POSITIVITY PROPERTIES ASSOCIATED WITH LINEAR DIFFERENTIAL OPERATORS

by

William Randolph Winfrey

Dissertation submitted to the Graduate Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

APPROVED:

J. K. Shaw, Chairman

J. R. Holub

J. A. Cochran

K. B. Hannsgen

R. A. McCoy

November, 1976

Blacksburg, Virginia

ACKNOWLEDGMENTS

It is a pleasure to acknowledge the encouragement and guidance of Professor J. K. Shaw in the conduct of these investigations and in the preparation of this dissertation.

I also would like to acknowledge the contributions of my parents, Howard and Henrietta Winfrey, to my formal education over these many years and thank them for their love and support.

TABLE OF CONTENTS

1. Introduction

The determination of the influence exerted on the analytic character of a real function $f \in C^\infty$ by the signs of its derivatives is a problem of long standing interest in classical analysis [5]. Most investigations of the problem have centered on extending the well known theorem of S. Bernstein (Widder [20]) which asserts that a function $f \in C^\infty$ with all derivatives non-negative on an interval I is necessarily real-analytic there; i.e., f is the restriction to I of a complex function analytic in a region containing I.

The scope of this dissertation is the study of analogous positivity results associated with linear differential operators of the form

$$(Ly)(t) = a_2(t)y''(t) + a_1(t)y'(t) + a_o(t)y(t),$$

where $a_2(t)$, $a_1(t)$ and $a_o(t)$ are real-analytic in some interval I and where $a_2(t) > 0$ for $t \in I$. We shall call a function $f \in C^\infty$ <u>L-positive</u> at $t_o \in I$ if it satisfies the "uniform" positivity condition $L^k f(t) \geq 0$, $t \in I$, $k = 0, 1, 2, \ldots$, plus the "pointwise" positivity condition $(L^k f)'(t_o) \geq 0$, $k = 0, 1, 2, \ldots$ ($L^o f = f$, $L^k f = L(L^{k-1}f)$, $k \geq 1$). Our principal result is that L-positivity of f implies analyticity of f in a neighborhood of t_o. If $Ly = y''$, this reduces to Bernstein's theorem.

We shall prove our result using a generalized Taylor Series Expansion known as the L-series. The L-series expansion about $t = t_o$ for a function $f \in C^\infty$ is:

$$\sum_{k=0}^{\infty} L^k f(t_o)\ \phi_{2k}(t) + \sqrt{a_2(t_o)}\ (L^k f)'(t_o)\ \phi_{2k+1}(t).$$

The "L-basis" functions $\{\phi_n(t)\}_{n=0}^{\infty}$ are defined by:

$$L\phi_o \equiv L\phi_1 \equiv 0,\ \phi_o(t_o) = 1,\ \phi_o{}'(t_o) = 0,\ \phi_1(t_o) = 0,$$

(1.1) $\sqrt{a_2(t_o)}\ \phi_1{}'(t_o) = 1$

and $L\phi_{n+2} = \phi_n,\ \phi_{n+2}(t_o) = \phi'_{n+2}(t_o) = 0,\ n \geq 0.$

Our technique will be to show that L-positivity of f implies the convergence of the above series to f(t). Then we observe that the analyticity of a_2, a_1, and a_o implies the analyticity of the ϕ's and thus, in view of uniform convergence of the series, the analyticity of the sum, f(t).

We shall also show that the same conditions on a_2, a_1 and a_o allow any function f, analytic in a neighborhood of t_o, to be represented by an L-series. If $a_2(t) \equiv 1$, the sequence $\{n!\phi_n(t)\}_{n=0}^{\infty}$ provides a heretofore unobserved example of a Pincherle basis.

2. A Brief Survey of the Literature

The results we obtain have several analogs in a vast literature on the subject of positivity. To give an overview of one portion of the literature, we now present some typical results.

The prototype of the results we mention is the theorem of S. Bernstein (Widder [20]).

Theorem (Bernstein): A function $f\varepsilon C^{\infty}(a,b)$ having $f^{(k)}(x)$ continuous for $x\varepsilon[a,b)$ and $k = 0, 1, 2, \ldots$ which satisfies the condition $f^{(k)}(x)\geq 0$ for $x\varepsilon[a,b)$ and $k = 0, 1, 2, \ldots$, can be continued analytically into the complex disc $|z-a|<b-a$.

The concept in Bernstein's theorem has been extended basically in two directions. One extension has been to restrict the number of sign changes of the derivatives in an interval rather than requiring the derivatives all to be non-negative. The other mode of extension, which was taken in this dissertation, has been to require non-negativity only of certain derivatives or to require non-negativity of the functions resulting from the repeated application of certain linear differential operators to a given function.

An early result of the first type is due to G. Polya and N. Wiener [16]. Using a Fourier series argument they were able to prove the following theorem.

Theorem (Polya and Wiener): Let $f(x)$ be a real valued periodic function of period 2π defined for all real values of x and possessing derivatives of all orders. Let N_k denote the number of changes of sign of $f^{(k)}(x)$ in a period and consider the order of magnitude of N_k as $k\to\infty$.

(I) If $N_k = O(1)$, $f(x)$ is a trigonometric polynomial.

(II) If $N_k = O(k^{\delta})$ where δ is fixed, $0<\delta<\frac{1}{2}$, $f(x)$ is an entire function of finite order not exceeding $(1-\delta)/(1-2\delta)$.

(III) If $N_k = o(k^{\frac{1}{2}})$, $f(x)$ is an entire function.

G. Szegö [19] developed a new proof of this theorem and strengthened other results in the Polya and Wiener paper. Further generalizations of this result, including analogous results for certain second order differential operators have been developed by E. Hille [12].

Results of the second type fall roughly into two cases: (1) analyticity resulting from restrictions on the signs of a certain sequence of derivatives of a function, and (2) analyticity resulting from the non-negativity of functions obtained by the repeated application of certain linear differential operators to a given function.

Typical of Case 1 are the theorems of D. V. Widder [20] and R. P. Boas [4]. Widder's result is stated in terms of completely convex functions.

Definition: A function $f(x)$ is completely convex in an interval (a,b) if it has derivatives of all orders there (which are continuous on $[a,b]$) and if, in that interval, $(-1)^k f^{(2k)}(x)\geq 0$ for $k=0, 1, 2, \ldots$.

Theorem (Widder): If $f(x)$ is completely convex in an interval (a,b), it may be extended analytically into the z-plane to an entire function $f(z)$.

Boas proved the following theorem.

Theorem (Boas): Let $n_1, n_2, \ldots$ be an increasing sequence of integers. Let $f \epsilon C^{\infty}(-1,1)$ be such that, for all p, $f^{(n_p)}(x) \geq 0$ for $x \epsilon (-1,1)$. If the ratio $\frac{n_{p+1}}{n_p}$ is bounded, then f may be extended analytically into a complex neighborhood of $(-1,1)$.

The simplest illustration of this result is the case $f^{(2n)}(x) \geq 0$, $n = 0, 1, 2, \ldots, -1<x<1$, with the conclusion that f continues analytically into a finite complex neighborhood of $(-1,1)$. This differs rather dramatically from the case of alternating signs in Widder's theorem where f extends to an entire function.

For Case 2, the main result is a theorem due to J. K. Shaw [17]. Shaw's theorem is stated in terms of L-B positive functions.

Definition: Let $Ly = -\big(p(x)y'(x)\big)' + q(x)y(x)$ where p and q are real-analytic on $[a,b]$ and $p(x)>0$ on $[a,b]$ (so that the lead coefficient of L is negative). Let $B_a y = \alpha y(a) + \alpha' y'(a)$ and $B_b y = \beta y(b) + \beta' y'(b)$. A function $f \epsilon C^{\infty}[a,b]$ is L-B positive if

$$(L^k f)(x) \geq 0 \text{ for } x \epsilon [a,b],\ k = 0, 1, 2, \ldots,$$

and $$B_a L^k f \geq 0,\ B_b L^k f \geq 0 \text{ for } k = 0, 1, 2, \ldots.$$

Theorem (Shaw): If the eigenvalue problem $Ly = \lambda y$, $B_a y = B_b y = 0$ is self-adjoint and has only positive eigenvalues, then each L-B positive function f(x) is the restriction to $[a,b]$ of a function analytic in some complex neighborhood of $[a,b]$.

Thus the results for the case $a_2(t)<0$, at least for self-adjoint operators, are more or less complete. In contrast, our techniques are restricted to, and use in a crucial way, the hypothesis $a_2(t)>0$.

Moreover, we do not require any assumptions about self-adjointness.

The principal tool used in our proofs is the L-series, which appears to have been developed first by M. K. Fage [9, 10]. He derived a number of its properties and proved a theorem equivalent to our Theorem 1 on Pincherle bases (see also the comments at the beginning of the next section). Subsequently, the L-series has been investigated by a number of Soviet mathematicians. Most of these studies have been concerned with obtaining estimates of the L-basis functions (I. F. Grigorčuk [11] and N. I. Sidenko [18]) and with extending to L-series certain phenomena, in particular quasianalyticity, which are associated with Taylor series (V. G. Hryptun [14]).

Finally, we observe that L-positive functions need not resemble absolutely monotonic functions nor, in fact, any class of functions for which a sequence of derivatives, $f^{(n_k)}$, is non-negative on an interval. For example, we observe that the function $F(x) = \frac{x^2}{1+x^2}$ is trivially L-positive for the operator $Ly = y'' - (\frac{F''}{F})y$, but the zeros of successive derivatives cluster everywhere on the real axis.

3. Definitions, Preliminary Results and Examples

Let L be an n-th order linear differential operator defined on I, an open interval of regular points of L, by

$$(Ly)(t) = a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \ldots + a_o(t)y(t)\; t\varepsilon I$$

where $a_n(t)$ is normalized so that $a_n(t)>0$ $t\varepsilon I$. One may define the L-series based at $t_o\varepsilon I$ (termed an initial value series by the author) for $f\varepsilon C^\infty$ as

$$\sum_{k=0}^{\infty}\sum_{p=0}^{n-1} (L^k f)^{(p)}(t_o)\ (\sqrt[n]{a_n(t_o)}\ )^p\ \phi_{nk+p}(t)$$

where $\quad L\phi_p(t) \equiv 0,\ (\sqrt[n]{a_n(t_o)}\ )^p\ \phi_p^{(j)}(t_o) = \delta_{jp}\quad 0 \le p,\ j \le n-1$

and $\quad L\phi_{n(k+1)+p}(t) = \phi_{nk+p}(t) \qquad k = 0, 1, 2, \ldots$

$$\phi_{n(k+1)}^{(j)}(t_o) = 0 \qquad 0 \le p,j \le n-1$$

As was mentioned in Section 2, the L-series, with $a_n(t)\equiv 1$, appears to have been introduced by M. K. Fage, who derived a number of its properties under the assumption that the a_i's satisfy certain smoothness conditions. In the case of analytic coefficients, he shows that every function analytic at t_o admits a convergent L-series representation in a neighborhood of t_o. Some of the lemmas in this section may also be found in Fage's paper. However, since Fage's work is available only in Russian, we will provide our own, independent proofs. Moreover, our approach also leads to a new proof of the representation of analytic functions. The result is obtained as a corollary to a theorem of Boas [3] on Pincherle bases.

Hereafter, all statements and proofs will be given for the second order case, $n = 2$. All of the results of the present section have obvious analogs for general n. Thus, from now on L will be a second order operator with $a_2(t)>0$.

The L-series is closely associated with the initial value problem for L. If $r(t)$ is a real, continuous function in a neighborhood of t_o, the problem

$$(3.1) \qquad Ly = r, \; y(t_o) = y_o, \text{ and } y'(t_o) = y_o'$$

has the (unique) solution

$$(3.2) \qquad y(t) = y_o\phi_o(t) + y_o'\sqrt{a_2(t_o)} \;\; \phi_1(t) + \int_{t_o}^{t} G(t,\tau)\, r(\tau)\, d\tau$$

of t_o. The function

$$(3.3) \qquad G(t,\tau) = \frac{1}{a_2(\tau)} \; \frac{\phi_o(\tau)\ \phi_1(t) - \phi_1(\tau)\ \phi_o(t)}{\phi_o(\tau)\ \phi_1'(\tau) - \phi_1(\tau)\ \phi_o'(\tau)}$$

is the Green's function for the initial value problem.

Define the operator G by $(Gr)(t) = \int_{t_o}^{t} G(t,\tau) \cdot r(\tau)\, d\tau$ and its iterates G^n by $G^n r = G(G^{n-1}r)$, $n = 2, 3, 4, \ldots$ and $G^o r = r$. Note that

$$(3.4) \qquad LGr = r$$

and that the solution (3.2) may be expressed as

$$(3.5) \qquad y(t) = y(t_o)\phi_o(t) + y'(t_o) \;\; a_2(t_o)\phi_1(t) + (G_r)(t).$$

Let $f \in C^{\infty}(I)$. If we let $r(t)=(Lf)(t)$ and take as initial conditions $y_o=f(t_o)$ and $y_o'=f'(t_o)$ then the solution of (3.1) is $y(t)=f(t)$. Then (3.5) becomes

$$f(t) = f(t_o)\phi_o(t) + f'(t_o)\sqrt{a_2(t_o)}\ \phi_1(t) + GLf,$$

an identity for all f. The same identity will, of course, hold for Lf:

$$Lf(t) = Lf(t_o)\phi_o(t) + (Lf)'(t_o)\sqrt{a_2(t_o)}\ \phi_1(t) + GL^2f.$$

Combination of these two identities yields:

$$\begin{aligned} f(t) = {} & f(t_o)\phi_o(t) + f'(t_o)\sqrt{a_2(t_o)}\ \phi_1(t) \\ & + Lf(t_o)G\phi_o(t) + (Lf)'(t_o)\sqrt{a_2(t_o)}G\phi_1(t) + G^2L^2f(t)\ . \end{aligned}$$

Repetition of this process yields the <u>finite L-series</u>

$$f(t) = \sum_{k=0}^{n} L^kf(t_o)G^k\phi_o(t) + (L^kf)'(t_o)\sqrt{a_2(t_o)}G^k\phi_1(t) + G^{n+1}L^{n+1}f(t), \tag{3.6}$$

valid for each positive integer n.

From the definition of the functions $\{\phi_n(t)\}_{n=2}^{\infty}$, (1.1), it is clear that they may be realized as

$$\phi_{2k}(t) = G^k\phi_o(t),\ \phi_{2k+1}(t) = G^k\phi_1(t) \quad k = 1, 2, 3, \ldots \tag{3.7}$$

or recursively as

$$\phi_n(t) = G\phi_{n-2}(t) \text{ for } n \geq 2. \tag{3.8}$$

Thus, the finite L-series may be expressed as:

$$f(t) = \sum_{k=0}^{n} L^kf(t_o)\phi_{2k}(t) + (L^kf)'(t_o)\sqrt{a_2(t_o)}\phi_{2k+1}(t) + G^{n+1}L^{n+1}f(t) \tag{3.9}$$

The case $Ly = y''$ yields the familiar Taylor expansion about $t = t_o$. Moreover, the general L-series has many properties in common with the Taylor expansion. The lemmas which we shall now prove will, in addition to their utility, serve to illustrate the similarities between the two expansions.

To simplify the discussion we shall prove the lemmas and theorems under the restriction $a_2(t) \equiv 1$. The results for the case $a_2(t)>0$ may be obtained by a change of independent variable.

Definition: A right neighborhood of t_o, $N(t_o)$, is an open interval of the form $(t_o, t_o + b)$ for some $b>0$.

Lemma 1: For each $n \geq 1$, $\phi_n(t)$ is a positive, monotonically increasing function for t in some right neighborhood, $N_o(t_o)$, of t_o.

Proof: Let $b_o>0$ be the largest number such that (i) $\phi_o(t) > 0$ for $t\varepsilon(t_o, t_o + b_o)$, (ii) $\phi_o(\tau)\phi_1'(t) - \phi_1(\tau)\phi_o'(t)>0$ in the triangle $t_o \leq \tau \leq t< t_o + b_o$ and (iii) $(t_o,t_o + b_o)\subset I$. We shall show that $N_o(t_o)$ exists and, in fact, may be chosen as (t_o,t_o+b_o).

To establish positivity we shall make use of the Green's function

$$(3.10) \qquad G(t,\tau) = \frac{\phi_o(\tau)\ \phi_1(t) - \phi_1(\tau)\ \phi_o(t)}{\phi_o(\tau)\ \phi_1'(\tau) - \phi_o'(\tau)\ \phi_1(\tau)}$$

showing that it is non-negative for $t_o \leq \tau \leq t< t_o + b_o$ and drawing the conclusion from that fact. The denominator, $\phi_o(\tau)\phi_1'(\tau) - \phi_o'(\tau)\ \phi_1(\tau)$, is simply the Wronskian of ϕ_o and ϕ_1 and must always be of one sign. Since $\phi_o(t_o)\phi_1'(t_o) - \phi_o'(t_o)\phi_1(t_o) = 1\cdot 1-0\cdot 0 = 1>0$, we must have $\phi_o(\tau)\phi_1'(\tau) - \phi_o'(\tau)\phi_1(\tau) > 0$ for all τ under consideration. Consequently, the sign of $G(t,\tau)$ is determined solely by the expression in the numerator.

Let $g_t(\tau) = \phi_o(\tau)\phi_1(t) - \phi_1(\tau)\phi_o(t)$ where $t\varepsilon(t_o, t_o + b_o)$ is fixed. We will argue that $g_t(\tau) > 0$ for $t_o<\tau<t$. We begin by observing that $g_t(t_o) = \phi_1(t)>0$ and $g_t(t) = 0$. (The assertion that $\phi_1(t) > 0$ for $t_o<t<t_o + b_o$ is a consequence of the interlacing property of zeros of independent solutions of the homogeneous equation $Ly = 0$ (see, for example, Birkhoff and Rota [2]). Specifically, if there existed a $t^*\varepsilon(t_o, t_o + b_o)$ such that $\phi_1(t^*) = 0$, the vanishing of ϕ_1 at t_o and t^* would force ϕ_o to vanish between these two points, contradicting the choice of b_o). Suppose that there is a $\tau^*\varepsilon(t_o, t)$ such that $g_t(\tau^*) = 0$. Since, for fixed t, $g_t(\tau)$ is a solution of the homogeneous equation, $(Ly)(\tau)\equiv 0$, which vanishes at $\tau=\tau^*$ and $\tau=t$, it must be that $\phi_o(\tau)$, an independent solution, vanishes between $\tau=\tau^*$ and $\tau=t$ (by the interlacing property again). This contradicts the choice of b_o. Therefore $g_t(\tau)>0$ for $t_o<\tau<t<t_o + b_o$ implying $G(t,\tau)\geq 0$ for $t_o\leq\tau\leq t<t_o + b_o$.

Positivity of the ϕ's follows from noting that $\phi_o(t)>0$ and $\phi_1(t)>0$ for $t\varepsilon(t_o, t_o + b_o)$ and recalling that (3.8)may be expressed as

$$\phi_n(t) = \int_{t_o}^{t} G(t,\tau)\phi_{n-2}(\tau)\, d\tau \qquad n = 2, 3, 4, \ldots . \tag{3.11}$$

The application of a trivial induction argument shows that $\phi_n(t)>0$ for $t_o<t<t_o + b_o$ and $n = 2, 3, 4, \ldots$

To establish <u>monotonicity</u> we shall show that $\phi_n'(t)>0$ for $t\varepsilon(t_o, t_o + b_o)$ and $n \geq 2$. Differentiating both sides of (3.11) and using $G(t,t) = 0$ we have

$$\phi_n'(t) = \int_{t_o}^{t} \frac{\partial}{\partial t} G(t,\tau)\phi_{n-2}(\tau)d\tau$$

Explicitly, this is

$$\phi_n'(t) = \int_{t_o}^{t} \frac{\phi_o(\tau)\phi_1'(t) - \phi_1(\tau)\phi_o'(t)}{\phi_o(\tau)\phi_1'(\tau) - \phi_1(\tau)\phi_o'(\tau)} \phi_{n-2}(\tau)d\tau$$

We have already observed that the denominator, the Wronskian, is positive and, by the choice of b_o, we are assured that the numerator is positive for $t_o<\tau<t<t_o + b_o$. It has just been shown that $\phi_{n-2}(\tau)>0$ for $t_o<\tau<t_o + b_o$. Therefore, it must be true that $\phi_n'(t)>0$ for $t\varepsilon(t_o, t_o + b_o)$ and $n\geq 2$.

If $N_o(t_o)$ is defined to be $(t_o, t_o + b_o)$, the proof is complete.

There are three corollaries to the proof of Lemma 1.

<u>Corollary</u> 1: For $t_o\leq\tau\leq t\leq t_o + b_o$, $G(t,\tau)\geq 0$.

<u>Corollary</u> 2: If $f(t)\geq 0$ for $t\varepsilon N_o(t_o)$, then $G^n f(t)\geq 0$ for all n and $t\varepsilon N_o(t_o)$.

<u>Corollary</u> 3: For $t_o\leq\tau\leq t\leq t_o + b_o$, $\frac{\partial}{\partial t} G(t,\tau)\geq 0$

<u>Lemma</u> 2: Let $\{a_n\}_{n=o}^{\infty}$ be a sequence of non-negative real numbers. If $\sum_{n=o}^{\infty} a_n \phi_n(t^*)$ converges for some $t^*\varepsilon N_o(t_o)$, then $\sum_n a_n\phi_n(t)$ converges uniformly for $0\leq t - t_o\leq t^* - t_o$.

<u>Proof</u>: By Lemma 1 we have

$$0\leq a_n\phi_n(t)\leq a_n\phi_n(t^*) \text{ for } t^*\varepsilon N_o(t_o) \text{ and } n\geq 2.$$

Therefore $\sum_n a_n\phi_n(t)$ converges uniformly for $t\varepsilon[t_o,t^*]$ by the Weierstrass M-test.

Lemma 3: There exist positive constants C_1 (≤ 1) and C_2 (≥ 1) and a right neighborhood, $N_1(t_o)$, of t_o such that

$$C_1^2(t-\tau)\leq G(t,\tau)\leq C_2^2(t-\tau) \text{ for } t,\tau\varepsilon N_1(t_o),\ \tau\leq t.$$

Proof: Consider the quotient $\frac{G(t,\tau)}{t-\tau}$. By the discussion in Lemma 1, we know that $G(t,\tau)>0$ for $t_o<\tau<t<t_o + b_o$ so

$$\frac{G(t,\tau)}{t-\tau} > 0 \quad \text{for } t_o<\tau<t<t_o + b_o$$

Pick $b_1>0$ so that $b_1<b_o$. Define the open triangle T in the (t,τ) plane by $T = \{(t,\tau) \mid t_o <\tau<t<t_o + b_1\}$. (Note that the choice of b_1 forces T to be bounded). Since ϕ_o and ϕ_1 are continuous, $\frac{G(t,\tau)}{t-\tau}$ is continuous in t and τ for $(t,\tau)\varepsilon T$.

Define C_1 and C_2 by

$$C_1 = \left(\inf_{(t,\tau)\varepsilon T} \frac{G(t,\tau)}{t-\tau}\right)^{1/2} \quad\text{and}\quad C_2 = \left(\sup_{(t,\tau)\varepsilon T} \frac{G(t,\tau)}{t-\tau}\right)^{1/2} \tag{3.12}$$

The lemma will be proved once we show that $0 < C_1 \leq 1 \leq C_2 < \infty$. The function is well defined and positive on the sides of T which coincide with the lines $\tau=t_o$ and $t=t_o+b_1$. Thus, any difficulties must occur on the hypotenuse, which coincides with the line $\tau=t$. Thus, questions concerning the positivity of C_1 and finiteness of C_2 may be answered by showing that

$$\lim_{(t^*,\tau^*)\to(t,t)} \frac{G(t^*,\tau^*)}{t^*-\tau^*} \qquad \left((t^*,\tau^*)\varepsilon T,\ t_o\leq t\leq t_o + b_1\right)$$

exists and is positive.

From (3.10), the denominator of G, the Wronskian, will not affect the existence of the limit or its positivity. Therefore, the existence and properties of the limit depend entirely upon

$$\frac{\phi_o(\tau^*)\phi_1(t^*) - \phi_o(t^*)\phi_1(\tau^*)}{t^* - \tau^*} \tag{3.13}$$

Adding and subtracting a term of the form $\phi_o(\tau^*)\phi_1(\tau^*)$ in the numerator, we may express (3.13) as

$$\phi_o(\tau^*)\,\frac{\phi_1(t^*) - \phi_1(\tau^*)}{t^* - \tau^*} - \phi_1(\tau^*)\,\frac{\phi_o(t^*) - \phi_o(\tau^*)}{t^* - \tau^*} \tag{3.14}$$

Thus, the limit of (3.13) may be obtained by taking the limit of (3.14). In taking the limit of (3.14) we need only be concerned with the limits of

$$\frac{\phi_1(t^*) - \phi_1(\tau^*)}{t^* - \tau^*} \quad \text{and} \quad \frac{\phi_o(t^*) - \phi_o(\tau^*)}{t^* - \tau^*}$$

Examining the first of these, we have

$$\phi_1(t^*) = \phi_1(t) + \int_t^{t^*} \phi_1{}'(x)dx,$$

and

$$\phi_1(\tau^*) = \phi_1(t) + \int_t^{\tau^*} \phi_1{}'(x)dx.$$

Then $\phi_1(t^*) - \phi_1(\tau^*) = \int_{\tau^*}^{t^*} \phi_1{}'(x)dx.$

By the Mean Value Theorem for integrals we have

$$\int_{\tau^*}^{t^*} \phi_1{}'(x)dx = \phi_1{}'(x_{t^*\tau^*})\ (t^* - \tau^*)$$

where $x_{t^*\tau^*}$ is between t^* and τ^*.

Then

$$\lim_{(t^*,\tau^*)\to(t,t)} \frac{\phi_1(t^*) - \phi_1(\tau^*)}{t^* - \tau^*} = \lim_{(t^*,\tau^*)\to(t,t)} \phi_1{}'(x_{t^*\tau^*}) = \phi_1{}'(t).$$

By similar reasoning we have

$$\lim_{(t^*,\tau^*)\to(t,t)} \frac{\phi_o(t^*) - \phi_o(\tau^*)}{t^* - \tau^*} = \phi_o{}'(t)$$

Therefore the limit of (3.14), and thus of (3.13), is

$$\phi_o(t)\phi_1{}'(t) - \phi_1(t)\phi_o{}'(t) .$$

Therefore $\lim_{(t^*,\tau^*)\to(t,t)} \frac{G(t^*,\tau^*)}{t^* - \tau^*} = 1$.

Thus the boundedness of C_2 and the positivity of C_1 have been shown. Furthermore, it is clear that $C_1 \leq 1$ and $C_2 \geq 1$.

By the definition of C_1 and C_2 we have

$$C_1{}^2 \leq \frac{G(t,\tau)}{t-\tau} \leq C_2{}^2 \qquad t_o \leq \tau < t \leq t_o + b_1$$

which implies that

$$C_1{}^2(t-\tau) \leq G(t,\tau) \leq C_2{}^2(t-\tau) \qquad t_o \leq \tau \leq t \leq t_o + b_1 .$$

To complete the proof define $N_1(t_o) = (t_o, t_o + b_1)$.

Remark: Note that C_1 and C_2 can be made arbitrarily close to 1 by taking b_1 sufficiently small. Also, note that the only requirement made of ϕ_i is $\phi_i \varepsilon C^1(I)$ rather than analyticity.

Lemma 4: There exist positive constants d_1 (≤ 1) and d_2 (≥ 1) such that

$$(3.15)\qquad d_1 \frac{C_1^n(t-t_o)^n}{n!} \le \phi_n(t) \le d_2 \frac{C_2^n(t-t_o)^n}{n!}$$

for $n = 0, 1, 2, \ldots$ and $t \varepsilon N_1(t_o)$.

Proof: Before embarking on the proof we need to make some observations. If $\hat{G}(t,\tau)$ is defined by $\hat{G}(t,\tau) = t-\tau$ and if the integral operator $\hat{G}$ is defined by

$$(\hat{G}y)(t) = \int_{t_o}^{t} (t-\tau)y(\tau)d\tau,$$

one may easily verify that

$$(\hat{G}^n y)(t) = \int_{t_o}^{t} \frac{(t-\tau)^{2n-1}}{(2n-1)!} y(\tau)d\tau$$

In particular, if $y(\tau) = \frac{(\tau-t_o)^p}{p!}$ then

$$(\hat{G}^n y)(t) = \int_{t_o}^{t} \frac{(t-\tau)^{2n-1}}{(2n-1)!} \frac{(\tau-t_o)^p}{p!} d\tau = \frac{(t-t_o)^{2n+p}}{(2n+p)!}$$

Define positive constants g_1, g_2, h_1 and h_2 by

$$g_1 = \inf_{t\varepsilon N_1} \phi_o(t), \qquad g_2 = \sup_{t\varepsilon N_1} \phi_o(t)$$

$$h_1 = \inf_{t\varepsilon N_1} \frac{\phi_1(t)}{t-t_o}, \quad h_2 = \sup_{t\varepsilon N_1} \frac{\phi_1(t)}{t-t_o}$$

Then we have

$$g_1 \le \phi_o(\tau) \le g_2$$

and

$$h_1(\tau-t_o) \le \phi_1(\tau) \le h_2(\tau-t_o), \quad \tau\varepsilon N_1(t_o).$$

By Lemma 3 we know that

$$C_1^2(t-\tau)\le G(t,\tau)\le C_2^2(t-\tau) \qquad t_o\le\tau\le t\le t_o + b_1 \ .$$

One may combine these inequalities to obtain

$$g_1C_1^2(t-\tau)\le G(t,\tau)\phi_o(\tau)\le g_2C_2^2(t-\tau), \quad t_o\le\tau\le t\le t_o + b_1.$$

Integrating with respect to τ from t_o to t, we obtain

$$g_1 C_1^2 \frac{(t-t_o)^2}{2} \le \phi_2(t) \le g_2C_2^2 \frac{(t-t_o)^2}{2} \quad t\epsilon N_1(t_o).$$

Combining this inequality with the estimates on $G(t,\tau)$ we obtain

$$g_1C_1^4(t-\tau) \frac{(\tau-t_o)^2}{2} \le G(t,\tau)\phi_2(\tau) \le g_2C_2^4(t-\tau) \frac{(\tau-t_o)^2}{2}$$

Integrating with respect to τ from t_o to t, we obtain

$$g_1C_1^4 \frac{(t-t_o)^4}{4!} \le \phi_4(t) \le g_2C_2^4 \frac{(t-t_o)^4}{4!} \quad t\epsilon N_1(t_o).$$

Continuing in this fashion yields the inequalities:

$$g_1C_1^{2k} \frac{(t-t_o)^{2k}}{(2k)!} \le \phi_{2k}(t) \le g_2C_2^{2k} \frac{(t-t_o)^{2k}}{(2k)!} \quad t\epsilon N_1(t_o)$$

$k = 0, 1, 2, \ldots .$

Application of the same reasoning will show that

$$h_1C_1^{2k} \frac{(t-t_o)^{2k+1}}{(2k)!} \le \phi_{2k+1}(t) \le h_2C_2^{2k} \frac{(t-t_o)^{2k+1}}{(2k+1)!}$$

$t\epsilon N_1(t_o)$ $k = 0, 1, 2, \ldots .$

If d_1 and d_2 are defined by

$$d_1 = \min(g_1, \frac{h_1}{C_1}) \text{ and } d_2 = \max\ (g_2, \frac{h_2}{C_2})$$

($d_1 \le 1$ since $g_1 \le 1$, $d_2 \ge 1$ since $g_2 \ge 1$), then the estimates on ϕ_{2k} and ϕ_{2k+1} may be expressed together as

$$d_1 \frac{C_1{}^n(t-t_o)^n}{n!} \leq \phi_n(t) \leq d_2 \frac{C_2{}^n(t-t_o)^n}{n!} \qquad t \epsilon N_1(t_o).$$

Remark: Note that g_1, g_2, h_1, h_2, d_1 and d_2 can be made arbitrarily close to 1 by making b_1 sufficiently small.

Lemma 4 has the following corollaries

Corollary 1: For each n, $\phi_n(t) = \frac{(t-t_o)^n}{n!} \psi_n(t)$ where ψ_n is continuous on I and satisfies $\psi_n(t_o) = 1$.

Proof: The conclusion of Lemma 4, (3.15), may be written as

$$d_1 C_1{}^n \leq \frac{n!\phi_n(t)}{(t-t_o)^n} \leq d_2 C_2{}^n \qquad t \epsilon N_1(t_o)$$

For $t \neq t_o$, define $\psi_n(t) = \frac{n!\phi_n(t)}{(t-t_o)^n}$

For these values of t, $\psi_n(t)$ is continuous wherever $\phi_n(t)$ is. The above estimates then say that

$$d_1 C_1{}^n \leq \psi_n(t) \leq d_2 C_2{}^n \qquad t \epsilon N_1(t_o)\,.$$

If we define $\psi_n(t_o) = 1$ and recall the remarks following Lemma 3 and Lemma 4, we see that $\psi_n(t)$ will be continuous on I.

Corollary 2: For each n, $\phi_n{}^{(n)}(t_o) = 1$.

Lemma 5: Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of non-negative real numbers. If $\sum_{n=0}^{\infty} a_n\phi_n(t)$ converges for $0 \leq t-t_o \leq t^*-t_o$ where $t^* \epsilon N_1(t_o)$ then $\sum_n a_n \frac{(t-t_o)^n}{n!}$ converges for $0 \leq (t-t_o) \leq C_1(t^*-t_o)$. Conversely, if $\sum_{n=0}^{\infty} a_n \frac{(t-t_o)^n}{n!}$ converges for $0 \leq t-t_o \leq C_2(t^*-t_o)$ where $(C_2(t^*-t_o) + t_o) \; \epsilon N_1(t_o)$ then $\sum_n a_n\phi_n(t)$ converges for $0 \leq t-t_o \leq t^*-t_o$.

Proof: Suppose that $\sum_n a_n\phi_n(t)$ converges for $0 \leq t-t_o \leq t^*-t_o$; then, in particular, $\sum_n a_n\phi_n(t^*)$ converges. Since $t^* \epsilon N_1(t_o)$ we may conclude from Lemma 4, (3.15), that

$$0 \le d_1 a_n \frac{C_1^n (t^*-t_o)^n}{n!} \le a_n \phi_n(t^*).$$

Consequently, convergence of $\sum_n a_n \phi_n(t^*)$ implies the convergence of $\sum_n a_n \frac{C_1^n(t^*-t_o)^n}{n!}$ which, in turn, implies the convergence of $\sum_n a_n \frac{(t-t_o)^n}{n!}$ for $0 \le t-t_o \le C_1(t^*-t_o)$.

Now suppose that $\sum_n a_n \frac{(t-t_o)^n}{n!}$ converges for $0 \le t-t_o \le C_2(t^*-t_o)$ then, in particular, $\sum_n a_n \left(C_2(t-t_o)\right)^n/n!$ converges. Since $C_2 \ge 1$ and $\left(C_2(t^*-t_o) + t_o\right) \varepsilon N_1(t_o)$ we have $t^* \varepsilon N_1(t_o)$. So we may conclude from Lemma 4, (3.15), that

$$0 \le a_n \phi_n(t^*) \le d_2 a_n \frac{C_2^n (t^*-t_o)^n}{n!} .$$

Therefore, $\sum_n a_n \phi_n(t^*)$ converges. By Lemma 2 we may conclude that $\sum_n a_n \phi_n(t)$ converges for $0 \le t-t_o \le t^*-t_o$.

<u>Definition</u>: $N_2(t_o)$ is the largest right neighborhood of t_o, contained in $N_o(t_o)$, such that $\phi_1(t) \le \phi_o(t)$ for $t \varepsilon N_2(t)$.

<u>Lemma</u> 6: $\phi_{2k+1}(t) \le \phi_{2k}(t)$ for $t \varepsilon N_2(t_o)$ and $k = 0, 1, 2, \ldots$.

<u>Proof</u>: Let $b_2 > 0$ be such that $N_2(t_o) = (t_o, t_o + b_2)$. Since $N_2(t_o) \subset N_o(t_o)$, we know (from Corollary 1 of Lemma 1) that $G(t,\tau) \ge 0$ for $t_o \le \tau \le t \le t_o + b_2$. Therefore it must be true that

$$G(t,\tau)\phi_1(\tau) \le G(t,\tau)\phi_o(\tau) \qquad t_o \le \tau \le t \le t_o + b_2$$

Integrating from t_o to t we have

$$\phi_3(t) \le \phi_2(t) \qquad t \varepsilon N_2(t_o)$$

A trivial induction argument produces the general result.

<u>Lemma</u> 7: Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers. If $|a_n\phi_n(t^*)|\leq M$ for all n and some $t^*\varepsilon N_1(t_o)$, then $\sum_n a_n\phi_n(t)$ converges absolutely for $0\leq t-t_o<\frac{C_1}{C_2}(t^*-t_o)$ and uniformly on closed subintervals of $[t_o, \frac{C_1}{C_2}(t^*-t_o)+t_o)$.

<u>Proof</u>: Let $0<t-t_o<t^*-t_o$.

Then $$|a_n\phi_n(t)| = |a_n\phi_n(t^*)|\left|\frac{\phi_n(t)}{\phi_n(t^*)}\right| \leq M\frac{\phi_n(t)}{\phi_n(t^*)}.$$

By Lemma 4, we know that $\phi_n(t)\leq d_2\frac{C_2{}^n(t-t_o)^n}{n!}$ and $\phi_n(t^*)\geq d_1\frac{C_1{}^n(t^*-t_o)^n}{n!}$.

Then $$\frac{\phi_n(t)}{\phi_n(t^*)} \leq \frac{d_2}{d_1}\left(\frac{C_2{}^n(t-t_o)^n}{C_1{}^n(t^*-t_o)^n}\right) = \frac{d_2}{d_1}\left(\frac{t-t_o}{\frac{C_1}{C_2}(t^*-t_o)}\right)^n$$

Therefore $$|a_n\phi_n(t)| \leq M\frac{d_2}{d_1}\left(\frac{t-t_o}{\frac{C_1}{C_2}(t^*-t_o)}\right)^n$$

and $\sum_n\left(\frac{t-t_o}{\frac{C_1}{C_2}(t^*-t_o)}\right)^n$ converges for $0\leq t-t_o<\frac{C_1}{C_2}(t^*-t_o)$.

Therefore, $\sum_n a_n\phi_n(t)$ converges absolutely for $0\leq t-t_o<\frac{C_1}{C_2}(t^*-t_o)$. The uniformity of convergence follows from Lemma 2.

<u>Lemma</u> 8: If $\sum_{n=0}^{\infty} a_n\phi_n(t^*)$ converges for some $t^*\varepsilon N_1(t_o)$ then $\sum_n a_{n+q}\phi_n(t)$ converges for $0\leq t-t_o<\frac{C_1}{C_2}(t^*-t_o)$, $q = 1, 2, 3, \ldots$. Furthermore, the convergence is absolute and uniform on closed subintervals of $[t_o, t_o+\frac{C_1}{C_2}(t^*-t_o))$.

<u>Proof</u>: In view of Lemma 7, we may assume, without loss of generality, that $a_n\geq 0$ for all n.

By Lemma 4 we have

$$d_1\frac{(C_1(t^*-t_o))^n}{n!} \leq \phi_n(t^*).$$

The convergence of $\sum_n a_n\phi_n(t^*)$ implies the convergence of $\sum_n a_n \frac{(C_1(t^*-t_o))^n}{n!}$ which, in turn, implies the convergence of

$$(3.16) \qquad \sum_n a_{n+q} \frac{(C_1(t^*-t_o))^{n+q}}{(n+q)!} \qquad q = 1, 2, 3, \ldots .$$

To exploit the convergence of (3.16) we first observe that

$$a_{n+q} \frac{(C_1(t-t_o))^n}{n!} = a_{n+q} \frac{(C_1(t-t_o))^{n+q}}{(n+q)!} \cdot \frac{(n+q)!}{n!(C_1(t-t_o))^q},$$

and then note that

$$\frac{(n+q)!}{n!} = (n+q)(n+q-1)\ldots(n+1)<(n+q)^q.$$

Thus

$$a_{n+q} \frac{(C_1(t-t_o))^n}{n!} \leq \frac{(n+q)^q}{(C_1(t-t_o))^q} \cdot a_{n+q} \frac{(C_1(t-t_o))^{n+q}}{(n+q)!}$$

The convergence of (3.16) implies the convergence of

$$\sum_n a_{n+q} \frac{(C_1(t-t_o))^{n+q}}{(n+q)!}$$

and of $\quad \sum_n a_{n+q} \cdot (n+q)^q \cdot \frac{(C_1(t-t_o))^{n+q}}{(n+q)!}, \quad 0\leq t-t_o<t^*-t_o.$

Therefore $\sum_n a_{n+q} \frac{(C_1(t-t_o))^n}{n!}$ converges for $0\leq t-t_o<t^*-t_o$.
We may now argue that convergence of this series implies the convergence of $\sum_n a_{n+q} \frac{(t-t_o)^n}{n!}$ for $0\leq t-t_o<\frac{1}{C_1}(t^*-t_o)$ which, by Lemma 5, implies the convergence of $\sum a_{n+q}\phi_n(t)$ for $0\leq t-t_o<\frac{1}{C_1C_2}(t^*-t_o)$. Since $\frac{C_1}{C_2} \leq \frac{1}{C_1C_2}$, we see that the convergence asserted in the statement of the lemma has been proved. Furthermore, by Lemma 2, the convergence is uniform on closed subintervals of $[t_o, \frac{C_1}{C_2}(t^*-t_o) + t_o)$.

<u>Lemma</u> 9: Let $f(t) = \sum_{n=0}^{\infty} a_n\phi_n(t)$ for $0\leq t-t_o\leq t^*-t_o$ where $t^*\varepsilon N_1(t_o)$. Then $L^p f(t) = \sum_n a_{n+2p}\phi_n(t)$ for $0\leq t-t_o<\frac{C_1}{C_2}(t^*-t_o)$ and $p = 0, 1, 2, \ldots .$

Proof: By Lemma 7, the series for $f(t)$ converges absolutely for $0 \le t-t_o < \frac{C1}{C2}(t^*-t_o)$, which allows the terms of the series to be grouped in any desired fashion. In particular, we may write, for $0 \le t-t_o < \frac{C1}{C2}(t^*-t_o)$,

$$\begin{aligned}
f(t) &= \sum_{k=0}^{p-1} a_{2k}\phi_{2k}(t) + a_{2k+1}\phi_{2k+1}(t) \\
&\qquad + \sum_{k=p}^{\infty} a_{2k}\phi_{2k}(t) + a_{2k+1}\phi_{2k+1}(t) \\
&= \sum_{k=0}^{p-1} a_{2k}\phi_{2k}(t) + a_{2k+1}\phi_{2k+1}(t) \\
&\qquad + \sum_{k=o}^{\infty} a_{2k+2p}\phi_{2k+2p}(t) + a_{2k+2p+1}\phi_{2k+2p+1}(t) \\
&= \sum_{k=0}^{p-1} a_{2k}\phi_{2k}(t) + a_{2k+1}\phi_{2k+1}(t) \\
&\qquad + \sum_{k=0}^{\infty} a_{2k+2p}G^p\phi_{2k}(t) + a_{2k+2p+1}G^p\phi_{2k+1}(t) \\
&= \sum_{k=0}^{p-1} a_{2k}\phi_{2k}(t) + a_{2k+1}\phi_{2k+1}(t) \\
&\qquad + G^p\left(\sum_{k=0}^{\infty} a_{2k+2p}\phi_{2k}(t) + a_{2k+2p+1}\phi_{2k+1}(t)\right).
\end{aligned}$$

The last step is justified by Lemma 8, which guarantees the uniform convergence of the series upon which G^p operates.

Operating on both sides of the above equation with L^p and recalling (3.4) we obtain

$$L^p f(t) = \sum_{k=0}^{\infty} a_{2k+2p}\phi_{2k}(t) + a_{2k+2p+1}\phi_{2k+2p+1}(t)$$

for $0 \le t-t_o < \frac{C1}{C2}(t^*-t_o)$.

<u>Lemma</u> 10: If $f(t) = \sum_{n=0}^{\infty} a_n\phi_n(t)$ for $0\leq t-t_o\leq t^*-t_o$

where $t^*\varepsilon No(t_o)$ then $f'(t) = \sum_n a_n\phi_n'(t)$

for $0\leq t-t_o< \frac{C1}{C2}(t^*-t_o)$.

<u>Proof</u>: Since $\phi_n(t) = \int_{to}^{t} G(t,\tau)\phi_{n-2}(\tau)d\tau \quad n=2, 3, 4, \ldots$

then $\phi_n'(t) = \int_{to}^{t} \frac{\partial}{\partial t} G(t,\tau)\phi_{n-2}(\tau)d\tau.$

By Corollary 3 of Lemma 1, we have

$$\frac{\partial}{\partial t} G(t,\tau)\geq 0 \qquad t_o\leq\tau\leq t<b_o + t_o$$

and $$\phi_n'(t)\geq 0 \qquad 0\leq t-t_o<b_o,\ n\geq 1.$$

Define the non-negative function $M(t)$, $t\varepsilon[t_o, t_o + b_o]$, by

$$M(t) = \sup_{t_o\leq\tau\leq t} \frac{\partial}{\partial t} G(t,\tau).$$

Then $$0\leq\phi_n'(t)\leq M(t)\int_{t_o}^{t} \phi_{n-2}(\tau)d\tau,\ n\geq 2.$$

By Lemma 1, $\phi_{n-2}(\tau)$ is a monotonically increasing function of $\tau\varepsilon N_o(t_o)$ for $n>2$, so

$$\int_{t_o}^{t} \phi_{n-2}(\tau)d\tau\leq(t-t_o)\phi_{n-2}(t) \text{ for } t\varepsilon N_o(t_o),\ n>2.$$

Therefore $0\leq\phi_n'(t)\leq M(t)(t-t_o)\phi_{n-2}(t)$, $t\varepsilon N_o(t_o)$, $n>2$.

By Lemma 8, $\sum_{n=3}^{\infty} a_n\phi_{n-2}(t)$ converges uniformly for $0\leq t-t_o<\frac{C1}{C2}(t^*-t_o)$.

Therefore $\sum_n a_n\phi_n'(t)$ converges uniformly for $0\leq t-t_o<\frac{C1}{C2}(t^*-t_o)$ and term by term differentiation of the series for f is justified.

Therefore $f'(t) = \sum_{n=0}^{\infty} a_n\phi_n'(t)$ for $0 \leq t-t_o < \frac{C1}{C2}(t^*-t_o)$.

Since $N_1(t_o) \subset N_o(t_o)$, the results of Lemmas 9 and 10 may be combined as:

Lemma 11: Let $f(t) = \sum_{n=0}^{\infty} a_n\phi_n(t)$ for $0 \leq t-t_o \leq t^*-t_o$

where $t^* \varepsilon N_1(t_o)$. Then

$$L^p f(t) = \sum_{n=0}^{\infty} a_{n+2p}\phi_n(t)$$

and $$(L^p f)'(t) = \sum_{n=0}^{\infty} a_{n+2p}\phi_n'(t)$$

for $0 \leq t-t_o < \frac{C1}{C2}(t^*-t_o)$ and $p = 0, 1, 2, \ldots.$

Lemma 11 has the following important corollary.

Corollary (Uniqueness of coefficients): Let $f(t) = \sum_{n=0}^{\infty} a_n\phi_n(t)$ for $0 \leq t-t_o \leq t^*-t_o$, where $t^* \varepsilon N_1(t_o)$. Then

$$a_{2k} = L^k f(t_o) \text{ and } a_{2k+1} = (L^k f)'(t_o),\ k = 0, 1, 2, \ldots.$$

Lemmas 1 through 11 have described the local (in a neighborhood of t_o) properties of the set $\{\phi_n(t)\}_{n=0}^{\infty}$, the mode of convergence of the L-series, and the behavior of the L-series under certain term-by-term operations. The reader should note that no use has been made of the analytic properties of the coefficients of L. Indeed, the above lemmas (with the exception of Corollary 2 of Lemma 4) have been proved under the assumption that the ϕ's are twice continuously differentiable, which does not require analyticity of the coefficients. However, the generality which could be so obtained will not be of any consequence in this paper.

At this point, we want to turn our attention away from the abstract and focus on some specific examples of L-series. Much of our effort will be directed toward actually generating the set $\{\phi_n(t)\}_{n=0}^{\infty}$ for a particular example. For this purpose (3.8) is ill-suited and inefficient. Rather, the <u>generating function</u>, $g(t,\lambda)$, often offers the most sensible means for explicit determination of the set $\{\phi_n(t)\}_{n=0}^{\infty}$.

<u>Definition</u>: The generating function is $g(t,\lambda) = \sum_{n=0}^{\infty} \phi_n(t)\lambda^n$.

We observe that in the special case of the Taylor series expansion about $t = t_o$ the generating function is $g(t,\lambda) = \exp((t-t_o)\lambda)$. Then the generating function is an entire function of order 1 and type $(t-t_o)$. This description of the generating function is almost true in the general case. More precisely we have the following.

<u>Lemma</u> 12: The generating function, $g(t,\lambda)$, is an entire function of λ of order 1 and type at most $C_2(t-t_o)$ and at least $C_1(t-t_o)$ for $t\varepsilon N_1(t_o)$.

<u>Proof</u>: By Lemma 4 we have

$$d_1 \frac{(C_1(t-t_o))^n}{n!} \leq \phi_n(t) \leq d_2 \frac{(C_2(t-t_o))^n}{n!}, \quad t\varepsilon N_1(t_o),$$

and so $$d_1 \frac{(C_1(t-t_o))^n|\lambda|^n}{n!} \leq \phi_n(t)|\lambda|^n \leq d_2 \frac{(C_2(t-t_o))^n|\lambda|^n}{n!}.$$

This implies that, for all complex λ,

$$d_1\exp(C_1(t-t_o)|\lambda|) \leq \sum_{n=0}^{\infty} \phi_n(t)|\lambda|^n \leq d_2\exp(C_2(t-t_o)|\lambda|),$$

$$t\varepsilon N_1(t_o).$$

Consider each half of the inequality. From

$$\sum_{n=0}^{\infty} \phi_n(t)|\lambda|^n \le d_2 \exp(C_2(t-t_o)|\lambda|)$$

we see that $g(t,\lambda)$ has order at most 1 and type at most $C_2(t-t_o)$. If λ is real and positive the other half of the inequality becomes

$$d_1 \exp(C_1(t-t_o)\lambda) \le \sum_{n=0}^{\infty} \phi_n(t)\lambda^n$$

So $g(t,\lambda)$ has order at least 1 and type at least $C_1(t-t_o)$.

The generating function would, of course, be of little value without a practical scheme for its computation. That scheme is contained in the following.

Lemma 13: The generating function, $g(t,\lambda)$, may be realized as the solution of the initial value problem

$$Ly = \lambda^2 y, \; y(t_o) = 1, \; y'(t_o) = \lambda.$$

Proof: By (3.5) the solution for this problem is

$$y(t) = \phi_o(t_o) + \lambda\phi_1(t) + \lambda^2(Gy)(t).$$

This is a linear Volterra integral equation for y which we can solve by means of the Neumann series (successive substitutions, see Cochran [7]) to obtain

$$\begin{aligned} y(t) &= \sum_{k=0}^{\infty} \lambda^{2k} G^k(\phi_o + \lambda\phi_1)(t) \\ &= \sum_{k=0}^{\infty} \lambda^{2k}(G^k\phi_o)(t) + \lambda^{2k+1}(G^k\phi_1)(t) \\ &= \sum_{n=0}^{\infty} \lambda^n \phi_n(t) = g(t,\lambda). \end{aligned}$$

Remark: The fact that $g(t,\lambda)$ is an entire function of λ may also be deduced from the properties of the resolvent kernel of $G(t,\tau)$ (Cochran [7]).

Remark: In the case $a_2(t) \not\equiv 1$, $g(t,\lambda)$ may be obtained by solving

$$Ly = \lambda^2 y,\ y(t_o) = 1,\ \sqrt{a_2(t_o)}\ y'(t_o) = \lambda.$$

With the aid of the generating function we will examine some specific examples of L-series. The reference for all definitions and properties of special functions used in the examples is Abramowitz and Stegun [1].

Example 1: The simplest example of an L-series is the Taylor series expansion about the point $t = t_o$. For this example, the above lemmas yield either trivial or well known properties of the Taylor series.

Example 2: Let $Ly = y'' + y$ and $t_o = 0$. Then $\phi_o(t) = \cos t$ and $\phi_1(t) = \sin t$. The Green's function is

$$G(t,\tau) = \frac{\cos\tau\sin t - \cos t\sin\tau}{\cos\tau\cos\tau - (-\sin\tau)\sin\tau}$$

$$= \cos\tau\sin t - \cos t\sin\tau = \sin(t-\tau).$$

The generating function is obtained by solving the initial value problem

$$y'' + y = \lambda^2 y,\ y(o) = 1,\ y'(o) = \lambda,$$

to obtain

$$g(t,\lambda) = \lambda\frac{\sin\left(\sqrt{1-\lambda^2}\ t\right)}{\sqrt{1-\lambda^2}} + \cos\left(\sqrt{1-\lambda^2}\ t\right).$$

The generating function splits naturally into even and odd functions of λ. This implies that

$$\{\phi_{2k}(t)\}_{k=0}^{\infty} \text{ is generated by } \cos\left(\sqrt{1-\lambda^2}\, t\right)$$

and $$\{\phi_{2k+1}(t)\}_{k=0}^{\infty} \text{ is generated by } \lambda \frac{\sin\left(\sqrt{1-\lambda^2}\, t\right)}{\sqrt{1-\lambda^2}}$$

To obtain the ϕ's we make use of the familiar expansions

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{(2n)!} \quad \text{and} \quad \frac{\sin z}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}$$

to expand the components of the generating function. The resulting series are rearranged to obtain expansions in terms of powers of λ, i.e.,

$$\cos\left(\sqrt{1-\lambda^2}\, t\right) = \sum_{k=0}^{\infty} \left(\sum_{p=0}^{\infty} \binom{p+k}{k} \frac{(-1)^p t^{2p+2k}}{(2p+2k)!} \right) \lambda^{2k}$$

and $$\frac{\lambda \sin\left(\sqrt{1-\lambda^2}\, t\right)}{\sqrt{1-\lambda^2}} = \sum_{k=0}^{\infty} \left(\sum_{p=0}^{\infty} \binom{p+k}{k} \frac{(-1)^p t^{2p+2k+1}}{(2p+2k+1)!} \right) \lambda^{2k+1} .$$

Therefore $$\phi_{2k}(t) = \sum_{p=0}^{\infty} \binom{p+k}{k} \frac{(-1)^p t^{2p+2k}}{(2p+2k)!}$$

and $$\phi_{2k+1}(t) = \sum_{p=0}^{\infty} \binom{p+k}{k} \frac{(-1)^p t^{2p+2k+1}}{(2p+2k+1)!}, \quad k = 0, 1, 2, \ldots .$$

By means of a Gamma function identity (Legendre's duplication formula) and some simplifications we may express the ϕ's as

$$\phi_{2k}(t) = \frac{\sqrt{\pi}}{k!} \left(\frac{t}{2}\right)^{k+1/2} J_{k-1/2}(t)$$

and $$\phi_{2k+1}(t) = \frac{\sqrt{\pi}}{k!} \left(\frac{t}{2}\right)^{k+1/2} J_{k+1/2}(t)$$

where $J_\nu(t)$ is the Bessel function of the first kind of order ν defined by the series

$$J_\nu(t) = \sum_{p=0}^{\infty} \frac{(-1)^p (\frac{t}{2})^{2p+\nu}}{p!\Gamma(p+\nu+1)} .$$

Example 3: Let $Ly = y'' - y$ and $t_0 = 0$. Then $\phi_0(t) = \cosh t$ and $\phi_1(t) = \sinh t$. The Green's function is

$$G(t,\tau) = \frac{\cosh\tau \sinh t - \cosh t \sinh\tau}{\cosh\tau \cosh\tau - \sinh\tau \sinh\tau}$$
$$= \sinh(t-\tau).$$

The generating function is obtained by solving the initial value problem

$$y'' - y = \lambda^2 y,\ y(o) = 1,\ y'(o) = \lambda,$$

to obtain

$$g(t,\lambda) = \lambda \frac{\sinh(\sqrt{1+\lambda^2}\, t)}{\sqrt{1+\lambda^2}} + \cosh(\sqrt{1+\lambda^2}\, t).$$

As in the previous example, the generating function splits naturally into even and odd functions of λ. Thus

$$\{\phi_{2k}(t)\}_{k=0}^{\infty} \text{ is generated by } \cosh(\sqrt{1+\lambda^2}\, t)$$

and $$\{\phi_{2k+1}(t)\}_{k=0}^{\infty} \text{ is generated by } \lambda \frac{\sinh(\sqrt{1+\lambda^2}\, t)}{\sqrt{1+\lambda^2}} .$$

To obtain the ϕ's we make use of the Maclaurin expansions for $\cosh z$ and $\frac{\sinh z}{z}$ to expand the components of the generating function. The resulting series are rearranged to obtain expansions in terms of powers of λ from which one may read off the ϕ's. This procedure yields

$$\phi_{2k}(t) = \sum_{p=0}^{\infty} \binom{p+k}{k} \frac{t^{2p+2k}}{(2p+2k)!}$$

and
$$\phi_{2k+1}(t) = \sum_{p=0}^{\infty} \binom{p+k}{k} \frac{t^{2p+2k+1}}{(2p+2k+1)!} \quad .$$

By means of the same Gamma function identities used in the previous example we may express the ϕ's as

$$\phi_{2k}(t) = \frac{\sqrt{\pi}}{k!} \left(\frac{t}{2}\right)^{k+1/2} I_{k-1/2}(t)$$

and
$$\phi_{2k+1}(t) = \frac{\sqrt{\pi}}{k!} \left(\frac{t}{2}\right)^{k+1/2} I_{k+1/2}(t)$$

where $I_\nu(t)$ is the modified Bessel function of the first kind of order ν defined by the series

$$I_\nu(t) = \sum_{p=0}^{\infty} \frac{\left(\frac{t}{2}\right)^{2p+\nu}}{p!\,\Gamma(p+\nu+1)} \, .$$

Example 4: Let $Ly = t^2 y''$ and $t_0 = 1$. (Note that in this example $a_2(t) = t^2$ so that we shall have to make use of (3.3) and the second remark following Lemma 13.) Then $\phi_0(t) \equiv 1$ and $\phi_1(t) = t-1$. The Green's function is

$$G(t,\tau) = \frac{1}{\tau^2} \; \frac{1\cdot(t-1) - 1\cdot(\tau-1)}{1\cdot 1 - 0\cdot 1}$$

$$= \frac{t-\tau}{\tau^2} \quad .$$

The generating function is obtained by solving the initial value problem

$$t^2 y'' = \lambda^2 y \qquad y(1) = 1,\; y'(1) = \lambda$$

to obtain

$$g(t,\lambda) = t^{\lambda} = e^{\lambda \log t}.$$

The generating function may easily be expanded in powers of λ to yield

$$g(t,\lambda) = \sum_{n=0}^{\infty} \frac{(\log t)^n}{n!} \lambda^n.$$

Therefore $\phi_n(t) = \dfrac{(\log t)^n}{n!}$ for all n.

<u>Remarks on the Examples</u>: There are two basic observations that we want to make, one involving the asymptotics of the ϕ's and the other concerning the region of convergence (in the complex plane) of the L-series.

In examples 2, 3, and 4 we see that the similarity between the ϕ's and the Taylor polynomials, $\dfrac{(t-t_o)^n}{n!}$, as developed in the lemmas is truly of a local, rather than global, nature. (Although the lemmas have not been proved herein for $a_2(t) \not\equiv 1$, the results in that case are similar to those stated in the lemmas.) In general, when t is significantly different from t_o, $\phi_n(t)$ no longer resembles $\dfrac{(t-t_o)^n}{n!}$. This point is brought out dramatically by the examples. Making use of the well known asymptotics for Bessel functions

$$J_\nu(t) = \left(\frac{2}{\pi t}\right)^{1/2} \cos\left(t-\frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{t}\right) \qquad t\to\infty$$

and $$I_\nu(t) = (2\pi t)^{-1/2} e^t \left(1 + O\left(\frac{1}{t}\right)\right) \quad t\to\infty$$

We see that the ϕ's of Example 2 oscillate with an amplitude that grows as a power of t as $t \to \infty$ while the ϕ's of Example 3 grow exponentially as $t \to \infty$. On the other hand, in Example 4, $\phi_n(t) = \dfrac{(\log t)^n}{n!}$ which, for a given n, grows more slowly than t as $t \to \infty$.

The especially simple form of the ϕ's in Example 4 allows one to observe an interesting contrast between the region of convergence of the Taylor series expansion about t_o and the region of convergence of the L-series expansion about t_o (the regions mentioned are regions in the complex plane). This contrast is readily brought out by considering the expansion of the function $f(t) = t^{1/2}$ about the point $t = 1$. (The branch cut is taken along the negative real axis and the branch chosen is the one which produces positive values for $t^{1/2}$ when t is positive.) As is well known, the Taylor series converges inside a disc of radius 1 centered at $t = 1$. Since

$$t^{1/2} = e^{1/2\ \log t} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \frac{(\log t)^n}{n!}$$

(the branch of logt is chosen to make logt real when t is positive) we see that the L-series for f converges to f(t) in the entire complex t plane with the negative real axis deleted.

4. Expansion of Analytic Functions

In this section we will prove the first of our major results, a theorem on the expansion of analytic functions. In part, the result is motivated by consideration of the corollary to Lemma 4 which says that, for each n, $n!\phi_n(t)$ is asymptotic to $(t-t_o)^n$ as $t\to t_o$. Since the set $\{(t-t_o)^n\}_{n=0}^{\infty}$ is the best known basis for the expansion of functions analytic in a (complex) neighborhood of t_o, one is led to speculate on the possibility of expanding analytic functions in terms of a set of functions which are, in a suitable sense, "sufficiently like" the functions $\{(t-t_o)^n\}_{n=0}^{\infty}$. The possibility of expansion in terms of such a set was first established by S. Pincherle (Boas [3]) and such expansion sets are termed Pincherle bases.

Definition: Let z_o be a complex number. A Pincherle basis at z_o is a set of functions $\{g_n(z)\}_{n=0}^{\infty}$, analytic at z_o, having, for all n, the form

$$g_n(z) = (z-z_o)^n \left(1+h_n(z)\right) \tag{4.1}$$

where $$h_n(z) = \sum_{k=1}^{\infty} \gamma_k^{(n)} (z-z_o)^k, \quad (|z-z_o|<r_o,\ r_o>0),$$

which has the property that any function f(z), analytic in a neighborhood of z_o, may be expanded as

$$f(z) = \sum_n C_n g_n(z), \text{ for } |z-z_o|<r_1, (r_1\le r_0).$$

Our result, Theorem 1, will say that the set $\{n!\phi_n(z)\}_{n=0}^{\infty}$ forms a Pincherle basis at $t_o \epsilon I$. (If $Ly = y'' + a_1(t)y' + a_0(t)y$, where a_1 and a_0 are analytic in $|t-t_o|<R$, and if $r(t)$ is analytic in $|t-t_o|<R$, then any solution of $Ly = r$ is analytic in $|t-t_o|<R$ (Coddington and Levinson [8]). In view of (1.1), this implies that each $\phi_n(t)$ is analytic in $|t-t_o|<R$.) The proof will use a theorem on Pincherle bases due to Boas [3]. To state Boas' theorem we need a definition.

<u>Definition</u>: A <u>common majorant</u> of $\{h_n(z)\}_{n=0}^{\infty}$ (4.1) is a series

$$(4.2) \qquad h(z) = \sum_{k=1}^{\infty} \delta_k (z-z_o)^k \text{ (convergent for } |z-z_o|<r_1 \le r_0)$$

where $\quad |\gamma_k^{(n)}| \le \delta_k$ for $k = 1, 2, 3, \ldots$ and $n = 0, 1, 2, \ldots$.

<u>Theorem</u> (Boas): A function $f(t)$, analytic in a disc $|t-t_o|<s$, may be expanded in terms of the functions $g_n(t)$ (4.1) provided that

$$(4.3) \qquad h(s+t_o)<1.$$

Note that $h(t_o) = 0$. Thus, to show that each f admits a representation in terms of the g_n's in <u>some</u> neighborhood of t_o, it is sufficient to prove that h has a positive radius of convergence.

To prove Theorem 1 we shall take the set $\{g_n(t)\}_{n=0}^{\infty}$ to be

$$(4.4) \qquad g_n(t) = n!\phi_n(t)$$

for all n. Keeping (4.1) in mind, we shall also take

$$(4.5) \qquad h_n(t) = \frac{n!\phi_n(t)}{(t-t_o)^n} - 1 = \sum_{k=1}^{\infty} \frac{n!\phi_n^{(n+k)}(t_o)}{(n+k)!} (t-t_o)^k$$

for all n and

$$(4.6) \qquad \gamma_k^{(n)} = \frac{n!\phi_n^{(n+k)}(t_o)}{(n+k)!}$$

for all $n\geq 0$ and $k\geq 1$. The proof will involve the generation of a common majorant for $\{h_n(t)\}_{n=0}^{\infty}$ and the application of Boas' theorem.

At this point we shall prove a number of rather technical lemmas basically involving estimates on certain of the γ's, used in the proof of Theorem 1. The lemmas will, again, be proved for the case $a_2(t) \equiv 1$.

Definition: Let R be the common radius of convergence of the Taylor series expansions, about t_o, of $a_1(t)$ and $a_0(t)$. The real number σ is defined to be the smallest number such that $\sigma\geq 1/R$, $\sigma\geq 1$ and

$$(4.7) \qquad \left|\frac{a_1^{(j)}(t_o)}{j!}\right| \leq \sigma^{j+1} \text{ and } \left|\frac{a_0^{(j)}(t_o)}{j!}\right| \leq \sigma^{j+1}$$

for all $j \geq 0$.

Remark: We shall show later that the important inequality

$$(4.8) \qquad |\gamma_k^{(n)}| \leq \sigma^k$$

is satisfied for $n \geq 0$ and $k \geq 1$.

Lemma 14: If $p = 0, 1$ and $k \geq 2$, then

$$(4.9) \qquad \left|\frac{\phi_p^{(p+k)}(t_o)}{(p+k)!}\right| \leq \frac{1}{(p+k)(p+k-1)} \sum_{i=0}^{p+k-2} \{(i+1) \left|\frac{a_1^{(p+k-2-i)}(t_o)}{(p+k-2-i)!}\right| \cdot \left|\frac{\phi_p^{(i+1)}(t_o)}{(i+1)!}\right| + \left|\frac{a_0^{(p+k-2-i)}(t_o)}{(p+k-2-i)!}\right| \cdot \left|\frac{\phi_p^{(i)}(t_o)}{i!}\right|\}.$$

Proof: Since ϕ_0 and ϕ_1 are solutions of the homogeneous equation we have

$$\phi_p^{(2)}(t) = -a_1(t)\phi_p^{(1)}(t) - a_0(t)\phi_p(t)$$

for $p = 0, 1$. Applying Leibniz' rule for the successive derivatives of a product, we have

$$\phi_p^{(p+k)}(t) = -\sum_{i=0}^{p+k-2} \binom{p+k-2}{i} a_1^{(p+k-2-i)}(t)\phi_p^{(i+1)}(t)$$

$$-\sum_{i=0}^{p+k-2} \binom{p+k-2}{i} a_0^{(p+k-2-i)}(t)\phi_p^{(i)}(t)$$

for $k \geq 2$. Division of both sides by $(p+k)!$ and some cancellation will yield

$$(4.10) \qquad \frac{\phi_p^{(p+k)}(t)}{(p+k)!} = -\frac{1}{(p+k)(p+k-1)} \sum_{i=0}^{p+k-2} (i+1) \frac{a_1^{(p+k-2-i)}(t)}{(p+k-2-i)!}$$

$$\cdot \frac{\phi_p^{(i+1)}(t)}{(i+1)!} - \frac{1}{(p+k)(p+k-1)} \sum_{i=0}^{p+k-2} \frac{a_0^{(p+k-2-i)}(t)}{(p+k-2-i)!} \cdot \frac{\phi_p^{(i)}(t)}{i!} .$$

To complete the proof we take absolute values of both sides of (4.10), apply the triangle inequality to the right side and evaluate the expressions at $t = t_o$ to obtain (4.9).

<u>Lemma</u> 15: If $p = 0, 1$ and $k \geq 1$, then

$$\left| \frac{\phi_p^{(p+k)}(t_o)}{(p+k)!} \right| \leq \sigma^k.$$

<u>Proof</u>: By definition of ϕ_0, we have $\phi_0^{(1)}(t_o) = 0$. By definition of ϕ_1 we have

$$\phi_1^{(2)}(t_o) = -a_1(t_o)\phi_1^{(1)}(t_o) - a_0(t_o)\phi_1(t_o) = -a_1(t_o).$$

Thus the inequality is satisfied for k = 1. For the case $k \geq 2$ we will use the inequality (4.9) of Lemma 14 and an induction argument. To perform the induction, we assume that

$$\left|\frac{\phi_p^{(p+j)}(t_o)}{(p+j)!}\right| \leq \sigma^j \text{ for } j = 1, 2, \ldots, k-1 \ (k \geq 2).$$

By definition of σ we have $\left|\frac{a_0^{(j)}(t_o)}{j!}\right| \leq \sigma^{j+1}$ and $\left|\frac{a_1^{(j)}(t_o)}{j!}\right| \leq \sigma^{j+1}$ for all j (4.7). The use of these estimates in (4.9) implies that

$$\left|\frac{\phi_p^{(p+k)}(t_o)}{(p+k)!}\right| \leq \frac{1}{(p+k)(p+k-1)} \sum_{i=0}^{p+k-2} \{(i+1)\sigma^{p+k-2-i+1} \cdot \sigma^{i+1-p}$$

$$+ \sigma^{p+k-2-i+1} \cdot \sigma^{i-p}\}$$

$$= \frac{\sigma^{k-1}}{(p+k)(p+k-1)} \sum_{i=0}^{p+k-2} \{\sigma(i+1) + 1\}$$

$$= \frac{\sigma^{k-1}}{(p+k)(p+k-1)} \{\frac{\sigma}{2}(p+k-1)(p+k) + (p+k-1)\}$$

$$= \frac{\sigma^k}{2} + \frac{\sigma^{k-1}}{p+k}.$$

Since $k \geq 2$ and $\sigma \geq 1$, $\frac{1}{p+k} \leq \frac{\sigma}{2}$; therefore

$$\left|\frac{\phi_p^{(p+k)}(t_o)}{(p+k)!}\right| \leq \sigma^k,$$

and the proof is complete.

Lemma 16: For $n \geq 2$ and $k \geq 2$ we have

$$\left|\frac{n!\phi_n^{(n+k)}(t_o)}{(n+k)!}\right| \le \frac{1}{(n+k)(n+k-1)}\Big\{\sum_{i=0}^{k-1}(n+i)\left|\frac{a_1^{(k-1-i)}(t_o)}{(k-1-i)!}\right| \tag{4.11}$$

$$\cdot \left|\frac{n!\phi_n^{(n+i)}(t_o)}{(n+i)!}\right| + \sum_{i=0}^{k-2}\left|\frac{a_0^{(k-2-i)}(t_o)}{(k-2-i)!}\right| \cdot \left|\frac{n!\phi_n^{(n+i)}(t_o)}{(n+i)!}\right|$$

$$+ n(n-1) \cdot \left|\frac{(n-2)!\phi_{n-2}^{(n-2+k)}(t_o)}{(n-2+k)!}\right|\Big\}.$$

Remark: The reader should note that we require $n \geq 2$. If the ϕ_{n-2} term is suppressed and n is set equal to 0, inequality (4.9) is not obtained.

Proof: From (1.1) we have

$$\phi_n^{(2)}(t) = -a_1(t)\phi_n^{(1)}(t) - a_0(t)\phi_n(t) + \phi_{n-2}(t)$$

for $n \geq 2$. By Leibniz' rule we have

$$\phi_n^{(n+k)}(t) = -\sum_{i=0}^{n+k-2} \binom{n+k-2}{i} a_1^{(n+k-2-i)}(t)\phi_n^{(i+1)}(t) \tag{4.12}$$

$$-\sum_{i=0}^{n+k-2} \binom{n+k-2}{i} a_0^{(n+k-2-i)}(t)\phi_n^{(i)}(t) + \phi_{n-2}^{(n-2+k)}(t).$$

If we evaluate this expression at $t = t_o$, and realize that $\phi_n^{(j)}(t_o) = 0$ when $j < n$, then we obtain

$$\phi_n^{(n+k)}(t_o) = -\sum_{i=n-1}^{n+k-2} \binom{n+k-2}{i} a_1^{(n+k-2-i)}(t_o)\, \phi_n^{(i+1)}(t_o)$$

$$-\sum_{i=n}^{n+k-2} \binom{n+k-2}{i} a_0^{(n+k-2-i)}(t_o)\, \phi_n^{(i)}(t_o) + \phi_{n-2}^{(n-2+k)}(t_o)$$

$$= - \sum_{i=0}^{k-1} \binom{n+k-2}{n-1+i} a_1^{(k-1-i)}(t_o)\, \phi_n^{(n+i)}(t_o)$$

$$- \sum_{i=0}^{k-2} \binom{n+k-2}{n+i} a_0^{(k-2-i)}(t_o)\, \phi_n^{(n+i)}(t_o)$$

$$+ \phi_{n-2}^{(n-2+k)}(t_o).$$

Multiplication of both sides by $\frac{n!}{(n+k)!}$ will yield

$$\frac{n!\phi_n^{(n+k)}(t_o)}{(n+k)!} = \frac{1}{(n+k)(n+k-1)} \{- \sum_{i=0}^{k-1} (n+1) \frac{a_1^{(k-1-i)}(t_o)}{(k-1-i)!} \cdot \frac{n!\phi_n^{(n+i)}(t_o)}{(n+i)!}$$

$$(4.13) \qquad - \sum_{i=0}^{k-2} \frac{a_0^{(k-2-i)}(t_o)}{(k-2-i)!} \cdot \frac{n!\phi_n^{(n+i)}(t_o)}{(n+i)!}$$

$$+ n(n-1) \cdot \frac{(n-2)!\, \phi_{n-2}^{(n-2+k)}(t_o)}{(n-2+k)!} \} .$$

Finally, taking absolute values of both sides of (4.13) and applying the triangle inequality on the right we have (4.11) for $n \geq 2$ and $k \geq 2$.

Lemma 17: If $n \geq 0$ and $k = 1, 2$, then

$$\left| \frac{n!\, \phi_n^{(n+k)}(t_o)}{(n+k)!} \right| \leq \sigma^k .$$

Proof: We first note that the cases $n = 0$ and $n = 1$ have been proved in Lemma 15, so we need only be concerned with the case $n \geq 2$ and will automatically assume that we are dealing with such n for the remainder of the proof. In both cases, $k = 1$ and $k = 2$, we will use an induction argument based upon an inequality similar to (4.11) for the case $k = 1$ and upon (4.11) itself for the case $k = 2$.

Case $k = 1$. First we must derive an inequality that will be used in the induction argument. If (4.12) is evaluated at $t = t_o$ for

$k = 1$, the second sum vanishes ($\phi_n^{(j)}(t_o) = 0$ when $j < n$) and the first sum reduces to a single term, producing the simple equation

$$\phi_n^{(n+1)}(t_o) = -a_1(t_o)\ \phi_n^{(n)}(t_o) + \phi_{n-2}^{(n-1)}(t_o)$$

$$= -\ a_1(t_o) + \phi_{n-2}^{(n-1)}(t_o).$$

Multiplication of both sides by $\frac{n!}{(n+1)!}$ and application of the triangle inequality will produce

(4.14)
$$\left| \frac{n!\ \phi_n^{(n+1)}(t_o)}{(n+1)!} \right| \leq \frac{|a_1(t_o)|}{n+1} + \frac{n-1}{n+1} \left| \frac{(n-2)!\ \phi_{n-2}^{(n-2+1)}(t_o)}{(n-2+1)!} \right|$$

which will play a key role in the induction argument.

For that argument we shall show that if

$$\left| \frac{p!\ \phi_p^{(p+1)}(t_o)}{(p+1)!} \right| \leq \sigma$$

is satisfied for $p = n-2$, then it is satisfied for $p = n$. So, we shall assume that it is true for $p = n-2$. Recalling from (4.7) that $|a_1(t_o)|<\sigma$, we may deduce, via (4.14), that

$$\left| \frac{n!\ \phi_n^{(n+1)}(t_o)}{(n+1)!} \right| \leq \frac{\sigma}{n+1} + \frac{n-1}{n+1}\sigma = \left(\frac{n}{n+1}\right)\sigma < \sigma.$$

Case k=2. As above, we shall show that if

$$\left| \frac{p!\ \phi_p^{(p+2)}(t_o)}{(p+2)!} \right| \leq \sigma^2$$

is satisfied for $p = n-2$, then it is satisfied for $p = n$. So, for the induction hypothesis, we shall assume that it is satisfied for

$p = n - 2$. For $k = 2$, (4.11) becomes

$$\left| \frac{n!\, \phi_n^{(n+2)}(t_o)}{(n+2)!} \right| \le \frac{1}{(n+2)(n+1)} \Big\{ n \cdot \left| \frac{a_1^{(1)}(t_o)}{1!} \right| \cdot \left| \frac{n!\, \phi_n^{(n)}(t_o)}{n!} \right|$$

$$+ (n+1) \cdot \left| \frac{a_1(t_o)}{0!} \right| \cdot \left| \frac{n!\phi_n^{(n+1)}(t_o)}{(n+1)!} \right|$$

$$+ \left| \frac{a_0(t_o)}{0!} \right| \cdot \left| \frac{n!\, \phi_n^{(n)}(t_o)}{n!} \right|$$

$$+ n(n-1) \cdot \left| \frac{(n-2)!\, \phi_{n-2}^{(n-2+2)}(t_o)}{(n-2+2)!} \right| \Big\}.$$

We shall now apply, to this inequality, the induction hypothesis, the results of the case $k = 1$, and (4.7) to obtain

$$\left| \frac{n!\, \phi_n^{(n+2)}(t_o)}{(n+2)!} \right| \le \frac{1}{(n+2)(n+1)} \{ n \cdot \sigma^2 + (n+1) \cdot \sigma \cdot \sigma$$

$$+ \sigma + n(n-1)\sigma^2 \}$$

$$= \frac{1}{(n+2)(n+1)} \{ (n^2 + n + 1)\sigma^2 + \sigma \}$$

$$\le \sigma^2 \frac{n^2+n+2}{n^2+3n+2} < \sigma^2. \text{ (Since } \sigma \ge 1)$$

<u>Lemma</u> 18: If $n \ge 2$ and $k \ge 2$, then

$$\text{(4.15)} \qquad \frac{n(n+k-1) + (\frac{1}{2}k+1)(k-1)}{(n+k)(n+k-1)} < 1 .$$

<u>Proof</u>: The inequality will be proved by a fairly classical technique, i.e. working backwards until we reach an inequality which

is obviously true. We will apply to (4.15) the following reversible sequence of steps:

$$\frac{n(n+k-1) + (\frac{1}{2}k+1)(k-1)}{(n+k)(n+k-1)} < 1,$$

$$n(n+k-1) + (\tfrac{1}{2}k+1)(k-1) < (n+k)(n+k-1),$$

$$(\tfrac{1}{2}k+1)(k-1) < k(n+k-1),$$

$$(\tfrac{1}{2}k+1)(k-1) < k\cdot n + k(k-1)$$

and $$0 < k \cdot n + (\tfrac{1}{2}k-1)(k-1).$$

Since the last inequality is obviously true for $n \geq 2$ and $k \geq 2$, we may reverse the order of the steps to prove (4.15).

This completes the task of proving the lemmas required for the proof of Theorem 1. Thus, we are now in a position to state and prove Theorem 1, one of our main results.

Theorem 1: The set of functions $\{n!\phi_n(t)\}_{n=0}^{\infty}$ forms a Pincherle basis at $t = t_o$.

Proof: The proof will consist of an induction argument to show that (4.8) is valid for $n \geq 0$ and $k \geq 1$, then applying Boas' theorem utilizing the majorant

$$h(t) = \sum_{k=1}^{\infty} \sigma^k (t-t_o)^k$$

which has radius of convergence $\frac{1}{\sigma}$. The notation used in the proof is defined by equations (4.1), (4.2), (4.4), (4.5) and (4.6). In the course of the proof, the reader may find it helpful to consult Table 1

Table 1

Taylor Coefficients of the Functions $\{h_n(t)\}$

	k=1	k=2	k=3	k=4	k=5	
n=0	$\phi_0^{(1)}(t_o)$	$\frac{\phi_0^{(2)}(t_o)}{2!}$	$\frac{\phi_0^{(3)}(t_o)}{3!}$	$\frac{\phi_0^{(4)}(t_o)}{4!}$	$\frac{\phi_0^{(5)}(t_o)}{5!}$	...
n=1	$\frac{\phi_1^{(2)}(t_o)}{2!}$	$\frac{\phi_1^{(3)}(t_o)}{3!}$	$\frac{\phi_1^{(4)}(t_o)}{4!}$	$\frac{\phi_1^{(5)}(t_o)}{5!}$	$\frac{\phi_1^{(6)}(t_o)}{6!}$	...
n=2	$\frac{2!\ \phi_2^{(3)}(t_o)}{3!}$	$\frac{2!\ \phi_2^{(4)}(t_o)}{4!}$	$\frac{2!\ \phi_2^{(5)}(t_o)}{5!}$	$\frac{2!\ \phi_2^{(6)}(t_o)}{6!}$	$\frac{2!\ \phi_2^{(7)}(t_o)}{7!}$	...
n=3	$\frac{3!\ \phi_3^{(4)}(t_o)}{4!}$	$\frac{3!\ \phi_3^{(5)}(t_o)}{5!}$	$\frac{3!\ \phi_3^{(6)}(t_o)}{6!}$	$\frac{3!\ \phi_3^{(7)}(t_o)}{7!}$	$\frac{3!\ \phi_3^{(8)}(t_o)}{8!}$	...
n=4	$\frac{4!\ \phi_4^{(5)}(t_o)}{5!}$	$\frac{4!\ \phi_4^{(6)}(t_o)}{6!}$	$\frac{4!\ \phi_4^{(7)}(t_o)}{7!}$	$\frac{4!\ \phi_4^{(8)}(t_o)}{8!}$	$\frac{4!\ \phi_4^{(9)}(t_o)}{9!}$	...
	⋮	⋮	⋮	⋮	⋮	

which lists the first few Taylor coefficients of the first few h_n's.

Inequality (4.8) has already been established for the first two rows of the table (Lemma 15) and the first two columns of the table (Lemma 17). Thus, the proof of the theorem hinges upon the proof of the inequality for the remainder of the table ($n \geq 2$, $k \geq 3$).

We enumerate the entries in Table 1 by the following rule. If $n \geq 0$ and $k \geq 1$, assign to the pair (n,k) the integer

$$\mu(n,k) = \frac{(n+k)(n+k-1)}{2} - n.$$

This is the "standard" enumeration of positive integer pairs. We shall prove (4.8) by induction on $\mu(n,k)$. If (n,k) is given we assume that

$$\left| \frac{m!\ \phi_m^{(m+j)}(t_o)}{(m+j)!} \right| < \sigma^j$$

whenever $m + j < n + k$, or when $m + j = n + k$ but $m > n$. Then by (4.11), (4.7) and the induction hypothesis, we have

$$\left| \frac{n!\ \phi_n^{(n+k)}(t_o)}{(n+k)!} \right| \leq \frac{1}{(n+k)(n+k-1)} \{ \sum_{i=0}^{k-1} (n+i)\sigma^{k-i} \cdot \sigma^i$$

$$+ \sum_{i=0}^{k-2} \sigma^{k-1-i} \cdot \sigma^i + n(n-1) \cdot \sigma^k \}$$

$$= \frac{1}{(n+k)(n+k-1)} \{ \sigma^k \left(n \cdot k + \frac{(k-1)k}{2}\right)$$

$$+ \sigma^{k-1}(k-1) + n(n-1)\sigma^k \}$$

$$\leq \frac{\sigma^k}{(n+k)(n+k-1)} \{ n \cdot k + \frac{k}{2}(k-1) + (k-1) + n(n-1) \}$$

(Since $\sigma \geq 1$)

$$= \sigma^k \frac{\left(n(n+k-1) + (\frac{k}{2}+1)(k-1)\right)}{(n+k)(n+k-1)}$$

$$< \sigma^k,$$

the last step being a consequence of Lemma 18.

This completes the proof of (4.8) and, by virtue of our remarks at the beginning of the section, Theorem 1.

Remark: The series $h(t)$ may be expressed in closed form as

$$h(t) = \frac{\sigma(t-t_o)}{1-\sigma(t-t_o)} \quad \text{for } |t-t_o| < \frac{1}{\sigma}$$

Then, for $s > 0$, $h(s+t_o) = \frac{\sigma s}{1-\sigma s} < 1$ if $s < \frac{1}{2}\frac{1}{\sigma}$. Therefore, by Boas' theorem, a function $f(t)$ analytic in the disc $|t-t_o| < s < \frac{1}{2\sigma}$ may be expanded as

$$(4.16) \qquad f(t) = \sum_{n=0}^{\infty} b_n g_n(t) = \sum_{n=0}^{\infty} b_n\, n!\, \phi_n(t) = \sum_{n=0}^{\infty} a_n \phi_n(t)$$

where $a_n = b_n \cdot n!$ for all n.

There is an immediate corollary to the proof of Theorem 1.

Corollary: If $n \geq 0$ and if $|t-t_o| < \frac{1}{\sigma}$, then

$$(4.17) \qquad |\phi_n(t)| \leq \frac{|t-t_o|^n}{n!} \; \frac{1}{1-\sigma|t-t_o|} \,.$$

We want to make two observations on the results developed so far. The first observation concerns the analyticity of the sum of a convergent L-series and the second concerns the explicit form of the coefficients a_n in (4.16).

Remark: The corollary implies that the convergence of an L-series in a disc centered at t_o will cause the same L-series to be uniformly convergent in a slightly smaller disc centered at t_o. (Briefly, the argument involves realizing that the series must converge for some real $t^* > t_o$, applying the first half of Lemma 5 and then appealing to the above corollary.) Since all of the ϕ's are analytic in some fixed neighborhood of t_o, the uniform convergence of the series implies that the sum of the series is analytic in a neighborhood of t_o. This, in turn, implies that the <u>only</u> functions which are expandable in an L-series are those which are analytic in a neighborhood of t_o. For functions of a real variable, this means that any function expandable in an L-series may be extended into the complex plane to a function analytic in a complex neighborhood of t_o.

<u>Remark</u>: The previously noted uniform convergence implied by (4.17) permits the term-by-term differentiation of (4.16) an arbitrary number of times (Hille [13]). Thus, we will be justified in applying L and its iterates term-by-term and in differentiating the resulting series. In consequence of this, we may deduce a complex analog of Lemma 11 and conclude that the coefficients in the expansion (4.16) are given by

$$a_{2k} = L^k f(t_o) \text{ and } a_{2k+1} = (L^k f)'(t_o)$$

for $\quad k = 0, 1, 2, \ldots .$

The results of this section have, until now, been obtained under the restriction that $a_2(t) \equiv 1$. We now wish to remove that restriction.

Theorem 1': Let $(Ly)(t) = a_2(t)y''(t) + a_1(t)y'(t) + a_o(t)y(t)$ where a_2, a_1 and a_0 are real-analytic on I and $a_2(t) > 0$ for $t\varepsilon I$. If $f(t)$ is analytic in a (complex) neighborhood of t_o, then in some neighborhood of t_o f may be expanded as

$$f(t) = \sum_{n=0}^{\infty} a_n \phi_n(t)$$

where $a_{2k} = L^k f(t_o)$ and $a_{2k+1} = \sqrt{a_2(t_o)}\ (L^k f)'(t_o)$ for $k = 0, 1, 2, \ldots$.

Proof: The basic idea in this proof is the introduction of a change of independent variable which will transform the problem to one for which Theorem 1 applies. The new independent variable will be such that Ly is transformed into an expression with 1 as the coefficient of the second derivative term. We shall see that changing to the variable T, defined by

$$(4.18) \qquad T = \int_{t_o}^{t} \frac{du}{\sqrt{a_2(u)}} ,$$

will accomplish this. (The branch of $\sqrt{z}$ is the one which is positive when z is positive. This allows T to be real when t is.) This change of variable is invertible. Indeed, we see that in a neighborhood of t_o, T is an analytic function of t and $T(t_o) = 0$, so we may apply the Inverse Function Theorem (Hille [13]) to conclude that t is an analytic function of T in a neighborhood of $T = 0$. Thus, if $p(t)$ is an analytic function of t in a neighborhood of $t = t_o$, then $P(T) = p\big(t(T)\big)$ is an analytic function of T in a neighborhood of $T = 0$.

We shall now verify that this change of variable proves Theorem 1'. Define the functions A_0, A_1, A_2 and Y by

(4.19)
$$A_0(T) = a_0\big(t(T)\big) \ , \ A_1(T) = a_1\big(t(T)\big) \ , \ A_2(T) = a_2\big(t(T)\big) \ ,$$

and
$$Y(T) = y\big(t(T)\big) .$$

By the chain rule we have

$$y'(t) = \frac{dY}{dT} \cdot \frac{dT}{dt} = \frac{dY}{dT} \cdot \frac{1}{\sqrt{a_2(t)}} = \frac{1}{\sqrt{A_2(T)}} \cdot \frac{dY}{dT}$$

and
$$y''(t) = \frac{d}{dT}\big(y'(t)\big) \cdot \frac{dT}{dt}$$

$$= \frac{1}{A_2(T)} \cdot \frac{d^2Y}{dT^2} - \frac{1}{2} \frac{A_2'(T)}{\big(A_2(T)\big)^2} \frac{dY}{dT}.$$

If we apply these facts to the transformation of Ly, we have

$$Ly = a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t)$$

$$= \frac{d^2Y}{dT^2} + \left(\frac{A_1(T)}{\sqrt{A_2(T)}} - \frac{1}{2}\frac{A_2'(T)}{A_2(T)}\right)\frac{dY}{dT} + A_0(T)Y$$

$$= L_T Y$$

Formally, the operator L_T is of the proper form. Indeed, $A_0(T)$ is obviously real analytic in a neighborhood of $T = 0$ and, since $A_2(T)$ is nonzero in a neighborhood of $T = 0$, the coefficient of $\frac{dY}{dT}$ is also real analytic in a neighborhood of $T=0$. Therefore, we are justified in invoking all of our previous results in the discussion of L_T.

Before proceeding any further we must define the set of functions $\{\Phi_n(T)\}_{n=0}^{\infty}$ by

$$L_T\Phi_0 = L_T\Phi_1 = 0,\ \Phi_0(0) = 1,\ \Phi_0'(0) = 0,\ \Phi_1(0) = 0,\ \Phi_1'(0) = 1$$

and $$L_T\Phi_n = \Phi_{n-2},\ \Phi_n(0) = \Phi_n'(0) = 0 \text{ for } n \geq 2.$$

These are simply the L-basis functions for L_T. The reader should note that if the change of variable, $t \to T$, is introduced in (1.1) the above equations are obtained. That is, $\Phi_n(T) = \phi_n\big(t(T)\big)$ for all n.

Since f(t) is analytic in a neighborhood of t_o, then $F(T) = f\big(t(T)\big)$ is analytic in a neighborhood of T=0. Therefore, we may invoke Theorem 1 and the remarks following it to conclude that

$$F(T) = \sum_{k=0}^{\infty} \left(L_T^k F(T)\big|_{T=0}\right) \Phi_{2k}(T) + \left(\frac{d}{dT} L_T^k F(T)\big|_{T=0}\right) \Phi_{2k+1}(T)$$

in a neighborhood of T=0. Inverting the change of variables we obtain

$$f(t) = \sum_{k=0}^{\infty} L^k f(t_o)\phi_{2k}(t) + \sqrt{a_2(t_o)}\ (L^k f)'(t_o)\phi_{2k+1}(t)$$

in a neighborhood of t_o.

Some Consequences of Theorem 1 (and 1'): With the results of Theorem 1' in hand, one may derive an interesting integral representation for analytic functions and a generalization of the Laurent expansion.

Integral Representation: We shall derive an integral representation which reduces to the Cauchy Integral Formula when Ly = y". Although we have not proved herein the analog of Lemma 5 for the case

$a_2(t) \not\equiv 1$ the result is basically the same as the one stated. Thus we assert that, if f is analytic in a neighborhood of t_o, the series

$$\sum_{k=0}^{\infty} L^k f(t_o) \frac{z^{2k}}{(2k)!} + \sqrt{a_2(t_o)} \; (L^k f)'(t_o) \frac{z^{2k+1}}{(2k+1)!} = \hat{f}(z)$$

converges uniformly for $|z|$ sufficiently small. We define the function

$$\hat{g}(t,z) = \sum_{n=0}^{\infty} \phi_n(t) \frac{n!}{z^{n+1}}$$

(which is simply the Laplace transform of $g(t,\lambda)$ with respect to λ) and note that in view of the asymptotics of the ϕ's, the series converges for $|t-t_o|$ sufficiently small.

Let A be an annulus in the z plane centered at $z = 0$ which is such that the series defining $\hat{f}(z)$ converges for all $z \epsilon A$. Choose $r > 0$ so that the series which defines $\hat{g}(t,z)$ converges for $z \epsilon A$ whenever $|t-t_o| < r$.

If C is a circle contained in the interior of A and centered at $z = 0$, then we may apply the complex convolution to evaluate the integral $\oint \hat{f}(z)\hat{g}(t,z)dz$ and obtain

$$\frac{1}{2\pi i} \oint_C \hat{f}(z)\hat{g}(t,z)dz = \sum_{k=0}^{\infty} L^k f(t_o)\phi_{2k}(t) + \sqrt{a_2(t_o)} \; (L^k f)'(t_o)\phi_{2k+1}(t)$$

$$= f(t).$$

Thus, we have the integral representation

$$f(t) = \frac{1}{2\pi i} \oint_C \hat{f}(z)\hat{g}(t,z)dz$$

which, in the case Ly=y", reduces to Cauchy's integral formula.

Generalized Laurent Expansion: We shall now derive an expansion which is a generalization of the classical Laurent expansion. The derivation is based upon the expansion of the Cauchy kernel, $\frac{1}{z-t}$, in an L-series.

The Cauchy kernel is an analytic function of t for $t \neq z$ and, in particular, for $0 \leq |t-t_o| < |z-t_o|$. Thus, by Theorem 1', we have

$$(4.20)\qquad \frac{1}{z-t} = \sum_{n=0}^{\infty} C_n(z)\phi_n(t)$$

for t in some neighborhood of t_o. (The coefficients, $C_n(z)$, are polynomials in $\frac{1}{z-t_o}$ of degree n+1 for all n.)

The Cauchy kernel is also an analytic function of z, so by Theorem 1' we have

$$(4.21)\qquad \frac{1}{z-t} = -\sum_{n=0}^{\infty} C_n(t)\phi_n(z)$$

for z in a neighborhood of t_o.

The derivation will involve the careful use of both of these expansions for the Cauchy kernel. We choose the constant r so that

$$\sum_n C_n(z)\phi_n(t)$$

converges for $|t-t_o| < |z-t_o| < r$ and let A be an annulus, $A = \{w: r_1 < |w-t_o| < r_2\}$, where $r_2 < r$ and $r_1 > 0$. Let c_1 and c_2 be circles centered at t_o with radii r_1^* and r_2^* respectively, where $r_1 < r_1^* < r_2^* < r_2$. If f(t) is analytic for $t \epsilon A$ then we may represent f as

$$f(t) = \frac{1}{2\pi i}\oint_{c_2} \frac{f(z)}{z-t}\,dz - \frac{1}{2\pi i}\oint_{c_1} \frac{f(z)}{z-t}\,dz$$

for $r_1^* < |t-t_o| < r_2^*$. Now replace the Cauchy kernels by their L-series expansions, using (4.20) in the integral over c_2 and (4.21) in the integral over c_1. Then we have

$$\begin{aligned} f(t) &= \frac{1}{2\pi i}\oint_{c2} f(z)\left(\sum_{n=0}^{\infty} C_n(z)\phi_n(t)\right) dz \\ &\qquad + \frac{1}{2\pi i}\oint_{c1} f(z)\left(\sum_{n=0}^{\infty} C_n(t)\phi_n(z)\right) dz \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{c_2} f(z)C_n(z)\,dz\right)\phi_n(t) \\ &\qquad + \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{c1} f(z)\phi_n(z)\,dz\right) C_n(t). \end{aligned}$$

If we define the constants α_n and β_n by

$$\alpha_n = \frac{1}{2\pi i}\oint_{c2} f(z)C_n(z)dz \text{ and } \beta_n = \frac{1}{2\pi i}\oint_{c_1} f(z)\,\phi_n(z)dz,$$

then the expansion may be written as

$$f(t) = \sum_{n=0}^{\infty}\alpha_n\phi_n(t) + \sum_{n=0}^{\infty}\beta_n C_n(t)$$

for $r_1^* < |t-t_o| < r_2^*$. In the case Ly=y" this reduces to the classical Laurent expansion for f.

5. Analyticity of L-Positive Functions

In this section we shall prove our other major result, L-positivity of a function f implies analyticity of f. For the convenience of the reader we shall repeat the definition of L-positivity.

Definition: A function $f(t)\varepsilon C^{\infty}(I)$ is termed L-positive at $t_o \varepsilon I$ if it satisfies the two positivity conditions, $L^k f(t) \geq 0$, $t\varepsilon I$, $k = 0, 1, 2, \ldots$ and $(L^k f)'(t_o) \geq 0$, $k = 0, 1, 2, \ldots$.

We will prove two forms of Theorem 2, first a weak version, and then a strong version. The weak version, which requires $(L^k f)'(t) \geq 0$ on an interval, will be used to prove the strong version. As in Theorem 1 we will do the proof with the restriction $a_2(t) \equiv 1$, then remove the restriction by an appropriate change of independent variable. The proof itself will depend very heavily upon a theorem on generalized convexity proved independently by M. M. Peixoto [15] and F. F. Bonsall [6]. To state the theorem, we shall impose a condition on I, which, until now, has been an arbitrary open interval on the real line. We shall require that I be such that if $t^* \varepsilon I$ and $\phi(t)$ is a solution of $Ly = 0$ satisfying $\phi(t^*) = 0$ then ϕ does not vanish at any other point of I. (For example, if $Ly = y'' + y$ this condition will force the length of I to be less than π. This restriction will pose no problem since the results being proved are of a local nature.)

<u>Theorem</u> (Peixoto-Bonsall): Let $f(t)\varepsilon C^2(I)$ and $Lf(t) \geq 0$ for $t\varepsilon I$. Then for $t_o\varepsilon I$ and $t_1\varepsilon I$, with $t_1 > t_o$, we have

$$f(t) \leq f(t_o)\phi_0(t) + \frac{f(t_1) - f(t_o)\phi_0(t_1)}{\phi_1(t_1)}\phi_1(t)$$

for $t_o \leq t \leq t_1$. (The functions ϕ_0 and ϕ_1 have their usual meaning (1.1).)

<u>Remark</u>: For the case $Ly = y''$, this becomes the familiar result that if $f''(t) \geq 0$ for $t\varepsilon I$, then for $t_o \leq t \leq t_1$, the graph of $f(t)$ lies below line joining the points $\left(t_o, f(t_o)\right)$ and $\left(t_1, f(t_1)\right)$.

<u>Remark</u>: In view of the remark following Theorem 1, we need only show that L-positivity of f will imply that f is represented by its L-series.

<u>Theorem</u> 2 (Weak Version): Let $f(t)\varepsilon C^\infty(I)$ be such that $L^kf(t) \geq 0$ and $(L^kf)'(t) \geq 0$ for all $t\varepsilon I$ and $k = 0, 1, 2, \ldots$. Then there exists $\varepsilon > 0$ such that

$$f(t) = \sum_{k=0}^{\infty} L^kf(t_o)\phi_{2k}(t) + (L^kf)'(t_o)\phi_{2k+1}(t)$$

for $t\varepsilon[t_o, t_o+\varepsilon]$.

<u>Proof</u>: We will start by considering the finite L-series

$$f(t) = \sum_{k=0}^{n} L^kf(t_o)\phi_{2k}(t) + (L^kf)'(t_o)\phi_{2k+1}(t) + G^{n+1}L^{n+1}f$$

for $t\varepsilon N_3(t_o)$, where $N_3(t_o) = (t_o, t_o+b_3)$ is the intersection of the right neighborhoods $N_o(t_o)$, $N_1(t_o)$ and $N_2(t_o)$. By the hypotheses and by Lemma 1 we have $L^kf(t_o)\phi_{2k}(t) \geq 0$ and $(L^kf)'(t_o)\phi_{2k+1}(t) \geq 0$

for $t \epsilon N_3(t_o)$ and for all k. By the hypothesis, $L^{n+1}f(t) \geq 0$, and by Corollary 2 of Lemma 1 we have

$$G^{n+1}L^{n+1}f(t) \geq 0$$

for $t \epsilon N_3(t_o)$ and for all n. Therefore,

$$\sum_{k=0}^{n} L^k f(t_o)\phi_{2k}(t) + (L^k f)'(t_o)\phi_{2k+1}(t)$$

is a series of non-negative terms which is bounded above by f(t) for $t \epsilon N_3(t_o)$. This forces the series to converge as $n \to \infty$. Consequently, we may allow n to tend to infinity in the finite L-series and obtain, for $t \epsilon N_3(t_o)$,

$$f(t) = \sum_{k=0}^{\infty} L^k f(t_o)\phi_{2k}(t) + (L^k f)'(t_o)\phi_{2k+1}(t) + R(t)$$

where $R(t) = \lim\limits_{n\to\infty} G^{n+1}L^{n+1}f(t)$. The proof of the theorem, then, will simply involve showing that $R(t) \equiv 0$ for t in some right neighborhood of t_o. This will be accomplished by obtaining some estimates on $G^{n+1}L^{n+1}f(t)$ and showing that the estimates tend to 0 as $n \to \infty$.

Since $L^{n+1}f(t) = L(L^n f)(t) \geq 0$, for $t \epsilon I$, the Peixoto-Bonsall Theorem implies that

$$(5.1) \quad L^n f(t) \leq L^n f(t_o)\phi_0(t) + \frac{L^n f(t_1) - L^n f(t_o)\phi_o(t_1)}{\phi_1(t_1)}\,\phi_1(t)$$

for $t_o \leq t \leq t_1$ where t_1 is an arbitrary, but fixed, number in $N_3(t_o)$. By Corollary 1 of Lemma 1, we know that $G(t,\tau) \geq 0$ for $t_o \leq \tau \leq t \leq t_1$.

Thus we may operate repeatedly on both sides of (5.1) with G, and use (3.7), to obtain

$$(5.2)\quad 0 \leq G^n L^n f(t) \leq L^n f(t_o)\phi_{2n}(t) + \left(\frac{L^n f(t_1) - L^n f(t_o)\phi_o(t_1)}{\phi_1(t_1)}\right)\phi_{2n+1}(t)$$

for $t_o \leq t \leq t_1$ and for all n. Our goal is to show that each term on the right side of (5.2) tends to 0 as $n \to \infty$.

The convergence of the L-series forces $\lim_{n\to\infty} L^n f(t_o)\phi_{2n}(t) = 0$ for $t \in [t_o, t_1]$. By Lemma 6 we have $\phi_{2n+1}(t) \leq \phi_{2n}(t)$ for $t_o \leq t \leq t_1$ and for all n. Thus we have

$$0 \leq L^n f(t_o)\phi_{2n+1}(t) \leq L^n f(t_o)\phi_{2n}(t).$$

Consequently, it is true that $\lim_{n\to\infty} L^n f(t_o)\phi_{2n+1}(t) = 0$ for $t_o \leq t \leq t_1$

Thus the only term of (5.2) that can possibly present any difficulties is $L^n f(t_1)\phi_{2n+1}(t)$. By Lemma 6 we have $L^n f(t_1)\phi_{2n+1}(t) \leq L^n f(t_1)\phi_{2n}(t)$ for $t_o \leq t \leq t_1$ and for all n. Finally, we apply Lemma 4 to obtain

$$(5.3)\quad L^n f(t_1)\phi_{2n}(t) \leq L^n f(t_1) d_2 \frac{\left(C_2(t-t_o)\right)^{2n}}{(2n)!}$$

for $t_o \leq t \leq t_1$. Thus, the problem has been reduced to showing that

$$\lim_{n\to\infty} L^n f(t_1) \frac{\left(C_2(t-t_o)\right)^{2n}}{(2n)!} = 0$$

for t sufficiently close to t_o.

The technique for showing this will involve consideration of the L-series expansion for f about $t = t_1$ and deduction of the above limit from the convergence of that expansion.

By the hypotheses we have $L^n f(t) \geq 0$ and $(L^n f)'(t) \geq 0$ for $t \varepsilon (t_1, t_o + b_3)$, so we may repeat the arguments used at the beginning of the proof to conclude that the L-series expansion, about $t = t_1$, for f, converges in some right neighborhood of t_1. In the remainder of the proof we will identify quantities associated with the expansion about $t = t_1$ by *. Then the convergence of the expansion implies that

$$(5.4) \quad \lim_{n\to\infty} L^n f(t_1)\phi_{2n}^*(t) = 0$$

for t in some right neighborhood of t_1. By means of Lemma 4, we have

$$d_1^* \frac{\left(C_1^*(t-t_1)\right)^{2n}}{(2n)!} \leq \phi_{2n}^*(t)$$

in a right neighborhood of t_1. Then (5.4) implies that

$$\lim_{n\to\infty} L^n f(t_1) \frac{\left(C_1^*(t-t_1)\right)^{2n}}{(2n)!} = 0$$

for t in a right neighborhood of t_1. Let $b^* > t_1$ be an arbitrary, but fixed, number such that

$$\lim_{n\to\infty} L^n f(t_1) \frac{\left(C_1^*(b^*-t_1)\right)^{2n}}{(2n)!} = 0$$

Define $\varepsilon = \min\left(t_1 - t_o, \frac{C_1^*(b^*-t_1)}{C_2}\right)$. Then for $0 \leq t - t_o \leq \varepsilon$ we have, from (5.3),

$$L^n f(t_1)\phi_{2n}(t) \leq L^n f(t_1) d_2 \frac{\left(C_2(t-t_o)\right)^{2n}}{(2n)!}$$

$$\leq L^n f(t_1) d_2 \frac{\left(C_1^*(b^*-t_1)\right)^{2n}}{(2n)!}$$

and the last quantity has limit 0 as $n \to \infty$.

Therefore, $\lim_{n\to\infty} G^n L^n f(t) \equiv 0$ for $0 \leq t-t_o \leq \varepsilon$, and we have the expansion

$$f(t) = \sum_{k=0}^{\infty} L^k f(t_o)\phi_{2k}(t) + (L^k f)'(t_o)\phi_{2k+1}(t)$$

for $0 \leq t-t_o \leq \varepsilon$.

Theorem 2 (Strong Version): Let f be L-positive at $t_o \varepsilon I$. Then there exists an $\varepsilon > 0$ such that

$$f(t) = \sum_{k=0}^{\infty} L^k f(t_o)\phi_{2k}(t) + (L^k f)'(t_o)\phi_{2k+1}(t)$$

for $0 \leq t-t_o \leq \varepsilon$.

Proof: Take $N_3(t_o)$ and t_1 as in the proof of the weak version. Precisely the same arguments used at the beginning of that proof may be applied here to conclude that

$$f(t) = \sum_{k=0}^{\infty} L^k f(t_o)\phi_{2k}(t) + (L^k f)'(t_o)\phi_{2k+1}(t) + R(t)$$

for $t_o \leq t \leq t_1$, where $R(t) = \lim_{n\to\infty} G^n L^n f(t)$. If we define

$$\hat{f}(t) = f(t) - \sum_{k=0}^{\infty} L^k f(t_o)\phi_{2k}(t)$$

then, of course,

$$\hat{f}(t) = \sum_{k=0}^{\infty} (L^k f)'(t_o)\phi_{2k+1}(t) + R(t). \tag{5.5}$$

Our method consists of applying the weak version of the theorem to show that

$$\hat{f}(t) = \sum_{k=0}^{\infty} (L^k f)'(t_o)\phi_{2k+1}(t).$$

To satisfy the hypotheses of the weak version we must show that $L^k\hat{f}(t) \geq 0$ and $(L^k\hat{f})'(t) \geq 0$ for all k and for t sufficiently close to t_o. In view of Lemma 11 and the L-positivity of f it is sufficient to show that $L^kR(t) \geq 0$ and $(L^kR)'(t) \geq 0$ for all k and for t sufficiently close to t_o.

To establish the inequality $L^kR(t) \geq 0$ we shall use the equation

$$(5.6) \qquad R(t) = f(t) - \sum_{k=0}^{\infty} \left(L^k f(t_o)\phi_{2k}(t) + (L^k f)'(t_o)\phi_{2k+1}(t) \right).$$

<u>Claim</u>: $L^P R(t) \geq 0$ for p = 0, 1, 2, . . . and $0 \leq t-t_o < \frac{C_1}{C_2}(t_1-t_o)$. We operate on both sides of (5.6) with L^P, for $0 \leq t-t_o < \frac{C_1}{C_2}(t_1-t_o)$, use Lemma 11 to justify operating term by term with L^P, and obtain

$$(5.7) \qquad L^P R(t) = L^P f(t) - \sum_{k=0}^{\infty} \left(L^{p+k} f(t_o)\phi_{2k}(t) + (L^{p+k} f)'(t_o)\phi_{2k+1}(t) \right).$$

The proof of the claim will be complete if we can show that the right hand side of (5.7) is non-negative. To accomplish this, we shall first apply L^P to the finite L-series (3.6) to obtain

$$(5.8) \qquad L^P f(t) = \sum_{k=0}^{m} L^{p+k} f(t_o)\phi_{2k}(t) + (L^{p+k} f)'(t_o)\phi_{2k+1}(t) + G^{m+1-p}L^{m+1}f$$

By hypothesis, $L^q f(t) \geq 0$ and $(L^q f)'(t_o) \geq 0$ for all q, so that the left side of (5.8) is non-negative, the remainder is non-negative, and each term of the series is non-negative.

Therefore, we may let m tend to infinity in (5.8) to obtain, for $t \varepsilon N_3(t_o)$.

$$L^P f(t) = \sum_{k=0}^{\infty} L^{p+k} f(t_o)\phi_{2k}(t) + (L^{p+k} f)'(t_o)\phi_{2k+1}(t) + R_p(t)$$

where $R_p(t) = \lim_{m\to\infty} G^{m+1}L^{m+1+p}f$. Since $G^{m+1}L^{m+1+p}f(t) \geq 0$ it must be true that $R_p(t) \geq 0$. We may express $R_p(t)$ as

$$(5.9) \qquad R_p(t) = L^p f(t) - \sum_{k=0}^{\infty}\Big(L^{p+k}f(t_o)\phi_{2k}(t) + (L^{p+k}f)'(t_o)\phi_{2k+1}(t)\Big)$$

and observe that the right side of (5.9) is identical to the right side of (5.7). Therefore, $L^pR(t) = R_p(t)$, and so $L^pR(t) \geq 0$ for $p = 0, 1, 2, \ldots$ and $0 \leq t-t_o < \frac{C_1}{C_2}(t_1-t_o)$.

Claim: $(L^PR)'(t) \geq 0$ for $p = 0, 1, 2, \ldots$ and $0 \leq t-t_o < \frac{C_1}{C_2}(t_1-t_o)$.
In the argument above we have shown that

$$L^PR(t) = \lim_{m\to\infty} G^{m+1}L^{m+1+p}f(t)$$

for $0 \leq t-t_o < \frac{C_1}{C_2}(t_1-t_o)$. Our scheme now will be to show that

$$\frac{d}{dt}G^{m+1}L^{m+1+p}f(t) \geq 0$$

and that $(L^PR)'(t) = \lim_{m\to\infty} \frac{d}{dt} G^{m+1}L^{m+1+p}f(t)$. This will prove the present claim.

We know that

$$(5.10) \qquad G^{m+1}L^{m+1+p}f(t) = \int_{t_o}^{t} G(t,\tau)G^mL^{m+1+p}f(\tau)d\tau.$$

Differentiating both sides of (5.10) we obtain

$$\frac{d}{dt} G^{m+1}L^{m+1+p}f(t) = \int_{t_o}^{t} \frac{\partial}{\partial t} G(t,\tau)G^mL^{m+1+p}f(\tau)d\tau.$$

From Corollary 3 of Lemma 1 we have $\frac{\partial}{\partial t} G(t,\tau) \geq 0$ for $t_o \leq \tau \leq t \leq t_1$.
Therefore,

$$\frac{d}{dt}\, G^{m+1} L^{m+1+p} f(t) \geq 0$$

for $0 \leq t-t_o \leq t_1-t_o$ and for all m and p. We now need to argue that

$$(L^P R)'(t) = \lim_{m\to\infty} \frac{d}{dt}\, G^{m+1} L^{m+1+p} f(t)$$

for $0 \leq t-t_o < \frac{C_1}{C_2}(t_1-t_o)$. It will be sufficient to show that the sequence

$$\{\frac{d}{dt}\, G^{m+1} L^{m+1+p} f(t)\}_{m=0}^{\infty}$$

is uniformly Cauchy. We note that the sequence is non-increasing as $m\to\infty$ (take derivatives of both sides of (5.8) and apply the hypothesis.) so that

$$\left| \frac{d}{dt}\, G^{m+1} L^{m+1+p} f(t) - \frac{d}{dt}\, G^{m+j+1} L^{m+j+1+p} f(t) \right|$$

$$= \frac{d}{dt}\, G^{m+1} L^{m+1+p} f(t) - \frac{d}{dt}\, G^{m+j+1} L^{m+j+1+p} f(t)$$

for all $j \geq 0$.

Then we have

$$\left| \frac{d}{dt}\, G^{m+1} L^{m+1+p} f(t) - \frac{d}{dt}\, G^{m+j+1} L^{m+j+1+p} f(t) \right|$$

$$= \int_{t_o}^{t} \frac{\partial}{\partial t} G(t,\tau)\ \{G^m L^{m+1+p} f(\tau) - G^{m+j} L^{m+j+1+p} f(\tau)\}\ d\tau$$

$$\leq\ M_t \int_{t_o}^{t} \{G^m L^{m+1+p} f(\tau) - G^{m+j} L^{m+j+1+p} f(\tau)\} d\tau$$

where $\quad M_t = \sup_{t_o \leq \tau \leq t} \frac{\partial}{\partial t} G(t,\tau).$

We now estimate the quantity under the integral sign. Observe that $G^mL^{m+1+p}f(t)$ converges to $R_{p+1}(t)$ as $m\to\infty$ and that the convergence is uniform. Therefore the sequence

$$\{G^mL^{m+1+p}f(t)\}_{m=0}^{\infty}$$

is uniformly Cauchy for $0 \leq t-t_o < \frac{C_1}{C_2}(t_1-t_o)$. Therefore, we have, for any $\varepsilon_1 > 0$,

$$\int_{t_o}^{t}\left(G^mL^{m+p+1}f(\tau) - G^{m+j}L^{m+j+p+1}f(\tau)\right) d\tau \leq \varepsilon_1(t-t_o)$$

for all m sufficiently large, and for $0 \leq t-t_o < \frac{C_1}{C_2}(t_1-t_o)$. For such m we then have

$$\left|\frac{d}{dt}G^{m+1}L^{m+1+p}f(t) - \frac{d}{dt}G^{m+j+1}L^{m+j+1+p}f(t)\right| \leq M_t\varepsilon_1(t-t_o)$$

for $j > 0$ and $t-t_o < \frac{C_1}{C_2}(t_1-t_o)$.

Therefore

$$\{\frac{d}{dt}G^{m+1}L^{m+1+p}f(t)\}_{m=0}^{\infty}$$

is uniformly Cauchy for $0 \leq t-t_o < \frac{C_1}{C_2}(t_1-t_o)$, and hence

$$(L^PR)'(t) = \lim_{m\to\infty}\frac{d}{dt}G^{m+1}L^{m+1+p}f(t).$$

Therefore $(L^PR)'(t) \geq 0$ for $0 \leq t-t_o < \frac{C_1}{C_2}(t_1-t_o)$.

We observe that, since $L^PR(t) = \lim_{m\to\infty} G^{m+1}L^{m+1+p}f(t)$ and $G^{m+1}L^{m+1+p}f(t_o) = 0$ for all m, it must be true that $L^PR(t_o) = 0$ for all p. Since

$$\frac{d}{dt} G^{m+1} L^{m+1+p} f(t_o) = 0$$

for all m and p we also have $(L^P R)'(t_o) = 0$.

This observation, together with (5.5), allows us to conclude that $L^k \hat{f}(t_o) = 0$ and $(L^k \hat{f})'(t_o) = (L^k f)'(t_o)$ for all k. Therefore, the formal L-series for $\hat{f}(t)$ is

$$\sum_{k=0}^{\infty} (L^k f)'(t_o) \phi_{2k+1}(t).$$

For t sufficiently close to t_o, we have, by the claims,

$$L^p \hat{f}(t) \geq 0 \text{ and } (L^p \hat{f})'(t) \geq 0.$$

The weak version of the theorem then implies the existence of an $\varepsilon > 0$ such that

$$\hat{f}(t) = \sum_{k=0}^{\infty} (L^k f)'(t_o) \phi_{2k+1}(t)$$

for $t_o \leq t \leq t_o + \varepsilon$. Therefore, $R(t) \equiv 0$ and

$$f(t) = \sum_{k=0}^{\infty} L^k f(t_o) \phi_{2k}(t) + (L^k f)'(t_o) \phi_{2k+1}(t)$$

for $t_o \leq t \leq t_o + \varepsilon$. This completes the proof of Theorem 2.

The result, L-positivity implies analyticity, has now been established for the case $a_2(t) \equiv 1$. We shall now prove the theorem for the general case $a_2(t) > 0$. The method will be to introduce a new independent variable which will convert the problem to one which can be solved by the machinery already developed.

<u>Theorem</u> 2': Let $a_2(t) > 0$ for $t \varepsilon I$ and let $Ly(t) = a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t)$. If $f(t)$ is L-positive at $t_o \varepsilon I$ then there exists an $\varepsilon > 0$ such that

$$f(t) = \sum_{k=0}^{\infty} L^k f(t_o)\phi_{2k}(t) + \sqrt{a_2(t_o)}\ (L^k f)'(t_o)\phi_{2k+1}(t)$$

for $t_o \leq t \leq t_o + \varepsilon$.

Proof: We shall use the same change of variable (4.18) and notation (4.19) employed in the proof of Theorem 1'. Then we have

$$T = \int_{t_o}^{t} \frac{du}{\sqrt{a_2(u)}} .$$

Let $b \varepsilon I$ be an arbitrary, but fixed, number such that $b > t_o$. Since the integrand is positive, T is an increasing function of t for $t \varepsilon [t_o, b]$ and the transformation may be inverted to give t as a function of T for $T \varepsilon [0, T(b)]$. Let $F(T) = f\big(t(T)\big)$. We see that, as in the proof of Theorem 1', this change of variables transforms the operator L into the operator L_T. Then,

$$Lf(t) = L_T F(T),$$

so that, if $Lf(t) \geq 0$ for $t \varepsilon [t_o, b]$ then $L_T F(T) \geq 0$ for $T \varepsilon [0, T(b)]$. One may make a trivial induction argument to show that if $L^k f(t) \geq 0$ for all k and $t \varepsilon [t_o, b]$ then $L_T^k F(T) \geq 0$ for all k and for $T \varepsilon [0, T(b)]$. Since $\sqrt{a_2(t_o)}\ (Lf)'(t_o) = \frac{d}{dt}\Big(L_T^k F(T)\Big)\Big|_{T=0}$ we can argue that $\frac{d}{dt}\Big(L_T^k F(T)\Big)\Big|_{T=0} \geq 0$. Therefore, Theorem 1 may be invoked to assert the existence of $\hat{\varepsilon} \varepsilon \big(0, T(b)\big)$ such that

$$F(T) = \sum_{k=0}^{\infty} L_T^k F(0)\Phi_{2k}(T) + (L^k F)'(0)\Phi_{2k+1}(T)$$

for $T \varepsilon [0, \hat{\varepsilon}]$. If we invert the change of variable and take $\varepsilon = T^{-1}(\hat{\varepsilon})$ we may conclude that

$$f(t) = \sum_{k=0}^{\infty} L^k f(t_o)\phi_{2k}(t) + \sqrt{a_2(t_o)}\ (L^k f)'(t_o)\phi_{2k+1}(t)$$

for $t_o \le t \le t_o + \varepsilon$.

Remark: The condition $(L^k f)'(t_o) \geq 0$ has played a very crucial role in the proof of Theorem 2. However, there is reason to suspect that its necessity is more a function of the method used than of the problem itself. We conjecture that the conclusion of Theorem 2' is still true assuming only $L^k f(t) \geq 0$ for $t \varepsilon I$. The suspicion is based, in part, on some simple examples.

Consider first the operator $Ly = y''$. Here $L^k f(t) \geq 0$ becomes simply $f^{(2k)}(t) \geq 0$. Then Boas' theorem (Section 2) implies that $f(t)$ is analytic in a neighborhood of the interval. A second example involves the operator $Ly = y'' - y$. If $L^k f(t) \geq 0$ for all k, then $Lf \geq 0$ implies $f^{(2)}(t) - f(t) \geq 0$ or $f^{(2)}(t) \geq f(t) \geq 0$. Also $L^2 f \geq 0$ implies $(Lf)'' - Lf \geq 0$, $f^{(4)}(t) - f^{(2)}(t) - Lf \geq 0$, and hence $f^{(4)}(t) \geq f^{(2)}(t) + Lf \geq 0$. Continuing in this fashion, we see that all the even derivatives of f are non-negative. Thus f must be analytic.

Thus, while the condition $(L^k f)'(t_o) \geq 0$, in conjunction with $L^k f(t) \geq 0$, is sufficient to guarantee analyticity of f, it is not always necessary. The question of dispensing with the hypothesis $(L^k f)'(t_o) \geq 0$ is open and there does not seem to be a way of answering it in the context of L-series.

REFERENCES

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover, New York, 1968.

[2] G. Birkhoff and G.-C. Rota, Ordinary Differential Equations, Ginn, Boston, 1962.

[3] R. P. Boas, "Expansions of analytic functions," Trans. Amer. Math. Soc., 48 (1940), pp. 467-487.

[4] ________, "Functions with positive derivatives," Duke Math. J., 8 (1941), pp. 163-172.

[5] ________, "Signs of derivatives and analytic behavior," Amer. Math. Monthly, 78 (1971), pp. 1085-1093.

[6] F. F. Bonsall, "The characterization of generalized convex functions," Quart. J. Math., Oxford Ser (2), 1 (1950), pp. 100-111.

[7] J. A. Cochran, The Analysis of Linear Integral Equations, McGraw Hill, New York, 1972.

[8] E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw Hill, New York, 1955.

[9] M. K. Fage, "Operator-analytic functions of one independent variable," (Russian), Doklady Akad. Nauk SSSR (N.S.), 112 (1957), pp. 1008-1011.

[10] ________, "Operator-analytic functions of one independent variable" (Russian), Trudy Moskov. Mat. Obšč., 7 (1958), pp. 227-268.

[11] I. F. Grigorčuk, "Estimates of L-basis functions and their application to certain problems of function theory" (Russian), Dopovidi Akad. Nauk Ukrain. RSR 1965, pp. 840-843.

[12] E. Hille, "On the oscillation of differential transforms. II Characteristic series of boundary value problems," Trans. Am. Math. Soc., 52 (1942), pp. 463-497.

[13] ________, Analytic Function Theory, Volume I, Ginn, Boston, 1959.

[14] V. G. Hryptun, "Classes of functions quasianalytic with respect to an ordinary linear differential operator and their applications" (Russian), Doklady Akad. Nauk SSSR 177 (1967), pp. 528-530.

[15] M. M. Peixoto, "Generalized convex functions and second order differential inequalities," Bull. Am. Math. Soc., 55 (1949), pp. 563-572.

[16] G. Polya and N. Wiener, "On the oscillation of the derivatives of a periodic function," Trans. Am. Math. Soc. 52 (1942), pp. 249-256.

[17] J. K. Shaw, "Analytic properties of generalized completely convex functions," SIAM J. Math. Anal., to appear.

[18] N. I. Sidenko, "The representation of the functions of an L-basis" (Russian), Ukrain. Mat. Z, 26 (1974), pp. 106-112, 143.

[19] G. Szegö, "On the oscillation of differential transforms. I," Trans. Am. Math. Soc., 52 (1942), pp. 450-462.

[20] D. V. Widder, The Laplace Transform, Princeton University Press, Princeton, 1941.

The vita has been removed from the scanned document

POSITIVITY PROPERTIES ASSOCIATED WITH

LINEAR DIFFERENTIAL OPERATORS

by

William Randolph Winfrey

(ABSTRACT)

The determination of the influence exerted on the analytic character of a real function $f\epsilon C^{\infty}$ by the signs of its derivatives is a problem of longstanding interest in classical analysis. Most investigations of the problem have centered on extending the well known theorem of S. Bernstein which asserts that a function $f\epsilon C^{\infty}$ with all derivatives non-negative on an interval I is necessarily real-analytic there; i.e., f is the restriction to I of a complex function analytic in a region containing I.

The scope of this dissertation is the study of analogous positivity results associated with linear differential operators of the form

$$(Ly)(t) = a_2(t)y''(t) + a_1(t)y'(t) + a_o(t)y(t),$$

where $a_2(t)$, $a_1(t)$ and $a_o(t)$ are real-analytic in some interval I and where $a_2(t) > 0$ for $t \epsilon I$. We call a function $f \epsilon C^{\infty}$ _L-positive_ at $t_o \epsilon I$ if it satisfies the "uniform" positivity condition $L^k f(t) \geq 0$, $t \epsilon I$, $k = 0, 1, 2, \ldots$, plus the "pointwise" positivity condition $(L^k f)'(t_o) \geq 0$, $k = 0, 1, 2, \ldots$ ($L^o f = f$, $L^k f = L$

$(L^{k-1}f)$, $k \geq 1$). Our principal result is that L-positivity of f implies analyticity of f in a neighborhood of t_o. If $Ly = y''$, this reduces to Bernstein's theorem.

We prove our result using a generalized Taylor Series Expansion known as the L-series. The L-series expansion about $t = t_o$ for a function $f \epsilon C^\infty$ is:

$$\sum_{k=0}^{\infty} L^k f(t_o)\phi_{2k}(t) + \sqrt{a_2(t_o)}\ (L^k f)'(t_o)\phi_{2k+1}(t).$$

The "L-basis" functions $\{\phi_n(t)\}_{n=0}^{\infty}$ are defined by:

$$L\phi_o \equiv L\phi_1 \equiv 0,\ \phi_o(t_o) = 1,\ \phi_o'(t_o) = 0,\ \phi_1(t_o) = 0,$$
$$\sqrt{a_2(t_o)}\phi_1(t_o) = 1$$

and $\quad L\phi_{n+2} = \phi_n,\ \phi_{n+2}(t_o) = \phi'_{n+2}(t_o) = 0,\ n \geq 0.$

Our technique is to show that L-positivity of f implies the convergence of the above series to f(t). Then we observe that the analyticity of a_2, a_1, and a_o implies the analyticity of the ϕ's and thus the analyticity of the sum, f(t), of the series.

We shall also show that the same conditions on a_2, a_1, and a_o allow any function f, analytic in a neighborhood of t_o, to be represented by an L-series. If $a_2(t) \equiv 1$, the sequence $\{n!\phi_n(t)\}_{n=0}^{\infty}$ provides a heretofore unobserved example of a Pincherle basis.

The problem of dispensing with the hypothesis $(L^k f)'(t_o) \geq 0$ in our result, L-positivity implies analyticity, is still open and does not seem to be solvable by our methods.