and

(2.1.15) λ by $\lambda(W-W')$

in the first part of equations (2.1.9). Therefore, the combined estimators for treatment effects are given by

$$(2.1.16) \qquad t'_{ij} = \frac{k(WQ_{ij}+W'Q'_{ij})}{v\lambda W+(r-\lambda)W'},$$

$i = 1,\ldots,m;\ j = 1,\ldots,n.$

Suppose one has two factors, A and C, at m and n levels, respectively. The $(ij)^{th}$ treatment is now the factorial treatment combination of the i^{th} level of factor A with the j^{th} level of factor C. Kramer and Bradley [14, 15] and Cornish [7] have shown that the factorial treatment combinations can be placed in the incomplete block design and analyzed by replacing τ_{ij} in the model by

$$(2.1.17) \qquad \tau_{ij} = \alpha_i + \gamma_j + \delta_{ij},$$

where α_i, $i = 1,\ldots,m$, represent the effects of the m levels of A, γ_j, $j = 1,\ldots,n$, represent the effects of the n levels of C, and δ_{ij} represent the interaction effects of the two factors.

The intra-block model for factorial treatments is given by

$$(2.1.8) \qquad y_{ijs} = \mu + \alpha_i + \gamma_j + \delta_{ij} + \beta_s + \epsilon_{ijs}.$$

In obtaining unbiased estimators we impose the restrictions

$$(2.1.19) \qquad \sum_{ij}\sum\tau_{ij} = 0,\ \sum_i\alpha_i = 0,\ \sum_j\gamma_j = 0,\ \sum_i\delta_{ij} = \sum_j\delta_{ij} = 0,\ \sum_s\beta_s = 0.$$

In view of equation (2.1.17) and the restrictions given by (2.1.19), the combined estimators for the factorial effects, obtained from equations (2.1.16) are

(2.1.20) $$a'_i = \frac{1}{n} \sum_j t'_{ij} = \bar{t}'_{i.} \quad ,$$

(2.1.21) $$c'_j = \frac{1}{m} \sum_i t'_{ij} = \bar{t}'_{.j} \quad ,$$

and

(2.1.22) $$d'_{ij} = t'_{ij} - \bar{t}'_{i.} - \bar{t}'_{.j} \quad ,$$

which are easily computed from a two-way table of values of t'_{ij} in the same way as described by Kramer and Bradley.

2.2 Variances and Covariances of the Estimators

Rao [18] has shown that the variances and covariances for the combined intra- and inter-block varietal estimators may be obtained from the intra-block formulae by making the substitutions (2.1.13), (2.1.14), and (2.1.15), and omitting the multiplier σ^2. Bose and Nair [3] have shown that the variances and covariances of the Q_{ij}'s are

(2.2.1) $$\frac{r(k-1)\sigma^2}{k}$$

and

(2.2.2) $$-\frac{\lambda\sigma^2}{k} \, ,$$

respectively. From equations (2.1.9), (2.2.1), and (2.2.2), it follows that

(2.2.3) $$V(t_{ij}) = \frac{kr(k-1)\sigma^2}{(rk-r+\lambda)^2}$$

and

$$\text{(2.2.4)} \qquad \mathrm{Cov}(t_{ij}, t_{i',j'}) = \frac{-k\lambda\sigma^2}{(rk-r+\lambda)^2} \ .$$

The variances of the combined treatment estimators, obtained from equation (2.2.3) by making the substitutions (2.1.13), (2.1.14), and (2.1.15), and omitting σ^2, are

$$\text{(2.2.5)} \qquad V(t'_{ij}) = \frac{k(k-1)r\left[W+W'/(k-1)\right]}{\left[r(k-1)W+rW'+\lambda(W-W')\right]^2}$$

$$= \frac{k\left[r(k-1)W+rW'\right]}{\left[(rk-r+\lambda)W+(r-\lambda)W'\right]^2}$$

$$= \frac{k(v-1)\lambda W+krW'}{\left[v\lambda W+(r-\lambda)W'\right]^2} \ ,$$

$i = 1,\ldots,m$; $j = 1,\ldots,n$. Similarly, the covariances of the combined treatment estimators are

$$\text{(2.2.6)} \qquad \mathrm{Cov}(t'_{ij}, t'_{i'j'}) = \frac{-k\lambda(W-W')}{\left[r(k-1)W+rW'+\lambda(W-W')\right]^2}$$

$$= \frac{-k\lambda(W-W')}{\left[v\lambda W+(r-\lambda)W'\right]^2} \ ,$$

i, $i' = 1,\ldots,m$; j, $j' = 1,\ldots,n$. From equations (2.2.5) and (2.2.6) the variance of the difference between two estimators is

$$(2.2.7) \quad V(t'_{ij}-t'_{i'j'}) = V(t'_{ij}) + V(t'_{i'j'}) - 2\,\mathrm{Cov}(t'_{ij},t'_{i'j'})$$

$$= \frac{2k(v-1)\lambda W+2krW'+2k\lambda(W-W')}{\left[v\lambda W+(r-\lambda)W'\right]^2}$$

$$= \frac{2k\left[v\lambda W+(r-\lambda)W'\right]}{\left[v\lambda W+(r-\lambda)W'\right]^2}$$

$$= \frac{2k}{\left[v\lambda W+(r-\lambda)W'\right]} .$$

To estimate the weights, W and W', we form the usual auxiliary table for inter-block analysis of variance as described by Bose, Clatworthy, and Shrikhande [2]. This method for estimating the weights was first discussed by Yates [22]. If we denote the mean square for error by E and for blocks adjusted by B, then the estimates of W and W' are

$$(2.2.8) \qquad w = \frac{1}{E}, \quad w' = \frac{bk-v}{k(b-1)B-(v-k)E} .$$

Using equations (2.2.5) and (2.2.6), we may derive the variances and covariances of the combined intra- and inter-block estimators of the factorial effects. For the A-factor we have

$$(2.2.9)\qquad V(a_i') = V(\bar{t}_{i.}') = \frac{1}{n^2}\,V(\sum_j t_{ij}')$$

$$= \frac{1}{n^2}\left[\sum_j V(t_{ij}') + \sum_{\substack{jj' \\ j'\neq j}}\sum \mathrm{Cov}\,(t_{ij}',t_{ij'}')\right]$$

$$= \frac{1}{n^2}\left\{n\,\frac{\left[k(v-1)\lambda W+krW'\right]}{\left[v\lambda W+(r-\lambda)W'\right]^2} + n(n-1)\,\frac{\left[-k\lambda(W-W')\right]}{\left[v\lambda W+(r-\lambda)W'\right]^2}\right\}$$

$$= \frac{(kv\lambda-nk\lambda)W+(kr+nk\lambda-k\lambda)W'}{n\left[v\lambda W+(r-\lambda)W'\right]^2}$$

$$= \frac{mnk\lambda(m-1)W+mk(r+n\lambda-\lambda)W'}{mn\left[v\lambda W+(r-\lambda)W'\right]^2}\ .$$

If we recall that $v\lambda = rk-r+\lambda$ from equation (2.12), the coefficient of W' in (2.2.9) may be written

$$(2.2.10)\qquad mk(r+n\lambda-\lambda) = mkr + rk^2 - rk + \lambda k - mk\lambda$$

$$= k^2r + k(m-1)(r-\lambda).$$

Substituting equation (2.2.10) into equation (2.2.9), we obtain the result

$$(2.2.11)\qquad V(a_i') = \frac{vk\lambda(m-1)W+\left[k^2r+k(m-1)(r-\lambda)\right]W'}{v\left[v\lambda W+(r-\lambda)W'\right]^2}\ .$$

Similarly, for a C-factor we have

$$(2.2.12)\qquad V(c_j') = \frac{vk\lambda(n-1)W+\left[k^2r+k(n-1)(r-\lambda)\right]W'}{v\left[v\lambda W+(r-\lambda)W'\right]^2}\ .$$

The covariances of two combined intra- and inter-block estimators for A-factor effects are of the form

$$(2.2.13)\qquad \mathrm{Cov}(a_i^{\prime}, a_{i^{\prime}}^{\prime}) = \mathrm{Cov}(\bar{t}_{i.}^{\prime}, \bar{t}_{i^{\prime}.}^{\prime})$$

$$= \frac{1}{n^2}\ \mathrm{Cov}(\sum_j t_{ij}^{\prime}, \sum_{j^{\prime}} t_{i^{\prime}j^{\prime}}^{\prime})$$

$$= \mathrm{Cov}(t_{ij}^{\prime}, t_{i^{\prime}j^{\prime}}^{\prime})$$

$$= \frac{-k\lambda(W-W^{\prime})}{[v\lambda W+(r-\lambda)W^{\prime}]^2}\ ,$$

$i \neq i^{\prime}$; $i, i^{\prime} = 1,\ldots,m$, from equation (2.2.6).

Similarly for C-factor effects the covariances of the combined intra- and inter-block estimators are of the form

$$(2.2.14)\qquad \mathrm{Cov}(c_j^{\prime}, c_{j^{\prime}}^{\prime}) = \frac{-k\lambda(W-W^{\prime})}{[v\lambda W+(r-\lambda)W^{\prime}]^2}\ ,$$

$j \neq j^{\prime}$; $j, j^{\prime} = 1,\ldots,n$. The covariance of an A-factor estimator with a C-factor estimator is no longer zero as in the intra-block analysis. We now find that

$$(2.2.15)\qquad \mathrm{Cov}(a_i^{\prime}, c_j^{\prime}) = \mathrm{Cov}(\bar{t}_{i.}^{\prime}, \bar{t}_{.j}^{\prime})$$

$$= \mathrm{Cov}(\frac{1}{n}\sum_j t_{ij}^{\prime}, \frac{1}{m}\sum_i t_{ij}^{\prime})$$

$$= \frac{1}{mn}\,V(t_{ij}^{\prime}) + \frac{1}{nm}\sum_{\substack{i^{\prime} \\ i^{\prime}\neq i}}\sum_{j^{\prime}} \mathrm{Cov}(t_{ij^{\prime}}^{\prime}, t_{i^{\prime}j}^{\prime}) + \frac{1}{nm}\sum_{\substack{j^{\prime} \\ j^{\prime}\neq j}} \mathrm{Cov}(t_{ij^{\prime}}^{\prime}, t_{ij}^{\prime})$$

$$= \frac{k(v-1)\lambda W+krW^{\prime}-[n(m-1)+(n-1)]k\lambda(W-W^{\prime})}{nm[v\lambda W-(r-\lambda)W^{\prime}]^2}$$

$$= \frac{k^2rW^{\prime}}{v[v\lambda W-(r-\lambda)W^{\prime}]^2}\ ,$$

by making use of equation (2.1.2).

From equations (2.2.5), (2.2.6), (2.2.10), (2.2.11), and (2.2.15) we can obtain the variance for the combined intra- and inter-block estimator of an AC-effect. We have

$$(2.2.16) \qquad V(d'_{ij}) = V(t'_{ij}-a'_i-c'_j)$$

$$= V(t'_{ij}) + V(a'_i) + V(c'_j) - 2Cov(t'_{ij},a'_i)$$

$$- 2Cov(t'_{ij},c'_j) + 2Cov(a'_i,c'_j) \; .$$

Now,

$$(2.2.17) \qquad Cov(t'_{ij},a'_i) = Cov(t'_{ij},\frac{1}{n}\sum_j t'_{ij})$$

$$= \frac{1}{n} V(t'_{ij}) + \frac{n-1}{n} Cov(t'_{ij},t'_{ij'}) \; .$$

Similarly,

$$(2.2.18) \qquad Cov(t'_{ij},c'_j) = \frac{1}{m} V(t'_{ij}) + \frac{m-1}{m} Cov(t'_{ij},t'_{i'j}) \; .$$

Therefore, substituting (2.2.17) and (2.2.18) in (2.2.16), we have

$$(2.2.19) \qquad V(d'_{ij}) = (1 - \frac{2}{n} - \frac{2}{m})V(t'_{ij}) + V(a'_i) + V(c'_j)$$

$$- 2(\frac{n-1}{n} + \frac{m-1}{m})Cov(t'_{ij},t'_{i'j'}) + 2Cov(a'_i,c'_j)$$

$$= \frac{(nm-2m-2n)\left[k(v-1)\lambda W+krW'\right]}{v\left[v\lambda W+(r-\lambda)W'\right]^2}$$

$$+ \frac{vk(m-1)\lambda W+\left[k^2r+k(m-1)(r-\lambda)\right]W'}{v\left[v\lambda W+(r-\lambda)W'\right]^2}$$

$$+ \frac{vk(n-1)\lambda W+\left[k^2r+k(n-1)(r-\lambda)\right]W'}{v\left[v\lambda W+(r-\lambda)W'\right]^2}$$

$$+ \frac{(4mn-2m-2n)k\lambda(W-W')}{v\left[v\lambda W+(r-\lambda)W'\right]^2}$$

$$+ \frac{2k^2rW'}{v\left[v\lambda W+(r-\lambda)W'\right]^2} \quad .$$

The coefficient of W simplifies to the form

$$(m-1)(n-1)kv\lambda \ ,$$

and the coefficient of W' becomes

$$k^2r + k(m-1)(n-1)(r-\lambda) \ .$$

Therefore,

$$(2.2.20) \qquad V(d'_{ij}) = \frac{kv\lambda(m-1)(n-1)W+\left[k^2r+k(m-1)(n-1)(r-\lambda)\right]W'}{v\left[v\lambda W+(r-\lambda)W'\right]^2} \quad .$$

The covariances of the combined intra- and inter-block estimators for the interaction effects are easily obtained by using equations (2.2.5) and (2.2.6). We have

$$(2.2.21)\qquad \mathrm{Cov}(d'_{ij},d'_{i'j'}) = \mathrm{Cov}(t'_{ij}-\bar{t}'_{i.}-\bar{t}'_{.j},t'_{i'j'}-\bar{t}'_{i'.}-\bar{t}'_{.j'})$$

$$= \frac{2}{mn}\,\mathrm{Var}(t'_{ij}) + \frac{(mn-2)}{mn}\,\mathrm{Cov}(t'_{ij},t'_{i'j'})$$

$$= \frac{2k(v-1)\lambda W+2krW'-(mn-2)k\lambda(W-W')}{v\left[v\lambda W+(r-\lambda)W'\right]^2}$$

$$= \frac{kv\lambda W+\left[rk^2+k(r-\lambda)\right]W'}{v\left[v\lambda W+(r-\lambda)W'\right]^2}\,,$$

$i \neq i'$, $j \neq j'$. Similarly, it may easily be shown that

$$(2.2.22)\qquad \mathrm{Cov}(d'_{ij},d'_{ij'}) = \frac{-kv\lambda(m-1)W+\left[rk^2-k(m-1)(r-\lambda)\right]W'}{v\left[v\lambda W+(r-\lambda)W'\right]^2}$$

for all $j \neq j'$, and

$$(2.2.23)\qquad \mathrm{Cov}(d'_{ij},d'_{i'j}) = \frac{-kv\lambda(n-1)W+\left[rk^2-k(n-1)(r-\lambda)\right]W'}{v\left[v\lambda W+(r-\lambda)W'\right]^2}$$

for all $i \neq i'$.

All covariances arising from an estimator of a main effect with an estimator of an interaction effect are of the form

$$(2.2.24)\qquad \mathrm{Cov}(a'_i,d'_{i'j}) = \mathrm{Cov}(\bar{t}'_{i.},t'_{i'j}-\bar{t}'_{i'.}-\bar{t}'_{.j})$$

$$= \frac{-1}{mn}\,V(t'_{ij}) - \frac{(mn-1)}{mn}\,\mathrm{Cov}(t'_{ij},t'_{i'j})$$

$$= \frac{-\left[k(v-1)\lambda W+krW'\right]+(mn-1)k\lambda(W-W')}{v\left[v\lambda W+(r-\lambda)W'\right]^2}$$

$$= \frac{-k^2rW'}{v\left[v\lambda W+(r-\lambda)W'\right]^2}\,,$$

for all i, $i' = 1,\ldots,m$, and $j = 1,\ldots,n$.

From equations (2.2.9) and (2.2.13) the variance of the difference between estimators of the A-effects is

$$(2.2.25) \qquad V(a_i'-a_{i'}') = \frac{2k}{n[v\lambda W+(r-\lambda)W']}$$

$i \neq i'$. Likewise, the variance of the difference between estimators of the C-effects, obtained from equations (2.2.12) and (2.2.14), is

$$(2.2.26) \qquad V(c_j'-c_{j'}') = \frac{2k}{m[v\lambda W+(r-\lambda)W']} \quad ,$$

$j \neq j'$.

2.3 Tests of Significance

If W and W' are known without error, then Rao [19] has shown that a test of the equality of treatment means for the combined intra- and inter-block analysis is based on the statistic

$$(2.3.1) \qquad \chi_T^2 = \sum_i\sum_j t_{ij}'(WQ_{ij}+W'Q_{ij}'),$$

which can be used as a χ^2-variate with (v-1) degrees of freedom. If the computed value obtained from equation (2.3.1) exceeds the tabled value of χ^2 with (v-1) degrees of freedom at the α level of significance, we reject the hypothesis that the treatment means are equal. This test may be used as an approximation if W and W' are estimated with a large number of degrees of freedom. From equation (2.1.11) and the restrictions (2.1.19), equation (2.3.1) may be put in the form

$$(2.3.2) \qquad \chi_T^2 = [v\lambda W+(r-\lambda)W']\sum_i\sum_j t_{ij}'^2/k \ .$$

To test the null hypothesis of no A-effects in the intra-block analysis, Kramer and Bradley [14, 15] used a statistic based on the fact that

(2.3.3) $$\chi_A^2 = \frac{1}{\sigma^2} \sum_{i}\sum_{j} a_i Q_{ij} = \frac{nv\lambda}{k\sigma^2} \sum_{i} a_i^2$$

is a χ^2-variate with m-1 degrees of freedom. If we make the substitutions (2.1.13), (2.1.14), and (2.1.15) in equation (2.3.3), then the statistic

(2.3.4) $$\chi_A^2 = \sum_{i}\sum_{j} a_i'(WQ_{ij}+W'Q_{ij}')$$

is also a χ^2-variate with (m-1) degrees of freedom which can be used as an approximation to test the null hypothesis of no A-effects for the combined intra- and inter-block analysis. From equation (2.1.11) and the restrictions (2.1.19), equation (2.3.4) may likewise be put in the form

(2.3.5) $$\chi_A^2 = \left[(rk-r+\lambda)W+(r-\lambda)W'\right] \sum_{i}\sum_{j} t_{i.}' t_{ij}'/nk$$

$$= \left[v\lambda W+(r-\lambda)W'\right] \sum_{i} t_{i.}'^2/nk ,$$

or

(2.3.6) $$\chi_A^2 = \left[nv\lambda W+n(r-\lambda)W'\right] \sum_{i} a_i'^2/k .$$

A similar argument is sufficient to derive χ^2-statistics to test the null hypotheses of no C-effects and no interaction effects. To test the hypothesis of no C-effects, we use the statistic

(2.3.7) $$\chi_C^2 = \sum_{i}\sum_{j} c_j'(WQ_{ij}+W'Q_{ij}'),$$

which is approximately distributed as a χ^2-variate with (n-1) degrees of freedom. From equation (2.1.11) and the restrictions (2.1.19), equation (2.3.7) may be written as

$$(2.3.8) \qquad \chi_C^2 = \left[(rk-r+\lambda)W+(r-\lambda)W'\right] \sum_{i}\sum_{j} t'_{\cdot j}t'_{ij}/mk$$

$$= \left[v\lambda W+(r-\lambda)W'\right] \sum_{j} t'^{2}_{\cdot j}/mk,$$

or

$$(2.3.9) \qquad \chi_C^2 = \left[mv\lambda W+m(r-\lambda)W'\right] \sum_{j} c'^{2}_{j}/k.$$

To test the hypothesis of no interaction effects we use the statsitic

$$(2.3.10) \qquad \chi_{AC}^2 = \sum_{i}\sum_{j} d'_{ij}(WQ_{ij}+W'Q'_{ij}),$$

which is approximately distributed as a χ^2-variate with (m-1)(n-1) degrees of freedom. From equation (2.1.11) and the restrictions (2.1.19), equation (2.3.10) may also be written as

$$(2.3.11) \qquad \chi_{AC}^2 = \left[(rk-r+\lambda)W+(r-\lambda)W'\right] \sum_{i}\sum_{j}(t'_{ij}-\bar{t}'_{i\cdot}-\bar{t}'_{\cdot j})^2/k,$$

or

$$(2.3.12) \qquad \chi_{AC}^2 = \left[(rk-r+\lambda)W+(r-\lambda)W'\right] \sum_{i}\sum_{j} d'^{2}_{ij}/k.$$

We will now show that

$$(2.3.13) \qquad \chi_T^2 = \chi_A^2 + \chi_C^2 + \chi_{AC}^2,$$

and the degrees of freedom add up to v-1. From equations (2.3.4), (2.3.7), and (2.3.10), the right side of equation (2.3.13) is

$$\sum_{i}\sum_{j} a_i'(WQ_{ij}+W'Q_{ij}') + \sum_{i}\sum_{j} c_j'(WQ_{ij}+W'Q_{ij}') + \sum_{i}\sum_{j} d_{ij}'(WQ_{ij}+W'Q_{ij}') \tag{2.3.14}$$

$$= \sum_{i}\sum_{j}(a_i'+c_j'+d_{ij}')(WQ_{ij}+W'Q_{ij}')$$

$$= \sum_{i}\sum_{j} t_{ij}'(WQ_{ij}+W'Q_{ij}'),$$

which is equal to the left side of equation (2.3.13) by equation (2.3.1). The sum of the degrees of freedom is

$$(m-1) + (n-1) + (m-1)(n-1) = mn - 1 = v - 1, \tag{2.3.15}$$

which is equal to the total number of degrees of freedom for treatments. Cochran's theorem [5] is sufficient to demonstrate the independence of all the χ^2-variates.

To test the significance of the difference between pairs of treatment estimators or factorial estimators the t-test may be used as an approximation.

2.4 Individual Comparisons and Multi-factor Factorials

Frequently in experimental work we wish to know the answers to certain questions about the treatments which can not be obtained from the complete treatment mean square. By an extension of the analysis of variance, we can sub-divide the treatment sum of squares into a number of components that are more relevant to the individual questions. While orthogonal comparisons are desired to perform tests of significance, this is not a necessary restriction.

Individual or single-degree-of-freedom comparisons are possible for the combined intra- and inter-block analysis in the same way as for the intra-block analysis, given by Kramer and Bradley [14, 15]. Let ξ be an (m-1) by m orthogonal matrix, and η an (n-1) by n orthogonal matrix used to transform the a_i''s and c_j''s to (m-1) and (n-1) individual contrasts, respectively, each yielding an adjusted sum of squares with one degree of freedom. Contrasts on A-factor effects would then be

(2.4.1) $$I_u = \sum_i \xi_{iu} a_i', \quad u = 1, \ldots, m-1,$$

and on C-factor effects

(2.4.2) $$J_v = \sum_j \eta_{vj} c_j', \quad v = 1, \ldots, n-1.$$

To test the hypothesis that $\sum_i \xi_{iu}\alpha_i = 0$ against the hypothesis that $\sum_i \xi_{iu}\alpha_i \neq 0$, we use the statistic

(2.4.3) $$\chi^2_{I_u} = [nv\lambda W + n(r-\lambda)W'](\sum_i \xi_{iu} a_i')^2 / k\sum_i \xi_{iu}^2$$

$$= [nv\lambda W + n(r-\lambda)W'](\sum_i \xi_{iu} \bar{t}_{i.}')^2 / k\sum_i \xi_{iu}^2,$$

which is easily derived from equation (2.4.1) and the multiplier of equation (2.3.6).

Similarly to test the hypothesis that $\sum_j \eta_{vj}\gamma_j = 0$ against the hypothesis that $\sum_j \eta_{vj}\gamma_j \neq 0$, we use the statistic

(2.4.4) $$\chi^2_{J_v} = [mv\lambda W + m(r-\lambda)W'](\sum_j \eta_{vj} c_j')^2 / k\sum_j \eta_{vj}^2$$

$$= [mv\lambda W + m(r-\lambda)W'](\sum_j \eta_{vj} \bar{t}_{.j}')^2 / k\sum_j \eta_{vj}^2$$

which is obtained from equation (2.4.2) and the multiplier of equation (2.3.9).

The adjusted interaction sum of squares may also be partitioned. The (m-1)(n-1) orthogonal contrasts for the interaction of I_u and J_v, obtained from the matrices ξ and η, are

$$(2.4.5) \qquad \begin{aligned} (IJ)_{uv} &= \sum_i\sum_j \xi_{ij}\eta_{vj}d'_{ij} \\ &= \sum_i\sum_j \xi_{iu}\eta_{vj}(t'_{ij}-a'_i-c'_j) \\ &= \sum_i\sum_j \xi_{iu}\eta_{vj}t'_{ij}, \end{aligned}$$

since $\sum_i \xi_{iu} = \sum_j \eta_{vj} = 0$. To test the hypothesis that $\sum_i\sum_j \xi_{iu}\eta_{vj}\delta_{ij} = 0$, we use the statistic

$$(2.4.6) \qquad \chi^2_{(IJ)_{uv}} = \left[v\lambda W + (r-\lambda)W'\right]\Big(\sum_i\sum_j \xi_{iu}\eta_{vj}t'_{ij}\Big)^2 \Big/ k\sum_i\sum_j(\xi_{iu}\eta_{vj})^2$$

which is obtained from equations (2.4.5) and the multiplier of equation (2.3.12).

From the manner in which we have constructed the single-degree-of-freedom contrasts, it is clear that the resulting sums of squares add up to the total adjusted sum of squares for treatments, which has been shown to be distributed as a χ^2-variate with (v-1) degrees of freedom. Since the degrees of freedom for the individual contrasts add up to (v-1), we may conclude by Cochran's theorem [5] that the corresponding sums of squares are independently distributed as χ^2-variates, each with one degree of freedom.

Special definition of the matrices, ξ and η, permits the use of special contrasts. For example, rows of ξ and η may be defined such that contrasts on A-factor and C-factor effects measure trends (linear, quadratic, cubic,...) over the factor levels.

Suppose the A-factor has levels which themselves are factorial combinations of other factors. Let there be p such factors $A_1,\ldots,A_p$, with levels $m_1,\ldots,m_p$, such that $m = \prod_{i=1}^{p} m_i$. Then ξ may be chosen in the obvious way so that the contrasts defined may be grouped to obtain main-effect and interaction comparisons for the subfactors of A. The corresponding adjusted sums of squares, each with one degree of freedom, may be grouped together to give adjusted sums of squares for the various subfactors of A. These sums of squares will be distributed as χ^2-variates since we are grouping sums of squares which themselves are distributed independently as a χ^2-variate. It now will be possible to test the hypotheses of no main effects or interaction effects among the subfactors of A. Similarly, the C-factor may consist of factorial combinations of q factors $C_1,\ldots,C_q$, with levels $n_1,\ldots,n_q$, such that $n = \prod_{j=1}^{q} n_j$. Appropriate contrasts and adjusted sums of squares may be obtained with proper selection of the rows of η. When ξ and η have been defined, the corresponding contrasts for interaction of A-factor and C-factor contrasts follow immediately. These in turn yield adjusted sums of squares that may be grouped to yield sums of squares for the various interactions of the subfactors of A with those of C.

Alternately we could obtain the combined intra- and inter-block estimators of all the factorial factors by generalizing equation

(2.1.17) so that τ_{ij} is a function of all the main and interaction effects. Imposing the restrictions that the sums of the various main effects are zero and the sums of the interaction effects over any one or more subscripts are zero, we could obtain the combined intra- and inter-block estimators of the factorial effects by considering a table of the t'_{ij}'s.

The use of fractional factorials also is possible in exactly the same way as carried out by Kramer and Bradley [14, 15].

III. GROUP DIVISIBLE DESIGNS

3.1 Properties of Group Divisible Designs

Bose, Clatworthy, and Shrikhande [2] list the following properties of group divisible designs with two associate classes:

(i) The requirements for partially balanced designs as outlined in Section 1.3 are satisfied.

(ii) There are $v = mn$ treatments, and the treatments can be divided into m groups of n each such that any two treatments of the same group are first associates while two treatments from different groups are second associates.

(iii) Each treatment has exactly n-1 first associates and n(m-1) second associates.

(iv) The design parameters are related so that

$$(n-1)\lambda_1 + n(m-1)\lambda_2 = r(k-1) \tag{3.1.1}$$

or

$$rk - v\lambda_2 = r - \lambda_1 + n(\lambda_1 - \lambda_2).$$

(v) In matrix notation P_1 and P_2 of Section 1.3 may be written

$$P_1 = \begin{bmatrix} (n-2) & 0 \\ 0 & n(m-1) \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} 0 & (n-1) \\ (n-1) & n(m-2) \end{bmatrix}$$

(vi) The inequalities, $r \geq \lambda_1$, $rk - \lambda_2 v \geq 0$, hold.

Group divisible designs have been divided into three subclasses:

(a) Singular, if $r = \lambda_1$,

(b) Semi-regular, if $r > \lambda_1$ and $rk - v\lambda_2 = 0$,

(c) Regular, if $r > \lambda_1$ and $rk - v\lambda_2 > 0$.

Let V_{ij} denote the j^{th} treatment of the i^{th} group noted in (ii), $i = 1,\ldots,m$; $j = 1,\ldots,n$. Then the usual association scheme is given by the matrix V with elements V_{ij}, such that two treatments in the same group or row (treatments with common first subscripts) are first associates and two treatments not in the same row (treatments with different first subscripts) are second associates.

3.2 Combined Intra- and Inter-block Treatment Estimators

The intra-block estimators of the treatment effects are obtained from equation (2.1.3) by minimizing the error sum of squares given by (2.1.6). Equations (2.1.3) and (2.1.6) still hold for group divisible designs; results differ owing to different values of δ^s_{ij} as sums are taken over treatment and block associations. The resulting normal equations were shown by Bose and Shimamoto [4] to be of the form

$$\text{(3.2.1)} \qquad \frac{r(k-1)}{k} t_{ij} - \frac{\lambda_1}{k} \sum_{\substack{j' \\ j' \neq j}} t_{ij'} - \frac{\lambda_2}{k} \sum_{j'} \sum_{\substack{i' \\ i' \neq i}} t_{i'j'} = Q_{ij},$$

where $i = 1,\ldots,m$; $j = 1,\ldots,n$.

If W and W' are assumed to be known without error, then, as in Chapter II, Sprott [20] shows that the combined intra- and inter-block estimators of the treatment effects can be obtained by minimizing the weighted sum of squares of deviations still symbolically given by

(2.1.10), subject to the condition that $\sum_{i}\sum_{j} \tau_{ij} = 0$. Therefore we must minimize

$$(3.2.2) \qquad W\sum_{s}\sum_{i}\sum_{j} \delta_{ij}^{s}\left(y_{ijs} - \frac{B_s}{k} - \tau_{ij} + \frac{\tau_{..s}}{k}\right)^2 + \frac{W'}{k}\sum_{s}(B_s - k\mu - \tau_{..s})^2$$

$$+ 2\lambda\sum_{i}\sum_{j} \tau_{ij},$$

where λ is a Lagrange multiplier and symbols have definitions as used earlier. Taking the partial derivative with respect to μ in (3.2.2) and setting it equal to zero, we obtain

$$(3.2.3) \qquad \sum_{s}(B_s - km - t'_{..s}) = 0,$$

from which it follows that

$$(3.2.4) \qquad m = \frac{G}{bk}$$

since $\sum_{s} t'_{..s} = \sum_{i}\sum_{j} t'_{ij} = 0$; t'_{ij} is the combined inter- and intra-block estimator of τ_{ij} as before.

Taking the partial derivative with respect to τ_{ij} in (3.2.2) and setting the resulting expression equal to zero lets us write the equation

$$(3.2.5) \qquad W\sum_{s} \delta_{ij}^{s}\left(y_{ijs} - \frac{B_s}{k} - t'_{ij} + \frac{t'_{..s}}{k}\right)$$

$$- W\sum_{s} \sum_{\substack{i' \\ \text{in} \\ s}} \sum_{j'} \delta_{ij}^{s}\left(y_{i'j'} - \frac{B_s}{k} - t'_{i'j'} + \frac{t'_{..s}}{k}\right)$$

$$+ \frac{W'}{k}\sum_{s} \delta_{ij}^{s}(B_s - km - t'_{..s}) - \lambda = 0,$$

where $\delta^s_{ij} = 1$ if treatment V_{ij} occurs in block s,

$= 0$, otherwise. Therefore, we have

$$(3.2.6) \qquad W\sum_s \delta^s_{ij}\left(y_{ijs} - \frac{B_s}{k} - t'_{ij} + \frac{t'_{..s}}{k}\right) + \frac{W'}{k}\sum_s \delta^s_{ij}(B_s - km - t'_{..s}) - \lambda = 0,$$

since

$$\sum_s \sum_{\substack{i' \\ \text{in} \\ s}} \sum_{j'} \delta^s_{ij}\left(y_{i'j'} - \frac{B_s}{k} - t'_{i'j'} + \frac{t'_{..s}}{k}\right)$$

$$= \sum_s \sum_{\substack{i' \\ \text{in} \\ s}} \sum_{j'} \delta^s_{ij} y_{i'j'} - \sum_s \sum_{\substack{i' \\ \text{in} \\ s}} \sum_{j'} \delta^s_{ij}\frac{B_s}{k} - \sum_s \sum_{\substack{i' \\ \text{in} \\ s}} \sum_{j'} \delta^s_{ij} t'_{i'j'}$$

$$+ \sum_s \sum_{\substack{i' \\ \text{in} \\ s}} \sum_{j'} \delta^s_{ij}\frac{t'_{..s}}{k}$$

$$= B_{ij.} - k\frac{B_{ij.}}{k} - \sum_s \delta^s_{ij} t'_{..s} + k\sum_s \frac{t'_{..s}}{k}$$

$$= 0.$$

It follows that

$$(3.2.7) \qquad W\left(T_{ij} - \frac{B_{ij.}}{k} - rt'_{ij} + \sum_s \delta^s_{ij}\frac{t'_{..s}}{k}\right) + \frac{W'}{k}\left(B_{ij.} - rkm - \sum_s \delta^s_{ij} t'_{..s}\right) - \lambda = 0,$$

$i = 1,\ldots,m;\ j = 1,\ldots,n$, where T_{ij} is the total for treatment V_{ij} and $B_{ij.}$ is the total of block totals for blocks containing treatment V_{ij}. Summing equation (3.2.7) over all treatments, we find $\lambda = 0$ and, therefore,

$$W\left(T_{ij} - \frac{B_{ij.}}{k} - rt'_{ij} + \sum_s \delta^s_{ij} \frac{t'_{..s}}{k}\right) + \frac{W'}{k}\left(B_{ij.} - rkm - \sum_s \delta^s_{ij} t'_{..s}\right) = 0, \tag{3.2.8}$$

$i = 1,\ldots,m$; $j = 1,\ldots,n$. Now

$$\sum_s \delta^s_{ij} t'_{..s} = rt'_{ij} + \lambda_1 \sum_{\substack{j' \\ j' \neq j}} t'_{ij'} + \lambda_2 \sum_{\substack{i' \\ i' \neq i}} \sum_j t'_{i'j}, \tag{3.2.9}$$

and substitution in equation (3.2.8) gives

$$r\left[W + W'/(k-1)\right] \left(\frac{k-1}{k}\right) t'_{ij} - \frac{\lambda_1 (W-W')}{k} \sum_{\substack{j' \\ j' \neq j}} t'_{ij'} - \frac{\lambda_2 (W-W')}{k} \sum_{\substack{i' \\ i' \neq i}} \sum_j t'_{i'j} = WQ_{ij} + W'Q'_{ij}, \tag{3.2.10}$$

$i = 1,\ldots,m$; $j = 1,\ldots,n$, where Q_{ij} and Q'_{ij} are defined as by equations (2.1.8) and (2.1.12), respectively.

It is now clear that the solutions for equations (3.2.10) may be obtained from the solutions of equations (3.2.1) if we replace

$$Q_{ij} \text{ by } WQ_{ij} + W'Q'_{ij}, \tag{3.2.11}$$

$$r \text{ by } r\left[W + W'/(k-1)\right], \tag{3.2.12}$$

and

$$\lambda_i \text{ by } \lambda_i (W-W'), \tag{3.2.13}$$

where $i = 1, 2$.

We may also obtain the combined intra- and inter-block treatment estimators, equivalent to the treatment estimators obtained by Bose, Clatworthy, and Shrikhande [2] but in a different form, by solving the equations (3.2.10). Applying the condition that $\sum_{i}\sum_{j} t'_{ij} = 0$, it follows that

$$\sum_{\substack{i' \\ i' \neq i}} \sum_{j} t'_{i'j} = -\sum_{j} t'_{ij}. \tag{3.2.14}$$

Substituting (3.2.14) in (3.2.10), we obtain

$$\frac{W(rk-r+\lambda_2)+W'(r-\lambda_2)}{k}\, t'_{ij} + \frac{(W-W')(\lambda_2-\lambda_1)}{k} \sum_{\substack{j' \\ j' \neq j}} t'_{ij'} = WQ_{ij} + W'Q'_{ij}, \tag{3.2.15}$$

$i = 1,\ldots,m$; $j = 1,\ldots,n$. If we denote the nxn square symmetric matrix of the coefficients of the t'_{ij}'s in (3.2.15) by K, the diagonal elements of K are

$$\frac{W(rk-r+\lambda_2)+W'(r-\lambda_2)}{k} \tag{3.2.16}$$

and the non-diagonal elements are

$$\frac{(W-W')(\lambda_2-\lambda_1)}{k}. \tag{3.2.17}$$

Therefore in matrix notation we have

$$Kt'_i = WQ_i + W'Q'_i, \tag{3.2.18}$$

where t'_i is the column vector of elements t'_{ij} ($j = 1,\ldots,n$) and $WQ_i + W'Q'_i$ is the column vector of elements $WQ_{ij} + W'Q'_{ij}$.

Once the elements of the inverse of the matrix K are obtained, the solutions of equation (3.2.16) are given by

$$t_i' = K^{-1}(WQ_i+W'Q_i'). \tag{3.2.19}$$

The inverse of an nxn matrix with a's on the diagonal and b's elsewhere has diagonal elements given by

$$x = \frac{a+(n-2)b}{a\left[a+(n-2)b\right]-(n-1)b^2} \tag{3.2.20}$$

and non-diagonal elements

$$y = \frac{-b}{a\left[a+(n-2)b\right]-(n-1)b^2} . \tag{3.2.21}$$

Substituting (3.2.16) and (3.2.17) for a and b, respectively, in equations (3.2.20) and (3.2.21), we see that K^{-1} will have diagonal elements

$$\frac{A''W+B''W'}{C'W^2+D'WW'+E'W'^2} , \tag{3.2.22}$$

where

$$A'' = k(rk-r+\lambda_2) + k(n-2)(\lambda_2-\lambda_1),$$

$$B'' = k(r-\lambda_2) - k(n-2)(\lambda_2-\lambda_1),$$

$$C' = r(k-1)\left[rk-r+n(\lambda_2-\lambda_1)+2\lambda_1\right] + n\lambda_1\lambda_2 - (n-1)\lambda_1^2,$$

$$D' = 2r^2(k-1) + (rkn-2rk-2rn+4r)\lambda_1 + (2rn-rkn)\lambda_2 - 2n\lambda_1\lambda_2 + 2(n-1)\lambda_1^2,$$

and

$$E' = r^2 - nr\lambda_2 + r(n-2)\lambda_1 + n\lambda_1\lambda_2 - (n-1)\lambda_1^2,$$

and non-diagonal elements

$$\frac{-k(\lambda_2-\lambda_1)(W-W')}{C'W^2+D'WW'+E'W'^2} \quad . \tag{3.2.23}$$

From equation (3.2.19) it follows that

$$t'_{ij} = \frac{A''W+B''W'}{C'W^2+D'WW'+E'W'^2}\,(WQ_{ij} + W'Q'_{ij}) - \frac{k(\lambda_2-\lambda_1)(W-W')}{C'W^2+D'WW'+E'W'^2}\,\Big(W \sum_{\substack{j'\\ j'\neq j}} Q_{ij'} + W'\sum_{\substack{j'\\ j'\neq j}} Q'_{ij'}\Big) \quad . \tag{3.2.24}$$

Equation (3.2.24) may also be written in the form

$$t'_{ij} = \frac{A'W+B'W'}{C'W^2+D'WW'+E'W'^2}\,(WQ_{ij}+W'Q'_{ij}) - \frac{k(\lambda_2-\lambda_1)(W-W')}{C'W^2+D'WW'+E'W'^2}\,(W\sum_{j'} Q_{ij'}+W'\sum_{j'} Q'_{ij'}), \tag{3.2.25}$$

where

$$A' = k(rk-r+\lambda_2) + k(n-1)\lambda_2-\lambda_1)$$

and

$$B' = k(r-\lambda_2) - k(n-1)(\lambda_2-\lambda_1).$$

The values of the constants, A', B', C', D', and E', may be further reduced by using equation (3.1.1) so that equation (3.2.25) may be put in the form

$$t'_{ij} = \frac{AW+BW'}{CW^2+DWW'+EW'^2}\,(WQ_{ij}+W'Q'_{ij}) + \frac{k(\lambda_1-\lambda_2)(W-W')}{CW^2+DWW'+EW'^2}\,(W\sum_{j'} Q_{ij'}+W'\sum_{j'} Q'_{ij'}), \tag{3.2.26}$$

$i = 1,\ldots,m$; $j = 1,\ldots,n$, where

$$A = kv\lambda_2,$$
$$B = k(rk-v\lambda_2),$$
$$C = v\lambda_2(\lambda_1+rk-r),$$
$$D = rk(rk-v\lambda_2) - (r-\lambda_1)(rk-2v\lambda_2),$$

and

$$E = (r-\lambda_1)(rk-v\lambda_2).$$

The intra-block estimators may be easily obtained by setting $W = 1$ and $W' = 0$ in equation (3.2.26). Therefore

$$(3.2.27) \qquad t_{ij} = \frac{k}{\lambda_1+rk-r} Q_{ij} + \frac{k(\lambda_1-\lambda_2)}{v\lambda_2(\lambda_1+rk-r)} \sum_{j'} Q_{ij'} ,$$

$i = 1,\ldots,m$; $j = 1,\ldots,n$, which are identical to the results shown by Kramer and Bradley [14]. The inter-block estimators may be obtained by setting $W = 0$ and $W' = 1$ in equations (3.2.26). In this case

$$(3.2.28) \qquad t^*_{ij} = \frac{k}{(r-\lambda_1)} \left[Q'_{ij} - \frac{(\lambda_1-\lambda_2)}{(rk-v\lambda_2)} \sum_{j'} Q'_{ij'} \right] ,$$

$i = 1,\ldots,m$; $j = 1,\ldots,n$, which are equivalent to the results obtained by Zelen [26].

The combined intra- and inter-block estimators of the treatment effects given by equations (3.2.26) are equivalent to the estimators obtained by Bose, Clatworthy, and Shrikhande [2] but in a more convenient form, especially in situations where the required design has not been catalogued and the constants, Δ, H, c_1, and c_2, as defined in Chapter I, have not been tabulated.

To incorporate factorial treatment combinations in group divisible designs, consider two factors,[1] A and C, at m and n levels, respectively. The treatment V_{ij} has now become the factorial treatment combination of the i^{th} level of A with the j^{th} level of C. Following the procedure of Bradley and Kramer [14, 15], we take

$$\tau_{ij} = \alpha_i + \gamma_j + \delta_{ij} \tag{3.2.29}$$

where α_i, γ_j, and δ_{ij} represent the effects as defined in Chapter II. In view of equation (3.2.29) and the restrictions given by (2.1.19), the combined intra- and inter-block estimators for the factorial effects, obtained from equations (3.2.26), are

$$a_i' = \frac{1}{n} \sum_j t_{ij}' = \bar{t}_{i.}', \tag{3.2.30}$$

$$c_j' = \frac{1}{m} \sum_i t_{ij}' = \bar{t}_{.j}', \tag{3.2.31}$$

and

$$d_{ij}' = t_{ij}' - \bar{t}_{i.}' - \bar{t}_{.j}' . \tag{3.2.32}$$

The estimators for the factorial effects are most easily obtained from a two-way table of values of t_{ij}''s.

3.3 Variances and Covariances of the Treatment Estimators

To facilitate the mathematical computations, it will be convenient to write equations (3.2.26) in the form

1 The factorial factors, A and C, never appear in formulas and should not be confused with the constants, A and C, of formula (3.2.26).

$$t'_{ij} = pP_{ij} + q \sum_{\substack{j' \\ j' \neq j}} P_{ij'}, \tag{3.3.1}$$

$i = 1,\ldots,m;\ j = 1,\ldots,n$, where

$$p = \frac{AW+BW'+k(\lambda_1-\lambda_2)(W-W')}{CW^2+DWW'+EW'^2}, \tag{3.3.2}$$

$$q = \frac{k(\lambda_1-\lambda_2)(W-W')}{CW^2+DWW'+EW'^2}, \tag{3.3.3}$$

and

$$P_{ij} = WQ_{ij} + W'Q'_{ij}. \tag{3.3.4}$$

If it is desirable to use the symbols employed by Bose, Clatworthy, and Shrikhande [2], we can write

$$\theta = \frac{k-d_2}{r[W'+W(k-1)]}, \tag{3.3.5}$$

and

$$\phi = \frac{d_1-d_2}{r[W'+W(k-1)]}, \tag{3.3.6}$$

where

$$d_i = \frac{c_i\Delta+r\lambda_i Z}{\Delta+rHZ+r^2Z^2} \quad (i = 1,2) \tag{3.3.7}$$

and

$$Z = \frac{W'}{W-W'}. \tag{3.3.8}$$

For group divisible designs $p = \theta$ and $q = \phi$.

The variances and the covariances of the P_{ij}'s will be used to determine the variances and covariances of the t'_{ij}'s. Bose and Nair [3] have shown for the intra-block analysis that

(3.3.9) $$V(Q_{ij}) = \frac{r(k-1)}{k},$$

and

(3.3.10) $$Cov(Q_{ij},Q_{i'j'}) = \frac{-\lambda_u}{k}$$

where $u = 1$ or 2 if Q_{ij} and $Q_{i'j'}$ are first or second associates, respectively. Therefore, by making the substitutions (3.2.11), (3.2.12), and (3.2.13), we obtain

(3.3.11) $$V(P_{ij}) = \frac{r[W(k-1)+W']}{k},$$

(3.3.12) $$Cov(P_{ij},P_{ij'}) = \frac{-\lambda_1(W-W')}{k}, \quad j \neq j',$$

for first associates and

(3.3.13) $$Cov(P_{ij},P_{i'j'}) = \frac{-\lambda_2(W-W')}{k}, \quad i \neq i',$$

for second associates.

Bose, Clatworthy, and Shrikhande [2] have shown for the combined intra- and inter-block analysis that the variance of the difference between two treatment estimators which are first associates is

(3.3.14) $$V(t'_{ij}-t'_{ij'}) = 2(p-q).$$

Likewise the variance of the difference between two treatments which are second associates was shown to be

(3.3.15) $$V(t'_{ij}-t'_{i'j'}) = 2p.$$

From equation (3.3.1), the covariance of any two treatment estimators which are second associates is

$$(3.3.16) \qquad \mathrm{Cov}(t'_{ij},t'_{i'j'}) = \mathrm{Cov}(pP_{ij}+q\sum_{\substack{j'\\ j'\neq j}} P_{ij'},pP_{i'j'}+q\sum_{\substack{j''\\ j''\neq j'}} P_{i'j''})$$

$$= \mathrm{Cov}\left[(p-q)P_{ij}+q\sum_j P_{ij},(p-q)P_{i'j'}+q\sum_{j'} P_{i'j'}\right]$$

$$= (p-q)^2\mathrm{Cov}(P_{ij},P_{i'j'}) + q^2\mathrm{Cov}(\sum_j P_{ij},\sum_{j'} P_{i'j'})$$

$$- 2(p-q)q\mathrm{Cov}(P_{ij},\sum_{j'} P_{i'j'})$$

$$= (p-q)^2\mathrm{Cov}(P_{ij},P_{i'j'}) + n^2q^2\mathrm{Cov}(P_{ij},P_{i'j'})$$

$$- 2n(p-q)q\mathrm{Cov}(P_{ij},P_{i'j'})$$

$$= \left[p+(n-1)q\right]^2\mathrm{Cov}(P_{ij},P_{i'j'}).$$

From equation (3.3.13) we have

$$(3.3.17) \qquad \mathrm{Cov}(t'_{ij},t'_{i'j'}) = -\frac{\left[p+(n-1)q\right]^2\lambda_2(W-W')}{k}.$$

If we write equation (3.3.15) in the form

$$(3.3.18) \qquad V(t'_{ij}) - \mathrm{Cov}(t'_{ij},t'_{i'j'}) = p,$$

and substitute from equation (3.3.17), we obtain

$$(3.3.19) \qquad V(t'_{ij}) = p - c,$$

where

$$(3.3.20 \qquad c = \frac{\left[p+(n-1)q\right]^2\lambda_2(W-W')}{k}.$$

Similarly, from equations (3.3.14) and (3.3.19), the covariance of any two treatment estimators which are first associates is

$$(3.3.21) \qquad \mathrm{Cov}(t'_{ij},t'_{ij'}) = q - c.$$

Using equations (3.3.17), (3.3.18), (3.3.20), and (3.3.21), the variances and covariances of the combined intra- and inter-block estimators of the factorial effects may be derived. The variance of an A-factor estimator is given by

$$(3.3.22) \qquad V(a_i') = V(\bar{t}_{i.}') = \frac{1}{n^2} V(\sum_j t_{ij}')$$

$$= \frac{1}{n}V(t_{ij}') + \frac{(n-1)}{n}Cov(t_{ij}', t_{ij'}')$$

$$= \frac{1}{n}(p-c) + \frac{(n-1)}{n}(q-c)$$

$$= \frac{p+(n-1)q}{n} - c.$$

Also

$$(3.3.23) \qquad V(c_j') = V(\bar{t}_{.j}') = \frac{1}{m^2}V(\sum_i t_{ij}')$$

$$= \frac{1}{m}V(t_{ij}') + \frac{(m-1)}{m}Cov(t_{ij}', t_{i'j}')$$

$$= \frac{1}{m}(p-c) - \frac{(m-1)}{m}c$$

$$= \frac{p}{m} - c.$$

The covariances of the combined intra- and inter-block estimators for the A-factor are of the form

$$(3.3.24) \qquad Cov(a_i', a_{i'}') = Cov(\bar{t}_{i.}', \bar{t}_{i'.}')$$

$$= \frac{1}{n^2}Cov(\sum_j t_{ij}', \sum_j t_{i'j}')$$

$$= Cov(t_{ij}', t_{i'j}')$$

$$= -c,$$

$i \neq i'$. For the C-factor estimators we have

$$(3.3.25) \qquad \mathrm{Cov}(c'_j, c'_{j'}) = \mathrm{Cov}(\bar{t}'_{.j}, \bar{t}'_{.j'})$$

$$= \frac{1}{m^2}\mathrm{Cov}(\sum_i t'_{ij}, \sum_i t'_{ij'})$$

$$= \frac{1}{m}\mathrm{Cov}(t'_{ij}, t'_{ij'}) + \frac{(m-1)}{m}\mathrm{Cov}(t'_{ij}, t'_{i'j'})$$

$$= \frac{(q-c)}{m} - \frac{(m-1)}{m}c$$

$$= \frac{q}{m} - c,$$

$i \neq i'$, $j \neq j'$, and for the covariance of an A-factor estimator with a C-factor estimator we have

$$(3.3.26) \qquad \mathrm{Cov}(a'_i, c'_j) = \mathrm{Cov}(\bar{t}'_{i.}, \bar{t}'_{.j})$$

$$= \frac{1}{nm}\mathrm{Cov}(\sum_j t'_{ij}, \sum_i t'_{ij})$$

$$= \frac{1}{nm}V(t'_{ij}) + \frac{(m-1)}{m}\mathrm{Cov}(t'_{ij'}, t'_{i'j})$$

$$+ \frac{(n-1)}{nm}\mathrm{Cov}(t'_{ij'}, t'_{ij})$$

$$= \frac{(p-c)}{nm} - \frac{(m-1)}{m}c + \frac{(n-1)}{nm}(q-c)$$

$$= \frac{p+(n-1)q}{nm} - c, \quad j \neq j'.$$

The variances of the difference between two main factorial effects, obtained from equations (3.3.22), (3.3.23), (3.3.24), and (3.3.25), are

(3.3.27) $$V(a'_i - a'_{i'}) = 2V(a'_i) - 2Cov(a'_i, a'_{i'})$$

$$= \frac{2[p+(n-1)q]}{n}$$

$i \neq i'$, and

(3.3.28) $$V(c'_j - c'_{j'}) = 2V(c'_j) - 2Cov(c'_j, c'_{j'})$$

$$= 2\frac{(p-q)}{m}, \; j \neq j'.$$

The variance of the combined intra- and inter-block estimators of an interaction effect is given by

(3.3.29) $$V(d'_{ij}) = V(t'_{ij} - a'_i - c'_j)$$

$$= V(t'_{ij}) + V(a'_i) + V(c'_j) - 2Cov(t'_{ij}, a'_i)$$

$$- 2Cov(t'_{ij}, c'_j) + 2Cov(a'_i, c'_j).$$

Since

(3.3.30) $$Cov(t'_{ij}, a'_i) = Cov(t'_{ij}, \bar{t}'_{i.})$$

$$= \frac{1}{n}V(t'_{ij}) + \frac{(n-1)}{n}Cov(t'_{ij}, t'_{ij'}),$$

$j \neq j'$, and

(3.3.31) $$Cov(t'_{ij}, c'_j) = Cov(t'_{ij}, \bar{t}'_{.j})$$

$$= \frac{1}{m}V(t'_{ij}) + \frac{(m-1)}{m}Cov(t'_{ij}, t'_{i'j}),$$

$i \neq i'$, it follows from equations (3.3.19), (3.3.22), (3.3.23), and (3.3.26), that

(3.3.32) $$V(d'_{ij}) = \frac{(m-1)(n-1)(p-q)}{mn} + \frac{p+(n-1)q}{mn} - c.$$

All covariances between the combined intra- and inter-block estimators of the interaction effects which do not appear in the same row or column of the association matrix are of the form

$$(3.3.33)\qquad \begin{aligned} \mathrm{Cov}(d'_{ij},d'_{i'j'}) &= \mathrm{Cov}(t'_{ij}-a'_i-c'_j,\,t'_{i'j'}-a'_{i'}-c'_{j'}) \\ &= \mathrm{Cov}(t'_{ij},t'_{i'j'}) - \mathrm{Cov}(t'_{ij},a'_{i'}) - \mathrm{Cov}(t'_{ij},c'_{j'}) \\ &\quad - \mathrm{Cov}(a'_i,t'_{i'j'}) + \mathrm{Cov}(a'_i,a'_{i'}) + \mathrm{Cov}(a'_i,c'_{j'}) \\ &\quad - \mathrm{Cov}(c'_j,t'_{i'j'}) + \mathrm{Cov}(c'_j,a'_{i'}) + \mathrm{Cov}(c'_j,c'_{j'}), \end{aligned}$$

$i \neq i'$, $j \neq j'$. Since

$$(3.3.34)\qquad \mathrm{Cov}(t'_{ij},a'_{i'}) = \frac{1}{n}\mathrm{Cov}(t'_{ij},\sum_j t'_{i'j}) = \mathrm{Cov}(t'_{ij},t'_{i'j}),$$

and

$$(3.3.35)\qquad \begin{aligned} \mathrm{Cov}(t'_{ij},c'_{j'}) &= \frac{1}{m}\mathrm{Cov}(t'_{ij},\sum_i t'_{ij'}) \\ &= \frac{1}{m}\mathrm{Cov}(t'_{ij},t'_{ij'}) + \frac{(m-1)}{m}\mathrm{Cov}(t'_{ij},t'_{i'j'}), \end{aligned}$$

then by substituting (3.3.34) and (3.3.35) into (3.3.33) and using equations (3.3.17), (3.3.21), (3.3.24), (3.3.25), and (3.3.26), we have

$$(3.3.36)\qquad \mathrm{Cov}(d'_{ij},d'_{i'j'}) = \frac{[2p+(n-1)q]}{mn} - c,$$

$i \neq i'$, $j \neq j'$. By a similar approach we obtain

$$(3.3.37)\qquad \mathrm{Cov}(d'_{ij},d'_{ij'}) = -\frac{(n-2)(p-q)}{mn} + \frac{q}{m} - c,$$

Finally, all covariances arising from the estimators of A-factor effects with the estimators of interaction effects are given by

$$(3.3.39) \qquad \mathrm{Cov}(a_i',d_{ij}') = \mathrm{Cov}(a_i',t_{ij}'-a_i'-c_j')$$

$$= \mathrm{Cov}(a_i',t_{ij}') - V(a_i') - \mathrm{Cov}(a_i',c_j')$$

$$= -\frac{p+(n-1)q}{mn} + c.$$

Similarly,

$$(3.3.40) \qquad \mathrm{Cov}(c_j',d_{ij'}') = -\frac{p+(n-1)q}{mn} + c.$$

The weights W and W', associated with p, q, and c, are estimated from the analysis of variance table by the relations (2.2.4).

3.4 The Efficiency of Group Divisible Designs Relative to Completely Randomized Designs

In order to obtain the efficiency for contrasts among A-factor effects of group divisible designs relative to completely randomized designs, we must find the ratio of the variance of the difference between two A-factor effects for a completely randomized design to the variance of the difference between two A-factor effects for a group divisible design. For a completely randomized design

$$(3.4.1) \qquad V(a_i'-a_{i'}') = V\Big[\sum_{s}\sum_{j} \delta_{ij}^{s} y_{ijs} - \sum_{s}\sum_{j} \delta_{i'j}^{s} y_{i'js}\Big]/rn.$$

Substituting from equation (2.1.18) we have

$$(3.4.2) \qquad V(a_i'-a_{i'}') = \frac{1}{r^2n^2} V\left[\sum_s\sum_j \delta_{ij}^s(\beta_s+\epsilon_{ijs})\right]$$

$$+ \frac{1}{r^2n^2} V\left[\sum_s\sum_j \delta_{i'j}^s(\beta_s+\epsilon_{i'js})\right]$$

$$- \frac{2}{r^2n^2}\mathrm{Cov}\left\{\left[\sum_s\sum_j \delta_{ij}^s(\beta_s+\epsilon_{ijs})\right], \left[\sum_s\sum_j \delta_{i'j}^s(\beta_s+\epsilon_{i'js})\right]\right\}$$

$$= \frac{\left[2nr+2n(n-1)\lambda_1\right]\sigma_b^2+2nr\sigma^2}{n^2r^2}$$

$$- \frac{2n^2\lambda_2\sigma_b^2}{n^2r^2}$$

$$= \frac{2\left[(rk-v\lambda_2)\sigma_b^2+r\sigma^2\right]}{nr^2}$$

$i \neq i'$, by using equation (3.1.1).

If, in the equation (3.4.2), we set $W = 1/\sigma^2$ and W' $1/(\sigma^2+k\sigma_b^2)$, then we obtain

$$(3.4.3) \qquad V(a_i'-a_{i'}') = \frac{2\left[(rk-v\lambda_2)(W-W')+rkW'\right]}{nkr^2WW'}$$

$i \neq i$, for a completely randomized design. The variance of the difference between two A-factor effects for a group divisible design is given by equation (3.3.27).

The efficiency for an A-factor contrast is given now by

$$(3.4.4) \qquad E_A = \frac{(rk-v\lambda_2)(W-W')+rkW'}{kr^2WW'\left[p+(n-1)q\right]} \ .$$

Substituting for p and q from equations (3.3.2) and (3.3.3) we obtain

$$(3.4.5) \qquad E_A = \frac{\left[(rk-v\lambda_2)(\gamma-1)+rk\right](C\gamma^2+D\gamma+E)}{k^2r^2\gamma\left[(rk-r+\lambda_1)\gamma+(r-\lambda_1)\right]} \ ,$$

where $\gamma = W/W'$.

Similarly, to obtain the efficiency for contrasts among C-factor effects, we must find the ratio of

$$(3.4.6) \qquad \frac{2\left[(r-\lambda_1)\sigma_b^2+r\sigma^2\right]}{mr^2}$$

to the variance of the difference between C-factor effects for a group divisible design. From equation (3.3.28) and (3.4.6) we obtain

$$(3.4.7) \qquad E_C = \frac{(r-\lambda_1)(W-W')+rkW'}{kr^2WW'(p-q)} .$$

Substituting for p and q from equation (3.3.2) and (3.3.3) we obtain

$$(3.4.8) \qquad E_C = \frac{\left[(r-\lambda_1)(\gamma-1)+rk\right](C\gamma^2+D\gamma+E)}{k^2r^2\gamma(v\lambda_2\gamma+rk-v\lambda_2)} .$$

The efficiencies of group divisible designs relative to completely randomized designs are given in Table I for different values of γ.

3.5 Tests of Significance

If W and W' are known without error, then Rao [19] has shown that a test of the equality of treatment means for the combined intra- and inter-block analysis is based on the statistic

$$(3.5.1) \qquad \chi_T^2 = \sum_{i}\sum_{j} t'_{ij} P_{ij},$$

which can be used as a χ^2-variate with (v-1) degrees of freedom. The test can be used as an approximation if W and W' are estimated with a large number of degrees of freedom, as described in Chapter II. From equation (3.2.15) and the restrictions (2.1.19), equation (3.5.1) may be put in the form

$$(3.5.2) \qquad \chi^2_T = \left[W(rk-r+\lambda_1)+W'(r-\lambda_1)\right] \sum_i\sum_j t'^2_{ij}/k$$

$$- (W-W')(\lambda_1-\lambda_2)\sum_i(\sum_j t'_{ij})^2/k.$$

Tests of significance for the factorial effects are established in exactly the same way as described in Section 2.3. To test the null hypothesis of no A-effects for the combined intra- and inter-block analysis we use the statistic

$$(3.5.3) \qquad \chi^2_A = \sum_i\sum_j a'_i P_{ij},$$

which is approximately distributed as a χ^2-variate with (n-1) degrees of freedom. From equation (3.2.15) and the restrictions (2.1.19), equation (3.5.3) may likewise be put in the form

$$(3.5.4) \qquad \chi^2_A = \left[v\lambda_2 W+(rk-v\lambda_2)W'\right] \sum_i t'^2_{i.}/nk,$$

or

$$(3.5.5) \qquad \chi^2_A = \left[nv\lambda_2 W+n(rk-v\lambda_2)W'\right] \sum_i a'^2_i/k.$$

Similarly, to test the null hypothesis of no C-effects, we use the statistic

$$(3.5.6) \qquad \chi^2_C = \sum_i\sum_j c'_j P_{ij},$$

which may be written as

$$(3.5.7) \qquad \chi^2_C = \left[(rk-r+\lambda_1)W+(r-\lambda_1)W'\right] \sum_j t'^2_{.j}/mk,$$

or

$$(3.5.8) \qquad \chi^2_C = \left[m(rk-r+\lambda_1)W+m(r-\lambda_1)W'\right] \sum_j c'^2_j/k,$$

and is approximately distributed as a χ^2-variate with (n-1) degrees of freedom. Finally, the null hypothesis of no interaction effects can be tested by the statistic

$$(3.5.9) \qquad \chi^2_{AC} = \sum_i\sum_j d'_{ij}P_{ij},$$

which is approximately distributed as a χ^2-variate with (m-1)(n-1) degrees of freedom. From equation (3.2.15) and the restrictions (2.1.19), equation (3.5.9) may also be written as

$$(3.5.10) \qquad \chi^2_{AC} = \left[(rk-r+\lambda_1)W+(r-\lambda_1)W'\right]\sum_i\sum_j(t'_{ij}-\bar{t}'_{i\cdot}-\bar{t}'_{\cdot j})^2/k,$$

or

$$(3.5.11) \qquad \chi^2_{AC} = \left[(rk-r+\lambda_1)W+(r-\lambda_1)W'\right]\sum_i\sum_j d'^2_{ij}/k.$$

From equations (3.5.1), (3.5.3), (3.5.6), and (3.5.9), it is clear that

$$(3.5.12) \qquad \chi^2_T = \chi^2_A + \chi^2_C + \chi^2_{AC},$$

and the degrees of freedom add up to (v-1). Cochran's theorem [5] is sufficient to demonstrate the independence of all the χ^2-variates.

To test the significance of the difference between pairs of treatment estimators or factorial estimators, the t-test may be used as an approximation.

3.6 Individual Comparisons and Multi-factor Factorials

Individual or single-degree-of-freedom comparisons are obtained in the same way as in Section 2.4. Let ξ be an (m-1) by m orthogonal matrix, and η an (n-1) by n orthogonal matrix used to transform the a_i''s and c_j''s to (m-1) and (n-1) individual contrasts, respectively, each yielding an adjusted sum of squares with one degree of freedom. Contrasts on A-factor effects then would be

$$I_u = \sum_i \xi_{iu} a_i', \quad u = 1,\ldots,m-1, \tag{3.6.1}$$

and on C-factor effects

$$J_v = \sum_j \eta_{vj} c_j', \quad v = 1,\ldots,n-1. \tag{3.6.2}$$

To test the hypothesis that $\sum_i \xi_{iu}\alpha_i = 0$ against the hypothesis that $\sum_i \xi_{iu}\alpha_i \neq 0$, we use the statistic

$$\begin{aligned} \chi^2_{I_u} &= \left[nv\lambda_2 W + n(rk-v\lambda_2)W'\right](\sum_i \xi_{iu}\bar{t}_{i.}')^2 / k\sum_i \xi^2_{iu} \\ &= \left[v\lambda_2 W + (rk-v\lambda_2)W'\right](\sum_{ij} \xi_{iu} t_{ij}')^2 / k\sum_{ij} \xi^2_{iu}, \end{aligned} \tag{3.6.3}$$

which follows from equation (3.6.1) and the multiplier of equation (3.5.5).

Similarly, to test the hypothesis that $\sum_j \eta_{vj}\gamma_j = 0$ against the hypothesis that $\sum_j \eta_{vj}\gamma_j \neq 0$, we use the statistic

$$\begin{aligned} \chi^2_{J_v} &= \left[m(rk-r+\lambda_1)W + m(r-\lambda_1)W'\right](\sum_j \eta_{vj}\bar{t}_{.j}')^2 / k\sum_j \eta^2_{vj} \\ &= \left[(rk-r+\lambda_1)W + (r-\lambda_1)W'\right](\sum_{ij} \eta_{vj} t_{ij}')^2 / k\sum_{ij} \eta^2_{vj}, \end{aligned} \tag{3.6.4}$$

which follows from equation (3.6.2) and the multiplier of equation (3.5.8).

The adjusted interaction sum of squares also may be partitioned. The (m-1)(n-1) orthogonal contrasts for the interaction of I_u and J_v, obtained from the matrices ξ and η, are

$$(IJ)_{uv} = \sum_{i}\sum_{j} \xi_{iu}\eta_{vj}t'_{ij}. \tag{3.6.5}$$

To test the hypothesis that $\sum_{i}\sum_{j} \xi_{iu}\eta_{vj}\delta_{ij} = 0$ we use the statistic

$$\chi^2_{(IJ)_{uv}} = \left[(rk-r+\lambda_1)W+(r-\lambda_1)W'\right] \cdot \left(\sum_{i}\sum_{j} \xi_{iu}\eta_{vj}t'_{ij}\right)/k\sum_{i}\sum_{j}(\xi_{iu}\eta_{vj})^2, \tag{3.6.6}$$

which follows from equation (3.6.5) and the multiplier of equation (3.5.11).

From the manner in which we have constructed the single-degree-of-freedom contrasts, it is clear that the resulting sums of squares add up to the total adjusted sum of squares for treatments, which has been shown to be distributed as a χ^2-variate with (v-1) degrees of freedom. Since the degrees of freedom for the individual contrasts add up to (v-1), we may conclude by Cochran's theorem [5] that the corresponding sums of squares are independently distributed as χ^2-variates, each with one degree of freedom.

Special definition of the matrices, ξ and η, as in Section 2.4, permits the use of special contrasts for measuring trends over the factor levels. By taking the A- and C- factors to have levels, which themselves are factorial combinations, we can again extend the two-factor factorial to the case of multi-factor factorials or fractional factorials as in Section 2.4.

IV. LATIN SQUARE SUB-TYPE L_2 PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS

4.1 Properties of Latin Square Sub-type L_2 Designs

Bose, Clatworthy, and Shrikhande [2] list the following properties of Latin Square, sub-type L_2, designs:

(i) The design is non-group-divisible with n^2 treatments arranged in a square array of n rows and n columns.

(ii) Two treatments are first associates if they occur in the same row or column of the array and are second associates otherwise.

(iii) Each treatment has exactly $2(n-1)$ first associates and $(n-1)^2$ second associates.

(iv) The relations

$$P_1 = \begin{bmatrix} n-2 & n-1 \\ n-1 & (n-2)(n-1) \end{bmatrix} ,$$

$$P_2 = \begin{bmatrix} 2 & 2(n-2) \\ 2(n-2) & (n-2)^2 \end{bmatrix} ,$$

hold.

(v) The design parameters are related so that

$$2(n-1)\lambda_1 + (n-1)^2\lambda_2 = r(k-1), \tag{4.1.1}$$

or

$$rk - v\lambda_2 = r - 2\lambda_1 + \lambda_2 + 2n(\lambda_1-\lambda_2).$$

4.2 Combined Intra- and Inter-block Treatment Estimators

The combined intra- and inter-block estimators of the τ_{ij}'s in a model like (2.1.3) are obtained by minimizing the weighted sum of squares of deviations given by (2.1.10), subject to the condition that $\sum_{i}\sum_{j} \tau_{ij} = 0$ and the treatment-to-blocks assignments of the L_2 designs. The resulting normal equations, given by equations like (3.2.8), are

$$W\left(T_{ij} - \frac{B_{ij.}}{k} - rt'_{ij} + \sum_{s} \delta_{ij}^{s} \frac{t'_{..s}}{k}\right) + \frac{W'}{k}\left(B_{ij.} - rkm - \sum_{s} \delta_{ij}^{s} t'_{..s}\right) = 0, \tag{4.2.1}$$

$i, j = 1,\dots,n$, and, substituting

$$\sum_{s} \delta_{ij}^{s} t'_{..s} = rt'_{ij} + \lambda_1\left(\sum_{\substack{i' \\ i'\neq i}} t'_{i'j} + \sum_{\substack{j' \\ j'\neq j}} t'_{ij'}\right) + \lambda_2 \sum_{\substack{i' \\ i'\neq i}} \sum_{\substack{j' \\ j'\neq j}} t'_{i'j'}\,, \tag{4.2.2}$$

we obtain the equations

$$r\left[W + W'/(k-1)\right]\left(\frac{k-1}{k}\right)t'_{ij} - \frac{\lambda_1(W-W')}{k}\left(\sum_{\substack{i' \\ i'\neq i}} t'_{i'j} + \sum_{\substack{j' \\ j'\neq j}} t'_{ij'}\right) - \frac{\lambda_2(W-W')}{k} \sum_{\substack{i' \\ i'\neq i}} \sum_{\substack{j' \\ j'\neq j}} t'_{i'j'} = P_{ij}, \tag{4.2.3}$$

where $i, j = 1,\dots,n$, and P_{ij} retains its former definition.

Applying the condition that $\sum_{i}\sum_{j} t'_{ij} = 0$, we obtain

$$\sum_{\substack{i' \\ i'\neq i}} \sum_{\substack{j' \\ j'\neq j}} t'_{i'j'} = -\left(t'_{ij} + \sum_{\substack{i' \\ i'\neq i}} t'_{i'j} + \sum_{\substack{j' \\ j'\neq j}} t'_{ij'}\right). \tag{4.2.4}$$

Substituting (4.2.4) in (4.2.3) we obtain

(4.2.5)
$$\frac{W(rk-r+\lambda_2)+W'(r-\lambda_2)}{k}\, t'_{ij} + \frac{(W-W')(\lambda_2-\lambda_1)}{k}\,\Big(\sum_{\substack{i'\\ i'\neq i}} t'_{i'j} + \sum_{\substack{j'\\ j'\neq j}} t'_{ij'}\Big) = P_{ij},$$

$i,\ j = 1,\ldots,n$. Equations (4.2.5) can now be written as

(4.2.6)
$$\frac{W(rk-r+2\lambda_1-\lambda_2)+W'(r-2\lambda_1+\lambda_2)}{k}\, t'_{ij} - \frac{(W-W')(\lambda_1-\lambda_2)}{k}\,(t'_{i.}+t'_{.j}) = P_{ij},$$

$i,\ j = 1,\ldots,n$. The solutions of equations (4.2.6) were shown by Bose, Clatworthy, and Shrikhande [2] to be

(4.2.7)
$$t'_{ij} = \theta P_{ij} + \phi\Big(\sum_{\substack{i'\\ i'\neq i}} P_{i'j} + \sum_{\substack{j'\\ j'\neq j}} P_{ij'}\Big),$$

$i,\ j = 1,\ldots,n$, where θ and ϕ are defined by equations (3.3.5) and (3.3.6), respectively. It is important that we do not confuse θ and ϕ with p and q as defined by equations (3.3.2) and (3.3.3). The values for p and q may be obtained from equations (3.3.5) and (3.3.6) by using the relation (3.1.1) and, therefore, are only valid for group divisible designs.

To incorporate factorial treatment combinations in the sub-type L_2 designs of the Latin Square type designs, consider two factors, A and C, both at n levels. The treatment, V_{ij}, has now become the factorial treatment combination of the i^{th} level of A with the j^{th} level of C. As before we take

$$\tau_{ij} = \alpha_i + \gamma_j + \delta_{ij}, \tag{4.2.8}$$

where α_i, γ_j, and δ_{ij} represent the effects as defined in Chapter II, and the restrictions (2.1.19) are imposed. The combined estimators for the factorial effects will be the same as before, that is,

$$a'_i = \frac{1}{n} \sum_j t'_{ij} = \bar{t}'_{i.} \ , \tag{4.2.9}$$

$$c'_j = \frac{1}{n} \sum_i t'_{ij} = \bar{t}'_{.j} \ , \tag{4.2.10}$$

and

$$d'_{ij} = t'_{ij} - \bar{t}'_{i.} - \bar{t}'_{.j} \ . \tag{4.2.11}$$

The estimates are most easily obtained from a two-way table of values of t'_{ij}.

4.3 Variances and Covariances of the Treatment Estimators

The variances and covariances of the P_{ij}'s will be used to determine the variances and covariances of the t'_{ij}'s. From equation (3.3.11) we have

$$V(P_{ij}) = \frac{r\left[W(k-1)+W'\right]}{k} \ . \tag{4.3.1}$$

From equations (3.3.12) and (3.3.13) we have

$$\mathrm{Cov}(P_{ij},P_{ij'}) = \mathrm{Cov}(P_{ij},P_{i'j}) = -\frac{\lambda_1(W-W')}{k} \ , \tag{4.3.2}$$

for first associates, and

(4.3.3) $$\mathrm{Cov}(P_{ij},P_{i'j'}) = -\frac{\lambda_2(W-W')}{k},$$

$i' \neq i$, $j' \neq j$, for second associates. The general equations of Chapter III referenced here still apply.

Bose, Clatworthy, and Shrikhande [2] have shown for the combined intra- and inter-block analysis that the variance of the difference between two treatment estimators, when the treatments are first associates, is

(4.3.4) $$V(t'_{ij}-t'_{i'j}) = V(t'_{ij}-t'_{ij'}) = 2(\theta-\phi).$$

Likewise the variance of the difference between two treatments which are second associates was shown to be

(4.3.5) $$V(t'_{ij}-t'_{i'j'}) = 2\theta.$$

From equation (4.2.7) the covariance of any two treatment estimators which are second associates is

(4.3.6) $$\mathrm{Cov}(t'_{ij},t'_{i'j'}) = \mathrm{Cov}\Big[\theta P_{ij}+\phi\big(\sum_{\substack{i' \\ i'\neq i}} P_{i'j} + \sum_{\substack{j' \\ j'\neq j}} P_{ij'}\big), \theta P_{i'j'} + \phi\big(\sum_{\substack{i'' \\ i''\neq i'}} P_{i''j'} + \sum_{\substack{j'' \\ j''\neq j'}} P_{i'j''}\big)\Big]$$

$$= \mathrm{Cov}\Big[(\theta-2\phi)P_{ij}+\phi\big(\sum_i P_{ij}+\sum_j P_{ij}\big), (\theta-2\phi)P_{i'j'}+\phi\big(\sum_{i'} P_{i'j'}+\sum_{j'} P_{i'j'}\big)\Big]$$

$$= (\theta-2\phi)^2\mathrm{Cov}(P_{ij},P_{i'j'}) + \phi^2\mathrm{Cov}\big(\sum_i P_{ij},\sum_{i'} P_{i'j'}\big)$$

$$+ 2\phi^2\mathrm{Cov}\big(\sum_i P_{ij},\sum_{j'} P_{i'j'}\big) + \phi^2\mathrm{Cov}\big(\sum_j P_{ij},\sum_{j'} P_{i'j'}\big)$$

$$+ 2\phi(\theta-2\phi)\mathrm{Cov}\big(P_{ij},\sum_{i'} P_{i'j'}\big) + 2\phi(\theta-2\phi)\mathrm{Cov}\big(P_{i'j'},\sum_j P_{ij}\big)$$

$$= (\theta-2\phi)^2\mathrm{Cov}(P_{ij},P_{i'j'}) + \phi^2\sum_{i}\mathrm{Cov}(P_{ij},P_{ij'}) + \phi^2\sum_{i}\sum_{\substack{i'\\ i\neq i'}}\mathrm{Cov}(P_{ij},P_{i'j'})$$

$$+ 2\phi^2 V(P_{ij}) + 2\phi^2\sum_{\substack{j'\\ j'\neq j}}\mathrm{Cov}(P_{ij},P_{ij'}) + 2\phi^2\sum_{\substack{i'\\ i'\neq i}}\mathrm{Cov}(P_{ij},P_{i'j})$$

$$+ 2\phi^2\sum_{\substack{i'\\ i'\neq i}}\ \sum_{\substack{j'\\ j'\neq j}}\mathrm{Cov}(P_{i'j},P_{ij'}) + \phi^2\sum_{j}\mathrm{Cov}(P_{ij},P_{i'j})$$

$$+ \phi^2\sum_{j}\sum_{\substack{j'\\ j\neq j'}}\mathrm{Cov}(P_{ij},P_{i'j'}) + 2\phi(\theta-2\phi)\mathrm{Cov}(P_{ij},P_{ij'})$$

$$+ 2\phi(\theta-2\phi)\sum_{\substack{i'\\ i'\neq i}}\mathrm{Cov}(P_{ij},P_{i'j'}) + 2\phi(\theta-2\phi)\mathrm{Cov}(P_{ij},P_{i'j})$$

$$+ 2\phi(\theta-2\phi)\sum_{\substack{j'\\ j'\neq j}}\mathrm{Cov}(P_{ij},P_{i'j'})$$

$$= 2\phi V(P_{ij}) + (6n\phi^2+4\theta\phi-12\phi^2)\mathrm{Cov}(P_{ij},P_{ij'})$$

$$+ \left[(\theta-2\phi)^2 + 2(n-1)\phi(2n\phi+2\theta-5\phi)\right]\mathrm{Cov}(P_{ij},P_{i'j'}),$$

$i = i'$, $j \neq j'$. From equations (4.3.1), (4.3.2), and (4.3.3) we have

$$\mathrm{Cov}(t'_{ij},t'_{i'j'}) = \left\{2\phi^2 r\left[W(k-1)+W'\right]-(6n\phi^2+4\theta\phi-12\phi^2)\right.$$
$$\left.\cdot\ \lambda_1(W-W')-\left[(\theta-2\phi)^2+2(n-1)\phi(2n\phi+2\theta-5\phi)\right]\lambda_2(W-W')\right\}/k = d, \text{ say,} \tag{4.3.7}$$

$i \neq i'$, $j \neq j'$. If we write equation (4.3.5) in the form

$$V(t'_{ij}) - \mathrm{Cov}(t'_{i'j'}) = \theta, \tag{4.3.8}$$

and substitute from equation (4.3.7), we obtain

$$V(t'_{ij}) = \theta + d. \tag{4.3.9}$$

Similarly, from equations (4.3.4) and (4.3.9) the covariance of any two treatment estimators which are first associates is

(4.3.10) $$\mathrm{Cov}(t'_{ij}, t'_{ij'}) = \mathrm{Cov}(t'_{ij}, t'_{i'j}) = \phi + d,$$

$i \neq i'$, $j \neq j'$.

Using equations (4.3.7), (4.3.9), and (4.3.10), we may derive the variances and covariances of the combined intra- and inter-block estimators of the factorial effects. The variance of an A-factor estimator is given by

(4.3.11) $$\begin{aligned} V(a'_i) = V(\bar{t}'_{i.}) &= \frac{1}{n^2} V(\sum_j t'_{ij}) \\ &= \frac{1}{n} V(t'_{ij}) + \frac{(n-1)}{n} \mathrm{Cov}(t'_{ij}, t'_{ij'}) \\ &= \frac{\theta + d}{n} + \frac{(n-1)(\phi + d)}{n} \\ &= \frac{\theta + (n-1)\phi}{n} + d. \end{aligned}$$

Also

(4.3.12) $$\begin{aligned} V(c'_j) = V(\bar{t}'_{.j}) &= \frac{1}{n^2} V(\sum_i t'_{ij}) \\ &= \frac{1}{n} V(t'_{ij}) + \frac{(n-1)}{n} \mathrm{Cov}(t'_{ij}, t'_{i'j}) \\ &= \frac{\theta + (n-1)\phi}{n} + d. \end{aligned}$$

The covariances of the combined intra- and inter-block estimators for the A-factor are of the form

$$(4.3.13) \qquad \mathrm{Cov}(a_i', a_{i'}') = \mathrm{Cov}(\bar{t}_{i.}', \bar{t}_{i'.}')$$

$$= \frac{1}{n^2}\,\mathrm{Cov}(\sum_j t_{ij}', \sum_j t_{i'j}')$$

$$= \frac{1}{n}\,\mathrm{Cov}(t_{ij}', t_{i'j}') + \frac{(n-1)}{n}\,\mathrm{Cov}(t_{ij}', t_{i'j'}')$$

$$= \frac{\phi+d}{n} + \frac{(n-1)d}{n}$$

$$= \frac{\phi}{n} + d,$$

$i \neq i'$. For the C-factor estimators we have

$$(4.3.14) \qquad \mathrm{Cov}(c_j', c_{j'}') = \mathrm{Cov}(\bar{t}_{.j}', \bar{t}_{.j'}')$$

$$= \frac{1}{n^2}\,\mathrm{Cov}(\sum_i t_{ij}', \sum_i t_{ij'}')$$

$$= \frac{1}{n}\,\mathrm{Cov}(t_{ij}', t_{ij'}') + \frac{(n-1)}{n}\,\mathrm{Cov}(t_{ij}', t_{i'j'}')$$

$$= \frac{\phi}{n} + d,$$

$j \neq j'$, and, for the covariance of an A-factor estimator with a C-factor estimator, we have

$$(4.3.15)\qquad \mathrm{Cov}(a_i',c_j') = \mathrm{Cov}(\bar{t}_{i.}',\bar{t}_{.j}')$$

$$= \frac{1}{n^2}\mathrm{Cov}(\sum_j t_{ij}',\sum_i t_{ij}')$$

$$= \frac{1}{n^2}V(t_{ij}') + \frac{(n-1)}{n^2}\mathrm{Cov}(t_{ij}',t_{i'j}')$$

$$+ \frac{(n-1)}{n^2}\mathrm{Cov}(t_{ij'}',t_{ij}') + \frac{(n-1)^2}{n^2}\mathrm{Cov}(t_{ij'}',t_{i'j}')$$

$$= \frac{\theta+d}{n^2} + \frac{2(n-1)(\phi+d)}{n^2} + \frac{(n-1)^2 d}{n^2}$$

$$= \frac{\theta+2(n-1)\phi}{n^2} + d.$$

The variances of the difference between two main factorial effects, obtained from equations (4.3.11), (4.3.12), (4.3.13), and (4.3.14), are

$$(4.3.16)\qquad V(a_i'-a_{i'}') = 2V(a_i') - 2\mathrm{Cov}(a_i',a_{i'}')$$

$$= \frac{2[\theta+(n-2)\phi]}{n},$$

$i \neq i'$, and

$$(4.3.17)\qquad V(c_j'-c_{j'}') = 2V(c_j') - 2\mathrm{Cov}(c_j',c_{j'}')$$

$$= \frac{2[\theta+(n-2)\phi]}{n}, \quad j \neq j'.$$

The variance of the combined intra- and inter-block estimators of the interaction effect is given by

$$(4.3.18)\qquad V(d_{ij}') = V(t_{ij}'-a_i'-c_j')$$

$$= V(t_{ij}') + V(a_i') + V(c_j') - 2\mathrm{Cov}(t_{ij}',a_i')$$

$$- 2\mathrm{Cov}(t_{ij}',c_j') + 2\mathrm{Cov}(a_i',c_j').$$

Since

$$(4.3.19) \qquad \mathrm{Cov}(t'_{ij},a'_i) = \mathrm{Cov}(t'_{ij},\bar{t}'_{i.})$$

$$= \frac{1}{n} V(t'_{ij}) + \frac{(n-1)}{n} \mathrm{Cov}(t'_{ij},t'_{ij'}),$$

and

$$(4.3.20) \qquad \mathrm{Cov}(t'_{ij},c'_j) = \mathrm{Cov}(t'_{ij},\bar{t}'_{.j})$$

$$= \frac{1}{n} V(t'_{ij}) + \frac{(n-1)}{n} \mathrm{Cov}(t'_{ij},t'_{i'j}),$$

it follows from equations (4.3.9), (4.3.11), (4.3.12), and (4.3.15), that

$$(4.3.21) \qquad V(d'_{ij}) = \frac{(n-1)^2(\theta-2\phi)}{n^2} + \frac{\theta+2(n-1)\phi}{n^2} + d.$$

All covariances between the combined intra- and inter-block estimators of the interaction effects, which are second associates, are of the form

$$(4.3.22) \qquad \mathrm{Cov}(d'_{ij},d'_{i'j'}) = \mathrm{Cov}(t'_{ij}-a'_i-c'_j,t'_{i'j'}-a'_{i'}-c'_{j'})$$

$$= \mathrm{Cov}(t'_{ij},t'_{i'j'}) - \mathrm{Cov}(t'_{ij},a'_{i'})$$

$$- \mathrm{Cov}(t'_{ij},c'_{j'}) - \mathrm{Cov}(a'_i,t'_{i'j'})$$

$$+ \mathrm{Cov}(a'_i,a'_{i'}) + \mathrm{Cov}(a'_i,c'_{j'})$$

$$- \mathrm{Cov}(c'_j,t'_{i'j'}) + \mathrm{Cov}(c'_j,a'_{i'})$$

$$+ \mathrm{Cov}(c'_j,c'_{j'}),$$

$i \neq i'$, $j \neq j'$. Since

$$(4.3.23) \qquad \mathrm{Cov}(t'_{ij},a'_{i'}) = \frac{1}{n} \mathrm{Cov}(t'_{ij},\sum_j t'_{i'j})$$

$$= \frac{1}{n} \mathrm{Cov}(t'_{ij},t'_{i'j}) + \frac{(n-1)}{n} \mathrm{Cov}(t'_{ij},t'_{i'j'})$$

and

$$(4.3.24) \qquad \mathrm{Cov}(t'_{ij}, c'_{j'}) = \frac{1}{n}\mathrm{Cov}(t'_{ij}, \sum_i t'_{ij'})$$

$$= \frac{1}{n}\mathrm{Cov}(t'_{ij}, t'_{ij'}) + \frac{(n-1)}{n}\mathrm{Cov}(t'_{ij}, t'_{i'j'}),$$

then by substituting (4.3.23) and (4.2.24) into (4.3.22) and using equations (4.3.7), (4.3.10), (4.3.13), (4.3.14), and (4.3.15), we have

$$(4.3.25) \qquad \mathrm{Cov}(d'_{ij}, d'_{i'j'}) = \frac{2[\theta+2(n-1)\phi]}{n^2} - \frac{2\phi}{n} + d,$$

$i \neq i'$, $j \neq j'$. By a similar approach we obtain

$$(4.3.26) \qquad \mathrm{Cov}(d'_{ij}, d'_{ij'}) = \frac{2[\theta+2(n-1)\phi]}{n^2} - \frac{\phi}{n} + d,$$

$j \neq j'$, and

$$(4.3.27) \qquad \mathrm{Cov}(d'_{ij}, d'_{i'j}) = \frac{2[\theta+2(n-1)\phi]}{n^2} - \frac{\theta}{n} + d, \ i \neq i'.$$

Finally, all covariances arising from the estimators of A-factor effects with the estimators of interaction effects are given by

$$(4.3.28) \qquad \mathrm{Cov}(a'_i, d'_{ij}) = \mathrm{Cov}(a'_i, t'_{ij} - a'_i - c'_j)$$

$$= \mathrm{Cov}(a'_i, t'_{ij}) - V(a'_i) - \mathrm{Cov}(a'_i, c'_j)$$

$$= -\frac{\theta+2(n-1)\phi}{n^2} - d.$$

Similarly,

$$(4.3.29) \qquad \mathrm{Cov}(c'_j, d'_{ij'}) = -\frac{\theta+2(n-1)\phi}{n^2} - d.$$

The weights, W and W', associated with θ, ϕ, and d, are estimated from the analysis of variance table by the relations (2.2.4).

In order to obtain the efficiency for contrasts among A-factor effects of Latin Square, sub-type L_2 designs relative to completely randomized designs, we must find the ratio of

$$(4.3.30) \qquad \frac{2\left[r-\lambda_1+(n-1)(\lambda_1-\lambda_2)\right]\sigma_b^2+2r\sigma^2}{nr^2},$$

the appropriate variance of the difference between two A-factor effects for the completely randomized design found similarly to (3.4.2), to the variance of the difference between two A-factor effects in the incomplete block design. Substituting $W = 1/\sigma^2$ and $W' = 1/(\sigma^2+k\sigma_b^2)$ in (4.3.30) we obtain

$$(4.3.31) \qquad \frac{2\left[r-\lambda_1+(n-1)(\lambda_1-\lambda_2)\right](W-W')+2krW'}{nkr^2WW'}.$$

The efficiency for an A-factor contrast, obtained from (4.3.16) and (4.3.31), is then given by

$$(4.3.32) \qquad E_A = \frac{\left[r-\lambda_1+(n-1)(\lambda_1-\lambda_2)\right](W-W')+krW'}{kr^2WW'\left[\theta+(n-2)\phi\right]}.$$

Similarly, the efficiency for a C-factor contrast is shown to be

$$(4.3.33) \qquad E_C = \frac{\left[r-\lambda_1+(n-1)(\lambda_1-\lambda_2)\right](W-W')+krW'}{kr^2WW'\left[\theta+(n-2)\phi\right]}.$$

4.4 Tests of Significance

If W and W' are known without error as in Chapter III, a test of the equality of treatment means for the combined intra- and inter-block analysis is based on the statistic

(4.4.1) $$\chi^2_T = \sum_i\sum_j t'_{ij}P_{ij},$$

which can be used as a χ^2-variate with (v-1) degrees of freedom. The test, as described in Section 2.3, can be used as an approximation if W and W' are estimated with a large number of degrees of freedom. From equation (4.2.6) and the restrictions (2.1.19), equation (4.4.1) may be put in the form

(4.4.2) $$\chi^2_T = \left[W(rk-r+2\lambda_1-\lambda_2)+W'(r-2\lambda_1+\lambda_2)\right] \sum_i\sum_j t'^2_{ij}/k$$
$$- (W-W')(\lambda_1-\lambda_2)(\sum_i t'^2_{i\cdot}+\sum_j t'^2_{\cdot j})/k.$$

Tests of significance for the factorial effects are established in exactly the same way as described in Section 2.3. To test the null hypothesis of no A-effects for the combined intra- and inter-block analysis we use the statistic

(4.4.3) $$\chi^2_A = \sum_i\sum_j a'_i P_{ij},$$

which is approximately distributed as a χ^2-variate with (n-1) degrees of freedom. From equation (4.2.6) and the restrictions (2.1.19), equation (4.4.3) may likewise be put in the form

(4.4.4) $$\chi^2_A = \left\{\left[v\lambda_2+n(\lambda_1-\lambda_2)\right](W-W')+rkW'\right\} \sum_i t'^2_{i\cdot}/nk,$$

or

(4.4.5) $$\chi^2_A = \left\{\left[nv\lambda_2+n^2(\lambda_1-\lambda_2)\right](W-W')+nrkW'\right\} \sum_i a'^2_i/k.$$

Similarly, to test the null hypothesis of no C-effects we use the statistic

(4.4.6) $$\chi^2_C = \sum_i\sum_j c'_j P_{ij},$$

which may be written as

(4.4.7) $$\chi_C^2 = \left\{ \left[v\lambda_2 + n(\lambda_1 - \lambda_2) \right](W-W') + rkW' \right\} \sum_j t'^2_{\cdot j}/nk,$$

or

(4.4.8) $$\chi_C^2 = \left\{ \left[nv\lambda_2 + n^2(\lambda_1 - \lambda_2) \right](W-W') + nrkW' \right\} \sum_j c'^2_j/k,$$

and is approximately distributed as a χ^2-variate with (n-1) degrees of freedom. Finally, the null hypothesis of no interaction effects can be tested by the statistic

(4.4.9) $$\chi_{AC}^2 = \sum_i \sum_j d'_{ij} P_{ij},$$

which is approximately distributed as a χ^2-variate with $(n-1)^2$ degrees of freedom. From equation (4.2.6) and the restrictions (2.1.19), equation (4.4.9) may also be written as

(4.4.10) $$\chi_{AC}^2 = \left\{ \left[v\lambda_2 + 2n(\lambda_1 - \lambda_2) \right](W-W') + rkW' \right\} \sum_i \sum_j (t'_{ij} - \bar{t}'_{i\cdot} - \bar{t}'_{\cdot j})^2/k,$$

or

(4.4.11) $$\chi_{AC}^2 = \left\{ \left[v\lambda_2 + 2n(\lambda_1 - \lambda_2) \right](W-W') + rkW' \right\} \sum_i \sum_j d'^2_{ij}/k.$$

From equations (4.4.1), (4.4.3), (4.4.6), and (4.4.9) it is clear that

(4.4.12) $$\chi_T^2 = \chi_A^2 + \chi_C^2 + \chi_{AC}^2,$$

and the degrees of freedom add up to (v-1). Cochran's theorem [5] is sufficient to demonstrate the independence of all the χ^2-variates.

To test the significance of the difference between pairs of treatment estimators or factorial estimators the t-test may be used as an approximation.

4.5 Individual Comparisons and Multi-factor Factorials

Individual or single-degree-of-freedom comparisons are obtained in the same way as in Section 2.4. Let ξ and η be two (n-1) by n orthogonal matrices used to transform the a_i''s and c_j''s, respectively, to individual contrasts, each yielding an adjusted sum of squares with one degree of freedom. Contrasts on A-factor effects would then be

$$(4.5.1) \qquad I_u = \sum_i \xi_{iu} a_i', \ u = 1,\ldots,n-1,$$

and on C-factor effects

$$(4.5.2) \qquad J_v = \sum_j \eta_{vj} c_j', \ v = 1,\ldots,n-1.$$

To test the hypothesis that $\sum_i \xi_{iu}\alpha_i = 0$ against the hypothesis that $\sum_i \xi_{iu}\alpha_i \neq 0$ we use the statistic

$$(4.5.3) \qquad \chi^2_{I_u} = \left\{\left[nv\lambda_2+n^2(\lambda_1-\lambda_2)\right](W-W')+nrkW'\right\}\left(\sum_i \xi_{iu}\bar{t}_{i.}'\right)^2/k\sum_i \xi^2_{iu}$$

$$= \left\{\left[v\lambda_2+n(\lambda_1-\lambda_2)\right](W-W')+rkW'\right\}\left(\sum_i\sum_j \xi_{iu}t_{ij}'\right)^2/k\sum_i\sum_j \xi^2_{iu},$$

which follows from equation (4.5.1) and the multiplier of equation (4.4.5).

Similarly, to test the hypothesis that $\sum_j \eta_{vj}\gamma_j = 0$ against the hypothesis that $\sum_j \eta_{vj}\gamma_j \neq 0$ we use the statistic

$$(4.5.4) \qquad \chi^2_{J_v} = \left\{\left[nv\lambda_2+n^2(\lambda_1-\lambda_2)\right](W-W')+nrkW'\right\}\left(\sum_j \eta_{vj}\bar{t}_{.j}'\right)^2/k\sum_j \eta^2_{vj}$$

$$= \left\{\left[v\lambda_2+n(\lambda_1-\lambda_2)\right](W-W')+rkW'\right\}\left(\sum_i\sum_j \eta_{vj}t_{ij}'\right)^2/k\sum_i\sum_j \eta^2_{vj},$$

which follows from equation (4.5.2) and the multiplier of equation

(4.4.8).

The adjusted interaction sum of squares may also be partitioned. The $(n-1)^2$ orthogonal contrasts for the interaction of I_u and J_v, obtained from the matrices ξ and η, are

(4.5.5) $$(IJ)_{uv} = \sum_{i}\sum_{j} \xi_{iu}\eta_{vj}t'_{ij} \cdot$$

To test the hypothesis that $\sum_{i}\sum_{j} \xi_{iu}\eta_{vj}\delta_{ij} = 0$ we use the statistic

(4.5.6) $$\chi^2_{(IJ)_{uv}} = \left\{ \left[v\lambda_2 + 2n(\lambda_1 - \lambda_2) \right](W-W') + rkW' \right\} \cdot \left(\sum_{i}\sum_{j} \xi_{iu}\eta_{vj}t'_{ij}\right)^2 \Big/ k\sum_{i}\sum_{j}(\xi_{iu}\eta_{vj})^2,$$

which follows from equation (4.5.5) and the multiplier of equation (4.4.11).

From the manner in which we have constructed the single-degree-of-freedom contrasts, it is clear that the resulting sums of squares add up to the total adjusted sum of squares for treatments, which has been shown to be distributed as a χ^2-variate with (v-1) degrees of freedom. Since the degrees of freedom for the individual contrasts add up to (v-1), we may conclude by Cochran's theorem [5] that the corresponding sums of squares are independently distributed as χ^2-variates, each with one degree of freedom.

Special definition of the matrices, ξ and η, as in section 2.4, permits the use of special contrasts for measuring trends over the factor levels. By taking the A- and C-factors to have levels, which themselves are factorial combinations, we can again extend the two-factor factorial to the case of multi-factor factorials or fractional factorials as in Section 2.4.

The results of Section 4.4 and 4.5 reduce to the intra-block formulas, obtained by Kramer in an unpublished paper, if we set $W = 1$ and $W' = 0$.

V. LATIN SQUARE SUB-TYPE L_3 PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS

5.1 Properties of Latin Square Sub-type L_3 Designs

Bose, Clatworthy, and Shrikhande [2] list the following properties of Latin Square sub-type L_3 designs:

(i) The designs are non-group-divisible with n^2 treatments arranged in a square array of n rows and n columns, and upon this array is imposed a Latin Square with letters.

(ii) Any two treatments are first associates if they occur in the same row or column of the array or correspond to the same letter, and are second associates otherwise.

(iii) Each treatment has exactly $3(n-1)$ first associates and $(n-1)(n-2)$ second associates.

(iv) The relations

$$P_1 = \begin{bmatrix} n & 2(n-2) \\ 2(n-2) & (n-3)(n-2) \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 6 & 3(n-3) \\ 3(n-3) & n^2-6n+10 \end{bmatrix}.,$$

hold.

(v) The design parameters are related so that

$$3(n-1)\lambda_1 + (n-1)(n-2)\lambda_2 = r(k-1) \tag{5.1.1}$$

or

$$rk - v\lambda_2 = r - 3\lambda_1 + 2\lambda_2 + 3n(\lambda_1-\lambda_2).$$

Let V_{ij} denote the treatments in the association scheme given by the matrix V. Therefore, any two treatments are first associates if corresponding subscripts are the same (in the same row or column) or if they correspond to the same letter, and are second associates otherwise.

5.2 Factorial Treatment Estimators

The model assumed for the sub-type L_3 designs is

(5.2.1) $$y_{ijs} = \mu + \tau_{ij} + \beta_s + \epsilon_{ijs},$$

i, j = 1,...,n; s = 1,...,b, where y_{ijs} is the observation on treatment V_{ij} in block s if that treatment occurs in block s, μ is the grand mean, τ_{ij} is the effect of V_{ij}, β_s is the effect of block s, and ϵ_{ijs} is the usual normal random error with mean zero and variance σ^2, the various ϵ_{ijs}'s being independent. The estimators, m, t_{ij}, and b_s, of the parameters, μ, τ_{ij}, and β_s, are found by minimizing

(5.2.2) $$\sum_s\sum_i\sum_j \delta^s_{ij}(y_{ijs}-\mu-\tau_{ij}-\beta_s)^2 - 2\lambda\sum_i\sum_j \tau_{ij} - 2\Gamma\sum_s\beta_s,$$

where λ and Γ are Lagrange multipliers associated with the usual restraints on the parameters, namely

(5.2.3) $$\sum_i\sum_j \tau_{ij} = 0,$$

and

(5.2.4) $$\sum_s \beta_s = 0,$$

and $\delta^s_{ij} = 1$ if V_{ij} is in block s and zero otherwise. After partially differentiating (5.2.2) and evaluating the Lagrange multipliers, we obtain the equations

(5.2.5) $$m = \frac{G}{bk},$$

(5.2.6) $$b_s = \frac{B_s}{k} - m - \frac{t_{..s}}{k},$$

and

(5.2.7) $$T_{ij} - rm - rt_{ij} - b_{ij.} = 0,$$

where $t_{..s}$ is the sum of the t_{ij}'s in block s, $b_{ij.}$ is the sum of the b_s's for blocks containing V_{ij}. These equations were obtained by Kramer and Bradley [14, 15] and are completely general for all incomplete block designs. Summing equation (5.2.6) over blocks containing V_{ij}, we obtain

(5.2.8) $$b_{ij.} = \frac{1}{k} B_{ij.} - rm - \sum_s \delta_{ij}^s \frac{t_{..s}}{k},$$

where $B_{ij.}$ is the total of block totals for blocks containing treatment V_{ij}. Now

(5.2.9) $$\sum_s \delta_{ij}^s t_{..s} = rt_{ij} + \lambda_1 S_1(t_{ij}) + \lambda_2 S_2(t_{ij}),$$

where $S_1(t_{ij})$ is the sum of the t_{ij}'s of all the first associates of treatment V_{ij}, and $S_2(t_{ij})$ is the sum of the t_{ij}'s of all the second associates of treatment V_{ij}. Also, since $\sum_i\sum_j t_{ij} = 0$, then

(5.2.10) $$t_{ij} + S_1(t_{ij}) + S_2(t_{ij}) = 0.$$

Therefore

(5.2.11) $$\sum_s \delta_{ij}^s t_{..s} = (r-\lambda_1)t_{ij} + (\lambda_2-\lambda_1)S_2(t_{ij}),$$

and substituting (5.2.8) and (5.2.11) in (5.2.7), we obtain

(5.2.12) $$\frac{(rk-r+\lambda_1)}{k} t_{ij} - \frac{(\lambda_2-\lambda_1)}{k} S_2(t_{ij}) = Q_{ij}.$$

Equation (5.2.12) may be written in the form

(5.2.13) $$\frac{(rk-r+\lambda_2)}{k} t_{ij} + \frac{(\lambda_2-\lambda_1)}{k} S_1(t_{ij}) = Q_{ij},$$

$i, j = 1,\ldots,n$, where $Q_{ij} = T_{ij} - B_{ij.}/k$. Solutions of equations (5.2.13) were shown by Bose, Clatworthy, and Shrikhande [2] to be

(5.2.14) $$t_{ij} = \frac{k-c_2}{r(k-1)} Q_{ij} + \frac{(c_1-c_2)}{r(k-1)} S_1(Q_{ij}),$$

$i, j = 1,\ldots,n$, where c_1 and c_2 have been defined by equations (1.3.12) and (1.3.13), and $S_1(Q_{ij})$ represents the sum of the adjusted yields for all the first associates of treatment V_{ij}. Values of c_1 and c_2 are tabled with catalogued designs; they may also be obtained from (1.3.12) and (1.3.13). Occasionally, explicit formulas will be helpful, and we note them as follows:

$$c_1 = \frac{k\lambda_1(rk-r+\lambda_2)+k(\lambda_1-\lambda_2)\left[2(n-2)\lambda_2-3(n-3)\lambda_1\right]}{(rk-r+\lambda_1)(rk-r+\lambda_2)+(\lambda_1-\lambda_2)\left[(rk-r)(5-n)+2(n-2)\lambda_2-3(n-3)\lambda_1\right]}$$

$$c_2 = \frac{r\lambda_2(rk-r+\lambda_1)+k(\lambda_1-\lambda_2)\left[2(n-2)\lambda_2-3(n-3)\lambda_1\right]}{(rk-r+\lambda_1)(rk-r+\lambda_2)+(\lambda_1-\lambda_2)\left[(rk-r)(5-n)+2(n-2)\lambda_2-3(n-3)\lambda_1\right]}.$$

To incorporate factorials in Latin Square sub-type L_3 designs consider factors A and C both with n levels providing $v = n^2$ treatment combinations associated with the V_{ij} so that

(5.2.15) $$\tau_{ij} = \alpha_i + \gamma_j + \delta_{ij}$$

with the restrictions (2.1.19) imposed. The change to factorial parameters may be regarded simply as a one-to-one transformation in the parameter space. It follows that

$$t_{ij} = a_i + c_j + d_{ij}. \tag{5.2.16}$$

From equation (5.2.16) we note that

$$a_i = \frac{1}{n} \sum_j t_{ij} = \bar{t}_{i.}, \tag{5.2.17}$$

$$c_j = \frac{1}{n} \sum_i t_{ij} = \bar{t}_{.j}, \tag{5.2.18}$$

and

$$d_{ij} = t_{ij} - \bar{t}_{i.} - \bar{t}_{.j} . \tag{5.2.19}$$

The combined intra- and inter-block estimators of the treatment effects can be obtained by minimizing the weighted sum of squares of deviations given in the form (2.1.10), subject to the condition that $\sum_{ij}\sum \tau_{ij} = 0$. The resulting normal equations, as given by equations (3.2.8), are

$$W(T_{ij} - \frac{B_{ij.}}{k} - rt'_{ij} + \sum_s \delta^s_{ij} \frac{t'_{..s}}{k}) + \frac{W'}{k}(B_{ij.} - rkm - \sum_s \delta^s_{ij} t'_{..s}) = 0, \tag{5.2.20}$$

$i, j = 1,\ldots,n$. Replacing t_{ij} by t'_{ij} in equation (5.2.11) and substituting in (5.2.20), we obtain the equations

$$\frac{W(rk-r+\lambda_1)+W'(r-\lambda_1)}{k} t'_{ij} - \frac{(W-W')(\lambda_2-\lambda_1)}{k} S_2(t'_{ij}) = P_{ij}, \tag{5.2.21}$$

$i, j = 1,\ldots,n$. Equations (5.2.21) can be written also as

$$\frac{W(rk-r+\lambda_2)+W'(r-\lambda_2)}{k} t'_{ij} + \frac{(W-W')(\lambda_2-\lambda_1)}{k} S_1(t'_{ij}) = P_{ij}, \tag{5.2.22}$$

$i, j = 1,\ldots,n$.

Bose, Clatworthy, and Shrikhande [2] have obtained the solution of equations (5.2.22) for the combined intra- and inter-block treatment estimators in the form

$$(5.2.23) \qquad t'_{ij} = \theta P_{ij} + \phi S_1(P_{ij}),$$

i, j = 1,...,n, where θ and ϕ are defined by equations (3.3.5) and (3.3.6), and $S_1(P_{ij})$ represents the sum of the P_{ij}'s for all the first associates of treatment V_{ij}.

By considering equation (5.2.15) and imposing the restrictions (2.1.19), we obtain

$$(5.2.24) \qquad a'_i = \frac{1}{n} \sum_j t'_{ij} = \bar{t}'_{i.} \; ,$$

$$(5.2.25) \qquad c'_j = \frac{1}{n} \sum_i t'_{ij} = \bar{t}'_{.j} \; ,$$

and

$$(5.2.26) \qquad d'_{ij} = t'_{ij} - \bar{t}'_{i.} - \bar{t}'_{.j} \; ,$$

all of which are easily obtained from a two-way table of values of t'_{ij}'s.

If W and W' are known without error, then Rao [19] has shown that a test of the equality of treatment means for the combined intra- and inter-block analysis is based on the statistic

$$(5.2.27) \qquad \chi^2_T = \sum_{ij}\sum t'_{ij} P_{ij},$$

which can be used as a χ^2-variate with (v-1) degrees of freedom. The test, as described in Chapter II, can be used as an approximation if W and W' are estimated with a large number of degrees of freedom. From equation (5.2.21) and the restrictions (2.1.19), equation

(5.2.27) may be put in the form

$$\chi_T^2 = \frac{W(rk-r+\lambda_1)+W'(r-\lambda_1)}{k} \sum_{ij}\sum t'^2_{ij} \tag{5.2.28}$$

$$- \frac{(W-W')(\lambda_2-\lambda_1)}{k} \sum_{ij}\sum t'_{ij} S_2(t'_{ij}).$$

The adjusted treatment sum of squares for the intra-block analysis, obtained by setting $W' = 0$ and $W = 1$ in equation (5.2.28), is given by

$$\text{SST(adj.)} = \frac{(rk-r+\lambda_1)}{k} \sum_{ij}\sum t^2_{ij} - \frac{(\lambda_2-\lambda_1)}{k} \sum_{ij}\sum t_{ij} S_2(t_{ij}). \tag{5.2.29}$$

The unadjusted sum of squares for blocks is computed in the usual way. Each block total depends on k observations, and we have

$$\text{SSB(unadj.)} = \frac{1}{k} \sum_s B_s^2 - \frac{G^2}{rv}. \tag{5.2.30}$$

The error sum of squares, obtained by subtraction, is given by

$$\text{Error SS} = \text{Total SS} - \text{SST(adj.)} - \text{SSB(unadj.)}, \tag{5.2.31}$$

where

$$\text{Total SS} = \sum_{sij}\sum\sum \delta^s_{ij} y^2_{ijs} - \frac{G^2}{rv}. \tag{5.2.32}$$

The analysis of variance in Table 1 can now be set up and the F-test carried out as indicated.

Table 1. Intra-block Analysis of Variance

for the General Model

Source	d.f.	S.S.	M.S.	F
Treatments (adj.)	v-1	$\frac{(rk-r+\lambda_1)}{k} \sum_{i}\sum_{j} t_{ij}^2 - \frac{(\lambda_2-\lambda_1)}{k} \sum_{i}\sum_{j} t_{ij}S_2(t_{ij})$	s_T^2	s_T^2/s_E^2
Blocks (unadj.)	b-1	$\frac{1}{k} \sum_{s} B_s^2 - \frac{G^2}{rv}$		
Intra-block error	$\left[v(r-1)-b+1\right]$	By Subtraction	s_E^2	
Total	rv-1	$\sum_{s}\sum_{i}\sum_{j} \delta_{ij}^{s} y_{ijs}^2 - \frac{G^2}{rv}$		

Tests of significance for the factorial effects are not possible in the same way as described in Section 2.3. It is now impossible to partition the total adjusted treatment sum of squares into independent sums of squares corresponding to the various factors. In situations where it is impossible to partition the total treatment sum of squares, we can always use the variances and covariances of Section 5.3 to make tests on comparisons among the factorial estimators.

5.3 Variances and Covariances of the Treatment Estimators

To facilitate the mathematical computations it will be convenient to write equations (5.2.14) in the form

(5.3.1) $$t_{ij} = \alpha Q_{ij} + \beta S_1(Q_{ij}),$$

$i, j = 1,\ldots,n$, where

(5.3.2) $$\alpha = \frac{k-c_2}{r(k-1)},$$

and

(5.3.3) $$\beta = \frac{(c_1-c_2)}{r(k-1)}.$$

To obtain the variances and covariances of the intra-block estimators, we shall make use of equations (3.3.9) and (3.3.10). Bose, Clatworthy, and Shrikhande [2] have shown that the variance of the difference between two treatment estimators which are first associates are

(5.3.4) $$V(t_{ij}-t_{i'j'}) = 2(\alpha-\beta)\sigma^2.$$

Likewise, the variance of the difference between two treatments which are second associates was shown to be

$$(5.3.5) \qquad V(t_{ij}-t_{i'j'}) = 2\alpha\sigma^2.$$

If, in equation (5.3.1), we denote the sum of all Q_{ij}'s falling on the same letter, ρ, in the association scheme as Q_{ij} by $Q_{ij\rho}$, ρ = A, B, C, ..., the covariance of any two treatment estimators which are second associates is

$$(5.3.6) \qquad \begin{aligned} Cov(t_{ij},t_{i'j'}) &= Cov\left[\alpha Q_{ij}+\beta S_1(Q_{ij}),\alpha Q_{i'j'} + \beta S_1(Q_{i'j'})\right] \\ &= Cov\left[(\alpha-3\beta)Q_{ij}+\beta(Q_{.j}+Q_{i.}+Q_{ij\rho}),(\alpha-3\beta)Q_{i'j'} + \beta(Q_{.j'}+Q_{i'.}+Q_{i'j'\rho})\right] \\ &= 6\beta^2V(Q_{ij}) + \left[12(\alpha-3\beta)\beta+24n\beta^2-18\beta^2\right]Cov(Q_{ij},Q_{ij'}) \\ &\quad + \left[(\alpha-3\beta)^2+3\beta(n-2)\right] \\ &\quad \cdot \left[\beta(3n-2)+2(\alpha-3\beta)\right]Cov(Q_{ij},Q_{i'j'}). \end{aligned}$$

From equations (3.3.9) and (3.3.10), we have

$$(5.3.7) \qquad \begin{aligned} Cov(t_{ij},t_{i'j'}) &= \{6\beta^2r(k-1)-(12\alpha\beta+24n\beta^2-54\beta^2)\lambda_1 \\ &\quad - \left[(\alpha-3\beta)^2+3\beta(n-2)(3n\beta+2\alpha-8\beta)\right]\lambda_2\}\sigma^2/k = e\sigma^2, \end{aligned}$$

say, where t_{ij} and $t_{i'j'}$ are second associates. If we write equation (5.3.5) in the form

$$(5.3.8) \qquad V(t_{ij}) - Cov(t_{ij},t_{i'j'}) = \alpha\sigma^2,$$

and substitute from equation (5.3.7), we obtain

$$(5.3.9) \qquad V(t_{ij}) = (\alpha+e)\sigma^2.$$

Similarly, from equations (5.3.4) and (5.3.9), the covariance of any two treatment estimators which are first associates is

$$\text{(5.3.10)} \qquad \operatorname{Cov}(t_{ij}, t_{i'j'}) = (\beta + e)\sigma^2.$$

Using equations (5.3.7), (5.3.8), and (5.3.10), we may derive the variances and covariances of the intra-block estimators of the factorial effects. The variance of an A-factor estimator is given by

$$\text{(5.3.11)} \qquad \begin{aligned} V(a_i) = V(\bar{t}_{i.}) &= \frac{1}{n^2} V(\sum_j t_{ij}) \\ &= \frac{1}{n} V(t_{ij}) + \frac{(n-1)}{n} \operatorname{Cov}(t_{ij}, t_{ij'}) \\ &= \frac{(\alpha+e)}{n} \sigma^2 + \frac{(n-1)}{n} (\beta+e)\sigma^2 \\ &= \left[\frac{\alpha+(n-1)\beta}{n} + e\right] \sigma^2. \end{aligned}$$

Also

$$\text{(5.3.12)} \qquad \begin{aligned} V(c_j) = V(\bar{t}_{.j}) &= \frac{1}{n^2} V(\sum_i t_{ij}) \\ &= \frac{1}{n} V(t_{ij}) + \frac{(n-1)}{n} \operatorname{Cov}(t_{ij}, t_{i'j}) \\ &= \left[\frac{\alpha+(n-1)\beta}{n} + e\right] \sigma^2 . \end{aligned}$$

The covariances of the intra-block estimators for the A-factor are of the form

$$(5.3.13) \qquad \begin{aligned} \mathrm{Cov}(a_i, a_{i'}) &= \mathrm{Cov}(\bar{t}_{i.}, \bar{t}_{i'.}) \\ &= \frac{1}{n^2} \mathrm{Cov}(\sum_j t_{ij}, \sum_j t_{i'j}) \\ &= \frac{2}{n} \mathrm{Cov}(t_{ij}, t_{i'j}) + \frac{(n-2)}{n} \mathrm{Cov}(t_{ij}, t_{i'j'}) \\ &= \frac{2(\beta+e)}{n} \sigma^2 + \frac{(n-2)e}{n} \sigma^2 \\ &= \left[\frac{2\beta}{n} + e\right] \sigma^2, \end{aligned}$$

$i \neq i'$. Similarly, for C-factor estimators, we have

$$(5.3.14) \qquad \mathrm{Cov}(c_j, c_{j'}) = \left[\frac{2\beta}{n} + e\right]\sigma^2,$$

$j \neq j'$. For the covariance of an A-factor estimator with a C-factor estimator, we have

$$(5.3.15) \qquad \begin{aligned} \mathrm{Cov}(a_i, c_j) = \mathrm{Cov}(\bar{t}_{i.}, \bar{t}_{.j}) &= \frac{1}{n^2} \mathrm{Cov}(\sum_j t_{ij}, \sum_i t_{ij}) \\ &= \frac{1}{n^2} V(t_{ij}) + \frac{3(n-1)}{n^2} \mathrm{Cov}(t_{ij}, t_{i'j}) \\ &\quad + \frac{(n-1)(n-2)}{n^2} \mathrm{Cov}(t_{ij}, t_{i'j'}) \\ &= \left[\frac{\alpha+e}{n^2} + \frac{3(n-1)(\beta+e)}{n^2} + \frac{(n-1)(n-2)e}{n^2}\right] \sigma^2 \\ &= \left[\frac{\alpha+3(n-1)\beta}{n^2} + e\right] \sigma^2 . \end{aligned}$$

The variance of the difference between two main factorial effects, obtained from equations (5.3.11), (5.3.12), (5.3.13), and (5.3.14), is

$$(5.3.16) \qquad \begin{aligned} V(a_i - a_{i'}) &= 2V(a_i) - 2\mathrm{Cov}(a_i, a_{i'}) \\ &= \frac{2[\alpha+(n-3)\beta]}{n} \sigma^2, \end{aligned}$$

$i \neq i'$, and

$$(5.3.17) \qquad V(c_j - c_{j'}) = 2V(c_j) - 2\mathrm{Cov}(c_j, c_{j'})$$
$$= \frac{2\left[\alpha + (n-3)\beta\right]}{n}\,\sigma^2$$

$j \neq j'$.

The variance of the intra-block estimators of the interaction effects is given by

$$(5.3.18) \qquad V(d_{ij}) = V(t_{ij} - a_i - c_j)$$
$$= V(t_{ij}) + V(a_i) + V(c_j) - 2\mathrm{Cov}(t_{ij}, a_i)$$
$$- 2\mathrm{Cov}(t_{ij}, c_j) + 2\mathrm{Cov}(a_i, c_j).$$

Since

$$(5.3.19) \qquad \mathrm{Cov}(t_{ij}, a_i) = \mathrm{Cov}(t_{ij}, \bar{t}_{i.})$$
$$= \frac{1}{n}\,V(t_{ij}) + \frac{(n-1)}{n}\,\mathrm{Cov}(t_{ij}, t_{ij'})$$

and

$$(5.3.20) \qquad \mathrm{Cov}(t_{ij}, c_j) = \mathrm{Cov}(t_{ij}, a_i),$$

it follows from equations (5.3.9), (5.3.11), (5.3.12), and (5.3.15), that

$$(5.3.21) \qquad V(d_{ij}) = \left[\frac{n(\alpha-2\beta)-2(\alpha-4\beta)}{n} \quad \frac{2(\alpha-3\beta)}{n^2} + e\right]\sigma^2.$$

All the covariances between the intra-block estimators of the interaction effects, which are first associates, are of the form

$$\text{(5.3.22)} \qquad \begin{aligned} \mathrm{Cov}(d_{ij},d_{ij'}) &= \mathrm{Cov}(t_{ij}-a_i-c_j,t_{ij'}-a_i-c_{j'}) \\ &= \mathrm{Cov}(t_{ij},t_{ij'}) - \mathrm{Cov}(t_{ij},a_i) \\ &\quad - \mathrm{Cov}(t_{ij},c_j') - \mathrm{Cov}(a_i,t_{ij'}) \\ &\quad + V(a_i) + \mathrm{Cov}(a_i,c_{j'}) - \mathrm{Cov}(c_j,t_{ij'}) \\ &\quad + \mathrm{Cov}(c_j,a_i) + \mathrm{Cov}(c_j,c_{j'}). \end{aligned}$$

Since

$$\text{(5.3.23)} \qquad \begin{aligned} \mathrm{Cov}(a_i,t_{ij'}) &= \frac{1}{n}\mathrm{Cov}(\sum_j t_{ij},t_{ij'}) \\ &= \frac{1}{n}V(t_{ij'}) + \frac{(n-1)}{n}\mathrm{Cov}(t_{ij},t_{ij'}) \end{aligned}$$

and

$$\text{(5.3.24)} \qquad \begin{aligned} \mathrm{Cov}(t_{ij},c_{j'}) &= \frac{1}{n}\mathrm{Cov}(t_{ij},\sum_i t_{ij'}) \\ &= \frac{2}{n}\mathrm{Cov}(t_{ij},t_{ij'}) \\ &\quad + \frac{(n-2)}{n}\mathrm{Cov}(t_{ij},t_{i'j'}), \end{aligned}$$

then by substituting (5.3.23) and (5.3.24) into (5.3.22) and using equations (5.3.7), (5.3.10), (5.3.11), (5.3.14), and (5.3.15), we have

$$\text{(5.3.25)} \qquad \mathrm{Cov}(d_{ij},d_{ij'}) = \left[\frac{2\alpha+(5n-6)\beta}{n^2} - \frac{\alpha}{n} + e\right]\sigma^2,$$

$j \neq j'$. This result holds for all first associates. By a similar approach, we obtain

$$\text{(5.3.26)} \qquad \mathrm{Cov}(d_{ij},d_{i'j'}) = \left[\frac{2\alpha+2(n-3)\beta}{n^2} + e\right]\sigma^2$$

for second associates.

Finally, all covariances arising from the estimators of A-factor effects with the estimators of interaction effects are given by

$$(5.3.27) \qquad \mathrm{Cov}(a_i,d_{ij}) = \mathrm{Cov}(a_i,t_{ij}-a_i-c_j)$$

$$= \mathrm{Cov}(a_i,t_{ij}) - V(a_i) - \mathrm{Cov}(a_i,c_j)$$

$$= -\left[\frac{\alpha+3(n-1)\beta}{n^2} + e\right]\sigma^2 .$$

Similarly

$$(5.3.28) \qquad \mathrm{Cov}(c_j,d_{ij'}) = -\left[\frac{\alpha+3(n-1)\beta}{n^2} + e\right]\sigma^2 .$$

Corresponding formulas for the variances and covariances of the combined intra- and inter-block estimators of the factorial effects may be obtained by replacing α and β by θ and ϕ, respectively, in the above equations, where θ and ϕ are defined by (3.3.5) and (3.3.6) and omitting σ^2. The weights, W and W', associated with θ, ϕ, and e, are estimated from the analysis of variance table by the relations (2.2.8).

In order to obtain the efficiency for contrasts among A-factor effects of Latin Square sub-type L_3 designs relative to completely randomized designs, using the recovery of inter-block information, we must find the ratio of

$$(5.3.29) \qquad \frac{2\left[r-\lambda_1+(n-2)(\lambda_1-\lambda_2)\right]\sigma_b^2+2r\sigma^2}{nr^2},$$

the variance of the difference between two A-factor effects for the completely randomized design found similarly to (3.4.2), to the variance of the difference between two A-factor effects in the incomplete block design. Substituting $W = 1/\sigma^2$ and $W' = 1/(\sigma^2+k\sigma_b^2)$ in (5.3.29), we obtain

$$(5.3.30) \qquad \frac{2\left[r-\lambda_1+(n-2)(\lambda_1-\lambda_2)\right](W-W')+2krW'}{nkr^2WW'} .$$

The efficiency for an A-factor contrast, obtained from (5.3.16), with α and β replaced by Θ and ϕ, and (5.3.30), is then given by

$$(5.3.31) \qquad E_A = \frac{\left[r-\lambda_1+(n-2)(\lambda_1-\lambda_2)\right](W-W')+krW'}{kr^2WW'\left[\Theta+(n-3)\phi\right]} .$$

Similarly, the efficiency for a C-factor contrast is shown to be

$$(5.3.32) \qquad E_C = \frac{\left[r-\lambda_1+(n-2)(\lambda_1-\lambda_2)\right](W-W')+krW'}{kr^2WW'\left[\Theta+(n-3)\phi\right]} .$$

The efficiencies of Latin Square type designs relative to completely randomized designs are given in Table II for different values of γ, where $\gamma = W/W'$.

5.4 Analysis of a Particular Design with n = 4

For the special cases of factorials or fractional factorials consisting of sixteen treatment combinations, it is possible to partition the treatment sum of squares into independent sums of squares corresponding to the various factors and perform the usual tests of significance provided that we arrange the treatment combinations in the association scheme so as to preserve orthogonality. This may be done if we use a set of three orthogonal 4 x 4 Latin Squares superimposed. The levels of the one factor are represented by the elements of one square and the levels of the other factor are represented by the elements of the second square. The third square is used to designate the association scheme. The three squares may be written as follows:

$$\begin{array}{ccc} \text{I} & \text{II} & \text{III} \\ \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{bmatrix} \end{array} .$$

Replacing the numbers of the first Latin Square by the letters, A, B, C, and D, and superimposing the three squares, we obtain the completely orthogonalized square

$$(5.4.1) \qquad \begin{bmatrix} A_{11} & B_{22} & C_{33} & D_{44} \\ B_{43} & A_{34} & D_{21} & C_{12} \\ C_{24} & D_{13} & A_{42} & B_{31} \\ D_{32} & C_{41} & B_{14} & A_{23} \end{bmatrix} .$$

Let us consider a basic two-factor factorial consisting of factors A and C at four levels each. Denote the levels of A and C by the numbers of the second and third Latin Squares, respectively. The sixteen treatment combinations, $V_{ij}(i, j = 1,\ldots,4)$, are then obtained from (5.4.1) and may be displayed by the array

$$\begin{bmatrix} V_{11}A & V_{22}B & V_{33}C & V_{44}D \\ V_{43}B & V_{34}A & V_{21}D & V_{12}C \\ V_{24}C & V_{13}D & V_{42}A & V_{31}B \\ V_{32}D & V_{41}C & V_{14}B & V_{23}A \end{bmatrix} .$$

For example, V_{42} represents the treatment combination consisting of the fourth level of factor A and the second level of factor C.

A design using the above association scheme and having the parameters, $\lambda_1 = 0$, $\lambda_2 = 2$, $r = 6$, $k = 3$, $v = 16$, $b = 32$, $n_1 = 9$, and $n_2 = 6$, has the plan shown below.

PLAN

V_{11} V_{21} V_{31}	V_{22} V_{12} V_{42}	V_{33} V_{13} V_{23}	V_{43} V_{13} V_{23}
V_{11} V_{21} V_{41}	V_{22} V_{12} V_{32}	V_{33} V_{31} V_{32}	V_{43} V_{42} V_{41}
V_{11} V_{12} V_{13}	V_{22} V_{24} V_{23}	V_{44} V_{43} V_{42}	V_{34} V_{24} V_{14}
V_{11} V_{12} V_{14}	V_{22} V_{42} V_{32}	V_{44} V_{43} V_{41}	V_{34} V_{31} V_{32}
V_{11} V_{13} V_{14}	V_{33} V_{43} V_{13}	V_{44} V_{34} V_{24}	V_{21} V_{24} V_{23}
V_{11} V_{31} V_{41}	V_{33} V_{43} V_{23}	V_{44} V_{34} V_{14}	V_{21} V_{31} V_{41}
V_{22} V_{21} V_{24}	V_{33} V_{34} V_{31}	V_{44} V_{24} V_{14}	V_{12} V_{13} V_{14}
V_{22} V_{21} V_{23}	V_{33} V_{34} V_{32}	V_{44} V_{42} V_{41}	V_{12} V_{42} V_{32}

General details on the construction of partially balanced, incomplete block designs are given by Bose and Nair [3].

Using the above association scheme we can write the last part of equation (5.2.12) in the form

$$(5.4.2) \qquad \sum_{ij}\sum t_{ij}S_2(t_{ij}) = \sum_{ij}\sum t_{ij}(t_{i.}+t_{.j}-2t_{ij})$$

$$= \sum_i t_{i.}^2 + \sum_j t_{.j}^2 - 2\sum_{ij}\sum t_{ij}^2 \ .$$

Therefore, using the design parameters, we have

$$(5.4.3) \qquad SST(adj.) = \frac{2}{3}(8\sum_{ij}\sum t_{ij}^2 - \sum_i t_{i.}^2 - \sum_j t_{.j}^2) \ ,$$

and substituting from (5.2.16) for the t_{ij}'s we obtain

$$(5.4.4) \qquad SST(adj.) = \frac{32}{3}\sum_i a_i^2 + \frac{32}{3}\sum_j c_j^2 + \frac{16}{3}\sum_{ij}\sum d_{ij}^2 \ .$$

From general regression theory we find, in a manner similar to that of Kramer and Bradley, that

$$(5.4.5) \qquad SSA(adj.) = \sum_{ij}\sum a_i Q_{ij} \ ,$$

$$(5.4.6) \qquad SSC(adj.) = \sum_{ij}\sum c_j Q_{ij} \ ,$$

and

$$(5.4.7) \qquad SS(AC)(adj.) = \sum_{ij}\sum d_{ij} Q_{ij} \ .$$

From equation (5.2.12), it then follows that

$$(5.4.8) \qquad SSA(adj.) = \frac{32}{3}\sum_i a_i^2 \ ,$$

$$(5.4.9) \qquad SSC(adj.) = \frac{32}{3}\sum_j c_j^2 \ ,$$

and

$$(5.4.10) \qquad SS(AC)(adj.) = \frac{16}{3}\sum_{ij}\sum d_{ij}^2$$

with three, three, and nine degrees of freedom, respectively. The complete analysis of variance for the two-factor factorial is given in Table 2.

Table 2. Analysis of Variance for the Basic

Two-factor Factorial

Source of Variation	Degrees of Freedom	Sum of Squares
Treatments (adjusted)	15	$\frac{2}{3}(8\sum_{i}\sum_{j} t_{ij}^2 - \sum_{i} t_{i.}^2 - \sum_{j} t_{.j}^2)$
A-factor (adjusted)	3	$\frac{32}{3} \sum_{i} a_i^2$
C-factor (adjusted)	3	$\frac{32}{3} \sum_{j} c_j^2$
AC-interaction (adjusted)	9	$\frac{16}{3} \sum_{i}\sum_{j} d_{ij}^2$
Blocks (unadjusted)	31	$\frac{1}{3} \sum_{s} B_s^2 - \frac{G^2}{96}$
Error	49	By Subtraction

Using the methods of Section 4.5 it is possible to obtain individual or single-degree-of-freedom comparisons. Let ξ and η be two 3×4 matrices used to transform A- and C-factor effects, respectively. Contrasts on A-factor effects for the intra-block analysis would then be

(5.4.11) $$I_u = \sum_{i} \xi_{iu} a_i, \quad u = 1, 2, 3,$$

and on C-factor effects

(5.4.12) $$J_v = \sum_{j} \eta_{vj} c_j, \quad v = 1, 2, 3.$$

To test the hypothesis that $\sum_{i} \xi_{iu} \alpha_i = 0$ we use the adjusted sum of squares given by

$$(5.4.13) \qquad \text{Adj. } SS(I_u) = 32(\sum_i \xi_{iu}\bar{t}_{i.})^2/3\sum_i \xi_{iu}^2,$$

which follows from equation (5.4.11) and the multiplier of equation (5.4.8).

Similarly, to test the hypothesis that $\sum_j \eta_{vj}\gamma_j = 0$, we use the adjusted sum of squares given by

$$(5.4.14) \qquad \text{Adj. } SS(J_v) = 32(\sum_j \eta_{vj}\bar{t}_{.j})^2/3\sum_j \eta_{vj}^2,$$

which follows from equation (5.4.12) and the multiplier of equation (5.4.9).

The adjusted interaction sum of squares may also be partitioned. The nine orthogonal contrasts for the interaction of I_u and J_v, obtained from the matrices ξ and η, are

$$(5.4.15) \qquad (IJ)_{uv} = \sum_{ij}\sum \xi_{iu}\eta_{vj}t_{ij}.$$

To test the hypothesis that $\sum_{ij}\sum \xi_{iu}\eta_{vj}\delta_{ij} = 0$ we use the adjusted sum of squares given by

$$(5.4.16) \qquad \text{Adj. } SS(IJ)_{uv} = 16(\sum_{ij}\sum \xi_{iu}\eta_{vj}t_{ij})^2/3\sum_{ij}\sum(\xi_{iu}\eta_{vj})^2,$$

which follows from equation (5.4.15) and the multiplier of equation (5.4.10).

Cochran's theorem [5] is sufficient to demonstrate the independence of all adjusted sums of squares, each with one degree of freedom. All F-tests are effected using the error mean square of Table 2.

Corresponding results for the combined intra- and inter-block analysis are obtained in exactly the same way as described in Chapter IV. Suitable X^2-statistics for testing the hypotheses of no main factorial effects and no interaction effects for the basic two-factor factorial are then given by

$$X_A^2 = (32W-14W')\sum_i a_i'^2/3, \tag{5.4.17}$$

$$X_C^2 = (32W-14W')\sum_j c_j'^2/3, \tag{5.4.18}$$

and

$$X_{AC}^2 = (16W+2W')\sum_i\sum_j d_{ij}'^2/3, \tag{5.4.19}$$

with three, three, and nine degrees of freedom, respectively.

To test hypotheses on linear contrasts among the effects we use the statistics

$$X_{I_u}^2 = (32W-14W')(\sum_i \xi_{iu}\bar{t}'_{i.})^2/3\sum_i \xi_{iu}^2, \tag{5.4.20}$$

$$X_{J_v}^2 = (32W-14W')(\sum_j \eta_{vj}\bar{t}'_{.j})^2/3\sum_j \eta_{vj}^2 \tag{5.4.21}$$

and

$$X_{(IJ)_{uv}}^2 = (16W+2W')(\sum_i\sum_j \xi_{iu}\eta_{vj}t'_{ij})^2/3\sum_i\sum_j(\xi_{iu}\eta_{vj})^2, \tag{5.4.22}$$

which follow from equations (4.5.1), (4.5.2), (4.5.5), and the multipliers of equations (5.4.17), (5.4.18), and (5.4.19), respectively.

Special definition of the matrices ξ and η, as in Section 2.4, permits the use of special contrasts for measuring trends over the factor levels. By taking the A- and C-factors to have levels, which themselves are factorial combinations, we can again extend the two-factor factorial to the case of multi-factor factorials or fractional factorials as in Section 2.4.

5.5 Discussion

In Sections 5.1, 5.2, and 5.3 we have discussed the L_3-type designs with a basic two-factor factorial assigned to the treatments so that factor levels correspond with rows and columns of the Latin Square association scheme. There V_{ij} is the treatment in the i^{th} row and j^{th} column of the Latin Square, and no cognizance is taken of the letters of the Latin Square in assigning factorial treatments through taking $V_{ij} = A_iC_j$. This basically seems awkward and the association of factorials with treatments of Section 5.4 was tried.

In Section 5.4 the use of three orthogonal 4 x 4 Latin Squares led to a simpler partition of the adjusted treatment sum of squares into components for A-factor, C-factor, and AC-interaction effects. In that section V_{ij} had subscripts corresponding to elements in two of the three orthogonal Latin Squares, a different association of subscripts from that of the earlier sections of this chapter. In developing this new association of factorials to treatments it was hoped that a general scheme for use of factorials in L_3-type designs would result . It turned out that the new scheme did produce a satisfactory analysis in the 4 x 4 case but did not result in any improvements or simplifications in general over the assignment of factorials used in Sections 5.1 to 5.4.

VI. NUMERICAL EXAMPLES

6.1 A Group Divisible Design

We shall illustrate the results developed in Chapter III by considering the plan for the design R27 as catalogued by Bose, Clatworthy, and Shrikhande [2]. The association scheme is given by

$$(6.1.1)\qquad \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \\ v_{41} & v_{42} & v_{43} \\ v_{51} & v_{52} & v_{53} \end{bmatrix} ,$$

from which the treatment combinations for a 5 x 3 factorial may be obtained and designated by

$$(6.1.2)\qquad \begin{bmatrix} A_1C_1 & A_1C_2 & A_1C_3 \\ A_2C_1 & A_2C_2 & A_2C_3 \\ A_3C_1 & A_3C_2 & A_3C_3 \\ A_4C_1 & A_4C_2 & A_4C_3 \\ A_5C_1 & A_5C_2 & A_5C_3 \end{bmatrix} .$$

The values $A_i(i = 1,\ldots,5)$ and $C_j(j = 1,\ldots,3)$, represent the levels of factors, A and C, respectively. The design parameters are $v = 15$, $r = 4$, $k = 4$, $b = 15$, $m = 5$, $n = 3$, $\lambda_1 = 0$, and $\lambda_2 = 1$.

The block plan and yields are given and analyzed by Bose, Clatworthy, and Shrikhande [2] for varietal trials. Assuming that their treatments actually came from factorial treatment combinations as given by (6.1.2), we can use many of their computations to illustrate our theory.

We tabulate the combined intra- and inter-block estimates of the treatment effects, as computed by Bose, Clatworthy, and Shrikhande, and the corresponding row and column totals, in Table 3.

The values of w and w', obtained from equations (2.2.8) by Bose, Clatworthy, and Shrikhande, are 10.7411 and 3.6350, respectively. Therefore, from (3.5.2),

$$\chi_T^2 = \left[W(rk-r+\lambda_1)+W'(r-\lambda_1)\right]\sum_i\sum_j t'^2_{ij}/k$$

$$- (W-W')(\lambda_1-\lambda_2)\sum_i(\sum_j t'_{ij})^2/k$$

and substitution of the design parameters and values in Table 3 yields

$$\chi_T^2 = 21.0345.$$

Also, using (3.5.4) and (3.5.7), we find

$$\chi_A^2 = \left[v\lambda_2 W+(rk-v\lambda_2)W'\right]\sum_i t'^2_{i.}/nk$$

$$= 4.3490,$$

and

$$\chi_C^2 = \left[(rk-r+\lambda_1)W+(r-\lambda_1)W'\right]\sum_j t'^2_{.j}/mk$$

$$= 2.1523,$$

with four and two degrees of freedom, respectively, both of which are insignificant at the five per cent level.

Table 3. Values of t'_{ij}

(Combined Intra- and Inter-block Estimates)

t'_{11} 0.0898	t'_{12} 0.1754	t'_{13} 0.1660	$t'_{1\cdot}$ 0.4312
t'_{21} -0.3785	t'_{22} 0.0570	t'_{23} 0.3225	$t'_{2\cdot}$ 0.0010
t'_{31} -0.2806	t'_{32} 0.0441	t'_{33} -0.1213	$t'_{3\cdot}$ -0.3578
t'_{41} 0.0547	t'_{42} 0.1527	t'_{43} -0.2376	$t'_{4\cdot}$ -0.0302
t'_{51} 0.0689	t'_{52} -0.2366	t'_{53} 0.1241	$t'_{5\cdot}$ -0.0436
$t'_{\cdot 1}$ -0.4457	$t'_{\cdot 2}$ 0.1926	$t'_{\cdot 3}$ 0.2537	

By subtraction, or (3.5.10), we have

$$\chi^2_{AC} = 14.5332$$

with eight degrees of freedom, which is also insignificant at the five per cent level.

To estimate the variance of the difference between two factorial effects, we require p and q in (3.3.2) and (3.3.3) based on A, B, C, D, and E following (3.2.26). We find

$$p = 0.0291,$$

and

$$q = -0.0012.$$

Referring back to (3.3.27) and (3.3.28) we now have

$$V(a_i'-a_{i'}') = \frac{2[p+(n-1)q]}{n} = 0.0178,$$

and

$$V(c_j'-c_{j'}') = \frac{2(p-q)}{m} = 0.0303.$$

Variance and covariances of the factorial estimators themselves may be easily obtained, if needed, from the definitions in Chapter III.

6.2 A Latin Square Sub-type L_2 Design

We shall illustrate the results of Chapter IV by considering the design LS12 as catalogued by Bose, Clatworthy, and Shrikhande [2]. It was necessary to make up the observations in order to indicate how the theory applies. For this reason no importance should be placed on the results of the analysis which may or may not indicate what would happen in an actual experiment.

The design LS12, with parameters $v = 16$, $r = 7$, $k = 4$, $b = 28$, $n_1 = 6$, $n_2 = 9$, $\lambda_1 = 2$, and $\lambda_2 = 1$, has the following scheme:

$$\begin{bmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{bmatrix}.$$

Recall that treatments in the same row or column are now first associates and are second associates otherwise. If we consider two

Table 4. Field Plan

Block	Treatments and yields				Block totals
1	V_{11} 33	V_{14} 38	V_{34} 36	V_{44} 48	155
2	V_{11} 36	V_{12} 47	V_{22} 35	V_{32} 47	165
3	V_{11} 31	V_{13} 42	V_{21} 30	V_{41} 43	146
4	V_{42} 44	V_{43} 46	V_{41} 46	V_{22} 41	177
5	V_{44} 50	V_{41} 47	V_{43} 43	V_{24} 35	175
6	V_{33} 42	V_{34} 35	V_{32} 46	V_{44} 51	174
7	V_{13} 43	V_{12} 44	V_{43} 44	V_{42} 41	172
8	V_{32} 45	V_{33} 43	V_{31} 32	V_{43} 42	162
9	V_{43} 40	V_{44} 48	V_{42} 40	V_{23} 41	169
10	V_{22} 37	V_{23} 38	V_{21} 33	V_{33} 40	148
11	V_{14} 40	V_{13} 44	V_{33} 45	V_{43} 42	171

Table 4 - continued

Block	Treatments and yields				Block totals
12	V_{14}	V_{12}	V_{24}	V_{44}	
	41	49	37	50	177
13	V_{31}	V_{32}	V_{34}	V_{42}	
	30	46	35	41	152
14	V_{24}	V_{21}	V_{23}	V_{31}	
	33	30	38	27	128
15	V_{13}	V_{11}	V_{23}	V_{43}	
	45	37	40	39	161
16	V_{14}	V_{11}	V_{21}	V_{31}	
	40	33	33	32	138
17	V_{12}	V_{11}	V_{31}	V_{41}	
	48	35	29	45	157
18	V_{41}	V_{42}	V_{44}	V_{21}	
	43	38	48	31	160
19	V_{21}	V_{22}	V_{24}	V_{32}	
	33	37	35	48	153
20	V_{34}	V_{31}	V_{33}	V_{41}	
	32	27	39	45	143
21	V_{13}	V_{14}	V_{24}	V_{34}	
	45	39	40	39	163
22	V_{12}	V_{14}	V_{22}	V_{42}	
	48	41	40	42	171

Table 4 - continued

Block	Treatments and yields				Block totals
23	V_{23} 39	V_{24} 33	V_{22} 38	V_{34} 33	143
24	V_{12} 48	V_{13} 45	V_{23} 40	V_{33} 44	177
25	V_{11} 37	V_{24} 35	V_{33} 45	V_{42} 44	161
26	V_{12} 48	V_{21} 33	V_{34} 31	V_{43} 39	151
27	V_{13} 42	V_{22} 38	V_{31} 27	V_{44} 49	156
28	V_{14} 40	V_{23} 42	V_{32} 45	V_{41} 45	172

Table 5. Values of T_{ij} and $B_{ij.}$

T_{11} 242	T_{12} 332	T_{13} 306	T_{14} 279
T_{21} 223	T_{22} 266	T_{23} 278	T_{24} 248
T_{31} 204	T_{32} 321	T_{33} 298	T_{34} 241
T_{41} 314	T_{42} 290	T_{43} 291	T_{44} 344

$B_{11.}$ 1083	$B_{12.}$ 1170	$B_{13.}$ 1146	$B_{14.}$ 1147
$B_{21.}$ 1024	$B_{22.}$ 1113	$B_{23.}$ 1098	$B_{24.}$ 1100
$B_{31.}$ 1036	$B_{32.}$ 1150	$B_{33.}$ 1136	$B_{34.}$ 1081
$B_{41.}$ 1130	$B_{42.}$ 1162	$B_{43.}$ 1166	$B_{44.}$ 1166

factors, A and C, at four levels each, then again the treatment V_{ij} of the association scheme represents the treatment combination of the i^{th} level of factor A and the j^{th} level of factor C.

The field plan, Table 4, shows the block numbers, treatments occurring in the blocks and their yields, and the block totals. To evaluate the estimators t_{ij}, and then t'_{ij}, it will be convenient to set up Table 5 giving treatment totals T_{ij} and values of $B_{ij.}$, the total of block totals for blocks containing V_{ij}.

From Table 5 we calculate $Q_{ij} = T_{ij} - B_{ij.}/k$. Values of Q_{ij} are given in Table 9. The intra-block estimators of the τ_{ij}'s are then obtained from the results of Bose, Clatworthy, and Shrikhande [2], by the formula

$$t_{ij} = [22Q_{ij}+S_1(Q_{ij})]/120.$$

These values, along with their row and column totals, are given in Table 6.

Table 6. Values of t_{ij} (Intra-block Estimates)

t_{11} -5.3146	t_{12} 7.2729	t_{13} 3.7417	t_{14} -1.2000	$t_{1\cdot}$ 4.5000
t_{21} -6.7833	t_{22} -2.1125	t_{23} 0.3146	t_{24} -5.1688	$t_{2\cdot}$ -13.7500
t_{31} -10.1833	t_{32} 5.7792	t_{33} 2.3313	t_{34} -5.2771	$t_{3\cdot}$ -7.3499
t_{41} 5.2313	t_{42} 1.1104	t_{43} 0.9125	t_{44} 9.3458	$t_{4\cdot}$ 16.6000
$t_{\cdot 1}$ -17.0499	$t_{\cdot 2}$ 12.0500	$t_{\cdot 3}$ 7.3001	$t_{\cdot 4}$ -2.3001	

Setting W = 1 and W' = 0 in formulas (4.4.2), (4.4.4), and (4.4.7), we obtain the intra-block analysis:

$$SST(adj.) = \frac{1}{4}(24\sum_{ij}\sum t_{ij}^2 - \sum_i t_{i\cdot}^2 - \sum_j t_{\cdot j}^2) = 2509.1539,$$

$$SSA(adj.) = \frac{5}{4}\sum_i t_{i\cdot}^2 = 673.6169,$$

and

$$SSC(adj.) = \frac{5}{4}\sum_j t_{\cdot j}^2 = 618.1044.$$

By subtraction

$$SS(AC)(adj.) = 1217.4326.$$

In the usual way, we have

$$SSB(unadj.) = \frac{1}{4}\sum_s B_s^2 - \frac{G^2}{112} = 1152.1696,$$

and

$$\text{Total SS} = \sum_{s}\sum_{i}\sum_{j} \delta_{ij}^2 y_{ijs}^2 - \frac{G^2}{112} = 3796.9196.$$

We summarize these results in Table 7.

Table 7. Intra-block Analysis of Variance
(Design LS12)

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square
Treatments	15	2509.1539	167.2769
A-factor (adj.)	3	673.6169	224.5389
C-factor (adj.)	3	618.1044	206.0348
AC-interaction (adj.)	9	1217.4326	135.2703
Blocks (unadj.)	27	1152.1696	
Error	69	135.5961	1.9652
Total	111	3796.9196	

To estimate the weights w and w' we must form the auxiliary table for inter-block analysis of variance. Therefore, we need the unadjusted sum of squares for treatments which is given by

$$\text{SST(unadj.)} = \frac{1}{7} \sum_{i}\sum_{j} T_{ij}^2 - \frac{G^2}{112} = 3438.9196.$$

Using the error and total sum of squares from Table 7 we obtain, by subtraction, the sum of squares for blocks adjusted. The results are listed in Table 8.

Table 8. Auxiliary Table for Inter-block

Analysis of Variance

(Design LS12)

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square
Treatments (unadj.)	15	3438.9196	
Blocks (adj.)	27	222.4039	B=8.2372
Error	69	135.5961	E=1.9652
Total	111	3796.9196	

We now find that

$$w = \frac{1}{E} = 0.5089$$

and

$$w' = \frac{bk-v}{k(b-1)B-(v-k)E} = 0.1108.$$

Using w and w' as estimates of W and W' in formula (3.3.7) we then have

$$d_1 = \frac{c_1\Delta+r\lambda_1 Z}{\Delta+rHZ+r^2Z^2} = 0.2518$$

and

$$d_2 = \frac{c_1\Delta+r\lambda_1 Z}{\Delta+rHZ+r^2Z^2} = 0.1668$$

where Z, Δ, H, c_1, and c_2 are defined in (3.3.8), (1.3.10), (1.3.11), (1.3.12), and (1.3.13), respectively.

To obtain the combined intra- and inter-block treatment estimates, we use (4.2.7), which reduces, for this design, to the form

$$t'_{ij} = 0.3344P_{ij} + 0.00742\Big(\sum_{\substack{i' \\ i' \neq i}} P_{i'j} + \sum_{\substack{j' \\ j' \neq j}} P_{ij'} \Big),$$

where $P_{ij} = WQ_{ij} + W'Q'_{ij}$, and $Q'_{ij} = B_{ij.}/k - rG/bk$. Table 9 gives the estimated values for Q'_{ij}, P_{ij}, t'_{ij}, and $S_1(P_{ij})$, where

$$S_1(P_{ij}) = \sum_{\substack{i' \\ i' \neq i}} P_{i'j} + \sum_{\substack{j' \\ j' \neq j}} P_{ij'}.$$

Table 9. Combined Intra- and Inter-block Estimates of the Treatment Effects (Design LS12)

Treatment	Q_{ij}	Q'_{ij}	P_{ij}	$S_1(P_{ij})$	t'_{ij}
V_{11}	-28.7500	-9.0625	-15.6350	-4.4030	-5.2610
V_{12}	39.5000	12.6875	21.5073	4.2766	7.2238
V_{13}	19.5000	6.6875	10.6645	12.5186	3.6591
V_{14}	-7.7500	6.9375	-3.1753	14.3306	-0.9555
V_{21}	-33.0000	-23.8125	-19.4321	-49.0906	-6.8623
V_{22}	-12.2500	-1.5625	-6.4072	7.8238	-2.0845
V_{23}	3.5000	-5.3125	1.1925	-16.0492	0.2797
V_{24}	-27.0000	-4.8125	-14.2735	-15.7548	-4.8900
V_{31}	-55.0000	-20.8125	-30.2955	-9.1954	-10.1990
V_{32}	33.5000	7.6875	17.8999	-22.6220	5.8179
V_{33}	14.0000	4.1875	7.5886	-15.4430	2.4230
V_{34}	-29.2500	-9.5625	-15.9449	5.7564	-5.2893
V_{41}	31.5000	2.6875	16.3281	-35.3802	5.1976
V_{42}	-0.5000	10.6875	0.9297	78.3808	0.8925
V_{43}	-0.5000	11.6875	1.0405	64.7156	0.8281
V_{44}	52.5000	11.6875	28.0122	-15.0954	9.2553

The values of the t'_{ij}'s, along with their row and column totals, are given in Table 10.

Table 10. Values of t'_{ij}

(Combined Intra- and Inter-block Estimates)

t'_{11} -5.2610	t'_{12} 7.2238	t'_{13} 3.6591	t'_{14} -0.9555	$t'_{1\cdot}$ 4.6664
t'_{21} -6.8623	t'_{22} -2.0845	t'_{23} 0.2797	t'_{24} -4.8900	$t'_{2\cdot}$ -13.5571
t'_{31} -10.1990	t'_{32} 5.8179	t'_{33} 2.4230	t'_{34} -5.2893	$t'_{3\cdot}$ -7.2474
t'_{41} 5.1976	t'_{42} 0.8925	t'_{43} 0.8281	t'_{44} 9.2553	$t'_{4\cdot}$ 16.1735
$t'_{\cdot 1}$ -17.1247	$t'_{\cdot 2}$ 11.8497	$t'_{\cdot 3}$ 7.1899	$t'_{\cdot 4}$ -1.8795	

Using equations (4.4.2), (4.4.4), (4.4.7), and (4.4.10) we obtain

$$\chi^2_T = 1340.7979,$$

$$\chi^2_A = 359.3568,$$

$$\chi^2_G = 338.0729,$$

and

$$\chi^2_{AC} = 643.3682,$$

with fifteen, three, three, and nine degrees of freedom, respectively, all of which are highly significant.

The variances of the various estimators may be determined from the proper formulas of Chapter IV, if required.

Suppose we take the A-factor to be a quantitative one and investigate the linear, quadratic, and cubic trends. The trend coefficients together with the sums of squares of the coefficients are given in Table 11.

Table 11. Trend Coefficients for Subdivision of X^2_A

Contrasts	Coefficients for $\bar{t}'_{1.}$ = 1.1666	$\bar{t}'_{2.}$ = -3.3893	$\bar{t}'_{3.}$ = -1.8119	$\bar{t}'_{4.}$ = 4.0434	Sums of Squared Coefficients
Linear A	-3	-1	+1	+3	20
Quadratic A	+1	-1	-1	+1	4
Cubic A	-1	+3	-3	+1	20

It now follows from equation (4.5.3)

$$X^2(\text{Linear A}) = \frac{44.2576}{4(20)} \left[(-3)(1.666) + \ldots + (3)(4.0434) \right]^2$$

$$= 57.6451.$$

Similarly

$$X^2(\text{Quad. A}) = 299.8261$$

and

$$X^2(\text{Cubic A}) = 2.2490.$$

If we assume the 4 x 4 factorial is now a 4 x 2^2 factorial by taking the levels of A to be made up of two levels of a factor N and two levels of a factor P, we can carry out the analysis in exactly the same way as described in Section 2.4.

6.3 A Latin Square Sub-type L_3 Design

Suppose we consider the design and block plan described in Section 5.4. The sixteen treatment combinations for the two factorial factors are given by the association scheme in Section 5.4. Once these treatment combinations have been properly assigned to the blocks according to the field plan the analysis is carried out in exactly the same way as for the previous example. To make the computations as simple as possible it is important for the design of Section 5.4 that the treatment estimates be arranged as in Tables 6 and 10 and not according to the association scheme.

Appropriate formulas of Chapter V yield the variances of the estimators. The extension to multi-factor factorials and the consideration of individual contrasts is carried out in the same manner as for the previous example.

For all other types of L_3 designs we obtain the factorial estimates from the formulas of Section 5.2 and make tests on comparisons among the factorial estimates by using the variances and covariances of Section 5.3.

VII. SUMMARY

The work of Kramer and Bradley has been extended to permit both the intra-block and combined intra- and inter-block analysis of factorials in balanced incomplete block designs and several classes of partially balanced incomplete block designs. In particular, we have obtained a combined intra- and inter-block analysis for factorials in balanced incomplete block designs, group divisible designs, and Latin Square type of partially balanced, incomplete block designs. For the class of Latin Square sub-type L_3 designs, both the intra-block and combined intra- and inter-block analyses have been considered. The only partially balanced incomplete block designs, catalogued by Bose, Clatworthy, and Shrikhande [2], that have not been considered in this dissertation are those whose treatment numbers are prime, such as the group of cyclic designs and those which must be treated individually rather than as a complete class.

Except for the special cases of 4 x 4 Latin Square sub-type L_3 designs, factorial treatment combinations were assigned to the association schemes by permitting the rows to represent the levels of one factor and the columns to represent the levels of a second factor. The extension to multi-factor factorials was then carried out by subdividing the levels of the basic two-factor factorial, the subdivisions representing the levels of the additional factors. This is possible only if the number of levels for the basic two-factor factorial is non-prime.

Estimators for the factorial effects have been obtained along with their variances and covariances. Sums of squares in terms of the factorial estimators have been derived and can be used to carry out tests of significance. These sums of squares were shown, for the combined intra- and inter-block analysis, to be independently distributed as X^2-variates with the appropriate number of degrees of freedom. Suitable sums of squares for tests of significance on the factorial effects are not possible in general for the Latin Square sub-type L_3 designs. In situations such as these, we can only consider contrasts among the estimates and use their variances to perform tests of significance.

For the special cases of 4 x 4 Latin Square sub-type L_3 designs, a complete analysis yielding the adjusted sums of squares, for the factorial effects, is possible if the factorial treatments are applied to the association scheme in a different manner. Using three orthogonal Latin Squares we obtained a satisfactory analysis by letting the levels of one factor be represented by the elements of one square and the levels of the second factor by the elements of the second square. The third square designates the association scheme.

Single-degree-of-freedom contrasts are obtained in much the usual way as in complete block designs. Main effects may be divided into trend contrasts and also interaction sum of squares may be partitioned. These partitions may be effected by using row and column averages or other appropriate functions of the original treatment estimators. The appropriate sums of squares for main effects and interactions of a multi-factor factorial are obtained either as functions of the original estimators or as functions of the row and column averages of the original treatment estimators. The method of incorporating a fractional replicate of a factorial is also considered.

Numerical examples have been worked in detail for a group divisible design and a Latin Square sub-type L_3 design. Using these two examples as a guide we can perform a combined intra- and inter-block analysis for factorials in balanced incomplete block designs. These examples also serve as guides for the special analysis of a 4×4 Latin Square sub-type L_3 design once the estimates of the t_{ij}'s are obtained and properly arranged in a two-way table so that row and column totals represent sums over first and second subscripts, respectively.

The problem of analyzing factorials in various types of lattice designs is being investigated at the present time. However, there are certain types of lattice designs which can be classified with the partially balanced incomplete block designs discussed in this dissertation. For example, the near balance rectangular lattices and the Latinized rectangular lattices belong to a subclass of the group divisible designs. Also, simple lattices may be classified as Latin Square sub-type L_2 designs while triple lattices belong to the sub-type L_3 class of Latin Square, partially balanced, incomplete block designs.

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IX. APPENDIX

Table I. Efficiencies of Group Divisible Designs Relative to Completely Randomized Designs[1,2]

Design No.	E_A: $\gamma = W/W'$ = 1	2	3	5	7	10
S1	100	109	125	160	196	252
S2	100	109	125	160	196	252
S3	100	109	125	160	196	252
S4	100	109	125	160	196	252
S5	100	109	125	160	196	252
S6	100	111	130	171	214	280
S7	100	105	113	132	151	180
S8	100	111	130	171	214	280
S9	100	105	113	132	151	180
S10	100	111	130	171	214	280
S11	100	105	113	132	151	180
S12	100	109	125	160	196	252
S13	100	109	125	160	196	252
S14	100	109	125	160	196	252
S15	100	109	125	160	196	252
S16	100	109	125	160	196	252
S17	100	112	131	175	220	290
S18	100	103	108	119	130	147
S19	100	107	119	144	171	213
S20	100	112	131	175	220	290
S21	100	103	108	119	130	147
S22	100	109	125	160	196	252
S23	100	111	130	171	214	280
S24	100	105	113	132	151	180
S25	100	109	125	160	196	252
S26	100	112	132	177	223	294
S27	100	108	121	151	182	230
S28	100	102	105	112	120	131
S29	100	111	130	171	214	280
S30	100	109	125	160	196	252
S31	100	105	113	132	151	180
S32	100	109	125	160	196	252
S33	100	111	130	171	214	280
S34	100	105	113	132	151	180
S35	100	112	132	177	223	294
S36	100	108	121	151	182	230

Table I - continued

Design No.	E_A $\gamma = W/W'$ 1	2	3	5	7	10
S37	100	109	125	160	196	252
S38	100	105	112	129	146	173
S39	100	102	105	112	120	131
S40	100	109	123	155	189	240
S41	100	105	115	135	156	189
S42	100	112	132	178	225	297
S43	100	109	123	155	189	240
S44	100	105	115	135	156	189
S45	100	109	123	155	189	240
S46	100	109	125	160	196	252
S47	100	112	131	175	220	290
S48	100	109	125	160	196	252
S49	100	107	119	144	171	213
S50	100	109	125	160	196	252
S51	100	112	131	175	220	290
S52	100	109	125	160	196	252
S53	100	109	125	160	196	252
S54	100	111	130	171	214	280
S55	100	111	130	171	214	280
S56	100	112	133	178	226	298
S57	100	106	116	139	163	199
S58	100	111	130	171	214	280
S59	100	109	125	160	196	252
S60	100	112	132	177	223	294
S61	100	108	121	151	182	230
S62	100	112	133	179	227	299
S63	100	109	125	160	196	252
S64	100	107	118	142	168	207
S65	100	112	132	177	223	294
S66	100	108	121	151	182	230
S67	100	105	112	129	146	173
S68	100	111	130	171	214	280
S69	100	112	131	175	220	290
S70	100	107	119	144	171	213
S71	100	111	130	171	214	280
S72	100	112	131	175	220	290
S73	100	112	133	179	227	300
S74	100	110	126	161	199	256
S75	100	111	130	171	214	280
S76	100	105	113	132	151	180
S77	100	109	123	155	189	240

Table I - continued

Design No.	E_A, $\gamma = W/W'$ = 1	2	3	5	7	10
S78	100	112	132	178	225	297
S79	100	109	123	155	189	240
S80	100	109	123	155	189	240
S81	100	105	114	134	154	186
S82	100	112	133	179	227	300
S83	100	105	114	134	154	186
S84	100	112	132	177	223	294
S85	100	112	133	178	226	298
S86	100	112	132	177	223	294
S87	100	112	131	175	220	290
S88	100	112	131	175	220	290
S89	100	108	120	149	178	223
S90	100	110	127	164	203	263
S91	100	108	120	149	178	223
S92	100	109	125	160	196	252
S93	100	112	133	179	227	299
S94	100	109	125	160	196	252
S95	100	112	132	178	225	297
S96	100	112	132	177	223	294
S97	100	110	127	165	205	265
S98	100	112	133	179	227	300
S99	100	110	126	161	199	256
S100	100	112	132	177	223	294
S101	100	108	121	151	182	230
S102	100	112	133	178	226	298
S103	100	108	121	151	182	230
S104	100	112	133	179	227	300
S105	100	112	132	178	225	297
S106	100	112	133	179	227	299
S107	100	110	128	167	207	269
S108	100	110	127	164	203	263
S109	100	112	133	178	226	298
S110	110	112	133	179	227	300
S111	100	107	118	143	169	209
S112	100	111	128	167	208	270
S113	100	107	118	143	169	209
S114	100	112	133	179	227	300
S115	100	110	127	165	205	265
S116	100	112	133	179	227	299
S117	100	107	119	144	171	213
S118	100	109	123	155	188	238

Table I - continued

Design No.	E_A $\gamma = W/W'$ 1	2	3	5	7	10
S119	100	112	133	179	227	300
S120	100	112	133	179	227	300
S121	100	109	123	155	189	240
S122	100	110	128	167	207	269
S123	100	111	128	167	208	270
S124	100	107	120	147	176	220

1 Note, that for the singular subclass of group divisible designs, $E_C = 1$ for all γ.

Table I - continued

Design No.	E_C $\gamma = W/W'$ 1	2	3	5	7	10
SR1	100	111	130	171	214	280
SR2	100	111	130	171	214	280
SR3	100	111	130	171	214	280
SR4	100	105	115	135	156	189
SR5	100	111	130	171	214	280
SR6	100	111	130	171	214	280
SR7	100	109	125	160	196	252
SR8	100	109	125	160	196	252
SR9	100	107	119	144	171	213
SR10	100	109	125	160	196	252
SR11	100	109	125	160	196	252
SR12	100	111	130	171	214	280
SR13	100	111	130	171	214	280
SR14	100	104	110	124	139	162
SR15	100	111	130	171	214	280
SR16	100	108	121	151	182	230
SR17	100	108	121	151	182	230
SR18	100	108	121	151	182	230
SR19	100	108	121	151	182	230
SR20	100	109	125	160	196	252
SR21	100	111	130	171	214	280
SR22	100	107	119	144	171	213
SR23	100	109	125	160	196	252
SR24	100	107	119	144	171	213
SR25	100	105	113	132	151	180
SR26	100	103	108	119	130	147
SR27	100	111	130	171	214	280
SR28	100	107	119	144	171	213
SR29	100	109	125	160	196	252
SR30	100	107	119	144	171	213
SR31	100	105	112	129	146	173
SR32	100	106	116	139	163	199
SR33	100	106	116	139	163	199
SR34	100	106	116	139	163	199
SR35	100	106	116	139	163	199
SR36	100	111	130	171	214	280
SR37	100	108	121	151	182	230
SR38	100	108	121	151	182	230
SR39	100	111	130	171	214	280
SR40	100	109	125	160	196	252
SR41	100	105	115	135	156	189

Table I - continued

Design No.	E_C $\gamma = W/W'$ 1	2	3	5	7	10
SR42	100	109	125	160	196	252
SR43	100	105	115	135	156	189
SR44	100	105	115	135	156	189
SR45	100	111	130	171	214	280
SR46	100	107	119	144	171	213
SR47	100	105	113	132	151	180
SR48	100	105	113	132	151	180
SR49	100	107	119	144	171	213
SR50	100	105	113	132	151	180
SR51	100	108	121	151	182	230
SR52	100	109	125	160	196	252
SR53	100	105	112	129	146	173
SR54	100	108	121	151	182	230
SR55	100	105	112	129	146	173
SR56	100	109	125	160	196	252
SR57	100	105	112	129	146	173
SR58	100	106	116	139	163	199
SR59	100	111	130	171	214	280
SR60	100	106	116	139	163	199
SR61	100	111	130	171	214	280
SR62	100	107	119	144	171	213
SR63	100	105	115	135	156	189
SR64	100	108	121	151	182	230
SR65	100	108	121	151	182	230
SR66	100	111	130	171	214	280
SR67	100	105	113	132	151	180
SR68	100	109	125	160	196	252
SR69	100	106	116	139	163	199
SR70	100	107	119	144	171	213
SR71	100	105	112	129	146	173
SR72	100	107	119	144	171	213
SR73	100	111	130	171	214	280
SR74	100	109	125	160	196	252
SR75	100	105	115	135	156	189
SR76	100	108	121	151	182	230
SR77	100	105	113	132	151	180
SR78	100	109	125	160	196	252
SR79	100	108	121	151	182	230
SR80	100	107	119	144	171	213
SR81	100	108	121	151	182	230
SR82	100	107	119	144	171	213

Table I - continued

Design No.	E_C					
	γ = W/W'					
	1	2	3	5	7	10
SR83	100	106	116	139	163	199
SR84	100	107	119	144	171	213
SR85	100	105	115	135	156	189
SR86	100	106	116	139	163	199
SR87	100	106	116	139	163	199
SR88	100	105	115	135	156	189
SR89	100	105	113	132	151	180
SR90	100	105	115	135	156	189
SR91	100	105	113	132	151	180

2 For the semi-regular subclass of group divisible designs, E_A = 1 for all γ.

Table I - continued

Design No.	E_A $\gamma = W/W'$ 1	2	3	5	7	10	E_C $\gamma = W/W'$ 1	2	3	5	7	10
R1	100	111	130	171	214	280	100	105	113	132	151	180
R2	100	109	125	160	196	252	100	103	108	119	130	147
R3	100	111	130	171	214	280	100	107	119	144	171	212
R4	100	103	108	119	130	147	100	105	115	135	156	189
R5	100	105	113	132	151	180	100	111	130	171	214	280
R6	100	105	113	132	151	180	100	111	130	171	214	280
R7	100	105	113	132	151	180	100	110	126	161	199	256
R8	100	112	133	179	227	299	100	103	108	119	130	147
R9	100	112	132	177	223	294	100	108	121	151	182	230
R10	100	112	133	180	229	302	100	107	119	144	171	212
R11	100	106	116	139	163	199	100	110	127	165	205	265
R12	100	111	130	171	214	280	100	109	123	155	189	240
R13	100	104	112	129	146	173	100	110	128	167	208	270
R14	100	103	108	119	130	147	100	109	125	160	196	252
R15	100	109	125	160	196	252	100	105	115	135	156	189
R16	100	108	121	151	182	230	100	111	130	171	214	280
R17	100	111	130	171	214	280	100	109	123	155	189	240
R18	100	112	133	178	226	298	100	109	124	158	193	247
R19	100	112	133	180	229	302	100	109	125	160	196	252
R20	100	105	113	132	151	180	100	111	130	171	214	280
R21	100	108	121	151	182	230	100	111	130	171	214	280
R22	100	108	121	151	182	230	100	110	128	167	208	270
R23	100	104	112	129	146	173	100	108	121	151	182	230
R24	100	105	115	135	156	189	100	109	125	160	196	252
R25	100	109	123	155	189	240	100	111	130	171	214	280
R26	100	105	115	135	156	189	100	109	125	160	196	252
R27	100	103	108	119	130	147	100	109	125	160	196	252
R28	100	107	119	144	171	212	100	111	130	171	214	280
R29	100	112	131	175	221	290	100	109	125	160	196	252
R30	100	103	108	119	130	147	100	109	125	160	196	252
R31	100	103	108	119	130	147	100	109	123	155	188	238
R32	100	112	133	179	227	300	100	110	126	161	199	256
R33	100	112	133	179	227	300	100	109	123	155	189	240
R34	100	104	112	129	146	173	100	107	118	143	169	209
R35	100	105	113	132	151	180	100	111	130	171	214	280
R36	100	111	130	171	214	280	100	107	119	144	171	212
R37	100	112	133	178	226	298	100	106	116	139	163	199
R38	100	112	132	177	224	296	100	110	126	161	199	256
R39	100	105	113	132	151	180	100	109	123	155	189	240
R40	100	109	125	160	196	252	100	111	130	171	214	280

Table I - continued

	E_A						E_C					
	γ = W/W'						γ = W/W'					
Design No.	1	2	3	5	7	10	1	2	3	5	7	10
R41	100	111	130	171	214	280	100	110	126	161	199	256
R42	100	110	126	161	199	256	100	111	130	171	214	280
R43	100	112	133	179	227	300	100	107	119	144	171	212
R44	100	109	123	155	189	240	100	111	130	171	214	280
R45	100	102	105	112	120	131	100	108	121	151	182	230
R46	100	106	116	139	163	199	100	109	125	160	196	252
R47	100	105	113	132	151	180	100	111	130	171	214	280
R48	100	108	121	151	182	230	100	111	130	171	214	280
R49	100	112	132	177	223	294	100	108	121	151	182	230
R50	100	102	105	112	120	131	100	108	121	151	182	230
R51	100	110	127	165	205	265	100	106	116	139	163	199
R52	100	112	131	175	221	290	100	105	115	135	156	189
R53	100	108	120	149	178	223	100	109	125	160	196	252
R54	100	107	118	142	168	207	100	109	125	160	196	252
R55	100	105	115	135	156	189	100	109	125	160	196	252
R56	100	110	128	167	208	270	100	108	121	151	182	230
R57	100	109	125	160	196	252	100	108	121	151	182	230
R58	100	111	129	170	213	278	100	105	112	129	147	174
R59	100	110	128	167	208	270	100	107	118	143	169	209
R60	100	109	123	155	188	239	100	107	118	143	169	209
R61	100	105	113	132	151	180	100	108	121	151	182	230
R62	100	104	112	129	146	173	100	108	121	151	182	230
R63	100	101	103	106	110	116	100	106	116	139	163	199
R64	100	109	123	155	189	240	100	105	113	132	151	180
R65	100	110	128	167	208	270	100	104	112	129	146	173
R66	100	101	102	105	108	112	100	105	115	135	156	189
R67	100	108	121	151	182	230	100	104	112	129	146	173
R68	100	101	102	104	106	110	100	105	113	132	151	180

Table II. Efficiencies of Latin Square Type Designs

Relative to Completely Randomized Designs

	$E_A = E_C$					
	$\gamma = W/W'$					
Design No.	1	2	3	5	7	10
LS1	100	103	107	119	130	147
LS2	100	103	107	119	130	147
LS3	100	107	119	144	171	213
LS4	100	105	115	135	156	189
LS5	100	108	121	151	182	230
LS6	100	112	133	180	229	302
LS7	100	112	133	180	229	302
LS8	100	105	115	135	156	189
LS9	100	105	115	135	156	189
LS10	100	102	105	112	120	131
LS11	100	102	105	112	120	131
LS12	100	110	127	165	205	265
LS13	100	111	130	171	214	280
LS14	100	105	113	132	151	180
LS15	100	109	125	160	196	252
LS16	100	114	135	183	231	305
LS17	100	102	105	112	120	131
LS18	100	101	104	108	114	122
LS19	100	101	103	106	110	116
LS20	100	101	102	104	106	110

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ABSTRACT

COMBINED INTRA- AND INTER-BLOCK ANALYSIS FOR FACTORIALS IN INCOMPLETE BLOCK DESIGNS

by

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The work of Kramer and Bradley on the use of factorials in incomplete block designs has been extended to permit both the intra-block and combined intra- and inter-block analyses of factorials in balanced and partially balanced incomplete block designs. In particular, we have obtained a combined intra- and inter-block analysis for factorials in balanced incomplete block designs, group divisible designs, and Latin Square types of partially balanced, incomplete block designs. For the class of Latin Square sub-type L_3 designs both the intra-block and combined intra- and inter-block analyses have been developed.

In general, factorial treatment combinations were assigned to the association schemes by permitting the rows of the association schemes to represent the levels of one factor and the columns to represent the levels of a second factor. The extension to multi-factor factorials was then carried out by sub-dividing the levels of the basic two-factor factorial, the levels in the sub-divisions representing the levels of the multi-factor factorials.

Estimators for the factorial effects have been obtained along with their variances and covariances. Sums of squares in terms of the factorial estimators have been derived and can be used to carry out tests of significance. These sums of squares were shown, for the combined intra- and inter-block analyses, to be independently distributed as X^2-variates with the appropriate numbers of degrees of freedom.

Suitable sums of squares for tests of significance are not possible in general for Latin Square sub-type L_3 designs. In situations such as these, we can only consider contrasts among the estimators and use their variances to perform tests of significance. However, for the special cases of factorials in the 4 x 4 Latin Square sub-type L_3 design, a complete analysis yielding the adjusted sums of squares for the factorial effects is possible if the factorial treatments are applied to the association scheme in a different manner.

Single-degree-of-freedom contrasts are obtained in much the usual way as for factorials in complete block designs. The method of incorporating a fractional replicate of a factorial into incomplete block designs is also considered.

Numerical examples have been worked in detail for a group divisible design and a Latin Square sub-type L_2 design. The procedure for analyzing a Latin Square sub-type L_3 design is also discussed.