ON THE MOTION OF A SYMMETRIC RIGID BODY WITH A

"YO-YO" DESPIN DEVICE ATTACHED

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IV. INTRODUCTION

In placing an artificial satellite into orbit it is sometimes necessary to spin the satellite and booster vehicle during launch to assure proper guidance and stability of the system. Whenever the final booster stage is separated from a spinning satellite payload, the rotation vector and the axis of symmetry of the satellite (assuming there is a single axis for this discussion) are not in general parallel. Therefore, coning or precession associated with the spinning is present. It is frequently desirable to reduce the satellite spin to a small value so that certain experiments may be carried out. Clearly, there are several means by which a satellite may be "despun." One method which has shown considerable merit is that which uses small weights attached to the satellite by cables, initially wound about the longitudinal spin axis. These cables are wound about the vehicle in a manner so that, when the weights are released they move outward, away from the body, and cause a tension in the cables which act on the satellite. The moment generated by these weights and cables acts on the body so that the spin of the satellite is reduced. The study of the motion of a symmetric rigid body with despin weights therefore has significant practical motivation in the design of satellite systems and is also of academic interest because of some mathematical and dynamical peculiarities which are discussed in this dissertation.

Several papers have been written on this subject and several modern textbooks on space mechanics include an analysis on this dynamical system.

However, a complete study of this problem has not yet been published. For instance, references 1, 2, 3, and 4 discuss and give some results for the two-dimensional motion or motion with the symmetry axis in a fixed direction, and with no coning. These studies consider both the tangential phase of the motion (when the cables are tangent to the body) and the fixed phase (where the cables swing away from the tangent until they are directed along a radius, at which time they are released). The effects of cable mass are also considered and the means for approximately accounting for this mass are discussed in reference 2. A further discussion of the yo-yo device combined with an elastic cable to reduce the effect of initial spin rate are to be found in references 5, 6, and 7. If the cables are elastic, they act as compensating devices to correct for overspin or underspin relative to the design or nominal value.

Reference 8 gives an estimation of the effects of initial precession and nutation for a slender body acted upon by a fixed torque always directed along the initial angular momentum vector of the system whenever this initial "coning" is small. However, these assumptions appear to be quite unrealistic for the case of the finite three-dimensional motion of a symmetric rigid body with a yo-yo despin device. Reference 9 is a study of the exact motion of this system during the tangential phase only and it does not contain an analysis of the motion from a fixed frame of reference. This reference should be consulted if one wishes to compare the results of this dissertation for the two-phase motion with the results obtained from studies limited to the tangential phase motion.

This dissertation presents a comprehensive study of the yo-yo despin problem. The two-dimensional case is thoroughly discussed for both phases and the three-dimensional equations are developed from the Euler equations for a rigid body and the application of Newton's Second Law. Several important conclusions have been reached in regard to the solution of the differential equations of motion and the constraints which are introduced through tacit assumption on the system configuration. The main item of academic interest is to note that the system as presented is not internally conservative for the three-dimensional case and hence the usual Lagrangian techniques are not valid.

A unique method of simplifying the exact equations of motion is presented which appears to be valid for fairly large (10°) initial coning angles. This method is discussed and suggestions for possible extension of the technique are given.

The results, as presented, are sufficient for at least rudimentary design computations and provide examples of the corrections which must be applied to two-dimensional computations so that they are fair estimates of the general motion.

V. SYMBOLS

$\overline{A}_i$	nondimensional acceleration of a despin weight
A_{ij}	Euler angle transformation matrix from the x_i to the X_i system
a_i	vector components directed to contact point of cable and body
a	magnitude of a_i
a_{ij}	coefficient matrix of equations (30)
$\overline{a}_{ij}$	inverse matrix of a_{ij}
$[a]$	determinant of a_{ij} matrix
B_{ij}	coefficient matrix of equations (55)
$\overline{B}_{ij}$	inverse matrix of B_{ij}
b_i	vector components defined by equations (33)
C_i	vector components defined by equations (55)
E	kinetic energy
$\overline{E}$	nondimensional kinetic energy $\Rightarrow \dfrac{P^2E}{I_1}$
F_r	complex functions defined by equation (98)
f	function defined in equation (92)
G	denoting the integral $\int_0^{\tau} f(x)dx$
g	function defined in equation (92)
H_i	components of the angular momentum vector for the system
$\overline{H}_i$	nondimensional form of $H_i \Rightarrow \dfrac{PH_i}{I_1}$
$\overline{H}$	nondimensional angular momentum for two dimensions
h_i	components of the angular momentum vector of the rigid body only

$\bar{h}_i$	nondimensional form of $h_i \Rightarrow \frac{Ph_i}{I_i}$
h	function defined in equations (92)
I_{ij}	inertia tensor of body
$\bar{I}_{ij}$	nondimensional inertia tensor $\Rightarrow \frac{I_{ij}}{I_1}$
I_1, I_3	mass moment of inertia about x_1 axis and x_3 axis
I	inertia ratio $\Rightarrow \frac{I_3}{I_1}$
K	inertia ratio $\Rightarrow \frac{2ma^2}{I_1}$
k	"control" constant defined by equation (1)
k_i	unit vector along the x_3 body axis
l	length of cable unwound
l_i	vector length of cable unwound
M_i	components of moment vector acting on body
$\bar{M}_i$	nondimensional form of $M_i \Rightarrow \frac{P^2 M_i}{I_i}$
M	magnitude of M_i; mass of body
m	mass of one despin weight
P	time for two-dimensional, tangential despin to zero angular rate $\Rightarrow \sqrt{1 + I/K}\,\frac{1}{\omega_{30}}$
R	force of reaction, due to cable-body interaction
r_i	radius vector of a despin weight given in body coordinates
r	magnitude of r_i
$S_1, S_2, S_3, \bar{S}_1, \bar{S}_2, \bar{S}_3$	representing vector products given in equations (34) and (37)
T_i	vector representing force due to cable tension acting on body
T	cable tension

$\bar{T}$	nondimensional form of $T \Rightarrow (1 + I/K) \dfrac{2aT}{I_1 \omega_{3o}^2}$
$\tilde{T}$	nondimensional form of $T \Rightarrow \sqrt{1 + I/K}\ \dfrac{2aT}{I_3 \omega_{3o}^2}$
U_1, U_2, U_3	representing vector products given in equation (35)
V_i	absolute velocity vector of a despin weight
$\bar{V}_i$	nondimensional form of $V_i \Rightarrow \dfrac{PV_i}{a}$
W	work done
X_i	inertial coordinates
x_i	body coordinates
x	function defined in equations (92)
Y_i	vector defined by equation (31)
y	function defined in equations (92)
Z_i	arbitrary vector in x_i coordinate system
$\bar{Z}_i$	arbitrary vector in X_i coordinate system
z_i	complex function defined in equation (96)
α_{ij}	transformation matrix form $x_i^{(o)}$ to $x_i^{(1)}$
α	angle cable is out of x_1, x_2 plane
$\tilde{\alpha}$	form of $\alpha \Rightarrow \alpha/\bar{\omega}_{3o}$
β_{ij}	transformation matrix from $x_i^{(1)}$ to x_i
β	angular measure to point of contact of unwinding cable and body in x_1, x_2 plane
$\tilde{\beta}$	form of $\beta \Rightarrow \beta/\bar{\omega}_{3o}$
Γ	angle between symmetry axis and angular momentum vector h_i
Γ_o	initial value of Γ

γ	angle in x_1, x_2 plane cable swings away from tangent
$\tilde{\gamma}$	form of $\gamma \Rightarrow \gamma/\bar{\omega}_{30}$
δ_{ij}	Kronecker delta $\Rightarrow \begin{cases} =1 & \text{if } i = j \\ =0 & \text{if } i \neq j \end{cases}$
δ	angle between symmetry axis and spin vector ω_i
ϵ_{ijk}	permutation symbol $\Rightarrow \begin{cases} = 0 & \text{if } i,j,k \text{ are not permutations of } 1,2,3 \\ = +1 & \text{if } i,j,k \text{ are even permutations of } 1,2,3 \\ = -1 & \text{if } i,j,k \text{ are odd permutations of } 1,2,3 \end{cases}$
η	nutation angle measured from x_1, x_2 plane, that is, $\pi/2 - \theta$
θ	Euler angle representing nutation angle
ϕ	Euler angle representing precession angle
ψ	Euler angle representing spin angle
$\tilde{\theta},\tilde{\phi},\tilde{\psi}$	form of $\theta,\phi,\psi \Rightarrow \theta/\bar{\omega}_{30}$, $\phi/\bar{\omega}_{30}$, $\psi/\bar{\omega}_{30}$
λ_i	parameter representing $\pm\sqrt{-1}$
ρ_i	nondimensional form of $r_i \Rightarrow r_i/a$
ρ	magnitude of ρ_i
τ	nondimensional time $\Rightarrow t/P$
ω_i	rotation vector
$\bar{\omega}_i$	nondimensional form of $\omega_i \Rightarrow P\omega_i$
$\tilde{\omega}_i$	nondimensional form of $\omega_i \Rightarrow \omega_i/\omega_{30}$
$\omega_{10},\omega_{20},\omega_{30}$	initial rotational rates x_1, x_2, x_3 axes
$\bar{\omega}_{30}$	$\Rightarrow \sqrt{1 + I/K}$

Subscripts

$i,j,k,..,r,s,t$, etc.	general index notation subscripts
B	refering to body
c	transition point
f	final time in fixed phase motion
s	system
α,β,γ	partial differentiation w.r.t. α,β,γ
$(\ )_{max}$	maximum value

Superscripts

o	coordinate system $x^{(o)}$
1	coordinate system $x^{(1)}$
$(\dot{\ }),(\ddot{\ })$	derivatives with respect to time in x_i coordinates
$(\ )',(\ )''$	nondimensional form of $(\dot{\ }),(\ddot{\ })$ => $P(\dot{\ }),P^2(\ddot{\ })$
$-,+$	immediately before and after transition point is reached

VI. EXACT ANALYSIS

A. General Considerations

In the past an investigator was restricted from delving too deeply into a problem because of the resulting complexity of the equations which would develop. At the present time, however, high-speed computers enable the investigator to solve problems to which, in the past, solutions could only be guessed. Of course, there are disadvantages to a "solution" obtained by the use of a digital (or analog) computer and such disadvantages are quite obvious. However, when it is not possible to obtain a complete analytical solution through some purely mathematical lever then, if there is to be a solution, it must be obtained via the computer. The exact analysis of the motion of a system composed of a rigid body and yo-yo despin device must be performed with the aid of a computer if one wishes to include in the analysis an initial coning angle (or precessional rate).

The equations of motion for the yo-yo problem are developed by applying the Euler equations for rigid-body motion and Newton's second law to the motion of the weights. Since the vector approach is used in developing the equations of motion, the question of whether or not the system is conservative is of little practical consequence. However, from an academic point of view it is interesting to note that the system is, in general, nonconservative. At first one is likely to make the assumption that the energy of the system has been conserved and that since there are no external effects this would result in a constant kinetic energy for the system. However, a nonconservative constraint is concealed in the assumption made on the unwinding of the cable from small grooves which restrains the cable from spiraling along the body. This constraint appears whenever the weights and cables move out of the initial winding plane, and does a negative amount of work which leads to a decrease of kinetic energy during the unwinding motion. A further development of the resulting energy effects follows later in this work.

In solving the present problem in terms of the Euler equations and coordinates describing the position of the weights relative to a rotating coordinate system fixed in the body, it is found that seven first order differential equations are required and an algebraic result developes naturally which gives the cable tension. It should be noted that three more differential equations are needed if the motion of the system is desired in inertial space and these may be the well known Euler angle-body rate equations. Initial conditions present no difficulty as they are easily obtained by setting the relative velocity of the despin weights to zero, initially. The fundamental matrix, resulting from the reduction of three second-order equations to first order ones, is singular at the initial instant but, as will be shown, does not present great difficulty.

B. Assumptions and Specifications

In order to determine the motion of this system certain assumptions are made which will simplify the development of the equations. First of all the rotating, rigid body will be assumed symmetric about one axis. This is done mainly because of the large amount of data which must be presented, although the simplification of the working equations is admittedly also a motivation. The despin device itself is chosen to be two diametrically opposed masses attached to inextensible, massless, cables which may take compression as well as tension and which cannot support any bending moment. The weights and cables are wound about a circle of radius (a), the plane of which is perpendicular to the symmetry axis and has its center pierced by this axis. Figure 1 shows how the center of this circle is placed at the mass center of the rigid body. The body set of Cartesian coordinates is placed with the origin at the mass center of the system and so that the plane of the x_1, x_2 axes contains the circle of winding; the x_1 axis passing through the initial position of one of the weights and the x_3 axis along the symmetry axis such that this component of the rotation vector (ω_3) is positive.

The cables are assumed to unwind from the body in two distinct phases of motion. First they unwind such that, while unwrapping, each cable lies in a plane tangent to the initial circle of winding and parallel with the symmetry axis. Also, it is assumed that while the cables are in contact with the body they must not move relative to the body. This assumption does not allow any "helical spiraling" of the cables along the body during their unwinding phase. Such unwinding would be accomplished if the cables unwound from small grooves about the periphery of the initial winding circle; the grooves providing sufficient longitudinal constraint to keep the cable from slipping along the rigid body. This portion of the motion will hereafter be referred to as the "tangential phase" of the motion. The second phase of the motion takes place after a specified amount of the cable is unwound. The unwinding ceases when the cables reach their specified length and at this time a three-dimensional hinge is assumed to connect the cables and the body. The cable swings about this point fixed in the rotating body. This second portion of the motion is referred to as the "fixed phase." These two phases of motion are shown for the two-dimensional case in figure 2. It is also assumed that, throughout the motion, a point on one cable has a corresponding point on the second cable reflected (symmetrically) through the origin of the coordinate system. It is of value to note that the configuration of the cables, weights, and mass center of the body are such that a single plane could contain them all.

The above assumptions concerning the tangential phase motion and the cable symmetry need further comment. The assumption that the cables move initially in the tangential plane is essentially a boundary condition since it arises from the lack of any radial component of the cable tension vector at the point of contact with the body. If the cable is not in a tangential plane then it must be swung out at some angle γ (see fig. 1); however, this would require a radial constraining force equivalent to the cable tensile force times the sine of the angle γ (see relation (6) in text) and since no constraint in this direction is assumed in the first phase of the motion the cable cannot be at the angle γ and therefore lies in the tangent plane. The assumption that the cables remain symmetric through the origin is intuitively obvious from the geometry of the problem. It is also easily reasoned from dynamical considerations; the initial conditions being the same for both weights (zero relative velocity) and the fact that the system mass center must remain fixed in inertial space if initially fixed. It is, of course, assumed that the system mass center is either initially fixed or moving at a constant velocity in an inertial reference frame.

It is found that the number of parameters in this problem are conveniently reduced by nondimensionalizing. This is done in the following analysis, but it should be pointed out here that the total number of parameters and initial conditions necessary to describe the motion of this yo-yo system is four: I, K, $\frac{\omega_{1o}}{\omega_{2o}}$, and Γ_o. The following text or symbol list may be consulted for a description of these terms.

In the derivation of the equations of motion which follows, the tangential and fixed phases of the motion will be considered simultaneously in order to conserve both the reader's and the writer's effort. An angle β is defined which represents the amount of cable unwound and which is a generalized coordinate for the tangential phase of the motion. An angle γ is defined to represent the angular distance which the cable has swung away from the tangent during the fixed phase of the motion. Therefore, γ is identically zero during the tangential phase of the motion, while β remains a fixed value during the fixed phase of the motion. In order that the equations of motion for the two phases may be developed simultaneously, a parameter k will be used which, when given the value zero or one, causes the appropriate quantities to vanish or remain depending on the phase of the motion. This constant k is then defined as

$$\left.\begin{array}{lll} k = 0 & \text{during tangential phase} & (\gamma \equiv 0) \\ k = 1 & \text{during fixed phase} & (\beta' \equiv 0) \end{array}\right\} \tag{1}$$

C. Derivation of Basic Equations

Consider a coordinate system fixed in the body as described earlier and shown in figure 1. It should be noted that the length of the cable is fixed by the previous definitions by the relation

$$l = a\beta \tag{2}$$

where a is the radius of the windings and β the angle through which the point of contact of the cable and the body has moved from its original position on the x_1 axis. It is known that a force (tension) will exist in the cable and the vector T_i is defined as this force. Because of the assumption of no bending moments in the cable, it will lie in a straight line and have length l. A vector describing the cable is defined by the relation

$$l_i = l \frac{T_i}{T} \tag{3}$$

If a_i is the radius vector to the cable-body contact point and if r_i is the position vector of a weight, the vector relationship for r_i is

$$r_i = a_i + l_i \tag{4}$$

It is found advantageous to normalize the vector r_i by the length (a) thereby defining the nondimensional position vector ρ_i as

$$\rho_i = \frac{1}{a} r_i \tag{5}$$

Figure 1 shows the coordinate angles (α, β, γ) which are the out of plane angle, the unwind angle and the swing angle, respectively. By means of these angles the vector components of the tension vector may be written in the (x_1, x_2, x_3) body axis system. The tension vector in the ($x_i^{(o)}$) coordinate system is

$$T_i^{(o)} \Rightarrow (T \sin \gamma, \; - T \cos \gamma, \; 0) \tag{6}$$

The transformation matrix from the $x_i^{(o)}$ coordinates to the $x_i^{(1)}$ coordinates is α_{ij} where

$$\alpha_{ij} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix}$$

and from $x_i^{(1)}$ to x_i is β_{ij} where

$$\beta_{ij} \Rightarrow \begin{pmatrix} \cos \beta & -\sin \beta & 0 \\ \sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with these transformations the tension vector is written in the x_i system as*

$$T_i = \beta_{ij}\alpha_{jk}T_k^{(o)} \tag{7}$$

or writing out these components

$$\left.\begin{aligned} T_1 &= T(\cos\alpha \sin\beta \cos\gamma + \cos\beta \sin\gamma) \\ T_2 &= T(-\cos\alpha \cos\beta \cos\gamma + \sin\beta \sin\gamma) \\ T_3 &= -T \sin\alpha \cos\gamma \end{aligned}\right\} \tag{8}$$

where the components are along the positive $(x_1,\ x_2,\ x_3)$ coordinate directions.

The vector a_i is easily seen to be

$$a_i \Rightarrow (a \cos\beta,\ a \sin\beta,\ 0) \tag{9}$$

From (6), (5), (4), (8), and (9) the nondimensional position vector ρ_i, has the components

$$\left.\begin{aligned} \rho_1 &= \cos\beta + \beta \cos\alpha \sin\beta \cos\gamma + \beta \cos\beta \sin\gamma \\ \rho_2 &= \sin\beta - \beta \cos\alpha \cos\beta \cos\gamma + \beta \sin\beta \sin\gamma \\ \rho_3 &= -\beta \sin\alpha \cos\gamma \end{aligned}\right\} \tag{10}$$

*The standard index notation will be used so that double indicies indicate a sum over the range 1, 2, 3.

The moment which acts on the rigid body, due to the tension T_i in the cable, is found by means of the vector product of a_i and T_i which is

$$M_i = 2\epsilon_{ijk} a_j T_k \tag{11}$$

(Note: Two force vectors (T_i) and ($-T_i$) act through a distance $2a$ to form the couple.) Performing the operation above leads to the components

$$\left.\begin{aligned} M_1 &= -2aT \sin\alpha \sin\beta \cos\gamma \\ M_2 &= 2aT \sin\alpha \cos\beta \cos\gamma \\ M_3 &= -2aT \cos\alpha \cos\gamma \end{aligned}\right\} \tag{12}$$

Certain nondimensional parameters may be defined which will be convenient in the following derivation of the equations of motion and useful in generalizing the results of the study.

First, two inertia ratios are defined which govern the geometry and size of the despin weights and body

$$K = \frac{2ma^2}{I_1} \tag{13}$$

$$I = \frac{I_3}{I_1} \tag{14}$$

It is also desirable to nondimensionalize the time and to do this the unit of time (or period) is defined as

$$P = \sqrt{\frac{K + I}{K}} \left(\frac{1}{\omega_{30}}\right) \tag{15}$$

where ω_{30} is the initial spin rate about the x_3 axis. It will be shown later this time P is the time required to reduce the spin rate ω to zero if the x_3 axis is fixed in direction ($\omega_1 \equiv \omega_2 \equiv 0$) and the cable unwound by the tangential phase only. The nondimensional time τ is then defined as

$$\tau = \frac{t}{P} \tag{16}$$

where t is the real time.

When taking derivatives with respect to τ the following relationships are used

$$(\dot{\ }) = \frac{1}{P}(\ )'; \ (\ddot{\ }) = \frac{1}{P^2}(\ )'' \tag{17}$$

where the primes denote differentiation with respect to the nondimensional time and dots, as usual, with respect to real time. When the derivatives are written in terms of dots or primes the derivative is always referred directly to the coordinate system being considered;

that is, the derivative is relative to this coordinate system. When considering absolute derivatives the notation $\frac{d}{dt}(\ )$ or $\frac{d}{d\tau}(\ )$ is used. The nondimensional angular rates are defined as

$$\bar{\omega} = P\omega = \sqrt{\frac{K + I}{K}}\left(\frac{\omega}{\omega_{3o}}\right) \tag{18}$$

The derivatives for angular rotation in the dimensional and dimensionless forms are therefore related by

$$\omega = \frac{1}{P}\bar{\omega} \qquad\qquad \dot{\omega} = \frac{1}{P^2}\bar{\omega}' \tag{19}$$

and similarly for other variables. The cable tension may also be conveniently nondimensionalized as follows. If $\bar{T}$ is the dimensionless quantity, then

$$\bar{T} = \left(\frac{2aP^2}{I_1}\right)T = \frac{K + I}{K}\,\frac{2aT}{I_1\omega_{3o}^{\ 2}} \tag{20}$$

Using nondimensional quantities the moments given in (12) may also be nondimensionalized as

$$\left.\begin{aligned} \bar{M}_1 &= \frac{P^2M_1}{I_1} = -\bar{T}\sin\alpha\sin\beta\cos\gamma \\ \bar{M}_2 &= \frac{P^2M_2}{I_1} = \bar{T}\sin\alpha\cos\beta\cos\gamma \\ \bar{M}_3 &= \frac{P^2M_3}{I_1} = -\bar{T}\cos\alpha\cos\gamma \end{aligned}\right\} \tag{21}$$

These nondimensional quantities will be used in the subsequent derivation of equations, however, a slightly altered nondimensional form will be used in presenting the results. This second form will be presented later.

Some basic quantities have been derived above which are easily pictured from the geometry of the system. The use of these quantities becomes evident when one writes down the governing equations for a rigid body. The components $(\omega_1, \omega_2, \omega_3)$ of the angular rotation vector are related dynamically to the moment components through the well known Euler equations for a rigid body

$$\bar{I}_{ij}\bar{\omega}'_j + \epsilon_{ijk}\bar{I}_{ks}\bar{\omega}_j\bar{\omega}_s = \bar{M}_i \tag{22}$$

Assuming the body to be symmetric about the x_3 axis (i.e., the principal moment of inertia about the x_2 axis (I_2) is equal to the principal moment of inertia about the x_1 axis (I_1)) the dimensionless inertia tensor becomes

$$\bar{I}_{ij} => \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I \end{pmatrix}$$

Equations (21) and (22) represent three ordinary differential equations for the seven unknown quantities $(\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \alpha, \beta, \gamma, \bar{T})$. Actually only six quantities are unknown since either β or γ is fixed during a finite interval of the motion while the other varies. Since six unknown functions can hardly be obtained from three differential equations, three other independent equations must be found. These three equations may be obtained by applying Newton's Second Law to a despin weight considering it as a free body in a rotating coordinate system acted upon by a force $-T_1$. Thus it is possible to write

$$m \frac{d^2 r_i}{dt^2} \equiv am \frac{d^2 \rho_i}{dt^2} = -T_i$$

or (23)

$$\frac{d^2 \rho_i}{d\tau^2} = -\frac{1}{K} \overline{T}_i$$

Now the derivatives of the vector ρ_i must be taken relative to an inertial coordinate system and since the (x_i) system rotates with the body at a rate ω_i, the derivatives become (after some manipulation)

$$\overline{V}_i = \frac{d\rho_i}{d\tau} = \rho_i' + \epsilon_{ijk} \overline{\omega}_j \rho_k \tag{24}$$

$$\overline{A}_i = \frac{d^2 \rho_i}{d\tau^2} = \rho_i'' + \epsilon_{ijk} \overline{\omega}_j' \rho_k + 2\epsilon_{ijk} \overline{\omega}_j \rho_k' + \overline{\omega}_i \overline{\omega}_s \rho_s - \overline{\omega}_r \overline{\omega}_r \rho_i \tag{25}$$

where the symbol ( )' is used to denote the (nondimensional) time derivative taken within the (x_i) coordinate system. The vectors ρ_i' and ρ_i'' will hereafter be referred to as the relative velocity and the relative acceleration of the weight. To find the components in (24) and (25) equations (10) must be differentiated, giving:

$$\rho_i' = \rho_{i\alpha}\alpha' + \left[\rho_{i\beta}(1 - k) + \rho_{i\gamma}(k)\right]\left[\beta'(1 - k) + \gamma'(k)\right] \tag{26}$$

where the k factor has been introduced (as mentioned earlier) to allow the use of a single set of equations for both the tangential and fixed phases of the motion. It should also be noted that the subscripts α, β, γ represent partial derivatives with respect to these quantities. These derivatives are set down below,

$$\left.\begin{aligned} \rho_{1\alpha} &\equiv \frac{\partial\rho_1}{\partial\alpha} = \rho_3 \sin\beta \\ \rho_{2\alpha} &\equiv \frac{\partial\rho_2}{\partial\alpha} = -\rho_3 \cos\beta \\ \rho_{3\alpha} &\equiv \frac{\partial\rho_3}{\partial\alpha} = -\beta \cos\alpha \cos\gamma \end{aligned}\right\} \tag{27a}$$

$$\left.\begin{aligned} \rho_{1\beta} &\equiv \frac{\partial\rho_1}{\partial\beta} = -\rho_2 + \cos\alpha \sin\beta \cos\gamma + \cos\beta \sin\gamma \\ \rho_{2\beta} &\equiv \frac{\partial\rho_2}{\partial\beta} = \rho_1 - \cos\alpha \cos\beta \cos\gamma + \sin\beta \sin\gamma \\ \rho_{3\beta} &\equiv \frac{\partial\beta_3}{\partial\beta} = -\sin\alpha \cos\gamma \end{aligned}\right\} \tag{27b}$$

$$\left.\begin{aligned} \rho_{1\gamma} &\equiv \frac{\partial\rho_1}{\partial\gamma} = -\beta \cos\alpha \sin\beta \sin\gamma + \beta \cos\beta \cos\gamma \\ \rho_{2\gamma} &\equiv \frac{\partial\rho_2}{\partial\gamma} = \beta \cos\alpha \cos\beta \sin\gamma + \beta \sin\beta \cos\gamma \\ \rho_{3\gamma} &\equiv \frac{\partial\rho_3}{\partial\gamma} = \beta \sin\alpha \sin\gamma \end{aligned}\right\} \tag{27c}$$

Now for the second derivatives ρ_i'' above, equations (27) will also need to be differentiated in time since

$$\rho_i'' = \rho_{i\alpha}\alpha'' + \left[\rho_{i\beta}(1 - k) + \rho_{i\gamma}(k)\right]\left[\beta''(1 - k) + \gamma''(k)\right] + \rho_{i\alpha}'\alpha' + \\ + \left[\rho_{i\beta}'(1 - k) + \rho_{i\gamma}'(k)\right]\left[\beta'(1 - k) + \gamma'(k)\right] \tag{28}$$

The derivatives of (27) follow:

$$\left.\begin{aligned}
\rho_{1\alpha}' &= (\rho_{3\alpha} \sin\beta)\alpha' + (\rho_{2\beta} \sin\beta + \rho_3 \cos\beta)\beta'(1 - k) + \\
&\quad + (\rho_{3\gamma} \sin\beta)\gamma'(k) \\
\rho_{2\alpha}' &= (-\rho_{3\beta} \cos\beta)\alpha' - (\rho_{3\beta} \cos\beta - \rho_3 \sin\beta)\beta'(1 - k) + \\
&\quad - (\rho_{3\gamma} \cos\beta)\gamma'(k) \\
\rho_{3\alpha}' &= -\rho_3\alpha' - (\cos\alpha \cos\gamma)\beta'(1 - k) + (\beta \cos\alpha \sin\gamma)\gamma'(k) \\
\rho_{1\beta}' &= -(\rho_{2\alpha} + \sin\alpha \sin\beta \cos\gamma)\alpha' + (\rho_1 - 2\rho_{2\beta})\beta'(1 - k) + \\
&\quad - (\cos\alpha \sin\beta \sin\gamma - \cos\beta \cos\gamma + \rho_{2\gamma})\gamma'(k) \\
\rho_{2\beta}' &= (\rho_{1\alpha} + \sin\alpha \cos\beta \cos\gamma)\alpha' + (\rho_{1\beta} + \cos\alpha \sin\beta \cos\gamma + \\
&\quad + \cos\beta \sin\gamma)\beta'(1 - k) + (\rho_{1\gamma} + \cos\alpha \cos\beta \sin\gamma + \\
&\quad + \sin\beta \cos\gamma)\gamma'(k) \\
\rho_{3\beta}' &= -(\cos\alpha \cos\gamma)\alpha' + (\sin\alpha \sin\gamma)\gamma'(k) \\
\rho_{1\gamma}' &= (\beta \sin\alpha \sin\beta \sin\gamma)\alpha' + \left[(\cos\beta - \beta \sin\beta) \cos\gamma + \right. \\
&\quad \left. - \cos\alpha(\beta \cos\beta + \sin\beta) \sin\gamma\right]\beta'(1 - k) + \\
&\quad - \beta(\cos\alpha \sin\beta \cos\gamma + \cos\beta \sin\gamma)\gamma'(k) \\
\rho_{2\gamma}' &= -(\beta \sin\alpha \cos\beta \sin\gamma)\alpha' + \left[(\sin\beta + \beta \cos\beta)\cos\gamma + \right. \\
&\quad \left. + \cos\alpha(\cos\beta - \beta \sin\beta)\sin\gamma\right]\beta'(1 - k) + \\
&\quad + \beta(\cos\alpha \cos\beta \cos\gamma - \sin\beta \sin\gamma)\gamma'(k) \\
\rho_{3\gamma}' &= (\beta \cos\alpha \sin\gamma)\alpha' + (\sin\alpha \sin\gamma)\beta'(1 - k) + \\
&\quad + (\beta \sin\alpha \cos\gamma)\gamma'(k)
\end{aligned}\right\} \tag{29}$$

The object in presenting these expressions is to show explicitly the terms required to write out equations (23) in scalar form, usable for programing into a computer. By combining equations (23), (20), (8), (25), (26), (28), (22) and (21)* the three equations of motion resulting from (23) may be written in the form,

$$a_{ij}Y_j = b_i \tag{30}$$

where for ease of writing one defines

$$Y_j \Rightarrow (\alpha'', \ \beta''(1 - k) + \gamma''(k), \ \bar{T}) \tag{31}$$

The matrix elements a_{ij} of equation (30), when written out are found to be,

$$\left.\begin{aligned}
a_{i1} &= \rho_{i\alpha} \\
a_{i2} &= \rho_{i\beta}(1 - k) + \rho_{i\gamma}(k) \\
a_{13} &= \frac{1}{K}(\cos\alpha \sin\beta \cos\gamma + \cos\beta \sin\gamma) + \rho_3 \sin\alpha \cos\beta \cos\gamma + \\
&\quad + \frac{1}{I}\rho_2 \cos\alpha \cos\gamma \\
a_{23} &= \frac{1}{K}(\sin\beta \sin\gamma - \cos\alpha \cos\beta \cos\gamma) - \frac{1}{I}\rho_1 \cos\alpha \cos\gamma + \\
&\quad + \rho_3 \sin\alpha \sin\beta \cos\gamma \\
a_{33} &= -\left(\frac{1}{K} + \rho_2 \sin\beta + \rho_1 \cos\beta\right) \sin\alpha \cos\gamma
\end{aligned}\right\} \tag{32}$$

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*Equations (22) and (21) are needed due to the appearance of ω_1' in equation (25).

and the b_i vector has components,

$$\left.\begin{aligned}
b_1 &= -\rho'_{1\alpha}\alpha' - \rho'_{1\beta}\beta'(1-k) - \rho'_{1\gamma}\gamma'(k) - \bar{S}_1 - U_1 + \\
&\quad -(I-1)\rho_3\bar{\omega}_1\bar{\omega}_3 \\
b_2 &= -\rho'_{2\alpha}\alpha' - \rho'_{2\beta}\beta'(1-k) - \rho'_{2\gamma}\gamma'(k) - \bar{S}_2 - U_2 + \\
&\quad -(I-1)\rho_3\bar{\omega}_2\bar{\omega}_3 \\
b_3 &= -\rho'_{3\alpha}\alpha' - \rho'_{3\beta}\beta'(1-k) - \rho'_{3\gamma}\gamma'(k) - \bar{S}_3 - U_3 + \\
&\quad +(I-1)\rho_1\bar{\omega}_1\bar{\omega}_3 + (I-1)\rho_2\bar{\omega}_2\bar{\omega}_3
\end{aligned}\right\} \tag{33}$$

The cross product terms may be written out in expanded form as

$$\left.\begin{aligned}
\bar{S}_1 &= \bar{\omega}_2\rho'_3 - \bar{\omega}_3\rho'_2 \\
\bar{S}_2 &= \bar{\omega}_3\rho'_1 - \bar{\omega}_1\rho'_3 \\
\bar{S}_3 &= \bar{\omega}_1\rho'_2 - \bar{\omega}_2\rho'_1
\end{aligned}\right\} \tag{34}$$

and

$$\left.\begin{aligned}
U_1 &= \bar{\omega}_2\bar{V}_3 - \bar{\omega}_3\bar{V}_2 \\
U_2 &= \bar{\omega}_3\bar{V}_1 - \bar{\omega}_1\bar{V}_3 \\
U_3 &= \bar{\omega}_1\bar{V}_2 - \bar{\omega}_2\bar{V}_1
\end{aligned}\right\} \tag{35}$$

The velocity vector components, which are also needed, are

$$\left.\begin{aligned} \bar{V}_1 &= \rho_1' + S_1 \\ \bar{V}_2 &= \rho_2' + S_2 \\ \bar{V}_3 &= \rho_3' + S_3 \end{aligned}\right\} \tag{36}$$

and

$$\left.\begin{aligned} S_1 &= \bar{\omega}_2\rho_3 - \bar{\omega}_3\rho_2 \\ S_2 &= \bar{\omega}_3\rho_1 - \bar{\omega}_1\rho_3 \\ S_3 &= \bar{\omega}_1\rho_2 - \bar{\omega}_2\rho_1 \end{aligned}\right\} \tag{37}$$

The several expressions above furnish sufficient information so that equations (22) and (30) may be solved for the highest order derivatives ($\bar{\omega}_1'$, $\bar{\omega}_2'$, $\bar{\omega}_3'$, α'', β'', γ''), and the cable tension $\bar{T}$, in terms of the lower order of quantities ($\bar{\omega}_1$, $\bar{\omega}_2$, $\bar{\omega}_2$, α', β', γ', α, β, γ). Equations (22) are in a form suitable for integration as they stand; however, equations (33) must be solved for the solution vector Y_j. This is easily done by Cramer's rule which yields the solution

$$Y_k = \bar{a}_{ki} b_i \tag{38}$$

where $\bar{a}_{ki}$ is the inverse matrix of a_{ij}, or:

$$\bar{a}_{ki} a_{ij} = \delta_{kj} \tag{39}$$

and δ_{kj} is the Kronecker delta. Equations (22) and (38) may now be integrated by standard procedures everywhere except at two singular points. The first is at the initial condition where ($t = 0$, $\beta = 0$, $\rho_i' = 0$), and the second is at the change in the phase of the motion (tangential to fixed). These singularities will be discussed separately.

D. Initial Conditions and the Tangential Phase of Motion

At the time $\tau = 0$; $\beta = 0$, and the position vector ρ_{io} is from (10):

$$\left.\begin{aligned} \rho_{1o} &= 1 \\ \rho_{2o} &= 0 \\ \rho_{3o} &= 0 \end{aligned}\right\} \tag{40}$$

The motion immediately following $\tau = 0$ is in the tangential phase thus $\gamma \equiv 0$. The despin weights are considered as being released from rest relative to the rotating body. Therefore, setting the relative velocity to zero leads to the following relations,

$$\left.\begin{aligned} \rho_{1o}' &\equiv 0 \\ \rho_{2o}' &= (1 - \cos\alpha_o)\beta_o' = 0 \\ \rho_{3o}' &= -(\sin\alpha_o)\beta_o' = 0 \end{aligned}\right\} \tag{41}$$

It can be shown from the first of equations (30) that $\beta_o' \neq 0$ and therefore, from (41)

$$\alpha_o = 2n\pi; \ n = 0, \ \pm 1, \ \pm 2, \ \ldots$$

For this problem $\alpha_o = 0$ is used.

The elements of the fundamental matrix a_{ij} can be written at the initial time as:

$$\left.\begin{aligned} a_{11} &= a_{12} = a_{13} = 0 \\ a_{21} &= a_{22} = 0 \\ a_{23} &= -\left(\frac{1}{K} + \frac{1}{I}\right) \\ a_{31} &= a_{32} = a_{33} = 0 \end{aligned}\right\} \qquad (42)$$

and the b_i at $t = 0$ are:

$$\left.\begin{aligned} b_1 &= -(\beta_o')^2 + (\bar{\omega}_{2o})^2 + (\bar{\omega}_{3o})^2 \\ b_2 &= -\bar{\omega}_{1o}\bar{\omega}_{2o} \\ b_3 &= 2\beta_o'\alpha_o' + (I - 2)\bar{\omega}_{1o}\bar{\omega}_{3o} \end{aligned}\right\} \qquad (43)$$

Now equations (22) and (30) must be satisfied at $t = 0$, these lead to:

$$\left.\begin{aligned} \bar{\omega}'_{1o} &= (1 - I)\bar{\omega}_{2o}\bar{\omega}_{3o} \\ \bar{\omega}'_{2o} &= (I - 1)\bar{\omega}_{1o}\bar{\omega}_{3o} \\ \bar{\omega}'_{3o} &= -\frac{\bar{T}_o}{I} \end{aligned}\right\} \tag{44a}$$

and:

$$\left.\begin{aligned} \beta'_o &= \sqrt{\bar{\omega}^2_{2o} + \bar{\omega}^2_{3o}} \\ \bar{T}_o &= \frac{K\,I}{K + I}\,\bar{\omega}_{1o}\bar{\omega}_{2o} \\ \alpha'_o &= \frac{(2 - I)\bar{\omega}_{1o}\bar{\omega}_{3o}}{2\beta'_o} \end{aligned}\right\} \tag{44b}$$

At first appearance it seems odd that the terms α'_o, β'_o are specified by equations (44b). This occurs because of the singularity in the matrix a_{ij} at $t = 0$ as shown by the expressions (42). This singularity, in effect, reduces the dimension of the space in which the "coordinates" ($\bar{\omega}_1$, $\bar{\omega}_2$, $\bar{\omega}_3$, α, β, α', β') are defined and the degeneracy results in the loss of freedom in the choice of α'_o and β'_o.

The singularity also leads to difficulty in the integration of the equations of motion [(22) and (38)], since the integration cannot be started at $t = 0$. In order to overcome this difficulty, one may return to the expression (30) and differentiate with τ, obtaining

$$a'_{ij}Y_j + a_{ij}Y'_j = b'_i \tag{45}$$

At $\tau = 0$ equations (45) reduce to the solutions for β''_o and α''_o, namely

$$\beta''_o = \frac{2}{3\beta'_o}\left(\bar{\omega}_{2o}\bar{\omega}'_{2o} + \bar{\omega}_{3o}\bar{\omega}'_{3o}\right) + \frac{4}{3}\alpha'_o\bar{\omega}_{2o} - \frac{1}{3K}\bar{T}_o \tag{46a}$$

and

$$\alpha''_o = -\frac{\alpha'_o}{\beta'_o}\beta'' + \frac{2 - I}{3\beta'_o}\left(\bar{\omega}'_{1o}\bar{\omega}_{3o} + \bar{\omega}_{1o}\bar{\omega}'_{3o}\right) - \frac{2\beta'_o}{3}\bar{\omega}_{2o} - \frac{1 + K}{K}\frac{\alpha'_o\bar{T}_o}{\beta'_o} \tag{46b}$$

and the superfluous result,

$$\bar{T}' = -\frac{KI}{K + I}b'_{2o} \tag{46c}$$

It is important to note that in order to integrate these equations of motion only the "geometrical parameters" I, K, and the initial conditions $\bar{\omega}_{1o}$, $\bar{\omega}_{2o}$, are needed as the quantity $\bar{\omega}_{3o}$ is given by equation (18). Later the initial coning angle will be introduced which replace either $\bar{\omega}_{1o}$ or $\bar{\omega}_{2o}$ as an initial condition.

The results which have been found thus far allow one to integrate the equations of motion from the initial conditions through the tangential phase up to the time (τ_c) where the cable is fixed. At this time a discontinuity occurs in α', β', $\hat{\omega}$ due to the sudden change in system constraints. The equations we have thus far derived apply to the first (tangential) phase of the motion and the last (fixed) phase of the motion if we use the k factor as defined. The point of transition ("change-over" point) will be denoted by subscripting the pertinent parameters with the quantity (c).

E. Transition Conditions and the Fixed Phase of Motion

At a predetermined time (τ_c), the tangential phase of the motion is assumed ended and β becomes fixed at β_c. The angle γ then begins to increase from zero until the body is despun, or at least until $\bar{\omega}_3 = 0$. Note that it is usually desirable to select the cable length so that β_c is precisely that angle needed to obtain $\bar{\omega}_3 = 0$ when $\gamma = 90^\circ$. In order to integrate from τ_c^- to τ_c^+ a value of $\gamma_c'^+$ must be obtained. Upon reflection, it seems likely that several variables will suffer discontinuities in this situation. Thus, the variables immediately before and after the transition time τ_c will be denoted as follows:

at	(before)	(after)
	$\tau = \tau_c(-)$:	$\tau = \tau_c(+)$:
	$\alpha = \alpha_c$	$\alpha = \alpha_c$
	$\beta = \beta_c$	$\beta = \beta_c$
	$\gamma = 0$	$\gamma = 0$
	$\alpha' = \alpha_c'(-)$	$\alpha' = \alpha_c'(+)$
	$\beta' = \beta_c'$	$\beta' = 0$
	$\gamma' = 0$	$\gamma' = \gamma_c'$
	$\bar{\omega}_1 = \bar{\omega}_1(-)$	$\bar{\omega}_1 = \bar{\omega}_1(+)$
	$\bar{\omega}_2 = \bar{\omega}_2(-)$	$\bar{\omega}_2 = \bar{\omega}_2(+)$
	$\bar{\omega}_3 = \bar{\omega}_3(-)$	$\bar{\omega}_3 = \bar{\omega}_3(+)$

In the body axis coordinate system the relative velocity at $\tau_c(-)$ has (in general) a component along the direction of the cable; calling the component $\rho'_{\tau i}$ it follows that

$$\rho'_{\tau i} = \left[\rho'_j \frac{\overline{T}_j}{T}\right]\left(\frac{\overline{T}_i}{T}\right) \tag{47}$$

During the fixed phase of the motion it is geometrically evident that there can be no relative velocity component along the cable so that the despin weight is constrained to lie on the surface of a sphere of radius $(a\beta_c)$. Therefore, an effective impulse is given to the system when the freely unwinding cable comes to the fixed pivot due to the sudden reduction of the relative velocity by $\rho'_{\tau i}$. Or stated mathematically

$$\rho'_i(+) = \rho'_i(-) - \rho'_{\tau i} \tag{48}$$

Equations (47), (26) and (8) are now used to obtain the scalar relations of (48). From (8) it is seen that

$$\frac{\overline{T}_i}{\overline{T}} \Rightarrow (\cos \alpha_c \sin \beta_c, \ - \cos \alpha_c \cos \beta_c, \ - \sin \alpha_c) \tag{49}$$

and performing the scalar product of (47) gives

$$\rho'(-)_i \frac{\overline{T}_i}{\overline{T}} = (1 - \cos \alpha_c)\beta'_c$$

and the components of $\rho'_{\tau i}$ become

$$\left.\begin{aligned}
\rho'_{\tau 1} &= \left[(1 - \cos\alpha_c)\cos\alpha_c \sin\beta_c\right]\beta'_c \\
\rho'_{\tau 2} &= -\left[(1 - \cos\alpha_c)\cos\alpha_c \cos\beta_c\right]\beta'_c \\
\rho'_{\tau 3} &= -\left[(1 - \cos\alpha_c)\sin\alpha_c\right]\beta'_c
\end{aligned}\right\} \tag{50}$$

The three scalar equations represented in (48) can be expanded with the aid of (50) as

$$\left.\begin{aligned}
\rho_{1\alpha}\alpha'(+) + \rho_{1\gamma}\gamma'_c &= \rho_{1\alpha}\alpha'(-) + \rho_{1\beta}\beta'_c - \rho'_{\tau 1} \\
\rho_{2\alpha}\alpha'(+) + \rho_{2\gamma}\gamma'_c &= \rho_{2\alpha}\alpha'(-) + \rho_{2\beta}\beta'_c - \rho'_{\tau 2} \\
\rho_{3\alpha}\alpha'(+) &= \rho_{3\alpha}\alpha'(-) + \rho_{3\beta}\beta' - \rho'_{\tau 3}
\end{aligned}\right\} \tag{51}$$

The last of (51) gives

$$\alpha'_c(+) = \alpha'_c(-) + \frac{\sin\alpha_c}{\beta_c}\beta' \tag{52}$$

and the first provides for

$$\gamma'_c = (\cos\alpha_c)\beta'_c \tag{53}$$

The second equation in (51) is identically satisfied after substitution of (52) and (53) into it.

Now the above relations have been developed from a kinematic point of view but they must also be solutions to the dynamic problem - that is, they must satisfy the equations of motion at $\tau = \tau_c(+)$. Equations (52), (53) must also be solutions to the first integrals of the equations of motion; that is, they must satisfy the conservation of angular momentum of the system. It may be observed that unless the rotation vector $\bar{\omega}_i$ also suffers a discontinuity this conservation cannot be valid. The conservation of angular momentum provides for a simple computation of the changes in the angular velocity components at $\tau = \tau_c$.

The nondimensional angular momentum of the system at $\tau = \tau_c(-)$ is

$$\bar{H}_i = \frac{PH_i(-)}{I_1} = I_{ij}\bar{\omega}_j(-) + K\epsilon_{ijk}\rho_j\bar{V}_k(-) \tag{54}$$

and since this vector is conserved

$$\bar{I}_{ij}\bar{\omega}_j(+) + K\epsilon_{ijk}\rho_j\bar{V}_k(+) = \bar{H}_i$$

from (24)

$$\bar{I}_{ij}\bar{\omega}_j(+) + K\epsilon_{ijk}\rho_j\epsilon_{krs}\bar{\omega}_r(+)\rho_s = \bar{H}_i - K\epsilon_{ijk}\rho_j\rho_k'(+)$$

which reduces to

$$B_{ij}\bar{\omega}_j(+) = C_i \tag{55}$$

where

$$B_{ij} = \bar{I}_{ij} + K(\rho_s\rho_s\delta_{ij} - \rho_i\rho_j)$$

and

$$C_i = \bar{H}_i - K\epsilon_{ijk}\rho_j\rho_k'(+)$$

Now if $\bar{B}_{ij}$ is the inverse matrix of B_{ij} then the "new" values of the $\bar{\omega}_i$ vector are

$$\bar{\omega}_i(+) = \bar{B}_{ij}C_j \tag{56}$$

and the integration of the equations of motion can be carried on into the fixed phase of the motion.

F. Discussion on Constraints

It has been mentioned that the Lagrange technique is not valid for the representation of the motion of this yo-yo system because of the non-conservative, internal constraints. This point will be elaborated on here and an attempt to give a reasonable physical interpretation of these nonconservative constraints will be made.

First of all consider the rate of work done on the system and the rate of change of the kinetic energy of the system. It may be reasoned directly from the second law of Newton that

$$\frac{d}{dt}(E_s) = \frac{d}{dt}(W_s)$$

where the subscript indicates system properties and E, W are, respectively, the kinetic energy and work done on the system. In order to investigate the work done on the system, it is in general necessary to consider each part of the system separately. Isolating the rigid body by considering the cable replaced by a force T_i (as we have done earlier), it is possible to compute both the kinetic energy rate and the rate of work for the rigid body during the tangential phase. These are,

$$\frac{d}{dt}(E_B) = \frac{1}{2} I_{ij} \frac{d}{dt}(\omega_i \omega_j) = I_{ij}\omega_i\dot{\omega}_j$$

and

$$\frac{d}{dt}(W_B) = M_i\omega_i = 2\epsilon_{ijk}\omega_i a_j T_k$$

Isolating the weights which have forces $-T_i$ acting on them, one finds

$$\frac{d}{dt}(E_w) = \frac{1}{2}\left(2m \frac{d}{dt} V_i V_i\right) = 2mV_i \frac{dV_i}{dt} \quad .$$

$$\frac{d}{dt}(W_w) = -2T_i V_i$$

where, as in the nondimensional equations (36),

$$V_i = \dot{r}_i + \epsilon_{ijk}\omega_j r_k$$

and then

$$\frac{d}{dt}(W_w) = -2T_i\dot{r}_i - 2\epsilon_{ijk}T_i\omega_j r_k$$

Now, noting that $r_i = a_i + l_i$ and also that

$$T_i(\epsilon_{ijk}\omega_j l_k) \equiv 0$$

since T_i and l_i are parallel, the work rate then becomes

$$\frac{d}{dt}(W_w) = -2T_i\dot{r}_i - 2\epsilon_{ijk}T_i\omega_j a_k$$

$$= -2T_i\dot{r}_i - 2\epsilon_{ijk}\omega_i a_j T_k$$

Combining the energy rates and work rates for the body and the weights gives

$$\frac{d}{dt}(E_s) = I_{ij}\omega_i\omega_j + 2mV_i \frac{dV_i}{dt}$$

and

$$\frac{d}{dt}(W_s) = -2T_i\dot{r}_i$$

Equating these, it is seen that

$$\frac{d}{dt}(E_s) = -2T_i\dot{r}_i$$

The product $T_i\dot{r}_i$ can be evaluated and is found to be

$$T_i\dot{r}_i = aT(1 - \cos\alpha)\dot{\beta}$$

Then nondimensionalizing, it is seen that

$$\frac{d}{dt}\bar{E}_s + \bar{T}(1 - \cos\alpha)\beta' = 0$$

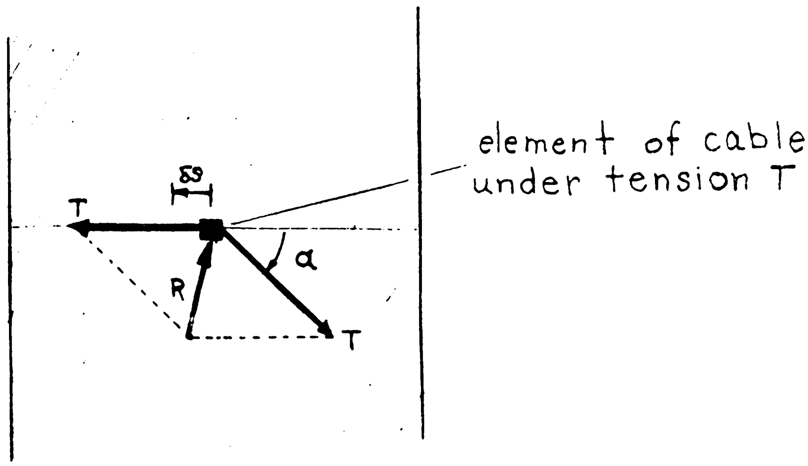
This expression reveals that the system kinetic energy is constant only for two practical cases. First, whenever $\alpha \equiv 0$ and second, whenever $\beta' \equiv 0$. Therefore, in two-dimensional studies, for which the assumption of $\alpha \equiv 0$ is valid, one may take advantage of the conservation of kinetic energy as a first integral in the solution of the problem. Also the kinetic energy is constant throughout the fixed phase motion in the general case as presented here since $\beta' \equiv 0$. However, if one assumes this in the general case when tangential phase motion exists, the results will be in error.

The yo-yo problem presented here is a good example of how one can be misled into performing an erroneous analysis by following intuition and neglecting or overlooking the basic hypotheses made in the development of the theory which governs the physics involved. Suppose this system of rigid-body and despin weights is analyzed in the tangential phase by the techniques of analytical mechanics (as distinguished from vectoral or Newtonian mechanics). Without regard to constraints, it is easily seen that three coordinates are needed to specify the configuration of the rigid body and three more for the weights (the position of the second weight, being symmetric through the origin to the first, is also determined by specifying these coordinates). These coordinates might be the Euler angles for the body (θ, ϕ, ψ), the angles (α, β), and the length l for the weight. The problem may be formulated in terms of five generalized coordinates (θ, ϕ, ψ, α, β) by eliminating l through the employment of the holonomic constraint $l = a\beta$. The kinetic energy is then found to be of the form,

$$E_s = E_s(\theta, \phi, \psi, \alpha, \beta, \dot{\theta}, \dot{\phi}, \dot{\psi}, \dot{\alpha}, \dot{\beta})$$

There is no potential energy for the system and no "applied" forces are acting on the system - recall here that the constraints have been included in the system of generalized coordinates. Under these assumptions the Lagrangian is equivalent to the kinetic energy. Also, without consideration of the work of constraint the system appears conservative and the kinetic energy appears as a constant of the motion, E = constant. We have already shown that this result is not true. The same error is easily made when considering Hamilton's equations. The basic difficulty here is not simply one of the occurrence of nonconservative forces, but of internal constraints or forces of reaction doing work; which is contrary to the hypothesis commonly made in analytical mechanics (i.e., ref. 10, page 15). It is not to be implied here that the analytical method cannot be used, but only that care must be taken to include the work of the constraints.

This "work of constraints" may be roughly visualized in the following manner: Consider a portion of the cable at the point of contact and let this point move through a virtual displacement δs as shown in the following sketch,



The horizontal component of the reaction R must therefore be equal to $(T - T \cos \alpha)$ and therefore the amount of work which is done by R is:

$$-T(1 - \cos \alpha)\delta s$$

G. Motion in a Rotating Reference System

It is known from the study of the force-free motion of a symmetric body that the motion is easily represented by consideration of the axis of symmetry x_3, the rotation vector $\bar{\omega}_i$ and the angular momentum vector of the body $\bar{h}_i$. If the axis of symmetry is designated by

$$k_i \Rightarrow (0, \ 0, \ 1)$$

it is seen that,

$$k_i(\epsilon_{ijk}\overline{\omega}_j\overline{h}_k) \equiv 0 \tag{57}$$

which implies that the three vectors k_i, $\overline{\omega}_i$, and $\overline{h}_i$ all lie in the same plane at any instant. Note that this result is valid for any external force system so long as the angular momentum of the body is given by

$$\overline{h}_i = \overline{I}_{ij}\overline{\omega}_j \tag{58}$$

where $\overline{I}_{ij}$ is

$$\overline{I}_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I \end{pmatrix}$$

The angular momentum vector $\overline{h}_i$, the rotation vector $\overline{\omega}_i$ and the symmetry vector k_i are important in describing the motion of the yo-yo, rigid body motion since, after the despin weights and cables are released, the rigid body behaves as an ordinary symmetric rigid body in force-free motion. That is, it moves such that the symmetry axis cones about the angular momentum vector which remains fixed in space. Reference 11 gives a complete and rather vivid account of this motion. The angle between the symmetry axis and the angular momentum vector is easily computed in the x_i coordinate axis system. This angle is referred to as the coning angle (Γ) and is given by

$$\Gamma = \tan^{-1}\left[\frac{\bar{\omega}_{12}}{I\bar{\omega}_3}\right] \tag{59}$$

where $\bar{\omega}_{12}$ is referred to as the cross axis spin rate, and is defined as

$$\bar{\omega}_{12} = \sqrt{\bar{\omega}_1^{\,2} + \bar{\omega}_2^{\,2}} \tag{60}$$

It is evident that $\bar{\omega}_{12}$ is the component of the rotation vector in the x_1, x_2 plane.

The angle between the symmetry axis and the rotation vector is called the spinning angle (δ) and is computed as

$$\delta = \tan^{-1}\left[\frac{\bar{\omega}_{12}}{\bar{\omega}_3}\right] \tag{61}$$

Several other important quantities may be computed from the study in relative coordinate axes. The length of cable required to despin is found from $l = a\beta$ and the cable tension is found from the nondimensional tension $\bar{T}$. These quantities (β, $\bar{T}$) are found from the integration of equations (22) and (38). The angles α, and γ while possibly only of academic interest are nevertheless also easily obtained from these same equations of motion. Hence, it is seen that considerable information may be obtained without referencing an inertial coordinate system.

H. Motion in an Inertial Reference System, Euler Angles

So far the analysis has been concerned with motion referred to a coordinate system attached to the rotating body (body axis system). Several interesting results and conclusions can be obtained by analyzing the problem in this manner. However, if a detailed study of the motion with respect to some fixed axis system is desired it is necessary to introduce a set of coordinates relating the rotating system to the desired fixed system. This is often done by introducing the Euler angles. This method is employed here and the particular Euler angles used are given in reference 10 and shown in figure 3. The fixed coordinate system is denoted by X_i where $X_i \Rightarrow (X_1, X_2, X_3)$.

Herein the angles ϕ, θ, ψ will be designated as

$\phi \sim$ precessional angle

$\theta \sim$ nutational angle

$\psi \sim$ spin angle

These angles may be obtained from the body rates $\bar{\omega}_i$ through the relations given in reference 10, equations (4-103),

$$
\begin{aligned}
\bar{\omega}_1 &= \phi' \sin\theta \sin\psi + \theta' \cos\psi \\
\bar{\omega}_2 &= \phi' \sin\theta \cos\psi - \theta' \sin\psi \qquad (62) \\
\bar{\omega}_3 &= \phi' \sin\theta + \psi'
\end{aligned}
$$

Upon inversion the desired form is obtained

$$\phi' = \frac{1}{\sin\theta}(\bar{\omega}_1 \sin\psi + \omega_2 \cos\psi)$$

$$\theta' = \bar{\omega}_1 \cos\psi - \bar{\omega}_2 \sin\psi \qquad (63)$$

$$\psi' = \bar{\omega}_3 - \phi' \cos\theta$$

The solution for ϕ, θ, and ψ describes the motion of the body with respect to the X_i. Equations (63) can be integrated digitally along with equations (22) and (38) so that very little additional computer effort is needed to obtain the motion in fixed space.

It is necessary to consider initial conditions for (63). These are defined by choosing the fixed system X_i at the initial instant such that the nondimensional angular momentum of the body, $\bar{h}_{io}$ lies along the X_3 axis while the symmetry axis x_3, and the rotation vector $\bar{\omega}_i$ lie in the X_2, X_3 plane. This may be accomplished by setting $\phi_o = 0$ and rotating θ until the angle $\theta_o = \Gamma_o$, which is the desired initial coning angle. Next the system is rotated about x_3 an amount ψ_o until the angular momentum vector $\bar{h}_i$ lies along X_3. It is then geometrically evident that

$$\psi_o = \tan^{-1}\left[\frac{\bar{h}_{1o}}{\bar{h}_{2o}}\right]$$

and since (see eq. (58))

$$\bar{h}_{io} \Rightarrow (\bar{\omega}_{1o},\ \bar{\omega}_{2o},\ I\bar{\omega}_{3o})$$

we have

$$\psi_o = \tan^{-1}\left[\frac{\bar{\omega}_{1o}}{\bar{\omega}_{2o}}\right] \tag{64}$$

Therefore the additional initial conditions needed to integrate (63) along with the basic equations of motion are $\phi_o = 0$, $\theta_o = \Gamma_o$ and ψ_o given by equation (64).

The introduction of the Euler angles not only provides a means for recording the motion of the body axis system (x_i) but also provides a handy method for transforming vector components from the body axis system to the inertial X_i system. If the components of a vector are Z_i in the x_i system then the components in the X_i system, $\bar{Z}_i$ are given by the Euler angle transformation found in reference 10, equations (4-47). The elements of the transformation matrix are rewritten here for purpose of reference.

$$\left.\begin{aligned}
A_{11} &= \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi \\
A_{12} &= -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi \\
A_{13} &= \sin\theta\sin\phi \\
A_{21} &= \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi \\
A_{22} &= -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi \\
A_{23} &= -\sin\theta\cos\phi \\
A_{31} &= \sin\theta\sin\psi \\
A_{32} &= \sin\theta\cos\psi \\
A_{33} &= \cos\theta
\end{aligned}\right\} \tag{65}$$

Thus the components in the fixed or space coordinates are

$$\bar{Z}_i = A_{ij} Z_j \tag{66}$$

Vector components known in the body axis system which are of interest for study in the fixed coordinate systems, are the angular momentum $\bar{h}_i$, the symmetry axis k_i, the rotational axis $\bar{\omega}_i$ and the normal to the plane containing the cables, mass center and weights. This normal lies along the negative moment vector, so that if this normal vector is denoted by S_i it may be found from:

$$S_i = -\bar{M}_i \tag{67}$$

Then these four vectors (k_i, $\bar{h}_i$, $\bar{\omega}_i$, S_i) may be transformed to the fixed axis system by use of equations (66).

The use of Euler angles to describe the motion of a rigid body has one major disadvantage. This is the singularity at $\theta = 0$ which is easily noticed in equations (63). If we return to equations (62) and set $\theta \equiv 0$, $\theta' \equiv 0$ we have,

$$\bar{\omega}_1 = \bar{\omega}_2 = 0$$

$$\bar{\omega}_3 = \phi' + \psi'$$

It is seen that, when $\theta \equiv 0$, the nutational rates and precessional rates are arbitrary to the extent that there sum must equal the symmetry axis spin rate. For studies in this thesis when $\Gamma_o = 0^o$ (two-dimensional case) the values $\phi \equiv 0$, $\psi_o' = \overline{\omega}_{3o}$ will be used. This singularity presents some difficulty when relating data with $\theta_o \equiv 0$ to data where $\theta_o \neq 0$ as will be noted later.

VII. TWO-DIMENSIONAL AND APPROXIMATE ANALYSIS

A. General Discussion

The exact equations presented in the previous chapter yield a solution only through a numerical procedure such as integration by a digital computer. This method of analysis is very useful since many computers perform a task such as this with relative ease. However, in spite of the enormous number of computations which may be accomplished by the computer, an approximate solution in terms of known analytic functions is often valuable in the study of a system's motion. A solution to the yo-yo problem can be obtained from the equations of motion when the direction of the axis of symmetry is fixed in inertial space and the motion is considered to be in the tangential phase. This situation was discussed in the introduction and was referred to as the two-dimensional case. The solution to this problem will be redeveloped here from the general results of chapter VI. The only two-dimensional results which are easily obtained for the fixed phase motion will also be given here for completeness of detail.

A general closed form, analytical solution for the three-dimensional problem has also been obtained with certain simplifying assumptions. This solution is presented following the discussion and derivation of the two-dimensional solution which follows.

B. Two-Dimensional Solution, Tangential-Phase Motion

If the motion is constrained so that the axis of symmetry remains fixed in space and the plane of the despin weights and mass center is always normal to this axis, then the motion may be termed "two-dimensional." In this case the angular momentum vector and the rotation vector both lie along the symmetry axis, and it follows that,

$$\alpha = \bar{\omega}_1 = \bar{\omega}_2 \equiv 0$$

Looking at the equations of motion after applying this simplification, it is found that, for the tangential phase ($\gamma \equiv 0$), a closed form solution is obtainable. The terms in equations (30) associated with (α, α', α'') can be disregarded along with the third equilibrium equation. Equations (30) then reduce to,

$$\left.\begin{aligned} a_{12}\beta'' + a_{13}\bar{T} = b_1 \\ a_{22}\beta'' + a_{23}\bar{T} = b_2 \end{aligned}\right\} \qquad (68)$$

The determinant of the coefficient matrix is

$$[a] = a_{12}a_{23} - a_{22}a_{13}$$

so that the inverse matrix is

$$a_{ij} \Rightarrow \frac{1}{[a]}\begin{pmatrix} a_{23} & -a_{13} \\ -a_{22} & a_{12} \end{pmatrix}$$

and the solutions for β'' and $\overline{T}$ are,

$$\left.\begin{aligned} \beta'' &= \frac{1}{[a]}\left(a_{23}b_1 - a_{13}b_2\right) \\ \overline{T} &= \frac{1}{[a]}\left(-a_{22}b_1 + a_{12}b_2\right) \end{aligned}\right\} \tag{69}$$

These are, of course, a direct application of Cramer's rule to (68).

Now in order to write (69) explicitly, we substitute $\alpha = \gamma = 0$ into the appropriate expressions of chapter VI and obtain,

$$a_{12} = \beta \cos\beta$$

$$a_{13} = \frac{K + I}{KI}\sin\beta - \frac{1}{I}\beta\cos\beta$$

$$a_{22} = \beta\sin\beta$$

$$a_{23} = -\frac{K + I}{KI}\cos\beta - \frac{1}{I}\beta\sin\beta$$

$$b_1 = -(\cos\beta - \beta\sin\beta)(\beta')^2 + (2\beta\sin\beta)\overline{\omega}_3\beta' + (\cos\beta + \beta\sin\beta)\overline{\omega}_3^2$$

$$b_2 = -(\sin\beta + \beta\cos\beta)(\beta')^2 - (2\beta\cos\beta)\overline{\omega}_3\beta' + (\sin\beta - \beta\cos\beta)\overline{\omega}_3^2$$

$$[a] = -\frac{K + I}{IK}\beta$$

These expressions applied to (69) produce the equations

$$\beta\beta'' + (1 - \overline{K}\beta^2)(\beta')^2 - (2\overline{K}\beta^2)\omega_3\beta' - (1 + \overline{K}\beta^2)\overline{\omega}_3^2 = 0 \tag{70}$$

$$\frac{1}{I}\,\bar{T} = \bar{K}\beta(\beta' + \bar{\omega}_3)^2 \tag{71}$$

Also equations (22), for the two-dimensional, tangential case, reduce to,

$$\bar{\omega}_3' = -\frac{1}{I}\,\bar{T} \tag{72}$$

Equations (71) and (72) represent a single differential equation,

$$\bar{\omega}' + \bar{K}\beta(\beta' + \bar{\omega}_3)^2 = 0 \tag{73}$$

where the term $\bar{K}$ is written for

$$\bar{K} = \frac{K}{K + I} = \frac{2ma^2}{2ma^2 + I_3} \tag{74}$$

from which it may be noted that I_1 disappears from the problem as it should.

Equations (70) and (73) are the equations which must be solved for the two-dimensional case. Although the system is of third order it is very nonlinear and also has a singularity since β appears as the coefficient of β'' in (70). The system is therefore degenerate at $\beta = 0$ and hence it is possible that only two arbitrary constants may be assigned at $\beta = 0$. This corresponds to the matrix singularity in the general case. Here it is found that β'_0 may not be arbitrary (initial condition) but must depend on $\bar{\omega}_{30}$.

The solutions of these equations - as found in reference 2 except for a change in notation and which may be verified by direct substitution - are,

$$\beta = \bar{\omega}_0 \tau \tag{75}$$

$$\bar{\omega} = \bar{\omega}_0 \frac{1 - \tau^2}{1 + \tau^2} \tag{76}$$

The cable tension, found from (72) is:

$$\frac{1}{I} \bar{T} = \frac{4\bar{\omega}_0 \tau}{\left[1 + \tau^2\right]^2} \tag{77}$$

In equation (76) the subscript (3) in $\bar{\omega}_3$ has been dropped for convenience.

The nondimensional time to despin to zero rate ($\bar{\omega} = 0$) is found directly from (76),

$$(\tau)_{\bar{\omega}=0} = 1 \tag{78}$$

which explains the reason for defining the period P as shown in equation (15). From the definition of the nondimensional time, equation (16) the true (two-dimensional) despin time is

$$(t)_{\omega_3=0} = P = \sqrt{\frac{1}{\bar{K}}}\,\frac{1}{\omega_{30}}$$

C. Two-Dimensional Results, Fixed Phase Motion

The equations of motion for the fixed phase of the two-dimensional motion are even more nonlinear and more difficult to solve than those for the tangential phase. In fact, as far as the author knows, no solution has been found for the 2-D fixed phase equations. The fixed phase motion is assumed to start at some time $\tau = \tau_c$ where $\beta = \beta_c$ as discussed in chapter VI - "Transition Conditions." Then the initial value of γ is

$$\gamma_c = 0$$

and for γ' (see eq. (53))

$$\gamma_c'(\tau_c^+) = \beta_c'(\tau_c^-) = \overline{\omega}_o$$

where, of course, $\beta' \equiv 0$ during the fixed phase motion and by definition the constant, k is unity. With these simplifications for the two-dimensional case the equations of motion (30) lead to,

$$\left(\beta_c \frac{K \cos^2 \gamma + I}{KI \cos \gamma}\right)\gamma'' - \frac{\beta_c}{I}(\beta_c + \sin \gamma)(\gamma')^2 - \frac{2\beta_c}{I}(\beta_c + \sin \gamma)\,\gamma'\overline{\omega} +$$

$$- \left[\frac{1}{K} + \frac{2}{I}(1 + \beta_c^2 + 2\beta_c \sin \gamma)\right]\overline{\omega}^2 = 0 \qquad (79)$$

$$\left(\frac{K \cos^2 \gamma + I}{KI \cos \gamma}\right)\overline{T} - \beta_c(\gamma')^2 - 2\beta_c\gamma'\overline{\omega} - (\beta_c + \sin \gamma)\overline{\omega}^2 = 0 \qquad (80)$$

Equations (79) and (80) above correspond to (70) and (71) of the tangential phase. The general equations (22) reduce in this case to:

$$\bar{\omega}' = -\frac{1}{I}\bar{T}\cos\gamma \tag{81}$$

which corresponds to (72) for the tangential case.

Although solutions to equations (79), (80), and (81) are not available at the present two first integrals are known in the form of well known conservation theorems. These are, of course, the conservation of momentum and the conservation of energy for the entire system.

The angular momentum of the system was written in nondimensional form in equation (54) and for the two-dimensional case this reduces to

$$\bar{H} = I\bar{\omega} + K(\rho_1\bar{V}_2 - \rho_2\bar{V}_1) \tag{82}$$

or writing out the velocity terms:

$$\bar{H} = I\bar{\omega} + K\left[\rho_1\rho_2' - \rho_2\rho_1' + (\rho_1^2 + \rho_2^2)\bar{\omega}\right] \tag{82a}$$

At the initial time ($\tau = 0$), (82a) reduces to

$$\bar{H} = (I + K)\bar{\omega}_o \tag{83}$$

One interesting result obtainable from a study of the system momentum and energy is the length of cable necessary to despin to $\bar{\omega} = 0$. This length which, in nondimensional quantities is the angle β_c, is easily found in this two-dimensional case if the end conditions desired are $\bar{\omega} = 0$ at $\gamma_f = 90^\circ$. When this requirement is employed it is found that,

$$\rho_{1f} = (1 + \beta_c) \cos \beta_c$$

$$\rho_{2f} = (1 + \beta_c) \sin \beta_c$$

$$\rho'_{1f} = -\beta_c(\sin \beta_c)\gamma'_f$$

$$\rho'_{2f} = \beta_c(\cos \beta_c)\gamma'_f$$

$$\bar{\omega}_f = 0$$

and that equation (82a) provides,

$$\bar{H} = K\beta_c(1 + \beta_c)\gamma'_f \tag{84}$$

Combining equations (83) and (84) yields

$$\gamma'_f = \frac{(I + K)\bar{\omega}_o}{K\beta_c(1 + \beta_c)} \tag{85}$$

The kinetic energy of the system E is expressed by

$$E = \frac{1}{2} I_3 \omega_3^2 + \frac{1}{2} (2m) V_i V_i$$

and nondimensionalizing,

$$\overline{E} \equiv \frac{P^2 E}{I_1} = \frac{1}{2} I \overline{\omega}^2 + \frac{1}{2} K(\rho_i' + \epsilon_{ijk} \overline{\omega}_j \rho_k)(\rho_i' + \epsilon_{irs} \overline{\omega}_r \rho_s)$$

At the initial instant ($\tau = 0$) this reduces to,

$$2\overline{E} = (I + K)\overline{\omega}_o^2 \tag{86}$$

At the final instant where $\gamma_f = 90^\circ$, $\overline{\omega}_f = 0$ and ρ_{1f}, ρ_{2f}, ρ_{1f}', ρ_{2f}' are as given just above the dimensionless kinetic energy is

$$2\overline{E} = K\beta_c^2(\gamma_f')^2 \tag{87}$$

Combining equations (86) and (87) gives

$$\gamma_f' = \sqrt{\frac{I + K}{K}} \frac{\overline{\omega}_o}{\beta_c} \tag{88}$$

eliminating γ_f' from (85) and (88) and solving for β_c gives

$$\beta_c = \sqrt{\frac{I}{K} + 1} - 1 \tag{89}$$

This result will be used to determine the "transition point" in the study of the general motion.

D. Approximate Solution in Three Dimensions

The two-dimensional solutions given in the preceding section are found to be very good estimates for cable length and tension even when there is a significant amount of initial coning. This estimation will be discussed in a subsequent section of this work. However, if one desires to study the three-dimensional motion, without resorting to the exact differential equations given earlier, an approximation must be made which will give at least a representative solution of the equations. The major difficulty in developing a closed form solution lies in the complex nonlinearity of the equilibrium equations for the weights, equations (38). The assumption to be made here is that a form of the functions $\alpha(\tau)$, $\beta(\tau)$, $\bar{\omega}_3(\tau)$ can be guessed with sufficient accuracy so that the errors involved in equations (38) are not significant. For instance, one might desire to choose the form of α, β, and $\bar{\omega}_3$ such that certain undefined multipliers or constants were contained in the chosen expressions; the Euler equations (22) might then be solved for $\bar{T}_1, \bar{\omega}_1$, and $\bar{\omega}_2$ with the yet arbitrary multipliers, and finally these expressions could be introduced into equations (38) where the multipliers might be determined so that the error in these equations is minimized.

In any event assume that $\alpha(\tau)$, $\beta(\tau)$ and $\bar{\omega}_3(\tau)$ are given functions, then the third of equations (22) give the solution for $\bar{T}$,

$$\bar{T} = -\frac{I\bar{\omega}_3'}{\cos\alpha} \tag{90}$$

and then $\bar{\omega}_1$ and $\bar{\omega}_2$ are found from the first and second differential equations of (22),

$$\bar{\omega}_1' = (1 - I)\bar{\omega}_3\bar{\omega}_2 - \bar{T}\sin\beta\sin\alpha \tag{91a}$$

$$\bar{\omega}_2' = -(1 - I)\bar{\omega}_3\bar{\omega}_1 + \bar{T}\cos\beta\sin\alpha \tag{91b}$$

To solve these equations it is convenient to make the following substitutions,

$$\left.\begin{aligned} x &\rightarrow \bar{\omega}_1 \\ y &\rightarrow \bar{\omega}_2 \\ f &\rightarrow (1 - I)\bar{\omega}_3 \\ g &\rightarrow -\bar{T}\sin\beta\sin\alpha \\ h &\rightarrow \bar{T}\cos\beta\sin\alpha \end{aligned}\right\} \tag{92}$$

Then equations (91) may be rewritten as:

$$x' - fy = g \tag{93a}$$

$$y' + fx = h \tag{93b}$$

Kamke, reference 12 page 66, D.G18.20, suggests a technique leading to the solution of equations (93).

Multiply the second equation by a nonzero constant λ and add to the first equation obtaining,

$$x' + \lambda y' + f(\lambda x - y) = g + \lambda h \qquad (94)$$

Now choose λ such that

$$\lambda x - y = \lambda(x + \lambda y)$$

or

$$(\lambda^2 + 1)y = 0$$

then λ has two possible values which must be considered:

$$\lambda_1 = i \qquad (95a)$$

$$\lambda_2 = -i \qquad (95b)$$

where $i = \sqrt{-1}$. Defining an auxiliary variable z_i as,

$$z_r = x_r + \lambda_r y_r \quad (\text{no-sum, } r = 1,2) \qquad (96)$$

then (94) may be written,

$$z_r' + \lambda_r f z_r = F_r \quad (\text{no-sum} \quad r = 1,2) \tag{97}$$

where:

$$F_r = g + \lambda_r h \tag{98}$$

The single, linear, first order equation (97) is easily integrated. Using the integrating factor,

$$e^{\lambda_r \int f d\tau}$$

equation (97) is directly integrable, giving

$$z_r = z_r(o)e^{-\lambda_r G} + e^{-\lambda_r G}\int_o^\tau e^{\lambda_r G(\xi)} F_r(\xi)d\xi \tag{99}$$

where,

$$G = \int_o^\tau f(x)dx$$

and from (95) it is noted that the exponentials can be written:

$$e^{-\lambda_r G} = \cos G - \lambda_r \sin G$$

$$e^{\lambda_r G} = \cos G + \lambda_r \sin G$$

and with,

$$z_r(o) = x_o + \lambda_r y_o$$

$$F_r = g + \lambda_r h$$

and (96) one obtains

$$x_r + \lambda_r y_r = x_o \cos G - \lambda_r^2 y_o \sin G + \lambda_r(y_o \cos G - x_o \sin G) +$$
$$+ (\cos G - \lambda_r \sin G) \int_o^\tau (\cos G + \lambda_r \sin G)(g + \lambda_r h)d\tau$$
$$\text{(no-sum on } r)$$

This relation leads to the same solution for either λ_r. Replacing the variables above by their original expressions given by equations (92) the solution becomes,

$$\bar{\omega}_1 = \bar{\omega}_{1o} \cos G + \bar{\omega}_{2o} \sin G - \cos G \int_o^\tau \bar{T} \sin \alpha \sin (\beta + G)d\tau +$$
$$+ \sin G \int_o^\tau \bar{T} \sin \alpha \cos (\beta + G)d\tau \qquad (100a)$$

$$\bar{\omega}_2 = \bar{\omega}_{2o} \cos G - \bar{\omega}_{1o} \sin G + \cos G \int_o^\tau \bar{T} \sin \alpha \cos (\beta + G)d\tau +$$
$$+ \sin G \int_o^\tau \bar{T} \sin \alpha \sin (\beta + G)d\tau \qquad (100b)$$

where,

$$G = (1 - I) \int_0^{\tau} \bar{\omega}_3 d\tau \tag{100c}$$

Equations (100) are exact expressions for the three-dimensional solution of the tangential phase motion. If it is assumed that $\gamma(\tau)$ is known then the above formulas may be adapted to the fixed phase motion in a similar manner. It is seen that unless α, β and $\bar{\omega}_3$ are chosen simply, equations (100) offer very little advantage over the original set of differential equations. In this work only the simplest form of α, β and $\bar{\omega}_3$ will be studied. It will be assumed that weights remain in the $x_1 x_2$ plane so that,

$$\alpha(\tau) \equiv 0 \tag{101}$$

and that the β and $\bar{\omega}$ functions will be taken directly from the two-dimensional solution, equations (75) and (76), that is

$$\beta = \bar{\omega}_{30} \tau \tag{102}$$

$$\bar{\omega}_3 = \bar{\omega}_{30} \frac{1 - \tau^2}{1 + \tau^2} \tag{103}$$

Then the tension is found from (90), (101), and (103) to be

$$\bar{T} = \frac{4I\bar{\omega}_{30}\tau}{(1 + \tau^2)^2} \tag{104}$$

which is the same as the two-dimensional result (77).

With these assumptions equations (100) become:

$$\bar{\omega}_1 = \bar{\omega}_{10} \cos G + \bar{\omega}_{20} \sin G \tag{105a}$$

$$\bar{\omega}_2 = \bar{\omega}_{20} \cos G - \bar{\omega}_{10} \sin G \tag{105b}$$

and

$$G = (1 - I)\bar{\omega}_{30} \int_0^\tau \frac{1 - x^2}{1 + x^2} dx$$

and upon integration,

$$G = (1 - I)\bar{\omega}_{30}(2 \tan^{-1}\tau - \tau) \tag{105c}$$

This solution for $\bar{\omega}_1$, $\bar{\omega}_2$ is of interest if we wish to estimate the motion in fixed space. Such an estimation is accomplished through the use of equations (63), but again a computer is necessary for the integration. It should be evident that the integration of these three equations is simpler to program than the 10 equations obtained in the exact analysis. An example case is presented in a subsequent section of this thesis.

VIII. FINAL ANALYSIS AND DISCUSSION OF RESULTS

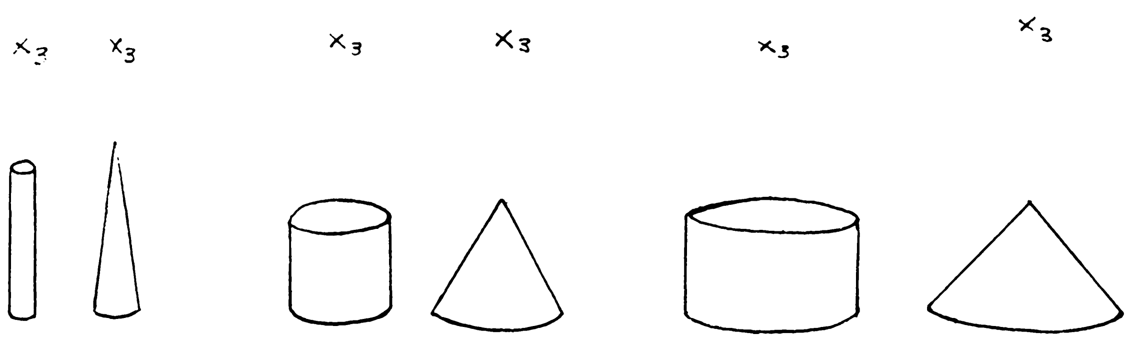
A. General

In chapters VI and VII the mathematical foundations were given by which the motion of a yo-yo rigid body, despin system can be studied and the effects of changes in initial conditions and design parameters on this motion can be determined. One objective sought in this chapter is to provide data necessary for a study of the general motion of the system; perhaps not so complete that some specific system may be fully designed, but at least so that design judgments may be made. Also, it is hoped that some insight into the general motion of this system might be gained by studying these data and the results.

Several curves are presented to show the influence of the parameters which naturally evolve in the development of the equations of motion. These parameters are the inertia ratio of the weights to the cross-axis body inertia (K) as defined in (13), the body inertia ratio (I) as given in (14), the ratio $\left(\frac{\bar{\omega}_{2o}}{\bar{\omega}_{1o}}\right)$, and initial coning angle Γ_o.

Although the solution of the general equations requires four parameters the two-dimensional cases require less. For instance, after proper nondimensionalization, the two-dimensional, tangential time history results are given as functions of τ only. The two-dimensional solutions also determine a more natural choice for the nondimensional form of the variables $(\bar{\omega}_i, \bar{T}, \beta)$ in the general case.

The choice of numerical values for the parameters I, K was made from quite simple reasoning. The values for the inertia ratio I were selected to represent three different geometrical regimes which are thought to be sufficiently representative for this study. These are, I = 0.1, 0.9, 1.3; which are respectively pencil-like, slightly under-spherical, and slightly over-spherical shapes. Sketches indicating the relative geometry of right-circular cylinders and right-circular cones are given below:



I=.1 I=.9 I=1.3

The choice for the K factor cannot be made quite so arbitrarily. In order to obtain reasonable values, a right-circular cone and a right-circular cylinder, both of constant density, are considered. From the definitions of I and K and the formulas for the inertias of these bodies in terms of their total mass (say M), simple relations are easily derived. These are,

$$K = 4 \frac{m}{M} I \qquad \text{(cylinder)}$$

$$K = \frac{15}{4} \frac{m}{M} I \qquad \text{(cone)}$$

These expressions allow the determination of K from the more familiar relationship of despin weight mass to body mass. In the time history studies, (K) is chosen so that the mass of the despin weights (2m) is approximately 1/2 percent of the vehicle mass (M).

In discussing the results the order of presentation is reversed from that of chapters VI and VII. This is done so that the reader is first familiarized with the simpler dynamics of the two-dimensional motion and consequently he is then better able to interpret the results of the three-dimensional cases. An example using the approximate three-dimensional equations as derived in chapter VI follows the discussion on the exact solutions.

It should be noted that throughout the study of the exact equations the condition $\Gamma_o = 0$ corresponds to the two-dimensional case and provides a basis of comparison. The result that $\Gamma_o = 0$ corresponds to two-dimensional motion is obvious from the symmetry of the system. A new form for nondimensionalizing the variables is a direct result of the desire to compare the three-dimensional solutions ($\Gamma_o \neq 0^o$) to the two-dimensional solutions ($\Gamma_o = 0^o$) in a uniform and yet simple manner.

B. Two-Dimensional Solution, Tangential Phase

The solutions of the equations of motion for the two-dimensional, tangential phase are given in equations (75), (76), and (77). It is interesting to note that these relations may be expressed in nondimensional forms which are entirely independent of the inertial ratios I, K; these are,

$$\tilde{\beta} \equiv \frac{\beta}{\bar{\omega}_{30}} = \tau \tag{106}$$

$$\tilde{\omega}_3 \equiv \frac{\bar{\omega}_3}{\bar{\omega}_{30}} = \frac{1 - \tau^2}{1 + \tau^2} \tag{107}$$

$$\tilde{T} \equiv \frac{\bar{T}}{I\bar{\omega}_{30}} = \frac{4\tau}{\left[1 + \tau^2\right]^2} \tag{108}$$

There are other variables of interest which can be easily obtained from these expressions, one is the angular distance which the body moves through during the despin. This is the integral of $\bar{\omega}_3$ in equation (107). Another quantity of interest is the maximum tension ($\tilde{T}_{max}$) which is found by use of equation (108). The integral of $\bar{\omega}_3$ is defined as the spin angle ψ and was essentially determined in equation (105c) of chapter VII. The spin angle is then found by the following integration,

$$\psi = \int_0^t \omega_3 dt = \int_0^\tau \bar{\omega}_3 d\tau = \bar{\omega}_{30} \int_0^\tau \frac{1 - x^2}{1 + x^2} dx$$

therefore, the nondimensional counterpart can be defined as,

$$\tilde{\psi} = \frac{\psi}{\bar{\omega}_{3o}} = 2 \tan^{-1} \tau - \tau \tag{109}$$

The value of this spin function after despin to $\bar{\omega}_3 = 0$, $(\tau = 1)$ is,

$$(\tilde{\psi})_{\omega_3=0} = \frac{\pi}{2} - 1 = 0.5708 \tag{110}$$

The maximum cable tension is obtained from the differentiation of equation (108),

$$\tilde{T}' = \frac{4(1 - 3\tau^2)}{(1 + \tau^2)^2} = 0$$

which gives the time at maximum $\tilde{T}$ as

$$(\tau)_{T_{max}} = \frac{1}{\sqrt{3}} = 0.5775 \tag{111}$$

and hence the maximum, nondimensional tension for the two-dimensional, tangential case is

$$(\tilde{T})_{max} = \frac{3\sqrt{3}}{4} = 1.299 \tag{112}$$

Equations (106), (107), (108), and (109) which represent $\tilde{\beta}$, $\tilde{\omega}$, $\tilde{T}$, and $\tilde{\psi}$ are plotted in figures 4(a), 4(b), 4(c), and 4(d). From these figures and the above equations and definitions a complete analysis for the two-dimensional tangential motion may be performed.

C. Two-Dimensional Solution, Fixed Phase

It was pointed out that the equations of motion for the two-dimensional fixed phase have not been solved; therefore, this study cannot be as precise as the tangential-phase study. However, an interesting and useful relation has been developed for the cable length (or unwind angle, β_c) required to despin the body to zero angular rate when the swing angle (γ) is 90°. This relation is given by equation (89) and can be written in the nondimensional form of tangential-phase section as

$$\tilde{\beta}_c = \frac{\beta_c}{\bar{\omega}_o} = 1 - \sqrt{\frac{K/I}{1 + K/I}} \tag{113}$$

It is seen that this relation is dependent only on the ratio K/I. The transition time τ_c is determined easily from the tangential-phase result, equation (75), and becomes

$$\tau_c = \frac{\beta_c}{\bar{\omega}_o} = \tilde{\beta}_c \tag{114}$$

thus, in this nondimensional form, the time, the cable length, and the unwind angle are equivalent.

Figure 5 presents a plot of equation (113). It should be noted that the difference in unwind angle (or cable length) required for a purely tangential despin as compared to a despin with fixed phase is simply $1 - \tilde{\beta}_c$, when expressed in these nondimensional terms.

It is possible to derive the expression for cable tension at the instant when $\gamma = 90^{\circ}$. Intuitively, for the two-dimensional case, the maximum cable tension during the fixed phase occurs at $\gamma = 90^{\circ}$. At this instant the tension is equivalent to the centrifugal force acting on a weight and therefore:

$$T = m\left[\omega^2 a + (\dot{\gamma} + \omega)^2 a\beta_c\right]$$

Considering only the case $\omega = 0$ (i.e., total despin) along with equations (88) and (89) and definitions for the nondimensional relations, the tension at this final state is expressed by,

$$(\tilde{T})_{max} = \frac{K}{I} \frac{\left(1 + \frac{1}{K/I}\right)^{3/2}}{\left(1 + \frac{1}{K/I}\right)^{1/2} - 1} \qquad \text{(fixed phase)} \tag{115}$$

It can be seen that the variables of equations (79), (80), and (81) can be reduced to the same nondimensional form as that given by equations (106), (107), and (108). With the addition of

$$\tilde{\gamma} = \frac{\gamma}{\bar{\omega}_{30}} \tag{116}$$

equations (79), (80), and (81) can be rewritten in terms of $\tilde{T}$, $\tilde{\gamma}$, $\tilde{\omega}$, $\tilde{\beta}_c$, the ratio K/I, and the dimensionless time τ. In order to study the systems motion the differential equations have been integrated numerically and the results are shown in figures 6(a), 6(b), 6(c), and 6(d) and figure 7.

The spin rate $\tilde{\omega}$ does not differ appreciably from the purely tangential result (fig. 4) for any of the K/I ratios presented in figure 6(a). The nondimensional-cable tension as shown in figure 6(b), can be greatest in either the tangential or fixed phase depending upon the value of K/I. Also presented in figure 6 are the swing angle $\tilde{\gamma}$ and the rotation of the body $\tilde{\psi}$. The swing angle shows large variation dependent upon the values of K/I, but the rotation as a function of time does not change greatly with K/I. In all of figures 6 the equations of motion have been integrated until $\omega_3 = 0$ and the cable length has been determined from equation (113) so that this final time also produces a γ of 90°. Therefore, the final value of $\tilde{\gamma}$ depends on K/I even though γ is 90° in every case.

Figures 7(a), 7(b), and 7(c) are the results of several time history integrations (such as those in figures 6) and show the time required to despin, the maximum cable tension during despin, and the final angle of rotation. The maximum cable tension is governed by the tangential maximum for values of K/I less than 0.04, but for larger values of K/I it increases in an almost linear fashion. In the range of greater K/I this maximum occurs during the fixed phase motion. The despin time and total rotation both have maxima within the range shown and vary significantly from the tangential results. The tangential values are indicated by dashed lines in figures 7(b) and 7(c) and $(\tau)_{\omega_3=0} = 1$ in figure 7(a).

Figures 5, 6, and 7 furnish the necessary information for analyzing a two-dimensional system when despinning to zero angular rate with a swing-out angle of 90°.

D. Three-Dimensional Solutions

The numerical integration of the equations of motion, (22) and (38) is started by choosing values for the parameters I, K, ω_{2o}/ω_{1o} and Γ_o. The initial conditions $\bar{\omega}_{1o}$ and $\bar{\omega}_{2o}$ needed to begin integration are found through the use of equation (59) to be

$$\bar{\omega}_{1o} = \sqrt{\frac{(I\bar{\omega}_{3o} \tan \Gamma_o)^2}{(\omega_{2o}/\omega_{1o})^2 + 1}} \tag{117}$$

$$\bar{\omega}_{2o} = \frac{\omega_{2o}}{\omega_{1o}} \bar{\omega}_{1o} \tag{118}$$

Using the initial conditions derived in (44) and (46), the parameters $\bar{\omega}_{1o}$, $\bar{\omega}_{2o}$ given by (117) and (118), and the values of I and K desired, the integration of the differential equation can be initiated. Once begun, the standard Runge-Kutta fourth-order integration technique is used. The equations are first integrated through the tangential phase where the constant k is zero. The tangential phase ends when the cable has unwound (if it does) to its prespecified length. The cable length was chosen to be the value necessary to despin a two-dimensional system. In nondimensional form the cable length is equal to $\tilde{\beta}_c$ which is given by equation (113).

When $\beta = \beta_c$ the transition point is reached where a slight discontinuity occurs in α', β', and γ' as indicated in chapter VI. The integration then proceeds on into the fixed phase. The integration is stopped whenever one of the following events occur:

(1) $\gamma = \pi/2$

(2) $\gamma = \gamma_{max} \leq \pi/2$

(3) $\beta = \beta_{max}$

(4) $\alpha = \pi/2$

(5) violent tumbling occurs or $\det\left[a_{ij}\right] \approx 0$

After the integration the forms of some of the variables are altered before plotting in order to compare these results with the two-dimensional forms obtained previously. The new forms obtained are,

$$\tilde{\omega}_{12} = \frac{\omega_{12}}{\omega_{30}} = \frac{\bar{\omega}_{12}}{\bar{\omega}_{30}}$$

$$\tilde{\omega}_3 = \frac{\omega_3}{\omega_{30}} = \frac{\bar{\omega}_3}{\bar{\omega}_{30}}$$

$$\tilde{\alpha},\ \tilde{\beta},\ \tilde{\gamma} \Rightarrow \alpha/\bar{\omega}_{30},\ \beta/\bar{\omega}_{30},\ \gamma/\bar{\omega}_{30}$$

$$\tilde{T} = \frac{\bar{T}}{I\bar{\omega}_{30}}$$

$$\tilde{\theta},\ \tilde{\phi},\ \tilde{\psi} \Rightarrow \theta/\bar{\omega}_{30},\ \phi/\bar{\omega}_{30},\ \psi/\bar{\omega}_{30}$$

These variables are then plotted for some of the solutions which have been obtained and are presented in figures 8 through 16.

In all the cases studied, the value of the input quantity ω_{20}/ω_{10} was chosen to be 1.0. Therefore, the initial ψ value (ψ_o) is 45^o in the general case. If this input is altered the results will, of course, be affected but the general trends and indications provided by the present study will not be significantly different.

Figures 8 and 9 are time histories obtained from relative and fixed reference frame integrations for the pencil like configuration $I = 0.1$ and for a fairly small despin weight size $K = 0.001$. Figure 8(a) shows that the cross-axis spin rate is nearly constant throughout the motion for $\Gamma_o \leq 30^o$ while figure 8(b) shows that in this same Γ_o range, the symmetry axis spin is very nearly the same as the two-dimensional value. The cable tension in curve 8c shows even a closer agreement with the two-dimensional results ($\Gamma_o = 0^o$). Figure 8(d) reveals a quite important characteristic of this motion. The out of plane angle in this case behaves quite well during the tangential phase, but in the fixed phase the weights rapidly approach $\alpha = 90^{\circ}$. In this particular example the swing out angle reaches its maximum just before $\alpha = 90^{\circ}$ ($\tilde{\alpha} = 0.1563$) for $\Gamma_o = 30^{\circ}, 40^{\circ}, 50^{\circ}$ but is stopped by the constraint $\alpha = 90^{\circ}$ just before $\gamma = \gamma_{max}$ for $\Gamma_o = 10^{\circ}, 20^{\circ}$. This plot points out the sensitivity of the variables in these equations. Figure 8(e) on the other hand shows that the unwind angle is very nearly linear (as in the 2-D case) for all Γ_o.

In figure 9(a), $\pi/2$ minus the nutation angle is plotted against time and appears well behaved. This shows that the nutation angle does increase with time but only by a small amount. It should be noted that this relationship for a symmetric, rigid body without despin weights would appear as a series of straight horizontal lines with the same initial points. The spin angle ψ is given in figure 9(b) and is seen to have the same general variation as the two-dimensional case but is somewhat lower due to a transfer of some of the rotation to the precession angle which is given in figure 9(c). The curve of $\tilde{\psi}$ for $\Gamma_o = 0^{\circ}$ is started at $\psi_o = 45^{\circ}$ instead of $\psi_o = 0$ as in the two-dimensional example of figure 6(d). This comes about because the choice for $\omega_{2o}/\omega_{1o} = 1.0$ for $\Gamma_o \neq 0$ always gives $\psi_o = 45^{\circ}$ ($\tilde{\psi} = 0.078$).

It is important to note that results using Euler angles in the manner presented here are not continuous as θ approaches zero due to the well known singularity in these angles at $\theta = 0$. In the cases studied, the condition $\theta_o = 0$ ($\Gamma_o = 0$) is handled as indicated in chapter VI under the article describing the Euler angles. Hence the curves of $\Gamma_o = 0^o$ and $\Gamma_o = 10^o, 20^o, \ldots, 50^o$ are not directly related. (In the two-dimensional case note that an arbitrary amount of the spin ω_3 may be assigned to $\dot{\psi}$ while the remaining portion becomes $\dot{\phi}$.) Again, for a symmetric, rigid body without despin weights the curves of figures 9(b) and 9(c) would be straight lines with the initial slopes as given in the figures.

Figures 10(a) through 10(e) correspond to figures 8(a) through 8(e), but for $I = 0.9$, $K = 0.1$. For this case the deviations from the two-dimensional case are large in both the spin rates and cable tension when the initial coning angle Γ_o is greater then 20^o. Note that the $\Gamma_o = 50^o$ case is stopped during the tangential phase when β begins to decrease which corresponds to one of the stopping conditions specified earlier $\beta = \beta_{max}$. The Euler angles plotted in figures 11(a), 11(b), and 11(c) show that, the position of the body is quite drastically changed during despin and for large Γ_o the body is seen to tilt nearly 90^o from the initial momentum vector.

Figures 12 and 13 are time histories for $I = 1.3$, $K = 0.015$ corresponding to figure 8 and 9. The deviations from $\Gamma_o = 0^o$ for these parameters is even greater than for the case of $I = 0.9$, $K = 0.01$. It is seen that reverse spin is actually given to the body. In the $\Gamma_o = 40^o, 50^o$ cases the symmetry axis is seen to dip below the fixed X_1, X_2 plane and the initial rotation is transfered almost entirely into precession leaving a negative spin angle ψ.

Figures 14 through 16 are a parameter study of the yo-yo problem. All these figures are for $\omega_{2o}/\omega_{1o} = 1.0$. They are presented to give an idea of how the basic variables of interest change with the parameters K and I. The variables plotted are values of $\tilde{\omega}_{12}$, $\tilde{\omega}_3$, $\tilde{\gamma}$ and τ when the integration is stopped for one of the reasons mentioned earlier. (See remarks following equation (118).) Also shown is the value of the maximum cable tension occurring during the despin process. These curves are sufficiently complete in themselves so that it seems only necessary here to point out that for $I = 0.9$ and 1.3 the design of a yo-yo device for small despin masses is quite critical whenever the coning angle is greater than 10°. It should be noticed (from figures 14(c), 15(c), and 16(c)) that only very rarely does the swing out angle get to 90° when $\Gamma_o \neq 0^\circ$ and when the two-dimensional result is used to determine the cable length. Hence, if a system is designed so that the cables and weights are to be released at $\gamma = 90^\circ$, based on two-dimensional considerations, and if there is actually some coning, the cables will not release properly.

E. Approximate Solutions

The approximate equations derived in chapter VII (equations (101) through (105)) have been solved for the special case of $I = 0.1$, $K = 0.001$, $\Gamma_o = 10^o$, and $\omega_{1o}/\omega_{2o} = 1.0$. The exact and approximate solutions for $\tilde{\alpha}(\tau)$, $\tilde{\beta}(\tau)$, $\tilde{\omega}_3$, and $\tilde{T}$ are found in figures 8(b), 8(c), 8(d), and 8(e), and figure 4 since the approximations for these variables are the two-dimensional, tangential solutions. Study of these curves show that the two-dimensional results are good representations of this three-dimensional case. The solutions for the three-dimensional approximations of $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are shown in figure 17 along with the exact solution and shows the accuracy which is obtained by these approximations.

In many instances it is important to know the body's change of position, in fixed space, due to the despin process. For instance, one may wish to study the aerodynamic effects on an atmospheric reentry vehicle being despun by this device. The Euler angles are obtained by a standard Runge-Kutta integration of equations (63) using the approximate rotational rates $\bar{\omega}_1$, $\bar{\omega}_2$, $\bar{\omega}_3$. The results are presented in figure 18 along with the exact solutions. As expected from a study of figures 8 and 17 the accuracy obtained by the use of these approximations is excellent at least for this case. Note that the exact solution (used for comparison) are those employing both the tangential phase and the fixed phase motions while the approximate equations employ only the tangential phase. If exact solutions for the tangential phase motion (without fixed phase considerations) are desired the reader should consult reference 9. The advantage of these approximations becomes evident when one contemplates the time needed to program and compute the exact equations and results.

IX. CONCLUDING REMARKS

Examination of the motion of a system comprised of a spinning rigid body and yo-yo despin device requires a numerical integration of the equations of motion since no general analytical solution to these equations has been found. This dissertation presents the governing equations and their derivation and notes that special precautions must be exercised at both the beginning of the integration where the fundamental matrix is singular and at the transition point between the tangential-phase motion and the fixed-phase motion where a "jump" or discontinuity in certain of the variables is found. These highly nonlinear equations are easily and quickly integrated by the standard Runge-Kutta integration technique, with difficulty occurring only in the final stages of the motion where tumbling sometimes occurs or sometimes when the cable begins violent departures from normal motions. A convenient means of nondimensionalizing the equations is also presented here which allows the reduction of the problem "input" parameters to four; namely: $\left(I, \; K, \; \Gamma_o, \; \frac{\omega_{2o}}{\omega_{1o}}\right)$. It is also of interest to note that this problem may not be attacked by the Lagrangian or Hamiltonian approach without special care since the basic postulate of workless internal constraints is violated.

A numerical analysis of several selected cases shows that the despin time, cable tension, and cable length to despin are very closely computed by simpler two-dimensional equations for initial coning angles as large as 30^o when the inertia ratio (I) is small (in the order of 0.1). For larger values of the inertia ratio (0.9, 1.3), the two-dimensional results are rather poor when the initial coning angle is larger than about 10°. A major difficulty encountered in the use of the two-dimensional equations appears to be the fact that only rarely will the cables move into a radial position for release if a significant amount of coning is present and if the two-dimensional value for cable length is used. This indicates that the cables should probably be a bit shorter than the two-dimensional value indicates if coning is expected and if a radial release of the weights is desired.

A study of the motion of the body relative to inertial coordinates reveals that the nutation angle increases during the despin process and that the increase becomes larger for increasing inertia ratios (I) and initial coning angles.

Some approximate relations are developed here so that the study of this inertial motion could be simplified. This approximate analysis compares well with the exact results for the case studied.

X. ACKNOWLEDGEMENTS

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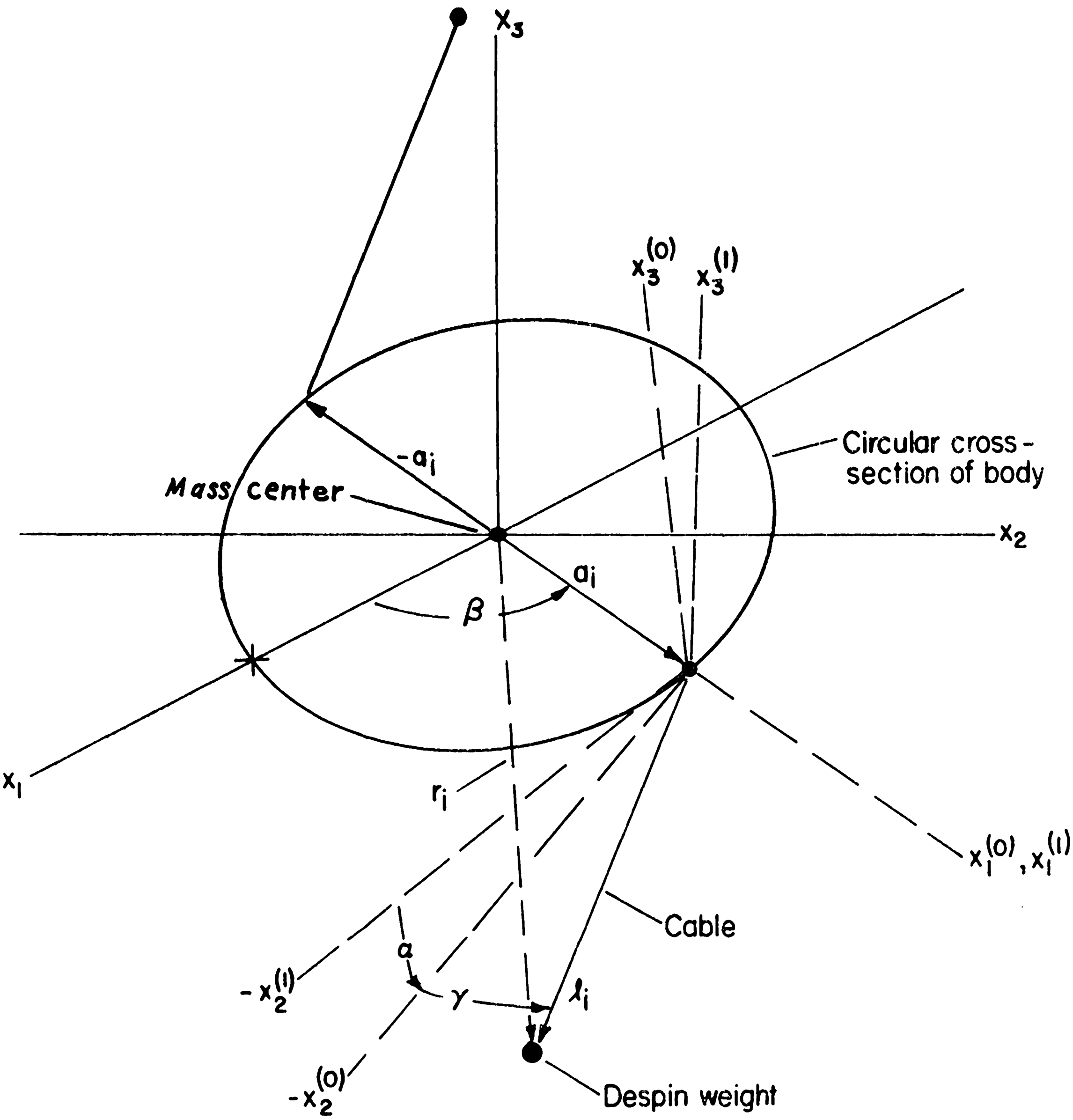


Figure 1.- Showing coordinate systems and other pertinent information. Note: β => rotation of cable contact point, α => rotation about $X_1^{(o)}$, γ => rotation about $X_3^{(o)}$.

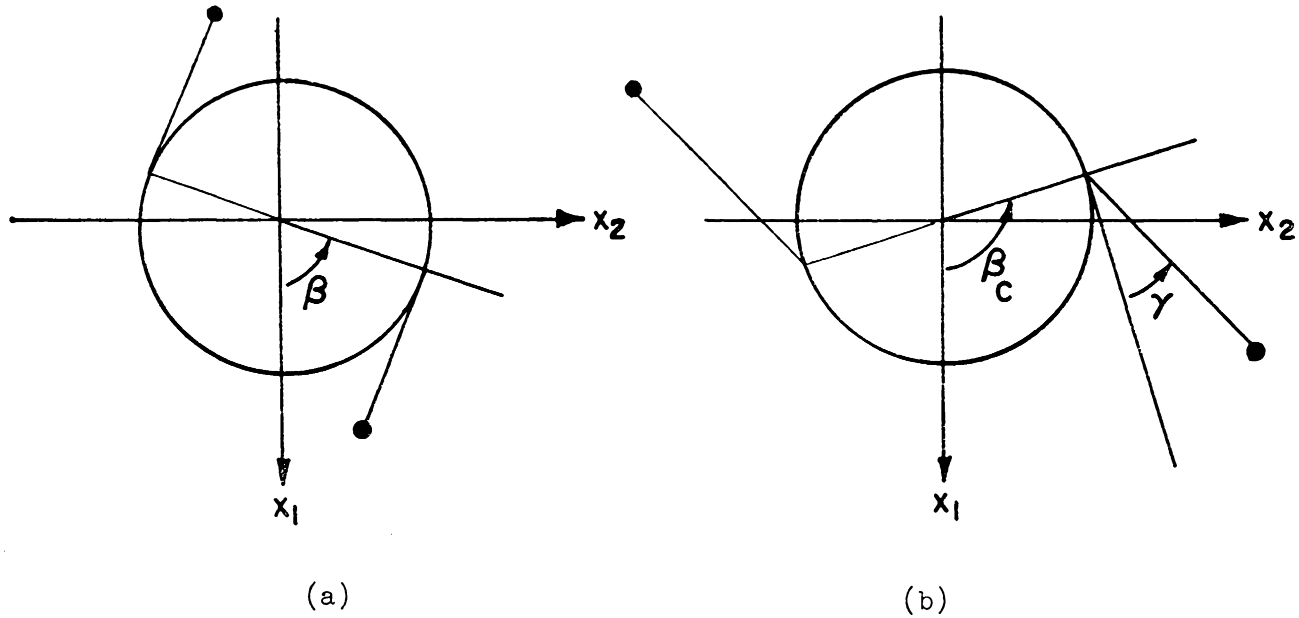


Figure 2.- Showing tangential phase (a), and fixed phase (b), motions for two-dimensional case.

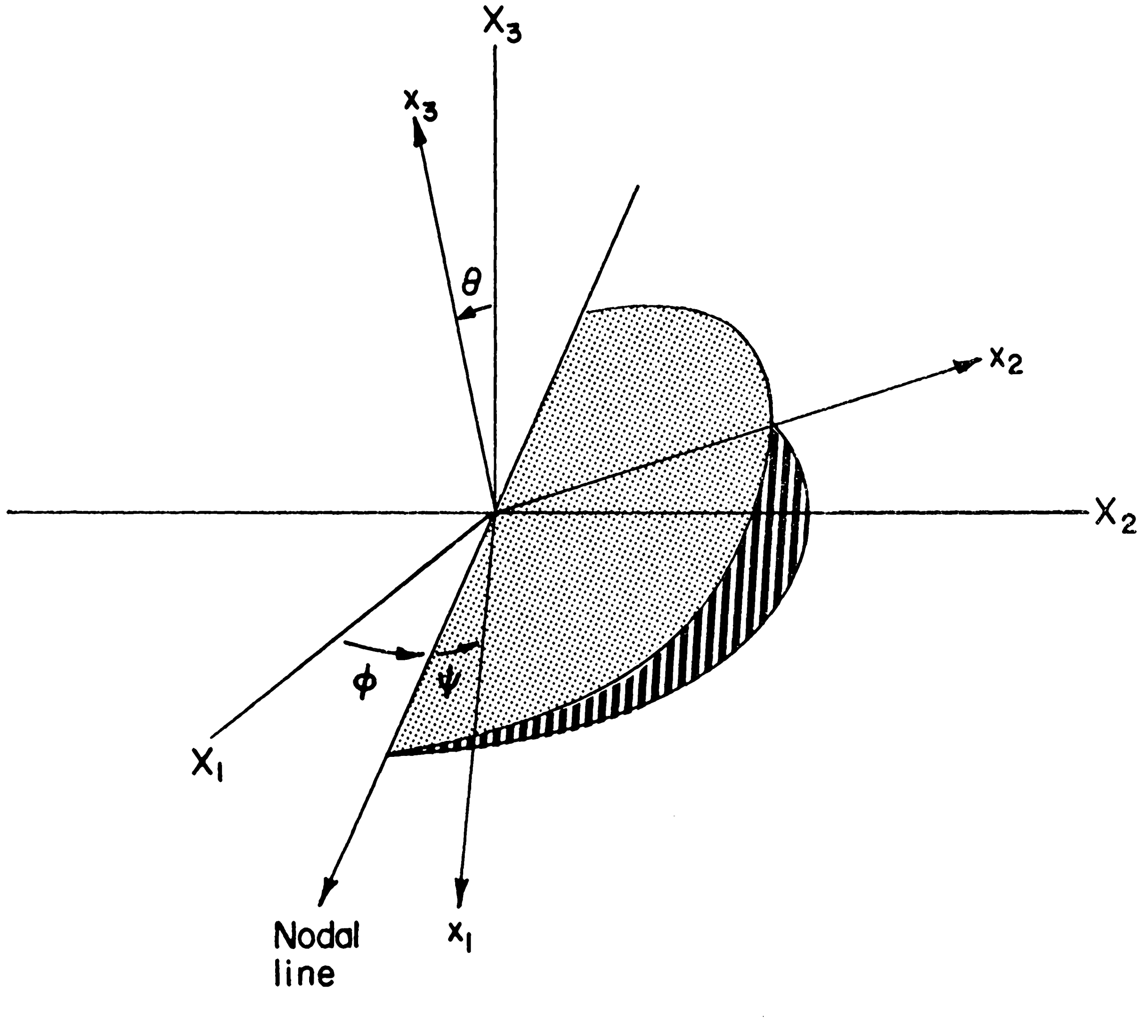


Figure 3.- Showing Euler angles adapted in study of motion in a fixed reference system. Order of rotation is ϕ, θ, ψ.

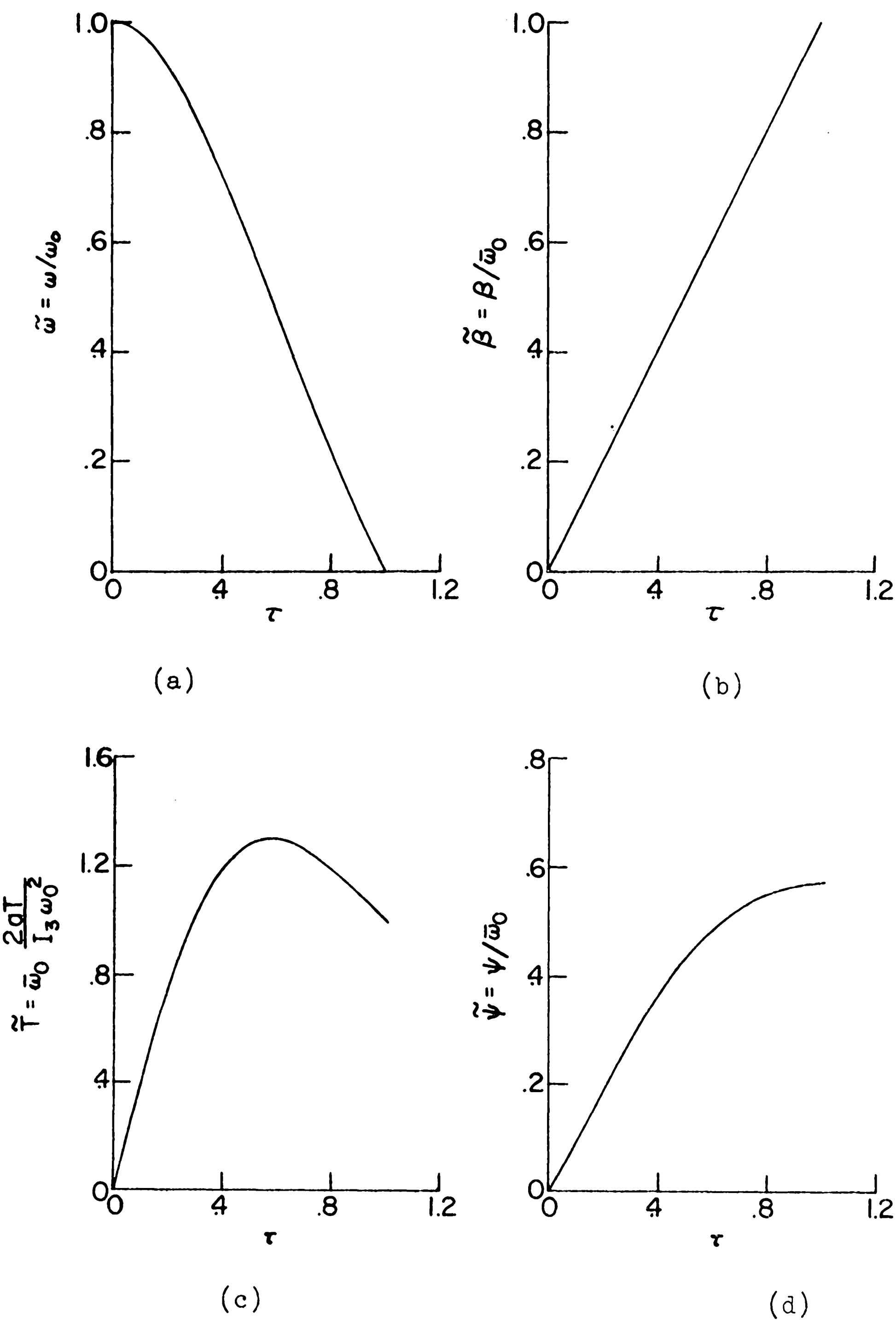


Figure 4.- Dimensionless spin rate, unwind angle, cable tension, and spin angle plotted against dimensionless time for the two-dimensional tangential phase motion.

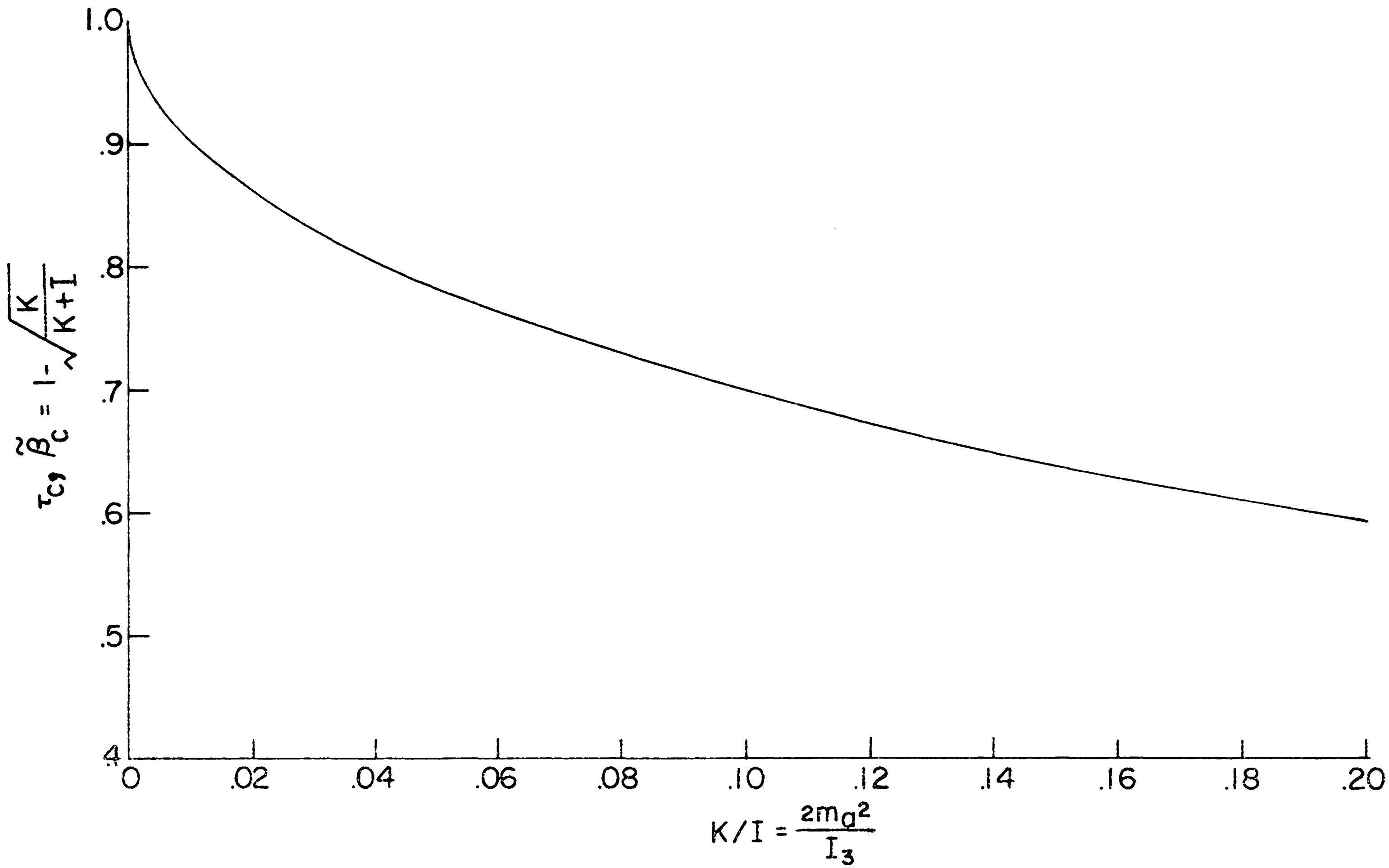
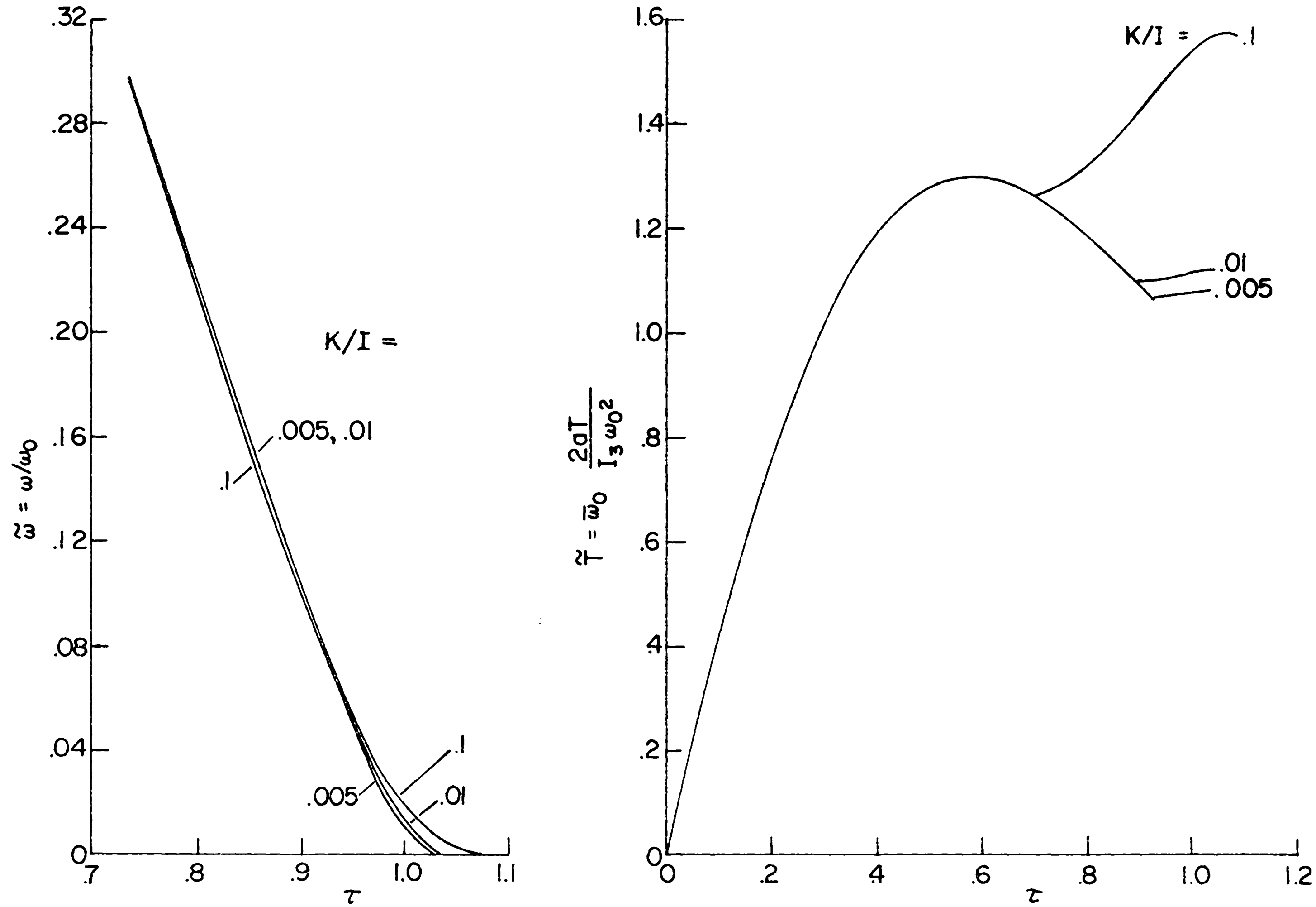


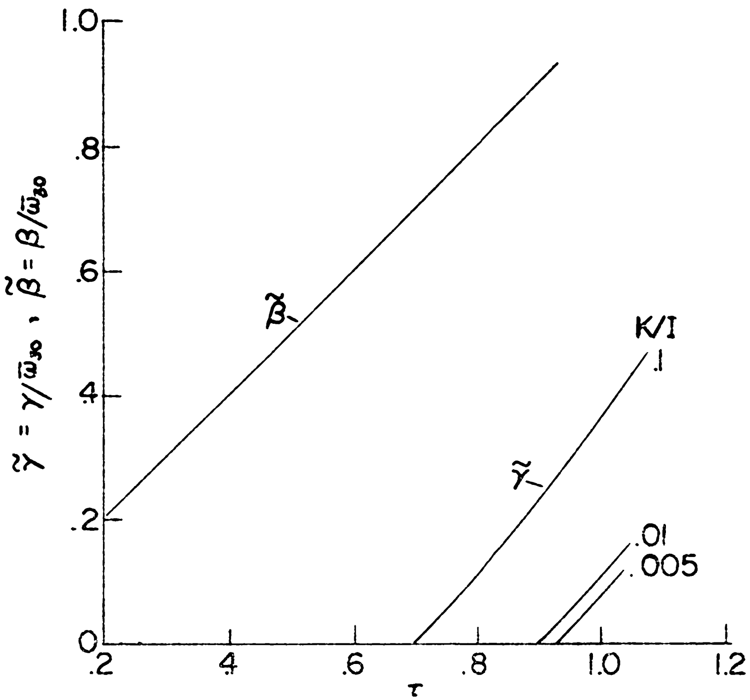
Figure 5.- Dimensionless time and cable length required at the end of the tangential phase, if the conditions $\omega = 0$, $\gamma = 90^{\circ}$ are to be met at end of fixed phase, plotted against the inertia ratio for two-dimensional motion.



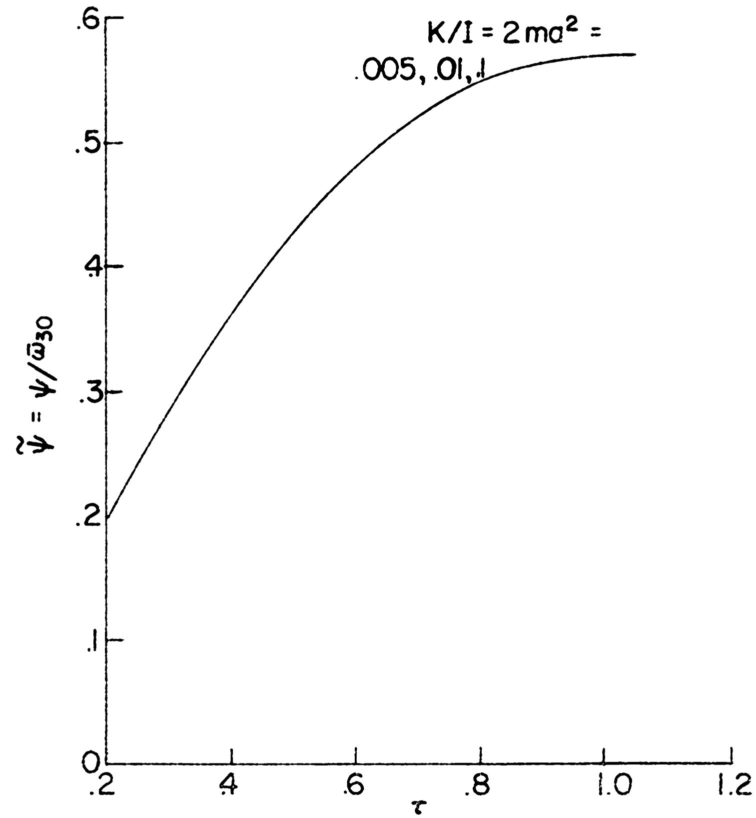
(a) Spin rate.

(b) Cable tension.

Figure 6.- Variation of significant quantities with time in nondimensional form representing two-dimensional motion with both tangential and fixed phase unwinding.



(c) Swingout and unwind angles.



(d) Spin angle.

Figure 6.- Concluded.

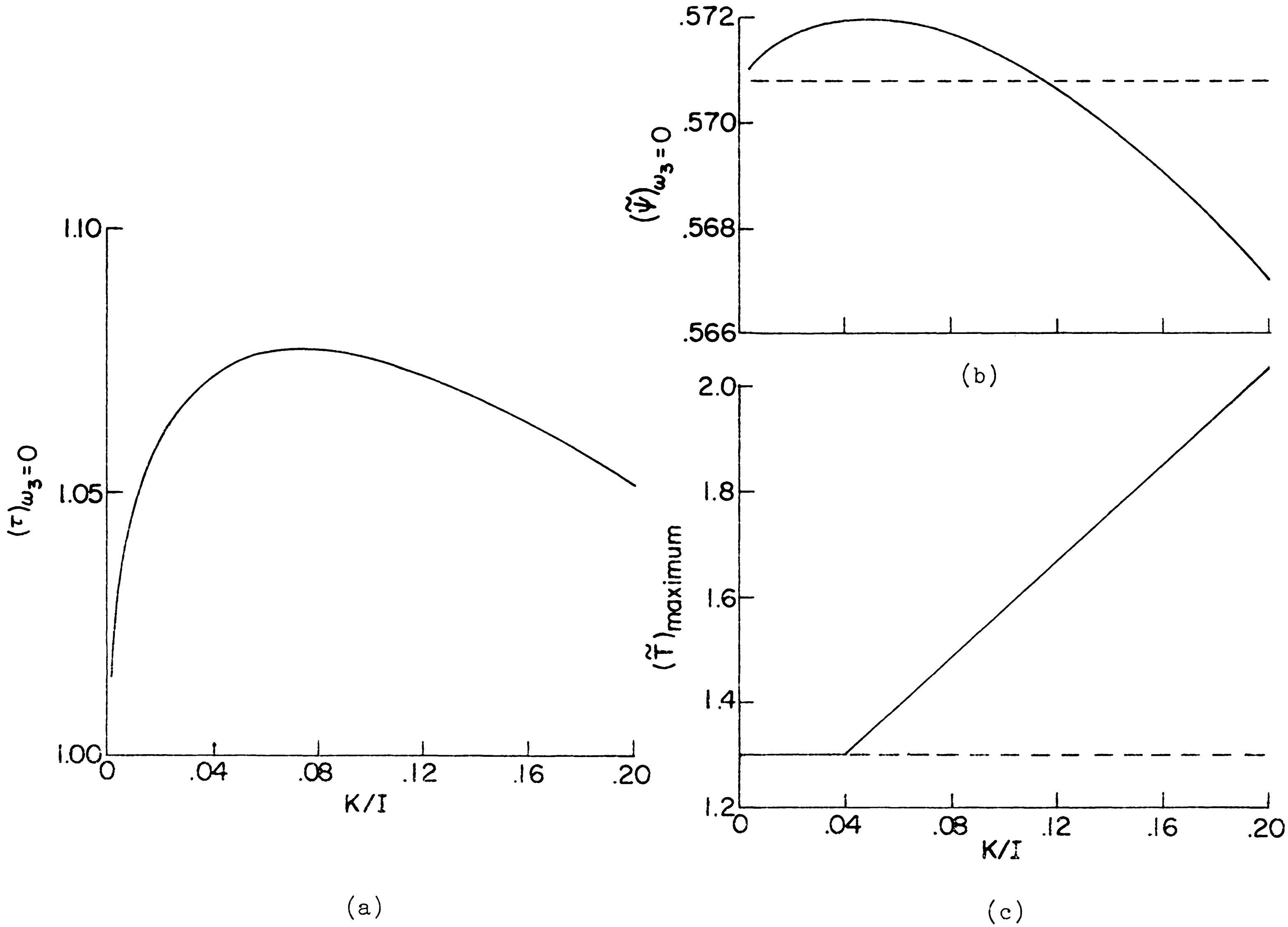
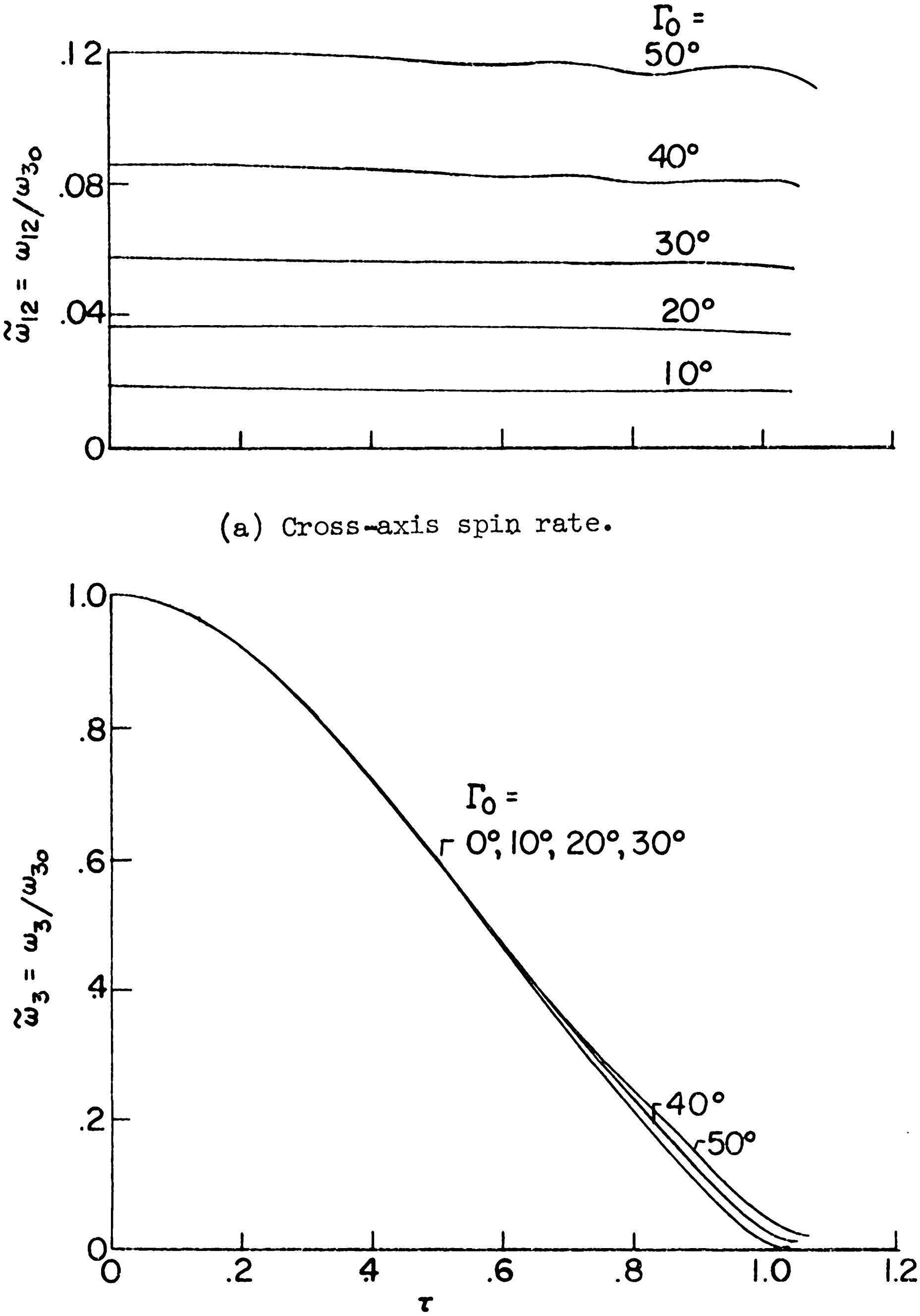


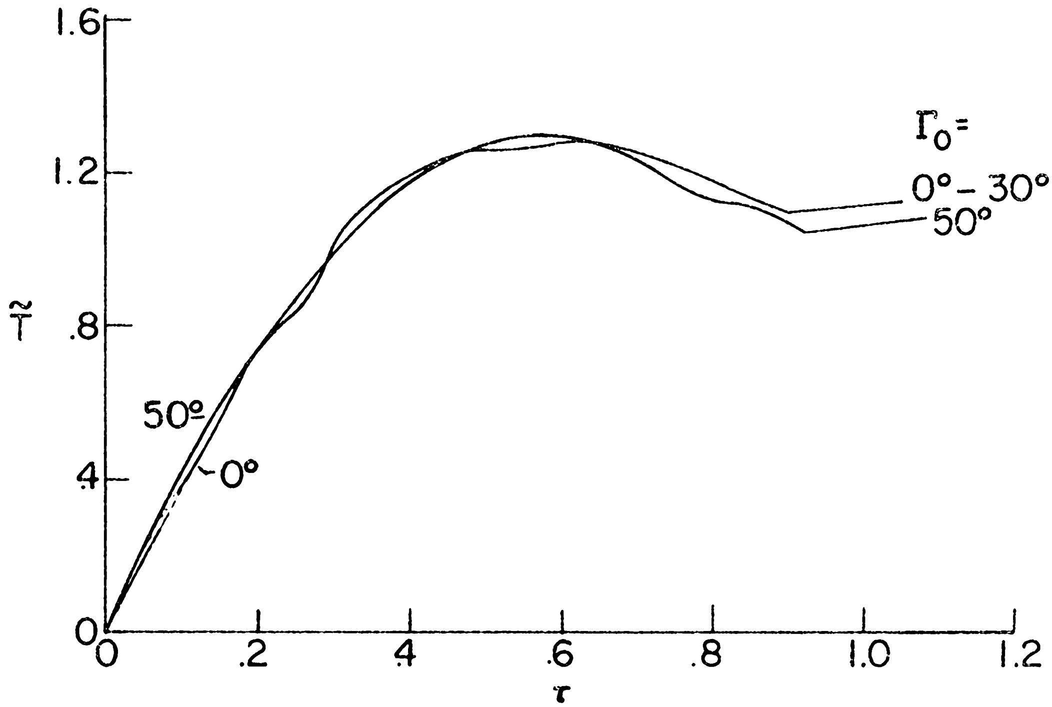
Figure 7.- Dimensionless time to despin, maximum tension during despin, and angle of body rotation plotted against inertia ratio, representing final conditions after despin to $\omega_3 = 0$, $\gamma = 90^{\circ}$ for two-dimensional motion.



(a) Cross-axis spin rate.

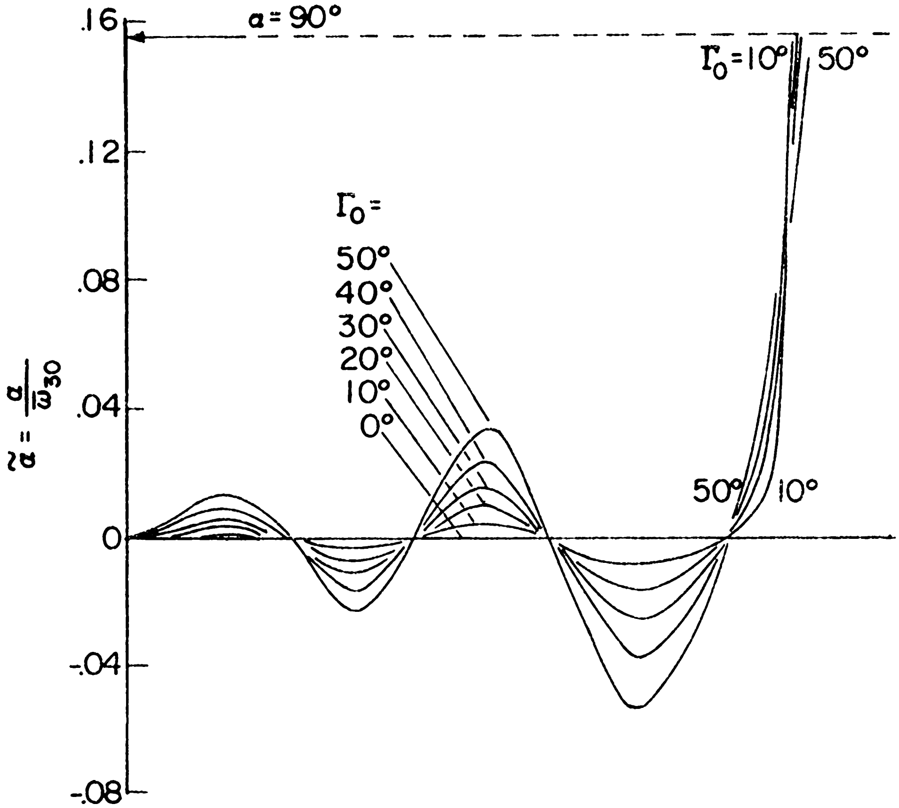
(b) Spin rate.

Figure 8.- Variation of significant quantities with time in nondimensional form for different values of the initial coning angle, and for $I = 0.1$ and $K = 0.001$.

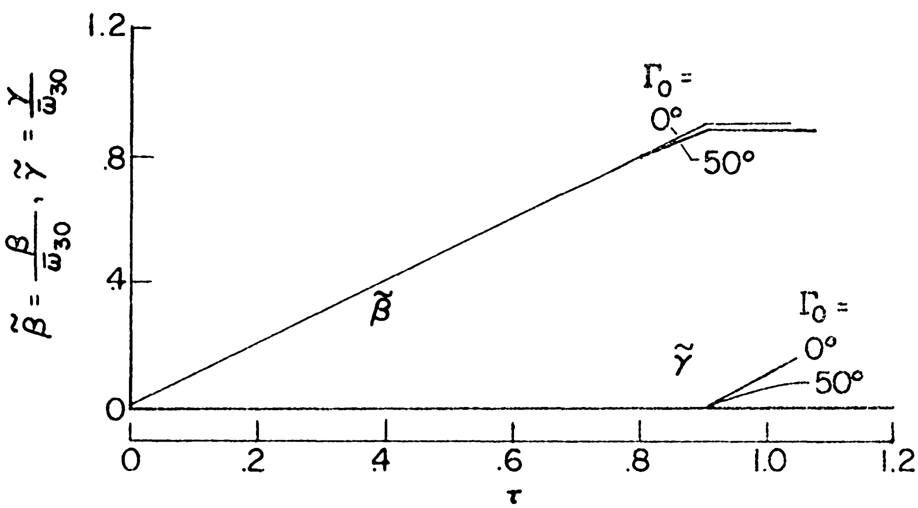


(c) Cable tension.

Figure 8.- Continued.

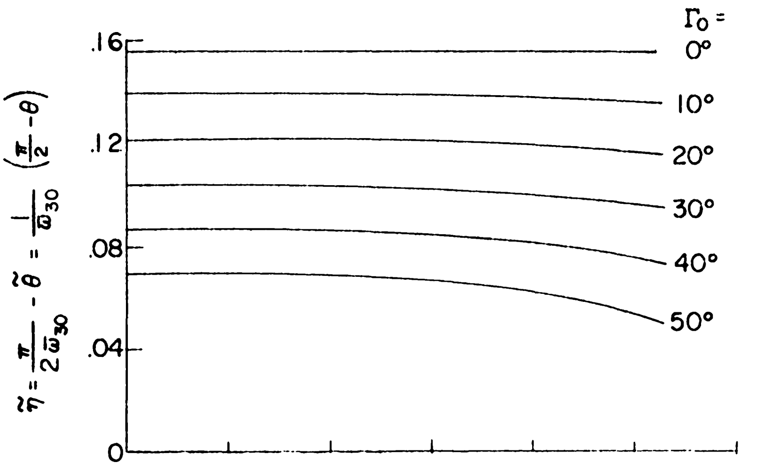


(d) Out of plane angle.

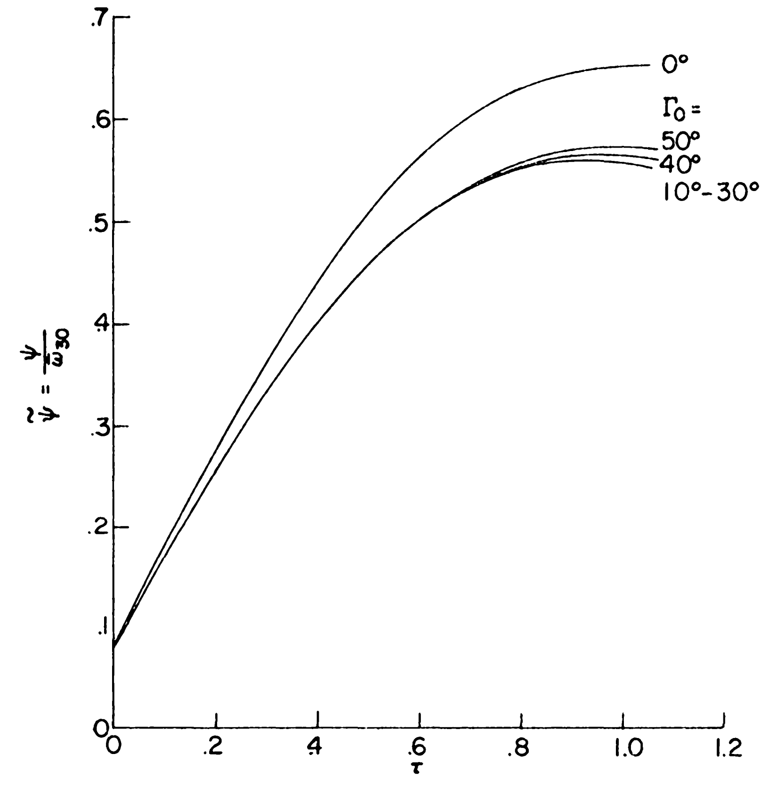


(e) Swingout and unwind angles.

Figure 8.- Concluded.

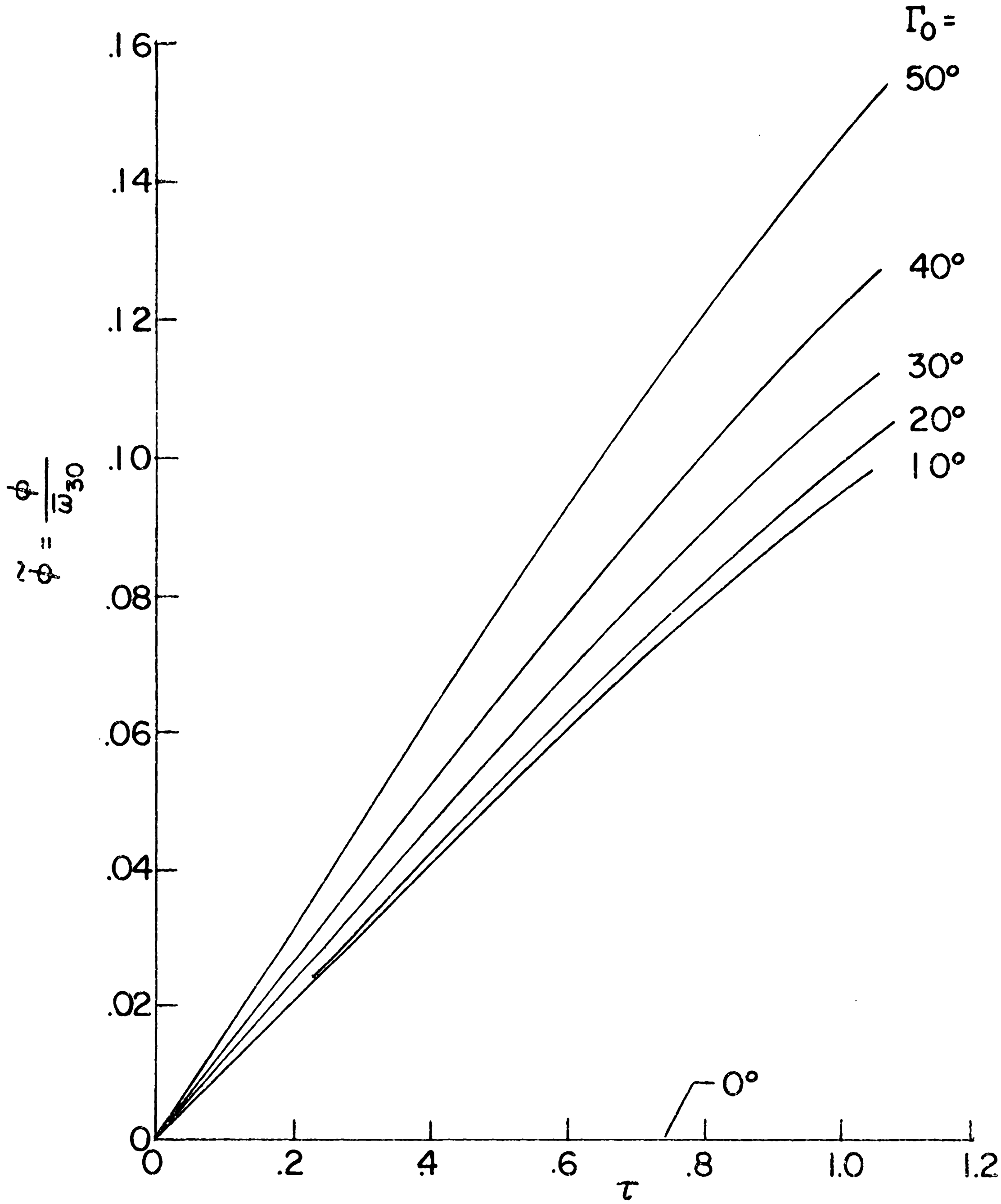


(a) Measure of the nutation angle.



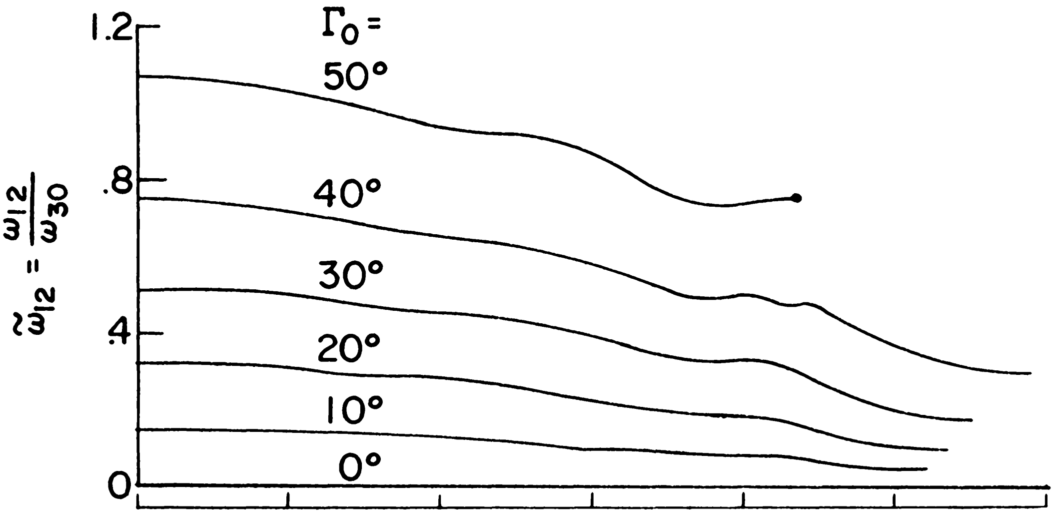
(b) Spin angle.

Figure 9.- Variation of Euler angles with time in nondimensional form for different values of the initial coning angle and for $I = 0.1$, $K = 0.001$.

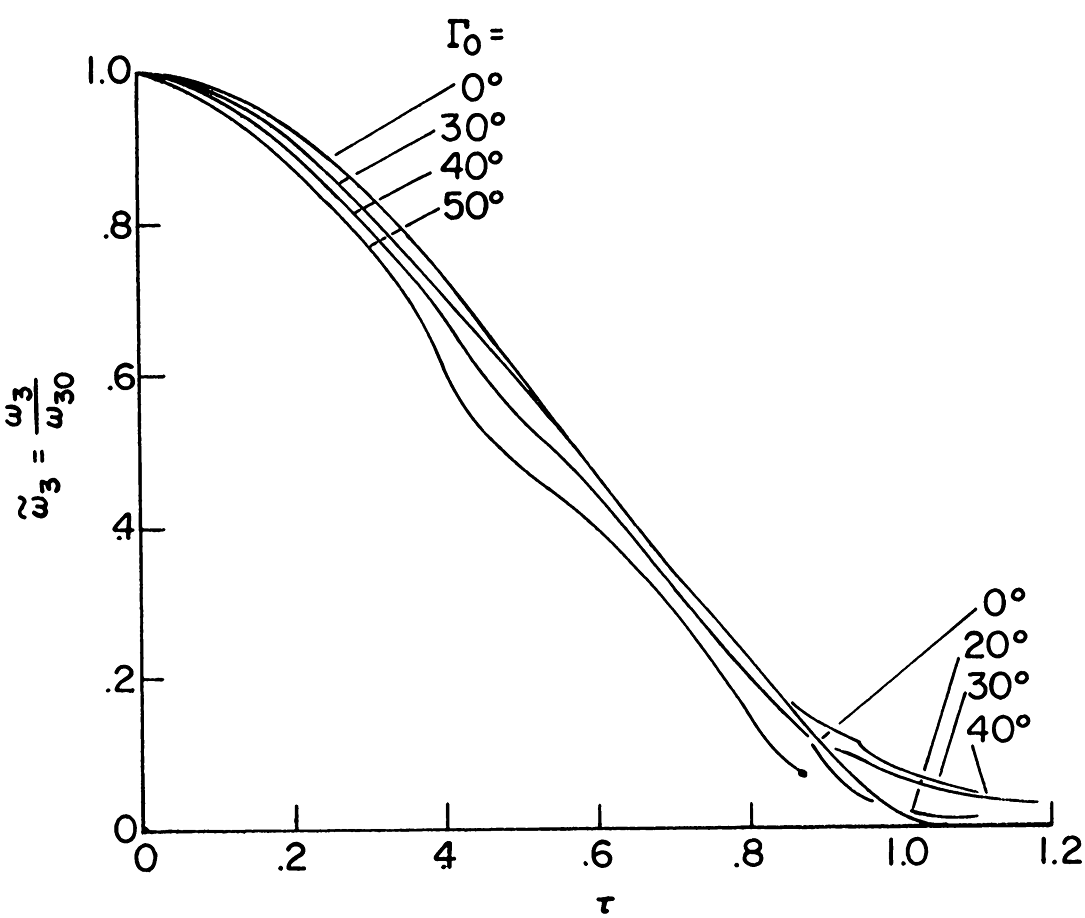


(c) Precession angle.

Figure 9.- Concluded.

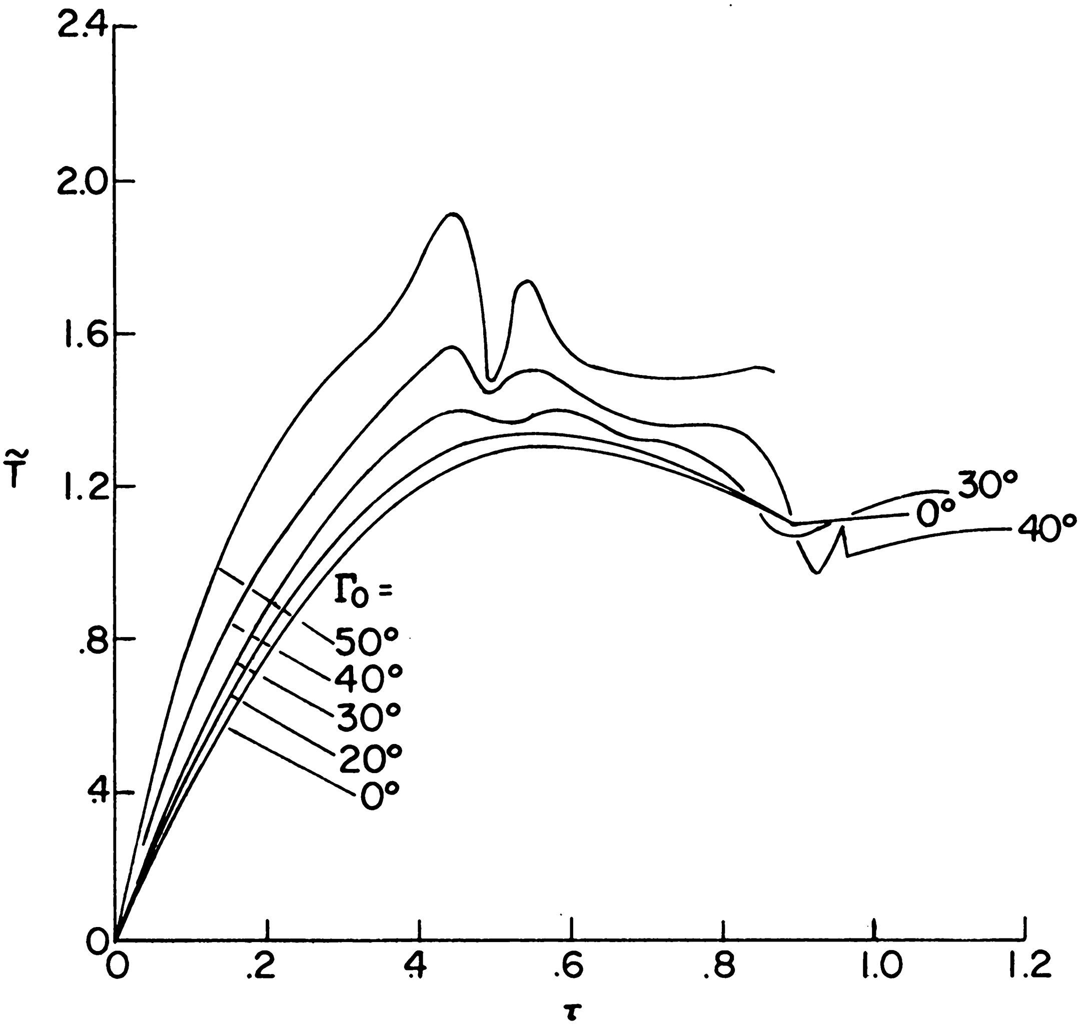


(a) Cross-axis spin rate.



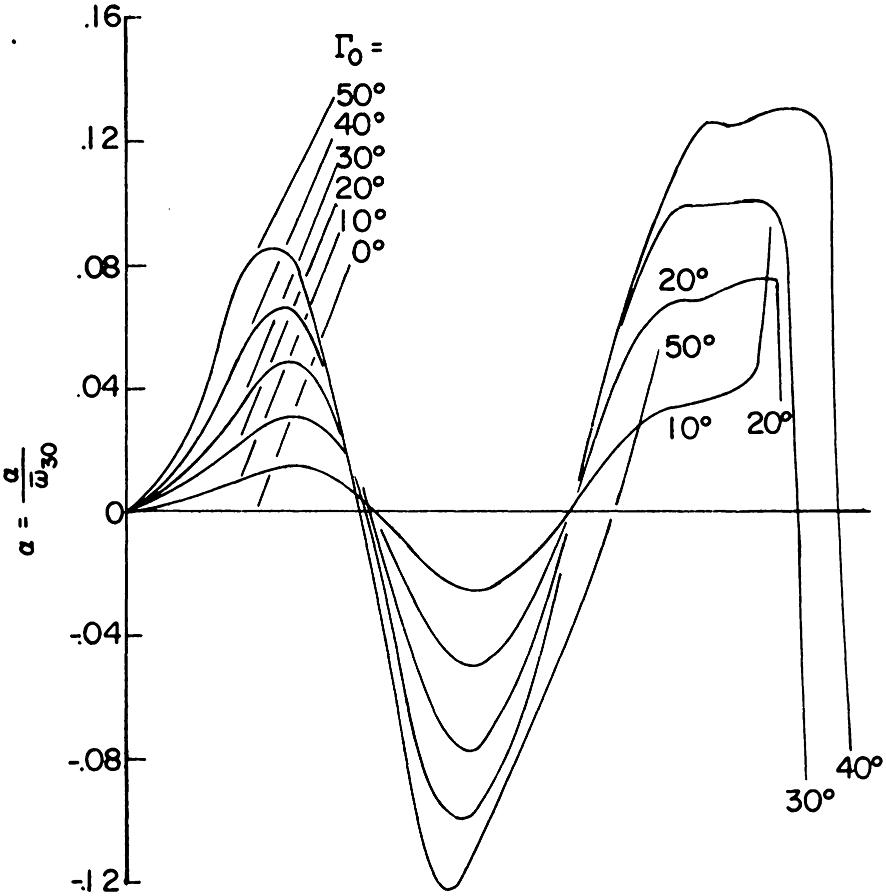
(b) Spin rate.

Figure 10.- Variation of significant quantities with time in nondimensional form for different values of the initial coning angle and for $I = 0.9$, $K = 0.01$.

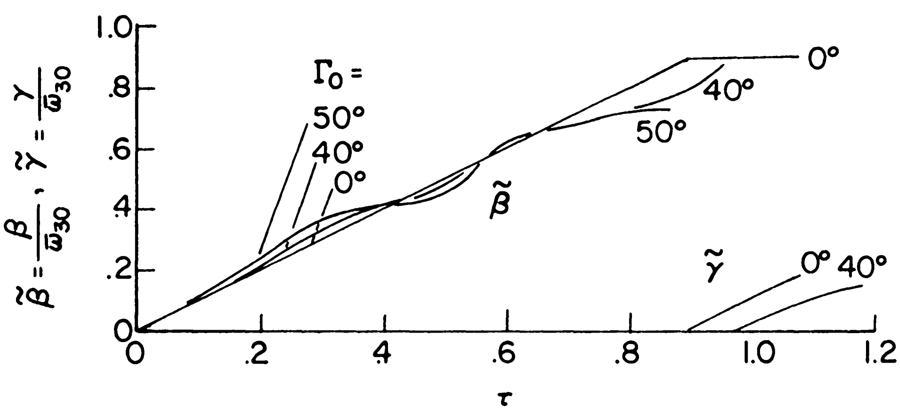


(c) Cable tension.

Figure 10.- Continued.

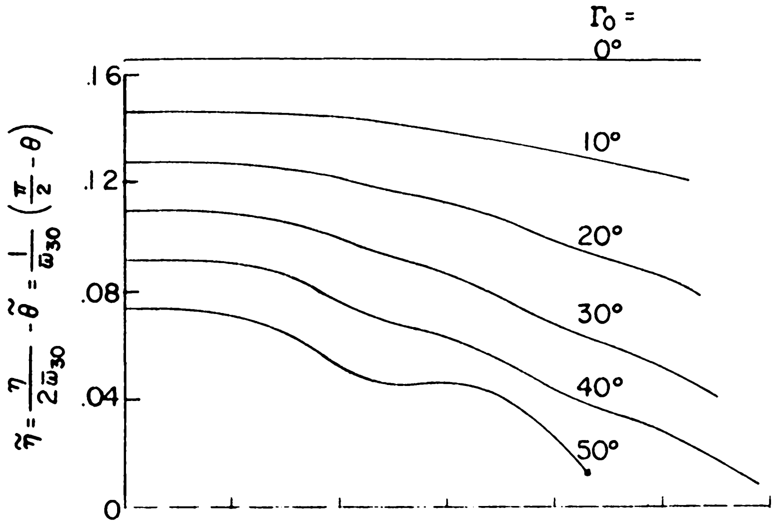


(d) Out of plane angle.

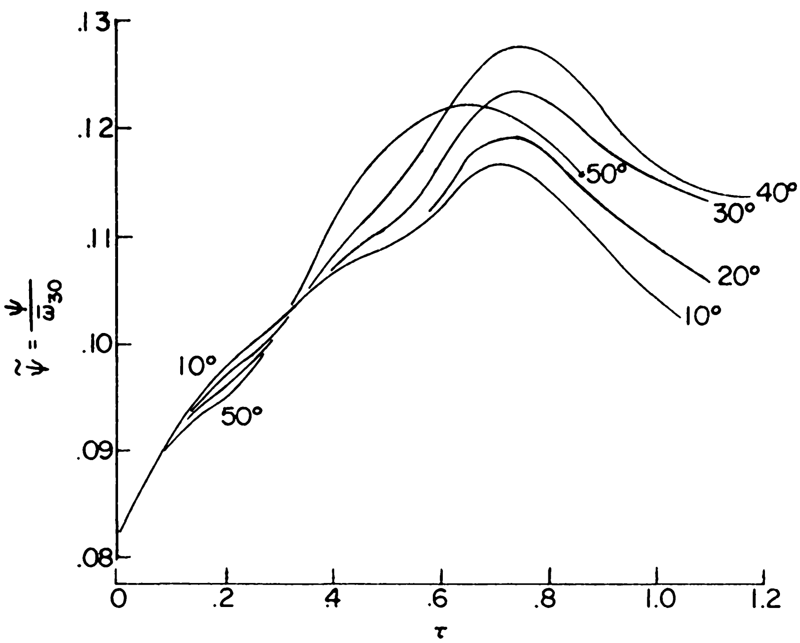


(e) Swingout and unwind angles.

Figure 10.- Concluded.

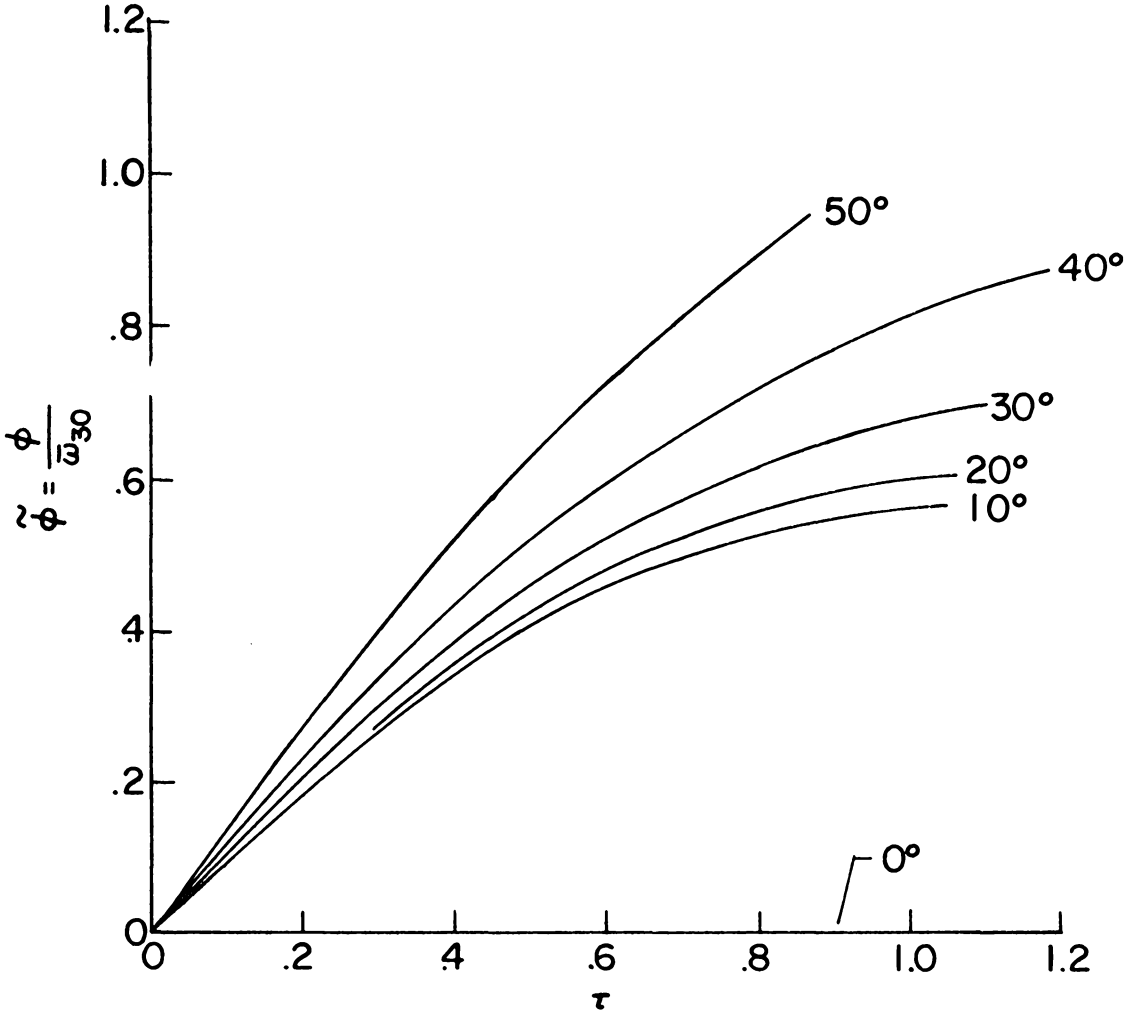


(a) Measure of the nutation angle.



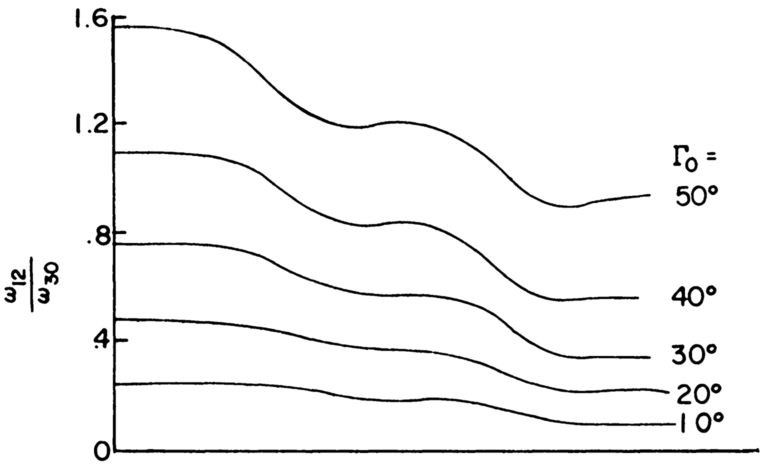
(b) Spin angle.

Figure 11.- Variation of Euler angles with time in nondimensional form for different values of the initial coning angle and for $I = 0.9$, $K = 0.01$.

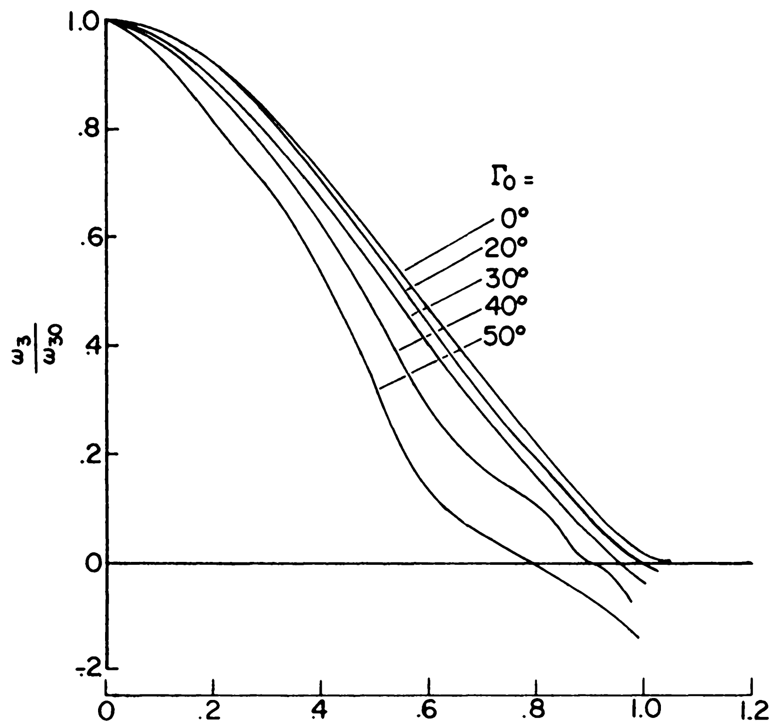


(c) Precession angle.

Figure 11.- Concluded.

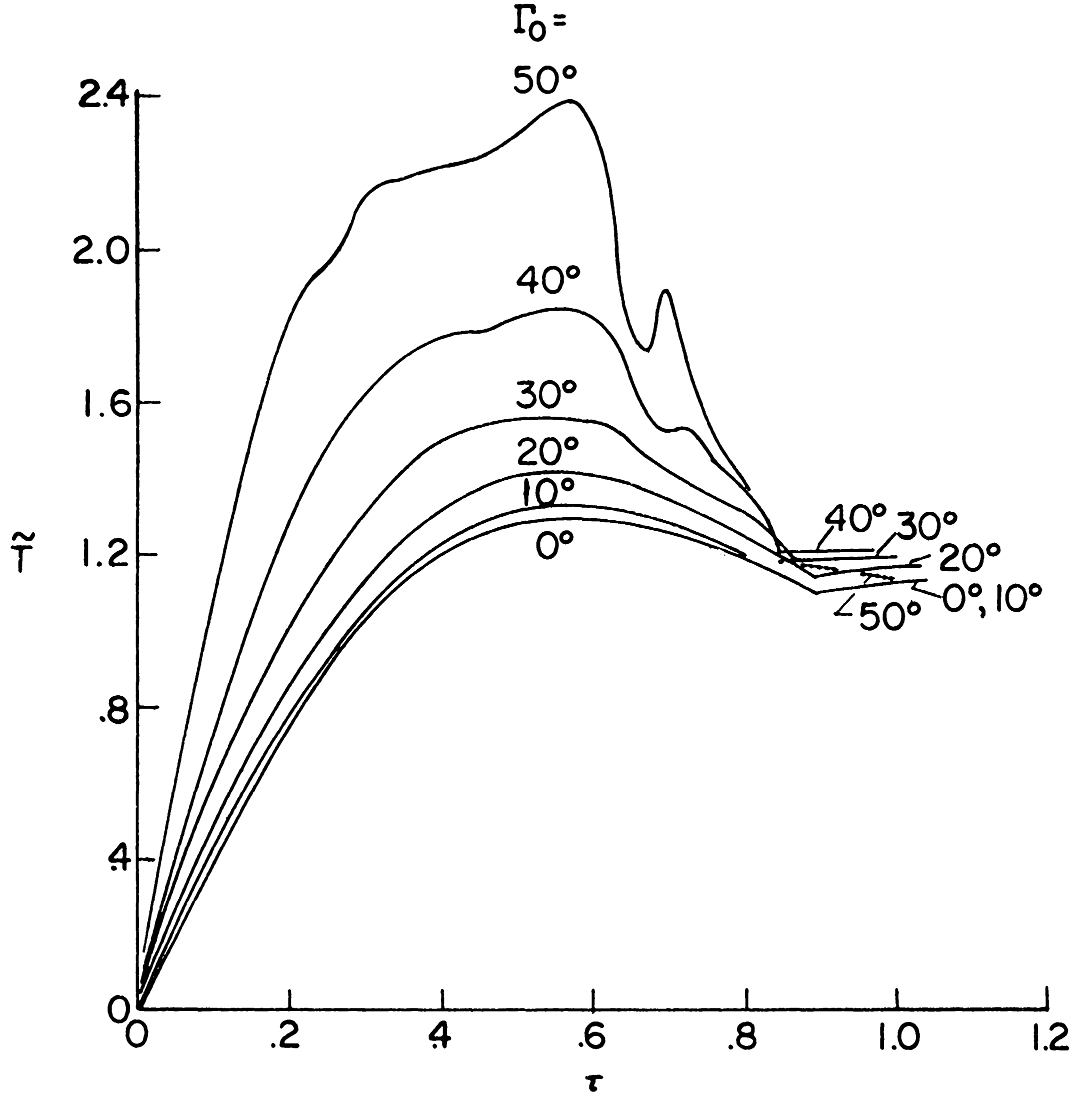


(a) Cross-axis spin rate.



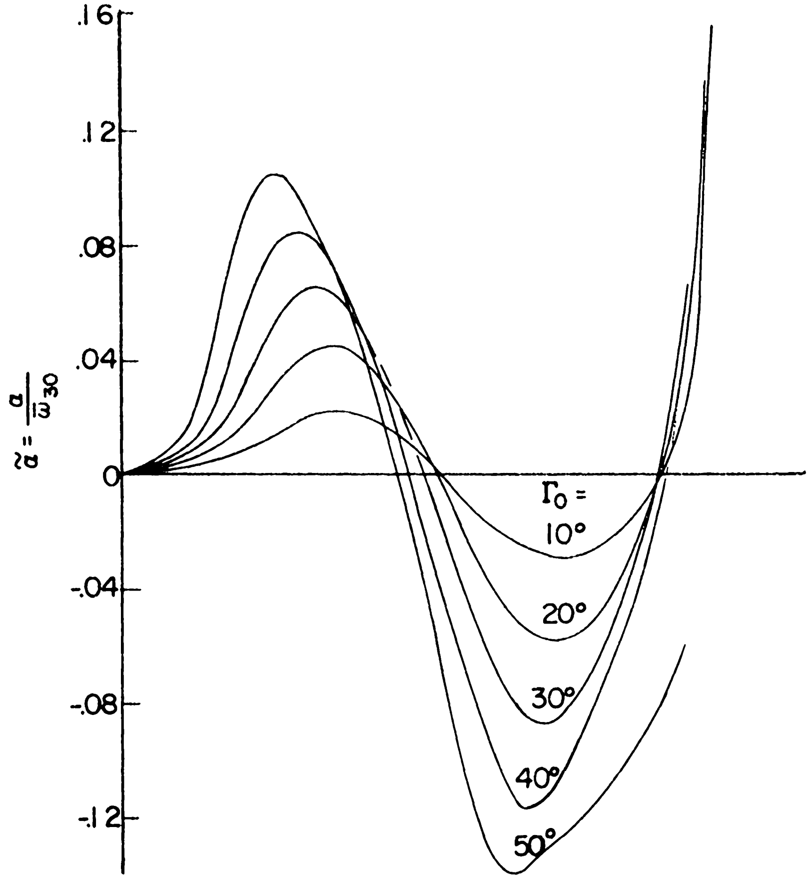
(b) Spin rate.

Figure 12.- Variation of significant quantities with time in nondimensional form for different values of the initial coning angle and for $I = 1.3$, $K = 0.015$.

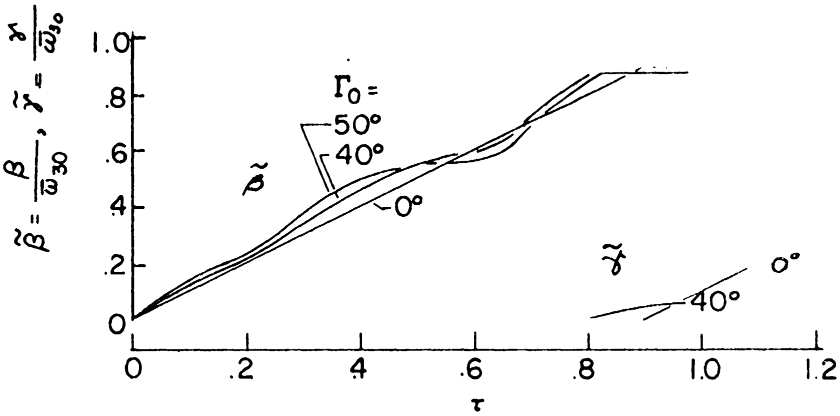


(c) Cable tension.

Figure 12.- Continued.

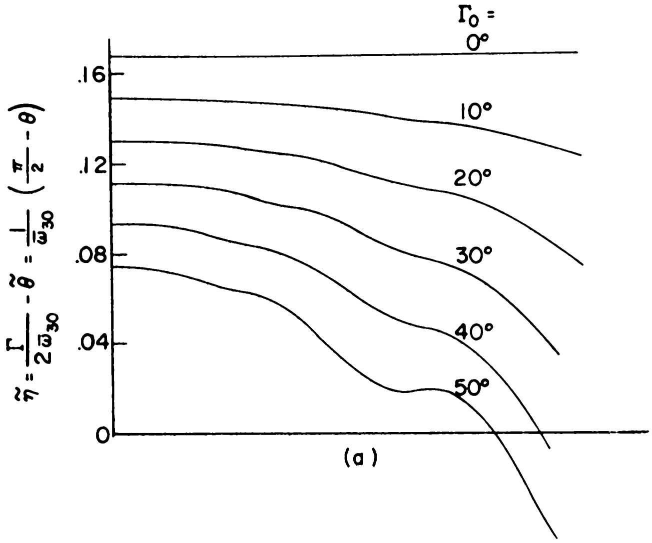


(d) Out of plane angle.

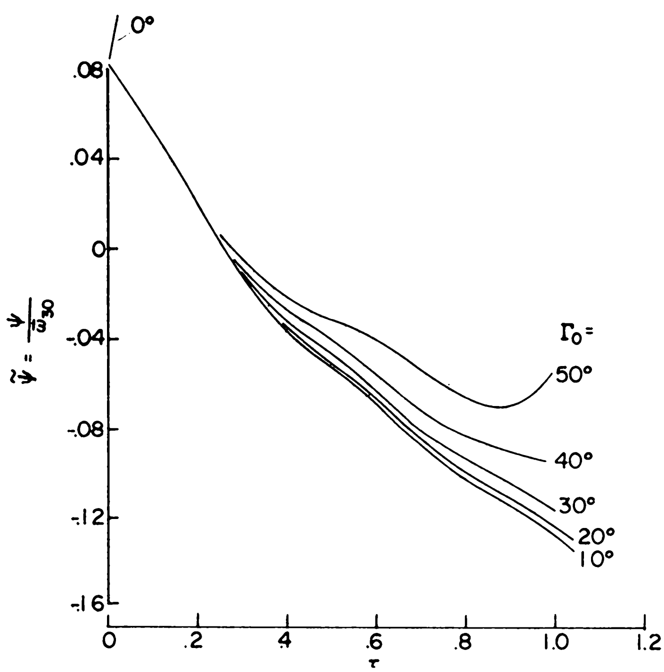


(e) Swingout and unwind angles.

Figure 12.- Concluded.

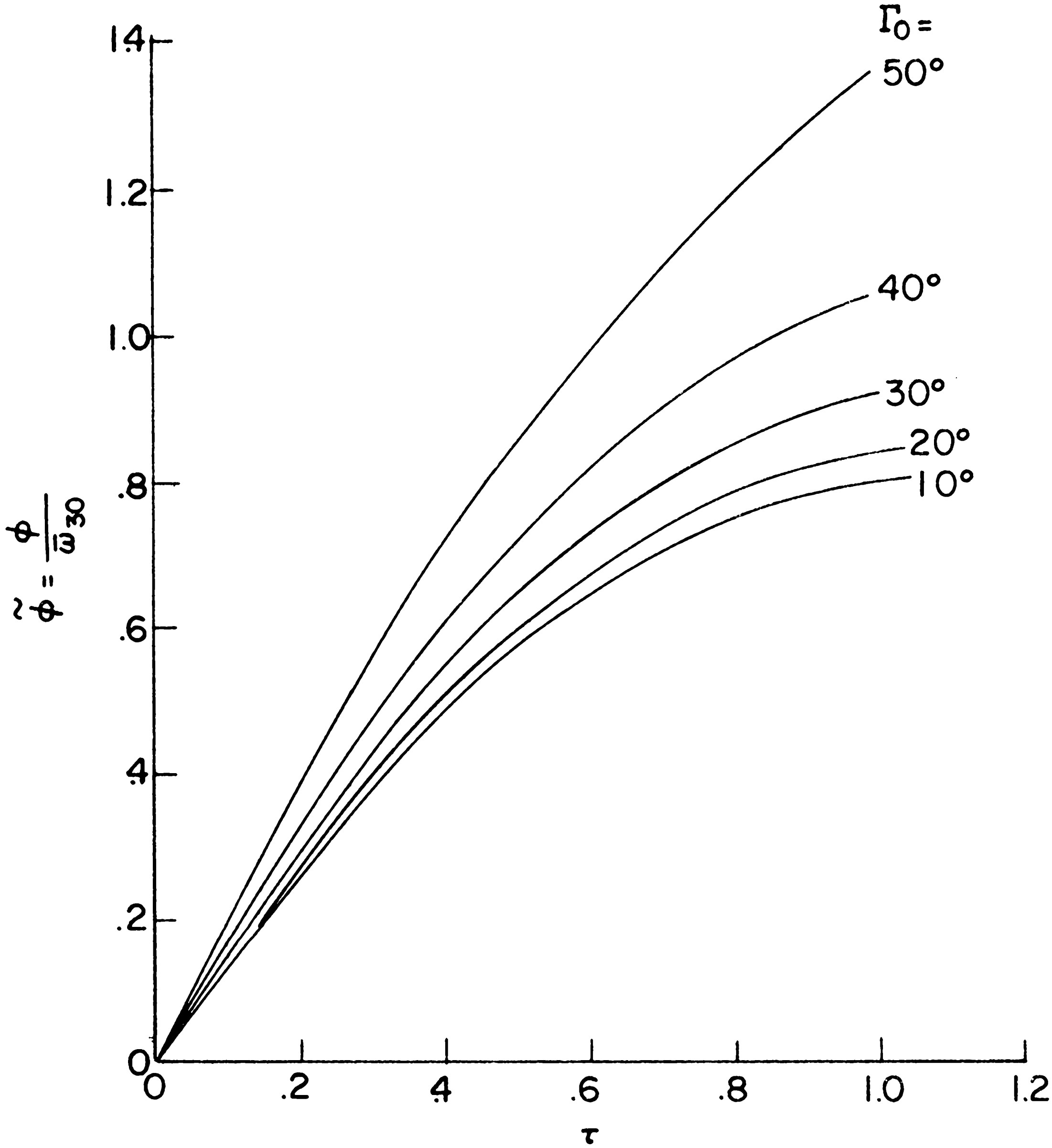


(a) Measure of the nutation angle.



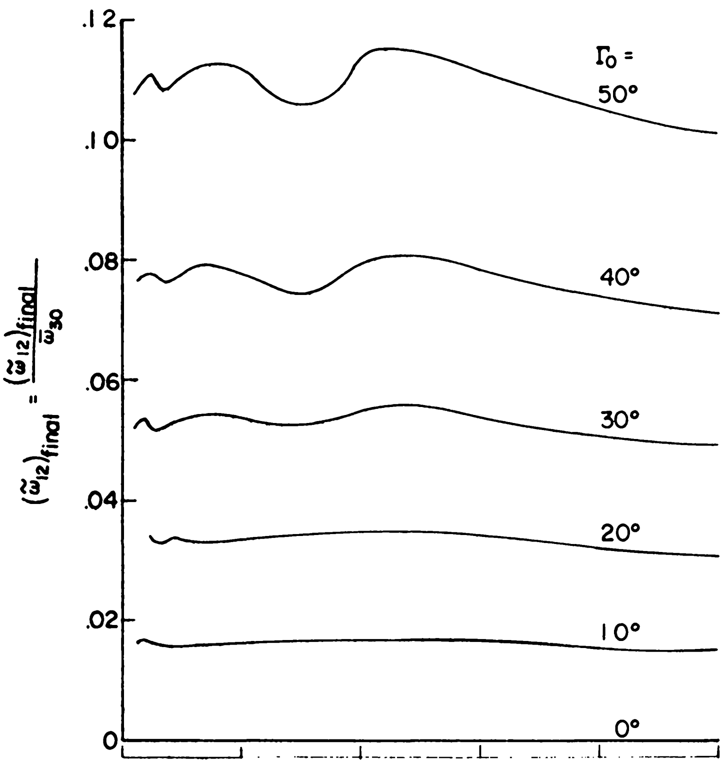
(b) Spin angle.

Figure 13.- Variation of Euler angles with time in nondimensional form for different values of the initial coning angle and for $I = 1.3$, $K = 0.015$.

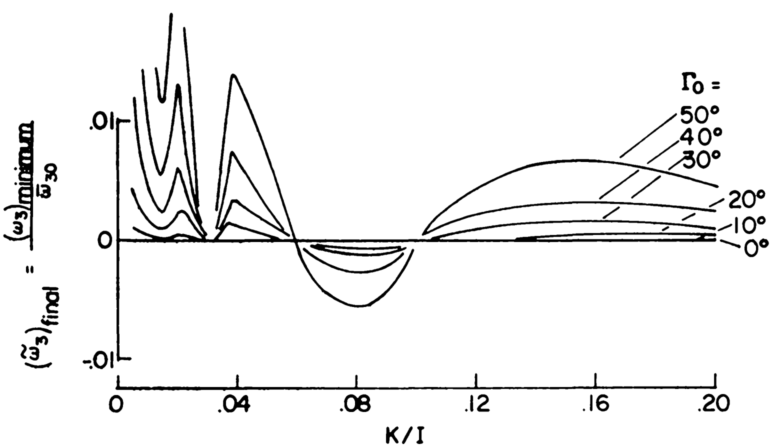


(c) Precession angle.

Figure 13.- Concluded.

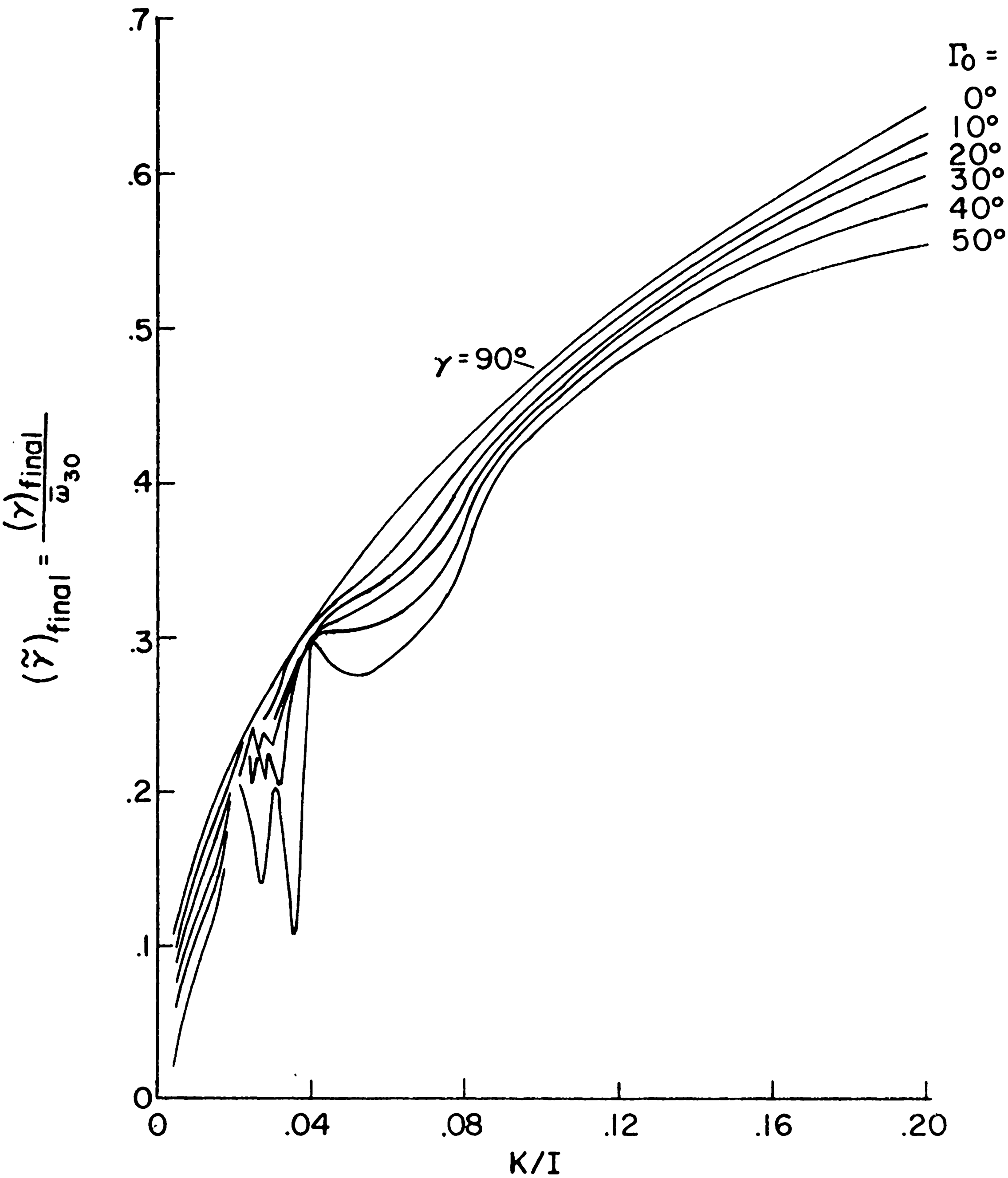


(a) Final cross-axis spin rate.



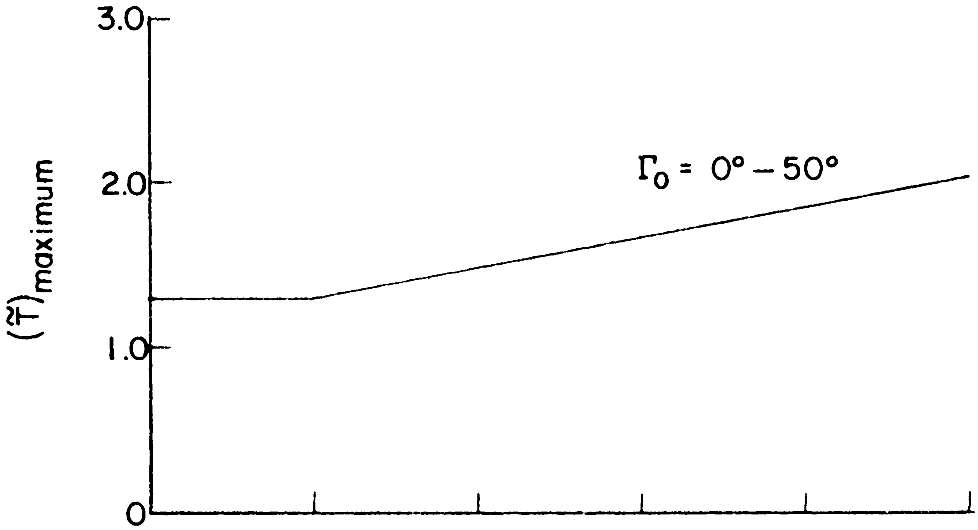
(b) Final spin rate.

Figure 14.- Variation of significant nondimensional design parameters with ratio of inertia factors for different values of the initial coning angle and for I = 0.1.

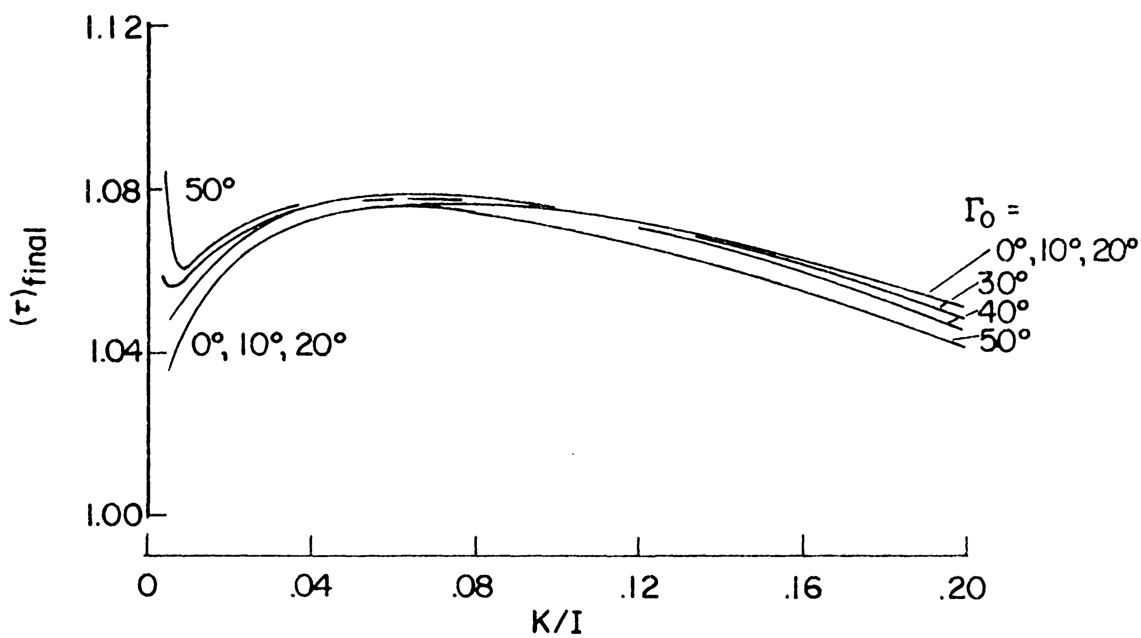


(c) Final swingout angle.

Figure 14.- Continued.

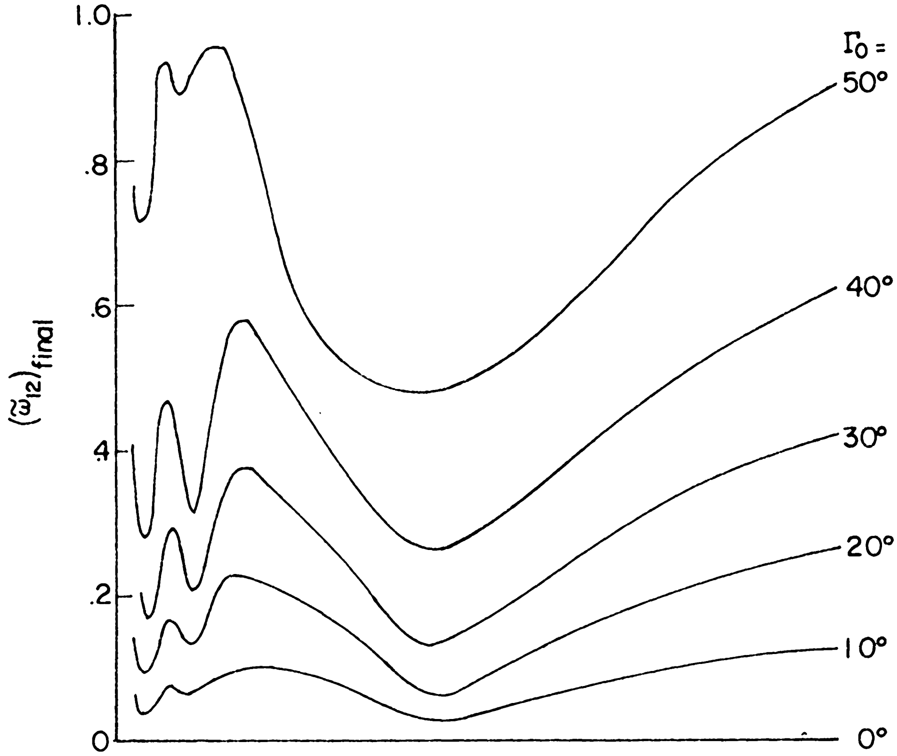


(d) Maximum cable tension.

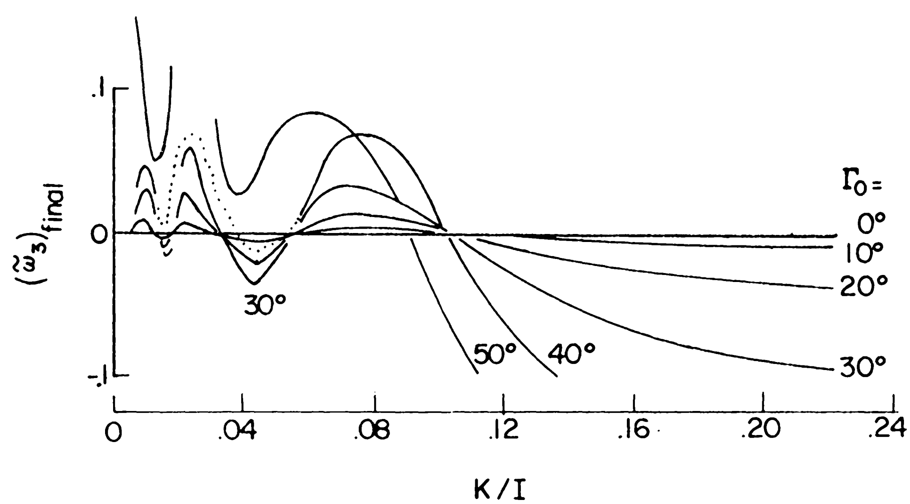


(e) Despin time.

Figure 14.- Concluded.

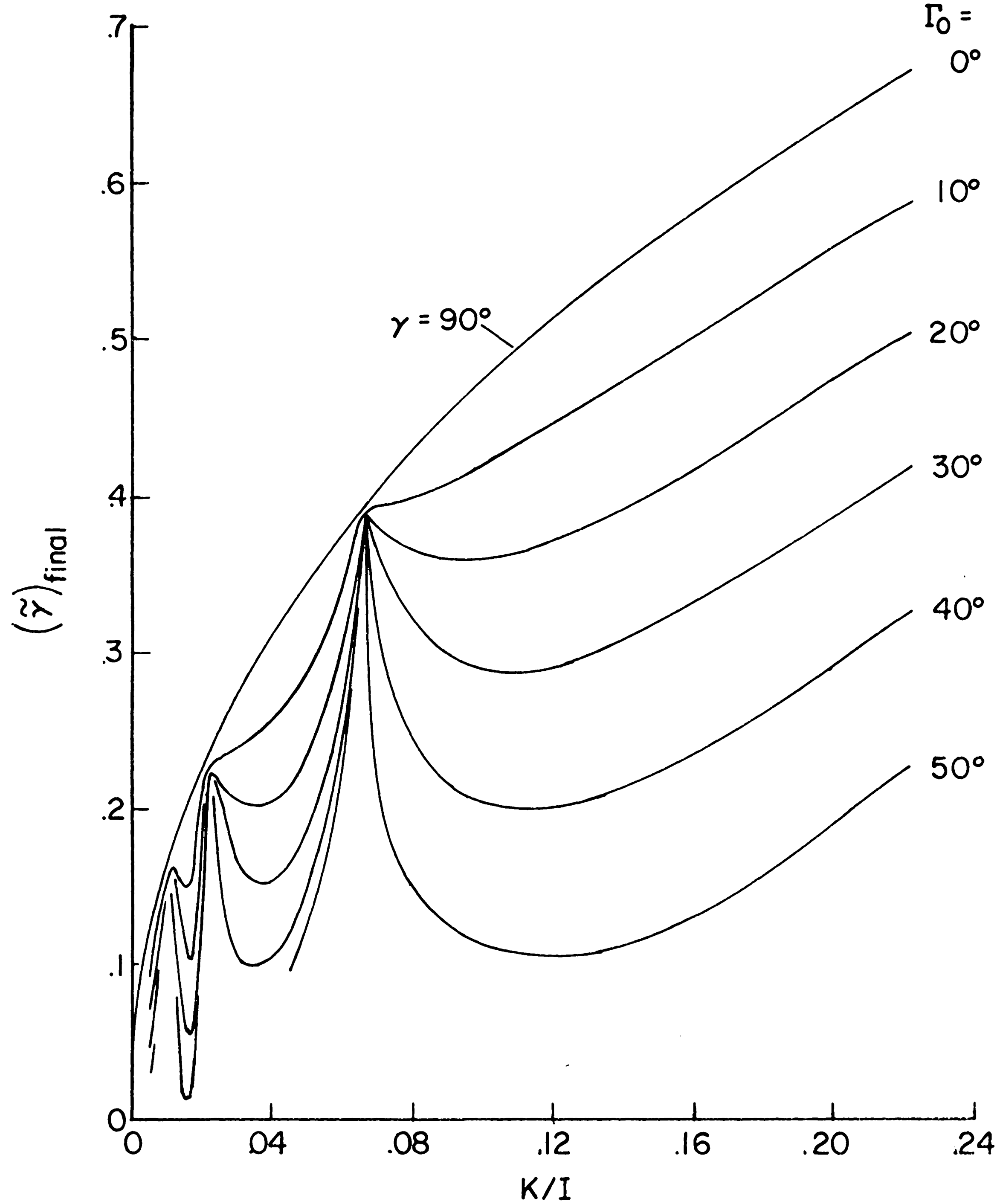


(a) Final cross-axis spin rate.



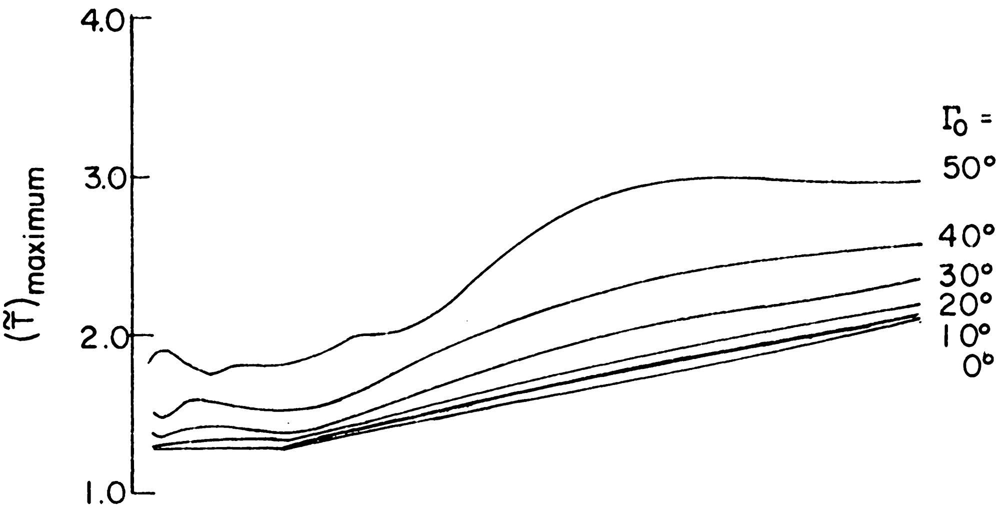
(b) Final spin rate.

Figure 15.- Variation of significant nondimensional design parameters with ratio of inertia factors for different values of the initial coning angle and for I = 0.9.

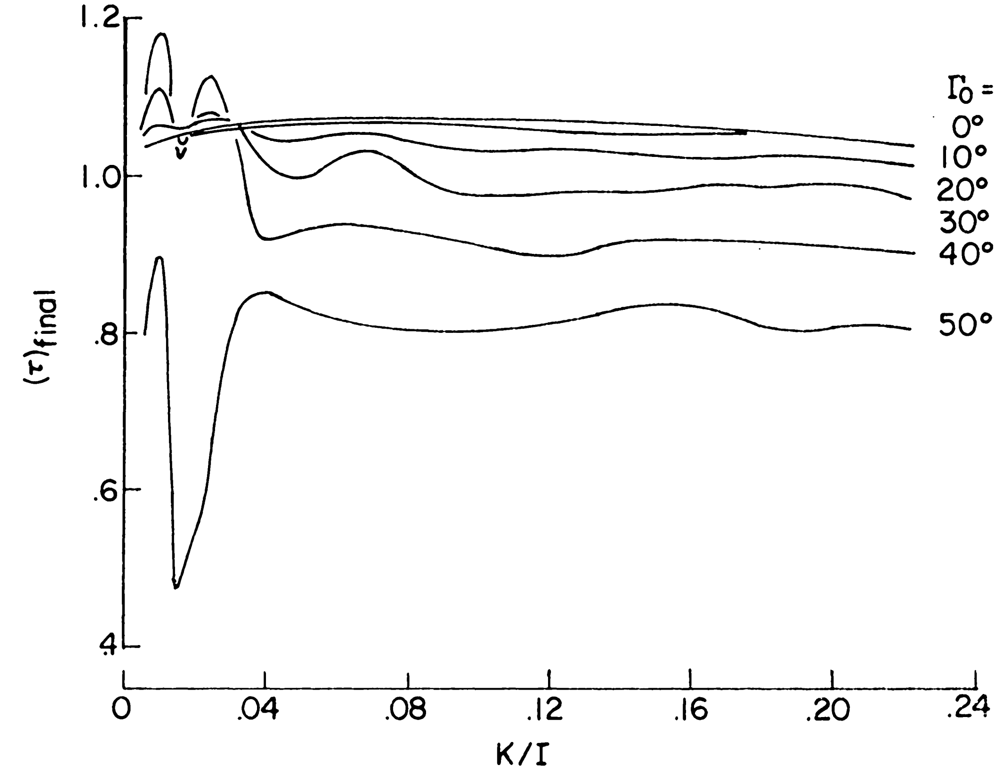


(c) Final swingout angle.

Figure 15.- Continued.

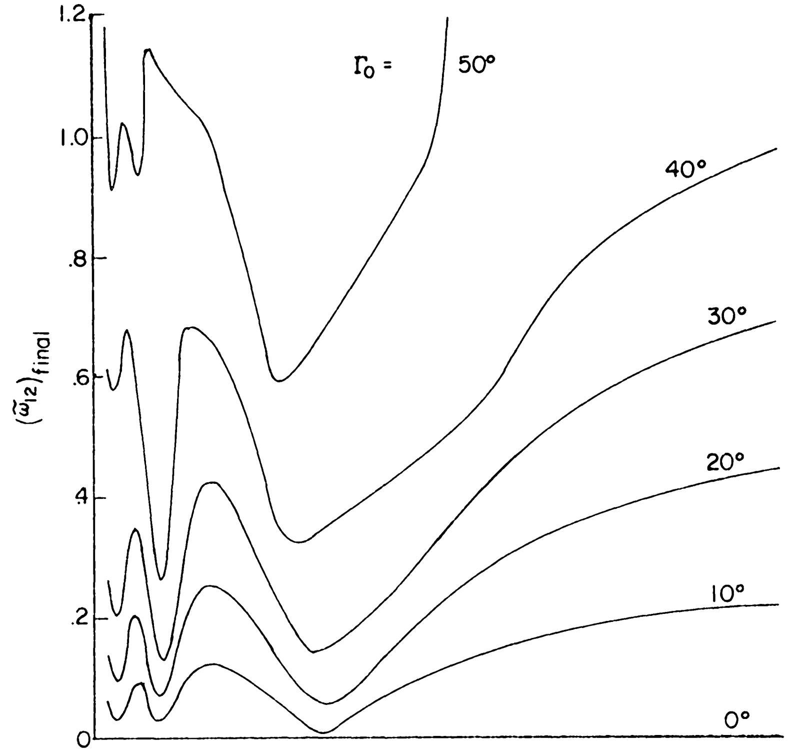


(d) Maximum cable tension.

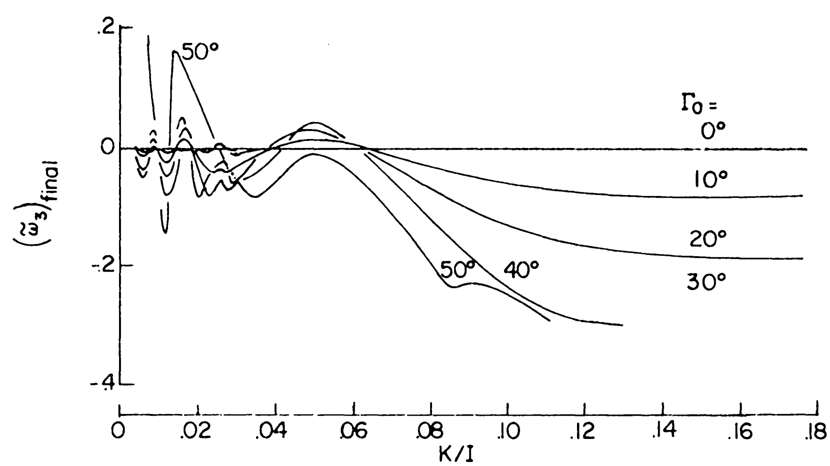


(e) Despin time.

Figure 15.- Concluded.

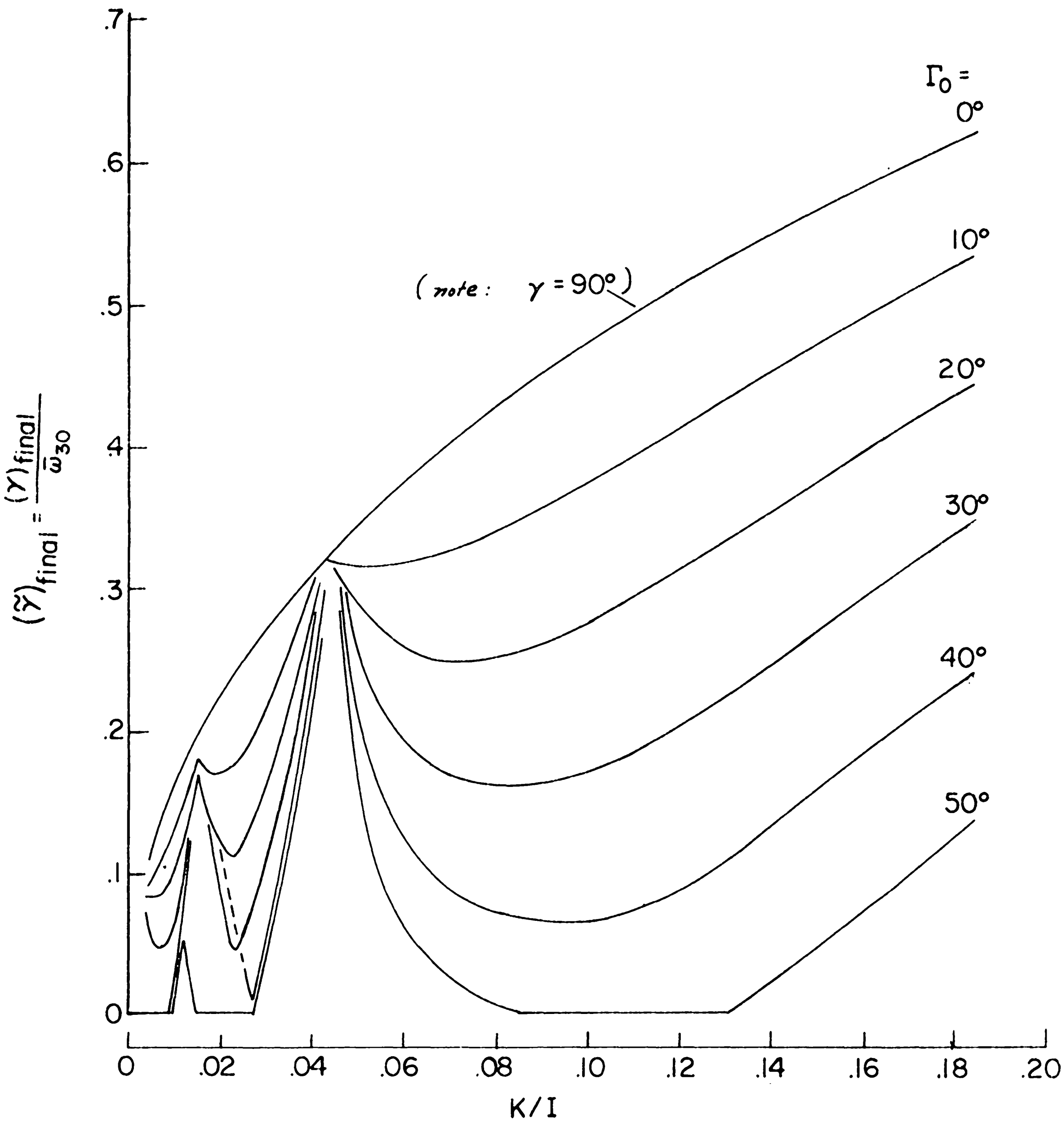


(a) Final cross-axis spin rate.



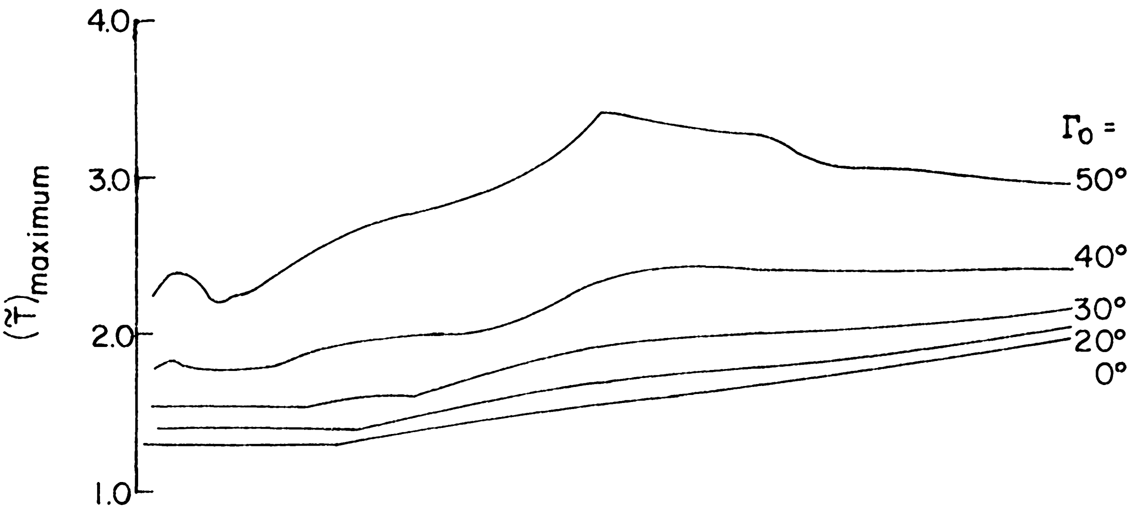
(b) Final spin rate.

Figure 16.- Variation of significant nondimensional design parameters with ratio of inertia factors for different values of the initial coning angle and for $I = 1.3$.

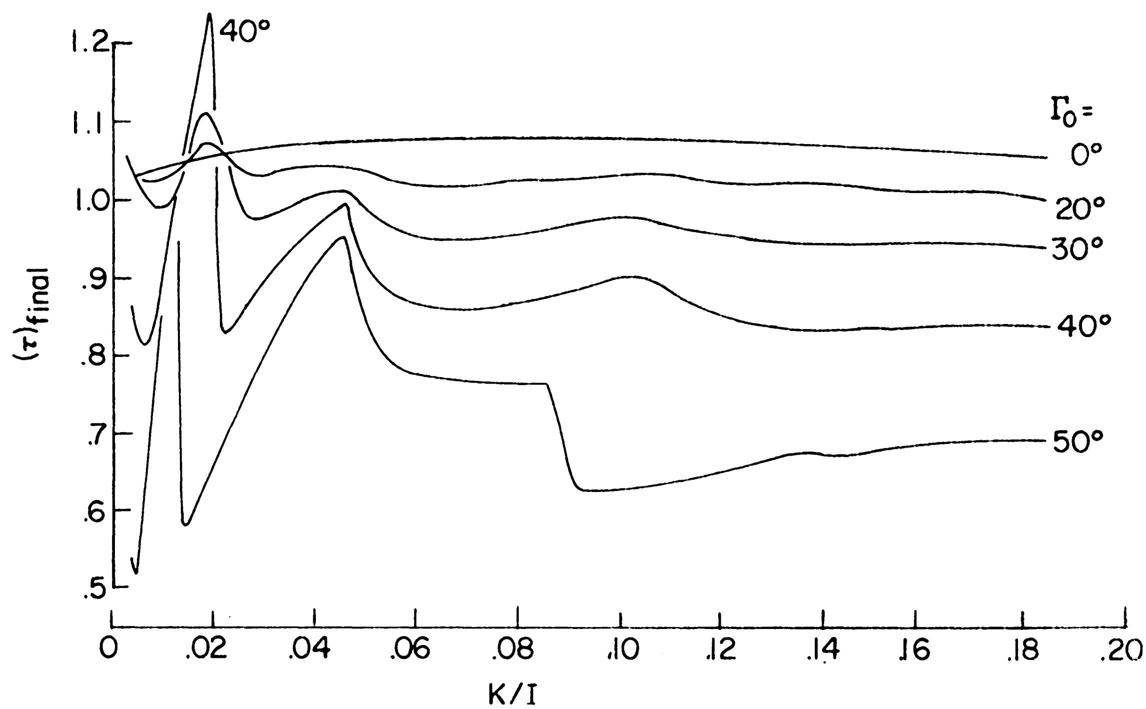


(c) Final swingout angle.

Figure 16.- Continued.



(d) Maximum cable tension.



(e) Despin time.

Figure 16.- Concluded.

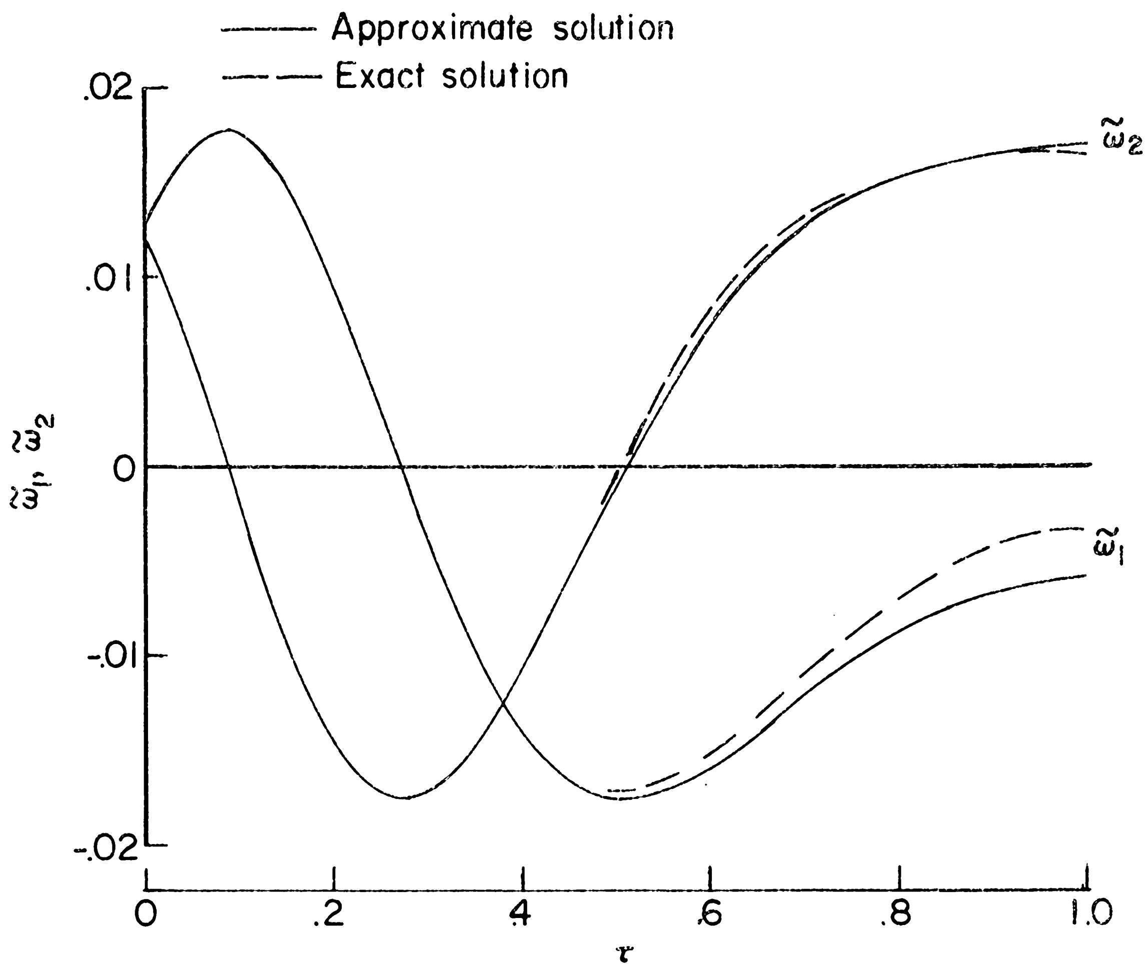


Figure 17.- Cross-axis rotational rates along x_1, x_2 body axes plotted against nondimensional time resulting from approximate equations with $I = 0.1$, $K = 0.001$, $\Gamma_o = 10^o$.

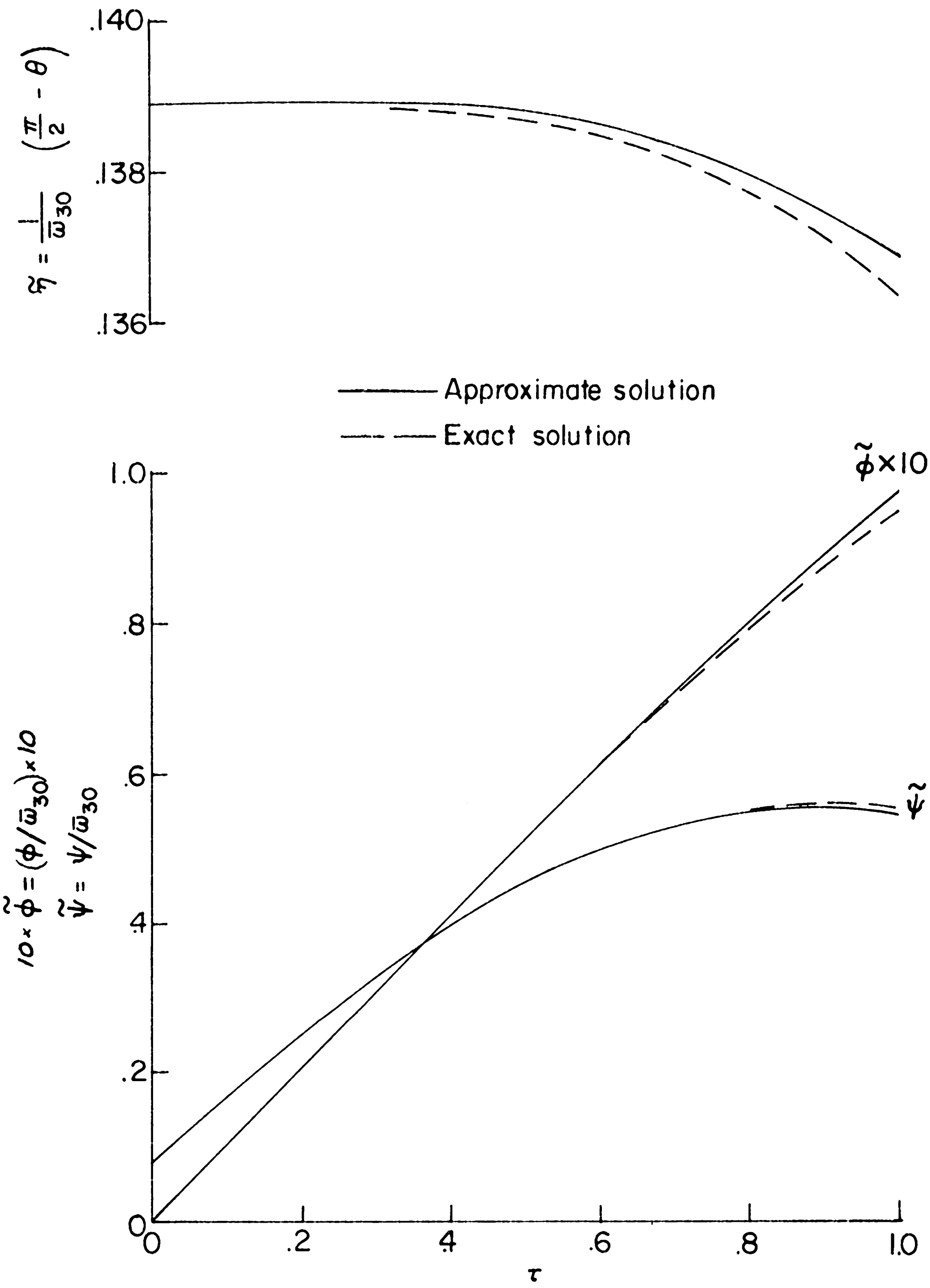


Figure 18.- Nondimensional Euler angles plotted against nondimensional time resulting from integration of approximate equations with $I = 0.1$, $K = 0.001$, $\Gamma_0 = 10^{\circ}$.

XIII. APPENDIX A

LISTING OF IBM 7094 (9) COMPUTER PROGRAM FOR EXACT EQUATIONS OF MOTION AND FOR DATA PRESENTED IN FIGURES 6 THROUGH 16

```
         YOYO                                                     01/26/66
      EXTERNAL FORMULA NUMBER   -   SOURCE STATEMENT    -   INTERNAL FORMULA NUMBER(S)

C       PROGRAM FOR SOLUTION OF EXACT EQUATIONS OF MOTION FOR YO YO
C       SATELLITE DESPINNING
C       R.L.COLLINS, SMD DEC,1965
        DIMENSIONX(12,5),AKK(12,4),C(4),CC(4),XF(12),Y(5,7,2),YIN(5,7),AM
       1(5,7),BAD(5,7)
      1 FORMAT(6X26HR.L.COLLINS SMD  DEC, 1965)
        RTD=57.2957795            ,1
      2 FORMAT(6X47HSOLUTION OF EXACT EQUATIONS FOR YO YO SATELLITE)
      3 FORMAT(1X1H )
      4 FORMAT(3X7E16.8)
      5 FORMAT(7E16.8)
      7 FORMAT(4E16.8)
    163 FORMAT(10X28HA-DETERMINANT IS NEAR ZERO =,E16.8)
    164 FORMAT(10X30HALPHA GREATER THAN 90 DEGREES=,E16.8)
    165 FORMAT(10X21HTRI GREATER THAN 100=,E16.8)
        WRITE(6,1)            ,2     ,3
        WRITE(6,2)            ,4     ,5
        WRITE(6,3)            ,6     ,7                         (.005)            (.025)          (1.5)
        READ(5,7)TINC,TPINC,TFINAL <----- ,8    ,9    ,10 --- Input: integration step size, printing increment, maximum integration value.
        WRITE(6,5)TINC,TPINC         ,11    ,12    ,13
    100 READ(5,7)AI,W2010R     <-- ,14  ,15  ,16 ---------- Input: I, W20/W10
        GAMOD=50.0         <-- ,17 ---------- 1st value of I0 to be considered
        READ(5,7)AKFST,AKLST,DELAK  <----- ,18   ,19   ,20 --- Input: K(1st value desired, last value, increment)
        AK=AKFST              ,21
    900 WRITE(6,3)            ,22    ,23
        WRITE(6,4)AI,AK,W2010R,GAMOD        ,24    ,25    ,26
        TESDET=0.0            ,27
        GLAG=0.0              ,28
        CAKE=0.0              ,29
        IF(W2010R)351,352,351         ,30
    351 EPS=W2010R/ABS(W2010R)        ,31
        GO TO 350             ,32
    352 EPS=1.0               ,33
    350 GAMO=GAMOD*3.14159/180.0     ,34
        TAUC=1.9              ,35
        CHECK2=0.0            ,36
        BETAC=SQRT((AI+AK)/AK)-1.0          ,37
        TAUO=0.0              ,38
        TAUP=TPINC            ,39
        TAUPIN=0.5*TINC           ,40
        OM30=SQRT((AK+AI)/AK)         ,41
        TAGAM=(SIN(GAMO)/COS(GAMO))/SQRT(1.0+W2010R**2)          ,42
```

```
      OM10=EPS*OM30*AI*TAGAM                    ,43
      OM20=W2010R*OM10                     ,44
      BETO=0.0                 ,45
      ALO=0.0                  ,46
      TO=OM10*OM20*AK*AI/(AK*AI)                    ,47
      BETOP=SQRT(OM20**2+OM30**2)                   ,48
      ALOP=OM10*OM30*(2.0-AI)/(2.0*BETOP)                ,49
      OM10P=OM20*OM30*(1.0-AI)                  ,50
      OM20P=-OM10*OM30*(1.0-AI)                     ,51
      OM30P=-TO/AI             ,52
      BETOPP=2.0*(OM20*OM20P+OM30*OM30P)/(3.0*BETOP)+4.0*ALOP*OM20/3.0
     1-TO/(3.0*AK)             ,53
      ALOPP=-ALOP*BETOPP/BETOP+(2.0-AI)*(OM10P*OM30+OM10*OM30P)/(3.0*
     1BETOP)-2.0*BETOP*OM20/3.0-(1.0+1.0/AK)*TO*ALOP/(3.0*BETOP)          ,54
      HO=SQRT(OM10**2+OM20**2+(AI*OM30)**2)                   ,55
      IF(GAMO)301,302,301                       ,56
  301 PHIO=0.0                 ,57
      THETO=GAMO               ,58
      SPSIO=OM10/(HO*SIN(GAMO))                     ,59
      CPSIO=OM20/(HO*SIN(GAMO))                     ,60
      PSIO=ATAN2(SPSIO,CPSIO)                   ,61
      PHIOP=(SPSIO*OM10+CPSIO*OM20)/SIN(THETO)                ,62
      THETOP=OM10*CPSIO-OM20*SPSIO                  ,63
      PSIOP=OM30-PHIOP*COS(THETO)                   ,64
      GO TO 303                ,65
  302 PSIO=0.0                 ,66
      PHIO=0.0                 ,67
      THETO=0.0                ,68
      PHIOP=0.0                ,69
      THETOP=OM10              ,70
      PSIOP=OM30               ,71
  303 CONTINUE                 ,72
      X(1,1)=OM10              ,73
      X(2,1)=OM20              ,74
      X(3,1)=OM30              ,75
      X(4,1)=BETO              ,76
      X(5,1)=ALO               ,77      initial values for integration
      X(6,1)=BETOP             ,78
      X(7,1)=ALOP              ,79
      X(8,1)=PHIO              ,80
      X(9,1)=THETO             ,81
      X(10,1)=PSIO             ,82
      X(11,1)=0.0              ,83
```

```
      X(12,1)=0.0         ,84
      I=0           ,85
 304  DO9I=1,12           ,86
   9  XF(I)=X(I,1)        ,87     ,88
      T=0.0         ,89
      AKK(1,1)=OM10P              ,90
      AKK(2,1)=OM20P              ,91
      AKK(3,1)=OM30P              ,92
      AKK(4,1)=BETOP              ,93
      AKK(5,1)=ALOP               ,94
      AKK(6,1)=BETOPP             ,95
      AKK(7,1)=ALOPP              ,96
      AKK(8,1)=PHIOP              ,97
      AKK(9,1)=THETOP             ,98
      AKK(10,1)=PSIOP             ,99
      AKK(11,1)=0.0               ,100
      AKK(12,1)=0.0               ,101
      OM120=SQRT(OM10**2+OM20**2)             ,102
      DELO=ATAN(OM120/OM30)             ,103
      DELOD=DELO*RTD              ,104
      OM10R=OM10/OM30             ,105
      OM20R=OM20/OM30             ,106
      OM30R=1.0           ,107
      OM120R=OM120/OM30           ,108
      H30R=AI             ,109
      TOR=TO/(AI*OM30)            ,110
      TESTR=TOR           ,111
      ALOD=ALO/OM30               ,112
      BETOD=BETO/OM30             ,113
      TESTB=BETOD         ,114
      TESTG=0.0           ,115
      TESTOM=1.0          ,116
      REAL M10R,M20R,M30R
      M10R=0.0            ,117
      M20R=0.0            ,118
      M30R=-TOR           ,119
      CALL TRANS1(0.0,0.0,1.0,XF(8),XF(9),XF(10),AXSYM1,AXSYM2,AXSYM3,AL
     1MBK,ETAK,AXK)               ,120
      CALL TRANS1(OM10R,OM20R,H30R,XF(8),XF(9),XF(10),H1RS,H2RS,H3RS,ALM
     1BH,ETAH,HR)         ,121
      CALL TRANS1(OM10R,OM20R,1.0,XF(8),XF(9),XF(10),OM1RS,OM2RS,OM3RS,A
     1LMBOM,ETAOM,OMR)            ,122
      CALL TRANS1(0.0,0.0,TOR,XF(8),XF(9),XF(10),S1RS,S2RS,S3RS,ALMBS,ET
```

initial values for integration (required due to singularity in equations of motion)

$\tilde{\tau}_0 = T_0 / I \bar{\omega}_{3_0}$

```
 1AS,SR)        ,123
  ALMBKD=ALMBK/OM30          ,124
  ETAKD=ETAK/OM30            ,125
  PHIOD=PHIO/OM30            ,126
  PSIOD=PSIO/OM30            ,127
  ALMBHD=ALMBH/OM30          ,128
  ETAHD=ETAH/OM30            ,129
  ALMBSD=ALMBS/OM30          ,130
  ETASD=ETAS/OM30            ,131
  ETAOMD=ETAOM/OM30          ,132
  ALMOMD=ALMBOM/OM30         ,133
  OM3RTD=OM30*RTD            ,134
  WRITE(6,3)          ,135   ,136
  WRITE(6,5)TAJD,TOR,M1OR,M2OR,M3OR,OM30R,OM12OR         ,137   ,138   ,139
  WRITE(6,5)AXSYM1,AXSYM2,AXSYM3,ALMBKD,ETAKD,PHIOD,PSIOD       ,140   ,141   ,142
  WRITE(6,5)H1RS,H2RS,H3RS,ALMBHD,ETAHD,GAMOD,HR         ,143   ,144   ,145
  WRITE(6,5)OM1RS,OM2RS,OM3RS,ALMOMD,ETAOMD,DELOD,OMR           ,146   ,147   ,148
  WRITE(6,5)S1RS,S2RS,S3RS,ALMBSD,ETASD,ALOD,BETOD       ,149   ,150   ,151
  WRITE(6,5)OM30,OM3RTD                ,152   ,153   ,154
  AAAA=0.0            ,155
  J=0         ,156
  B=0.0       ,157
  BBB=0.0             ,158
  TINCB=TINC          ,159
  CX11=1.0            ,160
  SX11=0.0            ,161
  RO1G=0.0            ,162
  RO2G=0.0            ,163
  RO3G=0.0            ,164
  RO1GP=0.0           ,165
  RO2GP=0.0           ,166
  RO3GP=0.0           ,167
10 DO1C1J=1,4         ,168
  CHECK=0.0           ,169
  IF(AAAA-0.5)30,20,20         ,170
20 BETARG=X(4,J)/6.2831853     ,171
  BETX=X(4,J)-AINT(BETARG)*6.2831853          ,172
  CX4=COS(BETX)              ,173
  SX4=SIN(BETX)              ,174
  ALPARG=X(5,J)/6.2831853      ,175
  ALPX=X(5,J)-AINT(ALPARG)*6.2831853          ,176
  CX5=COS(ALPX)              ,177
  SX5=SIN(ALPX)              ,178
```

print out initial quantities.

begin Runge-Kutta integration step

```
      IF(BBB-0.5)11,12,12                                ,179
   12 AGAM=X(11,J)/6.2831853                             ,180
      AGAMX=X(11,J)-AINT(AGAM)*6.2831853                        ,181
      CX11=COS(AGAMX)                          ,182
      SX11=SIN(AGAMX)                          ,183
      RO1G=X(4,J)*(CX4*CX11-SX4*CX5*SX11)                       ,184
      RO2G=X(4,J)*(SX4*CX11+CX4*CX5*SX11)                       ,185
      RO3G=X(4,J)*SX5*SX11                               ,186
      RO1GP=((CX4-X(4,J)*SX4)*CX11-(SX4*CX5+X(4,J)*CX4*CX5)*SX11)
     1*X(6,J)-X(4,J)*(CX4*SX11+SX4*CX11*CX5)*X(12,J)+X(4,J)*SX4*SX5
     1*SX11*X(7,J)                      ,187
      RO2GP=((SX4+X(4,J)*CX4)*CX11+(CX4*CX5-X(4,J)*SX4*CX5)*SX11)*
     1X(6,J)+X(4,J)*(CX4*CX5*CX11-SX4*SX11)*X(12,J)-X(4,J)*CX4*SX5
     1*SX11*X(7,J)                      ,188
      RO3GP=SX5*SX11*X(6,J)+X(4,J)*CX5*SX11*X(7,J)+X(4,J)*SX5*CX11*X(12,
     1J)                       ,189
   11 RO1=CX4+X(4,J)*SX4*CX5*CX11+X(4,J)*CX4*SX11                       ,190
      RO2=SX4-X(4,J)*CX4*CX5*CX11+X(4,J)*SX4*SX11                       ,191
      RO3=-X(4,J)*SX5*CX11                               ,192
      RO1B=SX4*CX5*CX11-RO2+CX4*SX11                ,193
      RO2B=RO1-CX4*CX5*CX11+SX4*SX11                ,194
      RO3B=-SX5*CX11                           ,195
      RO1A=RO3*SX4                      ,196
      RO2A=-RO3*CX4                            ,197
      RO3A=-X(4,J)*CX5*CX11                              ,198
      RO1P=RO1B*X(6,J)+RO1A*X(7,J)+RO1G*X(12,J)                   ,199
      RO2P=RO2B*X(6,J)+RO2A*X(7,J)+RO2G*X(12,J)                   ,200
      RO3P=RO3B*X(6,J)+RO3A*X(7,J)+RO3G*X(12,J)                   ,201
      RO1BP=(CX4*CX5*CX11-RO2B-SX4*SX11)*X(6,J)-(SX4*SX5*CX11+RO2A)*X(7,
     1J)
     1-(SX4*CX5*SX11+RO2G-CX4*CX11)*X(12,J)                       ,202
      RO2BP=(RO1B+SX4*CX5*CX11+CX4*SX11)*X(6,J)+(CX4*SX5*CX11+RO1A)*
     1X(7,J)+(CX4*CX5*SX11+RO1G+SX4*CX11)*X(12,J)                       ,203
      RO3BP=-CX5*CX11*X(7,J)+SX5*SX11*X(12,J)                     ,204
      RO1AP=(RO3B*SX4+RO3*CX4)*X(6,J)+(RO3A*SX4)*X(7,J)+RO3G*SX4*
     1X(12,J)                           ,205
      RO2AP=(RO3*SX4-RO3B*CX4)*X(6,J)-RO3A*CX4*X(7,J)-RO3G*CX4*X(12,J)            ,206
      RO3AP=-CX5*CX11*X(6,J)+X(4,J)*SX5*CX11*X(7,J)+X(4,J)*CX5*SX11
     1*X(12,J)                          ,207
      V1=RO1P+RO3*X(2,J)-RO2*X(3,J)                 ,208
      V2=RO2P+RO1*X(3,J)-RO3*X(1,J)                 ,209
      V3=RO3P+RO2*X(1,J)-RO1*X(2,J)                 ,210
      XXX1=V3*X(2,J)-V2*X(3,J)                           ,211
```

```
      XXX2=V1*X(3,J)-V3*X(1,J)        ,212
      XXX3=V2*X(1,J)-V1*X(2,J)        ,213
      REAL L1,L2,L3
      L1=RO1BP*X(6,J)+RO1AP*X(7,J)+RO3P*X(2,J)-RO2P*X(3,J)+XXX1         ,214
      L2=RO2BP*X(6,J)+RO2AP*X(7,J)+RO1P*X(3,J)-RO3P*X(1,J)+XXX2         ,215
      L3=RO3BP*X(6,J)+RO3AP*X(7,J)+RO2P*X(1,J)-RO1P*X(2,J)+XXX3         ,216
      A11=RO1B*(1.0-B)+RO1G*B         ,217
      A12=RO1A          ,218
      A13=(RO3*CX4*SX5+(RO2*CX5/AI)+SX4*CX5/AK)*CX11+CX4*SX11
     1/AK        ,219
      A21=RO2B*(1.0-B)+RO2G*B         ,220
      A22=RO2A          ,221
      A23=(RO3*SX4*SX5-(RO1*CX5/AI)-CX4*CX5/AK)*CX11+SX4*SX11/AK        ,222
      A31=RO3B*(1.0-B)+RO3G*B         ,223
      A32=RO3A          ,224
      A33=-(RO2*SX4+RO1*CX4+1.0/AK)*SX5*CX11          ,225
      ADET=A11*A22*A33+A12*A23*A31+A13*A21*A32-A31*A22*A13-A32*A23*A11-A
     133*A21*A12        ,226
      ADETA=ABS(ADET)         ,227
      ADETMX=AMAX1(ADETA,TESDET)          ,228
      TESDET=ADETMX           ,229
      IF(ADETA-ADETMX+.00000010)160,161,161           ,230
  160 GLOP=ADETA/ADETMX       ,231
      IF(GLOP-.00001000)162,162,161       ,232
  162 WRITE(6,3)        ,233  ,234
      WRITE(6,163)ADET        ,235  ,236  ,237
      WRITE(6,3)        ,238  ,239
      GLAG=1.0          ,240
      AKEY=3.0          ,241
      AAAA=1.0          ,242
      K=2         ,243
      GOTO150           ,244
  161 CONTINUE          ,245
      A11I=(A22*A33-A32*A23)/ADET         ,246
      A12I=(A32*A13-A12*A33)/ADET         ,247
      A13I=(A12*A23-A22*A13)/ADET         ,248
      A21I=(A31*A23-A21*A33)/ADET         ,249
      A22I=(A11*A33-A31*A13)/ADET         ,250
      A23I=(A21*A13-A11*A23)/ADET         ,251
      A31I=(A21*A32-A31*A22)/ADET         ,252
      A32I=(A31*A12-A11*A32)/ADET         ,253
      A33I=(A11*A22-A21*A12)/ADET         ,254
      B1=RO3*(1.0-AI)*X(1,J)*X(3,J)-L1-RO1GP*X(12,J)         ,255
```

```
        EXTERNAL FORMULA NUMBER  -  SOURCE STATEMENT  -  INTERNAL FORMULA NUMBER(S)

      B2=RO3*(1.0-AI)*X(2,J)*X(3,J)-L2-RO2GP*X(12,J)          ,256
      B3=(AI-1.0)*(RO2*X(2,J)+RO1*X(1,J))*X(3,J)-L3-RO3GP*X(12,J)        ,257
      T=A31I*B1+A32I*B2+A33I*B3          ,258
   19 AKK(1,J)=(1.0-AI)*X(2,J)*X(3,J)-SX4*SX5*T*CX11          ,259
      AKK(2,J)=(AI-1.0)*X(1,J)*X(3,J)+CX4*SX5*T*CX11          ,260
      AKK(3,J)=-CX5*T*CX11/AI          ,261
      AKK(4,J)=X(6,J)*(1.0-B)          ,262
      AKK(5,J)=X(7,J)          ,263
      AKK(6,J)=(A11I*B1+A12I*B2+A13I*B3)*(1.0-B)          ,264
      AKK(7,J)=A21I*B1+A22I*B2+A23I*B3          ,265
      IF(GAMO)13,14,13          ,266
   14 AKK(8,J)=0.0          ,267
      AKK(9,J)=0.0          ,268
      GOTO15          ,269
   13 AKK(8,J)=(X(1,J)*SIN(X(10,J))+X(2,J)*COS(X(10,J)))/SIN(X(9,J))          ,270
      AKK(9,J)=X(1,J)*COS(X(10,J))-X(2,J)*SIN(X(10,J))          ,271
   15 AKK(10,J)=X(3,J)-AKK(8,J)*COS(X(9,J))          ,272
      AKK(11,J)=X(12,J)*B          ,273
      AKK(12,J)=(A11I*B1+A12I*B2+A13I*B3)*B          ,274
      IF(CHECK)16,30,16          ,275
   30 AJ=J          ,276
      C(J)=(1.0+(AJ-1.0)*(AJ-2.0)*0.5)*TINC*0.5          ,277
      CC(J)=(AJ-(AJ-1.0)*(AJ-2.0)/2.0)*TINC/6.0          ,278
      K=J+1          ,279
      I=0          ,280
      DO90I=1,12          ,281
      X(I,K)=X(I,1)+C(J)*AKK(I,J)          ,282
   90 XF(I)=XF(I)+CC(J)*AKK(I,J)          ,283  ,284
      IF(J-4)16,17,16          ,285
   17 DO18I=1,12          ,286
   18 X(I,4)=XF(I)          ,287  ,288
      CHECK=1.0          ,289
      GO TO 20          ,290
   16 AAAA=1.0          ,291
      IF((XF(3)/OM30)-0.2)410,410,411          ,292
  410 CAKE=CAKE+1.0          ,293
      IF(CAKE-1.5)412,412,411          ,294
  412 TINC=TINC/2.0          ,295
  411 TR1=T/(AI*OM30)          ,296
      IF(TR1-100.0)101,414,414          ,297
  414 AAAA=1.0          ,298
      K=2          ,299
      AKEY=7.0          ,300
```

```
        YOYO                                                          01/25/55
           EXTERNAL FORMULA NUMBER   -   SOURCE STATEMENT   -   INTERNAL FORMULA NUMBER(S)

       WRITE(6,165)TRI              ,301  ,302  ,303
       GLAG=1.0             ,304
       TINC=TINCB           ,305
       GOTO150              ,306
   101 CONTINUE             ,307  ,308                                    end Runge-Kutta integration step
       H1=XF(1)+AK*(RO2*V3-RO3*V2)              ,309
       H2=XF(2)+AK*(RO3*V1-RO1*V3)              ,310
       H3=XF(3)*AI+AK*(RO1*V2-RO2*V1)           ,311
   150 CONTINUE             ,312
       IF(GLAG-.5)107,107,402           ,313
   107 CONTINUE             ,314
       IF(XF(4)-BETAC)21,22,22          ,315
    22 IF(CHECK2-1.0)23,21,21           ,316
    23 TAUC=TAUO+TINC*(BETAC-X(4,1))/(XF(4)-X(4,1))            ,317
       CHECK2=2.0           ,318
       DO24I=1,12           ,319
    24 XF(I)=X(I,1)         ,320  ,321
       GOTO25         ,322
    21 TAUF=TAUO+TINC               ,323
       TAUX=TAUP-TAUPIN             ,324
       OM12=SQRT(XF(1)**2+XF(2)**2)             ,325
       GAM=ATAN(OM12/(AI*XF(3)))                ,326
       DEL=ATAN(OM12/XF(3))             ,327
       ALD=XF(5)/OM30               ,328
       BETD=XF(4)/OM30              ,329
       GAMD=GAM*RTD         ,330
       DELD=DEL*RTD         ,331
       AGAMD=XF(11)*B/OM30              ,332
       OM1R=XF(1)/OM30              ,333
       OM2R=XF(2)/OM30              ,334
       OM3R=XF(3)/OM30              ,335
       OM12R=OM12/OM30              ,336
       TR=T/(AI*OM30)               ,337
       TRMAX=AMAX1(TR,TESTR)            ,338
       TESTR=TRMAX          ,339
       BETDMX=AMAX1(BETD,TESTB)         ,340
       TESTB=BETDMX         ,341
       AGAMX=AMAX1(AGAMD,TESTG)         ,342
       TESTG=AGAMX          ,343
       OM3RMN=AMIN1(OM3R,TESTOM)                ,344
       TESTOM=OM3RMN                ,345
       H3R=AI*OM3R          ,346
       REAL M1R,M2R,M3R
```

```
      M1R=-SX4*SX5*TR*CX11                    ,347
      M2R=CX4*SX5*TR*CX11                     ,348
      M3R=-CX5*TR*CX11               ,349
      PSID=XF(10)/OM30               ,350
      THETD=XF(9)/OM30               ,351
      PHID=XF(8)/OM30                ,352
      S1R=-M1R              ,353
      S2R=-M2R              ,354
      S3R=-M3R              ,355
      CALL TRANS1(H1,H2,H3,XF(8),XF(9),XF(10),H1S,H2S,H3S,BLMBH,BETAH,
     1HHR)         ,356
      CALL TRANS1(0.0,0.0,1.0,XF(8),XF(9),XF(10),AXSYM1,AXSYM2,AXSYM3,AL
     1MBK,ETAK,AXK)                  ,357
      CALL TRANS1(OM1R,OM2R,H3R,XF(8),XF(9),XF(10),H1RS,H2RS,H3RS,ALMBH,
     1ETAH,HR)              ,358
      CALL TRANS1(OM1R,OM2R,OM3R,XF(8),XF(9),XF(10),OM1RS,OM2RS,OM3RS,A
     1LMBOM,ETAOM,OMR)               ,359
      CALL TRANS1(S1R,S2R,S3R,XF(8),XF(9),XF(10),S1RS,S2RS,S3RS,ALMBS,
     1ETAS,SR)              ,360
      ALMBKD=ALMBK/OM30              ,361
      ALMBHD=ALMBH/OM30              ,362
      ALMBOD=ALMBOM/OM30             ,363
      ALMBSD=ALMBS/OM30              ,364
      ETAKD=ETAK/OM30                ,365
      ETAHD=ETAH/OM30                ,366
      ETAOMD=ETAOM/OM30              ,367
      ETASD=ETAS/OM30                ,368
      IF(AGAMD-TESTG+.00000100)102,103,103              ,369
  102 AAAA=1.0              ,370
      K=2          ,371
      AKEY=4.0              ,372
      GOTO400               ,373
  103 CONTINUE              ,374
      IF(BETD-TESTB+.00000100)104,109,109               ,375
  104 AAAA=1.0              ,376
      K=2          ,377
      AKEY=5.0              ,378
      GOTO400               ,379
  109 IF(TAUF-TAUX)110,120,120                ,380
  110 AAAA=2.0              ,381
      GOTO130               ,382
  120 AAAA=1.0              ,383
      TAUP=TAUP+TPINC                ,384
```

```
               YOYO                                                          01/26/66
            EXTERNAL FORMULA NUMBER   -   SOURCE STATEMENT   -   INTERNAL FORMULA NUMBER(S)

 130 IF(TAUF-TFINAL)210,220,220              ,385
 210 IF(1.57079633-XF(11))230,212,212              ,386
 212 K=1          ,387
     AKEY=1.0           ,388
     GO TO 400          ,389
 220 AAAA=1.0           ,390
     K=2          ,391
     GO TO 400          ,392
 230 AAAA=1.0           ,393
     K=2          ,394
     AKEY=2.0           ,395
 400 CONTINUE           ,396
     IF(XF(5)-1.57079633)402,403,403               ,397
 403 WRITE(6,164)ALD          ,398  ,399  ,400
     AAAA=1.0           ,401
     K=2          ,402
     AKEY=6.0           ,403
 402 Y(1,1,K)=TAUF            ,404
     Y(1,2,K)=TR        ,405
     Y(1,3,K)=M1R       ,406
     Y(1,4,K)=M2R       ,407
     Y(1,5,K)=M3R       ,408
     Y(1,6,K)=OM3R            ,409
     Y(1,7,K)=OM12R           ,410
     Y(2,1,K)=AXSYM1          ,411
     Y(2,2,K)=AXSYM2          ,412
     Y(2,3,K)=AXSYM3          ,413
     Y(2,4,K)=ALMBKD          ,414
     Y(2,5,K)=ETAKD           ,415
     Y(2,6,K)=PHID            ,416
     Y(2,7,K)=PSID            ,417
     Y(3,1,K)=H1RS            ,418
     Y(3,2,K)=H2RS            ,419
     Y(3,3,K)=H3RS            ,420
     Y(3,4,K)=ALMBHD          ,421
     Y(3,5,K)=ETAHD           ,422
     Y(3,6,K)=GAMD            ,423
     Y(3,7,K)=HR        ,424
     Y(4,1,K)=OM1RS           ,425
     Y(4,2,K)=OM2RS           ,426
     Y(4,3,K)=OM3RS           ,427
     Y(4,4,K)=ALMBOD          ,428
     Y(4,5,K)=ETAOMD          ,429
```

assigned for ease of printing

```
        Y(4,6,K)=DELD               ,430
        Y(4,7,K)=UMR          ,431
        Y(5,1,K)=S1RS               ,432
        Y(5,2,K)=S2RS               ,433
        Y(5,3,K)=S3RS               ,434
        Y(5,4,K)=ALMBSD             ,435
        Y(5,5,K)=ETASD              ,436
        Y(5,6,K)=ALD          ,437
        Y(5,7,K)=BETD*(1.0-B)+AGAMD*B           ,438
        IF(AAAA-1.5)211,213,213          ,439
    211 WRITE(6,3)            ,440  ,441
        I=0             ,442
        DO 214 I=1,5          ,443
    214 WRITE(6,5)Y(I,1,K),Y(I,2,K),Y(I,3,K),Y(I,4,K),Y(I,5,K),Y(I,6,K),Y(
       1I,7,K)          ,444  ,445  ,446  ,447
        WRITE(6,5)H1S,H2S,H3S            ,448  ,449  ,450
     50 WRITE(6,3)            ,451  ,452
    213 IF(FLOAT(K)-1.5)215,215,216         ,453
    215 I=0             ,454
        DO217I=1,12           ,455
    217 X(I,1)=XF(I)          ,456  ,457
        TAUO=TAUF             ,458
     25 IF(TAUO+TINC-TAUC)10,501,501        ,459
    501 IF(BBB-.5)502,503,503            ,460
    502 TINC=TAUC-TAUO              ,461
        BBB=1.0               ,462
        GOTO10          ,463
    503 IF(BBB-1.5)504,505,505           ,464
    505 TINC=TINCB            ,465
        GOTO10          ,466
    504 TINC=TINCB-TINC             ,467
        B=1.0           ,468
        BBB=2.0               ,469
        X(11,1)=0.0           ,470
        X(12,1)=CX5*XF(5)           ,471
        X(6,1)=0.0            ,472
        X(7,1)=XF(7)+XF(6)*SX5/XF(4)        ,473
        B11=1.0+AK*(RO2**2+RO3**2)          ,474
        B12=-AK*RO1*RO2             ,475
        B13=-AK*RO1*RO3             ,476
        B22=1.0+AK*(RO1**2+RO3**2)          ,477
        B23=-AK*RO2*RO3             ,478
        B33=AI+AK*(RO1**2+RO2**2)           ,479
```

Computations for variable "jumps" at transition point

EXTERNAL FORMULA NUMBER - SOURCE STATEMENT - INTERNAL FORMULA NUMBER(S)

```
      RO1P=RO1A*X(7,1)+RO1G*X(12,1)          ,480
      RO2P=RO2A*X(7,1)+RO2G*X(12,1)          ,481
      RO3P=RO3A*X(7,1)+RO3G*X(12,1)          ,482
      CC1=H1-AK*(RO2*RO3P-RO3*RO2P)          ,483
      CC2=H2-AK*(RO3*RO1P-RO1*RO3P)          ,484
      CC3=H3-AK*(RO1*RO2P-RO2*RO1P)          ,485
      BDET=B11*B22*B33+B12*B23*B13+B13*B12*B23-B13*B22*B13-B23*B23*
     1B11-B33*B12*B12          ,486
      B11I=(B22*B33-B23*B23)/BDET            ,487
      B12I=(B23*B13-B12*B33)/BDET            ,488
      B13I=(B12*B23-B22*B13)/BDET            ,489
      B22I=(B11*B33-B13*B13)/BDET            ,490
      B23I=(B12*B13-B11*B23)/BDET            ,491
      B33I=(B11*B22-B12*B12)/BDET            ,492
      X(1,1)=B11I*CC1+B12I*CC2+B13I*CC3          ,493
      X(2,1)=B12I*CC1+B22I*CC2+B23I*CC3          ,494
      X(3,1)=B13I*CC1+B23I*CC2+B33I*CC3          ,495
      I=0          ,496                                              A
      DO98I=1,12          ,497
   98 XF(I)=X(I,1)          ,498   ,499
      GO TO 10          ,500
  216 I=0          ,501
      J=0          ,502
      IF(AKEY-2.5)105,105,106          ,503
  105 CONTINUE          ,504
      AM(5,7)=(Y(5,7,2)-Y(5,7,1))/(Y(1,1,2)-Y(1,1,1))          ,505
      BAD(5,7)=Y(5,7,1)-AM(5,7)*Y(1,1,1)          ,506
      DO 219 I=1,5          ,507
      DO 218 J=1,7          ,508
      AM(I,J)=(Y(I,J,2)-Y(I,J,1))/(Y(1,1,2)-Y(1,1,1))          ,509
      BAD(I,J)=Y(I,J,1)-AM(I,J)*Y(1,1,1)          ,510
  218 YIN(I,J)=(AM(I,J)*(1.57079633/OM30-BAD(5,7))/AM(5,7))+BAD(I,J)          ,511   ,512
  219 CONTINUE          ,513   ,514
      I=0          ,515
      WRITE(6,3)          ,516   ,517
      DO 401 I=1,5          ,518
  401 WRITE(6,5)YIN(I,1),YIN(I,2),YIN(I,3),YIN(I,4),YIN(I,5),YIN(I,6),YI
     1N(I,7)          ,519   ,520   ,521   ,522
  106 CONTINUE          ,523
      WRITE(6,5)TRMAX,AGAMX,AKEY,OM3RMN,BETDMX          ,524   ,525   ,526
      AK=AK+DELAK          ,527          <------  increment K
      IF(AK-AKLST)140,140,901          ,528          <---  check K
  140 TINC=TINCB          ,529
```

```
        EXTERNAL FORMULA NUMBER   -   SOURCE STATEMENT   -   INTERNAL FORMULA NUMBER(S)

      GOTO900              ,530
901   GAMOD=GAMOD+10.0            ,531          ← increment Io
      AK=AKFST             ,532                 ← reset K
      IF(GAMOD-10.0)142,142,141         ,533    ← check Io against last desired value
141   TINC=TINCB           ,534
      GOTO100              ,535
142   TINC=TINCB           ;536
      GOTO900              ,537
      END          ,538
```

```
TRANS1                                                          01/26/65
        EXTERNAL FORMULA NUMBER   -   SOURCE STATEMENT   -   INTERNAL FORMULA NUMBER(S)
                          input from program          output to program
      SUBROUTINE TRANS1(C1,C2,C3,PHI,THET,PSI,C1S,C2S,C3S,ALMB,ETA,CS)
      CPHI=COS(PHI)           ,1
      SPHI=SIN(PHI)           ,2
      CTHET=COS(THET)         ,3
      STHET=SIN(THET)         ,4
      CPSI=COS(PSI)           ,5
      SPSI=SIN(PSI)           ,6
      AA11=CPSI*CPHI-CTHET*SPHI*SPSI          ,7
      AA12=-SPSI*CPHI-CTHET*SPHI*CPSI            ,8
      AA13=STHET*SPHI         ,9
      AA21=CPSI*SPHI+CTHET*CPHI*SPSI          ,10
      AA22=-SPSI*SPHI+CTHET*CPHI*CPSI            ,11
      AA31=STHET*SPSI         ,12
      AA23=-STHET*CPHI        ,13
      AA32=STHET*CPSI         ,14
      AA33=CTHET           ,15
      C1S=AA11*C1+AA12*C2+AA13*C3             ,16
      C2S=AA21*C1+AA22*C2+AA23*C3             ,17
      C3S=AA31*C1+AA32*C2+AA33*C3             ,18
      C12S=SQRT(C1S**2+C2S**2)             ,19
      IF(C12S)1000,1001,1000               ,20
 1001 ALMB=0.0             ,21
      ETA=1.57079633          ,22
      GOTO1002             ,23
 1000 SLMB=C2S/C12S           ,24
      CLMB=C1S/C12S           ,25
      ALMB=ATAN2(SLMB,CLMB)                ,26
      ETA=ATAN(C3S/C12S)      ,27
 1002 CS=SQRT(C1S**2+C2S**2+C3S**2)           ,28
      RETURN            ,29
      END               ,30
```

Determines Vector components (C1S, C2S, C3S) in fixed space from components (C1, C2, C3) in relative system where ϕ, θ, ψ are the Euler Angles.

XIV. APPENDIX B

LISTING OF IBM 7094 (9) COMPUTER PROGRAM FOR APPROXIMATE EQUATIONS OF MOTION AND FOR DATA PRESENTED IN FIGURES 17 and 18

```
          YOYO                                                                  02/04/66
        EXTERNAL FORMULA NUMBER   -    SOURCE STATEMENT     -    INTERNAL FORMULA NUMBER(S)

C       PROGRAM FOR SOLUTION OF APPROXIMATE EQUATIONS OF MOTION FOR YO-YO
C       SATELLITE DESPINNING
C       R.L.COLLINS, SMD JAN. 1966
        DIMENSION Z(3,5),ZF(3),A(3,5),C(5),CC(5)
      1 FORMAT(6X26HR.L.COLLINS SMD  JAN. 1966)
      2 FORMAT(6X47HSOLUTION OF APROX EQUATIONS FOR YO YO SATELLITE)
      3 FORMAT(1X1H )
      5 FORMAT(7E16.8)
        WRITE(6,1)          ,1     ,2
        WRITE(6,2)          ,3     ,4
        WRITE(6,3)          ,5     ,6
        READ(5,5)TINC,TPINC,TFINAL               ,7     ,8     ,9      <- Input: computing increment, printing increment, stopping time
        WRITE(6,5)TINC,TPINC,TFINAL              ,10    ,11    ,12
    100 READ(5,5)AI,AK,W2010R,GAMOD              ,13    ,14    ,15     <- Input: (I, K, w20/w10 , Γ0 (degrees))
        WRITE(6,3)          ,16    ,17
        WRITE(6,3)          ,18    ,19
        WRITE(6,5)AI,AK,W2010R,GAMOD             ,20    ,21    ,22
        GAMO=GAMOD/57.2957795              ,23
        OM30=SQRT((AK+AI)/AK)              ,24
        PHIRO=0.0           ,25
        THETRO=GAMO/OM30           ,26
        PSIRO=(ATAN(1.0/W2010R))/OM30            ,27
        OM12RO=AI*SIN(GAMO)/COS(GAMO)            ,28
        OM1RO=OM12RO/SQRT(1.0+W2010R**2)                ,29
        OM2RO=W2010R*OM1RO         ,30
        Z(1,1)=PHIRO        ,31
        Z(2,1)=THETRO              ,32
        Z(3,1)=PSIRO        ,33
        ZF(1)=Z(1,1)        ,34
        ZF(2)=Z(2,1)        ,35
        ZF(3)=Z(3,1)        ,36
        TAUO=0.0            ,37
        TR=0.0       ,38
        TAUP=TPINC          ,39
        TAUPIN=0.5*TINC            ,40
        OM3RO=1.0           ,41
        TAU=TAUO            ,42
        WRITE(6,3)          ,43    ,44
        WRITE(6,5)TAUO,TR,OM1RO,OM2RO,OM3RO             ,45    ,46    ,47
        WRITE(6,5)THETRO,PHIRO,PSIRO             ,48    ,49    ,50
        WRITE(6,3)          ,51    ,52
     10 DO101J=1,4          ,53                                        begin integration step
```

```
      CHECK=0.0            ,54
   20 G=(1.0-AI)*OM30*(2.0*ATAN(TAU)-TAU)              ,55
      OM1=OM1RO*COS(G)+OM2RO*SIN(G)            ,56
      OM2=OM2RO*COS(G)-OM1RO*SIN(G)            ,57
      OM3=(1.0-TAU**2)/(1.0+TAU**2)            ,58
      A(1,J)=(OM1*SIN(OM30*Z(3,J))+OM2*COS(OM30*Z(3,J)))/SIN(OM30*Z(2,J)
     1)             ,59
      A(2,J)=OM1*COS(OM30*Z(3,J))-OM2*SIN(OM30*Z(3,J))          ,60
      A(3,J)=OM3-A(1,J)*COS(OM30*Z(2,J))              ,61
      IF(CHECK-0.5)21,101,101        ,62
   21 AJ=J          ,63
      C(J)=(1.0+(AJ-1.0)*(AJ-2.0)*0.5)*TINC*0.5          ,64
      CC(J)=(AJ-(AJ-1.0)*(AJ-2.0)/2.0)*TINC/6.0          ,65
      K=J+1         ,66
      DO23I=1,3            ,67
      Z(I,K)=Z(I,1)+C(J)*A(I,J)                ,68
   23 ZF(I)=ZF(I)+CC(J)*A(I,J)         ,69     ,70
      IF(4-J)22,22,24            ,71
   22 CHECK=1.0            ,72
      DO40I=1,3            ,73
   40 Z(I,4)=ZF(I)         ,74     ,75
      GO TO 20             ,76
   24 TAU=TAUO+C(J)              ,77
  101 CONTINUE             ,78     ,79
      TAUF=TAUO+TINC             ,80
      TAUX=TAUP-TAUPIN           ,81
      TR=4.0*TAUF/((1.0+TAUF**2)**2)           ,82
      IF(TAUF-TAUX)26,25,25            ,83
   26 AAAA=2.0             ,84
      GO TO 27             ,85
   25 AAAA=1.0             ,86
      TAUP=TAUP+TPINC            ,87
   27 IF(TAUF-TFINAL)29,28,28          ,88
   29 CONTINUE             ,89
      K=1           ,90
      GO TO 30             ,91
   28 AAAA=1.0             ,92
      K=2           ,93
   30 IF(AAAA-1.5)31,31,32             ,94
   31 WRITE(6,3)           ,95     ,96
      WRITE(6,5)TAUF,TR,OM1,OM2,OM3            ,97    ,98    ,99
      WRITE(6,5)ZF(2),ZF(1),ZF(3),G            ,100   ,101   ,102
   32 IF(FLOAT(K)-1.5)33,33,100                ,103
```

end integration step

```
            YOYO                                                                          02/04/65
        EXTERNAL FORMULA NUMBER   -   SOURCE STATEMENT   -   INTERNAL FORMULA NUMBER(S)

33 DO34I=1,3           ,104
34 Z(I,1)=ZF(I)        ,105   ,106
   TAUO=TAUF           ,107
   TAU=TAUO            ,108
   GOTO10        ,109
   END           ,110
```

ON THE MOTION OF A SYMMETRIC RIGID BODY WITH A
"YO-YO" DESPIN DEVICE ATTACHED

by

Robert L. Collins, Jr.

ABSTRACT

A novel method of reducing the spin of a rotating symmetric body, similar to many earth orbit satellites, is by allowing small, despin weights to unwind from about the satellite so that they absorb some, or all, of the satellite angular momentum. This technique which has been used successfully on several U.S. satellites is commonly referred to as yo-yo despin. Several studies of the motion of a system such as this have been published where it was assumed that the motion was two-dimensional (i.e., without coning).

This dissertation presents a comprehensive study of the yo-yo despin problem which includes a derivation of two-dimensional results as well as a three-dimensional or exact solution. The results presented are sufficient for rudimentary design computations and provide examples of the corrections necessary to apply to two-dimensional computations for their applications as estimates for the general motion. An approximate solution of the three-dimensional equations of motion is also presented along with an example of the accuracy obtained by the approximation.

The equations of motion are derived in a straightforward manner using the vectorial methods of Newtonian mechanics. The Euler equations for a rigid body are used to describe the motion of the rigid body itself.

The moment acting on the body through the tension in the yo-yo cables is unknown and it is necessary to apply the second law of Newton to a despin weight so that sufficient independent differential equations are available for the solution of the problem variables. These relations give three first-order differential equations and two second-order ones. An expression for the cable tension is also obtained. This system of equations is integrated numerically by a standard Runge-Kutta process. Two singularities require special attention: first, at the initial instant the fundamental inversion matrix for the Newton equations is singular; and second, special care must be taken at a point in the integration where a discontinuity is found to occur. Outside of these special points, the integration process is quite routine although some cases require precautions near the end of the despinning process in order that the integration is stopped before violent tumbling occurs. In order to discuss the motion relative to a fixed reference axis the Euler angles, and Euler angle rate equations are also integrated. A point of interest concerning the derivation of the equations of motion is that the Lagrange technique cannot be used without modification due to internal constraints which do work.

After a numerical study of several typical examples, one concludes that for initial coning angles of less than 10° a two-dimensional analysis is sufficient for determining many important design variables such as maximum cable tension and despin time, although the cable length is somewhat overestimated and problems may occur in the release of the weights if only a two-dimensional analysis is considered. If one desires information on the angular trajectory of the body in inertial coordinates, a study of the problem must be made using the exact three-dimensional relations or the approximate three-dimensional relations. The approximate expressions save the investigator a great deal of effort and apparently provides excellent results.