NONPARAMETRIC PROCEDURES FOR PROCESS CONTROL

WHEN THE CONTROL VALUE IS NOT SPECIFIED

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Dissertation submitted to the Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

STATISTICS

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August, 1984

Blacksburg, Virginia

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(ABSTRACT)

In industrial production processes, control charts have been developed to detect changes in the parameters specifying the quality of the production so that some rectifying action can be taken to restore the parameters to satisfactory values. Examples of the control charts are the Shewhart chart and the cumulative sum control chart (CUSUM chart). In designing a control chart, the exact distribution of the observations, e.g. normal distribution, is usually assumed to be known. But, when there is not sufficient information in determining the distribution, nonparametric procedures are appropriate. In such cases, the control value for the parameter may not be given because of insufficient information.

To costruct a control chart when the control value is not given, a standard sample must be obtained when the process is known to be under control so that the quality of the product can be maintained at the same level as that of

the standard sample. For this purpose, samples of fixed size are observed sequentially, and at each time a sample is observed a two-sample nonparametric statistic is obtained from the standard sample and the sequentially observed sample. With these sequentially obtained statistics, the usual process control procedure can be done. The truncation point is applied to denote the finite run length or the time at which sufficient information about the distribution of the observations and/or the control value is obtained so that the procedure may be switched to a parametric procedure or a nonparametric procedure with a control value.

To lessen the difficulties in the dependent structure of the statistics we use the fact that conditioned on the standard sample the statistics are i.i.d. random variables. Upper and lower bounds of the run length distribution are obtained for the Shewhart chart. A Brownian motion process is used to approximate the discrete time process of the CUSUM chart. The exact run length distribution of the approximated CUSUM chart is derived by using the inverse Laplace transform. Applying an appropriate correction to the boundary improves the approximation.

ACKNOWLEDGMENTS

The author wishes to express his thanks and appreciation to the following people.

To his advisor, Dr. Marion R. Reynolds, Jr., for his advice and encouragement given so generously throughout the study.

To Dr. Donald R. Jensen for co-reading the manuscript and his valuable comments.

To Dr. Walter R. Pirie for his guidance and help in many nonparametric problems.

To Drs. Jefferey B. Birch and Klaus Hinkelmann for serving as members of the Graduate Committee and for their suggestions.

To Drs. Jesse C. Arnold and Robert S. Schulman for serving on the examining committee in place of Drs. Walter R. Pirie and Jeffrey B. Birch.

To his wife Gilsoo and his parents for their love, support and encouragement.

TABLE OF CONTENTS

TABLE OF CONTENTS (cont)

TABLE OF CONTENTS (cont)

LIST OF TABLES

LIST OF TABLES (cont)

LIST OF FIGURES

Chapter I

INTRODUCTION

In an industrial production process, statistical procedures are needed to detect changes in the parameters specifying the quality of the product - the mean, the variance, the proportion of defectives, etc. - so that some rectifying action can be taken to restore the parameters to satisfactory values after changes occur. For this purpose control charts have been developed to detect any shift of the parameters from specified control values.

The most widely used control charts are the Shewhart charts originated by Shewhart (1931). The basic rule of these charts is to take samples of fixed size at regular time intervals and to take a rectifying action as soon as the test statistic computed from a sample falls outside control limits. The control limits are lines drawn on a chart denoting a significant departure of the statistic from its satisfactory values. Page (1961) investigated some modified Shewhart charts with warning lines, which are drawn in less extreme positions than the control limits, to improve sensitivity to small changes in the parameters. Also Page (1954) proposed a cumulative sum (CUSUM) control chart which at each point uses all the previously obtained samples instead of the last few samples as in the standard Shewhart chart or the Shewhart chart with warning lines.

In monitoring a process, samples are observed successively to determine whether the quality of the product remains at the specified level, and usually these successive samples are assumed to be mutually independent. The run length of a control chart is the random time at which the chart signals, and the expectation of the run length is called the average run length (ARL). The ARL is one of the most commonly used measures for evaluating a process control procedure.

When the parameters monitored remain as specified the process is said to be under control. Otherwise, the process is said to be out of control. To make the analysis easier, the properties of control charts for processes not in control are derived here assuming that the parameter shifts from the control value at the time at which monitoring started. Also the amount of the change in the parameter is assumed to remain constant during the time the process is monitored.

When the process is out of control the ARL gives the expected time elapsed until a signal, and thus measures the amount of scrap produced before a rectifying action is taken. On the other hand, when the process is under control any signal by the control chart is a false alarm, and thus the ARL is a measure of the frequency of false alarms. As long as the process is under control, the ARL should be large so that production may continue uninterrupted as long as possible, but if the process is out of control the ARL should be small so that the change is detected quickly.

When two different control charts are to be compared, the usual procedure is to compare the ARL's for a certain amount of shift when the ARL's under control are set to be the same. If the first control chart has a smaller ARL than the second, then it is said that the first chart is more efficient than the second for that amount of shift. This method of comparison is similar to that of comparing the powers (or probabilities of Type II errors) of two tests having the same probability of a Type I error in a hypothesis testing problem.

Most standard control procedures are designed on the assumption that the distribution of the observations is of a specified form, usually the normal distribution. But, in some applications, there is insufficient information to completely determine the distribution. Also, if the true distribution of the observations is quite different from the specified form, the properties of the procedures can be affected significantly. In such cases, therefore, it may be appropriate to use nonparametric procedures for process control requiring fewer assumptions than parametric ones, but there appear to be few nonparametric procedures available.

Parent (1965) proposed a nonparametric procedure based on the signed sequential ranks of the observations; such a procedure is studied also by Reynolds (1972). Bakir and Reynolds (1979), and Reynolds and Bakir (1982), suggested some nonparametric control procedures for the location parameter when a specified control value is given.

Usually the control value for a process parameter is derived from past experience or is selected by the management to attain certain objectives. When neither source gives a clear specification of the control value and the distribution of the observations is unknown, there is a need for nonparametric control charts which can be applied to the control problem when the control value is not specified.

The purpose of this dissertation is to study some nonparametric process control procedures for the location and the scale parameters of a process when the control values for these parameters are not specified. These procedures use versions of Shewhart charts and CUSUM charts based on some well known two-sample nonparametric statistics.

When the control value of a process is unknown, a standard sample is required when the process is under control so that quality can be maintained at the same level as the standard. For this purpose, random samples of fixed size are observed successively, and on each occasion a two-sample nonparametric test statistic is computed for comparing each sample to the standard. We call these two samples the observed sample and the standard sample to distinguish them. With this sequence of statistics, versions of the usual Shewhart charts and CUSUM charts can be constructed.

A two-sample nonparametric statistic is chosen appropriately for the parameter to be monitored. In this dissertation the median placement statistic and the Wilcoxon-Mann-Whitney statistic are used to detect changes in the location parameter of the process. The sum of the squared ranks statistic is used to detect changes in the scale parameter of a process.

When the control value is given, the successive statistics are independent since in this case the statistics are one-sample statistics from independent samples. The properties of a procedure can be obtained easily relative to the case when the control value is not given. For the latter case the successive statistics are not independent because they depend on the same standard sample. This dependency among successive statistics makes the properties of the proposed procedure more difficult to evaluate. For example, the run length of a Shewhart chart with the control value given follows a geometric distribution while this is not true for the proposed control charts. To lessen the difficulty of the dependent structure, we use the fact that the obtained statistics are conditionally i.i.d. random variables, conditioned on the standard sample. Then the run length distribution of a procedure can be obtained conditionally as if the successive statistics are i.i.d., and unconditionally as an expectation with respect to the standard sample.

In practical applications, it may be useful to use a nonparametric control chart without a specified control value until information on the distribution of the observations is obtained. Later there may be a reasonable idea about the distribution of the observations or the control value or both, and then an appropriate control procedure, either parametric or nonparametric, can be selected according to the obtained information. The time necessary for obtaining such information can be regarded as determining a truncation point for the control chart used initially. If only the control value is determined at the truncation point then it may be desirable to use a nonparametric control chart with a specified control value until the distribution is determined. If both the control value and the distribution are determined at the truncation point, then a parametric control chart if avaiable may be more efficient than nonparametric procedures for future control of the process.

In the proposed control charts, the ARL need not be finite if there is no limit on the run length, which is possible only when the process can be continued forever. In practice, every industrial process has a finite time to run. Hence, in each of the proposed charts, a truncation point can also be used to denote the finite length of the production run. Therefore the truncation point of the proposed control charts is used either as the amount of time required to obtain information regarding the distribution of the observations, or the finite time which the process has to run.

This dissertation is arranged as follows. Chapter 2 gives a general review of some control charts . Two main parametric control charts, the Shewhart and CUSUM charts, and some nonparametric control charts are reviewed when the control value is given.

Chapter 3 discusses the general structure of process control procedures when the control value is not given. It also provides notations, definitions, and theorems which are used throughout the dissertation.

Chapter 4 is devoted to Shewhart charts using some nonparametric statistics when the control value is not given. Exact expressions or bounds for the run length distribution and the ARL of Shewhart charts are obtained for each test statistic.

In Chapter 5, some CUSUM charts are studied using nonparametric statistics when the control value is not given. The Brownian motion process is used to approximate the discrete time process of the CUSUM chart. The exact run length distribution of the approximated CUSUM chart is obtained by taking the inverse of the Laplace transform of the run length distribution which was derived by Reynolds (1972). Optimal choices for the reference value and the boundary of the CUSUM charts are discussed also. Although the Brownian motion approximation was found to underestimate the actual ARL, it is improved by using an appropriate continuity correction to the boundary of the process.

In Chapter 6, the ARL's of Shewhart charts and CUSUM charts are compared using parametric and nonparametric statistics when the control value is not specified, for cases where the observations are assumed to follow some well known continuous distributions. The ARL's are calculated by using exact expressions or by simulation.

Finally, in Chapter 7, the proposed nonparametric control charts are summarized, and some conclusions and recommendations are given. A control procedure requiring further study is suggested also.

FORTRAN computer programs which are used in calculating the ARL are documented and listed in the Appendix.

Chapter II

REVIEW OF THE PROCESS CONTROL PROCEDURES

2.1 INTRODUCTION

In this chapter, some well known parametric and nonparametric process control procedures are reviewed with emphasis on the case where the control value is given. The parametric procedures include Shewhart charts with and without warning lines and CUSUM charts. Much study has been done for the parametric procedures while little has been done for the nonparametric procedures.

2.2 THE SHEWHART CHARTS

Suppose that i.i.d. random samples of fixed size are observed at regular time intervals during the process and the goal is to maintain the quality of the product at a certain level. Let $\underline{X}_i=(X_{i1},\ldots,X_{in})$, $i=1,2,\ldots$, be the i-th observed sample and let θ be the parameter which specifies the quality of the product. Also suppose that a statistic $S_i=S(\underline{X}_i,\theta_0)$, which is a function of $\underline{X}_i$ and the given control value θ_0 of θ, is obtained each time a sample is observed. Then the criterion of the Shewhart chart is to decide whether a change in θ has occurred by examining the statistic S_i. The Shewhart chart signals at the first i for which

$$S_i \le c_1 \text{ or } S_i \ge c_2, \quad c_1<c_2 \tag{2.2.1}$$

for some constants c_1 and c_2 which are called the lower and upper control limits or action lines.

The run length of a control chart is the number of samples required to give a signal. The run length N of the Shewhart chart follows a geometric distribution with a parameter $P(c_1<S<c_2)$. That is, the run length distribution of the Shewhart chart is, for i=1,2,...,

$$P(N=i)=[P(c_1<S<c_2)]^{i-1}P(S\le c_1 \text{ or } S\ge c_2) \tag{2.2.2}$$

The ARL of the Shewhart chart is simply the mean of a geometric random variable with the parameter $P(c_1<S<c_2)$, i.e.

$$EN = 1/P(S\le c_1 \text{ or } S\ge c_2) \tag{2.2.3}$$

One standard example of the Shewhart chart is the $\overline{X}$-chart for detecting shifts in the process mean. The Shewhart $\overline{X}$-chart signals that the process mean has shifted from the control value θ_0 at the first i for which

$$\sqrt{n}\,|\overline{X}_i-\theta_0|/\sigma \ge k \tag{2.2.4}$$

where $\overline{X}_i$ is the sample mean of the i-th observed sample, σ^2 is the variance of X and k is a constant which is frequently taken to be 3.

Properties of the Shewhart $\overline{X}$-chart for controlling the mean of a process with a normal distribution when the variance σ^2 must be estimated from initial sample data were investigated by Ghosh, Reynolds and Hui (1981). In this case, unlike the case when the variance is known, the $\overline{X}$-chart is not equivalent to a sequence of independent tests and it was found that the effect of using small samples in estimating the variance is to increase the ARL and the variance of the run length distribution.

Shewhart control charts have been constructed using the fraction defective (P-chart), the number of defectives per unit (C-chart, U-chart), the sample range (R-chart) or the sample standard deviation (S-chart) as the test statistic. All these control charts are classed as Shewhart charts. For further details, refer to Duncan (1974).

Page (1955) investigated some Shewhart type control charts for monitoring the mean, modified with warning lines inside the action lines in order to use the information from the last few samples. The rule of one of his procedures is to take samples of fixed size and to take action if a fixed number ($d>1$) of consecutive points fall between the warning and action lines or if any point falls outside the action lines. Although the standard Shewhart charts are simple to use and their properties are easily determined, Page (1955) showed by comparing the ARL's of the two control charts that they are less efficient than the Shewhart charts with warning lines for detecting small shifts. This is because the Shewhart charts use information from only one sample at a time in determining whether the next sample should be observed or not.

Page (1962b) also showed that the control chart with warning lines, which is designed for detecting a change in the mean, is generally better than Moore's (1958) run rules, which take action only when a specified number of successive sample points exceed the control limit. The performance of the two rules is very similar for small departures from the target mean. But the ARL of the control chart with warning lines is markedly smaller than that of Moore's procedure as the departure from the target mean increases.

2.3 THE CUSUM CHARTS

A significant development in control procedures is the introduction of the cumulative sum (CUSUM) control charts by Page (1954). One advantage of the CUSUM charts over the Shewhart X-charts is that they may pick up a sudden and persistent change in the process more rapidly than the ordinary Shewhart charts especially if the change is not large. Let Δ denote the amount of shift of the parameter θ from the control value θ_0 for the observed samples. We assume that large values of S tend to indicate positive shifts and small values of S tend to indicate negative shifts. The stopping rule for the CUSUM chart is to signal at the first i for which

$$\Sigma_{j=1}^{i}(S_j-k) - \min_{0\le \ell\le i} \Sigma_{j=1}^{\ell}(S_j-k) \ge h \qquad (2.3.1)$$

for detecting shifts in the positive direction, or

$$\max_{0\le \ell\le i} \Sigma_{j=1}^{\ell}(S_j+k) - \Sigma_{j=1}^{i}(S_j+k) \ge h \qquad (2.3.2)$$

for detecting shifts in the negative direction, where $\Sigma_{j=1}^{0}(S_j-k)=\Sigma_{j=1}^{0}(S_j+k)=0$ and k (the reference value) and h (the boundary) are parameters of the CUSUM chart which are determined so as to produce a CUSUM procedure which is optimal in some sense. The stopping rule (2.3.1) can be explained in a different way. We plot the cumulative sum of (S_j-k), i.e. $\Sigma_{j=1}^{i}(S_j-k)$, until the cumulative sum is $\ge h$ or ≤ 0. If the cumulative sum is $\ge h$, then the procedure signals, but if the cumulative sum is ≤ 0, then we start plotting the cumulative sum over again with the next observed statistic. Thus the rule (2.3.1) breaks up into a sequence of sequential probability ratio tests (SPRT's) [see Wald (1947)] with boundaries at (0,h) and initial score 0. The test is reapplied when the previous test ends on the lower boundary and an action is taken when a test ends on the upper boundary.

Let $p^+(z)$ represent the probability that the SPRT with boundary (0,h) and the initial score z ends on the upper boundary. Also let $L(z)$, $L_1(z)$, and $L_2(z)$ be the average sample numbers, unconditional, conditional upon the test ending on the lower boundary, and conditional upon the test ending on the upper boundary, respectively. A test which ends on the lower boundary is called an acceptance test and on the upper boundary a rejection test. Using the fact that the probability of r acceptance tests before a rejection test is $[1-p^+(0)]^r p^+(0)$, the expected number of acceptance tests is

$$\Sigma_{r=1}^{\infty} r[1-p^+(0)]^r p^+(0) = [1-p^+(0)]/p^+(0). \qquad (2.3.3)$$

Hence the ARL for the rule (2.3.1) is

$$\begin{aligned} EN &= \{[1-p^+(0)]/p^+(0)\}L_1(0) + L_2(0) \\ &= L(0)/p^+(0). \qquad (2.3.4) \end{aligned}$$

In a one-sided CUSUM chart, it has been shown by Page (1954) and Ewan and Kemp (1960) that the run length distribution is approximately geometric when the ARL is large, i.e. the number of acceptance tests before action is taken is large. The number of repetitions of the SPRT has a geometric distribution and when the ARL is large this dominates the form of the distribution of the total number of samples. Page (1954) showed that the run length distribution is approximated for large i by

$$P(N \leq i) \simeq 1 - [1-p^{+}(0)]^{v+1} \tag{2.3.5}$$

where $v=i/L_1(0)$ and $\simeq$ denotes 'approximately equal to'. If we use the result (2.3.4) and put $L_1(0) \simeq L(0)$, we obtain

$$P(N \leq i) \simeq 1 - e^{-i/EN} \tag{2.3.6}$$

which is Ewan and Kemp's result. Thus for the one-sided CUSUM chart, at least for large i and EN, the run length distribution is specified approximately by the ARL and this gives a good justification for using the ARL to choose and compare control schemes.

For detecting shifts in both directions, a signal is given as soon as either (2.3.1) or (2.3.2) holds. If two separate rules of one-sided CUSUM charts are operated simultaneously, the run length of the combination is distributed as the minimum of two run lengths. Barnard (1959) presented the two-sided CUSUM scheme using a V-shaped mask to carry out the procedure in a simple manner.

A two-sided CUSUM chart is composed of two one-sided CUSUM charts, one for detecting shifts in the positive direction and the other in the negative direction. If the same parameters k and h are used in the two one-sided CUSUM charts, it is called a symmetrical two-sided CUSUM chart. The ARL for a two-sided CUSUM chart can be derived from the ARL's for the two one-sided CUSUM charts. Let $E_\theta N^+$ be the ARL at θ of a CUSUM chart for detecting positive deviations, let $E_\theta N^-$ be the ARL at θ of a CUSUM chart for detecting negative deviations, and let $E_\theta N$ be the ARL at θ of a two-sided CUSUM chart obtained by using the two one-sided CUSUM charts. Then it has been shown by Kemp (1961) that

$$1/E_\theta N = 1/E_\theta N^+ + 1/E_\theta N^-. \qquad (2.3.7)$$

If the process is under control, i.e. if θ is at the control value, and a symmetrical two-sided CUSUM chart is used, it is easy to see that, from (2.3.4) and (2.3.7),

$$E_\theta N = (1/2)E_\theta N^+ = (1/2)E_\theta N^-. \qquad (2.3.8)$$

If the process is out of control and the amount of shift is fairly large in the positive direction (or in the negative direction), the ARL for the two-sided CUSUM chart is practically equal to the ARL for the positive direction (or the negative direction), i.e.

$$E_\theta N \simeq E_\theta N^+ \text{ (or } E_\theta N^-) \qquad (2.3.9)$$

since $1/E_\theta N^-$ (or $1/E_\theta N^+$) is negligible.

Page (1961) indicated that the CUSUM schemes for monitoring the mean are much more sensitive than the ordinary Shewhart charts, especially to relatively small deviations from the target. The properties of the CUSUM scheme have been studied by many other authors. Ewan (1963) reviewed and described CUSUM charts with special emphasis on the practical problems of successful application. Johnson (1961) obtained some approximate formulas for the ARL, which may be useful in constructing the V-shaped mask proposed by Barnard (1959). Page (1962a) and Munford (1980) developed CUSUM schemes for monitoring the mean and standard deviation using gauges, where the range of the test statistic is divided into a fixed number of sections. Hawkins (1981) proposed a technique based on an appropriately chosen statistic for monitoring the variance. Brook and Evans (1972) used a different approach in which the operation of the scheme is regarded as forming a Markov chain. Nadler and Robbins (1971) applied a Brownian motion approximation in order to obtain the run length distribution of a two-sided CUSUM procedure without the reference value in the CUSUM chart. Process control problems for the case where the assumption of independent successive samples is violated have been studied by only a few authors. Goldsmith and Whitfield (1961), Johnson and Bagshaw (1974), and Bagshaw and Johnson (1975) studied the effect of serial correlation on the performance of CUSUM charts. The presence of serial correlation was found to have a quite major influence on the type of the run length distribution. Thus, the CUSUM charts are not robust with respect to departures from independence, whereas the Shewhart charts in many cases are. Acknowledging that independence of successive samples in a time-dependent production process may be tenuous assumption for many processes, one should be careful when applying procedures which are not robust to this independent assumption.

Additional properties of CUSUM charts, including a proper choice of the parameters k and h and the approximation of the run length distribution when the control value is given, are discussed in the following section.

2.4 A BROWNIAN MOTION APPROXIMATION TO THE CUSUM CHARTS

Usually the properties of a CUSUM chart are difficult to evaluate because there has been no exact expression available for the run length distribution. One of the techniques used in evaluating the properties of CUSUM charts is to approximate the discrete time CUSUM process by a continuous time Brownian motion process. When the statistics are i.i.d. random variables with finite second moments, it was shown by Reynolds (1975), using Donsker's [Donsker (1951)] theorem, that the continuous Brownian motion process can be used in place of the cumulative sums $\Sigma_{j=1}^{i}(S_j-k)$ or $\Sigma_{j=1}^{i}(S_j+k)$ to derive approximations for the properties of the procedures (2.3.1) and (2.3.2). If $X(t)$ is a Brownian motion process with mean $\mu' t$ and variance $\sigma^2 t$, where $\mu'=ES$ and $\sigma^2=VarS$, the following two procedures are considered to approximate the procedures (2.3.1) and (2.3.2).

Stop at the smallest t for which

$$(X(t)-kt) - \inf_{0<s<t} (X(s)-ks) \geq h \tag{2.4.1}$$

for detecting shifts in the positive direction, or

$$\sup_{0<s<t} (X(s)+ks) - (X(t)+kt) \geq h \tag{2.4.2}$$

for detecting shifts in the negative direction.

For detecting shifts in either direction, the procedures (2.4.1) and (2.4.2) are used simultaneously.

Consider the procedure for detecting shifts in the negative direction with the stopping rule given by (2.4.2). Let $\mu=\mu'+k$ and define the stopping time

$$\tau = \inf\{t: \sup_{0<s<t} (X(s)+ks)-(X(t)+kt)\geq h\}.$$

Let the probability density function of τ be

$$f_\tau(t) = \partial P(\tau\leq t)/\partial t,$$

and let the Laplace transform of $f_\tau(t)$ be

$$L_\tau(\lambda) = \int_0^\infty e^{-\lambda t} f_\tau(t)dt. \qquad (2.4.3)$$

The existence of a density for τ has been established by Fortet (1943). It was found by Reynolds (1972), using Theorem 3.1 in Darling and Siegert (1953), that

$$L_\tau(\lambda)=\beta e^{-\mu h/\sigma^2} / \{\beta\cosh(h\beta/\sigma^2)-\mu\sinh(h\beta/\sigma^2)\} \qquad (2.4.4)$$

where $\beta=\sqrt{\mu^2+2\sigma^2\lambda}$. Using (2.4.4), the expectation of τ is

$$\begin{aligned} E\tau &= -[\lim_{\lambda\to 0}\partial L_\tau(\lambda)/\partial\lambda] \\ &= -[h-\sigma^2(e^{2\mu h/\sigma^2}-1)/(2\mu)]/\mu, && \mu\neq 0 \\ &= h^2/\sigma^2, && \mu=0. \qquad (2.4.5) \end{aligned}$$

Analogously, when the stopping rule (2.4.1) is used for detecting shifts in the positive direction,

$$\begin{aligned} E\tau &= [h+\sigma^2(e^{-2\mu h/\sigma^2}-1)/(2\mu)]/\mu, && \mu\neq 0 \\ &= h^2/\sigma^2, && \mu=0. \qquad (2.4.6) \end{aligned}$$

Without loss of generality, assume that μt is the mean of the process $X(t)$ instead of $\mu' t$. The usual criterion for choosing the parameters k and h is to pick a desired ARL at $\mu=\mu_0$ and then try to minimize the ARL at some specified $\mu=\mu_1(\neq\mu_0)$, where $\mu_0=E_{\Delta=0}S$, $\mu_1=E_{\Delta=\Delta_1}S$, and Δ_1 is the amount of shift of the parameter θ at which it becomes important to detect a shift of this magnitude. The choice of the parameters k and h of the CUSUM chart has been discussed by many authors. However, the problem of choosing the parameters k and h, satisfying the criterion given above, has not been completely solved. The fact that there has been no analytical expression available for the ARL makes this problem difficult. Reynolds (1975) used (2.4.6) as an approximation to the ARL to obtain the proper choice of k and h. He chose the values k and h which minimize $ARL_{\mu=\mu_1}$ subject to $ARL_{\mu=\mu_0}$ fixed at some constant, say A_0. Using the Lagrange multiplier method, it was found that $k=(\mu_0+\mu_1)/2$ and h is chosen so that $ARL_{\mu=\mu_0}=A_0$. This result coincides with that of Ewan and Kemp (1960) and Johnson (1961), where it was assumed that the distribution of the observations was normal. One desirable consequence of this result is that the choice of the optimal k does not depend on A_0, which makes the problem very simple. When $\mu_1<\mu_0$, $k=-(\mu_0+\mu_1)/2$ by a similar method.

2.5 NONPARAMETRIC CONTROL CHARTS

A nonparametric procedure based on the signed sequential ranks of the observations was proposed by Parent (1965) and also studied by Reynolds (1972). If $Y_1, Y_2, \ldots$ is a sequence of independent random variables, then the signed sequential rank of Y_i relative to $Y_1, Y_2, \ldots, Y_i$ is defined as the product of the rank of $|Y_i|$ relative to $|Y_1|, \ldots, |Y_i|$ and $\text{sign}(Y_i)$, where $\text{sign}(x)=1,-1$ for $x \geq, <0$. The procedure is to signal whenever the sum of the signed sequential ranks of Y_i divided by the time i falls outside given boundaries. But this procedure is not very efficient until a certain amount of information is observed. Also it requires the ranking of long sequences of observations and the ARL is obtained only approximately.

Bakir and Reynolds (1979) proposed a nonparametric CUSUM procedure for controlling the process mean based on within-group rankings. The Wilcoxon signed rank statistic is obtained for each group and a CUSUM type stopping rule is applied; if a single observation is taken at each point, the observations must be artificially divided into groups. By ranking within groups the work in determining the ranks is greatly simplified and the exact ARL can be obtained using the mean time to absorption in a discrete Markov chain. It appears that the nonparametric procedure based on within-group rankings is only slightly less efficient than the parametric procedures under the normality assumption and it can be considerably more efficient than the parametric procedures for non-normal distributions.

Reynolds and Bakir (1982) studied some additional nonparametric procedures for controlling the process mean. These procedures are Shewhart charts, using the sign and the signed rank statistics with or without warning lines, and the CUSUM chart using the sign test statistic. Also the effect of nonnormality on $\overline{X}$-charts was studied by comparing the ARL's for certain well known nonnormal distributions and the ARL calculated under the normality assumption. The comparison indicated that some of the nonparametric procedures are almost as efficient as the parametric ones under normality and can be more efficient for non-normal distributions. An additional advantage of these nonparametric procedures is that the variance of the process does not need to be known or estimated in order to apply the control chart, moreover the variance need not to be stationary.

Chapter III

BASIC RESULTS ON A NONPARAMETRIC CONTROL CHART WHEN THE CONTROL VALUE IS NOT SPECIFIED

3.1 THE GENERAL STRUCTURE OF A NONPARAMETRIC CONTROL CHART

In this chapter we develop a basic nonparametric control procedure for use when the control value is not specified. This procedure is as follows: Obtain a standard sample $\underline{X}=(X_1,...,X_m)$ when the process is known to be under control and observe samples of fixed size $\underline{Y}_i=(Y_{i1},...,Y_{in})$, $i=1,2,...$, sequentially thereafter. Each sample $\underline{Y}_i$ yields a two-sample nonparametric statistic $S_i=S(\underline{X},\underline{Y}_i)$ which is chosen according to the parameter of interest. These sequentially obtained statistics then are used to construct Shewhart or CUSUM charts as appropriate. The purpose is to find shifts quickly when the quality of the product differs from that of the standard sample. The run length and the truncation point of a control chart are denoted by N and T, respectively.

If groups of size n are observed at each time point, the grouping is done naturally. But if a single observation is taken at each time point, then the observations must be divided into groups of size n. Assume that $X_1,...,X_m$ are i.i.d. random variables from a continuous distribution with distribution function F, and that $Y_{i1},\ldots,Y_{in}$, i=1,2,..., are i.i.d. random variables from a continuous distribution with distribution function G. Also assume that the random vectors $\underline{X}$, $\underline{Y}_1$, $\underline{Y}_2,\ldots$ are independent. Most of the previous work on the properties of process control procedures has been done under the assumption that the successive samples $\underline{Y}_1$, $\underline{Y}_2,\ldots$ are i.i.d. and the study in this dissertation is also restricted to i.i.d. samples. In many cases the size m of the standard sample will be considerably larger than the size n of the subsequent samples. With this in mind, throughout this dissertation we assume that $m \geq n$.

Unlike the case when the control value is known, the successive statistics are not independent because each statistic depends on the same standard sample. These dependencies make the analysis difficult. If, however, we first condition on the standard sample, then the statistics are conditionally i.i.d. random variables and many conditional properties can be derived easily. The unconditional properties, such as the run length distribution and the ARL, then may be obtained by taking expectation of the conditional properties with respect to the standard sample. The finite truncation point of the process ensures the existence of the unconditional ARL because the conditional ARL is always less than or equal to the truncation point.

3.2 BASIC RESULTS ON NONPARAMETRIC STATISTICS

Orban and Wolfe(1982) proposed a class of two-sample nonparametric statistics which is appropriate when one sample is large relative to the other in a two-sample distribution-free test. Initial interest in this type of statistic came as a result of an earlier study by Orban and Wolfe(1980) on properties of partially sequential two sample procedures for comparing treatments with a control.

Now we define a class of two-sample nonparametric statistics. Let $\underline{U}_i=(U_{i1},...,U_{in})=(F_m(Y_{i1}),...,F_m(Y_{in}))$, where F_m is the sample distribution function of the standard sample.

Definition 3.2.1.(Orban and Wolfe,1982) : The quantity

$$mU_{ij} = mF_m(Y_{ij}) = [\text{number of X's} \leq Y_{ij}]$$

is called the placement of Y_{ij} among the X's. ‖

Definition 3.2.2.(Orban and Wolfe,1982) : Let C be the class of real-valued Lebesgue measurable functions defined on

[0,1]. The class of linear placement statistics is defined to be the collection of all statistics of the form

$$S_i = \Sigma_{j=1}^{n} \phi_m(U_{ij}) \quad .$$

for some $\phi_m(.) \varepsilon C$. The function $\phi_m(.)$ is called a score function. ‖

The Wilcoxon-Mann-Whitney statistic [see Wilcoxon (1945), Mann and Whitney (1947)] is equivalent to a linear placement statistic S_i^W, corresponding to the identity score function $\phi_m(x)=x$, i.e.,

$$S_i^W = \Sigma_{j=1}^{n} U_{ij} = \Sigma_{j=1}^{n} F_m(Y_{ij}). \qquad (3.2.1)$$

The median placement statistic,

$$S_i^M = [\text{ number of } Y_i\text{'s} \geq X_{(M)} \text{ }]$$

where $X_{(M)}$ denotes the median of the standard sample, corresponds to the score function

$$\begin{aligned} \phi_m(x) &= \quad 1 \quad \text{if } x \geq 1/2 \\ &= \quad 0 \quad \text{if } x < 1/2, \end{aligned} \qquad (3.2.2)$$

i.e., the median placement statistic is of the form

$$S_i^M = \Sigma_{j=1}^{n} \Psi(Y_{ij} - X_{(M)}) \qquad (3.2.3)$$

where $\Psi(x)=0,1$ for $x<,\geq 0$.

The following theorem gives the null (F=G) distribution of the linear placement statistic. Let $\underline{r}=(r_1,\ldots,r_n)$ be any vector of integers containing exactly n_j values of j, $j=0,1,\ldots,m$, where $0\leq n_j\leq n$ and $n_0+\ldots+n_m=n$. Then, simple combinatorial arguments lead to the following result.

<u>Theorem 3.2.1.(Orban and Wolfe,1982)</u> : If F=G is true, then

$$P[m\underline{U}_i=\underline{r}] = [(m!)\Pi_{j=0}^{m}(n_j!)]/[(m+n)!]$$

for any vector $\underline{r}$ containing n_j values of j, $j=0,1,\ldots,m$ with $0\leq n_j\leq n$ and $\Sigma_{j=0}^{m}n_j=n$, and $P[m\underline{U}_i=\underline{r}]=0$ elsewhere.‖

From this theorem we can see that the linear placement statistic is nonparametric distribution-free when the two distributions F and G are equal. The null (F=G) mean and variance of a linear placement statistic (Definition 3.2.2) are then immediate consequences of the joint distribution of the placements in Theorem 3.2.1 and are given by

$$E_0(S)=n\phi_m$$

and

$$Var_0(S) \qquad (3.2.4)$$

$$=[n(n+m+1)/(m+1)(m+2)][\Sigma_{j=1}^{m}\phi_m^2(j/m)-(m+1)\bar{\phi}_m^2]$$

where $\bar{\phi}_m=[\Sigma_{j=1}^{m}\phi_m(j/m)]/(m+1)$. From (3.2.4), the null mean and variance of the Wilcoxon-Mann-Whitney statistic and the median placement statistic are given by

$$E_0(S^W)=n/2, \quad Var_0(S^W)=n(n+m+1)/12m$$

and (3.2.5)

$$E_0(S^M)=n/2, \quad Var_0(S^M)=n(n+m+1)/4(m+2).$$

In the following chapters, the Wilcoxon-Mann-Whitney statistic (3.2.1) and the median placement statistic (3.2.3) are used as the test statistics of control charts to detect shifts of a location parameter.

We now describe another general class of two-sample nonparametric statistics. Let $\underline{R}_i=(R_{i1},...,R_{iL})$, L=m+n, denote a vector of joint ranks of $(Y_{i1},...,Y_{in},X_1,...,X_m)$ and let a(1),...,a(L) be a set of L constants which are not all the same.

Definition 3.2.3.(Randles and Wolfe,1979) : A statistic of the form

$$S_i = \Sigma_{j=1}^{n}a(R_{ij})$$

is called a linear rank statistic. The function $a(.)$ is called a score function. ||

Any linear rank statistic is nonparametric distribution-free because the ranks are uniformly distributed over $[1,...,L]$ for any continuous distribution if $F=G$. The Wilcoxon-Mann-Whitney statistic is the only one that is both a linear placement statistic and a linear rank statistic. The null mean and variance of a linear rank statistic are given by

$$E_0(S)=na$$

and (3.2.6)

$$Var_0(S)=[mn/L(L-1)]\Sigma_{j=1}^{L}[a(j)-a]^2$$

where $a=(1/L)\Sigma_{j=1}^{L}a(j)$.

It is well known that for testing hypotheses about the variance of a distribution it may be disastrous to base a test on the chi-square or F distribution because of extreme sensitivity of the distribution of the test statistic to nonnormality of the observations. Thus many nonparametric tests using linear rank statistics have been proposed to deal with the scale testing problem. One popular choice among the class of linear rank statistics is the sum of the squared ranks statistic proposed by Taha (1964) which is of the form, for the score function $a(x)=x^2/L^2$,

$$S_i^S = \Sigma_{j=1}^{n}(R_{ij}/L)^2. \qquad (3.2.7)$$

From (3.2.6), the null mean and variance of the sum of the squared ranks statistic are given by

$$E_0(S^S)=n(L+1)(2L+1)/6L^2$$

and (3.2.8)

$$Var_0(S^S)=mn(L+1)(2L+1)(8L+11)/180L^4.$$

The sum of the squared ranks test has been studied by several authors including Duran and Mielke (1968), Whiteside, Duran and Boullion (1975), and Conover and Iman (1978). It is almost as easy to use as the Wilcoxon rank sum test, and has greater power when two populations differ in their scale parameters rather than in their location parameters especially for certain asymmetrical one-sided distributions (e.g. $F(x)=0$ if $x\leq 0$). In many process control problems, asymmetrical distributions tend to appear more frequently than symmetrical ones. Thus, the sum of the squared ranks statistic is used in this dissertation as the test statistic of the control chart to detect shifts in the scale parameter. For asymmetrical one-sided distributions when all parameters except the scale parameter are fixed, the mean of the distribution also changes when the variance changes, so the sum of the squared ranks statistic may be used to detect simultaneous changes of the mean and the variance.

An approximation to the Wilcoxon-Mann-Whitney null distribution based on the sum of independent uniform random variables has been studied by Buckle, Kraft and Eeden (1969). This approximation is based on the fact that if one of the sample sizes approaches infinity while the other is fixed, it is intuitively clear that the distribution of the sum of ranks for the smaller sample is asymptotically that of a sum of independent uniform random variables. They have shown by numerical comparison that the uniform approximation is almost uniformly better than the normal approximation. The generalized approximation theory for the linear placement statistic was established by Orban and Wolfe (1982).

Orban and Wolfe (1982) also established the one-sample limiting ($m \to \infty$) distribution of a linear placement statistic and derived an approximation to its exact null distribution. Two assumptions (Assumptions 1 and 2) on the score function are made to insure convergence, as $m \to \infty$, of the corresponding statistic. The assumptions are satisfied by many important score functions, including the median placement statistic and the Wilcoxon-Mann-Whitney statistic.

Assumption 1. : $\phi(x)$ is a real-valued function on [0,1], with at most a finite number of discontinuities. Let δ be the discontinuity set for $\phi(.)$. ‖

Assumption 2. : The score functions of the linear placement statistic $\{\phi_m(x)\ ;\ m=1,2,\ldots\}$ is a sequence of real-valued functions on [0,1] that converges uniformly (in x) to $\phi(x)$ on every closed interval [a,b] [0,1]-δ. ‖

Theorem 3.2.2.(Orban and Wolfe,1982) :Let $\{S_m\}_{m=1}^{\infty}$ be a sequence of linear placement statistics with score functions $\{\phi_m(x)\}$ satisfying Assumptions 1 and 2. If F and G are absolutely continuous distribution functions, then

$$(S_m - S') \to 0 \text{ in probability as } m \to \infty,$$

where $S'=\Sigma_{j=1}^{n}\phi[F(Y_j)]$ and $\phi(x)=\lim_{m\to\infty}\phi_m(x)$. ‖

Corollary 3.2.2.(Orban and Wolfe,1982) :Let $\{S_m\}_{m=1}^{\infty}$ be a sequence of linear placement statistics with score functions $\{\phi_m(x)\}$ satisfying Assumptions 1 and 2. If F and G are absolutely continuous distribution functions, then

$$\lim_{m\to\infty} P(S_m \leq x) = P(S' \leq x),$$

where S' is given in Theorem 3.2.2. If, in addition, H_0: $F=G$ is true and $F(.)$ is absolutely continuous, then the limiting null distribution of S_m is

$$\lim_{m\to\infty} P_0(S_m \le x) = P[\Sigma_{j=1}^{n} \phi(W_j) \le x]$$

where $W_1, \ldots, W_n$ is a random sample from a uniform (0,1) distribution. ‖

The one-sample limiting ($m\to\infty$) distribution of a linear rank statistic also can be derived by making an assumption about the score function, which is satisfied by many important score functions of the linear rank statistics.

Assumption 3. : The score function $a(j)$ of the linear rank statistic has the form

$$a(j) = e_L \alpha[j/(L+1)] + f_L, \quad j=1,\ldots,n$$

where $\alpha(.)$ is a real valued function on (0,1) and does not depend on L, and $\{e_L\}$ and $\{f_L\}$ are sequences of constants with $e_L > 0$ for each L. ‖

By using Assumption 3, results similar to Theorem 3.2.2 and Corollary 3.2.2 are obtained for the linear rank statistic.

Theorem 3.2.3. : Let $\{S_m\}_{m=1}^{\infty}$ be a sequence of linear rank statistics with score functions $\{a(x)\}$ satisfying Assumption 3, and let the limits $f=\lim_{m\to\infty} f_L$ and $e=\lim_{m\to\infty} e_L$ exist. Then

$$\lim_{m\to\infty} S_m = e\Sigma_{j=1}^{n}\alpha[F(Y_j)] + nf \qquad \text{w.p.1.}$$

Proof : The joint rank R_j can be expressed as

$$R_j = \{\#\text{ of } X\text{'s} \le Y_j\} + \{\#\text{ of } Y\text{'s} \le Y_j\}$$

$$= mF_m(Y_j) + nG_n(Y_j)$$

where F_m and G_n are the sample distribution functions of $X_1,\ldots,X_m$ and $Y_1,\ldots,Y_n$, respectively. Hence,

$$S_m = \Sigma_{j=1}^{n}\{e_L\alpha[R_j/(L+1)]+f_L\}$$

$$= e_L\Sigma_{j=1}^{n}\alpha[mF_m(Y_j)/(L+1)+nG_n(Y_j)/(L+1)] + nf_L.$$

Therefore,

$$\lim_{m\to\infty} S_m = e\Sigma_{j=1}^{n}\alpha[F(Y_j)] + nf \qquad \text{w.p.1.} \quad \|$$

Corollary 3.2.3. : Let $\{S_m\}_{m=1}^{\infty}$ be a sequence of linear rank statistics with score functions $\{a(x)\}$ satisfying Assumption 3, and let $f=\lim_{m\to\infty} f_L$ and $e=\lim_{m\to\infty} e_L$. If F and G are absolutely continuous distributions, then

$$\lim_{m\to\infty} P(S_m \le x) = P\{e\Sigma_{j=1}^{n}\alpha[F(Y_j)]+nf \le x\}.$$

If, in addition, H_0: F=G is true and F(.) is absolutely continuous, then the limiting null distribution of S_m is

$$\lim_{m\to\infty} P_0(S_m \leq x) = P[e\Sigma_{j=1}^{n}\alpha(W_j)+nf \leq x]$$

where $W_1,\ldots,W_n$ is a random sample from a uniform (0,1) distribution.

Proof : The proof follows at once from Theorem 3.2.3. ‖

3.3 GENERAL PROPERTIES OF A NONPARAMETRIC CONTROL CHART

Suppose that there is a control procedure using only the statistic S, either a linear placement statistic or a linear rank statistic. In the structure of the control chart when the control value is not given, the S_i's are not independent and it is not obvious whether or not the control chart is nonparametric distribution-free. The following theorem shows that the properties are still nonparametric distribution-free when the process is under control even though the S_i's are not independent. We let the notation '$\equiv$' denote 'equal in distribution' throughout this dissertation.

Theorem 3.3.1. : Let $\underline{X}=(X_1,\ldots,X_m)$ and $\underline{Y}_i=(Y_{i1},\ldots,Y_{in})$, $i=1,2,\ldots$, be independent random samples with the same continuous distribution function F, and let

$$\underline{V}=(V_1,\ldots,V_m)=(F(X_1),\ldots,F(X_m))$$

and

$$\underline{W}_i=(W_{i1},\ldots,W_{in})=(F(Y_{i1}),\ldots,F(Y_{in})).$$

If the test statistic S satisfies

$$S(\underline{X},\underline{Y}_i)|\underline{X} \equiv S(\underline{V},\underline{W}_i)|\underline{V} \text{ for } i\geq 1, \tag{3.3.1}$$

where $S(\underline{X},\underline{Y}_i)|\underline{X}$ and $S(\underline{V},\underline{W}_i)|\underline{V}$ denote the statistic S conditioned on $\underline{X}$ and $\underline{V}$, respectively, then $P(S(\underline{X},\underline{Y}_1)<s_1,\ldots,S(\underline{X},\underline{Y}_k)<s_k)$ is the same for all continuous distributions F.

Proof : Let $S_i=S(\underline{X},\underline{Y}_i)$ and $T_i=S(\underline{V},\underline{W}_i)$ for $i=1,2,\ldots,k$. Then, for every $k\geq 1$,

$$P(S_1<s_1,\ldots,S_k<s_k)$$

$$= \int\ldots\int P(S_1<s_1,\ldots,S_k<s_k|\underline{x})dF(x_1)\ldots dF(x_m)$$

$$= \int\ldots\int P(S_1<s_1|\underline{x})\ldots P(S_k<s_k|\underline{x})dF(x_1)\ldots dF(x_m)$$

by independence of S_i's given $\underline{X}$

$$= \int\ldots\int P(T_1<s_1|\underline{v})\ldots P(T_k<s_k|\underline{v})dv_1\ldots dv_m$$

by the given condition (3.3.1)

$$= P(T_1<s_1,\ldots,T_k<s_k).$$

The random variables V_i and W_{ij} follow a uniform (0,1) distribution for every continuous function F. Therefore the T_i's are always functions only of uniform (0,1) random variables through the statistic S and the proof is done. ‖

The condition (3.3.1) implies that the distribution of the statistic based on $\underline{X}$ and $\underline{Y}_i$ given $\underline{X}$ is invariant to the transformation F of $\underline{X}$ and $\underline{Y}_i$. If the statistic S_i uses only the joint ranks of $\underline{X}$ and $\underline{Y}_i$ or the placements of $\underline{Y}_i$ among the X's, then the condition is satisfied because neither the ranks nor the placements are changed by the transformation F of $\underline{X}$ and $\underline{Y}_i$. This theorem ensures that the run length distribution of any control chart, which uses only the statistic S_i for the stopping rule, is always the same regardless of the underlying distribution when the process is under control. This is one of the two main advantages of the nonparametric control charts; the other is that these charts are more efficient than the parametric control charts, based on normality assumptions, for heavy-tailed distributions as we show in later developments.

The following theorem can be applied to obtain the asymptotic properties of the proposed control charts when only the standard sample size goes to infinity. As indicated before, the standard sample size is usually considerably larger than or the observed sample size, since it is necessary to obtain precise information about the process in a base period when the process is under control. Hence, it is natural to investigate the asymptotic properties for the case when only the standard sample size goes to infinity instead of the case when both sample sizes go to infinity.

Theorem 3.3.2. : For either the linear placement statistic or the linear rank statistic, the statistics $\{S_i; i=1,2,..\}$ are asymptotically independent as $m\to\infty$.

Proof : The linear placement statistic is a function only of $F_m(Y)$'s and the linear rank statistic is a function only of $F_m(Y)$ and $G_n(Y)$; the joint rank can be expressed as $mF_m(Y)+nG_n(Y)$ where G_n is the sample distribution of the observed sample. Therefore the proof is complete if we show that $F_m(Y_i)$ and $F_m(Y_j)$ are asymptotically independent for $i\neq j$.

For $0\leq a,b\leq 1$,

$$\lim_{m\to\infty} P(F_m(Y_i)\leq a, F_m(Y_j)\leq b)$$

$$=\lim_{m\to\infty}\iint P(F_m(Y_i)\leq a, F_m(Y_j)\leq b \mid Y_i=v, Y_j=w)dG(v)dG(w)$$

$$=\iint\lim_{m\to\infty} P(F_m(v)\leq a, F_m(w)\leq b)dG(v)dG(w)$$

by the dominated convergence theorem (Theorem 3.4.3)

$$=\int\int P(F(v)\leq a, F(w)\leq b)dG(v)dG(w)$$

by Glivenko-Cantelli (Theorem 3.4.4)

$$=\int P(F(v)\leq a)dG(v)\ \int P(F(w)\leq b)dG(w)$$

and

$$\lim_{m\to\infty} P(F_m(Y)\leq a)=\int P(F(v)\leq a)dG(v)$$

by a similar method.

Thus $F_m(Y_i)$ and $F_m(Y_j)$ are asymptotically independent for $i\neq j$. ‖

3.4 SOME MISCELLANEOUS THEOREMS

This section contains some miscellaneous theorems which will be needed in the sequel.

Theorem 3.4.1.(Jensen's inequality) : If k(.) is a convex function and X is a real-valued random variable such that E[X] exists, then

$$E[k(X)] \geq k[E(X)]. \quad \|$$

Theorem 3.4.2. (Monotone convergence theorem) : if $0 \leq f_n \uparrow f$ almost everywhere, then

$$\int f_n \, d\mu \uparrow \int f \, d\mu$$

where μ is a probability measure. ‖

Theorem 3.4.3. (Dominated convergence theorem) : If $|f_n| \leq g$ almost everywhere, where g is integrable, and if $f_n \to f$ almost everywhere, then f and f_n are integrable and

$$\int f_n \, d\mu \to \int f \, d\mu$$

where μ is a probability measure. ‖

Theorem 3.4.4. (Glivenko-Cantelli theorem) : Suppose that X_1, $X_2, \ldots$ are independent and have a common distribution function F. Let $D_n = \sup_x |F_n(x) - F(x)|$ for $F_n(x) = \Sigma_{k=1}^{n} I_{(-\infty, x]}(X_k)$. Then $D_n \to 0$ with probability 1 and also $F_n(x)$ converges uniformly (in x) to $F(x)$. ‖

Theorem 3.4.5. (Theorem 16.6, Billingsley, 1979) : If $f_n \geq 0$, then $\int \Sigma_n f_n d\mu = \Sigma_n \int f_n d\mu$

where μ is a probability measure. ‖

Let a stochastic process $\{X(t), t \geq 0\}$ with a continuous time parameter possess stationary independent increments with $P(X(0)=0|\mu)=0$ for all μ. Suppose that the SPRT accepts or rejects $H_0:\mu=\mu_0$ according as the lower or the upper inequality in $b<X(t)<a$ is first violated. If both strict inequalities hold, the test continues.

The operating characteristic (OC) function is defined as the probability that the SPRT accepts H_0, and the average sample time (AST) is defined as the expectation of the sample time, which is the random time at which the SPRT ceases to observe $X(t)$ by accepting or rejecting H_0. Then the OC function and the AST of the sequential test can be approximated using the Wald approximations [see Wald(1947)].

<u>Theorem</u> <u>3.4.6.</u>(<u>Ghosh</u>, <u>1970</u>) : For $b<0<a$, the OC function is

$$Q(\mu) \simeq \{e^{ah(\mu)}-1\}/\{e^{ah(\mu)}-e^{bh(\mu)}\}, \quad E(X(1)|\mu)\neq 0$$

$$\simeq a/(a-b) \qquad\qquad , \quad E(X(1)|\mu)=0$$

and the AST is

$$E(t|\mu) \simeq \{a-(a-b)Q(\mu)\}/E\{X(1)|\mu\}, \quad E(X(1)|\mu)\neq 0$$

$$\simeq -ab/E\{X^2(1)|\mu\} \qquad , \quad E(X(1)|\mu)=0$$

where $h(\mu)$ is the unique nonzero root h of the moment generating function of $X(1)$, i.e. $E\{e^{hX(1)}|\mu\}$. ‖

Let z stand for any of a set of complex numbers and $i=\sqrt{-1}$. The residue α of $f(z)$ at the singular point a can be obtained as

$$\alpha = \lim_{z \to a} f(z)(z-a).$$

<u>Theorem</u> <u>3.4.7.</u>(<u>The</u> <u>residue</u> <u>theorem</u>) [see Spiegel(1964)] : Let $f(z)$ be single-valued and analytic inside and on a simple closed curve C except at the singularities $a,b,c,\ldots$ inside C which have residues given by $\alpha,\beta,\gamma,\ldots$. Then

$$\int_C f(z)\,dz = 2\Pi i(\alpha+\beta+\gamma+\ldots).\quad \|$$

Chapter IV

A NONPARAMETRIC SHEWHART CHART

4.1 INTRODUCTION AND BASIC RESULTS

In this chapter, a nonparametric Shewhart chart, based on two-sample nonparametric statistics calculated from the standard sample $\underline{X}$ and the sequentially observed sample $\underline{Y}_i$, is discussed when the control value is not specified. Temporarily we assume that there is no truncation point in the control chart.

The procedure of the Shewhart chart is to obtain a test statistic $S_i=S(\underline{X},\underline{Y}_i)$ at each successive time point, and to signal at the first time that the statistic falls outside the control limits. The region outside the control limits is called the action region and is denoted by C. The statistics considered in this dissertation are either linear placement statistics or linear rank statistics. The choice of the statistic depends on the parameter to be monitored.

When a control value is known (i.e. $\underline{X}$ is unobserved and $S_i=S(\underline{Y}_i)$), the procedure of the corresponding Shewhart chart is equivalent to a sequence of independent tests and the run length follows a geometric distribution. However, when a specified control value is not given, the procedure entails a sequence of dependent test statistics $\{S(\underline{X},\underline{Y}_i); i=1,2,\ldots\}$. Hence the run length distribution is not geometric and can not be characterized in terms of the probability of a signal at each point.

From the fact that the conditional distribution of the test statistic S_i given the standard sample entails an independent sequence, the run length N satisfies

$$\begin{aligned} P(N>t) &= P(S_i \notin C,\ i=1,2,\ldots,t) \\ &= \int [P(S_i \notin C \mid \underline{X}=\underline{x})]^t \, dF_{\underline{X}}(\underline{x}) \\ &= E[P(S \notin C \mid \underline{X})^t] \end{aligned} \tag{4.1.1}$$

where $F_{\underline{X}}$ is the distribution function of $\underline{X}$. From the relation $E[P(S \notin C|\underline{X})^t] \geq [EP(S \notin C|\underline{X})]^t$ obtained from Jensen's inequality (Theorem 3.4.1), the following inequality holds.

$$\begin{aligned} P(N>t) &\geq [EP(S \notin C \mid \underline{X})]^t \\ &= [P(S \notin C)]^t. \end{aligned} \tag{4.1.2}$$

An expression for the ARL can be obtained easily by

$$\begin{aligned} EN &= \Sigma_{t=0}^{\infty} P(N>t) \\ &= \Sigma_{t=0}^{\infty} E[P(S \notin C \mid \underline{X})^t] \\ &= E\{1/P(S \varepsilon C \mid \underline{X})\} \end{aligned} \tag{4.1.3}$$

by Theorem 3.4.5 and summing the geometric series.

Also by Jensen's inequality,

$$EN \geq 1/E\{P(S\varepsilon C|\underline{X})\}$$

$$= 1/P(S\varepsilon C). \tag{4.1.4}$$

The expressions (4.1.2) and (4.1.4) correspond to (2.2.2) and (2.2.3) for the geometric case and can be used as lower bounds for P(N>t) and the ARL, respectively.

In the following Theorem 4.1.1, the asymptotic distribution of the run length is derived when only the standard sample size approaches infinity. Assumption 3 for the linear rank statistic and the Assumptions 1 and 2 for the linear placement statistic are used to insure the convergence of the distribution of the run length.

<u>Theorem</u> <u>4.1.1.</u> :

i) If $P(S\varepsilon C|\underline{X})>0$ for almost every $\underline{X}$, then $P(N<\infty)=1$.

ii) Suppose that Assumptions 1, 2 and 3 hold for the corresponding statistics. Then, for fixed n, the run length distribution satisfies the relation

$$\lim_{m\to\infty} P(N>t) = [\lim_{m\to\infty} P(S\notin C)]^t$$

$$= [P(S'\notin C)]^t$$

for the linear placement statistic, and

$$\lim_{m\to\infty} P(N>t) = [P(S''\notin C)]^t$$

for the linear rank statistic, where $S' = \Sigma_{j=1}^{n} \phi[F(Y_j)]$ and $S'' = e\Sigma_{j=1}^{n} \alpha[F(Y_j)] + nf$.

Proof :

i) For any positive integer t,

$$P(N>t) = \int [P(S\notin C|\underline{X}=\underline{x})]^t \, dF_{\underline{X}}(\underline{x}).$$

Since $[P(S\notin C|\underline{X}=\underline{x})]^t \to 0$ as $t\to\infty$ and $[P(S\notin C|\underline{X}=\underline{x})]^t < 1$ for almost every $\underline{x}$, it follows from the dominated convergence theorem (Theorem 3.4.3) that

$$\lim_{t\to\infty} P(N>t) = \int \lim_{t\to\infty} [P(S\varepsilon C|\underline{X}=\underline{x})]^t \, dF_{\underline{X}}(\underline{x})$$

$$= 0.$$

Hence $P(N<\infty)=1$.

ii) From Theorem 3.3.2

$$\lim_{m\to\infty} P(N>t) = \lim_{m\to\infty} P(S_i \notin C,\ i=1,\ldots,t)$$

$$= \lim_{m\to\infty} [P(S\notin C)]^t$$

$$= [\lim_{m\to\infty} P(S\notin C)]^t$$

The rest of the theorem follows immediately by applying the Corollary 3.2.2 and 3.2.3. ‖

Theorem 4.1.1.i) tells that, when a specified control value is not given, the Shewhart chart will eventually signal if the probability that S belongs to C is positive for almost every $\underline{X}$, which is also true for the case when the control value is given. Theorem 4.1.1.ii) tells that, as the standard sample size m approaches to infinity, the run length distribution converges to a geometric distribution with a parameter $P(S' \varepsilon C)$ or $P(S'' \varepsilon C)$ for the linear placement statistic or the linear rank statistic, respectively, and the lower bound (4.1.2) of $P(N>t)$ is attained. Similar results were obtained for the Shewhart X-chart by Ghosh, Reynolds and Hui (1981) when the unknown variance is estimated from the standard sample.

The following theorem shows that the ARL of the Shewhart chart may not be finite for some action regions.

Theorem 4.1.2. : The ARL under control does not exist if the action region is contained in any one of the three events,

$$\{\underline{Y} \mid \text{all } Y\text{'s} > X_{(k)} \text{ for any } k=m-n+1,\ldots,m\} \qquad (1)$$

$$\{\underline{Y} \mid \text{all } Y\text{'s} < X_{(k)} \text{ for any } k=1,\ldots,n\} \qquad (2)$$

$\{\underline{Y} | q\ Y's > X_{(k)}$ and $(n-q)\ Y's < X_{(\ell)}$, $\ell < k$ for any

$k = m-q+1, \ldots, m$ and any $\ell = 1, \ldots, n-q\}$ (3)

where $X_{(k)}$ denotes the k-th order statistic of $\underline{X}$.

Proof : When the action region belongs to (3), the ARL under control satisfies

$$E_0 N = E[E_0(N | \underline{X})]$$

$$= E_0\{1/P(q\ Y's > X_{(k)},\ (n-q)\ Y's < X_{(\ell)})\}$$

$$= [m!/\{(\ell-1)!(k-\ell-1)!(m-k)!\binom{n}{q}\}] \int_0^1 y_1^{\ell-n+q-1}$$

$$\int_0^1 (y_2-y_1)^{k-\ell-1}(1-y_2)^{m-k-q}\, dy_2 dy_1$$

This integral does not exist if $m-k-q<0$ or $\ell-n+q-1<0$. The rest of the theorem can be easily shown using the fact that the action region (1) or (2) is a special case of action region (3) for $q=n$ or 0. ||

This theorem shows that, when the sequence of the test statistics is dependent, the ARL under control is not finite for some small action regions. Ghosh, Reynolds and Hui (1981) also showed that, under certain conditions, the ARL of the Shewhart $\overline{X}$-chart with estimated variance is not finite when the distribution is normal.

The fact that the ARL is not finite is possible only when we assume that the process will be continued forever. Thus we now apply the truncation point T to the proposed control charts. In the presence of the truncation point, the stopping rule of the control chart is changed to the following rule:

Obtain the standard sample $\underline{X}$, and signal at the first i for which

$$S_i \varepsilon C \quad \text{if} \quad i<T$$

or stop at T if $S_i \notin C$ for all $i<T$

From now on, the character T will appear as a superscript of each parameter of the control chart (e.g. N^T, ARL^T) to denote the case where the truncation point is used. Because of the truncation point, the equations (4.1.1), (4.1.2), (4.1.3), (4.1.4) are changed to

$$P(N^T>t) = \begin{cases} E[P(S\notin C|\underline{X})]^t, & t\leq T-1 \\ 0 & , t\geq T \end{cases} \tag{4.1.5}$$

$$\geq [P(S\notin C)]^t, \quad t\leq T-1 \tag{4.1.6}$$

$$EN^T = \Sigma_{t=0}^{T-1} P(N^T>t)$$

$$= \Sigma_{t=0}^{T-1} E[P(S\notin C|\underline{X})]^t \tag{4.1.7}$$

$$\geq \{1-[P(S\notin C)]^T\}/P(S\varepsilon C) \tag{4.1.8}$$

Also the expressions (4.1.6) and (4.1.8) can be used as lower bounds for $P(N^T>t)$ and the ARL^T, respectively. It is easy to see that Theorem 4.1.1 also holds for the Shewhart chart with the truncation point T. The inequaliy (4.1.6) implies that the run length N^T is stochastically larger than a geometric random variable with the parameter $P(S\varepsilon C)$.

It is of interest to see the behavior of the difference $EN^T-\{1-[P(S\notin C)]^T\}/P(S\varepsilon C)$ according to the value of the truncation point and the size of the standard sample. The difference may be large when there is no truncation point (i.e. at $T=\infty$) or the size of the standard sample is small. The following theorem shows how the difference is related to the trunction point T and the sample size m.

<u>Theorem</u> <u>4.1.3.</u> : In the nonparametric Shewhart chart using the statistic S with the truncation point T,

i) $EN^T-\{1-[P(S\notin C)]^T\}/P(S\varepsilon C)$ is a nondecreasing function of T.

ii) $\lim_{m\to\infty} EN^T = \{1-[P(S'\notin C)]^t\}/P(S'\varepsilon C)$ for the linear placement statistic, and

$\lim_{m\to\infty} EN^T = \{1-[P(S''\varepsilon C)]^t\}/P(S''\varepsilon C)$ for the linear rank statistic, where S' and S" are defined in Theorem 4.1.1.

<u>Proof</u> :

i) Let $D^T = EN^T-\{1-[P(S\notin C)]^T/P(S\varepsilon C)$, then by (4.1.7)

$$D^T = \Sigma_{t=0}^{T-1}\{E[P(S\notin C|\underline{X})]^t-[P(S\notin C)]^t\}$$

and

$$D^{T-1} = \Sigma_{t=0}^{T-2}\{E[P(S\varepsilon C|\underline{X})]^t-[P(S\notin C)]^t\}$$

Hence,

$$D^T-D^{T-1} = E[P(S\varepsilon C|\underline{X})]^{T-1}-[P(S\notin C)]^{T-1}$$

$$\geq 0$$

by Jensen's inequality. Therefore D^T is a nondecreasing function of T.

ii) $$\lim_{m\to\infty} EN^T = \lim_{m\to\infty} \Sigma_{t=0}^{T-1}P(N^T>t)$$

$$= \Sigma_{t=0}^{T-1} \lim_{m\to\infty}P(N^T>t)$$

$$= \Sigma_{t=0}^{T-1} [\lim_{m\to\infty}P(S\notin C)]^T$$

by Theorem 4.1.1.ii)

$$= \{1-[\lim_{m\to\infty}P(S\notin C)]^T\}/[\lim_{m\to\infty}P(S\varepsilon C)].$$

The rest of the theorem follows immediately by applying Corollaries 3.2.2 and 3.2.3. ‖

This theorem implies that the ARL of the Shewhart chart using a dependent sequence of statistics will be closer to the lower bound (4.1.8) as the truncation point gets smaller. In another words, (4.1.8) becomes a sharper lower bound as T decreases. Also, this theorem tells that the ARL approaches to the lower bound (4.1.8) as the standard sample size goes to infinity.

When the parameter of the process shifts from that of the standard sample, a reasonable control chart should signal faster than when the parameter of the process remains unchanged. Hence it is desired that the ARL out of control should be smaller than the ARL under control. In the following theorem, the stochastic order of the run lengths of two processes is found when the two processes have different amounts of shift. Stochastic order of run lengths in monitoring multivariate processes was studied by Hui (1980). Hui studied the problem based on the assumption that the observed samples are i.i.d. and the same assumption is used in the following theorem. Let the parameter of interest be θ (either the location parameter or the scale parameter) and suppose that only positive shifts of θ are going to be detected and the control limit of the test statistic S is c.

<u>Theorem</u> <u>4.1.4.</u> : Suppose that two processes are monitored by a Shewhart chart with i.i.d. observed samples. Let the run lengths and the amounts of the shift of the parameter θ for the two processes be denoted by N_1 and N_2, and Δ_1 and Δ_2, respectively, where $\Delta_1<\Delta_2$. If the statistic $S(\underline{x},\underline{y}+e\underline{1})$ [or $S(\underline{x},\underline{y}e),e>0$] is a nondecreasing function of e for the location parameter (or the scale parameter) θ, where $\underline{1}=(1,\ldots,1)$, then N_1 is stochastically larger than N_2.

<u>Proof</u> : Suppose that $\underline{Y}_i$ and $\underline{Z}_i$, i=1,2,..., are the i-th observed samples from processes having the shifts Δ_1 and Δ_2, respectively, of the location parameter θ. Then,

$$\underline{Y}_i + (\Delta_2-\Delta_1)\underline{1} \equiv \underline{Z}_i .$$

Let $d_i=S[\underline{X},\underline{Y}_i+(\Delta_2-\Delta_1)\underline{1}]-S(\underline{X},\underline{Y}_i)$, i=1,2,..., and by the nondecreasing property of S, $d_i\geq 0$ for i=1,2,....
The fact that measurable functions of two random variables which are 'equal in distribution' are also 'equal in distribution' gives, for i=1,2,...,

$$S(\underline{X},\underline{Z}_i)\equiv S(\underline{X},\underline{Y}_i+(\Delta_2-\Delta_1)\underline{1})$$

$$=S(\underline{X},\underline{Y}_i)+d_i.$$

Thus again,

$$\max_{1\le i\le t}\{S(\underline{X},\underline{Z}_i)\}$$

$$\equiv \max_{1\le i\le t}\{S(\underline{X},\underline{Y}_i)+d_i\}$$

$$\ge \max_{1\le i\le t}\{S(\underline{X},\underline{Y}_i)\}.$$

Therefore, for every $t>0$,

$$P[N_2\le t]=P[\max_{1\le i\le t}\{S(\underline{X},\underline{Z}_i)\}\ge c]$$

$$=P[\max_{1\le i\le t}\{S(\underline{X},\underline{Y}_i+d_i\underline{1})\}\ge c]$$

$$\ge P[\max_{1\le i\le t}\{S(\underline{X},\underline{Y}_i)\}\ge c]$$

$$=P[N_1\le t].$$

For the scale parameter θ, let $\underline{Y}_i$ and $\underline{Z}_i$ be the same as before and let $d_i=S[\underline{X},\underline{Y}_i(\Delta_2/\Delta_1)]-S(\underline{X},\underline{Y}_i)$, for $i=1,2,\ldots.$ Then

$$\underline{Y}_i(\Delta_2/\Delta_1) \equiv Z_i$$

and the rest of the proof can be done easily by the same method as before. $\|$

It is natural to assume that the nondecreasing property of S is satisfied if an appropriate statistic for detection of the parameter θ is selected. One easy way of selecting an appropriate statistic for a control chart is to choose a test which is used in the hypothesis testing problem for the parameter θ.

Now, procedures using some well known nonparametric statistics will be discussed. In each case, the run length distribution and the ARL are studied. The usual parameters of interest are the process mean and the variance. For detecting shifts in the process mean, the Wilcoxon-Mann-Whitney statistic (3.2.1) and the median placement statistic (3.2.3) are used, and for detecting shifts of the process variance, the sum of the squared ranks statistic (3.2.7) is used. If the parameter of interest is the mean, then the usual objective of process control is to detect shifts in the positive or negative or both directions. If the parameter of interest is the variance, then shifts only in the positive direction usually are to be detected. But, for convenience, for both the mean and variance we consider only one sided Shewhart charts for detecting positive changes. Detecting negative shifts can be done by using a lower control limit, and detecting shifts in both directions can be obtained readily from the one sided procedure. Nonparametric Shewhart charts using the aforementioned statistics are compared to the corresponding parametric procedures in terms of the ARL in Chapter 6. Chapter 6 also contains comparisons with CUSUM charts.

4.2 THE WILCOXON-MANN-WHITNEY STATISTIC

Suppose that the parameter of interest is the location parameter of a process and that a standard sample is given instead of a specified control value. One natural nonparametric approach to process control for the location parameter is to use the Wilcoxon-Mann-Whitney test, i.e. at every time a sample is observed, to apply the Wilcoxon-Mann-Whitney test for the hypothesis

$$H_0\colon G(x) = F(x) \quad \text{vs.} \quad H_1\colon G(x) = F(x-\Delta) \qquad (4.2.1)$$

for all x and $\Delta>0$ and to continue sampling until the null hypothesis is rejected or until the truncation point is reached. At the first time the null hypothesis is rejected, the process will be stopped and a rectifying action will be taken.

The test statisitc S_i^W (3.2.1) is equal to $\{\Sigma_{j=1}^{n} R_{ij} - n(n+1)/2\}/m$, where $\Sigma_{j=1}^{n} R_{ij}$ is the Wilcoxon rank sum statistic (R_{ij} is the joint rank of Y_{ij} among $(Y_{i1},\ldots,Y_{in},X_1,\ldots,X_m)$). Hence this statistic is a special case of the linear placement statistic and the linear rank statistic.

The run length distribution (4.1.5) and the ARL^T (4.1.7) can be written in terms of multiple integrals with respect to the standard sample $\underline{X}$:

$$P(N^T \le t) = 1 - E[P(S^W < c \mid \underline{X})]^t$$

by (4.1.5), so that

$$P(N^T \le t) = 1 - \int [P(S^W < c \mid \underline{x})]^t dF_X(\underline{x}) \qquad (4.2.2)$$

if $t \le T-1$ and $P(N^T \le t) = 1$ elsewhere. Moreover,

$$EN^T = \Sigma_{t=0}^{T-1} E[P(S^W < c) \mid \underline{X})]^t$$

by (4.1.7), so that

$$EN^T = \Sigma_{t=0}^{T-1} \int [P(S^W < c \mid \underline{x})]^t dF_X(\underline{x}). \qquad (4.2.3)$$

The asymptotic distribution ($m \to \infty$) of the Wilcoxon-Mann-Whitney statistic is, from Theorem 4.1.1,

$$\lim_{m\to\infty} P(N^T \le t) = 1 - [P(\Sigma_{j=1}^{n} F(Y_j) < c)]^t$$

$$= 1 - [P(\Sigma_{j=1}^{n} W_j < c)]^t, \qquad (4.2.4)$$

if $\Delta=0$ where W_j, $j=1,2,..,n$, is a uniform (0,1) random variable. Also from Theorem 4.1.3.ii),

$$\lim_{m\to\infty} EN^T = \{1 - [P(\Sigma_{j=1}^{n} F(Y_j) < c)]^T\} / P(\Sigma_{j=1}^{n} F(Y_j) < c)$$

$$=\{1-[P(\Sigma_{j=1}^{n}W_j<c)]^T\}/P(\Sigma_{j=1}^{n}W_j<c), \qquad (4.2.5)$$

if $\Delta=0$. A Table for the distribution of the sum of uniform random variables is in Buckle, Kraft and Eeden (1969). Also the exact expression for the density function of the sum of uniform random variables was obtained by Cramer (1945).

To obtain the unconditional run length distribution (4.2.2) and ARL (4.2.3), the conditional probability $P(S_i^W<c|\underline{X})$ should be expressed as a function only of $\underline{X}$. For this, however, all combinations of $\underline{X}$ and $\underline{Y}_i$ should be considered and it becomes more complicated as the sample sizes increase. Therefore, although analytic expressions for the run length distribution and the ARL are available, the calculations are practically impossible except for a few cases.

An example of one of the exceptions is the case when c=n. If c=n, then

$$P(S_i^W<c|\underline{X}) = 1-P(\text{all } Y_i\text{'s}>X_{(m)})$$

$$= 1-[1-F(X_{(m)}-\Delta)]^n$$

where $X_{(m)}$ is the m-th order statistic of $\underline{X}$.

Thus, for $t\leq T-1$,

$$P(N^T\leq t)=1-\int\{1-[1-F(y-\Delta)]^n\}^t m[F(y)]^{m-1}f(y)dy$$

$$=1-\int_0^1\{1-[1-F(F^{-1}(z)-\Delta)]^n\}^t mz^{m-1}dz \qquad (4.2.6)$$

and if $\Delta=0$

$$P(N^T\leq t)=1-\int_0^1\{1-[1-z]^n\}^t mz^{m-1}dz. \qquad (4.2.6')$$

Further,

$$EN^T=\Sigma_{t=0}^{T-1}\int\{1-[1-F(y-\Delta)]^n\}^t m[F(y)]^{m-1}f(y)dy$$

$$=\Sigma_{t=0}^{T-1}\int_0^1\{1-[1-F(F^{-1}(z)-\Delta)]^n\}^t mz^{m-1}dz \qquad (4.2.7)$$

and if $\Delta=0$

$$EN^T=\Sigma_{t=0}^{T-1}\int_0^1\{1-[1-z]^n\}^t mz^{m-1}dz. \qquad (4.2.7')$$

The calculations of (4.2.6) and (4.2.7) can be obtained easily by numerical methods.

For a general idea of the run length distribution and the ARL, bounds for $P(N^T\leq t)$ and the ARL^T can be obtained by simply obtaining bounds for $P(S^W<c|\underline{X})$.

Since mU_{ij} = {# of X's $\leq Y_{ij}$},

$$mS_i^W = m\Sigma_{j=1}^n F_m(Y_{ij})$$

$$= \Sigma_{j=1}^n\{\#\text{ of }X\text{'s} \leq Y_{ij}\}$$

and, thus, for $c=n-k/m$, $k=0,1,2,\ldots,mn$

$$P(S_i^W\geq n-k/m|\underline{X})=P(mS_i^W\geq mn-k|\underline{X})$$

$$=P(\Sigma_{j=1}^n\{\#.\text{of}.X\text{'s}\leq Y_{ij}\}\geq mn-k|\underline{X}) \qquad (4.2.8)$$

The following theorem uses the relation (4.2.8).

Theorem 4.2.1. :

For $k=0,n,2n,\ldots,(m-1)n$,

$$P(S^W \geq n-k/m|\underline{X}) \geq P(\text{all } Y\text{'s} \geq X_{(m-k/n)}). \qquad (4.2.9)$$

For $k=0,1,2,\ldots,m-1$,

$$P(S^W \geq n-k/m|\underline{X}) \leq P(\text{all } Y\text{'s} \geq X_{(m-k)}). \qquad (4.2.10)$$

Equality holds when $k=0$.

Proof : Let $a_{ij}=mU_{ij}$, $j=1,2,\ldots,n$. Then, from (4.2.8),

$$\{S_i^W \geq n-k/m|\underline{X}\} = \{\Sigma_{j=1}^n a_{ij} \geq mn-k|\underline{X}\}.$$

For $k=0,n,\ldots,(m-1)n$, if all a_{ij}'s are greater than or equal to $m-k/n$, then $\Sigma_{j=1}^n a_{ij}$ is always greater than or equal to $mn-k$, and thus from the above equality,

$$\{S_i^W \geq n-k/m|\underline{X}\} \supseteq \{\text{all } a_{ij}\text{'s} \geq m-k/n|\underline{X}\}.$$

Hence (4.2.9) holds. For $k=0,1,\ldots,m-1$, if Σa_{ij} is greater than or equal to $mn-k$, then the minimum possible value of a_{ij} is $m-k$, and thus all a_{ij}'s are greater than or equal to $m-k$. Hence, from the above equality,

$$\{S_i^W \geq n-k/m|\underline{X}\} \subseteq \{\text{all } a_{ij}\text{'s} \geq m-k|\underline{X}\}.$$

Hence (4.2.10) holds. From (4.2.9) and (4.2.10), when $k=0$,

$$P(S_i^W=n|\underline{X}) = P(\text{all } Y\text{'s}>X_{(m)}). \quad \|$$

Let $f(.)$ be the probability density function of X and $A=m!/[(m-k/n-1)!(k/n)!]$. From (4.2.9), for $k=0,n,2n,\ldots,(m-1)n$,

$$P(S_i^W<n-k/m|\underline{X}) \leq 1-[P(Y\geq X_{(m-k/n)})]^n$$

hence, from (4.2.2) and (4.2.3),

$$P(N^T\leq t)\geq 1-\int\{1-[P(Y\geq y)]^n\}^t$$

$$A[F(y)]^{m-k/n-1}[1-F(y)]^{k/n}f(y)dy$$

$$=1-\int_0^1\{1-[1-(F(F^{-1}(z)-\Delta))]^n\}^t$$

$$Az^{m-k/n-1}[1-z]^{k/n}dz \tag{4.2.11}$$

and if $\Delta=0$ the inequality becomes

$$P(N^T\leq t)\geq 1-\int_0^1\{1-(1-z)^n\}^t$$

$$Az^{m-k/n-1}[1-z]^{k/n}dz, \tag{4.2.11'}$$

Also

$$EN^T\leq\Sigma_{t=0}^{T-1}\int\{1-[P(Y\geq y)]^n\}^t$$

$$A[F(y)]^{m-k/n-1}[1-F(y)]^{k/n}f(y)dy$$

$$=\Sigma_{t=0}^{T-1}\int_0^1\{1-[1-F(F^{-1}(z)-\Delta)]^n\}^t$$

$$Az^{m-k/n-1}(1-z)^{k/n}dz \tag{4.2.12}$$

and if $\Delta=0$ the inequality becomes

$$EN^T\leq\Sigma_{t=0}^{T-1}\int_0^1\{1-[1-z]^n\}^t$$

$$Az^{m-k/n-1}(1-z)^{k/n}dz. \tag{4.2.12'}$$

Let $B=m!/[(m-k-1)!k!]$. From (4.2.10), for $k=0,1,2,\ldots,m-1$,

$$P(S^W<n-k/m|\underline{X})\geq1-[P(Y\geq X_{(m-k)})]^n$$

hence, from (4.2.2) and (4.2.3),

$$P(N^T\leq t)\leq1-\int_0^1\{1-[1-F(F^{-1}(z)-\Delta)]^n\}^t$$

$$Bz^{m-k-1}(1-z)^kdz \tag{4.2.13}$$

and if $\Delta=0$

$$P(N^T\leq t)\leq1-\int_0^1\{1-[1-z]^n\}^t$$

$$Bz^{m-k-1}(1-z)^kdz. \tag{4.2.13'}$$

Moreover,

$$EN^T\geq\Sigma_{t=0}^{T-1}\int_0^1\{1-[1-F(F^{-1}(z)-\Delta)]^n\}^t$$

$$Bz^{m-k-1}(1-z)^kdz \tag{4.2.14}$$

and if $\Delta=0$

$$EN^T \geq \Sigma_{t=0}^{T-1} \int_0^1 \{1-[1-z]^n\}^t$$

$$Bz^{m-k-1}(1-z)^k dz. \qquad (4.2.14')$$

The calculations for (4.2.11), (4.2.12), (4.2.13) and (4.2.14) also can be done by numerical methods.

The upper bound for $P(N^T \leq t)$ may be improved by taking the minimum of (4.2.13) and $1-[P(S^W<c)]^t$ from (4.1.6). Also the lower bound for the ARL^T may be improved by taking the maximum of (4.2.14) and (4.1.8).

Example 4.2.1. :

i) Let m=39, n=10, T=1000, c=10-100/39, Δ=0. From (4.2.9) in Theorem 4.2.1,

$$P[S^W \geq c] \geq P[\text{all } Y\text{'s} \geq X_{(29)}]$$

and by (4.2.12') using numerical methods,

$$EN^T \leq 845.57.$$

By (4.1.8) and $P(S^W \geq 10-100/39)=.008775$ (by interpolation from $P(S^W \geq 10-93/39)=.0051$ and $P(S^W \geq 10-101/39)=.0093$ in Selected Tables in Mathematical Statistics (1973)), we have

$$EN^T \geq 114.27.$$

A simulation result from 500 replications shows EN^T=411.23 and the asymptotic ARL^T, by (4.2.5) using the table given by Buckle, Kraft and Eeden (1969), is 299.83.

ii) Let m=5, n=5, c=5, T=1000, Δ=0. By (4.2.7') using numerical methods,

$$EN^T = 787.33. \quad \|$$

Unfortunately, the bounds calculated from (4.2.12), (4.2.14) and (4.1.8) do not set very tight limits on the values of the ARL. However, the lower and upper bounds come closer to each other when c approaches n. Thus the bounds can give at least an idea of the ARL, although such wide limits as in Example 4.2.1.i) are not going to be very helpful in practice. The asymptotic ARL^T is smaller than the simulated ARL^T and the difference is not negligible. Thus it seems that the sample size m=39 is not large enough to give a good approximation to the ARL^T.

4.3 THE MEDIAN PLACEMENT STATISTIC

Suppose shifts of the process mean in the positive direction are to be detected and a standard sample is given instead of a specified control value for the mean. The next approach to process control for the mean is to use the median placement statistic, i.e. obtain the median of the standard sample for use as the control value and apply the sign test for the hypothesis (4.2.1) and continue sampling until the null hypothesis is rejected.

In the Shewhart chart using the median placement statistic S_i^M given by (3.2.3), the exact calculations for $P(N^T \leq t)$ and the ARL^T are simple relative to the case when the Wilcoxon-Mann-Whitney statistic is used since the median placement statistic depends only on the median of the standard sample.

The probability $P(S_i^M < c|\underline{X})$ is the same as P(at most (c-1) Y's>$X_{(M)}$), (for convenience, let m be an odd number, i.e. $M=(m+1)/2$). Thus,

$$P(S_i^M < c|\underline{X}) = \Sigma_{k=0}^{c-1}\binom{n}{k}\{1-G(X_{(M)})\}^k\{G(X_{(M)})\}^{n-k}$$
$$= 1-\Sigma_{k=c}^{n}\binom{n}{k}\{1-G(X_{(M)})\}^k\{G(X_{(M)})\}^{n-k},$$

where $G(x)=F(x-\Delta)$. Hence, from (4.1.5) and (4.1.7),

$$P(N^T \leq t) = 1-E[P(S^M < c|\underline{X})]^t$$
$$= 1-\int\{1-\Sigma_{k=c}^{n}\binom{n}{k}[1-G(y)]^k[G(y)]^{n-k}\}^t$$
$$m!/[(M-1)!]^2\{[1-F(y)]F(y)\}^{M-1}f(y)dy$$
$$= 1-\int_0^1\{1-\Sigma_{k=c}^{n}\binom{n}{k}[1-F(F^{-1}(z)-\Delta)]^k$$
$$[F(F^{-1}(z)-\Delta)]^{n-k}\}^t$$

$$m!/[(M-1)!]^2\{[1-z]z\}^{M-1}dz, \tag{4.3.1}$$

and if $\Delta=0$

$$P(N^T\leq t)=1-\int_0^1\{1-\Sigma_{k=c}^{n}\binom{n}{k}[1-z]^k z^{n-k}\}^t$$

$$m!/[(M-1)!]^2\{[1-z]z\}^{M-1}dz. \tag{4.3.1'}$$

Moreover,

$$EN^T=\Sigma_{t=0}^{T-1}\int_0^1\{1-\Sigma_{k=c}^{n}\binom{n}{k}[1-F(F^{-1}(z)-\Delta)]^k[F(F^{-1}(z)-\Delta)]^{n-k}\}^t$$

$$m!/[(M-1)!]^2\{[1-z]z\}^{M-1}zdz, \tag{4.3.2}$$

and if $\Delta=0$

$$EN^T=\Sigma_{t=0}^{T-1}\int_0^1\{1-\Sigma_{k=c}^{n}\binom{n}{k}[1-z]^k z^{n-k}\}^t$$

$$m!/[(M-1)!]^2\{[1-z]z\}^{M-1}zdz. \tag{4.3.2'}$$

The asymptotic distribution ($m\to\infty$) of the median placement statistic is, from Theorem 4.1.1,

$$\lim_{m\to\infty}P(N^T\leq t)=1-[P(\Sigma_{j=1}^{n}\phi(F(Y_j))<c)]^t$$

$$=1-[P(B<c)]^t \tag{4.3.3}$$

where B is a binomial random variable with parameters n and $P(F(Y_j)\geq 1/2)$. Under control, B is a binomial random variable with parameters n and 1/2. Also, from Theorem 4.1.3.ii),

$$\lim_{m\to\infty} EN^T = \{1-[P(B<c)]^T\}/P(B\geq c) \qquad (4.3.4)$$

Example 4.2.1. :

Let m=39, n=10, T=100, c=9, Δ=0. By (4.3.2') using numerical methods,

$$EN^T=178.65$$

The asymptotic ARL^T, from (4.3.4) and P(B<9)=.989, is 90.91. As in Example 4.2.1.i), the asymptotic ARL^T is smaller than the true ARL^T, and the difference is not negligible either. Thus it also seems that the sample size m=39 is not large enough to give a good approximation to the ARL^T. ‖

4.4 THE SUM OF THE SQUARED RANKS STATISTIC

Suppose that the parameter of interest is the process variance and a standard sample is used instead of a specified control value for the variance. One approach among many nonparametric statistics for the scale testing problem is to use the sum of the squared ranks test S_i^S given by (3.2.7), i.e. at every time a sample is observed, apply the sum of the squared ranks test for the hypothesis

$$H_0: G(x) = F(x) \text{ vs. } H_1: G(x) = F(x/\Delta) \qquad (4.4.1)$$

for all x and $\Delta>1$ and continue sampling until the null hypothesis is rejected, at which time the process is stopped and a rectifying action is taken.

The expressions for the run length distribution and the ARL^T are the same as (4.2.2) and (4.2.3) for the Wilcoxon-Mann-Whitney statistic except S^W is changed to S^S. The expression for the conditional probability $P(S^S<c|\underline{X})$ is more complicated than that of the Wilcoxon-Mann-Whitney statistic because the statistic is the sum of the squared ranks while the Wilcoxon-Mann-Whitney statistic is simply the sum of the ranks. Hence, the exact calculation of the probability is practically impossible except for extreme values of the control limit c. Therefore, the study of the Shewhart chart using the sum of the squared ranks statistic will be done by obtaining the bounds and the asymptotic properties.

By (4.1.6), the upper bound for the probability of a signal by t is

$$P(N^T\leq t) \leq 1-[P(S^S<c)]^t \tag{4.4.2}$$

and by (4.1.8), the lower bound for the ARL^T is

$$EN^T \geq \{1-[P(S^S<c)]^T\}/P(S^S\geq c) \tag{4.4.3}$$

A table of exact probabilities for the sum of the squared ranks statistic is available for a few sample sizes in Whiteside, Duran and Boullion (1975).

The asymptotic distribution ($m \to \infty$) of the sum of the squared ranks statistic is, from Theorem 4.1.1,

$$\lim_{m\to\infty} P(N^T \leq t) = 1 - [P(\Sigma_{j=1}^{n} F^2(Y_j) < c)]^t, \qquad (4.4.4)$$

and if $\Delta=0$

$$\lim_{m\to\infty} P(N^T \leq t) = 1 - [P(\Sigma_{j=1}^{n} W_j^2 < c)]^t, \qquad (4.4.4')$$

where W_j, $j=1,2,..,n$, is a uniform (0,1) random variable. Also, from Theorem 4.1.3.ii),

$$\lim_{m\to\infty} EN^T = \{1 - [P(\Sigma_{j=1}^{n} F^2(Y_j) < c)]^T\} / P(\Sigma_{j=1}^{n} F^2(Y_j) < c), \qquad (4.4.5)$$

and if $\Delta=0$

$$\lim_{m\to\infty} EN^T = \{1 - [P(\Sigma_{j=1}^{n} W_j^2 < c)]^T\} / P(\Sigma_{j=1}^{n} W_j^2 < c). \qquad (4.4.5')$$

Unfortunately, there is no probability table available for the sum of the squared uniform (0,1) random variables for the calculation of (4.4.4') or (4.4.5'). Either a numerical method or the normal approximation may be used for the sum of the squared uniform random variables. The normal approximation can be easily applied using the null mean and variance of the sum of the squared ranks statistic (3.2.8).

Chapter V

A NONPARAMETRIC CUSUM CHART

5.1 THE DERIVATION OF THE RUN LENGTH DISTRIBUTION

In this chapter, the nonparametric CUSUM chart, based on a two-sample nonparametric statistic S calculated from the standard sample $\underline{X}$ and the sequentially observed sample $\underline{Y}_i$, is discussed when the control value is not specified. Before discussing the CUSUM chart when the control value is not given, first, consider the CUSUM chart for the case when the control value is given.

Suppose that the procedures (2.4.1) and (2.4.2) for the Brownian motion process are used to approximate the CUSUM procedures (2.3.1) and (2.3.2). Also suppose that the truncation point T is applied to the CUSUM procedures. Then the approximated procedure is to stop at the first time that the inequality (2.4.1) or (2.4.2) holds. If the inequality does not hold for all $i<T$, stop at T. An expression for the distribution of the stopping time τ of the approximated procedure had not been developed, and hence it was necessary to develop the distribution of the stopping time in order to investigate the properties of the CUSUM chart with the truncation point T.

When the truncation point T is used with the procedure, the expectation of τ can be obtained as

$$E\tau^T = \int_0^T t f_\tau(t)dt + T\int_T^\infty f_\tau(t)dt \qquad (5.1.1)$$

Thus, to derive $E\tau^T$ or other properties of the CUSUM chart with the truncation point T, it is necessary to obtain the p.d.f. of τ, $f_\tau(t)$, which can be obtained from the inverse Laplace transform which is defined as

$$f_\tau(t) = (1/2\Pi i)\int_{c-i\infty}^{c+i\infty} L_\tau(\lambda)e^{\lambda t}d\lambda, \qquad (5.1.2)$$

where $i=\sqrt{-1}$ and c is a real positive constant. This inverse Laplace transform is obtained from the residue theorem (Theorem 3.3.5) and is given in Theorem 5.1.1 below.

Theorem 5.1.1. : By means of the residue integral, the probability density function of τ is

$$\begin{aligned} f_\tau(t) &= \Sigma_{j=1}^\infty Q_j(t), && \text{if } \mu<\sigma^2/h \text{ and } \mu\neq 0 \\ &= \Sigma_{j=0}^\infty R_j(t), && \text{if } \mu=0 \qquad (5.1.3) \\ &= \Sigma_{j=0}^\infty Q_j(t), && \text{otherwise} \end{aligned}$$

where

$$Q_j(t) = (A_j/B_j)e^{-C_j t}, \; j=0,1,\ldots$$

$$R_j(t) = D_j e^{-E_j t}, \; j=0,1,\ldots$$

$A_0=q_0\sigma^2 e^{-\mu h/\sigma^2}$

$B_0=h^2\sinh(q_0)+h^2(\sigma^2-\mu h)\cosh(q_0)/(q_0\sigma^2)$

$C_0=(\mu^2h^2-q_0^2\sigma^4)/(2\sigma^2h^2)$

$A_j=q_j\sigma^2 e^{-\mu h/\sigma^2}$, $j=1,2....$

$B_j=h^2\sin(q_j)-h^2(\sigma^2-\mu h)\cos(q_j)/(q_j\sigma^2)$, $j=1,2...$

$C_j=(\mu^2h^2+q_j^2\sigma^4)/(2\sigma^2h^2)$, $j=1,2...$

$D_j=(-1)^j(j+1/2)\Pi^2\sigma^2/h^2$, $j=1,2...$

$E_j=(j+1/2)^2\Pi^2\sigma^2/2h^2$, $j=1,2...$

and q_0 is the solution of

$$\mu h\tanh(q_0) - \sigma^2 q_0 = 0, \qquad (5.1.4)$$

and the q_j's are the sequence of the solutions of

$$\mu h\tan(q_j) - \sigma^2 q_j = 0. \qquad (5.1.5)$$

Proof : Appendix A. ‖

Corollary 5.5.1. : The expectation of τ with the truncation point T is

$E\tau^T=\Sigma_{j=1}^{\infty}A_j(1-e^{-C_jT})/(B_jC_j^2)$, $\mu<\sigma^2/h$ and $\mu\neq 0$

$$=\Sigma_{j=0}^{\infty} D_j(1-e^{-E_jT})/E_j^2, \ \mu=0 \qquad (5.1.6)$$

$$=\Sigma_{j=0}^{\infty} A_j(1-e^{-C_jT})/(E_jC_j^2), \text{ otherwise}$$

where A_j, B_j, C_j, D_j and E_j are defined in Theorem 5.1.1.

Proof : Appendix A. ‖

The solutions for the nonlinear equations (5.1.4) and (5.1.5) can be easily obtained by a numerical method called the bisection method [see Conte and de Boor (1972)]. Because tan(x) is very sensitive to a small change of x when x is near $n\Pi/2$, $n=\pm1,\pm3,\pm5,\ldots$, it is better to use the equation

$$\mu h\sin(q_j) - \sigma^2 q_j\cos(q_j) = 0 \qquad (5.1.7)$$

instead of (5.1.5). In calculating $f_\tau(t)$ or $E\tau^T$ using (5.1.3) or (5.1.6), we add terms sequentially in the summation and stop adding as soon as the absolute value of the current term is less than some given tolerance level. The FORTRAN computer programs for the expectation of τ are listed in Appendix B.

The accuracy of the results can be checked for the case $T=\infty$ where a closed form for $E\tau$ is available. Table 1 gives $E\tau$ calculated by the equation (5.1.6) with $T=\infty$ and $E\tau$ calculated by the equation (2.4.5).

TABLE 1

The comparison of $E\tau$

μ	Exact	inv.Lap.	Iter
-0.9	5.5494	5.5494	389
-0.7	6.9086	6.9086	268
-0.5	9.1078	9.1078	185
-0.3	13.1433	13.1433	128
-0.1	21.9780	21.9780	88
0.1	46.2179	46.2180	61
0.3	131.1574	131.1150	19
0.5	501.3750	498.0826	13
0.7	2470.858	2478.332	8
0.9	13454.51	13211.60	5

$h=5.55$, $\sigma^2=1$, tolerance level$=10^{-5}$
Exact: $E\tau$ calculated by (2.4.5)
inv.Lap.: $E\tau$ calculated by (5.1.6) using the inverse Laplace transform
Iter: # of iterations used in inv.Lap.

In the procedures (2.3.1), (2.3.2), (2.4.1) and (2.4.2) it can be easily seen that when S_j and $X(t)$ are standardized (i.e. divided by the standard deviation σ) the procedure remains the same if standardized parameters are also used (i.e. the parameters k and h are divided by the standard deviation σ). Thus, without loss of generality, we can work with a Brownian motion process with mean μt and variance t. Table 1 shows that the values obtained by the inverse Laplace transform for the case $T=\infty$ are very accurate. When the mean of τ is positive, the expectations obtained by the inverse Laplace transform are close to the exact values and converge very quickly. When the mean is negative, the expectations are more accurate than when the means are positive but the series converges at a slow rate. Also, if a smaller tolerance level is given, the result can be more accurate.

The exact probability density function $f_\tau(t)$ of the stopping time τ of the procedure, which is designed for monitoring shifts in the negative direction, is obtained by the inverse Laplace transform and plotted in Figure 1 as a function of the mean μ.

In the figure the density tends to have more mass to the left as the mean decreases from 0.5 to -0.5. Hence, it can be easily seen that the control chart will signal more quickly as the mean decreases.

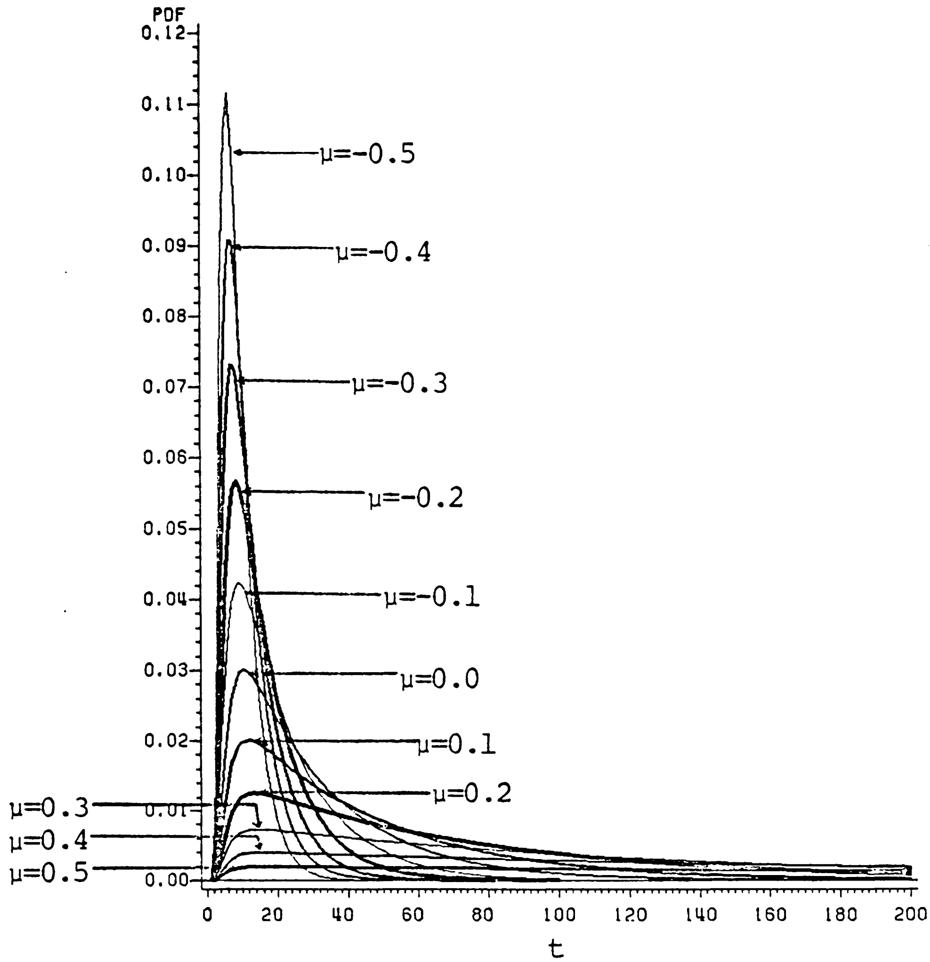

Figure 1: Probability density function of τ

5.2 A PROPER CHOICE OF THE REFERENCE VALUE

As is discussed in Chapter 3, one of the advantages of a nonparametric control chart is that the run length distribution is the same for all continuous distributions when the process is under control. However, if the parameter k is chosen as $\pm(\mu_0+\mu_1)/2$ from (2.4.8) to minimize $ARL_{\mu=\mu_1}$, then the whole control procedure is not nonparametric distribution-free. This is because $\mu_1(=E_{\Delta=\Delta_1}S)$ depends on the distribution of the observations when $\Delta\neq0$.

To attain the nonparametric distribution-free property, we instead seek a procedure which minimizes the ARL for an arbitrary small amount of shift. From (2.4.8), the value of k which minimizes the ARL at $\mu=\mu_0+\Delta_1$ is $(2\mu_0+\Delta_1)/2$ for the approximated CUSUM procedure (2.4.1). Thus if we let Δ_1 approach to 0 and choose the parameter k as

$$\begin{aligned} k &= \lim_{\Delta_1\to0} (2\mu_0+\Delta_1)/2 \\ &= \mu_0, \end{aligned} \tag{5.2.1}$$

then this choice of k minimizes the ARL of the procedure (2.4.1) in the neighborhood of the parameter value of the standard sample and also makes the CUSUM chart nonparametric distribution-free when the process is under control.

Therefore, from now on, $k=\mu_0$ is used for the CUSUM chart for detecting shifts in the positive direction and, similarly, $k=-\mu_0$ is used for detecting shifts in the negative direction. The idea of choosing k as (5.2.1) is that we simply choose the parameter k that is sensitive to small shifts, since they are hardest to detect. Typically, if a CUSUM chart with a reference value $k=\mu_0$ is sensitive to small shifts, although it may not be best among the CUSUM charts with other reference values, then it would still be sensitive to large shifts which are easier to detect than small shifts.

5.3 THE CONTINUITY-CORRECTED BOUNDARY

A defect of the Brownian motion approximation for the discrete random walk is that in many cases it is not sufficiently accurate. Especially in the case of the CUSUM procedure, the Brownian motion approximation underestimates the actual ARL. Using the fact that

$$ARL = AST/[1-Q(\mu)] \tag{5.3.1}$$

where AST is the average sample time and $Q(\mu)$ is the operating characteristic function in the SPRT (see Theorem 3.4.6), Kahn (1978) improved the Brownian motion approximation (2.4.6) by considering the excess over the boundaries (0,h) in the $Q(\mu)$ and the AST expressions. His refined ARL for detecting shifts in the positive direction is as follows:

$$ARL_k=[h+\xi(\mu)+\{e^{-2\mu h}-1\}/(2\mu)]/\mu, \quad \text{if } \mu>0$$

$$=[h+\xi(\mu) \qquad (5.3.2)$$

$$+q(\mu)\{\exp(-2\mu(h+\xi(\mu)))-1\}/(1-\eta(\mu))]/\mu, \quad \text{if } \mu<0$$

where $\xi(\mu)=\mu+\psi(\mu)/\Phi(\mu)$

$$\eta(\mu)=(1-\Phi(\mu))/\Phi(\mu)$$

$$q(\mu)=\mu-\psi(-\mu)/\Phi(-\mu)$$

$$\psi(\mu)=(1/\sqrt{2\Pi})e^{-\mu^2/2}$$

$$\Phi(\mu)=\int_{-\infty}^{\mu}\psi(x)dx.$$

There is no expression available when $\mu=0$.

Table 2 gives the ARL obtained by the Brownian motion approximation (2.4.6), Kahn's refined approximation (5.3.2) and exact calculation. The exact values for the ARL are calculated by van Dobben De Bruyn (1968) by solving an integral equation numerically for the case where the observations have a normal distribution.

According to van Dobben De Bruyn (1968), those values are correct to one unit in the last decimal place given.

TABLE 2

The comparison of the ARL

	μ	B.M.	Kahn	Exact
	1.2	40.18	528.95	610
	0.8	15.88	116.18	114
	0.4	7.35	31.14	28
h=2	0.0	4.00	-	10.0
	-0.4	2.51	4.91	5.06
	-0.8	1.75	3.21	3.24
	-1.2	1.32	2.51	2.38
	0.8	2321.84	14740.70	14000
	0.4	154.99	453.44	414
h=5	0.0	25.00	-	104
	-0.4	9.43	11.84	12.4
	-0.8	5.47	6.93	6.99
	-1.0	4.50	5.79	5.75
	0.4	9287.45	25776.00	200000
	0.0	100.00	-	126
h=10	-0.4	21.88	24.28	24.9
	-0.8	11.72	13.18	13.2
	-1.0	9.50	10.79	10.70

The ARL by the Browinian motion approximation is always smaller than the exact values and the discrepancy is serious, while the ARL by Kahn's refined approximation is very close to the exact values. Generally, the ARL by Kahn's refined approximation overestimates the ARL slightly. When the mean is very close to 0, however, Kahn's approximation should not be used. The reason is that ARL_k does not converge when the mean approaches to 0. i.e.

$$\lim_{\mu \to 0+} ARL_k = \infty$$

and (5.3.3)

$$\lim_{\mu \to 0-} ARL_k = \infty$$

These limits can be derived from (5.3.2) by L'Hospital's rule. Kahn's refined ARL in the neighborhood of $\mu=0$ is given in Table 3 which shows a very large ARL when μ is close to 0. Therefore, instead of using (5.3.2) in the neighborhood of $\mu=0$, interpolation is used for the ARL whose mean is close to 0.

From Table 3 it is seen that the ARL is overestimated when the mean is approximately in the region (0.5/h,-0.002). Hence, the interpolation of the ARL_k is calculated by

$$ARL_k(\mu)=wARL_k(0.5/h)+(1-w)ARL_k(-0.002) \qquad (5.3.4)$$

TABLE 3

Kahn's ARL when the mean is close to 0

μ	h=2	h=5	h=10
-0.1	14.1316	65.0020	322.9800
-0.05	12.6224	50.9658	197.1141
-0.01	11.5810	42.6545	140.5636
-0.002	11.4010	41.3706	132.3491
0.002	403.2971*	424.1443*	497.9893*
0.006	137.3143*	157.8573*	229.4638*
0.012	70.7920*	90.8845*	159.3131*
0.02	44.1552*	63.6732*	128.1598
0.05	20.0727*	37.6327	89.9025
0.1	11.8693*	26.7473	65.1201
0.14	9.4254	22.5730	53.5478
0.2	7.4920	18.5670	42.1043

* indicates the ARL is overestimated

for $-0.002<\mu<0.5/h$ where $w=(\mu+0.002)/(0.5/h+0.002)$ and $ARL_k(x)$ is the Kahn's refined ARL with the mean x.

Siegmund (1979) also used the continuity correction for the excess over the boundaries in obtaining several quantities similar to the operating characteristic function in a sequential probability ratio test. He suggested that one approximate these quantities by computing their analogues for a Brownian motion process with the continuity-corrected stopping boundaries. In the CUSUM chart, the continuity correction to the boundary h can be done in such a way that the ARL with a continuity-corrected boundary h' is the same as the Kahn's refined ARL with the original boundary h. That is, the continuity correcting procedure is to find h' such that

$$ARL(h')=ARL_k(h) \tag{5.3.5}$$

where $ARL(h')$ and $ARL_k(h)$ are the corresponding ARL's with boundaries h' and h. The continuity-corrected boundaries h' are obtained in Table 4 for different h and μ.

We see that the differences h'-h are almost the same for different h and change very slowly for different means. Thus Reynolds' (1975) artificial modification, to add approximately 1.2 to the original boundary h in order to improve the Brownian motion approximation, seems reasonable except for relatively small and large values of the mean.

TABLE 4

The continuity-corrected boundary of the CUSUM chart

μ	h=2	h=5	h=10
-1.9	3.9677	6.9676	11.9676
-1.6	3.7179	6.7173	11.7174
-1.3	3.4918	6.4897	11.4897
-1.0	3.2961	6.2876	11.2876
-0.7	3.1466	6.1124	11.1119
-0.4	3.1104	5.9743	10.9621
-0.1	3.8906	6.2391	10.9527
0.1	3.3416	6.3679	11.3789
0.4	3.2600	6.2726	11.2737
0.7	3.1774	6.1812	11.1813
1.0	3.1016	6.1026	11.1026
1.3	3.0345	6.0347	11.0347
1.6	2.9752	5.9753	10.9753
1.9	2.9222	5.9223	10.9223

The continuity-corrected boundary h' will be used in obtaining the properties of the CUSUM chart to improve the Brownian motion approximation.

5.4 A NONPARAMETRIC CUSUM CHART WHEN THE CONTROL VALUE IS NOT SPECIFIED

So far the CUSUM chart with the control value given has been discussed. Now we consider the case when the control value is not given.

As in Chapter 3, let $\underline{X}=(X_1,...,X_m)$ be the standard sample and $\underline{Y}_i=(Y_{i1},...,Y_{in})$, i=1,2,... be groups of independent observations taken sequentially. Assume $\underline{X}$ and $\underline{Y}_i$, i=1,2,... are independent random samples from continuous distributions with distribution functions F and G, respectively. The test statistic S belongs to either the class of the linear rank statistics or the class of the linear placement statistics. Suppose that the standard sample $\underline{X}$ has the parameter $\theta=\theta_0$, which is the unknown location or scale parameter, and let Δ denote the true amount of shift of the parameter θ of the observed samples $\underline{Y}_i$, i=1,2,... ($\Delta>0$ if θ is the scale parameter). Because the statistic S belongs to either a linear rank statistic or a linear placement statistic, the conditional mean and variance of S given the standard sample $\underline{X}$, denoted by

$\mu_\Delta(\underline{X})=E_\Delta(S|\underline{X})$ and $\sigma^2_\Delta(\underline{X})=Var_\Delta(S|\underline{X})$, respectively, can be expressed as functions of F, $\underline{X}$ and Δ. When there is a shift of size Δ in the parameter θ of $\underline{Y}_i$, then the distribution function G(x) is equivalent to either $F(x-\Delta)$ or $F(x/\Delta)$ depending on whether the shift occurred in the location parameter or the scale parameter.

The stopping rule for the CUSUM chart is the same as (2.3.1) and (2.3.2) except that the statistic S is a two-sample statistic rather than a one-sample statistic. Because each observed sample is compared to the same standard sample, the statistics S_i's are no longer independent. Thus there are difficulties in applying the Brownian motion approximation to the CUSUM chart using S_i. Fortunately, under the condition that the standard sample $\underline{X}$ is given, the statistics S_i's are i.i.d. random variables and the Brownian motion approximation can be applied on the condition that $\underline{X}$ is given. Denote these conditional statistics by $S_i(\underline{X})$ and the conditional parameter k by $k(\underline{X})$.

Now, given the standard sample $\underline{X}$, we consider the stopping rule which stops at the first i for which

$$\Sigma_{j=1}^{i}[S_j(\underline{X})-k(\underline{X})]-\min_{0\le \ell\le i}\Sigma_{j=1}^{\ell}[S_j(\underline{X})-k(\underline{X})]\underline{>}h \qquad (5.4.1)$$

for detecting shifts in the positive direction, and

$$\max_{0\le \ell\le i}\Sigma_{j=1}^{\ell}[S_j(\underline{X})+k(\underline{x})]-\Sigma_{j=1}^{i}[S_j(\underline{X})+k(\underline{X})]\underline{>}h \qquad (5.4.2)$$

for detecting shifts in the negative direction, where $\Sigma_{j=1}^{0}[S_j(\underline{X})-k(\underline{X})]=\Sigma_{j=1}^{0}[S_j(\underline{X})+k(\underline{X})]=0$. If the inequality (5.4.1) or (5.4.2) does not hold for all $i<T$, stop at T. For detecting shifts in both directions, we apply the rules (5.4.1) and (5.4.2) simultaneously. We assume that large values of $S(\underline{X})$ tend to indicate positive shifts and small values of $S(\underline{X})$ tend to indicate negative shifts.

If the conditional run length distribution $P(N^T\leq t|\underline{X})$ and the conditional ARL^T, $E(N^T|\underline{X})$, can be derived for the stopping rules (5.4.1) and (5.4.2), then the unconditional run length distribution and the ARL^T can be obtained as $P(N^T\leq t)=E[P(N^T\leq t|\underline{X})]$ and $EN^T=E[E(N^T|\underline{X})]$.

If Z(t) is a Brownian motion process with mean $\mu'(\underline{X})t=E[S(\underline{X})]t$ and the variance $\sigma^2(\underline{X})t=Var[\dot{S}(\underline{X})]t$, then the following two procedures are considered to approximate the conditional procedures (5.4.1) and (5.4.2):

Signal at the smallest t for which

$$[Z(t)-k(\underline{X})t]-\inf_{0<s<t}[Z(s)-k(\underline{X})s]\geq h'(\underline{X}) \qquad (5.4.3)$$

for detecting shifts in the positive direction, and

$$\sup_{0<s<t}[Z(s)+k(\underline{X})s]-[Z(t)+k(\underline{X})s]\geq h'(\underline{X}) \qquad (5.4.4)$$

for detecting shifts in the negative direction, where $h'(\underline{X})$ is the continuity-corrected boundary defined by (5.3.5). If the inequality (5.4.3) or (5.4.4) does not hold for all $t<T$, stop at T. Note that the continuity-corrected boundary $h'(\underline{X})$ depends on $\underline{X}$. This is because the Kahn's (1978) refined ARL depends on $\mu'(\underline{X})$ and thus a different continuity-corrected boundary will be obtained from (5.3.5).

Consider the procedures for detecting shifts in the negative direction with stopping rule given by (5.4.4). Let $\mu(\underline{X})=\mu'(\underline{X})+k(\underline{X})$ and define the stopping time

$$\tau(\underline{X})=\inf\{t: \sup_{0<s<t} [Z(s)+k(\underline{X})s]-[Z(t)+k(\underline{X})s]\geq h'(\underline{X})\} \quad (5.4.5)$$

and the conditional probability density function of $\tau(\underline{X})$

$$f_{\tau(X)}(t) = \partial P(\tau(\underline{X})\leq t)/\partial t. \quad (5.4.6)$$

Then the Laplace transform of $f_{\tau(X)}(t)$ is exactly the same as (2.4.5) except that μ and σ^2 are replaced by $\mu(\underline{X})$ and $\sigma^2(\underline{X})$. Therefore, the probability density function of $\tau(\underline{X})$ is given in Theorem 5.1.1 and the expectation of $\tau^T(\underline{X})$, which denotes $\tau(\underline{X})$ with the truncation point T, is given in Corollary 5.1.1 where μ and σ^2 are replaced by $\mu(\underline{X})$ and $\sigma^2(\underline{X})$.

Using these expressions, the unconditional probability density function of τ and expectation of τ^T can be expressed as

$$f_\tau(t) = E[f_{\tau(X)}(t)]$$

$$= \int f_{\tau(x)}(t)dF_{\underline{X}}(\underline{x}) \qquad (5.4.7)$$

and

$$E\tau^T = E\{E[\tau^T(\underline{X})]\}$$

$$= \int E\tau^T(\underline{x})dF_{\underline{X}}(\underline{x}). \qquad (5.4.8)$$

Detailed explanations about evaluating (5.4.7) and (5.4.8) for each statistic used will appear in the sections later in this chapter.

As in the Shewhart chart, there exists a stochastic order in the run lengths as given by the following theorem.

Theorem 5.4.1. : Suppose that two processes are monitored by a CUSUM chart with i.i.d. observed samples. Let the run lengths and the amounts of the shift of the parameter θ for the two processes be denoted by N_1 and N_2, and Δ_1 and Δ_2, respectively, where $\Delta_1<\Delta_2$. If the statistic $S(\underline{x},\underline{y}+e\underline{1})$ [or $S(\underline{x},\underline{y}e),e>0$] is a nondecreasing function of e for the location parameter (or the scale parameter) θ, where $\underline{1}=(1,..,1)$, then N_1 is stochastically larger than N_2.

Proof : Suppose that $\underline{Y}_i$ and $\underline{Z}_i$, i=1,2,..., are the i-th observed samples with the shifts of the location parameter θ of size Δ_1 and Δ_2, respectively. Then,

$$\underline{Y}_i + (\Delta_2-\Delta_1)\underline{1} \equiv \underline{Z}_i.$$

Let $d_i=S[\underline{X},\underline{Y}_i+(\Delta_2-\Delta_1)\underline{1}]-S(\underline{X},\underline{Y}_i)$, $i=1,2,...,$ then by the nondecreasing condition of S, $d_i\geq 0$ for $i=1,2,...$. For convenience, let $S_{1i}=S(\underline{X},\underline{Y}_i)$ and $S_{2i}=S(\underline{X},\underline{Z}_i)$ for $i=1,2,...$ The fact that measurable functions of two random variables which are 'equal in distribution' are also 'equal in distribution' gives, for $i=1,2,...,$

$$\begin{aligned} S_{2i} &\equiv S(\underline{X},\underline{Y}_i+(\Delta_2-\Delta_1)\underline{1}) \\ &= S_{1i}+d_i \end{aligned}$$

Let

$$A(t)=\max_{1\leq i\leq t}\{\Sigma_{j=1}^{i}[S_{2j}-k]-\min_{0\leq \ell\leq i}\Sigma_{j=1}^{\ell}[S_{2j}-k]\},$$

$$B(t)=\max_{1\leq i\leq t}\{\Sigma_{j=1}^{i}[S_{1j}+d_j-k]-\min_{0\leq \ell\leq i}\Sigma_{j=1}^{\ell}[S_{1j}+d_j-k]\},$$

and

$$C(t)=\max_{1\leq i\leq t}\{\Sigma_{j=1}^{i}[S_{1j}-k]-\min_{0\leq \ell\leq i}\Sigma_{j=1}^{\ell}[S_{1j}-k]\},$$

then again

$$A(t)\equiv B(t)\geq C(t).$$

Therefore, for every $t>0$,

$$\begin{aligned} P[N_2\leq t] &= P[A(t)\geq h] \\ &= P[B(t)\geq h] \end{aligned}$$

$$\geq P[C(t)\geq h]$$

$$= P[N_1\leq t].$$

For the scale parameter θ, let $\underline{Y}_i$ and $\underline{Z}_i$ be the same as before and let $d_i=S[\underline{X},\underline{Y}_i(\Delta_2/\Delta_1)]-S(\underline{X},\underline{Y}_i)$, for $i=1,2,\ldots.$ Then

$$\underline{Y}_i(\Delta_2/\Delta_1) \equiv \underline{Z}_i$$

and the rest of the proof can be done easily by the same method as before. $\|$

This theorem indicates that, when there is a shift, every percentile of the run length as well as the ARL are smaller than those for the case when the process is under control.

A proper choice of the parameter $k(\underline{X})$ can be done in a way similar to the case when the control value is given. For given $\underline{X}$, the reference value $k(\underline{X})$ is chosen as

$$k(\underline{X})=\mu_0(\underline{X})$$

for positive deviations, and as (5.4.9)

$$k(\underline{X})=-\mu_0(\underline{X})$$

for negative deviations, for the same reasons used in choosing the reference value k when the control value is given. After $k(\underline{X})$ is chosen, h will be determined to produce $ARL_{\mu=\mu_0}=A_0$ for given constant A_0.

As mentioned before, $\mu_\Delta(\underline{X})$ is a function only of $F,\underline{X}$ and Δ. Hence $\mu_o(\underline{X})$ is a function only of F and $\underline{X}$. However, the distribution function of the standard sample F was assumed to be unknown. Therefore the distribution function F must be estimated from the standard sample by some method such as a density estimation method. There have been many density estimation methods by many authors including Parzen (1962), Good and Gaskins (1971) and Theil and Laitinen (1980). The applications of the methods by Parzen, and Good and Gaskins are explained in Tapia and Thompson (1978). However, none of these methods gives a sufficiently good estimate for every distribution. One easy method is to use the sample distribution function of the standard sample. In this paper, we consider only the sample distribution function as an estimate of the distribution function F. The sample distribution function does not require any parameter values which might make the estimate unstable, as in the kernel method developed by Parzen or the penalized likelihood method of Good and Gaskins, and it is also easy to use in practice. Also the sample distribution function is much easier to calculate than the maximum entropy method of Theil and Laitinen, the kernel method or the penalized likelihood method.

For symmetry, the sample distribution function to be used in the sequel is defined as follows.

$$
\begin{aligned}
F_m(x) &= 0 && \text{if } x < X_{(1)} \\
&= j/m && \text{if } X_{(j)} < x < X_{(j+1)},\ j=2,..,m-1 \\
&= (j-0.5)/m && \text{if } x = X_{(j)},\ j=1,2,..,m \\
&= 1 && \text{if } x > X_{(m)}
\end{aligned}
\tag{5.4.10}
$$

where $X_{(j)}$ is the j-th order statistic of $\underline{X}$. Denote the choice of $k(\underline{X})$, where the sample distribution function F_m is used instead of F, by $k_m(\underline{X})$. The expression for $k_m(\underline{X})$ for each statistic is obtained in the section corresponding to that statistic.

The asymptotic ($m\to\infty$) properties of the CUSUM chart when the control value is not given also can be derived by Theorem 3.3.2. Because of the asymptotic independence of the S_i's, as $m\to\infty$, the conditional procedures (5.4.1) and (5.4.2) converge to the unconditional procedures (2.3.1) and (2.3.2) where $S_i=\lim_{m\to\infty}S(\underline{X},\underline{Y}_i)$ for a two-sample nonparametric distribution-free statistic S and $k=\lim_{m\to\infty}E_0S$ or $-\lim_{m\to\infty}E_0S$ according to the direction of the shift.

Hence, the asymptotic properties can be derived from the two procedures (2.4.1) and (2.4.2) where X(t) is a Brownian motion process with mean $(\lim_{m\to\infty} ES)t$ and the variance $(\lim_{m\to\infty} VarS)t$. Therefore the asymptotic run length distribution and the ARL^T can be obtained by the expressions (5.1.3) and (5.1.6) where $\mu=\lim_{m\to\infty}[ES-E_0S]$ and $\sigma^2=\lim_{m\to\infty} VarS$.

Now, the CUSUM procedures using some well known nonparametric statistics will be discussed. The statistics to be considered are the Wilcoxon-Mann-Whitney statistic (3.2.1), the median placement statistic (3.2.3) and the sum of the squared ranks statistic (3.2.7). For each of these statistics, an expression for the parameter $k(\underline{X})$ is obtained and hence, the run length distribution and the ARL^T are studied. As in the Shewhart chart, the Wilcoxon-Mann-Whitney statistic and the median placement statictic are used for detecting shifts of the process mean while the sum of the squared ranks statistic is used for detecting shifts of the process variance.

For convenience, we consider the CUSUM chart for detecting shifts only in the positive direction. Detecting shifts in the negative direction can be easily modified from the procedure for positive deviations as follows. If $S(\underline{X})$ has mean $\mu(\underline{X})$ and variance $\sigma^2(\underline{X})$, then $-S(\underline{X})$ has mean $-\mu(\underline{X})$ and variance $\sigma^2(\underline{X})$ and

$$\max_{1\le\ell\le i} \Sigma_{j=1}^{\ell} S_j(\underline{X}) = -\min_{1\le\ell\le i} \Sigma_{j=1}^{\ell}[-S_j(\underline{X})].$$

Thus,

$$\max_{1\le \ell \le i} \Sigma_{j=1}^{\ell} S_j(\underline{X}) - \Sigma_{j=1}^{i} S_j(\underline{X})$$

$$= \Sigma_{j=1}^{i}[-S_j(\underline{X})] - \min_{1\le \ell \le i} \Sigma_{j=1}^{\ell}[-S_j(\underline{X})] \tag{5.4.11}$$

which implies that the modification can be done by substituting $-[\mu(\underline{X})-k(\underline{X})]$ for $\mu(\underline{X})-k(\underline{X})$ in the expressions for the CUSUM chart for positive deviations.

The nonparametric CUSUM procedures using the three statistics are compared in Chapter 6 to the corresponding parametric procedures as well as the nonparametric Shewhart charts, using the same statistics, in terms of the ARL.

5.5 THE WILCOXON-MANN-WHITNEY STATISTIC

One natural approach to the CUSUM procedure is to use the Wilcoxon-Mann-Whitney statistic (3.2.1). It is not difficult to see that the mean and the variance of the Wilcoxon-Mann-Whitney statistic conditioned on the standard sample $\underline{X}$, $S^W(\underline{X})$, are

$$ES^W(\underline{X})=\mu_\Delta(\underline{X})=(n/m)\Sigma_{j=1}^{m}[1-F(X_j-\Delta)]$$

and (5.5.1)

$$\mathrm{Var}S^W(\underline{X})=\sigma_\Delta^2(\underline{X})=(n/m^2)\Sigma_{j=1}^{m}\Sigma_{k=1}^{m}$$

$$[1-F(\max(X_j,X_k)-\Delta)-(1-F(X_j-\Delta)F(X_k-\Delta)]$$

where Δ is the true amount of shift of the location parameter of the observed samples. Then, from (5.5.1) and (5.4.9), the reference value $k(\underline{X})$ is

$$k(\underline{X})=(n/m)\Sigma_{j=1}^{m}[1-F(X_j)] \qquad (5.5.2)$$

and the estimate of $k(\underline{X})$ using the sample distribution function is

$$k_m^W=k_m^W(\underline{X})=(n/m)\Sigma_{j=1}^{m}[1-F_m(X_j)]$$

$$=n/2. \qquad (5.5.3)$$

Note that the estimate of $k(\underline{X})$ does not depend on $\underline{X}$. This is always true if $k_m(\underline{X})$ can be expressed by the order statistics of $\underline{X}$ only through the sample distribution function F_m.

The run length of the procedure is approximated by a continuous random variable $\tau^T(\underline{X})$ which is the first time t for which

$$[Z^W(t)-(n/2)t]-\inf_{0<s<t}[Z^W(s)-(n/2)s]\geq h'(\underline{X}) \qquad (5.5.4)$$

if $t<T$, and $\tau^T(\underline{X})=T$ otherwise, where $Z^W(t)$ is a Brownian motion process with parameters $[ES^W(\underline{X})]t$ and $[VarS^W(\underline{X})]t$, and $h'(\underline{X})$ is the continuity-corrected boundary defined by (5.3.5).

The expressions for the unconditional run length distribution and the ARL^T, which are used for calculations, at the amount of shift Δ are

$$P_\Delta(\tau^T<t)=E\{P_\Delta[\tau^T(\underline{X})<t]\}$$
$$=\int P_\Delta[\tau^T(\underline{x})<t]dF_X(\underline{x})$$
$$=\int_I P_\Delta\{\tau^T[F^{-1}(\underline{y})]<t\}d\underline{y} \qquad (5.5.5)$$

by change of variables $y_j=F(x_j)$, $j=1,\ldots,m$ if $t<T$, otherwise $P_\Delta(\tau^T<t)=1$ and

$$E_\Delta\tau^T=E[E_\Delta\tau^T(\underline{X})]$$
$$=\int E_\Delta\tau^T(\underline{x})dF_X(\underline{x})$$
$$=\int_I E_\Delta\{\tau^T[F^{-1}(\underline{y})]\}d\underline{y} \qquad (5.5.6)$$

where $I=(0,1)^m$ and $F^{-1}(\underline{y})=[F^{-1}(y_1),\ldots,F^{-1}(y_m)]$. The advantage of the change of variable $y=F(x)$ is that the range of the integral reduces from $(-\infty,\infty)$ to $(0,1)$ which makes the numerical integration more accurate. The expressions $P_\Delta\{\tau^T[F^{-1}(\underline{y})]<t\}$ and $E_\Delta\{\tau^T[F^{-1}(\underline{y})]\}$ can be obtained from (5.1.3) and (5.1.6) where μ and σ^2 are replaced by

$$\mu_\Delta[F^{-1}(\underline{y})]-k_m^W=(n/m)\Sigma_{j=1}^m\{1-F[F^{-1}(y_j)-\Delta]\}-k_m^W, \qquad (5.5.7)$$

and if $\Delta=0$

$$\mu_\Delta[F^{-1}(\underline{y})]-k_m^W=(n/m)\Sigma_{j=1}^m(1-y_j)-K_m^W, \qquad (5.5.7')$$

also

$$\sigma_\Delta^2[F^{-1}(\underline{y})]=(n/m^2)\Sigma_{j=1}^m\Sigma_{k=1}^m$$
$$[1-F(F^{-1}(\max(y_j,y_k))-\Delta)-(1-F(F^{-1}(y_j)-\Delta))F(F^{-1}(y_k)-\Delta)] \qquad (5.5.8)$$

and if $\Delta=0$

$$\sigma_\Delta^2[F^{-1}(\underline{y})]=(n/m^2)\Sigma_{j=1}^m\Sigma_{k=1}^m[1-\max(y_j,y_k)-(1-y_j)y_k] \qquad (5.5.8')$$

To calculate the values of $P_\Delta(\tau^T<t)$ and $E_\Delta\tau^T$ using (5.5.5) and (5.5.6), it is necessary to evaluate the multiple integral of $P_\Delta\{\tau^T[F^{-1}(\underline{y})]<t\}$ and $E_\Delta\{\tau^T[F^{-1}(\underline{y})]\}$ with respect to $\underline{y}$ numerically. In using numerical integration, the difficulty in calculating the integral increases rapidly as the number of the variables increases. In the IMSL (International Mathematical and Statistical Libraries) subroutine package, the multiple integration subroutine can handle up to 20 independent variables.

The asymptotic ($m\to\infty$) run length distribution and the ARL^T are easily obtained by the expressions (5.1.3) and (5.1.6) where μ and σ^2 are replaced by

$$\mu = \lim_{m\to\infty}(ES^W-k_m^W)$$
$$= nE[F(Y)] - n/2 \qquad (5.5.9)$$

by Theorem 3.2.2 and (5.5.3), and if $\Delta=0$

$$\mu = 0. \tag{5.5.9'}$$

Moreover,

$$\sigma^2 = \lim_{m\to\infty} \mathrm{Var}S^W$$

$$= n\mathrm{Var}[F(Y)], \tag{5.5.10}$$

and if $\Delta=0$

$$\sigma^2 = n/12. \tag{5.5.10'}$$

For numerical calculation of $E[F(Y)]$ and $\mathrm{Var}[F(Y)]$ when $\Delta\neq 0$, the following expressions may be used instead:

$$E[F(Y)] = E\{F[F^{-1}(W)+\Delta]\}$$

and (5.5.11)

$$\mathrm{Var}[F(Y)] = E\{F[F^{-1}(W)+\Delta]\}^2 - \{E[F(Y)]\}^2$$

where W is a uniform (0,1) random variable.

5.6 THE MEDIAN PLACEMENT STATISTIC

The study of the CUSUM chart using the median placement statistic (3.2.3) can be done in exactly the same way as the Wilcoxon-Mann-Whitney statistic. The calculations for the median placement statistic are easier than the Wilcoxon-Mann-Whitney statistic because the median placement statistic depends only on the median of the standard sample.

As in the preceding section, suppose that the parameter of interest is the process mean and a standard sample is given instead of a specified control value for the mean. It can be easily seen that the mean and the variance of the median placement statistic, $S^M(X_{(M)})$, conditioned on the standard sample median $X_{(M)}$, are

$$ES^M(X_{(M)})=\mu_\Delta(X_{(M)})=n[1-F(X_{(M)}-\Delta)]$$

and

$$\mathrm{Var}S^M(X_{(M)})=\sigma_\Delta^2(X_{(M)})=n[1-F(X_{(M)}-\Delta)]F(X_{(M)}-\Delta) \tag{5.6.1}$$

where Δ is the true amount of shift of the location parameter of the observed samples. Then, from (5.6.1) and (5.4.9), the reference value $k(X_{(M)})$ is

$$k(X_{(M)})=n[1-F(X_{(M)})] \tag{5.6.2}$$

and the estimate of $k(X_{(M)})$ using the sample distribution function is

$$\begin{aligned}k_m^M=k_m(X_{(M)})&=n[1-F_m(X_{(M)})]\\&=n/2.\end{aligned} \tag{5.6.3}$$

Note that the estimate of $k(X_{(M)})$ does not depend on $X_{(M)}$.

The run length of the procedure is approximated by a continuous random variable $\tau^T(X_{(M)})$ which is the first time t for which

$$[Z^M(t)-(n/2)t]-\inf_{0<s<t}[Z^M(s)-(n/2)s]\geq h'(\underline{X}) \qquad (5.6.4)$$

if $t<T$, and $\tau^T(X_{(M)})=T$ otherwise, where $Z^M(t)$ is a Brownian motion process with parameters $[ES^M(X_{(M)})]t$ and $[VarS^M(X_{(M)})]t$, and $h'(\underline{X})$ is the continuity-corrected boundary defined by (5.3.5).

The expressions for the unconditional run length distribution and the ARL, which are used for calculations with m an odd number for convenience, are

$$\begin{aligned}
P_\Delta(\tau^T<t)&=E\{P_\Delta[\tau^T(X_{(M)})<t]\}\\
&=\int P_\Delta[\tau^T(x)<t]dF_{X_{(M)}}(x)\\
&=\int P_\Delta[\tau^T(x)<t]\\
&\quad\{m!/[(M-1)!]^2\}\{F(x)[1-F(x)]\}^{M-1}dx\\
&=\int_0^1 P_\Delta\{\tau^T[F^{-1}(y)]<t\}\\
&\quad\{m!/[(M-1)!]^2\}[y(1-y)]^{M-1}dy \qquad (5.6.5)
\end{aligned}$$

by change of variable $y=F(x)$, if $t<T$ and $P_\Delta(\tau<t)=1$ otherwise, where $F_{X_{(M)}}$ is the distribution function of $X_{(M)}$, and

$$\begin{aligned}
E_\Delta\tau^T&=E[E_\Delta\tau^T(X_{(M)})]\\
&=\int E_\Delta\tau^T(x)dF_{X_{(M)}}(x)\\
&=\int_0^1 E_\Delta\{\tau^T[F^{-1}(y)]\}
\end{aligned}$$

$$\{m!/[(M-1)!]^2\}[y(1-y)]^{M-1}dy. \qquad (5.6.6)$$

The expressions $P_\Delta\{\tau^T[F^{-1}(y)]<t\}$ and $E_\Delta\{\tau^T[F^{-1}(y)]\}$ can be obtained from (5.1.3) and (5.1.6) where μ and σ^2 are replaced by

$$\mu_\Delta[F^{-1}(y)]-k_m^M=n\{1-F[F^{-1}(y)-\Delta]\}-k_m^M, \qquad (5.6.7)$$

and if $\Delta=0$

$$\mu_\Delta[F^{-1}(y)]-k_m^M=n(1-y)-k_m^M. \qquad (5.6.7')$$

Further,

$$\sigma_\Delta^2[F^{-1}(y)]=n\{1-F[F^{-1}(y)-\Delta]\}F[F^{-1}(y)-\Delta], \qquad (5.6.8)$$

and if $\Delta=0$

$$\sigma_\Delta^2[F^{-1}(y)]=n(1-y)y. \qquad (5.6.8')$$

In calculating the unconditional distribution function and ARL^T using (5.6.5) and (5.6.6), a one-dimensional integration subroutine can be applied easily.

The asymptotic ($m\to\infty$) run length distribution and the ARL^T are easily obtained by the expressions (5.1.3) and (5.1.6) where μ and σ^2 are replaced by

$$\mu = \lim_{m\to\infty}(ES^M-k_m^M)$$

$$= nP[F(Y)>1/2]-n/2 \qquad (5.6.9)$$

by Theorem 3.2.2 and (5.6.3), and if $\Delta=0$

$$\mu = 0, \tag{5.6.9'}$$

also

$$\sigma^2 = \lim_{m\to\infty} \mathrm{Var}S^M$$

$$= nP[F(Y)>1/2]\{1-P[F(Y)>1/2]\}, \tag{5.6.10}$$

and if $\Delta=0$

$$\sigma^2 = n/4. \tag{5.6.10'}$$

For numerical calculations of $P[F(Y)>1/2]$ when $\Delta\neq 0$, the following expression may be used instead:

$$P[F(Y)>1/2] = 1-F[F^{-1}(1/2)+\Delta]. \tag{5.6.11}$$

5.7 THE SUM OF THE SQUARED RANKS STATISTIC

Suppose that the parameter of interest is the process variance and that a standard sample is given instead of a specified control value. One among many nonparametric approaches for this problem is to use the sum of the squared ranks, which is efficient especially for certain asymmetrical one-sided distributions. Here we use the sum of the squared ranks statistic (3.2.7) in a CUSUM chart.

The mean and the variance of the sum of the squared ranks statistic, $S^S(\underline{X})$, conditioned on the standard sample $\underline{X}$, are as follows; the derivations are in Appendix C. Let $M(i_1,..,i_a)=\max(X_{i_1},..,X_{i_a})$. Then

$$ES^S(\underline{X})=\mu_\Delta(\underline{X})$$

$$=(n/L^2)\{\Sigma_k\Sigma_{k'}[1-F(M(k,k')-\Delta)]$$

$$+(n+1)\Sigma_k[1-F(X_k-\Delta)]+n(n+1)(2n+1)/6\}$$

and (5.7.1)

$$\mathrm{Var}S^S(\underline{X})=\sigma^2_\Delta(\underline{X})$$

$$=(1/L^4)\{n\Sigma_k\Sigma_{k'}\Sigma_q\Sigma_{q'}[1-F(M(k,k',q,q,)-\Delta)]$$

$$-[1-F(M(k,k')-\Delta)][1-F(M(q,q')-\Delta)]$$

$$+2n(n+1)[\Sigma_k\Sigma_{k'}\Sigma_{k''}(1-F(M(k,k',k'')\_\Delta))$$

$$-\Sigma_k(1-F(M(k,k')-\Delta))]$$

$$+[2n(n+1)(2n+1)/3][\Sigma_k\Sigma_{k'}(1-F(M(k,k')-\Delta))$$

$$-(\Sigma_k(1-F(X_k-\Delta)))^2]\},$$

where $\Delta(>0)$ is the true amount of shift of the location parameter of the observed samples and all summations are from 1 to m.

From the above expression, the reference value $k(\underline{X})$ is

$$k(\underline{X})=(n/L^2)\{\Sigma_k\Sigma_{k'}[1-F(M(k,k'))]$$

$$+(n+1)\Sigma_k[1-F(X_k)]+n(n+1)(2n+1)/6\} \quad (5.7.2)$$

and the estimate of $k(\underline{X})$ using the sample distribution function is

$$k_m^S(\underline{X})=(n/6L^2)[2m^2+(n+1)(3m+2n+1)+1]. \quad (5.7.3)$$

The derivation of $k_m(\underline{X})$ using (5.7.2) is also in Appendix C. Moreover, the estimate for $k(\underline{X})$ does not depend on $\underline{X}$, as in the case for the Wilcoxon-Mann-Whitney statistic and the median placement statistic.

The rest of this section is arranged in a way similar to the section for the Wilcoxon-Mann-Whitney statistic. The run length of the procedure is approximated by a continuous random variable $\tau^T(\underline{X})$ which is the first time t for which

$$[Z^S(t)-k_m^S(t)]-\inf_{0<s<t}[Z^S(s)-k_m^S(s)]\geq h'(\underline{X}) \quad (5.7.4)$$

if $t<T$, and $\tau^T(\underline{X})=T$ otherwise, where $Z^S(t)$ is a Brownian motion process with parameters $[ES^S(\underline{X})]t$ and $[VarS^S(\underline{X})]t$, and $h'(\underline{X})$ is the continuity-corrected boundary defined by (5.3.5).

The expressions for the unconditional run length distribution and the ARL^T to be used for calculations are the same as the Wilcoxon-Mann-Whitney statistic given by (5.5.5) and (5.5.6).

The asymptotic ($m\to\infty$) run length distribution and the ARL^T are easily obtained by the expressions (5.1.3) and (5.1.6) where μ and σ^2 are replaced by

$$\mu = \lim_{m\to\infty}(ES^S - k_m^S)$$

$$= nE[F(Y)]^2 - n/3 \qquad (5.7.5)$$

by Theorem 3.2.3 and (5.7.3), and if $\Delta=0$

$$\mu = 0, \qquad (5.7.5')$$

also

$$\sigma^2 = \lim_{m\to\infty} \mathrm{Var}S^S$$

$$= n\mathrm{Var}[F(Y)]^2, \qquad (5.7.6)$$

and if $\Delta=0$

$$\sigma^2 = 4n/45. \qquad (5.7.6')$$

For numerical calculation of $E[F(Y)]^2$ and $\mathrm{Var}[F(Y)]^2$ when $\Delta\neq 0$, the following expressions may be used instead:

$$E[F(Y)]^2 = E\{F[F^{-1}(W)+\Delta]\}^2$$

and (5.7.7)

$$\mathrm{Var}[F(Y)]^2 = E\{F[F^{-1}(W)+\Delta]\}^4 - \{E[F(Y)]^2\}^2$$

where W is a uniform (0,1) random variable.

Chapter VI

A COMPARISON OF THE PARAMETRIC AND THE NONPARAMETRIC PROCEDURES

6.1 DEFINITIONS OF PARAMETRIC PROCEDURES

This chapter contains comparisons between the parametric and nonparametric procedures for both the Shewhart charts and the CUSUM charts. The comparisons are done in terms of the ARL.

First, we describe the parametric procedures for the Shewhart and the CUSUM charts. The statistic commonly used for monitoring the process mean in both the Shewhart and CUSUM charts is the sample mean, and the statistic commonly used for monitoring the process variance in both charts is the sample range. However, the sample variance is considered here instead of the sample range because the former is more efficient than the latter for normal data. In this section, parametric procedures for the Shewhart and CUSUM charts for use when the control value is not given are designed by a method similar to that used to create nonparametric procedures.

If a standard sample is given instead of the control value, then the parametric procedures for process control can be performed in exactly the same way as the nonparametric procedures except for the choice of the statistic to use. That is, we first select a control chart, either a Shewhart chart or a CUSUM chart, and then choose an appropriate parametric statistic according to the parameter of interest. Finally, control charts can be constructed using the selected statistics. Thus the only difference between the parametric and the nonparametric procedures will be in the choice of statistic.

Appropriate two-sample statistics for use with the control charts for monitoring the mean and the variance are the two-sample t statistic and the F statistic. The two-sample t statistic for comparing the standard sample $\underline{X}$ and the i-th observed sample $\underline{Y}_i$ is defined as

$$(\overline{Y}_i-\overline{X})/[s\sqrt{(1/n)+(1/m)}] \qquad (6.1.1)$$

where $s^2=[\Sigma_{j=1}^{n}(Y_{ij}-\overline{Y}_i)^2+\Sigma_{j=1}^{m}(X_j-\overline{X})^2]/(n+m-2)$, $\overline{X}=\Sigma_{j=1}^{m}X_j/m$, and $\overline{Y}_i=\Sigma_{j=1}^{n}Y_{ij}/n$.

The statistic (6.1.1) follows a t distribution with degrees of freedom (n+m-2) when the two variables X and Y follow normal distributions with the same variance. However, the variance of the process may change as well as the mean. Hence, the assumption of equal variances does not seem to be appropriate and the size of the observed sample is usually too small to provide a reasonable estimate for the variance. Therefore we use a modification of the statistic (6.1.1) which does not depend on the sample variance of the observed sample.

In the standard Shewhart charts, the mean and variance are estimated from the standard sample, and thus a control procedure corresponding to what is done in practice will be defined here for purposes of comparison with the nonparametric procedure. Suppose that we estimate the population mean and variance of the standard sample by its sample mean and variance, respectively. Then the standard one-sided Shewhart procedure is to signal if $Y_i \geq \dot{X} + cs_0/\sqrt{m}$ where c is a constant and s_0^2 is the standard sample variance. This is equivalent to signaling if $S_i^P \geq c$ where

$$S_i^P = \sqrt{m}(\overline{Y}_i - \overline{X})/s_0. \tag{6.1.2}$$

Thus the statistic S_i^P will be used as the statistic in the parametric Shewhart chart. By the same reason the parametric CUSUM procedure can be applied using S^P. The reference value for the CUSUM chart using S^P is obtained as the conditional expectation, i.e.

$$k^P(\underline{X}) = E(S^P | \underline{X})$$

$$= \sqrt{m}(E_0\overline{Y} - \overline{X})/s_0. \tag{6.1.3}$$

The statistic S^P is used in the parametric control chart for detection of the mean shifts and is compared to the Wilcoxon-Mann-Whitney statistic and the median placement statistic of the nonparametric control charts.

The F statistic is defined as

$$S_i^F = s_i^2/s_0^2 \tag{6.1.4}$$

where $s_i^2=\Sigma_{j=1}^n(Y_{ij}-\overline{Y}_i)^2/(n-1)$ and s_0^2 is defined above. The reference value for the CUSUM chart using S^F is obtained as

$$\begin{aligned} k^F(\underline{X}) &= E_0(S^F|\underline{X}) \\ &= \mathrm{Var}_0(Y)/s_0^2 \end{aligned} \tag{6.1.5}$$

The statistic S^F follows an F distribution with the degrees of freedom (n-1,m-1) when the two variables X and Y follow normal distributions. In the parametric control chart, the statistic S^F is used for detection of the variance shifts and is compared to the sum of the squared ranks statistic of the nonparametric control charts.

In asymmetrical one-sided distributions with all parameters except the scale parameter fixed, the mean of the distribution also changes when the variance changes. The standard procedure for the case where the objective is to monitor both the mean and variance is to use control charts using the sample mean and sample variance (or sample range) simultaneously. Thus, parametric procedures using both S^P and S^F is also frequently used in practice to detect variance changes in the process when the distribution is asymmetrical one-sided. The stopping rule of the combined procedure is to signal at the first time for which either

the procedure using S^P stops

or (6.1.6)

the procedure using S^F stops.

In the following two sections, the ARL's of the parametric and the nonparametric control charts are obtained when the parameters of interest are the process mean and variance. The sample sizes to be considered are (m=39,n=10) and (m=19,n=5) and the truncation point T is set to be 1000. To compare the ARL^T's for various shifts, the ARL^T's under control were set to be the same or at least approximately the same.

6.2 CONTROL CHARTS FOR MONITORING THE LOCATION PARAMETER

In this section, we consider the shifts of the location parameter of the observed samples. That is, the distribution function of the observed sample is $G(x)=F(x-\Delta)$ for the true amount of shift Δ of the location. Four well known distributions, the uniform, normal, double exponential, and Cauchy, which are often used in comparing parametric and nonparametric hypotheses tests, are used as the parent distributions. The variances of the uniform, normal and double exponential distributions are set to be 1. In practical applications where the variance of the observations is not one, the values of Δ can be thought of as being expressed in units of standard deviation. For the Cauchy distribution the scale parameter is chosen to have the probability 0.05 above $\Delta+1.645$, which corresponds to the normal distribution with the same median and variance 1. This scale factor was used by Arnold (1965) in comparing nonparametric tests for location against the t distribution when the distribution is Cauchy.

The probability density function of the normal distribution is

$$f(x) = (1/\sqrt{2\Pi})e^{-(x-\Delta)^2/2}, \quad -\infty<x<\infty. \qquad (6.2.1)$$

When simulation is used for generating the normal random variables, the IMSL subroutine GGNML is used.

The probability density function of the uniform distribution is

$$f(x) = 1/(2\sqrt{3}), \quad \Delta-\sqrt{3}<x<\Delta+\sqrt{3} \qquad (6.2.2)$$

and the IMSL subroutine GGUBS is used for generating the uniform (0,1) random variables.

The probability density function of the double exponential distribution is

$$f(x) = (1/\sqrt{2})e^{(-\sqrt{2}|x-\Delta|)}, \quad -\infty<x<\infty \qquad (6.2.3)$$

There is no subroutine available in the IMSL package program for generating the double exponential random numbers. However, the double exponential random variables can be generated by using the inverse distribution function and the uniform (0,1) random number generator GGUBS.

The distribution function of the double exponential random variable when $\Delta=0$ is

$$\begin{aligned} F(x) &= (1/2)e^{\sqrt{2}x}, &&\text{if } x<0 \\ &= 1-(1/2)e^{-\sqrt{2}x}, &&\text{if } x\geq 0 \end{aligned} \qquad (6.2.4)$$

Hence, the inverse distribution function of F(x) is

$$x = (1/\sqrt{2})\log[2F(x)], \qquad \text{if } F(x)<1/2$$

$$X = -(1/\sqrt{2})\log\{2[1-F(x)]\}, \text{ if } F(x)\geq 1/2. \qquad (6.2.5)$$

From the fact that $F_X(X)$ follows a uniform (0,1) distribution when X is a continuous random variable, the double exponential random variable can be generated as

$$\begin{aligned} X &= (1/\sqrt{2})\log(2W), && \text{if } W<1/2 \\ &= -(1/\sqrt{2})\log[2(1-W)], && \text{if } W\geq 1/2. \end{aligned} \qquad (6.2.6)$$

where W is the uniform (0,1) random variable generated by the IMSL subroutine GGUBS.

The probability density function of the Cauchy distribution is

$$f(x) = 0.2605/\{\Pi[0.2605^2+(x-\Delta)]\}, \quad -\infty<x<\infty. \qquad (6.2.7)$$

The scale parameter value 0.2605 gives the probability 0.05 above Δ+1.645 as noted. Also, there is no subroutine available in the IMSL package program for generating the Cauchy random numbers, but the Cauchy random numbers also can be generated by the inverse distribution function method. The distribution function of the Cauchy random variable is

$$F(x) = (1/\Pi)\tan^{-1}(x/0.2605)+1/2, \quad -\infty<x<\infty. \qquad (6.2.8)$$

Hence, the inverse distribution function of F(x) is

$$x = 0.2605\tan\{\Pi[F(x)-1/2]\}, \quad 0<F(x)<1 \tag{6.2.9}$$

and thus, the Cauchy random number can be generated as

$$X = 0.2605\tan[\Pi(W-1/2)], \quad 0<W<1 \tag{6.2.10}$$

where W is the uniform (0,1) random variable generated by the IMSL subroutine GGUBS.

The ARL^T's of the Shewhart chart using the median placement statistic are obtained by the integral equation (4.3.2). The ARL^T's of the Shewhart chart using the statistic S^P (6.1.2) for the normal distribution are obtained as follows. When the distributions of the observation are normal with constant variance equal to 1, it is known that $\overline{X}$, $\overline{Y}_i$ follow normal distributions with mean 0 and variance 1/m, and mean Δ and variance 1/n, respectively, and $(m-1)s_0^2$ follows a chi-squared distribution with m-1 degrees of freedom. Also $\overline{X}$ and s_0^2 are independent. Hence,

$$\begin{aligned} P[S^P(\underline{X})<c] &= P(\overline{Y}<\overline{X}+cs_0/\sqrt{m}) \\ &= \Phi\{\sqrt{n}[\overline{X}+(cs_0/\sqrt{m})-\Delta]\} \end{aligned} \tag{6.2.11}$$

and

$$\begin{aligned} E\{P[S^P(\underline{X})<c]\}^t = \int_0^\infty\int_{-\infty}^\infty \{\Phi[\sqrt{n}(v/\sqrt{m} \\ +c\sqrt{w}/\sqrt{m(m-1)}-\Delta)]\}^t d\Phi(v)dH(w) \end{aligned}$$

$$=\int_0^1\int_0^1\{\Phi[\sqrt{n}[\Phi^{-1}(v)/\sqrt{m}$$

$$+c\sqrt{H^{-1}(w)}/\sqrt{m(m-1)}-\Delta)]\}^t dvdw \qquad (6.2.12)$$

where $\Phi(.)$ is the standard normal distribution function, $H(.)$ is the chi-squared distribution function with degrees of freedom $(m-1)$, and c is the control limit of the Shewhart chart. Thus, the ARL^T can be obtained by the equation

$$EN^T=\Sigma_{t=0}^{T-1}E\{P[S^P(\underline{X})<c|\underline{X}]\}^t. \qquad (6.2.13)$$

The ARL^T's for the other cases were obtained by simulation. The 500 simulated run lengths were averaged to produce the ARL^T. The ARL^T's of the Shewhart charts for the four distributions are given in Tables 5, 6, 7, and 8 when the sample sizes are m=39, n=10. The ARL^T's of the CUSUM chart for the four distributions are given in Tables 9, 10, 11, and 12 when the sample sizes are m=39, n=10. Also the ARL^T's of the Shewhart and the CUSUM charts for the four distributions are listed in Tables 13, 14, 15, 16, 17, 18, 19, and 20 when the sample sizes are m=19, n=5. From the obtained results, it is seen that, for each of the distributions, the ARL^T of the CUSUM chart is generally smaller than that of the Shewhart chart when the amount of shift is relatively small (Δ=0.2, 0.4, 0.6) but the ARL^T of the CUSUM chart tends to be larger than that of the Shewhart chart when the amount of shift increases (Δ=0.8, 1.0). Recalling that the reference value k of the CUSUM chart was chosen to make the chart sensitive to an arbitrarily small amount of shift, it can be concluded that this CUSUM chart is useful for detecting small amounts of shift while the Shewhart chart can be used for relatively large amounts of shift.

When the distribution has a heavier tail than the normal distribution as in the double exponential and Cauchy distributions, the parametric procedures in both the Shewhart and the CUSUM charts do not seem to be as efficient as the nonparametric procedures. Especially when the distribution is Cauchy, the parametric procedures hardly detect the shift. However, both the nonparametric procedures using the Wilcoxon-Mann-Whitney statistic and the median placement statistic still can detect shifts very quickly.

For the uniform and the normal distributions, the parametric procedures are generally better than the nonparametric procedures. But the Wilcoxon-Mann-Whitney statistic is almost as good as the parametric procedures.

When the sample sizes are m=19, n=5, every ARL^T decreases more slowly as a function of Δ than when the sample sizes are m=39, n=10.

TABLE 5

The ARL^T of the Shewhart chart when the distribution is normal for m=39, n=10, and T=1000

shift	S^P(c=4.47)	S^M(c=9)	S^W(c=277/39)
0.0	178.71	178.65	178.24(15.21)
0.2	47.12*	59.64	57.68(6.71)
0.4	11.80*	19.34	12.29(1.35)
0.6	3.98*	7.51	4.92(0.45)
0.8	1.97*	3.70	2.07(0.21)
1.0	1.34*	2.25	1.49(0.14)

* indicates the smallest ARL^T among the three statistics

(.) is the standard deviation of estimator of the ARL^T

TABLE 6

The ARL^T of the Shewhart chart when the distribution is uniform for m=39, n=10, and T=1000

shift	S^P(c=4.35)	S^M(c=9)	S^W(c=277/39)
0.0	178.24(12.46)	178.65(13.04)	178.24(12.18)
0.2	46.17(5.39)	80.33(7.90)	39.77*(3.92)
0.4	8.95*(0.62)	34.14(5.27)	9.90(0.81)
0.6	3.25*(0.17)	15.15(3.18)	4.76(0.22)
0.8	1.79*(0.07)	7.41(0.67)	2.44(0.11)
1.0	1.33*(0.04)	4.03(0.18)	1.51(0.06)

TABLE 7

The ARL^T of the Shewhart chart when the distribution is double exponential for m=39, n=10, and T=1000

shift	S^P(c=4.64)	S^M(c=9)	S^W(c=277/39)
0.0	177.61(12.65)	178.65(12.44)	178.24(11.91)
0.2	58.03(6.92)	32.31*(5.09)	47.87(4.82)
0.4	21.45(4.07)	7.74*(2.43)	8.06(1.08)
0.6	7.59(1.67)	3.38(1.16)	3.07*(0.52)
0.8	2.36(0.81)	2.11(0.67)	1.62*(0.16)
1.0	1.54(0.64)	1.58(0.33)	1.24*(0.12)

TABLE 8

The ARL^T of the Shewhart chart when the distribution is Cauchy for m=39, n=10, and T=1000

shift	S^P(c=24)	S^M(c=9)	S^W(c=277/39)
0.0	182.22(13.71)	178.65(10.48)	178.24(11.59)
0.2	158.90(13.94)	8.78*(2.71)	17.73(3.58)
0.4	156.88(10.29)	2.48*(1.35)	2.59(1.31)
0.6	150.40(8.71)	1.64(0.98)	1.38*(0.27)
0.8	147.45(12.45)	1.37(0.14)	1.14*(0.23)
1.0	147.35(10.69)	1.24(0.50)	1.07*(0.18)

TABLE 9

The ARL^T of the CUSUM chart when the distribution is normal for m=39, n=10, and T=1000

shift	S^P(h=24.3)	S^M(h=7)	S^W(h=4.6)
0.0	176.95(12.87)	171.04(12.14)	180.88(14.56)
0.2	20.06*(3.68)	41.39(4.17)	22.73(3.34)
0.4	10.39(1.30)	7.66(0.42)	5.74*(0.24)
0.6	7.03(0.58)	4.48(0.07)	3.59*(0.06)
0.8	5.54(0.33)	3.10(0.06)	2.89*(0.04)
1.0	4.39(0.12)	2.49(0.05)	2.42*(0.03)

TABLE 10

The ARL^T of the CUSUM chart when the distribution is uniform for m=39, n=10, and T=1000

shift	S^P(h=23.7)	S^M(h=7)	S^W(h=4.6)
0.0	172.60(11.84)	171.04(12.14)	180.88(13.96)
0.2	19.50*(1.86)	67.96(6.28)	28.09(3.58)
0.4	10.21(1.31)	14.22(2.86)	7.73*(0.39)
0.6	6.98(0.47)	6.11(0.79)	3.75*(0.12)
0.8	5.27(0.20)	3.90(0.42)	2.96*(0.06)
1.0	4.39(0.08)	3.14(0.10)	2.51*(0.05)

TABLE 11

The ARL^T of the CUSUM chart when the distribution is double exponential for m=39, n=10, and T=1000

shift	S^P(h=24)	S^M(h=7)	S^W(h=4.6)
0.0	171.53(13.09)	171.04(12.14)	180.88(13.96)
0.2	19.20(3.97)	10.51*(0.93)	21.17(1.42)
0.4	10.30(1.04)	4.09*(0.32)	4.81(0.15)
0.6	6.99(0.38)	2.89*(0.14)	3.15(0.10)
0.8	5.21(0.27)	2.50(0.09)	2.46*(0.06)
1.0	4.34(0.08)	2.25(0.07)	2.15*(0.05)

TABLE 12

The ARL^T of the CUSUM chart when the distribution is Cauchy for m=39, n=10, and T=1000

shift	S^P(h=55)	S^M(h=7)	S^W(h=4.6)
0.0	178.43(15.84)	171.04(12.14)	180.88(13.96)
0.2	128.97(10.76)	4.67*(0.56)	5.22(0.21)
0.4	95.37(8.88)	2.64*(0.35)	2.94(0.30)
0.6	84.00(8.21)	2.29(0.44)	2.24*(0.21)
0.8	61.37(6.51)	2.13(0.16)	2.08*(0.09)
1.0	42.11(3.07)	2.09(0.08)	1.99*(0.08)

TABLE 13

The ARL^T of the Shewhart chart when the distribution is normal when m=19, n=5, and T=1000

shift	S^P(c=3.5)	S^M(c=5)	S^W(c=71/19)
0.0	71.03	71.60	72.24(6.61)
0.2	26.36*	31.09	34.53(4.44)
0.4	10.24*	14.26	14.84(1.29)
0.6	4.68*	7.36	6.33(0.44)
0.8	2.61*	4.33	3.27(0.15)
1.0	1.75*	2.87	2.02(0.07)

TABLE 14

The ARL^T of the Shewhart chart when the distribution is uniform for m=19, n=5, and T=1000

shift	S^P(c=3.39)	S^M(c=5)	S^W(c=71/19)
0.0	72.69(8.90)	71.60(7.41)	72.24(6.83)
0.2	18.42*(5.47)	37.34(6.57)	18.42*(4.12)
0.4	7.88*(4.13)	20.09(3.92)	9.39(2.99)
0.6	3.54*(2.41)	11.51(2.01)	5.27(0.41)
0.8	2.40*(0.27)	7.03(2.93)	3.20(1.26)
1.0	1.60*(0.35)	4.53(0.95)	2.89(1.16)

TABLE 15

The ARL^T of the Shewhart chart when the distribution is double exponential for m=19, n=5, and T=1000

shift	S^P(c=3.64)	S^M(c=5)	S^W(c=71/19)
0.0	70.92(7.43)	71.60(6.93)	72.24(7.67)
0.2	34.88(4.58)	22.31*(4.91)	34.59(4.25)
0.4	20.70(2.56)	8.16*(2.60)	12.22(2.62)
0.6	11.20(0.82)	4.09*(0.51)	4.80(0.39)
0.8	3.46(0.28)	2.67(0.32)	2.62*(0.23)
1.0	2.14(0.13)	2.02(0.12)	1.85*(0.13)

TABLE 16

The ARL^T of the Shewhart chart when the distribution is Cauchy for m=19, n=5, and T=1000

shift	S^P(c=7.8)	S^M(c=5)	S^W(c=71/19)
0.0	72.77(7.93)	71.60(8.01)	72.24(7.75)
0.2	70.87(4.98)	9.23*(2.94)	28.17(4.65)
0.4	67.74(3.59)	3.12*(0.51)	3.66(0.65)
0.6	64.50(3.53)	2.08*(0.92)	3.21(0.17)
0.8	58.62(3.19)	1.72*(0.35)	1.91(1.12)
1.0	54.27(2.97)	1.54*(0.27)	1.68(1.91)

TABLE 17

The ARL^T of the CUSUM chart when the distribution is normal for m=19, n=5, and T=1000

shift	S^P(h=15)	S^M(h=3.5)	S^W(h=1.95)
0.0	80.43(3.18)	79.98(8.77)	80.46(8.93)
0.2	16.90*(0.38)	29.13(3.93)	21.59(4.37)
0.4	9.24*(0.16)	11.74(1.87)	6.25(1.05)
0.6	6.11(0.09)	5.78(0.2)	4.12*(0.28)
0.8	4.89(0.06)	3.26(0.08)	2.57*(0.06)
1.0	3.97(0.05)	2.73*(0.05)	2.19*(0.04)

TABLE 18

The ARL^T of the CUSUM chart when the distribution is uniform for m=19, n=5, and T=1000

shift	S^P(h=15.4)	S^M(h=3.5)	S^W(h=1.95)
0.0	79.54(3.20)	79.98(8.77)	80.46(8.93)
0.2	17.84*(0.37)	32.45(4.81)	18.38(2.92)
0.4	9.40(0.15)	17.41(1.05)	6.19*(0.96)
0.6	6.39(0.08)	6.38(0.32)	3.72*(0.12)
0.8	5.06(0.06)	4.70(0.15)	2.69*(0.07)
1.0	4.06(0.04)	3.70(0.06)	2.28*(0.06)

TABLE 19

The ARL^T of the CUSUM chart when the distribution is double exponetial for m=19, n=5, and T=1000

shift	S^P(h=14.93)	S^M(h=3.5)	S^W(h=1.95)
0.0	79.44(3.67)	79.98(8.77)	80.46(8.93)
0.2	17.00(0.42)	13.02*(2.24)	18.30(2.11)
0.4	9.13(0.17)	7.03(1.03)	5.52*(1.18)
0.6	6.03(0.09)	3.26(0.08)	2.84*(0.21)
0.8	4.78(0.07)	2.64(0.04)	2.23*(0.08)
1.0	3.91(0.06)	2.41(0.03)	1.98*(0.04)

TABLE 20

The ARL^T of the CUSUM chart when the distribution is Cauchy for m=19, n=5, and T=1000

shift	S^P(h=16)	S^M(h=3.5)	S^W(h=1.95)
0.0	78.58(6.99)	79.98(9.21)	80.46(8.93)
0.2	56.66(4.30)	5.93*(1.01)	7.65(0.97)
0.4	33.02(3.01)	2.81(0.15)	2.80*(0.21)
0.6	23.52(2.96)	2.49(0.08)	2.12*(0.11)
0.8	19.61(1.45)	2.29(0.04)	1.78*(0.05)
1.0	13.65(2.53)	2.19(0.04)	1.56*(0.03)

6.3 CONTROL CHARTS FOR MONITORING THE SCALE PARAMETER

In this section, we consider changes of the scale parameter of observed samples from that of the standard sample. Thus, the distribution function of observed samples is $G(x)=F(x/\Delta)$ for the true amount of shift Δ of the scale. In practical applications, the values of Δ can be thought of as being expressed in units of the scale parameter value. Three asymmetric distributions, the Weibull, lognormal, and gamma, are chosen as the parent distributions.

The probability density function of the two-parameter Weibull distribution is

$$f(x) = (a/\Delta)(x/\Delta)^{a-1}e^{-(x/\Delta)^a}, \quad x>0, a>0, \Delta>0 \qquad (6.3.1)$$

where the shape parameter a is set to be 2. The mean and variance of the Weibull distribution when $\Delta=1$ are $\Pi/2$ and $1-\Pi/4$ respectively. The IMSL subroutine GGWIB is used for generating the Weibull random variables.

The probability density function of the lognormal distribution is

$$f(x) = (1/x\Delta\sqrt{2\Pi})e^{[-(\log x)^2/2\Delta^2]}, \quad x>0, \Delta>0. \qquad (6.3.2)$$

The mean and variance of the lognormal distribution when $\Delta=1$ are $\sqrt{e}$ and e^2-e, respectively. Using the fact that the exponential of normal random variable follows a lognormal distribution, the lognormal random variables are generated by taking the exponential of the normal random variables generated by the IMSL subroutine GGNML.

The probability density function of the gamma distribution is

$$f(x) = [x^{a-1}/\Gamma(a)\Delta^{a}]e^{-(x/\Delta)},\ x>0, a>0, \Delta>0 \qquad (6.3.3)$$

where the shape parameter a is set to be 2. Both the mean and variance of the gamma distribution when $\Delta=1$ are 2. These gamma random variables are generated by adding two exponential random variables which are generated by the IMSL subroutine GGEXN.

All the ARL^T's were obtained by simulation and 500 simulated run lengths were averaged to produce the ARL^T. In the three asymmetric distributions, whose distributions satisfy $F(x)=0$ if $x<0$, the skewness has a big role in determining the shapes of the distributions. The skewness [see Johnson and Kotz (1970) p115, 168, 252] of the three distributions, (6.3.1), (6.3.2), and (6.3.3), are 0.63111, 6.18 and $\sqrt{2}$, respectively.

The ARL^T's of the Shewhart chart for the three distributions are given in Tables 21, 22, and 23 when the sample sizes are m=39, n=10. The ARL^T's of the CUSUM chart for the three distributions are given in Tables 24, 25, and

26 when the sample sizes are m=39, n=10. Also the ARL^T's of the Shewhart and the CUSUM charts for the three distributions are listed in Tables 27, 28, 29, 30, 31, and 32 when the sample sizes are m=19, n=5.

For each of the three distributions, the ARL^T of the CUSUM chart is generally smaller than that of the corresponding Shewhart chart when the amount of shift is small (Δ=1.2, 1.4, 1.6) but the ARL^T of the CUSUM chart tends to be larger than that of the Shewhart chart when the amount of shift is large (Δ=1.8, 2.0). This result supports the general observation that the CUSUM chart may be used for detecting relatively small shifts while the Shewhart chart may be used for detecting large shifts. Thus, a CUSUM scheme may be selected if the amount of shift which is important to be detected is relatively small, whereas a Shewhart scheme may be used if a small amount of shift is of no importance to detect.

In all three distributions, the ARL^T of the parametric procedure using S^F only decreases more slowly as a function of Δ than the nonparametric procedure. Also, as the distributions have increasingly heavier tails in the order of Weibull, gamma and lognormal in terms of their skewness, the ARL^T of the parametric procedure using S^F only decreases more slowly than does that of the nonparametric procedure.

In general, the combined parametric procedure (6.1.6) improves the sensitivity of the parametric procedure using S^F only except for the Shewhart chart when the underlying distribution is lognormal. For the Weibull and gamma distribution, in general, the combined Shewhart procedure is more efficient than the nonparametric procedure whereas the combined CUSUM chart is less efficient than the nonparametric procedure. For the lognormal distribution, however, the combined procedure for both the Shewhart and CUSUM charts is still not as good as the nonparametric procedure although the combined CUSUM procedure is an improvement over the parametric CUSUM procedure using S^F only.

When the sample sizes are m=19, n=5, every ARL^T decreases more slowly as a function of Δ than when the sample sizes are m=39, n=10.

TABLE 21

The ARL^T of the Shewhart chart when the distribution is Weibull for m=39, n=10, and T=1000

shift	S^F(c=2.23)	S^FS^P(c=4.1)	S^S(c=5)
1.0	117.23(9.38)	112.92(10.03)	112.89(9.71)
1.2	13.30(3.17)	6.85*(0.63)	7.81(0.61)
1.4	3.82(0.61)	1.87*(0.09)	2.49(0.11)
1.6	2.18(0.20)	1.27*(0.08)	1.44(0.06)
1.8	1.54(0.11)	1.06*(0.08)	1.15(0.05)
2.0	1.29(0.08)	1.03*(0.05)	1.09(0.05)

* denotes the smallest ARL^T among the three statistics and S^FS^P denotes the combined procedure defined in (6.1.6)

TABLE 22

The ARL^T of the Shewhart chart when the distribution is lognormal for m=39, n=10, and T=1000

shift	S^F(c=10.5)	S^FS^P(c=10.8)	S^S(c=5)
1.0	102.20(9.01)	102.20(10.03)	112.89(9.71)
1.2	101.81(10.45)	90.36(8.85)	28.69*(2.49)
1.4	60.54(6.37)	50.36(5.11)	8.98*(1.55)
1.6	33.76(3.29)	41.88(4.30)	6.64*(0.83)
1.8	27.60(1.96)	24.14(2.16)	3.04*(0.54)
2.0	24.50(2.08)	18.25(1.27)	2.38*(0.21)

TABLE 23

The ARL^T of the Shewhart chart when the distribution is gamma for m=39, n=10, and T=1000

shift	S^F(c=3.2)	$S^F S^P$(c=4.55)	S^S(c=5)
1.0	112.33(9.33)	109.07(10.03)	112.89(9.71)
1.2	29.35(2.81)	17.10(2.70)	14.01*(1.82)
1.4	9.38(0.76)	3.82*(0.34)	5.85(0.57)
1.6	4.94(0.36)	1.90*(0.13)	2.50(0.10)
1.8	3.04(0.25)	1.47*(0.08)	1.59(0.06)
2.0	2.28(0.09)	1.17*(0.05)	1.30(0.04)

TABLE 24

The ARL^T of the CUSUM chart when the distribution is Weibull for m=39, n=10, and T=1000

shift	S^F(h=4.5)	$S^F S^P$(h=19)	S^S(h=5.3)
1.0	113.20(8.86)	117.52(10.34)	120.03(9.82)
1.2	11.58(0.76)	8.47(0.84)	6.98*(0.83)
1.4	5.45(0.39)	4.56(0.38)	3.77*(0.30)
1.6	3.79(0.21)	3.19(0.27)	2.87*(0.27)
1.8	2.71(0.07)	2.56(0.11)	2.50*(0.09)
2.0	2.23(0.05)	2.15*(0.08)	2.32(0.04)

TABLE 25

The ARL^T of the CUSUM chart when the distribution is lognormal for m=39, n=10, and T=1000

shift	S^F(h=13)	$S^F S^P$(h=17)	S^S(h=5.3)
1.0	112.96(9.90)	118.21(10.34)	120.03(9.82)
1.2	40.73(4.70)	20.20*(3.56)	30.30(3.80)
1.4	19.15(2.91)	10.87(1.62)	8.87*(1.21)
1.6	11.91(1.24)	7.35(0.65)	4.50*(0.35)
1.8	10.03(1.38)	5.69(0.27)	4.32*(0.28)
2.0	8.08(0.88)	4.76(0.18)	3.44*(0.09)

TABLE 26

The ARL^T of the CUSUM chart when the distribution is gamma for m=39, n=10, and T=1000

shift	S^F(h=6.4)	S^FS^P(c=19)	S^S(h=5.3)
1.0	115.83(9.25)	118.25(10.34)	120.03(9.82)
1.2	16.02(1.74)	11.49(1.24)	8.13*(0.85)
1.4	7.53(0.36)	5.98(0.28)	4.98*(0.23)
1.6	5.22(0.19)	4.22(0.13)	3.65*(0.11)
1.8	3.97(0.16)	3.24(0.08)	3.04*(0.07)
2.0	2.98(0.07)	2.71(0.06)	2.69*(0.06)

TABLE 27

The ARL^T of the Shewhart chart when the distribution is Weibull for m=19, n=5, and T=1000

shift	S^F(c=2.98)	S^FS^P(c=4.26)	S^S(c=2.9)
1.0	108.69(8.77)	109.60(9.33)	111.25(10.94)
1.2	27.47(4.21)	12.08*(2.04)	14.24(3.68)
1.4	8.50(0.95)	3.42*(0.16)	4.03(0.83)
1.6	3.71(0.23)	1.94*(0.08)	2.50(0.18)
1.8	2.60(0.08)	1.40*(0.07)	1.75(0.09)
2.0	2.06(0.06)	1.15*(0.05)	1.43(0.06)

TABLE 28

The ARL^T of the Shewhart chart when the distribution is lognormal for m=19, n=5, and T=1000

shift	S^F(c=11)	$S^F S^P$(c=12)	S^S(c=2.9)
1.0	101.47(8.80)	119.98(9.33)	111.25(10.42)
1.2	58.67(6.21)	70.54(6.19)	44.91*(4.87)
1.4	51.14(5.30)	52.42(4.21)	21.65*(2.55)
1.6	32.14(4.87)	31.47(4.38)	11.71*(1.71)
1.8	20.37(1.85)	25.09(2.91)	9.90*(0.37)
2.0	18.03(1.64)	21.35(2.04)	5.58*(0.20)

TABLE 29

The ARL^T of the Shewhart chart when the distribution is gamma for m=19, n=5, and T=1000

shift	S^F(c=4.1)	S^FS^P(c=5.1)	S^S(c=2.9)
1.0	109.46(9.28)	111.63(9.33)	111.25(10.42)
1.2	35.19(3.64)	31.24(2.48)	29.83*(2.57)
1.4	14.70(2.71)	9.11(0.96)	8.76*(0.71)
1.6	8.20(1.22)	3.93*(0.31)	4.77(0.28)
1.8	4.94(0.88)	2.59*(0.09)	2.91(0.11)
2.0	3.35(0.32)	1.90*(0.04)	2.16(0.08)

TABLE 30

The ARL^T of the CUSUM chart when the distribution is Weibull for m=19, n=5, and T=1000

shift	S^F(h=6.6)	S^FS^P(h=19)	S^S(h=2.8)
1.0	116.07(8.63)	116.64(4.69)	115.93(8.63)
1.2	16.28(0.95)	12.08(0.21)	9.27*(0.95)
1.4	8.14(0.63)	6.42(0.11)	4.16*(0.63)
1.6	5.53(0.13)	4.43(0.06)	3.11*(0.13)
1.8	4.09(0.08)	3.43(0.05)	2.69*(0.08)
2.0	3.06(0.06)	2.89(0.04)	2.39*(0.06)

TABLE 31

The ARL^T of the CUSUM chart when the distribution is lognormal for m=19, n=5, and T=1000

shift	S^F(h=16.5)	$S^F S^P$(h=23)	S^S(h=2.8)
1.0	117.92(9.51)	119.04(8.79)	115.93(13.04)
1.2	47.90(3.63)	27.85*(3.56)	31.57(7.08)
1.4	27.22(2.02)	14.78(1.06)	14.48*(1.08)
1.6	17.99(1.03)	10.63(0.84)	7.00*(0.60)
1.8	12.69(0.83)	8.23(0.63)	4.78*(0.15)
2.0	10.27(0.49)	7.08(0.50)	3.97*(0.08)

TABLE 32

The ARL^T of the CUSUM chart when the distribution is gamma for m=19, n=5, and T=1000

shift	S^F(h=8.7)	S^FS^P(h=19)	S^S(h=2.8)
1.0	117.22(7.22)	120.17(11.04)	115.93(11.71)
1.2	21.45(0.78)	14.96*(1.73)	16.23(3.16)
1.4	11.25(0.36)	8.00(0.60)	5.87*(0.33)
1.6	7.03(0.23)	5.78(0.18)	4.18*(0.08)
1.8	5.27(0.16)	4.40(0.15)	3.23*90.06)
2.0	4.34(0.13)	3.69(0.09)	2.85*(0.05)

Chapter VII

SUMMARY AND CONCLUSIONS

In this chapter, the main ideas and conclusions of nonparametric control procedures are summarized for use when the control value is not given and a topic for further study is suggested.

To monitor a process when the control value is not given, an initial standard sample is obtained when the process is known to be under control, and then samples of fixed size are observed sequentially and are compared to the standard sample to produce a sequence of two-sample nonparametric statistics. With this sequence of statistics, versions of the Shewhart and CUSUM charts are constructed. The two-sample nonparametric statistics considered in this dissertation are the Wilcoxon-Mann-Whitney statistic and the median placement statistic for monitoring the process mean, and the sum of the squared ranks statistic for monitoring the process variance.

The truncation point is used either as the finite time that the process runs or as the time at which enough information about the distribution of the observations is obtained so that the decision about the type of procedure, can be reevaluated for future control of the process.

In the Shewhart chart, lower and upper bounds for the run length distribution are obtained. In the CUSUM chart, the run length distribution of the process is approximated using a Brownian motion process and an appropriate correction is derived for the boundary to improve the approximation.

Comparisons between the parametric and nonparametric procedures for both the Shewhart charts and the CUSUM charts are done in terms of the ARL. For the comparison, parametric procedures are designed exactly the same as the nonparametric procedures except for the statistic used.

For both the parametric and nonparametric procedures, the CUSUM charts are generally more sensitive than the Shewhart charts to relatively small changes of the parameter, whereas the Shewhart charts are more sensitive than the CUSUM charts to large changes. In monitoring the process mean, the control charts using the median placement statistic and the Wilcoxon-Mann-Whitney statistic are more efficient than using the parametric statistic for the double exponential and Cauchy distributions. Moreover, the charts using the Wilcoxon-Mann-Whitney statistic are almost as efficient as the parametric charts for the normal and uniform distributions. Although, in general, the performance of the charts using the median placement statistic is not as good as the charts using the Wilcoxon-Mann-Whitney statistic, if the distribution is not heavy-tailed, the former is easier to apply than the latter and can detect small and moderate changes faster than the other procedures for the heavy-tailed distributions studied. In monitoring the process variance, the control charts using the sum of the squared ranks statistic are more efficient than the parametric charts for the Weibull, gamma and lognormal distributions, although the combined parametric procedure, which is designed for monitoring the variance as well as the mean, improves the sensitivity over the parametric procedure for monitoring the variance only.

From the study of the performance of the control charts, the following may be suggested for constructing a control chart in practice: When the control value is not given, the nonparametric Shewhart chart may be used if the amount of shift to be detected is relatively large and the nonparametric CUSUM chart can be used otherwise. The control limits of a Shewhart chart and the boundary of a CUSUM chart should be determined by considering the ARL under control, which is the same for all continuous distributions. In order to determine the ARL under control, either simulation or the expressions obtained in this dissertation can be used. When the distribution of the observations is known to be asymmetrical, the control chart using the sum of the squared ranks statistic may be used for monitoring the scale parameter of the process.

In Chapter 6, it was shown for the cases simulated that the Shewhart chart is efficient for large deviations and also that the CUSUM chart is efficient for small and moderate deviations. These results agree with those found when the control value is given. To retain these advantages of the two procedures simultaneously, Lucas (1982) described a composite control scheme that combines a Shewhart control scheme with a CUSUM control scheme. The composite scheme gives a signal if the most recent sample is out of the Shewhart control limits or if a CUSUM signal is given. Implementing the Shewhart-CUSUM scheme detects small shifts quickly while the addition of the Shewhart limits increases the speed of detecting large shifts. This type of composite control scheme may also be considered for further study when the control value is not given.

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Appendix A

Proof of Theorem 5.1.1 : The Theorem is proved by using the residue theorem(Theorem 3.4.7).

I. First consider the case $\mu \neq 0$. Let $a=\sqrt{\mu^2+2\sigma^2\lambda}$, so that $\lambda=(a^2-\mu^2)/(2\sigma^2)$ and

$$L_\tau(\lambda)=ae^{-\mu h/\sigma^2}/[a\cosh(ha/\sigma^2)-\mu\sinh(ha/\sigma^2)] \qquad (A.1)$$

1. First check to see if there is any branch cut. If we circle around $-\mu^2/(2\sigma^2)$ over 2Π, then $a \to -a$ but $L_\tau(\lambda) \to L_\tau(\lambda)$. Therefore there is no branch cut.

2. Let $ha/\sigma^2 = p + iq$ for $i=\sqrt{-1}$. (A.2)

Then $a=\sigma^2(p+iq)/h$ and it can be seen that those points such that $\cosh(ha/\sigma^2)=0$ are not singular points. Thus, the denominator of the integrand can be written as

$$\cosh(ha/\sigma^2)[a-\mu\tanh(ha/\sigma^2)] \qquad (A.3)$$

Hence, the singularity only occurs at

$$a = \mu\tanh(ha/\sigma^2) \qquad (A.4)$$

which is equivalent to

$$\begin{aligned} ha/\sigma^2 &= (\mu h/\sigma^2)\tanh(ha/\sigma^2) \\ &= (\mu h/\sigma^2)\tanh(p+iq) \end{aligned}$$

$$=\mu h[\sinh(2p)+i\ \sin(2q)]/(2\sigma^2 d) \qquad (A.5)$$

where $d=\cosh^2(p)-\sin^2(q)$

This implies that, by (A.1),

$$p=\mu h\ \sinh(2p)/\{2\sigma^2[\cosh^2(p)-\sin^2(q)]\}$$

and (A.6)

$$q=\mu h\ \sin(2q)/\{2\sigma^2[\cosh^2(p)-\sin^2(q)]\}$$

i) Suppose that $p\neq 0$ and $q\neq 0$. Then, by (A.6),

$$\sinh(2p)/(2p)=\sin(2q)/(2q) \qquad (A.7)$$

It can be easily seen from Figure 2 that the only case where the two curves $\sinh(x)/x$ and $\sin(x)/x$ join is when $x=0$. But this contradicts the assumption that $p\neq 0$ and $q\neq 0$. Therefore all singularities occur either on the real axis or on the imaginary axis of the a-plane.

ii) If $p=0$, then $ha/\sigma^2=iq$ by (A.2). Also by (A.4),

$$iq = (\mu h/\sigma^2)\tanh(iq)$$

$$= i(\mu h/\sigma^2)\tan(q)$$

which is equivalent to

$$q\sigma^2/(\mu h) = \tan(q) \qquad (A.8)$$

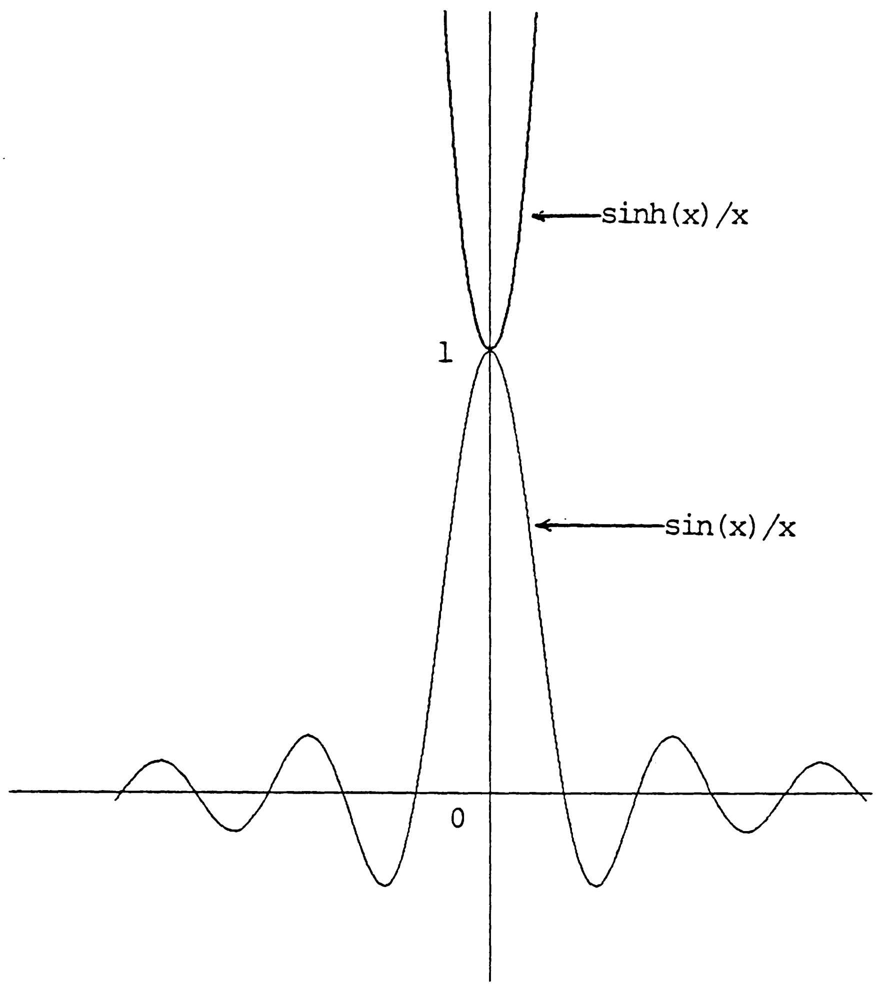

Figure 2: The joining point of sinh(x)/x and sin(x)/x

Let q_j, $j=1,2,\ldots$, be the points satisfying the equation (A.8) except for the point 0. Then the singular points in the a-plane are

$$(0,0),\ (0,\pm q_1\sigma^2/h),\ (0,\pm q_2\sigma^2/h),\ldots. \qquad \text{(A.9)}$$

The solutions of (A.8) are shown in Figure 3.

Note that if $\sigma^2>\mu h$, then there is a solution q_1 between 0 and $\Pi/2$.

iii) If $q=0$, then $ha/\sigma^2=p$ by (A.2). Also by (A.4),

$$p = (\mu h/\sigma^2)\tanh(p)$$

which is equivalent to

$$p\sigma^2/(\mu h) = \tanh(p) \qquad \text{(A.10)}$$

Let $\pm q_0$ be the points satisfying the equation (A.10) except for the point 0. Then the singular points in the a-plane are

$$(0,0),\ (0,\pm q_0\sigma^2/h) \qquad \text{(A.11)}$$

The solutions of (A.10) are shown in Figure 4.

Note that if $\sigma^2<\mu h$, then (0,0) is the only solution of (A.10).

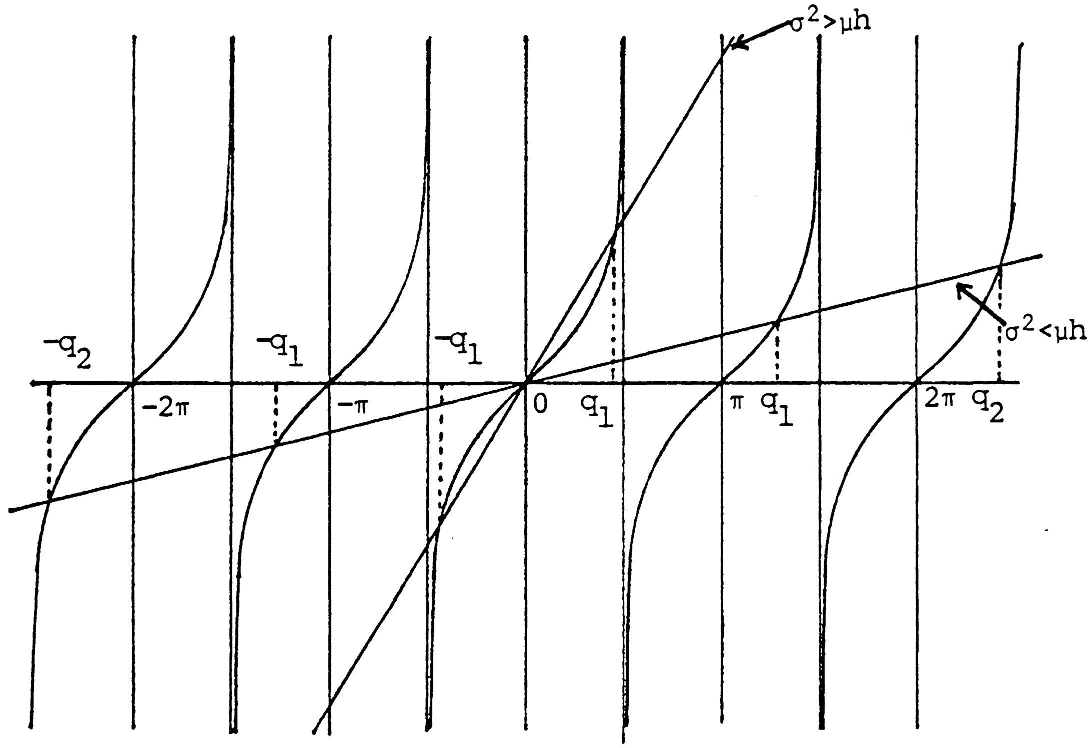

Figure 3: The singular points in the a-plane for p=0 when μ =0

From (A.9) and (A.11), the singular points in the a-plane are

$$(0,0),\ (0,\pm q_1\sigma^2/h),\ (0,\pm q_2\sigma^2/h),\ldots \quad \text{if } \sigma^2 \geq \mu h,$$

or (A.12)

$$(0,0),\ (0,\pm q_0\sigma^2/h),\ (0,\pm q_1\sigma^2/h),\ldots \quad \text{if } \sigma^2 < \mu h.$$

Then the singular points in the λ-plane are

$$(-[q_1^2\sigma^4-\mu^2h^2]/[2\sigma^2],0),\ (-[q_2^2\sigma^4-\mu^2h^2]/[2\sigma^2],0),$$

$$(-[q_3^2\sigma^4-\mu^2h^2]/[2\sigma^2],0),\ldots \qquad \text{if } \sigma^2 \geq \mu h,$$

or (A.13)

$$([q_0^2\sigma^4-\mu^2h^2]/[2\sigma^2],0),\ (-[q_1^2\sigma^4-\mu^2h^2]/[2\sigma^2],0),$$

$$(-[q_2^2\sigma^4-\mu^2h^2]/[2\sigma^2],0),\ldots \qquad \text{if } \sigma^2 < \mu h,$$

which are all on the real axis of the λ-plane.

By the residue theorem (Theorem 3.3.5), the integration of $L_\tau(\lambda)e^{\lambda t}$ along the simple closed curve from A→B→C→A in Figure 5 is $2\Pi i$ times the sum of the residues.

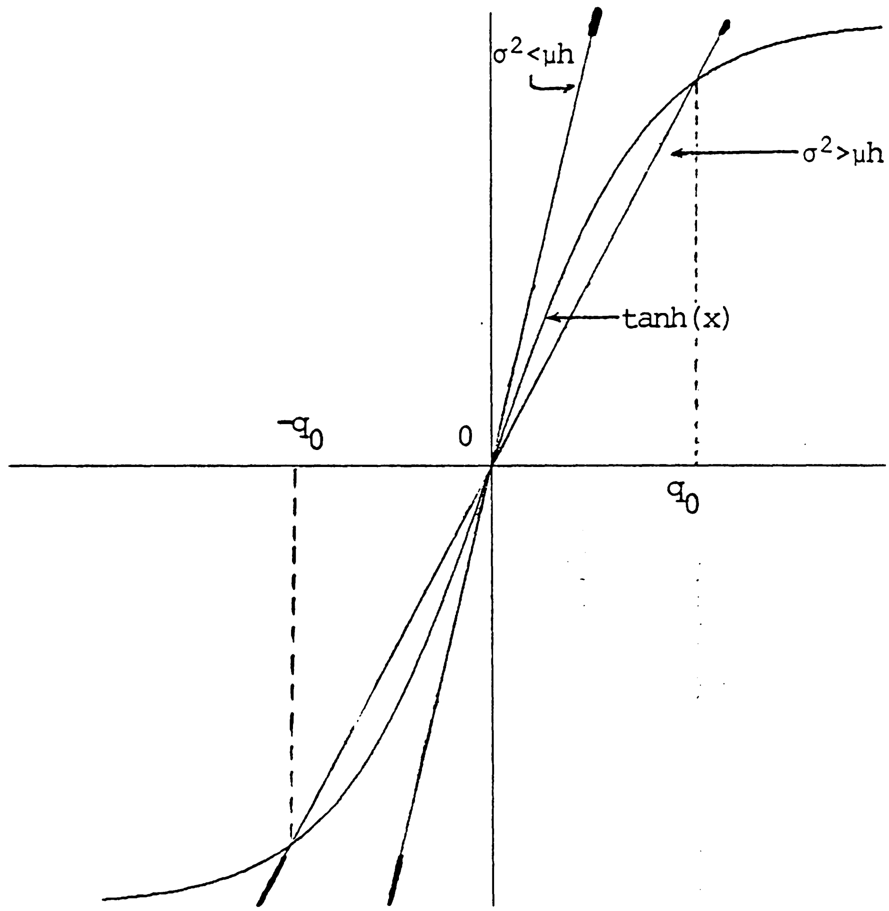

Figure 4: The singular points in the a-plane for q=0 when μ =0

Jordan's lemma [see Thomson (1950)] implies that if the integrand is analytic and converges uniformly to zero as λ increases indefinitely, then the integral along the infinite half circle is zero for $t>0$. Therefore the inverse Laplace tranform (5.1.2) is equal to the sum of the residues of $L_\tau(\lambda)e^{\lambda t}$.

Let the residue at each singular point r be R(r). Then

$$R(r)=\lim_{\lambda\to r}(\lambda-r)L_\tau(\lambda)e^{rt}$$

$$=be^{-[(\mu h/\sigma^2)+rt]}/\{[(\sigma^2-\mu h)/b]\cosh(hb/\sigma^2+h\sinh(hb/\sigma^2)\}$$

where $b=\sqrt{\mu^2+2\sigma^2 r}$. Therefore the inverse Laplace transform is the sum of the residues

$$f_\tau(t) = \Sigma_r R(r),$$

where Σ_r includes all singular points. Then

$$f_\tau(t) = \Sigma_{j=1}^{\infty} Q_j(t), \qquad \text{if } \sigma^2 \geq \mu h$$

$$\Sigma_{j=0}^{\infty} Q_j(t), \qquad \text{if } \sigma^2 < \mu h \qquad \text{(A.14)}$$

where

$$Q_j(t) = (A_j/B_j)e^{-C_j t},\ j=0,1,\ldots$$

$$A_0=q_0\sigma^2 e^{-\mu h/\sigma^2}$$

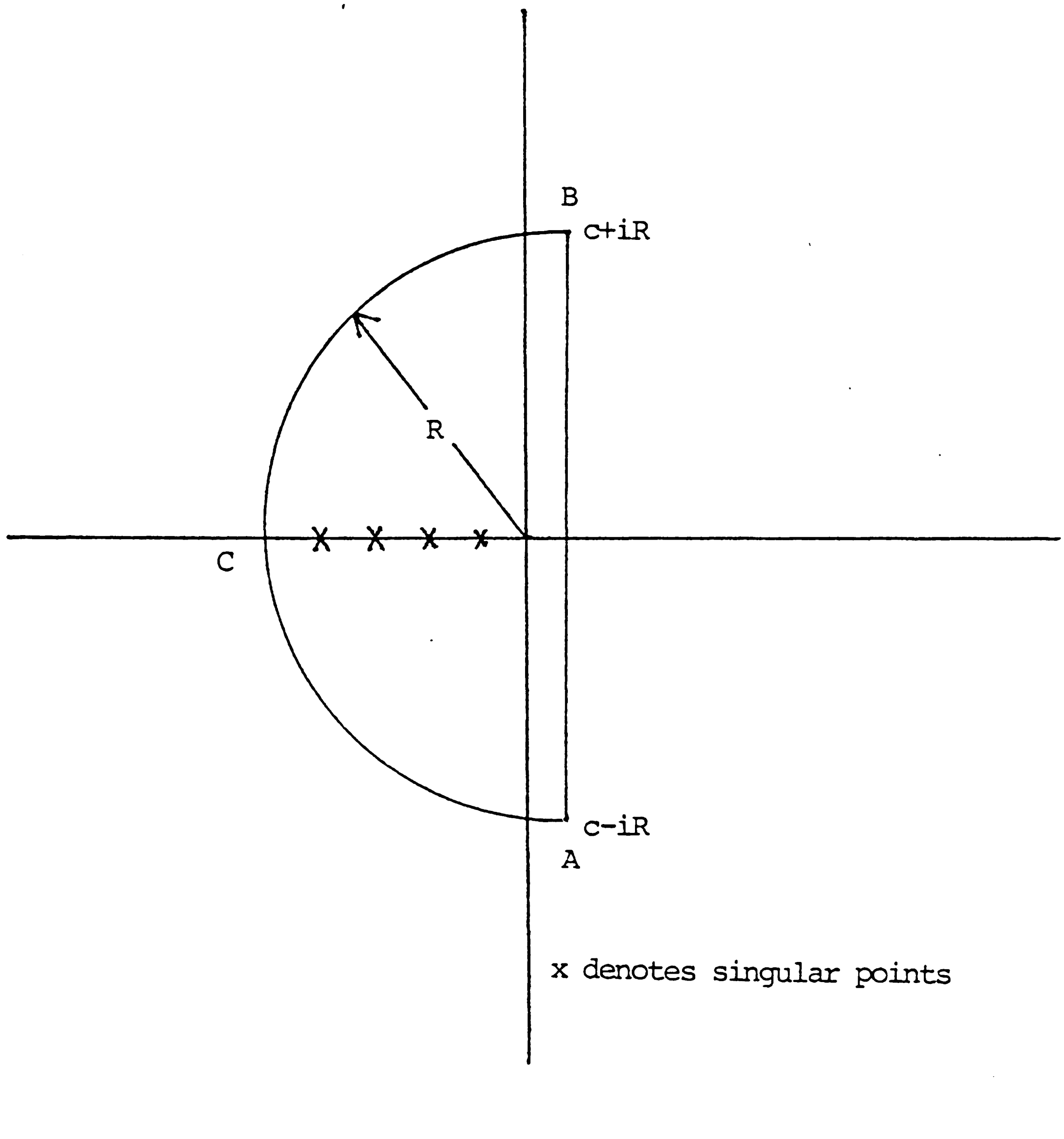

Figure 5: The singular points in the λ-plane and a simple closed half circle

$B_0 = h^2 \sinh(q_0) + h^2(\sigma^2 - \mu h)\cosh(q_0)/(q_0\sigma^2)$

$C_0 = (\mu^2 h^2 - q_0^2 \sigma^4)/(2\sigma^2 h^2)$

$A_j = q_j \sigma^2 e^{-\mu h/\sigma^2}$, $j=1,2....$

$B_j = h^2 \sin(q_j) - h^2(\sigma^2 - \mu h)\cos(q_j)/(q_j\sigma^2)$, $j=1,2...$

$C_j = (\mu^2 h^2 + q_j^2 \sigma^4)/(2\sigma^2 h^2)$, $j=1,2...$

and q_0 is the solution of

$$\mu h \tanh(q_0) - \sigma^2 q_0 = 0,$$

and the q_j's are the sequence of the solutions of

$$\mu h \tan(q_j) - \sigma^2 q_j = 0.$$

II. Next consider the case $\mu=0$.

If $\mu=0$, then

$$L_\tau(\lambda) = 1/\cosh[(h/\sigma)\sqrt{2\lambda}]. \qquad \text{(A.15)}$$

Let $a=\sqrt{2\lambda}$. Then $\lambda=a^2/2$ and

$$L_\tau(\lambda) = 1/\cosh(ha/\sigma)$$

1. There is no branch cut for the same reason as when $\mu \neq 0$.

2. By the same method for the case $\mu \neq 0$, we can obtain the singular points in the a-plane as

$$(0, \sigma[\Pi/2+\Pi j]/h), \; j=0, \pm 1, \pm 2, \ldots \qquad (A.16)$$

Thus the singular points in the λ-plane are

$$-\sigma^2(\Pi/2+\Pi j)^2/(2h^2), \; j=0,1,2,\ldots \qquad (A.17)$$

which are all on the real axis of the λ-plane. Hence, the residue at each singular point r is

$$R(r)=\lim_{\lambda \to r}(\lambda-r)L_\tau(\lambda)e^{rt}$$

$$=e^{rt}/\{(h/\sigma)(2\lambda)^{-1/2}\sinh[(h/\sigma)\sqrt{2\lambda}]\}.$$

Therefore the inverse Laplace transform is

$$f_\tau(t) = \Sigma_{j=0}^{\infty} R_j(t) \qquad (A.18)$$

where

$$R_j(t) = D_j e^{-E_j t}, \; j=0,1,\ldots$$

$$D_j=(-1)^j(j+1/2)\Pi^2\sigma^2/h^2, \; j=1,2\ldots$$

$$E_j=(j+1/2)^2\Pi^2\sigma^2/2h^2, \; j=1,2\ldots \qquad \|$$

<u>Proof of Corollary 5.1.1</u> : The expectation of τ with the truncation point T can be obtained, from the expression (5.1.1) and for a>0, b>0, as

$$\int_0^T t(ae^{-bt})dt+T\int_T^\infty ae^{-bt}dt=(a/b^2)(1-e^{-bt}). \quad (A.19)$$

Using the fact that $f_\tau(t)$ is of the form Σae^{-bt}, we can obtain the expectation of τ with the truncation point T as follows.

$$E\tau^T = \Sigma_{j=1}^\infty A_j(1-e^{-C_jT})/(B_jC_j^2),\ \sigma^2>\mu h,\ \mu\neq 0$$

$$\square\ \Sigma_{j=0}^\infty D_j(1-e^{-E_jT})/E_j^2,\ \mu=0 \qquad (A.20)$$

$$\square\ \Sigma_{j=0}^\infty A_j(1-e^{-C_jT})/(B_jC_j^2),\ \text{otherwise}\ \|$$

Appendix B

A FORTRAN function subprogram is listed for the ARL^T using the expression (5.1.6). The unconditional ARL^T can be obtained if the function subprogram FINTEG is integrated from 0 to 1 by using any integration routine.

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C   THIS IS A FUNCTION SUBPROGRAM TO OBTAIN THE AVERAGE RUN
C   LENGTH OF A CUSUM CHART WHEN THE CONTROL VALUE IS NOT
C   SPECIFIED.
C   THE PARAMETERS USED IN THIS PROGRAM ARE AS FOLLOWS.
C
C     T - TRUNCATION POINT
C
C     H - ORIGINAL BOUNDARY OF THE CUSUM CHART
C
C     HH - CONTINUITY-CORRECTED BOUNDARY
C
C     M - STANDARD SAMPLE SIZE
C
C     N - OBSERVED SAMPLE SIZE
C
C     U - MEAN OF THE STATISTIC USED IN THE CUSUM CHART
C
C     V - VARIANCE OF THE STATISTIC USED IN THE CUSUM CHART
C
C   IMSL SUBROUTINE MDNOR AND MDNRIS ARE USED TO GET THE
C   DISTRIBUTION FUNCTION AND THE INVERSE DISTRIBUTION
C   FUNCTION RESPECTIVELY. IF UNDERLYING DISTRIBUTION IS
C   DIFFERENT FROM THE NORMAL, THE CORRESPONDING
C   DISTRIBUTION AND THE INVERSE DISTRIBUTION FUNCTION
C   SHOULD BE USED.
C   FOR THE AVERAGE RUN LENGTH OF THE EXPRESSION (5.1.6)
C   WHEN THE CONTROL VALUE IS GIVEN, ONE OF THE SUBROUTINES
C   - ARLTGV, ARLTLV, ARLTEO, ARLTN - MAY BE USED ACCORDING
C   TO THE VALUE OF U.
C
C
      FUNCTION FINTEG(X)
      DATA H,T,M,N/5.55,1000.,39,10/
      COMMON U,V,H
```

```
      OSS=N
      CALL MDNRIS(X,FINX,IER)
      FINXMS=FINX-SHIFT
      CALL MDNOR(FINXMS,FFS)
      U1=OSS*(1.-FFS)
      U=U1-0.5*OSS
      VAR=U1*FFS
      STU=U/SQRT(VAR)
      STH=H/SQRT(VAR)
      IF(ABS(STU).GT.2.)GO TO 41
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C            THIS IS TO OBTAIN A MODIFIED H
C            NEW H IS CREATED INSTEAD OF ORIGINAL H
      CALL HMOD(STU,STH,STHM)
      H=STHM*SQRT(VAR)
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      GO TO 42
   41 IF(STU.LT.-2.)H=(STH+2.)*SQRT(VAR)
      IF(STU.GT.2.)H=(STH+.9)*SQRT(VAR)
   42 IF(U.LT.0.)GO TO 901
      IF(U.EQ.0.)GO TO 902
      UPH=U*H
      IF(UPH.LT.VAR.AND.UPH.GT.0.)GO TO 903
      CALL ARLTGV(U,VAR,H,T,E)
      GO TO 1110
  901 CALL ARLTN(U,VAR,H,T,E)
      GO TO 1110
  902 CALL ARLEO(H,VAR,T,E)
      GO TO 1110
  903 CALL ARLTLV(U,VAR,H,T,E)
 1110 SSS=M
      SSM=(SSS+1.)/2.
      SM1=SSM-1.
      C=FACT(SSS)/(FACT(SM1)**2)
      IF(E.GT.T)E=T
      FINTEG=E*C*(X*(1.-X))**SM1
1     RETURN
      END


C                    FUNCTION FOR FACTORIAL
      FUNCTION FACT(X)
      FACT=1.
      NNN=X
      DO 1 I=1,NNN
    1 FACT=FACT*I
      RETURN
      END
```

```
C          SUBROUTINE FOR SOLVING THE NONLINEAR EQUATION
C          USING THE BISECTION METHOD
      SUBROUTINE BISECT(F,A,B,XTOL,IFLAG)
      COMMON U,V,H
      IFLAG=0
      N=-1
      CALL F(A,FA)
CHECK FOR SIGN CHANGE
      CALL F(B,FB)
      IF(FA*FB.LE.0.) GO TO 5
      IFLAG=2
      WRITE(6,601)A,B
  601 FORMAT(/5X,'F(X) IS SAME SIGN AT TWO ENDPTS',2E15.7)
      RETURN
    5 ERROR=ABS(B-A)
    6 ERROR=ERROR/2.
CHECK FOR SUFFICIENTLY SMALL INTERVAL
      IF(ERROR.LE.XTOL) RETURN
      XM=(A+B)/2.
CHECK FOR UNREASONABLE ERROR REQUIREMENT
      IF(XM+ERROR.EQ.XM) GO TO 20
      CALL F(XM,FM)
      N=N+1
CHANGE TO NEW INTERVALS
      IF(FA*FM.LE.0.) GO TO 9
      A=XM
      FA=FM
      GO TO 6
    9 B=XM
      GO TO 6
   20 IFLAG=1
      RETURN
      END


C              FUNCTION SUBPROGRAM
C              FF(X)=U*H*SIN(X)-X*COS(X)
      SUBROUTINE FF(X,FFX)
      COMMON U,V,H
      TWOPHI=4.*ARSIN(1.)
      XX=AMOD(X,TWOPHI)
      FFX=U*H*SIN(XX)-X*V*COS(XX)
      RETURN
      END


C              FUNCTION SUBPROGRAM
C              FG(X)=UHSINH(X)-XVCOSH(X)
      SUBROUTINE FG(X,FGX)
      COMMON U,V,H
```

```
      FGX=U*H*SINH(X)-X*V*COSH(X)
      RETURN
      END


C               SUBROUTINE OF AVERAGE RUN LENGTH WITH
C               TRUNCATION POINT T FOR THE CASE
C               WHEN U*H> OR EQUAL TO VARIANCE
      SUBROUTINE ARLTGV(U,V,H,T,E)
      EXTERNAL FF,FG
      COUNT=0.
      AA=.00001
      BB=U*H/V+.0001
      IF(U*H.EQ.V)GO TO 300
      IF(U*H/V.GT.10.)GO TO 77
      CALL BISECT(FG,AA,BB,1.E-5,IFLAG)
      IF(IFLAG.GT.1)GO TO 1
      P=(AA+BB)/2.
      IF(P.GT.10.)P=10.
      GO TO 788
   77 P=U*H/V
  788 R1=(P*V/H)*EXP(-U*H/V)
      R2=(V-U*H)*(H/(P*V))*COSH(P)
      R3=H*SINH(P)
      R4=-(P*P*V*V-U*U*H*H)/(2.*H*H*V)
      IF(R2+R3.EQ.0.)GO TO 4444
      IF(R4.EQ.0.)GO TO 5555
      E=(R1/((R2+R3)*(R4*R4)))*(1.-EXP(-R4*T))
      GO TO 5556
 5555 E=R1/(R2+R3)
      GO TO 5556
 4444 PRINT 44444
44444 FORMAT(3X,'R2+R3=0')
      E=T
 5556 IF(ABS(E).LT.1.E-5)GO TO 1
      GO TO 400
  300 E=0.
  400 VV=-1.
      PHI=ARCOS(VV)
      A=0.
   50 B=1.5*PHI+PHI*COUNT
      A=A+PHI
      CALL BISECT(FF,A,B,1.E-5,IFLAG)
      IF(IFLAG.GT.1)GO TO 1
      Q=(A+B)/2.
      TWOPHI=2.*PHI
      AMODQ=AMOD(Q,TWOPHI)
      S1=-(Q*V/H)*EXP(-U*H/V)
      S2=(V-U*H)*(H/(Q*V))*COS(AMODQ)
      S3=H*SIN(AMODQ)
```

```
      S4=(Q*Q*V*V+U*U*H*H)/(2.*V*H*H)
      RES=(S1/((S2-S3)*(S4*S4)))*(1.-EXP(-S4*T))
      IF(ABS(RES).LT.1.E-5)GO TO 1
      E=E+RES
      COUNT=COUNT+1.
      GO TO 50
    1 RETURN
      END

C               SUBROUTINE OF AVERAGE RUN LENGTH WITH
C               TRUNCATION POINT T FOR THE CASE
C               WHEN U IS LESS THAN 0
      SUBROUTINE ARLTN(U,V,H,T,E)
      EXTERNAL FF
      COUNT=0.
      E=0.
      VV=-1.
      PHI=ARCOS(VV)
      B=0.
  150 A=PHI/2.+PHI*COUNT
      B=B+PHI
      CALL BISECT(FF,A,B,1.E-5,IFLAG)
      IF(IFLAG.GT.1.)GO TO 1
      Q=(A+B)/2.
      TWOPHI=2.*PHI
      AMODQ=AMOD(Q,TWOPHI)
      ARG1=-U*H/V
      S1=-(Q*V/H)*EXP(ARG1)
      S2=(V-U*H)*(H/(Q*V))*COS(AMODQ)
      S3=H*SIN(AMODQ)
      S4=(Q*Q*V*V+U*U*H*H)/(2.*V*H*H)
      ARG2=-S4*T
      RES=(S1/((S2-S3)*(S4*S4)))*(1.-EXP(ARG2))
      IF(ABS(RES).LT.1.E-5)GO TO 1
      E=E+RES
      COUNT=COUNT+1
      GO TO 150
   11 ARG3=2.*U*H/V
      E=(-1./U)*(H-(0.5*V/U)*(EXP(ARG3)-1.))
    1 RETURN
      END

C               SUBROUTINE OF AVERAGE RUN LENGTH WITH
C               TRUNCATION POINT T FOR THE CASE
C               WHEN U IS EQUAL TO 0
      SUBROUTINE ARLEO(H,V,T,E)
      H=H/SQRT(V)
      COUNT=0.
```

```
      E=0.
      VV=-1.
      PHI=ARCOS(VV)
      S2=-1.
  250 S1=PHI/2.+COUNT*PHI
      S2=S2*(-1.)
      S3=H*H
      S4=(S1*S1)/(2.*S3)
      RES=((S1*S2)/(S3*(S4*S4)))*(1.-EXP(-S4*T))
      IF(ABS(RES).LT.1.E-5) GO TO 1
      E=E+RES
      COUNT=COUNT+1.
      GO TO 250
    1 RETURN
      END

C              SUBROUTINE OF AVERAGE RUN LENGTH WITH
C              TRUNCATION POINT T FOR THE CASE
C              WHEN U*H IS BETWEEN 0 AND 1
      SUBROUTINE ARLTLV(U,V,H,T,E)
      EXTERNAL FF
      COUNT=0.
      VV=-1.
      E=0.
      PHI=ARCOS(VV)
      A=-PHI+.000001
  350 B=.5*PHI+PHI*COUNT
      A=A+PHI
      CALL BISECT(FF,A,B,1.E-5,IFLAG)
      IF(IFLAG.GT.1)GO TO 1
      Q=(A+B)/2.
      TWOPHI=2.*PHI
      AMODQ=AMOD(Q,TWOPHI)
      S1=-(Q*V/H)*EXP(-U*H/V)
      S2=(V-U*H)*(H/(Q*V))*COS(AMODQ)
      S3=H*SIN(AMODQ)
      S4=(Q*Q*V*V+U*U*H*H)/(2.*V*H*H)
      RES=(S1/((S2-S3)*(S4*S4)))*(1.-EXP(-S4*T))
      IF(ABS(RES).LT.1.E-5)GO TO 1
      E=E+RES
      COUNT=COUNT+1
      GO TO 350
    1 RETURN
      END

      SUBROUTINE FHMOD(U,H,HH,FHH)
      PHI=2.*ARSIN(1.)
      CALL MDNOR(-U,NDFN)
```

```
      CALL MDNOR(U,NDFP)
      CALL MDNOR(ABS(U),NDFA)
      CSHI=-U+EXP(-U**2/2.)/SQRT(2.*PHI)/NDFN
      Q=-U-EXP(-U**2/2.)/(SQRT(2.*PHI)*NDFP)
      ETA=(1.-NDFA)/NDFA
      IF(-U.GT.0.)GO TO 1
      RRL=H+CSHI+Q*(EXP(2.*U*(H+CSHI))-1.)/(1.-ETA)
      GO TO 2
    1 RRL=H+CSHI+(EXP(2.*U*H)-1.)/(-2.*U)
    2 ARL=HH+(EXP(2.*U*HH)-1.)/(-2.*U)
      FHH=RRL-ARL
      RETURN
      END


      SUBROUTINE HMOD(U,H,HM)
      EXTERNAL FHMOD
      A=H
      B=H+5.
      IF (U.GT.-0.002.AND.U.LT.5./H)GO TO 2
      CALL BISECT(FHMOD,A,B,1.E-5,IFLAG)
      HM=(A+B)/2.
      IF(IFLAG.GT.1)HM=B
    1 CONTINUE
      RETURN
    2 OU=U
      U=-0.002
      CALL BISECT(FHMOD,A,B,1.E-5,IFLAG)
      HM1=(A+B)/2.
      U=5./H
      CALL BISECT(FHMOD,A,B,1.E-5,IFLAG)
      HM2=(A+B)/2.
      W=(OU+0.002)/(5./H+0.002)
      HM=W*HM2+(1.-W)*HM1
      RETURN
      END
```

Appendix C

The conditional mean, the conditional variance, and the conditional reference value $k_m(\underline{X})$ for the sum of the ranks statistic, given the standard sample $\underline{X}$, are derived. The joint rank R_j can be expressed as

$$R_j=\Sigma_k(jk) + \Sigma_{j=1}^{n}\Psi(Y_j-Y_i) + 1 \qquad (C.1)$$

where $\Psi(x)=1,0$ for $x\geq,<0$ and $(jk)=\Psi(Y_j-X_k)$ and Σ_k is summation from k=1 to m.

Then

$$L^2S^S(\underline{X})=\Sigma_{j=1}^{n}R_j^2$$

$$=\Sigma_{j=1}^{n}[\Sigma_k(jk)+\Sigma_{i=1}^{n}\Psi(Y_j-Y_i)+1]^2$$

$$=\Sigma_{j=1}^{n}[\Sigma_k((j)k)+\Sigma_{i=1}^{n}\Psi(Y_{(j)}-Y_{(i)})+1]^2$$

$$=\Sigma_{j=1}^{n}\{[\Sigma_k((j)k)]^2+j^2+2j\Sigma_k((j)k)\}$$

$$=\Sigma_{j=1}^{n}\{[\Sigma_k((j)k)]^2+2I_{(j)}\Sigma_k((j)k)\}+n(n+1)(2n+1)/6$$

where I_j is j-th number of random permutation of $(1,2,\ldots,n)$

$$=\Sigma_{j=1}^{n}\{[\Sigma_k(jk)]^2+2I_j\Sigma_k(jk)\}+n(n+1)(2n+1)/6$$

$$=\Sigma_{j=1}^{n}[\Sigma_k\Sigma_{k'}(jk)(jk')+2I_j\Sigma_k(jk)]$$

$$+n(n+1)(2n+1)/6 \qquad (C.2)$$

Using the fact that $EI_j=(n+1)/2$, and I_j and (jk) are independent gives

$$L^2ES^S(\underline{X})=n[\Sigma_k\Sigma_{k'}P(Y_j>M(k,k'))+(n+1)\Sigma_kP(Y_j>X_k)]$$

$$+\ n(n+1)(2n+1)/6$$

where $M(k,k')=\max(X_k,X_{k'})$

Therefore,

$$ES^S(\underline{X})=(n/L^2)\{\Sigma_k\Sigma_{k'}[1-F(M(k,k')-\Delta)]$$

$$+(n+1)\Sigma_k[1-F(X_k-\Delta)]+(n+1)(2n+1)/6\} \quad (C.3)$$

From (C.2),

$$L^4VarS^S(\underline{X})=\Sigma_{j=1}^n\Sigma_{j'=1}^n Cov[\Sigma_k\Sigma_{k'}(jk)(jk')+2I_j\Sigma_k(jk),$$

$$\Sigma_k\Sigma_{k'}(j'k)(j'k')+2I_{j'}\Sigma_k(j'k)]$$

$$=\Sigma_{j=1}^n\Sigma_{j'=1}^n\{Cov[\Sigma_k\Sigma_{k'}(jk)(jk'),\Sigma_k\Sigma_{k'}(j'k)(j'k')]$$

$$+4Cov[I_j\Sigma_k(jk),\Sigma_k\Sigma_{k'}(j'k,j'k')]$$

$$+4Cov[I_j\Sigma_k(jk),I_{j'}\Sigma_k(j'k)]\}$$

$$=n\{Cov[\Sigma_k\Sigma_{k'}(jk)(jk'),\Sigma_k\Sigma_{k'}(jk)(jk')]$$

$$+4Cov[I_j\Sigma_k(jk),\Sigma_k\Sigma_{k'}(jk)(jk')]$$

$$+4var[I_j\Sigma_k(jk),I_j\Sigma_k(jk)]\}$$

$$+4\Sigma_{j=1}^{n}\Sigma_{j'=1}^{n}\mathrm{Cov}[I_j\Sigma_k(jk),I_{j'}\Sigma_k(j'k)].\quad (C.4)$$

Here,

$$\mathrm{Cov}[\Sigma_k\Sigma_{k'}(jk)(jk'),\Sigma_k\Sigma_{k'}(j'k)(j'k')]$$

$$=\Sigma_k\Sigma_{k'}\Sigma_q\Sigma_{q'}\mathrm{Cov}[(jk)(jk'),(jq)(jq')]$$

$$=\Sigma_k\Sigma_{k'}\Sigma_q\Sigma_{q'}[e(jk)(jk')(jq)(jq')-E(jk)(jk')E(jq)(jq')]$$

$$=\Sigma_k\Sigma_{k'}\Sigma_q\Sigma_{q'}\{1-F(M(k,k',q,q')-\Delta)$$

$$-[1-F(M(k,k')-\Delta)][1-F(M(q,q')-\Delta)]\}\quad (C.5)$$

and

$$\mathrm{Cov}[I_j\Sigma_k(jk),\Sigma_k\Sigma_{k'}(jk)(jk')]$$

$$=(n+1)/2[\Sigma_k\Sigma_{k'}\Sigma_{k''}E(jk)(jk')(jk'')-\Sigma_kE(jk)\Sigma_k\Sigma_{k'}E(jk)(jk')]$$

$$=(n+1)/2\{\Sigma_k\Sigma_{k'}\Sigma_{k''}[1-F(M(k,k',k'')-\Delta)]$$

$$-\Sigma_k[1-F(X_k-\Delta)]\Sigma_k\Sigma_{k'}[1-F(M(k,k')-\Delta)]\}\quad (C.6)$$

and

$$\mathrm{Var}[I_j\Sigma_k(jk)]=EI_j^2\Sigma_k\Sigma_{k'}E(jk)(jk')-[EI_j\Sigma_kE(jk)]^2$$

$$=(n+1)(2n+1)/6\Sigma_k\Sigma_{k'}[1-F(M(X_k,X_{k'})-\Delta)]$$

$$-(n+1)^2/4\{\Sigma_k[1-F(X_k-\Delta)]\}^2\quad (C.7)$$

and

$$\mathrm{Cov}[I_j\Sigma_k(jk),I_{j'}\Sigma_k(j'k)]$$

$$=EI_jI_{j'}\Sigma_k\Sigma_{k'}E(jk)(j'k')-EI_j\Sigma_kE(jk)\Sigma_{k'}E(j'k')$$

$$=\mathrm{Cov}(I_j,I_{j'})[\Sigma_kE(jk)]^2$$

$$=-(n+1)/12\{\Sigma_k[1-F(X_k-\Delta)]\}^2 \tag{C.8}$$

Substituting (C.5), (C.6), (C.7), (C.8) into (C.4) we obtain

$$\begin{aligned}\mathrm{Var}S^S(\underline{X})=(1/L^4)\{&n\Sigma_k\Sigma_{k'}\Sigma_q\Sigma_{q'}[1-F(M(k,k',q,q,)-\Delta)]\\&-[1-F(M(k,k')-\Delta)][1-F(M(q,q')-\Delta)]\\&+2n(n+1)[\Sigma_k\Sigma_{k'}\Sigma_{k''}(1-F(M(k,k',k'')\_\Delta))\\&\quad-\Sigma_k(1-F(M(k,k')-\Delta))]\\&+[2n(n+1)(2n+1)/3][\Sigma_k\Sigma_{k'}(1-F(M(k,k')-\Delta))\\&\quad-(\Sigma_k(1-F(X_k-\Delta)))^2]\}\end{aligned} \tag{C.9}$$

Now, for the sample distribution function F_m,

$$\Sigma_k\Sigma_{k'}[1-F_m(M(k,k'))]$$

$$=\Sigma_k[1-F_m(X_k)]+\Sigma_k\Sigma_{k'}[1-F_m(M(k,k'))]$$

$$=\Sigma_k[1-(k-0.5)/m]+\Sigma_k\Sigma_{k'}[1-(\max(k,k')-0.5)/m]$$

$$=m/2+m(m-1)+(m-1)/2-(1/m)\Sigma_k\Sigma_{k'}\max(k,k')$$

where $\Sigma_k \Sigma_{k'} \max(k,k') = (2/3)m(m+1)(m-1)$

$$=(2m^2+1)/6 \qquad (C.10)$$

Therefore, if we let $F=F_m$ and $\Delta=0$ in (C.3) we obtain

$$k_m^S(\underline{X})=(n/6L^2)[2m^2+(n+1)(3m+2n+1)+1] \quad \|$$

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