WELL-POSEDNESS QUESTIONS AND APPROXIMATION SCHEMES FOR A GENERAL CLASS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper we consider approximation schemes and questions of well-posedness for a general class of functional differential equations of neutral-type (NFDE) where the difference operator does not have an atom at zero. Equations of this type occur in the modeling of certain aeroelastic control problems and include many singular integro-differential equations.

We obtain general necessary and sufficient conditions for the well-posedness of functional differential equations of neutral-type on the Banach-spaces $\mathbb{R}^n \times L_p$. As an example the well-posedness of the non-atomic NFDE-system that arises in the study of aeroelasticity is established on $\mathbb{R}^n \times L_p$, $1 \leqslant p < 2$.

Employing the equivalence between generalized solutions of NFDEs and mild solutions of the

"corresponding" abstract Cauchy-problems, we make use of general approximation results for well-posed Cauchy-problems to establish and analyze the convergence of the "averaging projection" scheme on the Banach spaces $\mathbb{R}^n \times L_p$, $1<p<\infty$, for a class of problems with atomic difference operators.

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Dedicated to the memory of

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CHAPTER I.

INTRODUCTION AND NOTATION

1.1 Introduction

Approximation of functional differential equations by high order ordinary differential equations has received considerable attention during the past few years (see e.g. [3],[4],[5],[26],[28]). The general approach to the approximation problem is to develop an abstract state space model for the functional differential equation and use the basic structure provided by the resulting abstract evolution equation for the construction of approximating schemes. Concerning the choice of an appropriate state space, it was shown (see [3],[5] for retarded and [10],[33] for neutral functional differential equations) that certain classes of functional differential equations with appropriate initial data can be transformed into well-posed Cauchy problems in the product spaces $\mathbb{R}^n \times L_p$. The product space model also proved to be very useful in investigating a variety of control and identification problems for systems governed by functional differential equations.

In this paper we extend the results in [10] to a general class of functional differential equations of neutral type including equations where the difference operator does not have an atom at zero. Neutral type systems with nonatomic difference operators occur in the modeling of certain aeroelastic control problems and include many singular integro-differential equations.

The paper is organized as follows. In Chapter II, Section 2.1 we begin by recalling general results concerning the equivalence between generalized solutions of linear functional differential equations and mild solutions of the "corresponding" abstract Cauchy problems. In Section 2.2 we study general necessary conditions for the well-posedness of functional differential equations of neutral type on the Banach spaces $\mathbb{R}^n \times L_p$. In Section 2.3 we consider sufficient conditions for the well-posedness of these equations. We also show that the functional differential equation that has been used to model the elastic motions of a two-dimensional airfoil in unsteady flows is well-posed on $\mathbb{R}^n \times L_p$ for $1 \leqslant p < 2$. In Chapter III , Section 3.1 we consider general approximation schemes for linear abstract Cauchy problems on the Banach spaces $\mathbb{R}^n \times L_p$.

In Section 3.2 we analyze the convergence of the "averaging" scheme on the Banach spaces $\mathbb{R}^n \times L_p$ for a class of problems with atomic difference operators.

1.2 Notation

The notation used in this paper is fairly standard. For $1 \leqslant p < \infty$, $k=0,1,2,\ldots$, a closed interval I in $\mathbb{R}$ and a Banach-space X , the symbols $L_p(I;X)$, $W^{k,p}(I;X)$ and $C^k(I;X)$ will denote the spaces of functions $\varphi: I \longrightarrow X$ which are respectively, Lebesque integrable when raised to the p^{th} power, absolutely continuous with absolutely continuous derivatives of order j, $j=1,2,\ldots,k-1$ and the k^{th} derivative belongs to $L_p(I,X)$, and continuous with k continuous derivatives on I. Whenever $I=[-r,0]$ with $r>0$ and $X=\mathbb{R}^n$ we shall simply write L_p, $W^{k,p}$ and C^k for $L_p([-r,0];\mathbb{R}^n)$, $W^{k,p}([-r,0];\mathbb{R}^n)$ and $C^k([-r,0];\mathbb{R}^n)$, respectively. The symbols $L_{p,loc}([a,\infty);X)$ and $W^{k,p}_{loc}([a,\infty);X)$ stand for the spaces of functions $\varphi:[a,\infty) \longrightarrow X$, which belong to $L_p([a,b];X)$ or $W^{k,p}([a,b];X)$ for each $b>a$. When $a = 0$ and $X = \mathbb{R}^n$ we shall use the suppressed notations $L_{p,loc}$ and $W^{k,p}_{loc}$, respectively. For Banach spaces X and Y, the symbol $\mathbb{B}(X,Y)$ denotes the Banach space of all bounded linear operators from X into Y. The norm of an element x

contained in a normed linear space X is denoted by $\|x\|_X$, or more simply by $\|x\|$ when it is clear from the context which space is intended. Also, we shall denote $\|\ \|_{L_p}$ by $\|\ \|_p$ when it is convenient to do so. For a linear operator T we use the standard notation: $\mathbb{D}(T)$, $\mathbb{R}(T)$, $\mathbb{N}(T)$ for the domain, range and null space of T, respectively. By $\rho(T)$ we denote the resolvent set in the complex plane $\mathbb{C}$ of a linear operator T, and for $\lambda\in\rho(T)$, the symbol $R(\lambda;T)$ denotes the resolvent of T. If $x:[-r,a)\longrightarrow\mathbb{R}^n$ for some $0<a\leqslant\infty$, then we define the function $x_t:[-r,0]\longrightarrow\mathbb{R}^n$ for $0\leqslant t\leqslant a$ by $x_t(s)=x(t+s)$. The set of $\mathbb{R}^n$-valued measurable functions on $[-r,0]$ will be denoted by $\mathbb{M}$. Each theorem, lemma, remark,...etc. is labeled by a.b.c , where a, b and c are positive integers denoting the chapter, the section and the occurence within the section, respectively.

CHAPTER II

LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE ON $\mathbb{R}^n \times L_p$

2.1 Preliminaries

We consider the linear functional differential equation of neutral-type (NFDE)

$$\frac{d}{dt} Dx_t = Lx_t + f(t) \tag{2.1.1}$$

with initial data

$$Dx_o(\cdot)=\eta \quad ; \quad x_o(s)=\varphi(s) \ , \ -r\leqslant s<0 \tag{2.1.2}$$

where D and L are linear $\mathbb{R}^n$-valued operators with domains $\mathcal{D}(D)$ and $\mathcal{D}(L)$ subspaces of the Lebesque-measurable $\mathbb{R}^n$-valued functions on $[-r,0]$. We assume that $W^{1,p} \subseteq \mathcal{D}(D)\cap\mathcal{D}(L)$, $(\eta,\varphi)\in\mathbb{R}^n\times L_p$, $f\in L_{p,loc}$, $1\leqslant p<\infty$, $0\leqslant r<\infty$ and n is a positive integer.

Following Borisovic and Turbabin [6] a number of other authors have studied various classes of functional differential equations (FDE) in the context of functional analytic semigroup theory. The basic approach in this direction is to construct a generator for a C_0-semigroup which may be associated with the solutions of the FDE and then investigate the relations between this semigroup and solutions of the FDE.

In this paper we are interested in FDEs which generate C_0-semigroups on the product space $\mathbb{R}^n \times L_p$. The product space model was developed for retarded functional differential equations (RFDE), and it proved very useful in investigating certain control and identification problems for RFDE systems (see [3],[4],[5]). The concept that makes it possible to treat NFDEs on $\mathbb{R}^n \times L_p$ was introduced by Burns, Herdman and Stech in [9] and [10]. For the convenience of the reader, we recall some results concerning the use of $\mathbb{R}^n \times L_p$ as state space for NFDEs.

Define the linear operator $\mathcal{Q}$ with domain

$$\mathcal{D}(\mathcal{Q}) = \{(\eta,\varphi) \in \mathbb{R}^n \times L_p \; / \; \varphi \in W^{1,p} \;, \; D\varphi = \eta \} \tag{2.1.3}$$

by

$$\mathcal{Q}(\eta,\varphi) = (L\varphi,\dot{\varphi}). \tag{2.1.4}$$

It was shown in [10] that if $\mathcal{Q}$ defined by (2.1.3)-(2.1.4) generates a C_0-semigroup on $\mathbb{R}^n \times L_p$, then i) D and L must belong to $\mathcal{B}(W^{1,p};\mathbb{R}^n)$ and ii) $D \notin \mathcal{B}(L_p;\mathbb{R}^n)$. Moreover, under the stronger assumption that $D \in \mathcal{B}(C;\mathbb{R}^n)$ and D has an atom at zero, Burns, Herdman and Stech (see [10]) established the equivalence between generalized solutions to the NFDE (2.1.1) and mild solutions to the Cauchy-problem

$$\dot{z}(t) = \mathcal{Q}z(t) + (f(t),0), \tag{2.1.5}$$

where $\mathcal{Q}$ is defined by (2.1.3)-(2.1.4).

In the same paper the authors gave an example of a well-posed problem with a nonatomic D operator. They considered the integral equation

$$\int_{-r}^{0} |s|^{-\alpha} x(t+s)ds = \eta, \quad 0<t, \ 0<\alpha<1 \ , \tag{2.1.6}$$

with initial data

$$x(s) = \varphi(s) \ , \ -r \leqslant s \leqslant 0. \tag{2.1.7}$$

If D and L are defined by

$$L\varphi \equiv 0 \ , \ D\varphi = \int_{-r}^{0} |s|^{-\alpha} \varphi(s)ds, \tag{2.1.8}$$

then (2.1.6)-(2.1.7) can be considered as the integrated form of (2.1.1)-(2.1.2) with $f(t) \equiv 0$. It was shown in [10] that if D and L are given by (2.1.8), then $\mathcal{Q}$ generates a C_o-semigroup on $\mathbb{R}^n \times L_p$ for $1 \leqslant p < 1/(1-\alpha)$.

Kappel and Zhang (see [27],[37]) considered the <u>scalar</u> equation

$$Dx_t = D\varphi \ , \quad 0<t \tag{2.1.9}$$

with initial data

$$x_o(s) = \varphi(s) \ , \ -r \leqslant s \leqslant 0, \tag{2.1.10}$$

where $\varphi \in C$ and D is not atomic at $s = 0$. In [37] Zhang showed that if (2.1.9)-(2.1.10) has a unique continuous solution $x(t,\varphi)$ on $[-r,\infty)$ for each $\varphi \in C$ and $x(t,\varphi)$ depends continuously on φ, then D has to be <u>weakly</u> <u>atomic</u> at $s = 0$, i.e.

$$\lim_{\substack{\lambda \to \infty \\ \lambda \in \mathbb{R}}} | \lambda D(e^{\lambda \cdot}) | = \infty. \tag{2.1.11}$$

It was shown in [9] (see Theorem 3.2) that if (2.1.1)-(2.1.2) is well-posed in $\mathbb{R}^n \times L_p$, then

(2.1.1)-(2.1.2) leads to a well-posed problem on C. It follows that if the scalar version of (2.1.1)-(2.1.2) is well-posed in $\mathbb{R}\times L_p$, then D is weakly atomic at s = 0. For the scalar case, the assumption that D is weakly atomic at s=0 implies that for all real λ sufficiently large the operator $\mathcal{Q}$ has λ in its resolvent. However, for vector equations the problem becomes much more complex. Recall that $\lambda\epsilon\rho(\mathcal{Q})$ if and only if

$$\Delta(\lambda) = \lambda D(e^{\lambda\cdot}I) - L(e^{\lambda\cdot}I) \tag{2.1.12}$$

is nonsingular (see Lemma 2.3 in [10]). As the following example shows, condition (2.1.11) is not sufficient to imply that $\rho(\mathcal{Q})$ is non-empty in the case n = 2 .

<u>EXAMPLE</u> <u>2.1.1:</u> Let $L\varphi(\cdot) \equiv 0$ and $D\epsilon\mathcal{B}(C;\mathbb{R}^2)$ be defined by

$$D\varphi(\cdot) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\varphi(0).$$

Note that $\|\lambda D(e^{\lambda\cdot}I)\|\longrightarrow\infty$ but $\Delta(\lambda) = \lambda D(e^{\lambda\cdot}I)$ is singular for all λ. Consequently, by the Hille Yosida Theorem $\mathcal{Q}$ defined by (2.1.3)-(2.1.4) can not generate a C_o-semigroup on $\mathbb{R}^2\times L_p$. ∎

The main objective in this chapter is to extend the results in [10], i.e. to get stronger necessary conditions and weaker sufficient conditions for the well-posedness of the NFDE on $\mathbb{R}^n \times L_p$. In the remainder of this section we recall some definitions and equivalence results which we shall use throughout Chapter II.

<u>DEFINITION</u> <u>2.1.2:</u> A <u>classical</u> <u>solution</u> to the initial value problem (2.1.1)-(2.1.2) is a function $x:[-r,+\infty)\longrightarrow\mathbb{R}^n$ such that

i) $x \in W^{1,p}_{loc}$,

ii) $x_0(s) = \varphi(s)$ for $-r \leqslant s < 0$,

iii) $y(t) = Dx_t \in W^{1,p}_{loc}$,

iv) $y(0) = \eta$,

v) $\frac{d}{dt} Dx_t = Lx_t + f(t)$, a.e on $[0,\infty)$. ■

<u>DEFINITION</u> <u>2.1.3:</u> The system (2.1.1)-(2.1.2) is <u>well-posed</u> <u>in</u> <u>the</u> <u>strong</u> <u>sense</u> if given $x_0 = \varphi \in W^{1,p}$ and $y(0) = D\varphi$, there exists a unique classical solution to the homogeneous problem (i.e. to (2.1.1) with $f \equiv 0$), and the solution depends continuously on the initial function φ, relative to the respective topologies of $W^{1,p}$ and $W^{1,p}_{loc}$. ■

<u>DEFINITION</u> <u>2.1.4:</u> A <u>generalized</u> <u>solution</u> of the initial

value problem (2.1.1)-(2.1.2) is a pair of functions $y:[0,\infty)\longrightarrow\mathbb{R}^n$, $x:[-r,+\infty)\longrightarrow\mathbb{R}^n$ such that i) $y(\cdot)$ is continuous and $x_t(\cdot)\in L_p$, ii) $x_o(s) = \varphi(s)$ on $[-r,0]$, iii) the pair (y,x) satisfies the system of integral equations

$$y(t) = \eta + \int_0^t (Lx_s + f(s))ds \tag{2.1.14}$$

and

$$y(t) = Dx_t \tag{2.1.15}$$

for almost all $t \geqslant 0$. ∎

<u>DEFINITION</u> <u>2.1.5:</u> The system (2.1.1)-(2.1.2) is <u>well-posed in the weak sense</u>, if given initial data $(\eta,\varphi)\in \mathbb{R}^n \times L_p$, there exists a unique generalized solution which depends continuously on (η,φ). ∎

<u>LEMMA</u> <u>2.1.6:</u> (see [33], Theorem A) The following statements are equivalent

i) The operators D and L belong to $\mathcal{B}(W^{1,p};\mathbb{R}^n)$ and (2.1.1)-(2.1.2) is well-posed in the weak sense.

ii) D and L belong to $\mathcal{B}(W^{1,p};\mathbb{R}^n)$ and (2.1.1)-(2.1.2) is well-posed in the strong sense.

iii) The operator $\mathcal{Q}$, defined by (2.1.3)-(2.1.4) is the infinitesimal generator of a C_o-semigroup (i.e. (2.1.5) is well-posed in $\mathbb{R}^n \times L_p$). ∎

<u>LEMMA</u> <u>2.1.7:</u> (see [33], Theorem B) Assume that the system (2.1.1)-(2.1.2) is well-posed in the weak sense and $\mathcal{Q}$ is defined by (2.1.3)-(2.1.4) . If $z(t)$ is the mild solution to (2.1.5) corresponding to the initial data $z(0) = (\eta,\varphi)$ and forcing term f, then $z(t) = (y(t),x_t)$ where $(y(t),x_t)$ is the unique generalized solution of (2.1.1)-(2.1.2). Furthermore, if $x(t)$ is a classical solution of (2.1.1)-(2.1.2), then the corresponding solution $z(t)$ is strong and is given by $z(t) = (Dx_t,\ x_t)$. ∎

2.2 Necessary conditions for the well-posedness

In this section we consider general necessary conditions for the well-posedness in $\mathbb{R}^n \times L_p$ for problems of the form (2.1.1)-(2.1.2). Our approach to this problem is based on Lemma 2.1.6 that implies that if the NFDE (2.1.1)-(2.1.2) is well-posed in $\mathbb{R}^n \times L_p$, then $\mathcal{Q}$, defined by (2.1.3)-(2.1.4), is the infinitesimal generator of a C_o-semigroup $(S(t))_{t \geqslant 0}$. We then make use of general characterizations of the infinitesimal generators of C_o-semigroups (see [19],[23],[30]) to impose conditions on D and L in (2.1.1)-(2.1.2). The main result of this section (Theorem 2.2.7 below) extends the necessary conditions by Burns, Herdman and Stech [10] in that it specifies the asymptotic behavior of $D(e^{\lambda \cdot} I)$, i.e. it provides information on the "structure" of the operator D at $s = 0$ needed for the well-posedness.

If $\mathcal{Q}$ defined by (2.1.3)-(2.1.4) is the infinitesimal generator of a C_o-semigroup, then it follows (see Theorem 2.1 in [10]) that $D, L \in \mathcal{B}(W^{1,p};\mathbb{R}^n)$. For $\varphi \in W^{1,p}$ the following representations hold

$$L\varphi = \int_{-r}^{0}[F(s)\varphi(s) + G(s)\dot{\varphi}(s)]ds \tag{2.2.1}$$

and

$$D\varphi = \int_{-r}^{0}[A(s)\varphi(s) + B(s)\dot{\varphi}(s)]ds , \tag{2.2.2}$$

where $A(\cdot)$, $B(\cdot)$, $F(\cdot)$ and $G(\cdot)$ are $n\times n$ matrix valued functions whose column vectors belong to L_q, where q is the exponent conjugate to p, i.e. $1/p + 1/q = 1$.

Note that the matrix valued function B in the representation (2.2.2) can not be zero a.e. on $[-r,0]$, i.e. D can not be bounded as an operator on L_p (see Remark 2.1 in [10]). It follows from Theorem 2.2 in [10], that for $(\eta,\varphi)\in\mathcal{D}(\mathcal{Q})$

$$S(t)(\eta,\varphi) = (Dx_t, x_t), \tag{2.2.3}$$

where $x(\cdot)$ is the unique $W^{1,p}$ solution to the NFDE $(d/dt)Dx_t = Lx_t$ with initial data $x_0(s) = \varphi(s)$, $-r\leqslant s\leqslant 0$, or equivalently,

$$Dx_t = \eta + \int_0^t\int_{-r}^{0}F(u)x(s + u)duds \tag{2.2.4}$$

$$+ \int_0^t \int_{-r}^0 G(u)\dot{x}(s + u)duds.$$

Changing the order of integration in the last term of the right-hand side of (2.2.4) gives

$$\int_0^t \int_{-r}^0 G(u)\dot{x}(s + u)duds = \int_{-r}^0 G(u)\int_0^t \dot{x}(s + u)dsdu \qquad (2.2.5)$$

$$= \int_{-r}^0 G(u)(x(t + u) - \varphi(u))du.$$

Define the operators $\mathcal{F},\mathcal{G} : L_p \longrightarrow \mathbb{R}^n$ by

$$\mathcal{F}\varphi \equiv \int_{-r}^0 F(s)\varphi(s)ds \ , \ \mathcal{G}\varphi \equiv \int_{-r}^0 G(s)\varphi(s)ds \ ,$$

respectively. Equation (2.2.4) can be rewritten as

$$Dx_t = \eta + \int_0^t \mathcal{F}x_u du + \mathcal{G}x_t - \mathcal{G}\varphi. \qquad (2.2.6)$$

Recall that if $D \in \mathcal{B}(C;\mathbb{R}^n)$, then there exists an $n\times n$ matrix valued function $\mu(\cdot)$ whose entries are of bounded variation on $[-r,0]$ and such that if $\varphi \in C$, then

$$D\varphi = \int_{-r}^{0} d\mu(s)\varphi(s). \tag{2.2.7}$$

Throughout the remainder of this section we shall assume that

i) $D \in \mathcal{B}(C;\mathbb{R}^n)$

ii) $\mu(\cdot)$ is normalized to be right continuous on $(-r,0)$ with $\mu(-r) = 0$ and extend $\mu(\cdot)$ over $\mathbb{R}$ by $\mu(s) = \mu(0)$ for $s \geqslant 0$ and $\mu(s) = 0$ for $s \leqslant -r$

iii) there exists an $n \times n$ matrix A such that

$$D\varphi = A\varphi(0) + \int_{-r}^{0} d\mu(s)\varphi(s) \tag{2.2.8}$$

iv) the function μ satisfies

$$\lim_{\varepsilon \to 0} \mathrm{Var}_{[-\varepsilon,0]}(\mu) = 0, \tag{2.2.9}$$

where $\mathrm{Var}_{[a,b]}(\mu)$ denotes the total variation of $\mu(\cdot)$ on $[a,b]$

v) φ is extended to $\mathbb{R}$ by defining

$$\varphi(s) = 0 \text{ for } s \notin [-r,0]. \tag{2.2.10}$$

<u>REMARK</u> <u>2.2.1:</u> Recall that D is said to be atomic at $s \in [-r,0]$ if the jump at s, $J(\mu,s) \equiv \mu(s) - \mu(s^-)$ is

nonsingular. If D is atomic at zero then in (2.2.8) we can assume $A = I$, where I is the $n\times n$ identity matrix. Condition (2.2.9) implies that μ is continuous at $s = 0$. Thus, if A is a singular matrix in (2.2.8) then the D operator is not atomic at zero, and conversely. ∎

In the sequel we shall use certain results from the theory of Laplace-Stieltjes transforms. For the convenience of the reader we include the following remark.

REMARK 2.2.2: Let $\alpha(t)$ be a complex-valued function of the real variable t, defined for $0\leqslant t<\infty$. We assume that $\alpha(t)$ has bounded variation on the interval $0\leqslant t\leqslant R$ for every positive R. If the limit

$$\lim_{R\to\infty}\int_0^R e^{-\lambda t}d\alpha(t)$$

exists for some λ, then one defines

$$\int_0^\infty e^{-\lambda t}d\alpha(t) \equiv \lim_{R\to\infty}\int_0^R e^{-\lambda t}d\alpha(t).$$

The function

$$f(\lambda) = \int_0^\infty e^{-\lambda t}d\alpha(t)$$

is called the Laplace-Stieltjes transform of $\alpha(t)$. We recall the following standard results (see Widder [36]):

i) (see Theorem 2.1 on pp. 38 in [36]) If $\alpha(t) = O(e^{\gamma t})$ as $t \to \infty$ for some real number γ, then the integral for $f(\lambda)$ converges for $\sigma > \gamma$, where $\sigma = \mathrm{Re}\lambda$.

ii) (see Theorem 2.3a on pp. 41 in [36]) If the integral for $f(\lambda)$ converges, then $f(\lambda) = \lambda\hat{\alpha}(\lambda) - \alpha(0)$, where $\hat{\alpha}(\lambda)$ is the Laplace-transform of $\alpha(t)$, i.e.

$$\hat{\alpha}(\lambda) = \int_0^\infty e^{-\lambda t}\alpha(t)dt$$

iii) (see Corollary 1a on pp. 182 in [36]) If for some non-negative real number γ and nonzero constant K

$$\alpha(t) \sim (1/\Gamma(\gamma + 1))Kt^\gamma \text{ as } t \to 0^+, \tag{2.2.11}$$

$$\text{then } f(\lambda) \sim K/\lambda^\gamma \text{ as } \lambda \to \infty.$$

If in addition $\alpha(0) = 0$, then $\hat{\alpha}(\lambda) = (1/\lambda)f(\lambda) \sim K/\lambda^{(\gamma + 1)}$. If $K = 0$, then (2.2.11) is replaced by

$$\alpha(t) = o(t^\gamma) \text{ and } \hat{\alpha}(\lambda) = o(\lambda^{-1-\gamma}). \tag{2.2.12}$$

Note that the symbols "O","o" and "~" used in i),ii) and iii) are read as "big o","little o" and "asymptotic". Given two functions $f(\cdot)$ and $g(\cdot)$; $f(u) = O(g(u))$, $f(u) = o(g(u))$ or $f(u) \sim g(u)$ if $[f(u)/g(u)]$ is bounded, $[f(u)/g(u)] \longrightarrow 0$ or $[f(u)/g(u)] \longrightarrow 1$ as $u \longrightarrow u_o$, respectively. ∎

Note that (2.2.3) and Remark 2.2.2 immediately imply that if $(\eta,\varphi) \in \mathcal{D}(\mathcal{Q})$, then Dx_t and x_t have Laplace-transforms and an application of Holder's inequality shows the existence of the Laplace-transforms for $\mathcal{F}x_t$ and $\mathcal{G}x_t$.

In order to derive our necessary conditions we need some technical lemmas. The scalar versions of the following two results were proved in [37], assuming that $L\equiv0$ and that $A=0$ in (2.2.8).

<u>LEMMA</u> <u>2.2.3:</u> If $(\eta,\varphi)\in\mathcal{D}(\mathcal{Q})$ and $x(\cdot)$ is the unique $W^{1,p}$ solution to (2.2.4), then

$$\frac{\Delta(\lambda)}{\lambda}\hat{x}(\lambda) = \frac{D\varphi}{\lambda} - \frac{\mathcal{G}\varphi}{\lambda} - \mathcal{L}(D\varphi_t) + \frac{1}{\lambda}\mathcal{L}(\mathcal{F}\varphi_t) + \mathcal{L}(\mathcal{G}\varphi_t), \quad (2.2.13)$$

where $\hat{x}(\lambda)$ is the Laplace-transform of $x(\cdot)$,

$$\Delta(\lambda)\equiv \lambda D(e^{\lambda\cdot}I) - L(e^{\lambda\cdot}I), \quad (2.2.14)$$

$$D\varphi_t = \begin{cases} 0 & \text{for } t > r \\ \int_{-r}^{-t} [d\mu(s)]\varphi(t+s) & \text{for } 0 \leqslant t \leqslant r \end{cases},$$

$$\mathcal{F}\varphi_t = \begin{cases} 0 & \text{for } t > r \\ \int_{-r}^{-t} F(s)\varphi(t+s)ds & \text{for } 0 \leqslant t \leqslant r \end{cases},$$

$$\mathcal{G}\varphi_t = \begin{cases} 0 & \text{for } t > r \\ \int_{-r}^{-t} G(s)\varphi(t+s)ds & \text{for } 0 \leqslant t \leqslant r \end{cases}.$$

PROOF: First we note that the "growth" condition on x_t(i.e. $\|x_t\|_{L_p} \leqslant \alpha e^{\omega t}$) enforces a similar estimate on $x(\cdot)$. Therefore i) and ii) in Remark 2.2.2 imply the existence of $\hat{x}(\lambda)$, the Laplace-transform of $x(\cdot)$.

Simple calculations and an application of Fubini's theorem yield the equalities:

$$\mathcal{L}(Dx_t) = \int_0^\infty e^{-\lambda t} Dx_t dt$$

$$= \int_0^\infty e^{-\lambda t}(Ax(t) + \int_{-r}^0 d\mu(s)x(t+s))dt$$

$$= A\hat{x}(\lambda) + \int_{-r}^0 d\mu(s) \int_s^\infty e^{\lambda(s-u)}x(u)du$$

$$= A\hat{x}(\lambda) + \int_{-r}^{0} d\mu(s) \int_{0}^{\infty} e^{\lambda(s - u)} x(u)du$$

$$+ \int_{-r}^{0} d\mu(s) \int_{s}^{0} e^{\lambda(s - u)} x(u)du$$

$$= D(e^{\lambda \cdot}I)\hat{x}(\lambda) + \int_{0}^{r} e^{-\lambda t} \int_{-r}^{-t} d\mu(s)\varphi(t + s)dt$$

$$= D(e^{\lambda \cdot}I)\hat{x}(\lambda) + \int_{0}^{\infty} e^{-\lambda t} D\varphi_t dt$$

$$= D(e^{\lambda \cdot}I)\hat{x}(\lambda) + \mathcal{L}(D\varphi_t).$$

Similarly, it follows that

$$\mathcal{L}(\mathcal{F}x_t) = [\int_{-r}^{0} F(s)e^{\lambda s} ds]\hat{x}(\lambda) + \mathcal{L}(\mathcal{F}\varphi_t)$$

and

$$\mathcal{L}(\mathcal{G}x_t) = [\int_{-r}^{0} G(s)e^{\lambda s} ds]\hat{x}(\lambda) + \mathcal{L}(\mathcal{G}\varphi_t) .$$

Noting that

$$L(e^{\lambda\cdot}I) = \int_{-r}^{0} F(s)e^{\lambda s}ds + \lambda \int_{-r}^{0} G(s)e^{\lambda s}ds,$$

and by substituting the expressions for $\mathcal{L}(Dx_t)$, $\mathcal{L}(\mathcal{F}x_t)$ and $\mathcal{L}(\mathcal{G}x_t)$ into the $\mathcal{L}$aplace-transformed form of (2.2.6) we obtain

$$D(e^{\lambda\cdot}I)\hat{x}(\lambda) + \mathcal{L}(D\varphi_t) = (1/\lambda)(D\varphi - \mathcal{G}\varphi) + (1/\lambda)L(e^{\lambda\cdot}I)\hat{x}(\lambda)$$

$$+ (1/\lambda)\mathcal{L}(\mathcal{F}\varphi_t) + \mathcal{L}(\mathcal{G}\varphi_t),$$

or equivalently,

$$(1/\lambda)\Delta(\lambda)\hat{x}(\lambda) = (1/\lambda)(D\varphi - \mathcal{G}\varphi) - \mathcal{L}(D\varphi_t) + (1/\lambda)\mathcal{L}(\mathcal{F}\varphi_t)$$

$$+ \mathcal{L}(\mathcal{G}\varphi_t) \qquad \blacksquare$$

<u>LEMMA</u> <u>2.2.4:</u> If $\varphi \in C^1$ and $\varphi(0) = \dot{\varphi}(0) = 0$, then

$$\mathcal{L}(D\varphi - D\varphi_t) = -(1/\lambda^2)D\dot{\varphi} + o(1/\lambda^2) \text{ as } \lambda \longrightarrow \infty,\ \lambda \in \mathbb{R}.$$

<u>PROOF:</u> It follows from (2.2.10) and the assumption $\varphi(0) = \dot{\varphi}(0) = 0$ that $\varphi_t(\cdot) \in C^1$ for all $t \geq 0$. For $t > 0$ and

$s \in [-r,0]$ there exist $\xi_i \in (s, s + t)$ $(i = 1,2,\ldots, n)$ such that

$$\varphi_i(t + s) - \varphi_i(s) = \dot{\varphi}_i(\xi_i)t, \tag{2.2.15}$$

where φ_i denotes the i^{th} component of φ. Equation (2.2.15) can be rewritten as

$$\varphi_i(t + s) - \varphi_i(s) = [\dot{\varphi}_i(\xi_i) - \dot{\varphi}_i(s)]t + \dot{\varphi}_i(s)t. \tag{2.2.16}$$

Since $\dot{\varphi}_i$ is uniformly continuous on bounded intervals, we have

$$|\dot{\varphi}_i(\xi_i) - \dot{\varphi}_i(s)| = o(1)$$

as $t \longrightarrow 0^+$, uniformly for $s \in [-r,0]$. Therefore, it follows that

$$\varphi_i(t + s) - \varphi_i(s) = \dot{\varphi}_i(s)t + o(t) \tag{2.2.17}$$

and hence

$$\varphi(t + s) - \varphi(s) = \dot{\varphi}(s)t + o(t). \qquad (2.2.18)$$

Equation (2.2.18) implies that

$$D\varphi_t - D\varphi = (D\dot{\varphi})t + o(t) \quad \text{as } t \longrightarrow 0^+. \qquad (2.2.19)$$

By (2.2.11) and (2.2.12) we get

$$\mathcal{L}(D\varphi_t - D\varphi) = (1/\lambda^2)D\dot{\varphi} + o(1/\lambda^2) \qquad (2.2.20)$$

as $\lambda \longrightarrow \infty$, $\lambda \in \mathbb{R}$. ∎

If $\varphi \in C^1$ and $\varphi(0) = \dot{\varphi}(0) = 0$, then an argument similar to the proof of Lemma 2.2.4 yields analogous results for $\mathcal{G}$ and $\mathcal{F}$. Therefore, substituting (2.2.20) and the expressions for $\mathcal{L}(\mathcal{G}\varphi - \mathcal{G}\varphi_t)$ and $\mathcal{L}(\mathcal{F}\varphi - \mathcal{F}\varphi_t)$ into (2.2.13), we have that

$$\begin{aligned}(1/\lambda)\Delta(\lambda)\hat{x}(\lambda) &= \mathcal{L}(D\varphi - D\varphi_t) + (1/\lambda)\mathcal{L}(\mathcal{F}\varphi_t) \qquad (2.2.21)\\ &\quad - \mathcal{L}(\mathcal{G}\varphi - \mathcal{G}\varphi_t)\\ &= (1/\lambda^2)(-D\dot{\varphi} + \mathcal{G}\dot{\varphi} + \mathcal{F}\varphi)\end{aligned}$$

$$+ (1/\lambda)\mathcal{L}(\mathcal{F}\varphi_t - \mathcal{F}\varphi) + o(1/\lambda^2)$$

as $\lambda \longrightarrow \infty$, $\lambda \in \mathbb{R}$.

Noting that $(1/\lambda)\mathcal{L}(\mathcal{F}\varphi_t - \mathcal{F}\varphi) = o(1/\lambda^2)$ as $\lambda \longrightarrow \infty$, $\lambda \in \mathbb{R}$, we can rewrite (2.2.21) as

$$(1/\lambda)\Delta(\lambda)\ \hat{x}(\lambda) = (1/\lambda^2)(\ -D\dot{\varphi} + L\varphi) + o(1/\lambda^2) \qquad (2.2.22)$$

as $\lambda \longrightarrow \infty$, $\lambda \in \mathbb{R}$ $(\varphi \in C^1,\ \varphi(0) = \dot{\varphi}(0) = 0)$.

We now consider the term $-D\dot{\varphi} + L\varphi$ on the right-hand side of (2.2.22). By direct calculations we obtain

$$-D\dot{\varphi} + L\varphi = -A\dot{\varphi}(0) - \int_{-r}^{0} d\mu(s)\dot{\varphi}(s) \qquad (2.2.23)$$

$$+ \int_{-r}^{0} F(s)\varphi(s)ds + \int_{-r}^{0} G(s)\dot{\varphi}(s)ds$$

$$= - \int_{-r}^{0} d\mu(s)\dot{\varphi}(s)ds + H(0)\varphi(0) - H(-r)\varphi(-r)$$

$$- \int_{-r}^{0} H(s)\dot{\varphi}(s)ds + \int_{-r}^{0} G(s)\dot{\varphi}(s)ds$$

$$= \int_{-r}^{0} d\nu(s)\psi(s),$$

where

$$H(s) \equiv \int_{-r}^{s} F(u)du, \qquad (2.2.24)$$

$$\nu(s) \equiv \int_{-r}^{s} (G(u) - H(u))du - \mu(s), \qquad (2.2.25)$$

and $\psi(s) = \dot{\phi}(s)$ satisfies

$$\psi \in C, \ \psi(0) = 0. \qquad (2.2.26)$$

Clearly $\nu(\cdot)$ is an n×n matrix valued function of bounded variation on $[-r,0]$, $\nu(\cdot)$ is normalized and satisfies (2.2.9).

Define the set R by

$$R = \{ \ c \in \mathbb{R}^n, \ c = \int_{-r}^{0} d\nu(s)\psi(s), \ \psi \in C, \psi(0) = 0 \ \}.$$

Clearly R is a vector subspace of $\mathbb{R}^n$ with $0 \leqslant \dim(R) \leqslant n$. If $\dim(R) = k$, then without loss of generality we may assume that for $\psi \in C$, $\psi(0) = 0$

$$\int_{-r}^{0} d\nu(s)\psi(s) = \mathrm{col}[c_1, 0] , \tag{2.2.27}$$

where $c_1 \in \mathbb{R}^k$.

It will prove convenient to introduce the following notation. Let T be an $n\times m$ matrix and $0 \leqslant k = \dim(R) \leqslant n$. We shall partition T as

$$T = \begin{bmatrix} T_1 \\ \cdots\cdot \\ T_2 \end{bmatrix} \tag{2.2.28}$$

where T_1, T_2 are $k\times m$, $(n - k)\times m$ matrices, respectively (for example $\Delta_2(\lambda)$ denotes the lower $(n-k)\times n$ submatrix in $\Delta(\lambda)$).The following technical lemmas will be needed.

<u>LEMMA</u> <u>2.2.5:</u> If $\dim(R) = k$, then

i) $\nu_2(\cdot) = 0$

ii) $\Delta_2(\lambda) = \lambda A_2 - H_2(0)$.

<u>PROOF:</u> The statements are trivially true when $k = n$. Consider the case $k<n$. First we show that $\dim(R) = k$ implies

$$\int_{-r}^{0} d\nu_{ij}(s)\psi(s) = 0 , \quad k<i\leqslant n , \ 1\leqslant j\leqslant n, \tag{2.2.29}$$

where $\psi \in C([-r,0];\mathbb{R})$, $\psi(0) = 0$. Suppose that (2.2.29) does not hold, i.e.

$$\int_{-r}^{0} d\nu_{ij}(s)\psi(s) \neq 0 ,$$

for some $k < i \leqslant n$, $1 \leqslant j \leqslant n$ and $\psi \in C([-r,0];\mathbb{R})$, $\psi(0) = 0$. If $\hat{\psi}(s) = \mathrm{col}[0,0,\ldots,\psi(s),0,\ldots,0]$, where $\psi(s)$ is the j^{th} coordinate of $\hat{\psi}$, then

$$\int_{-r}^{0} d\nu(s)\hat{\psi}(s) = \mathrm{col}[c_1,c_2] ,$$

where $c_2 \neq 0$, contrary to our assumption that $\dim(R) = k$.

Now we show that (2.2.9) and (2.2.29) imply that

$$\int_{-r}^{0} d\nu_{ij}(s)\psi(s) = 0 , \quad k < i \leqslant n , \ 1 \leqslant j \leqslant n$$

for every $\psi \in C([-r,0];\mathbb{R})$.

Let ψ be an arbitrary function in $C([-r,0];\mathbb{R})$ and for $\epsilon > 0$ define ψ_ϵ as

$$\psi_\epsilon(s) = \begin{cases} \psi(s) & \text{for } s \in [-r,-\epsilon) \\ (1/(-\epsilon))\psi(-\epsilon)s & \text{for } s \in [-\epsilon,0] . \end{cases}$$

It follows that

$$\left|\int_{-r}^{0} d\nu_{ij}(s)\psi(s)\right| \leqslant \left|\int_{-r}^{0} d\nu_{ij}(s)(\psi(s) - \psi_\epsilon(s))\right|$$

$$+ \left|\int_{-r}^{0} d\nu_{ij}(s)\psi_\epsilon(s)\right|$$

$$= \left|\int_{-r}^{0} d\nu_{ij}(s)(\psi(s) - \psi_\epsilon(s))\right|$$

$$\leqslant \int_{-\varepsilon}^{0} |d\nu_{ij}(s)|(|\psi(s)| + |\psi(-\varepsilon)|)$$

$$\leqslant 2\| \psi \| \int_{-\epsilon}^{0} |d\nu_{ij}(s)| \text{ for all } \epsilon > 0.$$

We employ (2.2.9) together with the Riesz Representation Theorem to conclude that $\nu_{ij}(s) \equiv 0$ for $k < i \leqslant n$, $1 \leqslant j \leqslant n$. Hence, $\nu_2(s) \equiv 0$ and i) is established.

In order to establish ii), note that (2.2.25) yields

$$\nu_2(s) = \int_{-r}^{s} (G_2(u) - H_2(u))du - \mu_2(s)$$

and hence

$$\mu_2(s) = \int_{-r}^{s} (G_2(u) - H_2(u))du \ . \qquad (2.2.30)$$

It follows from (2.2.30) that

$$D_2(e^{\lambda\cdot}I) = A_2 + \int_{-r}^{0} d\mu_2(s)e^{\lambda s}$$

$$= A_2 + \int_{-r}^{0} (G_2(s) - H_2(s))e^{\lambda s}ds$$

$$= A_2 + \int_{-r}^{0} G_2(s)e^{\lambda s}ds - (1/\lambda)H_2(0)$$

$$+ (1/\lambda)\int_{-r}^{0} F_2(s)e^{\lambda s}ds$$

$$= A_2 + (1/\lambda)L_2(e^{\lambda\cdot}I) - (1/\lambda)H_2(0).$$

Therefore,

$$\Delta_2(\lambda) = \lambda D_2(e^{\lambda\cdot}I) - L_2(e^{\lambda\cdot}I)$$

$$= \lambda A_2 - H_2(0)$$

and the proof is complete. ∎

LEMMA 2.2.6: If $\dim(R)=k$, then $\mathrm{rank}(A_2) = n - k$.

PROOF: Following [10], we define the operator $B : \mathbb{R}^n \times W^{1,p} \longrightarrow \mathbb{R}^n$ by

$$B(\eta,\varphi) = \eta - D\varphi \tag{2.2.31}$$

and let $B_i(\eta,\varphi)$ denote the i^{th} coordinate of $B(\eta,\varphi)$, $i = 1, 2, \ldots, n$. It follows that $\mathcal{D}(\mathcal{A}) = \bigcap_{i=1}^{n} \mathcal{N}(B_i)$ and Lemma 2.1 in [10] implies that $\mathcal{D}(\mathcal{A})$ is dense in $\mathbb{R}^n \times L_p$ if and only if each nonzero linear combination $\Lambda = \sum_{i=1}^{n} \lambda_i B_i$ is unbounded on $\mathbb{R}^n \times L_p$.

Suppose that $\mathrm{rank}(A_2) < (n - k)$, i.e. the rows of A_2 are linearly dependent. In this case there exist constants λ_i, $i=k+1, k+2, \ldots, n$, not all zero such that

$$\sum_{i=k+1}^{n} \lambda_i A_i = 0,$$

where A_i denotes the i^{th} row of A. Using the same constants it follows that Λ defined by

$$\Lambda(\eta,\varphi) = \sum_{i=k+1}^{n} \lambda_i [\ \eta_i - \int_{-r}^{0} d\mu_i(s)\varphi(s)] \ , \qquad (2.2.32)$$

(where μ_i denotes the i^{th} row of the motive μ) is a nontrivial linear functional. Condition (2.2.30) implies that $d\mu_i(s) = f_i(s)ds$, with $f_i \in L_q$, $i = k + 1, k + 2, \ldots, n$, i.e.

$$\int_{-r}^{0} d\mu_i(s)\varphi(s) = \int_{-r}^{0} f_i(s)\varphi(s)ds,$$

$i = k+1, \ldots, n$. Consequently, Λ is bounded on $\mathbb{R}^n \times L_p$. ■

We are ready now to state the main result of this section.

<u>THEOREM</u> <u>2.2.7:</u> If $\mathcal{a}$ defined by (2.1.3)-(2.1.4) is the infinitesimal generator of a C_o-semigroup on $\mathbb{R}^n \times L_p$, then there is an $n \times n$ matrix valued function $H(\lambda)$ whose entries, $h_{ij}(\lambda)$ satisfy

$$\lim_{\substack{\lambda \to \infty \\ \lambda \in \mathbb{R}}} h_{ij}(\lambda) = 0$$

and such that

$$\lim_{\substack{\lambda \to \infty \\ \lambda \in \mathbb{R}}} D(e^{\lambda \cdot}I)\lambda^{1/p}H(\lambda) = I. \qquad (2.2.33)$$

PROOF: It follows from equations (2.2.23) and (2.2.27) that for $1 \leqslant j \leqslant k$ (where $k = \dim(R)$) there exists $\varphi^j \in C^1$ such that $\varphi^j(0) = \dot{\varphi}^j(0) = 0$ and

$$-D\dot{\varphi}^j + L\varphi^j = e^j,$$

where e^j is the j^{th} unit vector. Let $x^j(t)$ denote the unique $W^{1,p}$ solution to (2.2.4) with φ^j as initial data. By (2.2.22), $\hat{x}^j(\lambda)$, the Laplace-transform of $x^j(t)$, satisfies

$$(1/\lambda)\Delta(\lambda)\hat{x}^j(\lambda) = (1/\lambda^2)e^j + o(1/\lambda^2).$$

Define the $n \times k$ matrix $X(\lambda)$ by

$$X(\lambda) \equiv [\hat{x}^1(\lambda), \hat{x}^2(\lambda), \ldots, \hat{x}^k(\lambda)].$$

Then we have

$$(1/\lambda)\Delta(\lambda)X(\lambda) = (1/\lambda^2)\begin{bmatrix} I_{k\times k} \\ \cdots\cdots\cdots \\ 0_{(n-k)\times k} \end{bmatrix} + o(1/\lambda^2). \qquad (2.2.34)$$

Moreover, since $x^j(0) = \varphi^j(0) = 0$, we have for $t \geqslant 0$ that

$$x^j(t) = \int_0^t \dot{x}^j(u)du,$$

where $\dot{x}^j(\cdot) \in L_{p,loc}$. An application of Holder's inequality to the last equation yields

$$|\ x^j(t)\ | \leqslant t^{1/q}\|\dot{x}^j\|_{L_p([0,t];\mathbb{R}^n)} \cdot$$

Note that

$$\|\dot{x}^j\|_{L_p(0,t;\mathbb{R}^n)} \longrightarrow 0, \text{ as } t \longrightarrow 0^+.$$

It follows that

$$|\ x^j(t)\ | = o(t^{1/q}) \quad \text{as} \quad t \longrightarrow 0^+$$

and then by (2.2.12)

$$|\hat{x}^j(\lambda)| = o(1/\lambda^{(1 + 1/q)}) \quad \text{as } \lambda \longrightarrow \infty.$$

Multiplying (2.2.34) by λ^2 we get

$$\Delta(\lambda)\lambda X(\lambda) = \begin{bmatrix} I_{k\times k} \\ \cdots\cdots\cdots \\ 0_{(n-k)\times k} \end{bmatrix} + o(1) \quad \text{as } \lambda \longrightarrow \infty. \qquad (2.2.35)$$

It follows from Lemma 2.2.6 that $\text{rank}(A_2) = n - k$, and hence there exists an $n\times(n-k)$ matrix B such that $A_2B = I$, where I is the $(n-k)\times(n-k)$ identity matrix. Define the $n\times(n-k)$ matrix valued function $Y(\lambda)$ by $Y(\lambda) = 1/\lambda\ B$. Then we have

$$\Delta(\lambda)Y(\lambda) = \begin{bmatrix} C_{k\times(n-k)} \\ \cdots\cdots\cdots\cdots \\ I_{(n-k)\times(n-k)} \end{bmatrix} + o(1), \qquad (2.2.36)$$

as $\lambda \longrightarrow \infty$, $\lambda \in \mathbb{R}$ where C is a constant matrix. If $H(\lambda)$ is defined by

$$H(\lambda) \equiv \lambda^{1/q}[\lambda X(\lambda),\ Y(\lambda)] \begin{bmatrix} I_{k\times k} & -C_{k\times(n-k)} \\ 0 & I_{(n-k)\times(n-k)} \end{bmatrix},$$

then $h_{ij}(\lambda) \longrightarrow 0$ as $\lambda \longrightarrow \infty$, $\lambda \in \mathbb{R}$ and

$$\lim_{\substack{\lambda\to\infty \\ \lambda\in\mathbb{R}}} \lambda^{-1/q}\Delta(\lambda)H(\lambda) = I. \tag{2.2.37}$$

Since $L\in \mathbb{B}(W^{1,p},\mathbb{R}^n)$, it follows that

$$\|L(e^{\lambda\cdot}I)\| \leqslant \|L\|(1 + \lambda)(p\lambda)^{-1/p}. \tag{2.2.38}$$

Therefore, we have that

$$\|(1/\lambda^{1/q})L(e^{\lambda\cdot}I)H(\lambda)\| \leqslant \|L\|(1/p^{1/p})(1 + \lambda)/\lambda\| H(\lambda)\| ,$$

where $\|H(\lambda)\| \to 0$ as $\lambda \to \infty$. Consequently,

$$\lim_{\substack{\lambda\to\infty \\ \lambda\in\mathbb{R}}} \| (1/\lambda^{1/q})L(e^{\lambda\cdot}I)H(\lambda) \| \to 0 \text{ as } \lambda \to \infty,\ \lambda\in \mathbb{R} ,$$

and hence condition (2.2.33) follows immediately from (2.2.37). ∎

<u>REMARK</u> <u>2.2.8:</u> If D is atomic at $s = 0$, i.e. $k = 0$ and $\mathcal{Q}$ generates a C_o-semigroup on $\mathbb{R}^n\times L_p$, then condition (2.2.33) holds for any $p\in[1,\infty)$ with $H(\lambda) = (1/\lambda^{1/p})A^{-1}$ because in this case (2.2.8) and (2.2.9) imply that $D(e^{\lambda\cdot}I)\to A$ as $\lambda\to\infty$, $\lambda\in\mathbb{R}$. ∎

<u>REMARK</u> <u>2.2.9:</u> Consider the scalar equation (2.1.6)-(2.1.7). If D and L are defined by (2.1.8), then $\Delta(\lambda) = \lambda D(e^{\lambda\cdot}I)-0$, i.e.

$$\Delta(\lambda) = \lambda\int_{-r}^{0} e^{\lambda s}|s|^{-\alpha}ds$$

$$= \lambda\int_{0}^{r} e^{-\lambda t}t^{-\alpha}dt \ .$$

We see that $\Delta(\lambda) = \lambda\mathcal{L}(f(t))$, where $f(t) = t^{-\alpha}$ for $t\in(0,r]$, and $f(t) = 0$ for $t>r$. Applying (2.2.11) to $\beta(t) = (1/(1-\alpha))t^{(1-\alpha)}$ (i.e. $d\beta(t) = t^{-\alpha}dt$) we have

$$\Delta(\lambda) = \lambda\mathcal{L}(f(t)) \sim \lambda\Gamma(1 - \alpha)/\lambda^{(1 - \alpha)},$$

as $\lambda\longrightarrow\infty$, $\lambda\in\mathbb{R}$, or equivalently,

$$(1/\lambda^{1/q})\Delta(\lambda) = \lambda^{1/p}D(e^{\lambda\cdot}) \sim \lambda^{(1/p - (1 - \alpha))}\Gamma(1 - \alpha)$$

as $\lambda\longrightarrow\infty$, $\lambda\in\mathbb{R}$. Thus, for $(1/p - (1 - \alpha))< 0$ condition (2.2.33) is satisfied with

$$H(\lambda) = (1/\Gamma(1 - \alpha))\lambda^{-(1/p - (1 - \alpha))}.$$

On the other hand, if $(1/p - (1 - \alpha)) \geqslant 0$, then $\lambda^{1/p}D(e^{\lambda\cdot}) \longrightarrow K$(constant) as $\lambda \longrightarrow \infty$, $\lambda \in \mathbb{R}$. Therefore, there is no $H(\cdot)$ such that $H(\lambda) \longrightarrow 0$ and (2.2.33) holds as $\lambda \longrightarrow \infty$, $\lambda \in \mathbb{R}$. Consequently $\mathcal{Q}$ **defined by (2.1.3)-(2.1.4)** can be the infinitesimal generator of a C_o-semigroup on $\mathbb{R}^n \times L_p$ only if $p < 1/(1 - \alpha)$. ■

To conclude this section we note that throughout the remainder of this paper we shall use condition (2.2.33) to determine possible values for p so that $\mathcal{Q}$ defined by (2.1.3)-(2.1.4) generates a C_o-semigroup in $\mathbb{R}^n \times L_p$ (see e.g. Lemma 2.3.3, Example 2.3.12("aero-problem")).

2.3 Sufficient conditions for the well-posedness

In this section we consider a general class of functional differential equations of neutral-type (NFDE) and under weak conditions on D and L (see CONDITION H below) we establish the well-posedness of these equations on the product spaces $\mathbb{R}^n \times L_p$. Our results extend the paper by Burns, Herdman and Stech [10] in that we consider vector equations with general L without assuming that D is atomic at s = 0. Our investigation was motivated by a problem which occurs in aeroelasticity (see e.g. [8]), i.e. the class of NFDEs considered here contains the finite delay version of the functional differential equation which has been used in [8] to model the elastic motions of a two dimensional airfoil in unsteady flows (see Example 2.3.12 below for details).

We begin our presentation by stating precisely the class of NFDEs for which we derive well-posedness results. Consider the NFDE (2.1.1)-(2.1.2) with

$$D\varphi \equiv \int_{-r}^{0} [A d\alpha(s) + d\beta(s)]\varphi(s) \qquad (2.3.1)$$

and

$$L\varphi \equiv B\varphi(0) + \int_{-r}^{0} B(s)\dot{\varphi}(s)ds, \qquad (2.3.2)$$

where A, $\alpha(s)$, $\beta(s)$, B and B(s) satisfy the following condition.

<u>CONDITION</u> <u>H.</u>

H1) A and B are constant $n\times n$ matrices, A is nonsingular and has the form

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \qquad (2.3.3)$$

where A_{11}, A_{22} are $k\times k$, $m\times m$ matrices, respectively and $k + m = n$.

H2) $\alpha(\cdot)$ is an $n\times n$ matrix valued function on $[-r,0]$ with entries

$$\alpha_{ij}(s) = \begin{cases} 0 & ,\text{for } i\neq j \\ -\rho(-s) & ,\text{for } i=j, i\leqslant k \\ -(-s)^{(1-\alpha_i)}/(1-\alpha_i) & ,\text{for } i=j, i>k \end{cases}, \qquad (2.3.4)$$

where

$$\rho(0) = 0,\ \rho(s) = 1 \text{ for } s > 0 \qquad (2.3.5)$$

and the constants α_i satisfy

$$0 < \alpha_i < 1, \ i > k \ . \tag{2.3.6}$$

H3) $\beta(\cdot)$ is an $n \times n$ matrix valued function of bounded variation on $[-r,0]$, $\beta(0) = 0$ and $\beta(s)$ is left continuous for $-r \leq s < 0$. Moreover we assume that

$$\lim_{\epsilon \longrightarrow 0} \mathrm{Var}_{[-\epsilon,0]}(\beta) = 0. \tag{2.3.7}$$

H4) $B(\cdot)$ is an $n \times n$ matrix valued function on $[-r,0]$ with entries $b_{ij}(\cdot) \in L_q$, $(1/p + 1/q = 1)$.

H5) The $n \times n$ matrix valued function

$$\gamma(s) \equiv \beta(s) - \int_0^s B(u)du \tag{2.3.8}$$

has the form

$$\gamma(s) = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} ,$$

where γ_1, γ_2 are $k \times n$, $m \times n$ matrix valued functions, respectively. Moreover $\gamma_2(s)$ is absolutely continuous and

$\dot{\gamma}_2(s) = \delta_2(s)$ is of bounded variation on $[-r,0]$. ∎

<u>REMARK</u> <u>2.3.1:</u> It is clear that D and L defined by (2.3.1)and (2.3.2) belong to $\mathcal{B}(C;\mathbb{R}^n)$ and $\mathcal{B}(W^{1,p};\mathbb{R}^n)$, respectively. Note that if $k = n$ and $m=0$, i.e. $A = A_{11}$, then D is atomic at $s = 0$ and the sufficiency result in [10] applies (see Theorem 2.3 in [10]), i.e. the NFDE is well-posed on $\mathbb{R}^n \times L_p$ for $1 \leqslant p < \infty$.

The case $k = 0$, $m = n = 1$, $\beta(\cdot) \equiv 0$, $L\varphi \equiv 0$ and $f(\cdot) \equiv 0$ was also considered in [10] and in Theorem 4.1 Burns, Herdman and Stech established the well-posedness of (2.1.1)-(2.1.2) on $\mathbb{R} \times L_p$ for $1 \leqslant p < 1/(1-\alpha_1)$. ∎

<u>REMARK</u> <u>2.3.2:</u> We may assume without loss of generality $A = I$ in (2.3.1) where I is the $n\times n$ identity matrix. Otherwise, multiplying (2.1.1)-(2.1.2) by A^{-1} and introducing $\tilde{\beta}(\cdot) = A^{-1}\beta(\cdot)$, $\tilde{B} = A^{-1}B$ and $\tilde{B}(\cdot) = A^{-1}B(\cdot)$ we can always reduce the original problem to the case of $A = I$. ∎

<u>LEMMA</u> <u>2.3.3:</u> Let $1 \leqslant p < \infty$, $D \in \mathcal{B}(C;\mathbb{R}^n)$ have the representation (2.3.1) with $A = I$ and assume that CONDITION H is satisfied.

i) If $k = n$, then $D(e^{\lambda \cdot}I)$ satisfies condition (2.2.33) for $1 \leqslant p < \infty$.

ii) If $k \neq n$, then $D(e^{\lambda \cdot}I)$ satisfies condition (2.2.33) for $1 \leqslant p < 1/(1 - \alpha_{min})$, where

$$\alpha_{min} = \min_{i>k} (\alpha_i). \qquad (2.3.9)$$

PROOF: Statement i) follows immediately from Remark 2.2.8. In order to establish ii) we construct an $n \times n$ matrix valued function $H(\lambda)$, such that $H(\lambda) \longrightarrow 0$ as $\lambda \longrightarrow \infty$, $\lambda \in \mathbb{R}$, and

$$\lim_{\substack{\lambda \to \infty \\ \lambda \in \mathbb{R}}} \lambda^{1/p} D(e^{\lambda \cdot}I)H(\lambda) = I.$$

First we note that, by H3), $\beta(0) = 0$ and $\beta(\cdot)$ satisfies (2.3.7) which imply that $\beta(-t) \longrightarrow 0$ as $t \longrightarrow 0+$, or equivalently,

$$\left\| \int_{-r}^{0} e^{\lambda s} d\beta(s) \right\| \longrightarrow 0 \text{ as } \lambda \longrightarrow \infty, \ \lambda \in \mathbb{R}.$$

Moreover, by H4) and H5), $\beta_2(\cdot)$, the lower $m \times n$ submatrix in $\beta(\cdot)$, belongs to $W^{1,q}$ and hence we can write $d\beta_2(s) = \dot{\beta}_2(s)ds$, i.e.

$$\int_{-r}^{0} e^{\lambda s} d\beta_2(s) = \int_{-r}^{0} e^{\lambda s}\, \dot{\beta}_2(s) ds.$$

An application of Holder's inequality to the right-hand side of the last equation yields

$$\left\| \int_{-r}^{0} e^{\lambda s}\, \dot{\beta}_2(s) ds \right\| = \|\dot{\beta}_2\|_q \|e^{\lambda s}\|_p \leqslant K(1/\lambda^{1/p}).$$

For large values of λ, we define the $n\times n$ nonsingular matrix $Q(\lambda)$ by

$$Q(\lambda) \equiv \int_{-r}^{0} e^{\lambda s} d\alpha(s).$$

It follows that $q_{ij}(\lambda) = 0$, $i \neq j$, $q_{ii}(\lambda) = 1$, $i=1,2,\ldots,k$ and $q_{ii}(\lambda) = \mathcal{L}(f_i(t))$, $i=k+1,\ldots,n$, where $f_i(t) = t^{-\alpha_i}$ for $t \in (0,r]$, $f_i(t) = 0$ for $t > r$. Let $P(\lambda)$ denote the $n\times n$ matrix

$$P(\lambda) \equiv Q^{-1}(\lambda) \int_{-r}^{0} e^{\lambda s} d\beta(s).$$

For the matrix $P(\lambda)$ we have

$$P(\lambda) = \begin{bmatrix} I_{k\times k} & 0 \\ 0 & W(\lambda) \end{bmatrix} \begin{bmatrix} \int_{-r}^{0} e^{\lambda s} d\beta_1(s) \\ \int_{-r}^{0} e^{\lambda s} d\beta_2(s) \end{bmatrix}$$

$$= \begin{bmatrix} \int_{-r}^{0} e^{\lambda s} d\beta_1(s) \\ W(\lambda)\int_{-r}^{0} e^{\lambda s} d\beta_2(s) \end{bmatrix},$$

where $W(\lambda) = \mathrm{dia}[q_{ii}^{-1}(\lambda)]$. Thus for $p < 1/(1 - \alpha_{min}) \leqslant 1/(1 - \alpha_i)$, $i=k+1,\ldots,n$ we have $\|P(\lambda)\| \longrightarrow 0$ as $\lambda \longrightarrow \infty$. Here we employed the above estimate for $\|\int_{-r}^{0} e^{\lambda s}\dot{\beta}(s)ds\|$, together with the fact that $\|\int_{-r}^{0} e^{\lambda s} d\beta(s)\| \longrightarrow 0$ as $\lambda \longrightarrow \infty$. Using the matrices Q and P we have the following representation for $D(e^{\lambda\cdot}I)$

$$D(e^{\lambda\cdot}I) = Q(\lambda)(I + P(\lambda)).$$

Since $P(\lambda) \longrightarrow 0$ as $\lambda \longrightarrow \infty$, $\lambda \in \mathbb{R}$, the matrix

$$R(\lambda) \equiv (I + P(\lambda))^{-1}$$

is well-defined for sufficiently large values of λ and $\|R(\lambda)\|$ is bounded as $\lambda \longrightarrow \infty$.

Define $H(\lambda)$ by

$$H(\lambda) = R(\lambda)(1/\lambda^{1/p})Q^{-1}(\lambda),$$

then $H(\lambda) \longrightarrow 0$ and for sufficiently large values of λ

$$\lambda^{1/p}D(e^{\lambda\cdot}I)H(\lambda) = I. \qquad \blacksquare$$

<u>REMARK</u> <u>2.3.4:</u> Let $\bar{\alpha}(s) = -\alpha(-s)$, $\bar{\beta}(s) = -\beta(-s)$ and $\bar{\gamma}(s) = -\gamma(-s)$ for $0 \leqslant s \leqslant r$. It is easy to see that $\bar{\alpha},\bar{\beta},\bar{\gamma} \in NBV([0,r];\mathbb{R}^{n\times n})$, where $NBV([0,r];\mathbb{R}^{n\times n})$ denotes the space of $n\times n$ matrix valued functions which are of bounded variation on $[0,r]$, right continuous for $0 < s < r$, and which take the value 0 at $s = 0$. $\blacksquare$

In Theorem 2.3.5 below we establish the well-posedness of the NFDE (2.1.1)-(2.1.2) onthe product space $\mathbb{R}^n\times L_p$ for $1 \leqslant p < 1/(1-\alpha_{min})$.

<u>THEOREM</u> <u>2.3.5:</u> Let $1 \leqslant p < 1/(1-\alpha_{min})$, $D \in \mathcal{B}(C;\mathbb{R}^n)$ have the representation (2.3.1), $L \in \mathcal{B}(W^{1,p};\mathbb{R}^n)$ have the representation (2.3.2) and assume CONDITION H is satisfied. If $(\eta,\varphi) \in \mathbb{R}^n\times L_p$ and $f \in L_{p,loc}$, then:

i) **The initial value problem**

$$y(t) = \eta + \int_0^t [Lx_u + f(u)]du, \qquad (2.3.10)$$

$$Dx_t = y(t) \quad \text{a.e. on } [0,\infty)$$

and

$$x_o(s) = \varphi(s) \quad \text{for almost all } s \in [-r,0]$$

has a unique solution $y(t) = y(t;\eta,\varphi,f)$ on $[0,\infty)$, $x(t) = x(t;\eta,\varphi,f)$ on $[-r,\infty)$ such that $y(\cdot)$ is continuous, and $x_t(\cdot) \in L_p$.

ii) For $t_1>0$ the mapping $(\eta,\varphi,f)\longrightarrow(y(\cdot;\eta,\varphi,f),x(\cdot.;\eta,\varphi,f))$ from $\mathbb{R}^n\times L_p\times L_p([0,t_1];\mathbb{R}^n)$ into $C([0,t_1];\mathbb{R}^n)\times L_p([-r,t_1];\mathbb{R}^n)$ is continuous.

The proof of Theorem 2.3.5 requires some preparation. First we note that to establish i) it is sufficient to consider the problem

$$Dx_t = \eta + \int_0^t [Lx_u + f(u)]du \quad \text{a.e. on } [0,\infty) \qquad (2.3.11)$$

$$x(s) = \varphi(s) \quad \text{a.e on } [-r,0].$$

Using (2.3.1), (2.3.2) and Remark 2.3.2 we can rewrite (2.3.11) as

$$\int_{-r}^{0}[d\alpha(s) + d\beta(s)]x(t + s) = \eta + B\int_{0}^{t}x(u)du \qquad (2.3.12)$$

$$+ \int_{0}^{t}\int_{-r}^{0}B(s)\dot{x}(u + s)dsdu + \int_{0}^{t}f(u)du.$$

Changing the order of integration of the integral involving B(s), (2.3.12) becomes

$$\int_{-r}^{0}[d\alpha(s) + d\beta(s)]x(t + s) - \int_{-r}^{0}B(s)x(t + s)ds \qquad (2.2.13)$$

$$- B\int_{0}^{t}x(u)du = \eta - \int_{-r}^{0}B(s)\varphi(s)ds + \int_{0}^{t}f(u)du.$$

For $0 < t \leqslant r$ we can rewrite the left-hand side of (2.3.13) as

$$\int_{-r}^{-t}[d\alpha(s) + d\gamma(s)]x(t + s) \qquad (2.3.14)$$

$$+\int_{-t}^{0}[d\alpha(s) + d\mu(s)]x(t + s),$$

where $\gamma(\cdot)$ is given by (2.3.8) and

$$\mu(s) = \gamma(s) - Bs. \qquad (2.3.15)$$

By Remark 2.3.4, the second term in (2.3.14) can be written as

$$\int_{-t}^{0}[d\alpha(s) + d\mu(s)]x(t + s) \qquad (2.3.16)$$

$$= \int_{0}^{t}[d\bar{\alpha}(s) + d\bar{\mu}(s)]x(t - s)$$

with $\bar{\alpha},\bar{\mu} \in NBV([0,r];\mathbb{R}^{n\times n})$ and $\bar{\mu}(s) \equiv \bar{\gamma}(s) + Bs$. Finally, introducing

$$g(t) = \eta - \int_{-r}^{0}B(s)\varphi(s)ds + \int_{0}^{t}f(u)du \qquad (2.3.17)$$

$$- \int_{-r}^{-t}[d\alpha(s) + d\gamma(s)]\varphi(t + s),$$

and combining (2.3.13)-(2.3.17), it follows that (2.3.11) is equivalent to

$$\int_0^t [d\bar{\alpha}(s) + d\bar{\mu}(s)]x(t - s) = g(t), \qquad (2.3.18)$$

for $0 < t \leqslant r$.

<u>LEMMA</u> <u>2.3.6:</u> Suppose that $f \in L_p([0,r];\mathbb{R})$ and $0 < \alpha < 1$. If

$$y(t) \equiv (1/\Gamma(\alpha))\int_0^t (t - u)^{\alpha - 1} f(u)du \qquad (2.3.19)$$

belongs to $W^{1,p}([0,r];\mathbb{R})$, then $x(\cdot)$, the unique L_p solution to

$$\int_0^t (t - u)^{-\alpha} x(u)du = f(t), \qquad (2.3.20)$$

is given by

$$x(t) = \frac{\sin\alpha\pi}{\pi}\frac{d}{dt}\int_0^t (t - u)^{\alpha - 1} f(u)du. \qquad (2.3.21)$$

<u>PROOF:</u> First note that (2.3.19) implies (see [10; Lemma 4.1])

$$\int_0^t f(u)du = (1/\Gamma(1 - \alpha))\int_0^t (t - u)^{-\alpha} y(u)du \quad . \tag{2.3.22}$$

Differentiating (2.3.22), for almost all $t \in [0,r]$

$$f(t) = (1/\Gamma(1-\alpha)\Gamma(\alpha))\cdot \tag{2.3.23}$$

$$\int_0^t (t - u)^{-\alpha}(d/du)\int_0^u (u - s)^{\alpha-1} f(s)ds\ du$$

$$= \int_0^t (t - u)^{-\alpha} x(u)du,$$

where we have used the identity $\Gamma(\alpha)\Gamma(1 - \alpha) = \pi/(\sin\alpha\pi)$. Equality (2.3.23) implies that $x(\cdot)$ given by (2.3.21) satisfies (2.3.20).

Uniqueness follows from the fact that $z(\cdot)$, the difference of two solutions to (2.3.20) satisfies

$$\int_0^t (t - u)^{-\alpha} z(u)du = 0 \quad \text{a.e on} \quad [0,r]$$

which implies that

$$\int_0^t z(u)du = 0 \quad \text{for all } t \in [0,r],$$

and hence $z(t) =$ a.e. in $[0,r]$. ■

Define $h(\cdot) \in NBV([0,r);\mathbb{R}^{n\times n})$ by $h(\cdot) = h_{ij}(\cdot)$, $1 \leqslant i,j \leqslant n$, where for all $1 \leqslant j \leqslant n$

$$h_{ij}(s) = \begin{cases} \bar{\mu}_{ij}(s) & i \leqslant k, \\ \frac{d}{ds}\left[\frac{\sin\alpha_i\pi}{\pi}\int_0^s (s-u)^{\alpha_i - 1}\,\bar{\mu}_{ij}(u)du\right] & i > k. \end{cases} \tag{2.3.24}$$

REMARK 2.3.7: The functions $h_{ij}(\cdot)$, $i>k$, $1\leqslant j\leqslant n$ are the unique solutions of the Abel-type integral equations

$$\int_0^t (t-u)^{-\alpha_i} h_{ij}(u)du = \bar{\mu}_{ij}(t). \qquad ■$$

REMARK 2.3.8. In (2.3.24) we can carry out the differentiation to get

$$h_{ij}(s) = \frac{\sin\alpha_i\pi}{\pi}\int_0^s (s-u)^{\alpha_i-1}\dot{\bar{\mu}}_{ij}(u)du \quad i>k,\ 1\leqslant j\leqslant n, \tag{2.3.25}$$

because $\bar{\mu}_{ij}(0) = 0$ and $\bar{\mu}_{ij}(s)$ is absolutely continuous for $i>k$, $1\leqslant j\leqslant n$. Also $\dot{\bar{\mu}}_{ij}(\cdot)$ is of bounded variation on $[0,r]$ which implies that $h_{ij}(0) = 0$ and $h_{ij}(s)$ is absolutely continuous for $i>k$, $1\leqslant j\leqslant n$. ■

By (2.3.25)

$$\dot{\bar{\mu}}_{ij}(t) = \int_0^t (t-u)^{-\alpha_i} \dot{h}_{ij}(u)du$$

and therefore, substituting into (2.3.18), for $i>k$, $1\leqslant j\leqslant n$ we have

$$\int_0^t d\bar{\mu}_{ij}(s)x_j(t-s) = \int_0^t \dot{\bar{\mu}}_{ij}(s)x_j(t-s)ds$$

$$= \int_0^t \int_0^s (s-u)^{-\alpha_i} \dot{h}_{ij}(u)\, du\, x_j(t-s)ds.$$

Changing the order of integration and making several changes of variables we get

$$\int_0^t \int_0^s (s-u)^{-\alpha_i} \dot{h}_{ij}(u)du\, x_j(t-s)ds$$

$$= \int_0^t \int_0^s u^{-\alpha_i} \dot{h}_{ij}(s-u)du\, x_j(t-s)ds$$

$$= \int_0^t u^{-\alpha_i} \int_u^t \dot{h}_{ij}(s-u)x_j(t-s)ds\, du$$

$$= \int_0^t u^{-\alpha_i} \int_0^{t-u} \dot{h}_{ij}(v)\, x_j(t-u-v)\, dv\, du$$

$$= \int_0^t u^{-\alpha_i} \int_0^{t-u} dh_{ij}(v)\, x_j(t-v-u)\, du$$

$$= \int_0^t d\bar{\alpha}_{ii}(u) \int_0^{t-u} dh_{ij}(v)\; x_j(t-u-v).$$

Therefore, for $i>k$, $1\leqslant j\leqslant n$,

$$\int_0^t d\bar{\mu}_{ij}(s)x_j(t-s) = \int_0^t d\bar{\alpha}_{ii}(u)\int_0^{t-u} dh_{ij}(v)x_j(t-u-v). \quad (2.3.26)$$

For $i\leqslant k$, $1\leqslant j\leqslant n$ we have that

$$\int_0^t d\bar{\mu}_{ij}(s)x_j(t-s) = \int_0^t d\rho(u)\int_0^{t-u} dh_{ij}(v)x_j(t-u-v) \quad (2.3.27)$$

$$= \int_0^t d\bar{\alpha}_{ii}(u) \int_0^{t-u} dh_{ij}(v)\; x_j(t-u-v).$$

Combining (2.3.26) and (2.3.27) we have

$$\int_0^t d\bar{\mu}(s)x(t-s) = \int_0^t d\bar{\alpha}(s) \int_0^{t-s} dh(u)x(t-s-u) \quad (2.3.28)$$

and substituting into (2.3.18) it follows that

$$\int_0^t d\bar{\alpha}(s)[x(t-s) + \int_0^{t-s} dh(u)x(t-s-u)] = g(t), \quad (2.3.29)$$

where g is defined by (2.3.17). If $w(t)$ is defined by

$$w(t) = x(t) + \int_0^t dh(u)x(t-u), \qquad (2.3.30)$$

then (2.3.29) becomes

$$\int_0^t d\bar{\alpha}(s)w(t-s) = g(t). \qquad (2.3.31)$$

Concerning the existence, uniqueness, continuous dependence and representation of L_p - solutions to (2.3.30) we include the following standard result.

<u>LEMMA</u> <u>2.3.9:</u> If $w \in L_p$ ([0,r]; $\mathbb{R}^n$), then there exists a unique solution $x \in L_p$ [0,r]; $\mathbb{R}^n$) to (2.3.30). Furthermore, x depends continuously on w with respect to the L_p norm.

<u>PROOF:</u> Our assumptions guarantee that $h \in NBV([0,r]);$ $\mathbb{R}^{n\times n}$) and is continuous at 0 from the right, i.e.

$$\lim_{t\to 0^+} h(t) = 0. \qquad (2.3.32)$$

Equation (2.3.32) is a sufficient condition (see for example [25]) for the existence and uniqueness of the fundamental solution of (2.3.30). In particular, there exists a unique solution ζ of the equation

$$\zeta(t) + \int_0^t dh(s)\zeta(t-s) = \rho(t), \tag{2.3.33}$$

where $\rho(0) = 0$ and $\rho(t) = I$ for $t>0$. Moreover, $\zeta \in NBV([0,r];\mathbb{R}^{n\times n})$. Let $x \in L_p([0,r];\mathbb{R}^n)$ be given by

$$x(t) = \int_0^t d\zeta(s)w(t-s). \tag{2.3.34}$$

Substituting (2.3.34) into the right hand side of (2.3.30) it follows that

$$x(t) + \int_0^t dh(s)x(t-s)$$

$$= \int_0^t d\zeta(s)w(t-s) + \int_0^t dh(s) \int_0^{t-s} d\zeta(u)w(t-s-u)$$

$$= \frac{d}{dt} \int_0^t [\int_0^v d\zeta(s)w(v-s) + \int_0^v dh(s)\int_0^{v-s} d\zeta(u)w(v-s-u)]dv$$

$$= \frac{d}{dt} (I + II),$$

where I and II are defined below. Changing the order of integration, making several changes of variables, and integrating by parts we have that

$$I = \int_0^t \int_0^v d\zeta(s)\ w(v-s)\ dv$$

$$= \int_0^t d\zeta(s) \int_s^t w(v-s)\ dv$$

$$= \int_0^t d\zeta(s) \int_0^{t-s} w(z)dz$$

$$= \zeta(s)\int_0^{t-s} w(z)\ dz\ \Big|_0^t - \int_0^t \zeta(s)\ \frac{d}{ds}\ [\int_0^{t-s} w(z)dz]ds$$

$$= \int_0^t \zeta(s)w(t-s)ds$$

$$= \int_0^t \zeta(t-s)w(s)ds$$

and

$$II = \int_0^t \int_0^v dh(s) \int_0^{v-s} d\zeta(u)w(v-s-u)dv$$

$$= \int_0^t dh(s) \int_s^t \int_0^{v-s} d\zeta(u)w(v-s-u)\ dv$$

$$= \int_0^t dh(s) \int_0^{t-s} \int_0^z d\zeta(u)w(z-u)dz$$

$$= \int_0^t dh(s) \int_0^{t-s} d\zeta(u) \int_u^{t-s} w(z-u)dz$$

$$= \int_0^t dh(s) \int_0^{t-s} \zeta(u)w(t-s-u)du$$

$$= \int_0^t dh(s)\int_0^{t-s} \zeta(t-s-u)w(u)du$$

$$= \int_0^t \int_0^{t-u} dh(s)\zeta(t-s-u)w(u)du.$$

Therefore,

$$I + II = \int_0^t [\zeta(t-s) + \int_0^{t-s} dh(u)\zeta(t-s-u)]w(s)ds$$

$$= \int_0^t \rho(t-s)w(s)ds = \int_0^t \rho(s)\ w(t-s)ds$$

$$= + \int_0^t d\rho(s) \int_0^{t-s} w(u)du,$$

and hence

$$x(t) + \int_0^t dh(s)x(t-s) = \frac{d}{dt}(I + II)$$

$$= + \frac{d}{dt}\int_0^t d\rho(s) \int_0^{t-s} w(u)du$$

$$= \int_0^t d\rho(s)w(t-s) = w(t).$$

We conclude that $x(t)$ given by (2.3.34) solves (2.3.30). On the other hand if x is a solution of (2.3.30), then

$$\int_0^t d\zeta(s)w(t-s)$$

$$= \int_0^t d\zeta(s)\ [x(t-s) + \int_0^{t-s} dh(u)x(t-s-u)]$$

$$= \frac{d}{dt}\int_0^t [\zeta(t-s) + \int_0^{t-s} d\zeta(u)h(t-s-u)]x(s)ds$$

$$= \frac{d}{dt}\int_0^t \rho(t-s)x(s)ds = x(t),$$

where we have used the fact that ζ, the unique solution of (2.3.33), satisfies

$$\zeta(t) + \int_0^t d\zeta(u)\ h(t-u) = \rho(t). \qquad (2.3.35)$$

Continuous dependence of x on w with respect to the L_p-norm is an immediate consequence of (2.3.34) and the estimate

$$\| x \|_{L_p(0,t_1,\mathbb{R}^n)} \leqslant \mathrm{Var}_{[0,t_1]}(h)\ \| w \|_{L_p(0,t_1;\mathbb{R}^n)} \qquad (2.3.36)$$

for $0 \leqslant t_1 \leqslant r$. ∎

<u>REMARK</u> <u>2.3.10:</u> It is known (see Honig [25]) that the necessary and sufficient condition for the existence and uniqueness of the fundamental solution of (2.3.30) is that

$$1 \notin \sigma\ (H_0), \qquad (2.3.37)$$

where $H_0 \in \mathbb{R}^{n\times n}$ is given by

$$H_0 = \lim_{t \to 0^+} h(t). \qquad \blacksquare$$

Next we consider equation (2.3.31) in component form, i.e.

$$\int_0^t d\bar{\alpha}_{ii}(s) w_i(t-s) = g_i(t) \ ; \ 1 \leqslant i \leqslant n. \tag{2.3.38}$$

Using the special form of $\bar{\alpha}(\cdot)$, equation (2.3.38) implies that

$$w_i(t) = g_i(t) \ , \qquad i \leqslant k \tag{2.3.39}$$

and

$$\int_0^t s^{-\alpha_i} w_i(t-s) ds = g_i(t) \ , \ i > k \ . \tag{2.3.40}$$

For $t \in [0,r]$ define G_i by

$$G_i(t) \equiv \int_0^t (t-s)^{\alpha_i - 1} g_i(s) ds \ , \quad i > k. \tag{2.3.41}$$

We have the following result for the existence, uniqueness, continuous dependence, and representation of

L_p-solutions to (2.3.31).

<u>LEMMA</u> <u>2.3.11:</u> If $(\eta,\varphi) \in \mathbb{R}^n \times L_p$ and $f \in L_p(0,r;\ \mathbb{R}^n)$ for $1 \leqslant p < \frac{1}{1-\alpha_{min}}$, then the unique solution $w \in L_p\ (0,r;\ \mathbb{R}^n)$ of (2.3.31) is given by

$$w_i(t) = \begin{cases} g_i(t) & , \quad \text{for} \quad i \leqslant k \\ \frac{d}{dt}\left[\ \frac{\sin\alpha_i\pi}{\pi}\ G_i(t)\right], & \text{for } i > k. \end{cases} \tag{2.3.42}$$

Moreover, there exists a nonnegative, increasing function $M \in C(0,r,\mathbb{R})$ such that

$$\| w \|_{L_p(0,t,\mathbb{R}^n)} \leqslant M\ (t)\ \| (\eta,\varphi,f) \|_{\mathbb{R}^n \times L_p \times L_p(0,t,\mathbb{R}^n)}$$

for $t \in [0,r]$.

<u>PROOF:</u> In view of (2.3.39), (2.3.40) and Lemma 2.3.6, it follows that if $g_i \in L_p\ (0,r;\ \mathbb{R})$, $1 \leqslant i \leqslant n$, and $G_i \in W^{1,p}\ (0,r;\ \mathbb{R})$, $i>k$, then the components $w_i(\cdot)$ of the unique L_p-solution $w(\cdot)$ to (2.3.31) on $[0,r]$ are given by (2.3.42). Thus it suffices to show that $g_i \in L_p(0,r;\mathbb{R})$, $1 \leqslant i \leqslant n$ and $G_i \in W^{1,p}(0,r;\mathbb{R})$, $i>k$.

First we consider the case $i \leqslant k$. By (2.3.17) and

(2.3.42) we have

$$g_i(t) = w_i(t) = \eta_i - \sum_{j=1}^{n} [\int_{-r}^{0} b_{ij}(s)\varphi_j(s)ds$$

$$- \int_{-r}^{-t} d\gamma_{ij}(s)\varphi_j(t+s)]$$

$$+ \int_0^t f_i(u)du.$$

Clearly $w_i \in L_p(0,r;\mathbb{R})$, and for $t \in [0,r]$ an application of Holder's inequality yields

$$\|w_i\|_{L_p(0,t;\mathbb{R})} \leqslant t^{1/p}(|\eta_i| + \|f_i\|_{L_p(0,t;\mathbb{R})}\cdot t$$

$$+ \|B\|_q\|\varphi\|_p$$

$$+ Var_{[0,t]}(\gamma)\ \|\varphi\|_p)$$

$$\leqslant M_i(t)\|(\eta,\varphi,f)\|_{\mathbb{R}^n\times L_p\times L_p(o,t;\mathbb{R}^n)},$$

where

$$M_i(t) = t^{1/p}\ (\|B\|_q + 1 + Var_{[0,r]}(\gamma)) + t. \qquad (2.3.43)$$

Consider now the case $i>k$. By (2.3.17), (H2) and (H5) we have that

$$g_i(t) = \eta_i - \sum_{j=1}^{n} \int_{-r}^{0} b_{ij}(s)\varphi_j(s)ds \tag{2.3.44}$$

$$+ \int_0^t f_i(u)du$$

$$- \int_{-r}^{-t} (-s)^{-\alpha_i} \varphi_i(t+s)ds$$

$$- \sum_{j=1}^{n} \int_{-r}^{-t} \delta_{2ij}(s)\varphi_j(t+s)ds.$$

We claim that for $i>k$ the function

$$G_i(t) = \int_0^t (t-s)^{\alpha_i - 1} g_i(s)ds$$

belongs to $W^{1,p}(0,r;\mathbb{R})$ and its derivative is given by

$$\dot{G}_i(t) = t^{\alpha_i - 1}(\eta_i - \sum_{j=1}^{n}\int_{-r}^{0} d\beta_{ij}(s)\varphi_j(s) \tag{2.3.45}$$

$$+ \int_0^t (t-u)^{\alpha_i - 1} f_i(u)du)$$

$$- \int_{-r}^{0} \varphi_i(s) \frac{(t/-s)^{\alpha_i - 1}}{t-s} ds$$

$$+ \frac{1}{r^{\alpha_i - 1}} \int_{-r}^{t-r} \varphi_i(s) \frac{(t-s-r)^{\alpha_i - 1}}{(t-s)} ds$$

$$+ \int_0^t (t-u)^{\alpha_i - 1} (\sum_{j=1}^{n} \delta_{2ij}(-r)\varphi_j(u-r)$$

$$- \sum_{j=1}^{n} \int_{-r}^{-u} d\delta_{2ij}(v)\varphi_j(u+v))du.$$

To establish this result we substitute (2.3.44) into (2.3.41) giving

$$G_i(t) \equiv K_{i1}(t) + K_{i2}(t) + K_{i3}(t),$$

where

$$K_{i1}(t) = \int_0^t (t-s)^{\alpha_i - 1}(\eta_i - \sum_{j=1}^{n}\int_{-r}^{0} b_{ij}(u)\varphi_j(u)du$$

$$+ \int_0^s f_i(u)du)ds,$$

$$K_{i2}(t) = -\int_0^t (t-s)^{\alpha_i - 1}\int_{-r}^{-s}(-u)^{-\alpha_i}\varphi_i(s+u)duds$$

and

$$K_{i3}(t) = - \sum_{j=1}^{n}\int_0^t (t-s)^{\alpha_i - 1} \int_{-r}^{-s}\delta_{2ij}(u)\varphi_j(s+u)duds.$$

Observe that $\dot{K}_{i1}(t)$ exists and is given by

$$\dot{K}_{i1}(t) = \frac{d}{dt}\int_0^t (t-s)^{\alpha_i - 1}(\eta_i$$

$$- \sum_{j=1}^{n} \int_{-r}^{0} b_{ij}(u)\varphi_j(u)du$$

$$+ \int_0^s f_i(u)du)ds$$

$$= (\eta_i - \sum_{j=1}^{n}\int_{-r}^{0} b_{ij}(u)\varphi_j(u)du)t^{\alpha_i - 1}$$

$$+ \int_0^t (t-s)^{\alpha_i - 1} f_i(s)ds.$$

Moreover, $\dot{K}_{i1}$ is in $L_p(0,r;\mathbb{R})$ and for $t\epsilon[0,r]$

$$\|\dot{K}_{i1}\|_{L_p(0,t;\mathbb{R})} \leqslant |\eta_i|(\int_0^t u^{(\alpha_i-1)p}du)^{1/p} \tag{2.3.46}$$

$$+ \|B\|_q \;\|\varphi\|_p(\int_0^t u^{(\alpha_i-1)p}du)^{1/p}$$

$$+ \int_0^t u^{\alpha_i - 1} du \cdot \|f_i\|_{L_p(0,t;\mathbb{R})} \cdot$$

In order to establish the differentiability of K_{i2}, we define $P_i(t)$ by

$$P_i(t) \equiv -\int_{-r}^{0} \varphi_i(s)\ \frac{(t/-s)^{\alpha_i-1}}{t-s}\ ds \tag{2.3.47}$$

$$+\ \frac{1}{r^{\alpha_i-1}} \int_{-r}^{t-r} \varphi_i(s)\frac{(t-s-r)^{\alpha_i-1}}{t-s}ds$$

and show that $P_i(t)$ is in $L_p(0,r;\mathbb{R})$ and that

$$K_{i2}(t) = \int_0^t P_i(s)ds, \quad \text{for } 0\leqslant t\leqslant r.$$

For the second term on the right-hand side of (2.3.47), we have that

$$\frac{1}{r^{\alpha_i-1}} \int_{-r}^{t-r} \varphi_i(s)\ \frac{(t-s-r)^{\alpha_i-1}}{t-s}\ ds \tag{2.3.48}$$

$$= \frac{1}{r^{\alpha_i-1}} \int_0^t \varphi_i(t-u-r)\frac{u^{\alpha_i-1}}{r+u}du$$

The right-hand side of equation (2.3.48) is the convolution of $\varphi_i(\cdot - r) \in L_p(0,r;\mathbb{R})$ and $\frac{t^{\alpha_i-1}}{r+t} \in L_1(0,r;\mathbb{R})$. Therefore, for $t \in [0,r]$

$$\left\| \frac{1}{r^{\alpha_i - 1}} \int_{-r}^{t-r} \varphi_i(s) \frac{(t-s-r)^{\alpha_i - 1}}{t-s} ds \right\|_{L_p(0,t;\mathbb{R})} \tag{2.3.49}$$

$$\leqslant \frac{1}{r^{\alpha_i - 1}} \cdot \int_0^t \frac{u^{\alpha_i - 1}}{r+u} du \cdot \|\varphi_i\|_p .$$

Consider now the first term on the right hand side of (2.3.47)

$$\int_{-r}^{0} \varphi_i(s) \frac{(t/-s)^{\alpha_i - 1}}{t-s} ds$$

$$= \int_{-t}^{0} \varphi_i(s) \frac{(t/-s)^{\alpha_i - 1}}{t-s} ds + \int_{-r}^{-t} \varphi_i(s) \frac{(t/-s)^{\alpha_i - 1}}{t-s} ds$$

$$= Q_{i1}(t) + Q_{i2}(t).$$

Since $\frac{1}{t-s} \leqslant \frac{1}{t}$, we conclude that

$$\int_0^r |Q_{i1}(t)|^p dt \leqslant \int_0^r t^{(\alpha_i - 1)p} \left(\frac{1}{t} \int_{-t}^{0} |\varphi_i(s)| (-s)^{1-\alpha_i} ds\right)^p dt.$$

Recalling that $0 < \frac{1}{p} + \alpha_i - 1 < 1$, an application of a standard inequality by Hardy-Littlewood-Polya (see Adams [1; Lemma 7.23]) for Q_{i1} yields

$$\int_0^r |Q_{i1}(t)|^p dt \leqslant \frac{1}{(2-(1/p)-\alpha_i)^p} \|\varphi_i\|_p^p ,$$

i.e. for $t \in [0,r]$

$$\|Q_{i1}\|_{L_p(0,t;\mathbb{R})} \leqslant \frac{1}{2-(1/p)-\alpha_i} \|\varphi_i\|_p .$$

Note that for $q = \frac{p}{p-1}$ we have $0<\frac{1}{q} + 1 - \alpha_i<1$ and then the inequality by Hardy-Littlewood-Polya applied to $\psi \in L_q\ (0,r;\mathbb{R})$ implies

$$(\int_{-r}^0 (-s)^{(1-\alpha_i)q}(\frac{1}{-s}\int_0^{-s} |\Psi(t)|\ t^{\alpha_i-1} dt)^q ds)^{1/q} \tag{2.3.50}$$

$$\leqslant \frac{1}{\alpha_i-(1/q)} \|\Psi\|_q = \frac{1}{\frac{1}{p}+\alpha_i-1} \|\Psi\|_q .$$

Changing the order of integration, using the estimate $\frac{-s}{t-s} \leqslant 1$, Hölder's inequality, and (2.3.50) it follows that

$$|\int_0^r \Psi(t)Q_{i2}(t)dt| \leqslant \int_0^t |\Psi(t)| \int_{-r}^{-t} |\varphi_i(s)| \frac{(t/-s)^{\alpha_i-1}}{t-s} dsdt$$

$$= \int_{-r}^0 |\varphi_i(s)|(-s)^{-\alpha_i} \int_0^{-s} |\Psi(t)| \frac{-st^{\alpha_i-1}}{t-s} dtds$$

$$\leqslant \int_{-r}^{0} |\varphi_i(s)|(-s)^{-\alpha_i} \int_0^{-s} |\Psi(t)| t^{\alpha_i -1} dtds$$

$$\leqslant \|\varphi_i\|_p \; \{\int_{-r}^{0} (-s)^{(1-\alpha_i)q} [\frac{1}{-s}\int_0^{-s} |\Psi(t)| t^{\alpha_i -1} dt]^q ds\}^{1/q}$$

$$\leqslant \frac{1}{\frac{1}{p}+\alpha_i -1} \|\Psi\|_q \; \|\varphi_i\|_p .$$

Therefore, $Q_{i2} \in L_p(0,r;\mathbb{R})$ and for $t\in[0,r]$

$$\|Q_{i2}\|_{L_p(0,t;\mathbb{R})} \leqslant \frac{1}{\frac{1}{p}+\alpha_i -1} \|\varphi_i\|_p \; .$$

We only need to establish that $K_{i2}(t) = \int_0^t P_i(s)ds$.

Changing the order of integration and making several changes of variables we have

$$\int_0^t P_i(u)du = - \int_0^t \int_{-r}^{0} \varphi_i(s) \; \frac{(u/-s)^{\alpha_i -1}}{u-s} \; dsdu$$

$$+ \; \frac{1}{r^{\alpha_i -1}} \int_0^t \int_{-r}^{u-r} \varphi_i(s) \; \frac{(u-s-r)^{\alpha_i -1}}{u-s} dsdu$$

$$= - \int_{-r}^{0} \varphi_i(s) \int_0^t \frac{(u/-s)^{\alpha_i -1}}{u-s} duds$$

$$+ \frac{1}{r^{\alpha_i - 1}} \int_{-r}^{t-r} \varphi_i(s) \int_{r+s}^{t} \frac{(u-s-r)^{\alpha_i - 1}}{u-s} du ds$$

$$= - \int_{-r}^{0} \varphi_i(s) \int_{\frac{-s}{t-s}}^{1} (1-v)^{\alpha_i - 1} v^{-\alpha_i} dv ds$$

$$+ \int_{-r}^{t-r} \varphi_i(s) \int_{\frac{r}{t-s}}^{1} (1-v)^{\alpha_i - 1} v^{-\alpha_i} dv ds$$

$$= - \int_{-r}^{t-r} \varphi_i(s) \int_{\frac{-s}{t-s}}^{1} (1-v)^{\alpha_i - 1} v^{-\alpha_i} dv ds$$

$$- \int_{t-r}^{0} \varphi_i(s) \int_{\frac{-s}{t-s}}^{1} (1-v)^{\alpha_i - 1} v^{-\alpha_i} dv ds$$

$$+ \int_{-r}^{t-r} \varphi_i(s) \int_{\frac{r}{t-s}}^{1} (1-v)^{\alpha_i - 1} v^{-\alpha_i} dv ds$$

$$= - \left\{ \int_{t-r}^{0} \varphi_i(s) \int_{\frac{-s}{t-s}}^{1} (1-v)^{\alpha_i - 1} v^{-\alpha_i} dv ds \right.$$

$$\left. + \int_{-r}^{t-r} \varphi_i(s) \int_{\frac{-s}{t-s}}^{\frac{r}{t-s}} (1-v)^{\alpha_i - 1} v^{-\alpha_i} dv ds \right\}$$

$$= - \left\{ \int_{t-r}^{0} \varphi_i(s) \int_{0}^{t} (t-u)^{\alpha_i - 1} (u-s)^{-\alpha_i} du ds \right.$$

$$\left. + \int_{-r}^{t-r} \varphi_i(s) \int_{0}^{r+s} (t-u)^{\alpha_i - 1} (u-s)^{-\alpha_i} du ds \right\}$$

$$= - \int_{0}^{t} (t-u)^{\alpha_i - 1} \int_{-r}^{-u} (-s)^{-\alpha_i} \varphi_i(s+u) ds du$$

$$= K_{i2}(t) \ .$$

Therefore, $\dot{K}_{i2}(t) = P_i(t)$ and for $t \in [0,r]$

$$\|\dot{K}_{i2}\|_{L_p(0,t;\mathbb{R})} \leqslant \left(\frac{1}{r^{\alpha_i-1}}\int_0^t \frac{u^{\alpha_i-1}}{r+u}du \right. \tag{2.3.51}$$

$$\left. + \frac{1}{2-\frac{1}{p}-\alpha_i} + \frac{1}{\frac{1}{p}+\alpha_i-1}\right)\|\varphi_i\|_p \ .$$

In order to complete the proof it remains to show that

$$K_{i3}(t) = -\int_0^t (t-s)^{\alpha_i-1} H_i(s)ds,$$

where

$$H_i(t) = \sum_{j=1}^{n} \int_{-r}^{-t} \delta_{2ij}(u)\varphi_j(t+u)du$$

$$= H_i(0) - \int_0^t h_i(u)du,$$

with

$$H_i(0) = \sum_{j=1}^{n} \int_{-r}^{0} \delta_{2ij}(u)\varphi_j(u)du$$

and

$$h_i(t) = -\sum_{j=1}^{n} \delta_{2ij}(-r)\varphi_j(t-r)$$

$$+ \sum_{j=1}^{n} \int_{-r}^{-t} d\delta_{2ij}(u)\varphi_j(t+u) \ .$$

Integrating the very last equation we have

$$\int_0^t h_i(u)du = - \sum_{j=1}^{n} \delta_{2ij}(-r)\int_0^t \varphi_j(u-r)du$$

$$+ \sum_{j=1}^{n}\int_0^t\int_{-r}^{-u} d\delta_{2ij}(v)\varphi_j(u+v)du$$

$$= I_1 + I_2 \ .$$

It follows that

$$I_1 = - \sum_{j=1}^{n} \delta_{2ij}(-r) \int_0^t \varphi_j(u-r)du$$

$$= \sum_{j=1}^{n} \delta_{2ij}(-r)[R_j(t-r)-R_j(-r)],$$

where

$$R_j(s) = \int_0^s \varphi_j(u)du \ .$$

An application of Fubini's theorem together with simplifications yield

$$I_2 = \sum_{j=1}^{n} \int_0^t \int_{-r}^{-u} d\delta_{2ij}(v)\ \varphi_j(u+v)du$$

$$= \sum_{j=1}^{n} \int_{-r}^{-t} d\delta_{2ij}(v) \int_0^t \varphi_j(u+v)du$$

$$+ \sum_{j=1}^{n} \int_{-t}^{0} d\delta_{2ij}(v) \int_0^{-v} \varphi_j(u+v)du$$

$$= - \sum_{j=1}^{n} \int_{-r}^{0} d\delta_{2ij}(v)\ R_j(v)$$

$$+ \sum_{j=1}^{n} \int_{-r}^{-t} d\delta_{2ij}(v) R_j(t+v)\ .$$

Integrating by parts we get

$$\int_0^t h_i(u)du = \sum_{j=1}^{n} \delta_{2ij}(-r)[R_j(t-r)-R_j(-r)]$$

$$- \sum_{j=1}^{n} \int_{-r}^{0} d\delta_{2ij}(u)R_j(u)$$

$$+ \sum_{j=1}^{n} \int_{-r}^{-t} d\delta_{2ij}(u)R_j(t+u)$$

$$= \sum_{j=1}^{n} \int_{-r}^{0} \delta_{2ij}(u)\varphi_j(u)du$$

$$- \sum_{j=1}^{n} \int_{-r}^{-t} \delta_{2ij}(u)\varphi_j(t+u)du$$

$$= H_i(0)-H_i(t)\ .$$

Therefore

$$\dot{K}_{i3}(t) = -\frac{d}{dt}\int_0^t (t-s)^{\alpha_i - 1} H_i(s)ds \; .$$

$$= - H_i(0)t^{\alpha_i - 1} - \int_0^t (t-s)^{\alpha_i - 1} h_i(s)ds \; .$$

Note that $h_i \in L_p(0,r;\mathbb{R})$ and for $t \in [0,r]$

$$\|h_i\|_{L_p(0,t;\mathbb{R})} \leqslant C_{i1} \|\varphi\|_p \; ,$$

where

$$C_{i1} = \sum_{j=1}^{n} |\delta_{2ij}(-r)| + \mathrm{Var}_{[0,r]}(\delta) \; .$$

Also we have

$$|H_i(0)| \leqslant C_{i2} \|\varphi\|_p \; ,$$

where

$$C_{i2} = r^{1/q} \sum_{j=1}^{n} \max_{u\in[-r,0]} |\delta_{2ij}(u)| .$$

Therefore for $t \in [0,r]$

$$\|\dot{K}_{i3}\|_{L_p(0,t;\mathbb{R})} \leqslant C_{i2}\left(\int_0^t u^{(\alpha_i-1)p}du\right)^{1/p}\|\varphi\|_p \tag{2.3.52}$$

$$+ \; C_{i1}\int_0^t u^{\alpha_i-1}du\;\|\varphi\|_p$$

Recalling that by (H5)

$$\delta_{2ij}(u) = \dot{\gamma}_{2ij}(u) = \dot{\beta}_{2ij}(u) - b_{ij}(u)$$

and combining the expressions for $\dot{K}_{i1}$, $\dot{K}_{i2}$ and $\dot{K}_{i3}$ we get (2.3.45). Then Lemma 2.3.6 implies that w, the unique L_p-solution of (2.3.31) is given by (2.3.42).

To finish the proof of the Lemma we note that (2.3.42), (2.3.46), (2.3.51), and (2.3.52) imply for $t \in [0,r]$,$i>k$,

$$\|w_i\|_{L_p(0,t;\mathbb{R})} \leqslant M_i(t)\;\|(\eta,\varphi,f)\|_{\mathbb{R}^n\times L_p\times L_p(0,t;\mathbb{R}^n)} \;,$$

with

$$M_i(t) = \frac{\sin\alpha_i\pi}{\pi}\left\{\left(\int_0^t u^{(\alpha_i-1)p}du\right)^{1/p}(1+\|B\|_q+C_{i2}) \right. \tag{2.3.53}$$

$$\left. + \left(\int_0^t \frac{u^{\alpha_i-1}}{r+u}du\right)\;(1 + \frac{1}{r^{\alpha_i-1}} + C_{i1})\right.$$

$$+ \frac{1}{2-(1/p)-\alpha_i} + \frac{1}{(1/p)+\alpha_i-1} \Bigg\} .$$

Let $M(t) = \sum_{i=1}^{n} M_i(t)$. Then $M \in C(0,r;\mathbb{R})$; $M(t)$ is increasing, nonnegative and for $t \in [0,r]$

$$\|w\|_{L_p(0,t;\mathbb{R})} \leqslant M(t)\|(\eta,\varphi,f)\|_{\mathbb{R}^n \times L_p \times L_p(0,t;\mathbb{R}^n)} \qquad ■$$

Now we are in a position to prove the main result of this section.

PROOF OF THEOREM 2.3.5:

i) By (2,3,28)-(2.3.31), Lemma 2.3.9, and Lemma 2.3.11 the unique solution $x \in L_p(0,r;\mathbb{R}^n)$ of (2.3.18) is given by

$$x(t) = \int_0^t d\zeta(u)w(t-u) \tag{2.3.55}$$

Define $x^1 \in L_p(-r,r;\mathbb{R}^n)$ by

$$x^1(t) = \begin{cases} \varphi(t) & \text{a.e. on } [-r,0] \\ x(t) & \text{a.e. on } [0,r] \end{cases} \tag{2.3.56}$$

where x is given by (2.3.55) .

An easy calculation shows that x^1 satisfies (2.3.11) on $[-r,r]$.

On the other hand if we assume that x^1 and x^2 are different solutions of (2.3.11) on $[-r,r]$ then $z \in L_p(0,r;\mathbb{R}^n)$ defined by $z(t) = x^1(t) - x^2(t)$ a.e. on $[0,r]$ satisfies (2.3.18) with $g \equiv 0$ which leads to the contradiction $z = 0$ a.e. on $[0,r]$; i.e. x^1 is the unique solution of (2.3.11) on $[-r,r]$.

Using the "method of steps" x^1 can be extended to $[-r,\infty)$ as follows.

Assume that x^1 is given by (2.3.56) and for $k \geqslant 1$ define x^{k+1} by

$$x^{k+1}(t) = \begin{cases} x^k(t) & \text{a.e. on } [-r,kr] \\ x(t-kr) & \text{a.e. on } [kr,(k+1)r] \end{cases} \tag{2.3.57}$$

where x^k is the solution of (2.3.11) on $[-r,kr]$ and x is the solution of

$$Dx_t = \eta_k + \int_0^t Lx_u du + \int_0^t f_k(u)du \tag{2.3.58}$$

a.e. on $[0,r]$,

$$x(t) = \varphi_k(t) \qquad \text{a.e. on } [-r,0],$$

with

$$\eta_k = \eta + \int_0^{kr} Lx_u du + \int_0^{kr} f(u)du,$$

$$\varphi_k(t) = x^k(t+kr) \qquad \text{a.e. on } [-r,0],$$

and

$$f_k(t) = f(t+kr) \quad \text{a.e. on } [0,r].$$

Note that the unique solution $x \in L_p\ (0,r;\mathbb{R}^n)$ of (2.3.58) is the unique solution of (2.3.18) with

$$g(t) = \eta_k - \int_{-r}^{0} B(s)\varphi_k(s)ds + \int_0^r f_k(u)du$$

$$- \int_{-r}^{-t} [d\alpha(s) + d\gamma(s)]\ \varphi_k(t+s) \ .$$

(2.3.57) shows then that $x^k \in L_p\ (-r,kr,\mathbb{R}^n)$ for every k.

Let $\tilde{x}$ denote the unique solution of (2.3.11) on $[-r,\infty)$. Then $\tilde{x}_t \in L_p$ for all $t \geq 0$ and an application of the convolution theorem yields the continuity of $y(t)$.

ii) By (2.3.54) - (2.3.58) we have for $t \in [0,r]$, $k \geqslant 0$,

$$\|\tilde{x}\|_{L_p(kr,kr+t;\mathbb{R}^n)} \tag{2.3.59}$$

$$\leqslant L_1(t)\ \|(\eta_k,\varphi_k,f_k)\|_{R^n \times L_p \times L_p(0,t;\mathbb{R}^n)}$$

with

$$L_1(t) \equiv \mathrm{Var}_{[0,r]}(\zeta)M(t), \tag{2.3.60}$$

where we have used the notation $(\eta_0,\ \varphi_0,\ f_0)$ for (η,φ,f).

Hölder's inequality and the estimate (2.3.59) imply for $t \in [0,r]$, $k \geqslant 0$,

$$|y(kr+t)| \leqslant |\eta_k| + \|B\|_q t^{1/q}\ \|\tilde{x}\|_{L_p(kr,kr+t;\mathbb{R}^n)} \tag{2.3.61}$$

$$+ \|B\|_q \|\ \tilde{x}_{t+kr}\|_p + \|B\|_q \|\tilde{x}_{kr}\|_p$$

$$+ t^{1/q}\|f_k\|_p$$

$$\leqslant |\eta_k| + (\|B\|_q t^{1/q} + \|B\|_q)\|\tilde{x}\|_{L_p(kr,kr+t;\mathbb{R}^n)}$$

$$+ 2\|B\|_q \|\tilde{x}_{kr}\|_p + t^{1/q}\|f_k\|_p \leqslant |\eta_k|$$

$$+ (\|B\|_q t^{1/q} + \|B\|_q)\cdot$$

$$L_1(t)\|(\eta_k,\varphi_k,f_k)\|_{\mathbb{R}^n\times L_p\times L_p(0,t;\mathbb{R}^n)}$$

$$+ 2\ \|B\|_q\|\varphi_k\|_p$$

$$+ t^{1/q}\ \|f_k\|_p$$

$$\leqslant L_2(t)\ \|(\eta_k,\ \varphi_k,\ f_k)\|_{\mathbb{R}^n\times L_p\times L_p(0,t;\mathbb{R}^n)},$$

with

$$L_2(t) \equiv 1 + 2\|B\|_q + t^{1/q} + (\|B\|_q t^{1/q} + \|B\|_q)L_1(t) \qquad (2.3.62)$$

Noting that $\eta_{k+1} = y((k+1)r)$, $\varphi_{k+1} = \tilde{x}_{(k+1)r}$, and using the estimates (2.3.59), (2.3.61) we have

$$\|(\eta_{k+1},\ \varphi_{k+1},\ f)\|_{\mathbb{R}^n\times L_p\times L_p(0,(k+1)r;\mathbb{R}^n)} \qquad (2.3.63)$$

$$\leqslant |\eta_{k+1}| + \|f_{k+1}\|_p + \|f\|_{L_p(0,(k+1)r;\mathbb{R}^n)}$$

$$\leqslant (L_2(r) + L_1(r))\|(\eta_k,\ \varphi_k,\ f_k)\|_{\mathbb{R}^n\times L_p\times L_p(0,r;\mathbb{R}^n)}$$

$$+\|f\|_{L_p(0,(k+1)r;\mathbb{R}^n)}$$

$$\leqslant L(r)\|(\eta_k,\varphi_k,f)\|_{\mathbb{R}^n\times L_p\times L_p(0,(k+1)r;\mathbb{R}^n)}$$

with

$$L(r) = L_1(r) + L_2(r) + 1 \tag{2.3.64}$$

Therefore, for $t \in [0,r]$ we have

$$\|\tilde{x}\|_{L_p(-r,kr+t;\mathbb{R}^n)} \leqslant \sum_{i=0}^{k} \|\varphi_i\|_p + \|\tilde{x}\|_{L_p(kr,kr+t;\mathbb{R}^n)} \tag{2.3.65}$$

$$\leqslant \left(\sum_{i=0}^{k} L^i(r) + L^k(r)L_1(t)\right) \|(\eta,\varphi,f)\|_{R^n \times L_p \times L_p(0,kr+t;\mathbb{R}^n)}$$

$$\leqslant \left(\sum_{i=0}^{k} L^i(r) + L^k(r)L_1(r))\right)\|(\eta,\varphi,f)\|_{R^n \times L_p \times L_p(0,(k+1)r;\mathbb{R}^n)},$$

where in the last step we have used the fact that $L_1(\cdot)$ is an increasing, nonnegative, continuous function on $[0,r]$. Similarly

$$|y(t+kr)| \leqslant L_2(t)\|(\eta_k,\varphi_k,f_k)\|_{R^n \times L_p \times L_p(0,t;\mathbb{R}^n)} \tag{2.3.66}$$

$$\leqslant L_2(t)\ L^k(r)\|(\eta_k,\varphi_k,f)\|_{R^n \times L_p \times L_p(0,kr+t;\mathbb{R}^n)}$$

$$\leqslant L_2(r)L^k(r)\|(\eta_k,\ \varphi_k,f)\|_{R^n \times L_p \times L_p(0,(k+1)r;\mathbb{R}^n)},$$

where in the last step we have used the fact that $L_2(\cdot)$ is an increasing, nonnegative, continuous function on $[0,r]$.

Let $T(\cdot)$ be a continuous, increasing and nonnegative function on $t \geqslant 0$ such that

$$T(kr) = \sum_{i=0}^{k+1} L^i(r), \quad k=0,1,2,\ldots . \tag{2.3.67}$$

Then by (2.3.65) and (2.3.66) we get for $t \geqslant 0$

$$\|(y,x)\|_{C(0,t;\mathbb{R}^n)\times L_p(-r,t;\mathbb{R}^n)} \tag{2.3.68}$$

$$\leqslant T(t)\|(\eta,\varphi,f)\|_{R^n\times L_p\times L_p(0,t;\mathbb{R}^n)} ,$$

which implies that for any $t_1>0$ the mapping $(\eta,\varphi,f)\rightarrow(y(\cdot;\eta,\varphi,f),x(\cdot;\eta,\varphi,f))$ from $\mathbb{R}^n\times L_p\times L_p(0,t_1)$ into $C(0,t_1;\mathbb{R}^n) \times L_p(-r,t_1;\mathbb{R}^n)$ is continuous. ■

We conclude this section by establishing the well-posedness of the finite delay version of a particular FDE-system which has been proposed as a mathematical model for two-dimensional aeroelastic systems (see [8]). Note that the well-posedness of this problem was considered in [12] using a somewhat different approach.

<u>EXAMPLE 2.3.12 ("aero problem"):</u> Consider the NFDE (2.1.1)-(2.1.2) with

$$D\phi \equiv A_o\phi(0) + \int_{-r}^{0} A_1(s)\phi(s)ds \tag{2.3.69}$$

and

$$L\phi \equiv B_o\phi(0) + \int_{-r}^{0} B_1(s)\phi(s)ds, \tag{2.3.70}$$

where A_o, $A_1(s)$, B_o, $B_1(s)$ are $n\times n$ matrices satisfying the following condition.

CONDITION A.

i) $A_o = \text{dia}(1,1,\ldots,0)$, $B_{0nn}=0$

ii) $A_{1ij}(s)\equiv 0$, $B_{1ij}(s)\equiv 0$ $i=1,2,\ldots,n$,

$j=1,2,\ldots,n-1$

iii) $A_{1in}(s)$, $B_{1in}(s)$ are continuous on $[-r,0]$,

$i=1,2,\ldots,n-1$

iv) $A_{1nn}(s) = a|s|^{-\alpha} + \psi(s)$, $0 < \alpha < 1$ with $\psi(s)$ is of bounded variation on $[-r,0]$, $\psi(0)=0$ and $B_{1nn}(s)=0$.

Note that the FDE describing the aeroelastic system considered in [8] satisfies the above conditions with $\alpha=1/2$, $n=8$ and $\psi(s)=((1 - s)^{1/2})/(-s)^{1/2}$ for $-r<s<0$, $\psi(0)=0$. Let $A = \text{dia}(A_o, a)$, $\alpha_i(s) = -\rho(-s)$, $i=1,2,\ldots,n-1$, $\alpha_n(s)=-(-s)^{(1-\alpha)}/(1-\alpha)$, $\beta_{ij}(s)= 0$, $i=1,2,\ldots,n$, $j=1,2,\ldots,n-1$, $\beta_{in}(s)= \int_0^s A_{1in}(u)du$, $i=1,2,\ldots,n-1$,

$\beta_{nn}(s) = \int_0^s \psi(u)du$, $B = B_o - \int_{-r}^0 B_1(u)du$ and $B(s) = -\int_{-r}^s B_1(u)du$.

Then (2.3.69), (2.3.70) can be written as (2.3.2), (2.3.1), respectively. Moreover A, $\alpha(\cdot)$, $\beta(\cdot)$, B and $B(\cdot)$ satisfy CONDITION H. It follows by Theorem 2.3.5 that the "aero problem" is well-posed in $\mathbb{R}^n \times L_p$ for $p < 1/(1-\alpha)$.

On the other hand if $p \geqslant 1/(1-\alpha)$ then there is no $H(\cdot)$ such that $H(\lambda) \longrightarrow 0$ as $\lambda \longrightarrow \infty$ and (2.2.33) holds, because in the last row of the $n\times n$ matrix valued function $D(e^{\lambda \cdot} I)$ every element is identically zero except $D_{nn}(e^{\lambda \cdot} I)$ and $\lambda^{1/p} D_{nn}(e^{\lambda \cdot} I) \longrightarrow$constant as $\lambda \longrightarrow \infty$, $\lambda \in \mathbb{R}$ (see Remark 2.2.9). It follows that the aero problem is not well-posed in $\mathbb{R}^n \times L_p$ for $p \geqslant 1/(1-\alpha)$. As an important consequence, in the case $\alpha = 1/2$, the Hilbert space $\mathbb{R}^n \times L_2$ is not available for approximation purposes. ■

CHAPTER III

APPROXIMATION RESULTS

3.1 Preliminaries

In this chapter we will be concerned with the construction of approximate solutions to linear abstract Cauchy-problems in the Banach space $\mathbb{Z} = \mathbb{R}^n \times L_p$. In view of the equivalence between generalized solutions to a well-posed NFDE and mild solutions to the "corresponding" abstract Cauchy-problem sequences of approximate solutions that converge to the actual solution of the abstract Cauchy-problem provide convergent sequences of approximate solutions to the actual solution of the NFDE, assuming certain "structural" conditions on the D-operator and sufficiently smooth initial data. To explain this, note that in the case of an NFDE the numerical schemes give an approximation of Dx_t instead of $x(t)$, where $x(\cdot)$ is the solution of the NFDE, i.e. there is the possibility that one has to impose additional conditions in order to get an approximation of $x(t)$ (see[26]).

We begin by calling upon some results concerning approximation of C_o-semigroups of bounded linear operators on abstract spaces that will prove useful in our discussion below (see e.g. [3]-[5], [26], [28], [30]).

Let the linear operator $\mathcal{Q}$ be the infinitesimal generator of a C_o-semigroup $\{S(t)\}_{t\geq 0}$, on the Banach space $\mathbb{Z}$ and consider the homogeneous abstract Cauchy-problem

$$\dot{z}(t) = \mathcal{Q}z(t), \quad t \geq 0, \tag{3.1.1}$$

with initial data

$$z(0) = z_0 \in \mathbb{Z}. \tag{3.1.2}$$

We want to approximate the solutions of (3.1.1) - (3.1.2) by sequences of solutions to Cauchy-problems on $\mathbb{Z}^N$, where $\mathbb{Z}^N$ is a finite dimensional subspace of $\mathbb{Z}$ for N = 1,2,... . Accordingly, we consider for N = 1,2,... the initial value problems on $\mathbb{Z}^N$

$$\dot{z}^N(t) = \mathcal{Q}^N z^N(t), \quad t \geq 0, \tag{3.1.3}$$

$$\dot{z}^N(0) = P^N z_0, \quad z_0 \in Z, \tag{3.1.4}$$

where P^N: $\mathbb{Z} \to \mathbb{Z}^N$ is a projection and Q^N: $\mathbb{Z}^N \to \mathbb{Z}^N$ is linear operator.

We shall refer to the sequence $(\mathbb{Z}^N, P^N, Q^N)$ as an approximation scheme for (3.1.1.) - (3.1.2). Note that Q^N generates a C_o-semigroup $(S^N(t))_{t \geq 0}$, on $\mathbb{Z}^N$ which can be extended to a C_o-semigroup on $\mathbb{Z}$ defining Q^N by $Q^N z = Q^N P^N z$, $z \in \mathbb{Z}$. In this case we have

$$S^N(t)z_0 = e^{Q^N t} P^N z_0 + z_0 - P^N z_0, \; t \geq 0, \; z_0 \in \mathbb{Z}.$$

For the convenience of the reader we include the statement of the following theorem which we shall use in Section 3.2 to establish the convergence of the averaging projection scheme for certain classes of NFDE's on the Banach space $R^N \times L_p$.

<u>LEMMA</u> <u>3.1.1:</u> (see Theorem 4.1 in [28]) Suppose that the approximation scheme $(\mathbb{Z}^N, P^N, Q^N)$ satisfies the following hypotheses:

H1) $\lim_{N \to \infty} P^N z = z$ for all $z \in \mathbb{Z}$.

H2) There exist constants $\tilde{M} \geq 1$ and $\tilde{\omega} \in \mathbb{R}$ such that

$\|S^N(t)z\| \leq \tilde{M}e^{\tilde{\omega}t}\|z\|$ for all $t \geq 0$, $z \in \mathbb{Z}^N$ and

$N = 1,2,\ldots.$

H3) There exists a dense subset $D \subset \mathcal{D}(\mathcal{Q})$ which is invariant with respect to $(S(t))_{t\geqslant 0}$ such that

i) $\lim\limits_{N\to\infty} \mathcal{Q}^N P^N z = \mathcal{Q}z$ for all $z \in D$

and

ii) for any $z \in D$ there exists a function $m(\cdot,z) \in L_{1,loc}(0,\infty;\mathbb{R})$ such that $\|\mathcal{Q}^N P^N S(t)z\| \leqslant m(t;z)$ a.e. on $[0,\infty)$ for all N.

Then for all $z_0 \in \mathbb{Z}$

$$\lim_{N\to\infty} e^{\mathcal{Q}^N t} P^N z_0 = S(t) z_0$$

uniformly for t in bounded intervals. ∎

<u>REMARK</u> <u>3.1.2.:</u> (see pp. 25 in [28]) Condition H2) is equivalent to

H2*) For any N there exists a norm $\|\cdot\|_N$ on $\mathbb{Z}^N$ such that

i) for some constant $\tilde{M} \geqslant 1$, $\|z\| \leqslant \|z\|_N \leqslant \tilde{M}\,\|z\|$, $z \in \mathbb{Z}^N$, $N = 1,2\ldots$

and

ii) for some constants $\tilde{\omega} \in \mathbb{R}$ all operators $\mathcal{Q}^N - \tilde{\omega}I$ are dissipative on $(\mathbb{Z}^N, \|\cdot\|_N)$. ■

<u>REMARK</u> <u>3.1.3:</u> Let $\mathbb{Z}$ be a real Banach space and $\mathcal{Q}$ be a linear operator with domain $\mathcal{D}(\mathcal{Q})$ and range $\mathcal{R}(\mathcal{Q})$ both in $\mathbb{Z}$. Recall that $\mathcal{Q}$ is called dissipative if and only if for all $z \in \mathcal{D}(\mathcal{Q})$ and some $j \in J(z)$, $\langle \mathcal{Q}z, j\rangle \leqslant 0$, where $\langle\cdot,\cdot\rangle$ denotes the pairing between $\mathbb{Z}$ and $\mathbb{Z}^*$ and J is the duality mapping. $J: \mathbb{Z} \to 2^{\mathbb{Z}^*}$ defined by $j \in J(z)$ if and only if $\langle z,j\rangle = \|z\|^2 = \|j\|^2$ for $z \in \mathbb{Z}$. ■

Next we consider the nonhomogeneous problem

$$\dot{z}(t) = \mathcal{Q}z(t) + (f(t),0), \quad t \geqslant 0, \tag{3.1.5}$$

$$z(0) = z_0 \in \mathbb{Z}, \; f \in L_{p,loc}. \tag{3.1.6}$$

The unique mild solution of (3.1.5)-(3.1.6) is given by

$$z(t) = S(t)z_0 + \int_0^t S(t-s)(f(s),0))ds. \tag{3.1.7}$$

Similar to the homogeneous case we consider the approximating systems on $\mathbb{Z}^N$;

$$\dot{z}^N(t) = \mathcal{Q}^N z^N(t) + P^N(f(t),0) \tag{3.1.8}$$

$$z^N(0) = P^N z_0, \quad z_0 \in \mathbb{Z}. \tag{3.1.9}$$

The unique solution $z^N(t)$ of (3.1.8)-(3.1.9) is given by

$$z^N(t) = S^N(t)P^N z_0 + \int_0^t S^N(t-s)P^N(f(s),0)ds. \tag{3.1.10}$$

For further reference we include the statement of the following result.

<u>LEMMA</u> <u>3.1.4:</u> (see Theorem 4.2 in [28], Theorem 3.2 in [26]) Assume that $S^N(\cdot)$, $N=1,2,\ldots,$ and $S(\cdot)$ are C_0-semigroups on $\mathbb{Z}$ such that for constant $M \geqslant 1$, $\omega \in \mathbb{R}$, $\|S^N(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$, $N=1,2,\ldots,$ and for all $z_0 \in \mathbb{Z}$

$$\lim_{N\to\infty} S^N(t)P^N z_0 = S(t)z_0$$

uniformly on bounded t-intervals. Then,

i) for $z_0 \in \mathbb{Z}$ and $f \in L_{p,loc}$ we have

$$\lim_{N\to\infty} z^N(t) = z(t), \quad t \geqslant 0,$$

and for any $T>0$ the limit is uniform with respect to $t \in [0,T]$ and with respect to f in bounded subsets of $L_p(0,T;\mathbb{R}^n)$.

ii) If $\{f^k\}$ is a sequence in $L_p(0,T;\mathbb{R}^n)$ converging weakly to f, then

$$\lim_{N,k\to\infty} z^N(t,z_0,f^k) = z(t,z_0,f),$$

uniformly for $t \in [0,T]$. ∎

3.2 Averaging projections

Our objective in this section is to establish the convergence of the "averaging projections" scheme in the Banach spaces $\mathbb{R}^n \times L_p$ for a class of NFDEs with atomic difference operators. This approximation has been studied by a number of authors in the Hilbert space $\mathbb{R}^n \times L_2$ (see e.g. [4] for RFDEs, [26] for NFDEs). The consideration of the more general Banach space situation is motivated by certain well-posed NFDEs for which the Hilbert space $\mathbb{R}^n \times L_2$ is not available (see Section 2.3). Note that the discussion below relies heavily on the paper by Kappel (see [26]). Extending the ideas presented in [26] we show that the "averaging projection" scheme satisfies the hypotheses of Lemma 3.1.1 on the Banach space $\mathbb{Z} = \mathbb{R}^n \times L_p$.

Throughout this section we shall assume that the linear operators D and L have the following representations

$$D\varphi = \varphi(0) - \int_{-r}^{0} A(s)\ \varphi(s)ds \tag{3.2.1}$$

and

$$L\varphi = B\varphi(0) + \int_{-r}^{0} B(s)\varphi(s)ds, \tag{3.2.2}$$

where B is nxn constant matrix and $A(\cdot)$, $B(\cdot)$ belong to

$L_q[-r,0]; \mathbb{R}^{n\times n})$, $1/p + 1/q = 1$.

Note that (3.2.1)-(3.2.2) imply that $\mathcal{A}$ defined by (2.1.3)-(2.1.4) is the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on $\mathbb{Z} = \mathbb{R}^n \times L_p$, $1 < p < \infty$, where $\mathbb{Z}$ is supplied with the norm

$$\|(\eta,\varphi)\| = (|\eta|^p + \int_{-r}^{0} |\varphi(\theta)|^p d\theta)^{1/p}. \qquad (3.2.3)$$

Recall (see pp 25 in [19], Proposition 2.1 in [35]) that the duality mapping for $\mathbb{Z}$ is single valued for $1<p<\infty$, $J(0,0) = \{0\}$, and for $(\eta,\varphi) \neq (0,0)$, $(\eta,\varphi) \in \mathbb{Z}$, $j \in J(\eta,\varphi)$ if for all $(\kappa,\psi) \in \mathbb{Z}$ we have

$$\langle(\kappa,\psi),J\rangle = \|(\eta,\varphi)\|^{2-p}(\int_{-r}^{0} \{\psi,\varphi\}|\varphi|^{p-2}d\theta \qquad (3.2.4)$$

$$+ \{\kappa,\eta\}|\eta|^{p-2}),$$

where $\{\cdot,\cdot\}$ denotes the inner product in $\mathbb{R}^n$.

<u>LEMMA</u> <u>3.2.1</u>: The operator $\mathcal{A} - \omega I$ is dissipative in $\mathbb{Z}$ for some $\omega \in \mathbb{R}$.

<u>PROOF</u>: Let $(\eta,\varphi) \in \mathcal{D}(\mathcal{A})$ and $j \in J(\eta,\varphi)$ given by (3.2.3). Then

$$\langle \mathcal{A}(\eta,\varphi), j\rangle = \langle(L\varphi, \dot{\varphi}), j\rangle$$

$$= \|(\eta,\varphi)\|^{2-p} \left(\int_{-r}^{0} \langle\dot{\varphi}, \varphi\rangle|\varphi|^{p-2}d\theta\right.$$

$$\left. + \langle L\varphi,\eta\rangle|\eta|^{p-2}\right)$$

$$= \|(\eta,\varphi)\|^{2-p}(I+II).$$

Using the representations (3.2.1)-(3.2.2) and the condition $\eta = D\varphi$ we can estimate II as follows:

$$II = \langle L\varphi,\eta\rangle|\eta|^{p-2} = |\eta|^{p-2}\eta^T(B\varphi(0)+\int_{-r}^{0} B(s)\varphi(s)ds)$$

$$= |\eta|^{p-2}\eta^T(B\eta + B\int_{-r}^{0} A(s)\varphi(s)ds + \int_{-r}^{0} B(s)\varphi(s)ds)$$

$$\leqslant |\eta|^{p-2}|\eta|(|B|\ |\eta| + |B|\ \|A\|_q\|\varphi\|_p + \|B\|_q\|\varphi\|_p).$$

Noting that

$$|\eta|^{p-1}\|\varphi\|_p \leqslant \|(\eta,\varphi)\|^{p-1}\ \|(\eta,\varphi)\| = \|(\eta,\varphi)\|^p$$

and

$$|\eta|^p \leqslant \|(\eta,\varphi)\|^p$$

we get

$$II \leqslant (|B| + |B|\|A\|_q + \|B\|_q)\|(\eta,\varphi)\|^p.$$

On the other hand, I satisfies the inequality

$$I = \int_{-r}^{0} (\dot{\varphi}, \varphi) |\varphi|^{p-2} d\theta$$

$$= (1/p) \int_{-r}^{0} \frac{d}{d\theta} |\varphi|^{p} d\theta$$

$$= (1/p)(|\varphi(0)|^{p} - |\varphi(-r)|^{p})$$

$$\leqslant (1/p)|\varphi(0)|^{p}.$$

The representation (3.2.1), $\eta = D\varphi$ and simple calculations imply that

$$|\varphi(0)|^{p} \leqslant (|\eta| + |\int_{-r}^{0} A(s)\varphi(s)ds|)^{p}$$

$$\leqslant 2^{p}(|\eta|^{p} + \|A\|_{q}^{p} \|\varphi\|_{p}^{p})$$

$$\leqslant 2^{p}(1 + \|A\|_{q}^{p}) \|(\eta,\varphi)\|^{p}.$$

Using the estimates for I and II it follows that

$$\langle a(\eta,\varphi), j\rangle \leqslant \omega_0 \|(\eta,\varphi)\|^{2},$$

where

$$\omega_0 = (|B| + |B|\ \|A\|_q + \|B\|_q + 2^p(1 + \|A\|_q^p)),$$

which implies the dissipativeness of $\mathcal{a} - \omega I$ for $\omega \geq \omega_0$ ∎

Next we discuss the "<u>averaging projection</u>" scheme. For a given N, partition $[-r,0]$ into subintervals $[t_j^N, t_{j-1}^N]$ for $j = 1,2\ldots,N$, where $t_j^N = -\frac{r}{N}j$. Corresponding to this partition define:

i) The finite dimensional subspace $\mathbb{Z}^N$ by

$$\mathbb{Z}^N = \left\{(\eta,\varphi) \in \mathbb{Z},\ \varphi = \sum_{j=1}^{N} \varphi_j^N \chi_j,\ \varphi_j^N \in \mathbb{R}^n\right\},$$

where χ_j^N is the characteristic function of $[t_j^N, t_{j-1}^N)$,

ii) The projection P^N: $\mathbb{Z} \to \mathbb{Z}^N$ by

$$P^N(\eta,\varphi) = (\eta,\varphi^N),$$

where

$$\varphi^N = \sum_{j=1}^{N} \varphi_j^N \chi_j^N \text{ and } \varphi_j^N = \frac{N}{r}\int_{t_j^N}^{t_{j-1}^N} \varphi(s)ds,$$

$j = 1,2,\ldots,N$ and

iii) The linear operator $\mathcal{a}^N$: $\mathbb{Z} \to \mathbb{Z}^N$ by

$$\mathcal{A}^N(\eta,\varphi) = \mathcal{A}^N P^N(\eta,\varphi) = (L^N(\varphi_0^N,\varphi^N),\ D^N\varphi^N),$$

where

$$\varphi_0^N = \eta + \int_{-r}^{0} A(s)\ \varphi^N(s)ds,$$

$$[D^N\varphi^N]_j = \sum_{j=1}^{N} t_j^N [D^N\varphi^N]_j \text{ and } [D^N\varphi^N]_j = \frac{N}{r}(\varphi_{j-1}^N - \varphi_j^N),$$

$$j = 1,2,\ldots,N.$$

<u>LEMMA 3.2.2</u>: For some constant $\tilde{\omega}$ all operators $\mathcal{A}^N - \tilde{\omega}I$ are dissipative on $(\mathbb{Z}^N, \|\cdot\|)$, where $\|\cdot\|$ denotes the norm inherited from $\mathbb{Z}$.

<u>PROOF</u>. We have to estimate

$$\begin{aligned}\langle \mathcal{A}^N(\eta,\varphi^N), j\rangle &= \langle (L^N(\varphi_0^N,\ \varphi^N),\ D^N\varphi^N), j\rangle \\ &= \|(\eta,\varphi^N)\|^{2-p}\Big(\int_{-r}^{0} (D^N\varphi^N,\varphi^N)|\varphi^N|^{p-2}d\theta \\ &\quad + (L^N(\varphi_0^N,\varphi^N),\ \eta)\ |\eta|^{p-2}\Big) \\ &= \|(\eta,\varphi^N)\|^{2-p}(I + II).\end{aligned}$$

Proceeding as in the proof of Lemma 3.2.1 (except that now we use $\varphi_0^N = \eta + \int_{-r}^{0} A(s)\varphi^N(s)ds$) we get for II:

$$II = |\eta|^{p-2} \cdot \eta^T(B\varphi_0^N + \int_{-r}^{0} B(s)\ \varphi^N(s)ds)$$

$$= |\eta|^{p-2}\eta^T(B\eta + B\int_{-r}^{0} A(s)\ \varphi^N(s)ds$$

$$+ \int_{-r}^{0} B(s)\ \varphi^N(s)ds)$$

$$\leqslant |\eta|^{p-1}(|B||\eta| + |B|\|A\|_q\|\varphi^N\|_p$$

$$+ \|B\|_q\|\varphi^N\|_p)$$

$$\leqslant (|B| + |B|\|A\|_q + \|B\|_q)\|(\eta,\varphi)\|^p.$$

Using the definition of $D^N\varphi^N$ and computing the integral in I we obtain the estimate

$$I = \int_{-r}^{0} (D^N\varphi^N)^T\ \varphi^N\ |\varphi^N|^{p-2}d\theta$$

$$= \sum_{j=1}^{N}((\varphi_{j-1}^N)^T - (\varphi_j^N)^T)\ \varphi_j^N\ |\varphi_j^N|^{p-2}$$

$$= \sum_{j=1}^{N}((\varphi_{j-1}^N)^T\varphi_j^N\ |\varphi_j^N|^{p-2} - |\varphi_j^N|^p)$$

$$\leqslant \sum_{j=1}^{N}(|\varphi_{j-1}^N|\ |\varphi_j^N|^{p-1} - |\varphi_j^N|^p).$$

Let $a = |\varphi_{j-1}^N|^p$ and $b = |\varphi_j^N|^p$. An application of the well-known inequality (see Royden:Real Analysis, pp. 112),

$$a^{1/p} \cdot b^{1/q} \leqslant (1/p)a + (1/q)b,$$

implies that

$$|\varphi_{j-1}^N| \; |\varphi_j^N|^{p-1} \leqslant \frac{1}{p} |\varphi_{j-1}^N|^p + \frac{1}{q} |\varphi_j^N|^p.$$

Therefore, we have that

$$I \leqslant \sum_{j=1}^{N} \left((1/p) |\varphi_{j-1}^N|^p + \frac{1}{q} |\varphi_j^N|^p - |\varphi_j^N|^p \right)$$

$$= \sum_{j=1}^{N} \frac{1}{p}(|\varphi_{j-1}^N|^p - |\varphi_j^N|^p)$$

$$= (1/p)(|\varphi_0^N|^p - |\varphi_N^N|^p)$$

$$\leqslant 1/p \; |\varphi_0^N|^p.$$

Noting that

$$|\varphi_0^N|^p \leqslant 2^p(|\eta|^p + \|A\|_q^p \; \|\varphi^N\|_p^p)$$

$$\leqslant 2^p \; (1 + \|A\|_q^p)\|(\eta,\varphi)\|^p$$

and using the estimates for I and II we obtain

$$\langle \mathcal{a}^N(\eta,\varphi), \; j\rangle \leqslant \tilde{\omega}_0 \|(\eta,\varphi^N)\|^2,$$

where

$$\tilde{\omega}_0 = (|B| + |B| \, \|A\|_q + \|B\|_q + 2^p(1 + \|A\|_q^p)),$$

which implies the dissipativeness of $\mathcal{A}^N - \omega I$ for $\omega \geqslant \tilde{\omega}_0$. ∎

<u>LEMMA</u> <u>3.2.3</u>: If $z \in \mathbb{Z}$, then $P^N z \longrightarrow z$

<u>PROOF</u>: First we show that the sequence $\{P^N\}$ is uniformly bounded. Since $P^N(\eta,\varphi) = (\eta,\varphi^N)$, it follows that

$$\|P^N(\eta,\varphi)\| = \|(\eta,\varphi^N)\| = (|\eta|^p + \int_{-r}^{0} |\varphi^N|^p d\theta)^{1/p}.$$

Using the definition of φ^N, an application of Hölder's inequality yields the estimate

$$\int_{-r}^{0} |\varphi^N(\theta)|^p d\theta = \sum_{j=1}^{N} (\tfrac{N}{r})^p |\int_{t_j^N}^{t_{j-1}^N} \varphi(\theta) d\theta|^p \, \tfrac{r}{N}$$

$$\leqslant \sum_{j=1}^{N} (\tfrac{N}{r})^p (\tfrac{r}{N})^{p/q} \int_{t_j^N}^{t_{j-1}^N} |\varphi(\theta)|^p \, d\theta \, \tfrac{r}{N}$$

$$= \sum_{j=1}^{N} \int_{t_j^N}^{t_{j-1}^N} |\varphi(\theta)|^p \, d\theta = \int_{-r}^{0} |\varphi(\theta)|^p \, d\theta.$$

Consequently $\|P^N(\eta,\varphi)\| \leqslant \|(\eta,\varphi)\|$ and $\|P^N\| \leqslant 1$, for all $N=1,2,\ldots$.

We now show that $P^N z \to z$ on a dense subset $\mathbb{Z}_c \subseteq \mathbb{Z}$, defined by $\mathbb{Z}_c = \left\{(D\varphi,\varphi) \mid \varphi\epsilon C\right\}$. In order to estimate

$\| \varphi^N - \varphi\|_p$, we note that

$$\|\varphi^N - \varphi\|_p = \left(\int_{-r}^{0} |\varphi^N(\theta) - \varphi(\theta)|^p d\theta\right)^{1/p}$$

$$= \Big(\sum_{j=1}^{N} \int_{t_j^N}^{t_{j-1}^N} \Big| \frac{N}{r} \int_{t_j^N}^{t_{j-1}^N} \varphi(s)ds - \varphi(\theta)\Big|^p d\theta\Big)^{1/p}.$$

The mean value theorem implies that there exists $\tau_j \in [t_j^N, t_{j-1}^N]$ such that

$$\varphi(\tau_j) = \frac{N}{r} \int_{t_j^N}^{t_{j-1}^N} \varphi(s)ds.$$

Uniform continuity of φ implies that $\|\varphi^N-\varphi\|_p \longrightarrow 0$ as $N \longrightarrow \infty$ for all $(\eta,\varphi) \in \mathbb{Z}_c$.

Uniform boundedness of $\{P^N\}$ and density of $\mathbb{Z}_c$ in $\mathbb{Z}$ yield that $P^N z \longrightarrow z$ as $N \longrightarrow \infty$ for all $z \in \mathbb{Z}$. ∎

<u>LEMMA</u> <u>3.2.4</u>: Let $\mathbb{Z}_{C^1} = \left\{(D\varphi,\varphi) \mid \varphi\in C^1\right\}$. If $(\eta,\varphi) \in \mathbb{Z}_{C^1}$, then

$$\mathcal{Q}^N(\eta,\varphi) \to \mathcal{Q}(\eta,\varphi).$$

<u>PROOF</u>: Using the definitions of $\mathcal{Q}^N$, $\mathcal{Q}$, and the representations (3.2.1) - (3.2.2) we have

$$\|\mathcal{Q}(\eta,\varphi) - \mathcal{Q}^N(\eta,\varphi)\|^p = |L^N(\varphi_0^N,\varphi^N)-L\varphi|^p \tag{3.2.5}$$

$$+ \int_{-r}^{0} |[D^N\varphi^N](\theta) - \dot{\varphi}(\theta)|^p d\theta = I + II.$$

Consider

$$I = |L(\varphi_0^N, \varphi^N) - L\varphi|^p = |B(\varphi_0^N - \varphi(0)) \tag{3.2.6}$$

$$+ \int_{-r}^{0} B(\theta)\ [\varphi^N(\theta) - \varphi(\theta)]d\theta|^p$$

$$= |B\int_{-r}^{0} A(\theta)[\varphi^N(\theta) - \varphi(\theta)]d\theta$$

$$+ \int_{-r}^{0} B(\theta)\ [\varphi^N(\theta) - \varphi(\theta)]d\theta^p|^p$$

$$\leqslant K\ \|\varphi^N-\varphi\|_p^p \to 0 \quad \text{as } N \to \infty,$$

where $K = \|BA(\cdot) + B(\cdot)\|_q^p$. Thus $L(\varphi_0^N, \varphi^N) \to L\varphi$ in $\mathbb{R}^n$.

Next we show that

$$II = \int_{-r}^{0} |[D^N\varphi^N](\theta) - \dot{\varphi}(\theta)|^p d\theta \longrightarrow 0 \text{ as } N \longrightarrow \infty.$$

Since $\varphi \in C^1$, we have that for $u \in [t_j^N, t_{j-1}^N]$

$$\varphi(u) = \varphi(t_j^N) + \dot{\varphi}(t_j^N)(u-t_j^N) + \int_{t_j^n}^{u}(\dot{\varphi}(\theta) - \dot{\varphi}(t_j^N))\ d\theta$$

Consequently, it follows that

$$\varphi_j^N = \frac{N}{r}\int_{t_j^N}^{t_{j-1}^N} \varphi(\theta)d\theta$$

$$= \varphi(t_j^N) + \dot{\varphi}(t_j^N) \cdot \frac{N}{r} \int_{t_j^N}^{t_{j-1}^N} (u - t_j^N)\, du$$

$$+ \frac{N}{r} \int_{t_j^N}^{t_{j-1}^N} \int_{t_j^N}^{u} (\dot{\varphi}(\theta) - \dot{\varphi}(t_j^N))\, d\theta\, du$$

$$= \varphi(t_j^N) + \frac{1}{2} \frac{r}{N} \dot{\varphi}(t_j^N)$$

$$+ \frac{N}{r} \int_{t_j^N}^{t_{j-1}^N} \int_{t_j^N}^{u} [(\dot{\varphi}(\theta) - \dot{\varphi}(t_j^N)]\, d\theta\, du.$$

If $j > 1$, then

$$\left| [D^N \varphi^N]_j - \dot{\varphi}(\theta) \right| = \left| \frac{N}{r} (\varphi_{j-1}^N - \varphi_j^N) - \dot{\varphi}(\theta) \right|$$

$$= \left| \left(\frac{N}{r} (\varphi(t_{j-1}^N) - \varphi(t_j^N) \right) - \dot{\varphi}(\theta) \right.$$

$$+ \frac{1}{2} [\dot{\varphi}(t_{j-1}^N) - \dot{\varphi}(t_j^N)]$$

$$+ \frac{N^2}{r^2} \left(\int_{t_{j-1}^N}^{t_{j-2}^N} \int_{t_{j-1}^N}^{u} [\dot{\varphi}(\theta) - \dot{\varphi}(t_{j-1}^N)]\, d\theta\, du \right.$$

$$- \int_{t_{j-1}^N}^{t_j^N} [\dot\varphi(\theta) - \dot\varphi(t_j^N)]d\theta du\Big)\Big|$$

$$\leqslant \Big|\frac{N}{r}[\varphi(t_{j-1}^N) - \varphi(t_j^N)] - \dot\varphi(\theta)\Big|$$

$$+ \frac{1}{2}\,\Big|\,\dot\varphi(t_{j-1}^N) - \dot\varphi(t_j^N)\Big|$$

$$+ \frac{N^2}{r^2}\,\Big|\int_{t_{j-1}^N}^{t_{j-2}^N}\int_{t_{j-1}^N}^{u} [\dot\varphi(\theta) - \dot\varphi(t_{j-1}^N)]d\theta du\Big|$$

$$+\frac{N^2}{r^2}\,\Big|\int_{t_j^N}^{t_{j-1}^N}\int_{t_j^N}^{u}[\dot\varphi(\theta) - \dot\varphi(t_j^N)]d\theta du\Big|$$

$$= A + B + C + D.$$

Now we estimate the right-hand side of the last inequality. Let

$$\epsilon^N = \max_{1\leqslant j\leqslant N}\Big(\max_{t_j^N\leqslant\theta,\tau\leqslant t_{j-1}^N}\Big|\dot\varphi(\theta) - \dot\varphi(\theta)\Big|\Big),$$

and note that A satisfies

$$\left|\frac{N}{r}\left[\varphi(t_{j-1}^N) - \varphi(t_j^N)\right] - \dot{\varphi}(\theta)\right| = \left|\dot{\varphi}(\zeta) - \dot{\varphi}(\theta)\right| \leqslant \epsilon^N,$$

where $\zeta \in [t_j^N, t_{j-1}^N]$. It follows that

$$B = \frac{1}{2} \mid \dot{\varphi}(t_{j-1}^N) - \dot{\varphi}(t_j^N) \mid \leqslant \frac{1}{2} \epsilon^N.$$

and C satisfies the estimate

$$C = \frac{N^2}{r^2} \left| \int_{t_{j-1}^N}^{t_{j-2}^N} \int_{t_{j-1}^N}^{u} [\dot{\varphi}(\theta) - \dot{\varphi}(t_{j-1}^N)] d\theta du \right|$$

$$\leqslant \frac{N^2}{r^2} \epsilon^N \left| \int_{t_{j-1}^N}^{t_{j-2}^N} \mid (u - t_{j-1}^N) \, du \right|$$

$$= \frac{N^2}{r^2} \epsilon^N \cdot \frac{1}{2} \frac{r^2}{N^2} = \frac{1}{2} \epsilon^N$$

Finally, using a similar estimates for D we obtain

$$D = \frac{N^2}{r^2} \left| \int_{t_j^N}^{t_{j-1}^N} \int_{t_j^N}^{u} [\dot{\varphi}(\theta) - \dot{\varphi}(t_j^N)] d\theta du \right| \leqslant \frac{1}{2} \epsilon^N.$$

Therefore, if $j > 1$ we have shown that

$$|[D^N\varphi^N]_j - \dot{\varphi}(\theta)| \leqslant \frac{5}{2}\,\epsilon^N,$$

and hence

$$\int_{-r}^{-r/N} |[D^N\varphi^N]_j(\theta) - \dot{\varphi}(\theta)|^p\, d\theta \leqslant (\tfrac{5}{2})^p(\epsilon^N)^p \cdot r. \qquad (3.2.7)$$

If $j = 1$, then

$$[D^N\varphi^N]_1 = \frac{N}{r}\,(\varphi_0^N - \varphi_1^N)$$

$$= \frac{N}{r}\Big(\varphi(0) + \int_{-r}^{0} A(\theta)\,(\varphi^N(\theta) - \varphi(\theta))d\theta - \varphi(t_1^N)$$

$$- \frac{1}{2}\,\frac{r}{N}\,\dot{\varphi}(t_1^N) - \frac{N}{r}\int_{t_1^N}^{0}\int_{t_1^N}^{u} [\dot{\varphi}(\theta)-\varphi(t_1^N)]d\theta du\Big).$$

Therefore, we have

$$\Big|[D^N\varphi^N]_1 - \dot{\varphi}(\theta)\Big| \leqslant \Big|\frac{N}{r}\,(\varphi(0) - \varphi(t_1^N)) - \dot{\varphi}(\theta)\Big|$$

$$+ \frac{1}{2} \left|\dot{\varphi}(t_1^N)\right| + \frac{N^2}{r^2}\left|\int_{t_1^N}^{0} \int_{t_1^N}^{u}[\dot{\varphi}(\theta)-\varphi(t_1^N)]d\theta du\right|$$

$$+ \frac{N}{r}\left|\int_{-r}^{0} A(\theta)\left(\varphi^N(\theta)-\varphi(\theta)\right)d\theta\right|$$

$$\leqslant \epsilon^N + \frac{1}{2} | \dot{\varphi}(t_1^N)| + \frac{1}{2} \epsilon^N + \frac{N}{r} \|A\|_q \|\varphi^N-\varphi\|_p .$$

Observe that there exists $\zeta_j \in [t_j^N, t_{j-1}^N]$ such that

$$\|\varphi^N-\varphi\|_p = (\int_{-r}^{0} |\varphi^N(\theta) - \varphi(\theta)|^p d\theta)^{1/p}$$

$$= (\sum_{j=1}^{N} \int_{t_j^N}^{t_{j-1}^N} \left|\frac{N}{r} \int_{t_j^N}^{t_{j-1}^N} \varphi(s)ds - \varphi(\theta) \right|^p d\theta)^{1/p}$$

$$= \left(\sum_{j=1}^{N} \int_{t_j^N}^{t_{j-1}^N} | \varphi(\zeta_j) - \varphi(\theta)|^p d\theta\right)^{1/p}.$$

Moreover, if $M = \max_{\tau\in[-r,0]} \left|\dot{\varphi}(\tau)\right|$, then $|\varphi(\zeta_j)-\varphi(\theta)| \leqslant M\frac{r}{N}$,

and

$$\|\varphi^N - \varphi\|_p \leqslant \left(\sum_{j=1}^{N} M^p \left(\frac{r}{N}\right)^p \cdot \frac{r}{N}\right)^{1/p} = M \cdot \frac{r}{N} r^{1/p}.$$

Consequently, it follows that

$$\left|[D^N\varphi^N]_1 - \dot{\varphi}(\theta)\right| \leqslant \epsilon^N + \frac{1}{2} M + \frac{1}{2} \epsilon^N + Mr^{1/p} \leqslant M_1$$

and

$$\int_{\frac{-r}{N}}^{0} |[D^N\varphi^N]_1 - \dot{\varphi}(\theta)|^p d\theta \leqslant M_1^p \cdot \frac{r}{N} . \qquad (3.2.8)$$

Therefore, (3.2.7) and (3.2.8) yield the estimate

$$\int_{-r}^{0} |[D^N\varphi^N](\theta) - \dot{\varphi}(\theta)|^p d\theta \leqslant \left(\frac{5}{2}\right)^p (\epsilon^N)^p \cdot r + M_1^p \frac{r}{N} \qquad (3.2.9)$$

$$\rightarrow 0 \text{ as } N \rightarrow \infty.$$

and inequalities (3.2.6) and (3.2.9) imply that if $(\eta,\varphi) \in \mathbb{Z}_{C^1}$, then

$$\|Q^N(\eta,\varphi) - Q(\eta,\varphi)\| \longrightarrow 0 \text{ as } N \longrightarrow \infty$$

This completes the proof. ∎

Define the set $\tilde{D}$ by

$$\tilde{D} = \{(\eta,\varphi) \in \mathbb{R}^N \times L_p,\ D\varphi = \eta, \varphi \in W_{2,p}\}.$$

The following technical lemma will be needed.

<u>LEMMA</u> <u>3.2.5</u>: There exists a constant $\tilde{K}$ such that for any $(\eta,\varphi) \in \tilde{D}$ and $N = 1,2,\ldots$

$$\|\mathcal{a}^N P^N (\eta,\varphi) - \mathcal{a}(\eta,\varphi)\| \leqslant \tilde{K}\|\varphi\|_{W^{2,p}} .$$

<u>PROOF</u>:

$$\|\mathcal{a}^N P^N(\eta,\varphi) - \mathcal{a}(\eta,\varphi)\| \tag{3.2.10}$$

$$= \left|\left|\left(L(\varphi_0^N,\varphi^N),D^N\varphi^N\right) - \left(L\varphi,\dot{\varphi}\right)\right|\right|$$

$$= \left(\left|L(\varphi_0^N,\varphi^N) - L\varphi\right|^p + \int_{-r}^0 |[D^N\varphi^N](\theta) - \dot{\varphi}(\theta)|^p d\theta\right)^{1/p}.$$

Recall that by (2.3.6) the first term on the right-hand side of (3.2.10) satisfies the estimate

$$\left|L(\varphi_0^N,\varphi^N) - L\varphi\right|^p \leqslant K \|\varphi^N - \varphi\|_p^p,$$

where $K = \|BA(\cdot) + B(\cdot)\|_q^p$. Using the assumption that (η,φ)

$\in \tilde{D}$ an argument similar to the above yields the existence of $\zeta_j \in [t_j^N, t_{j-1}^N]$ such that

$$\|\varphi^N - \varphi\|_p = \left(\sum_{j=1}^{N} \int_{t_j^N}^{t_{j-1}^N} |\varphi(\zeta_j) - \varphi(\theta)|^p d\theta\right)^{1/p}$$

$$\leqslant \left(\sum_{j=1}^{N} \int_{t_j^N}^{t_{j-1}^N} \left|\int_{\zeta_j}^{\theta} \dot{\varphi}(u)du\right|^p d\theta\right)^{1/p}$$

$$\leqslant \left(\sum_{j=1}^{N} \int_{t_j^N}^{t_{j-1}^N} \left|\int_{\zeta_j}^{\theta} \left|\dot{\varphi}(u)\right| du\right|^p d\theta\right)^{1/p}$$

$$\leqslant \left(\sum_{j=1}^{N} \int_{t_j^N}^{t_{j-1}^N} \left|\int_{t_j^N}^{t_{j-1}^N} |\dot{\varphi}(u)| du\right|^p d\theta\right)^{1/p}$$

$$\leqslant \left(\sum_{j=1}^{N} \int_{t_j^N}^{t_{j-1}^N} \left(\frac{r}{N}\right)^{p-1} \|\dot{\varphi}\|^p_{p[t_j^N, t_{j-1}^N]} \, d\theta\right)^{1/p}$$

$$= \left(\sum_{j=1}^{N} \left(\frac{r}{N}\right)^p \|\dot{\varphi}\|^p_{p[t_j^N, \; t_{j-1}^N]}\right)^{1/p}$$

$$= \left(\frac{r}{N}\right)\cdot\left(\sum_{j=1}^{N} \|\dot{\varphi}\|^{p}_{p[t_j^N,\ t_{j-1}^N]}\right)^{1/p} = \frac{r}{N}\ \|\dot{\varphi}\|_p .$$

Therefore, it follows that

$$|L(\varphi_0^N,\varphi^N) - L\varphi|^p \leqslant K \cdot \frac{r}{N}\ \|\dot{\varphi}\|_p^p. \tag{3.2.11}$$

Now we consider the second term on the right-hand side of (3.2.10). Assume that $j>1$. Then

$$\left|[D^N\varphi^N]_j(\theta) - \dot{\varphi}(\theta)\right| = \left|\frac{N}{r}(\varphi_{j-1}^N - \varphi_j^N) - \dot{\varphi}(\theta)\right|$$

$$= \left|\frac{N}{r}\left(\varphi(t_{j-1}^N) - \varphi(t_j^N)\right) - \dot{\varphi}(\theta)\right.$$

$$+ \frac{1}{2}\left(\dot{\varphi}(t_{j-1}^N) - \dot{\varphi}(t_j^N)\right)$$

$$+ \frac{N^2}{r^2}\int_{t_{j-1}^N}^{t_{j-2}^N}\int_{t_{j-1}^N}^{u}\left(\dot{\varphi}(\theta)-\dot{\varphi}(t_{j-1}^N)\right)d\theta du\Big|$$

$$- \frac{N^2}{r^2}\int_{t_{j-1}^N}^{t_{j-2}^N}\int_{t_j^N}^{u}(\dot{\varphi}(\theta) - \dot{\varphi}(t_j^N))d\theta du\Big|$$

$$\leqslant E + F + G + H,$$

where

$$E = \left| \frac{N}{r}((\varphi(t_{j-1}^N) - \varphi(t_j^N)) - \dot{\varphi}\ (\theta)\right|$$

$$= |\dot{\varphi}(\zeta_j) - \dot{\varphi}(\theta)| = \left|\int_{\zeta_j}^{\theta} \ddot{\varphi}(u)du\right|$$

$$\leqslant \int_{\zeta_j}^{\theta} |\ddot{\varphi}(u)|du \leqslant \int_{t_j^N}^{t_{j-1}^N} \left|\ddot{\varphi}(u)\right|du$$

$$\leqslant (\frac{r}{N})^{1/q}\ \|\ddot{\varphi}\|_{p[t_j^N, t_{j-1}^N]}.$$

We estimate F by

$$F = \frac{1}{2}\ |\ \dot{\varphi}(t_{j-1}^N) - \dot{\varphi}(t_j^N)|$$

$$= \frac{1}{2}\ \left|\ \int_{t_j^N}^{t_{j-1}^N} g\ddot{u}(u)du\right|$$

$$\leqslant \frac{1}{2}(\frac{r}{N})^{1/q}\ .\ \|\ \ddot{\varphi}\|_{p[t_j^N, t_{j-1}^N]}.$$

and G has the estimate

$$G = \frac{N^2}{r^2} \left| \int_{t_{j-1}^N}^{t_{j-2}^N} \int_{t_{j-1}^N}^{u} (\dot{\varphi}(\theta) - \dot{\varphi}(t_{j-1}^N)) d\theta du \right|$$

$$= \frac{N^2}{r^2} \left| \int_{t_{j-1}^N}^{t_{j-2}^N} \int_{t_{j-1}^N}^{u} \int_{t_{j-1}^N}^{\theta} \ddot{\varphi}(v) dv \; d\theta du \right|$$

$$\leqslant \frac{N^2}{r^2} \int_{t_{j-1}^N}^{t_{j-2}^N} \int_{t_{j-1}^N}^{u} \int_{t_{j-1}^N}^{\theta} |\ddot{\varphi}(v)| dv \; d\theta du$$

$$\leqslant \frac{N^2}{r^2} \int_{t_{j-1}^N}^{t_{j-2}^N} \int_{t_{j-1}^N}^{t_{j-2}^N} \int_{t_{j-1}^N}^{t_{j-2}^N} | \ddot{\varphi}(v) | dv d\theta du$$

$$\leqslant \frac{N^2}{r^2} \int_{t_{j-1}^N}^{t_{j-2}^N} \int_{t_{j-1}^N}^{t_{j-2}^N} (\frac{r}{N})^{1/q} \, \|\ddot{\varphi}\|_{p[t_{j-1}^N, t_{j-2}^N]} \; d\theta du$$

$$\leqslant (\frac{r}{N})^{1/q} \, \|\ddot{\varphi}\|_{p[t_{j-1}^N, \; t_{j-2}^N]} .$$

Finally, we have that

$$H = \frac{N^2}{r^2} \left| \int_{t_j^N}^{t_{j-1}^N} \int_{t_i}^{u} ((\dot{\varphi}(\theta) - \dot{\varphi}(t_j^N))d\theta du \right|$$

$$\leqslant (\tfrac{r}{N})^{1/q} \ \|\ddot{\varphi}\|_{p[t_j^N, t_{j-1}^N]}.$$

Therefore, it follows that

$$|[D^N\varphi^N]_j(\theta) - \dot{\varphi}(\theta)| \leqslant \tfrac{5}{2}(\tfrac{r}{N})^{1/q} \ \|\ddot{\varphi}\|_{p[t_j^N, t_{j-1}^N]}$$

$$+ (\tfrac{r}{N})^{1/q} \ \|\ddot{\varphi}\|_{p[t_{j-1}^N, t_{j-2}^N]},$$

and hence

$$\int_{t_j^N}^{t_{j-1}^N} |[D^N\varphi^N](\theta) - \dot{\varphi}(\theta)\ |^p d\theta$$

$$\leqslant \int_{t_j^n}^{t_{j-1}^N} \left| \tfrac{5}{2} \ (\tfrac{r}{N})^{1/q} \ \|\ddot{\varphi}\|_{p[t_j^N, t_{j-1}^N]} \right.$$

$$\left. + (\tfrac{r}{N})^{1/q} \ \|\ddot{\varphi}\|_{p[t_{j-1}^N, t_{j-2}^N]} \right|^p d\theta$$

$$\leqslant \frac{r}{N}\, 2^{p} \left[\left(\frac{5}{2}\right)^{p} \left(\frac{r}{N}\right)^{p/q} \|\ddot{\varphi}\|^{p}_{p[t_j^N, t_{j-1}^N]} \right.$$

$$\left. + \left(\frac{r}{N}\right)^{p/q} \|\ddot{\varphi}\|^{p}_{p[t_{j-1}^N, t_{j-2}^N]} \right]$$

$$= 2^{p}\left(\frac{r}{N}\right)^{p}\left[\left(\frac{5}{2}\right)^{p} \|\ddot{\varphi}\|^{p}_{p[t_j^N, t_{j-1}^N]} + \|\ddot{\varphi}\|^{p}_{p[t_{j-1}^N,\ t_{j-2}^N]}\right].$$

Consequently, we have the estimate

$$\int_{-r}^{-r/N} |[D^N\varphi^N](\theta) - \dot{\varphi}(\theta)|^p d\theta$$

$$\leqslant 2^{p}\left(\frac{r}{N}\right)^{p}\left(\left(\frac{5}{2}\right)^{p} \sum_{j=2}^{N} (\|\ddot{\varphi}\|^{p}_{p[t_j^N, t_{j-1}^N]}\right.$$

$$\left. + \sum_{j=2}^{N} \|\ddot{\varphi}\|^{p}_{p[t_j^N, t_{j-2}^N]}\right)$$

$$= 2^{p}\left(\frac{r}{N}\right)^{p}\left(\left(\frac{5}{2}\right)^{p}\|\ddot{\varphi}\|^{p}_{p[-r,\ -\frac{r}{N}]} + \|\ddot{\varphi}\|^{p}_{p[-r\ +\ \frac{r}{N},\ 0]}\right)$$

$$\leqslant 2^{p}\left(\frac{r}{N}\right)^{p} \left[\left(\frac{5}{2}\right)^{p} + 1\right]\|\ddot{\varphi}\|^{p}_{p}.$$

If $j = 1$, then

$$\varphi_1^N = \frac{N}{r}\int_{t_1^N}^{0} \varphi(\theta)d\theta = \varphi(0) + \frac{N}{r}\int_{t_1^N}^{0}\int_{t_1^N}^{\theta} \dot{\varphi}(u)du d\theta$$

and

$$|[D^N\varphi^N]_1 - \dot{\varphi}(\theta)| = |\frac{N}{r}(\varphi_0^N - \varphi_1^N) - \dot{\varphi}(\theta)|$$

$$= |\frac{N}{r}(\varphi(0) - \varphi_1^N) - \dot{\varphi}(\theta)$$

$$+ \frac{N}{r}\int_{-r}^{0} A(\theta)(\varphi^N(\theta) - \varphi(\theta))d\theta|$$

$$= |\frac{N^2}{r^2}\int_{t_1^N}^{0}\int_{t_1^N}^{\theta}\dot{\varphi}(u)du d\theta - \dot{\varphi}(\theta)$$

$$+ \frac{N}{r}\int_{-r}^{0} A(\theta)(\varphi^N(\theta) - \varphi(\theta))d\theta|$$

$$\leqslant P + Q + R,$$

where

$$P = \frac{N^2}{r^2} \left| \int_{t_1^N}^{0} \int_{t_1^N}^{\theta} \dot{\varphi}(u)dud\theta \right| \leqslant \frac{N^2}{r^2} \int_{t_1^N}^{0} \int_{t_1^N}^{\theta} |\dot{\varphi}(u)|dud\theta$$

$$\leqslant \frac{N^2}{r^2} \int_{t_1^N}^{0}\int_{t_1^N}^{0} |\dot{\varphi}(u)|dud\theta$$

$$\leqslant \frac{N^2}{r^2} \cdot \frac{r}{N} \left(\frac{r}{N}\right)^{1/q} \|\dot{\varphi}\|_{p[t_1^N,0]}$$

$$= \frac{N}{r} \cdot \left(\frac{r}{N}\right)^{1/q} \|\dot{\varphi}\|_{p[t_1^n,0]},$$

$$Q = |\dot{\varphi}(\theta)|$$

and

$$R = \frac{N}{r} \left| \int_{-r}^{0} A(\theta)(\varphi^N(\theta) - \varphi(\theta))d\theta \right|$$

$$\leqslant \frac{N}{r} \|A\|_q \|\varphi^N - \varphi\|_p \leqslant \|A\|_q \|\dot{\varphi}\|_p .$$

Therefore, it follows that

$$\int_{t_1^N}^{0} |\psi_1^N - \dot{\varphi}(\theta)|^p d\theta$$

$$\leqslant \int_{t_1^N}^{0} |(\|A\|_q + \frac{N}{r}(\frac{r}{N})^{1/q})\|\dot{\varphi}\|_p + |\dot{\varphi}(\theta)| \; |^p \, d\theta$$

$$\leqslant 2^p \int_{t_1^N}^{0}((\|A\|_q + \frac{N}{r} \cdot (\frac{r}{N})^{1/q})^p \; \|\dot{\varphi}\|_p^p + |\dot{\varphi}(\theta)|^p) d\theta$$

$$= 2^p \frac{r}{N} (\|A\|_q + \frac{N}{r}(\frac{r}{N})^{1/q})^p \; \|\dot{\varphi}\|_p^p + 2^p \int_{t_1^N}^{0} |\dot{\varphi}(\theta|^p d\theta$$

$$= 2^p((\frac{r}{N})^{1/p}\|A\|_q + \frac{N}{r}(\frac{r}{N})^{1/p} \cdot (\frac{r}{N})^{1/q})^p \|\dot{\varphi}\|_p^p$$

$$+ 2^p \int_{t_1^N}^{0} |\dot{\varphi}(\theta)|^p d\theta$$

$$= 2^p((\frac{r}{N})^{1/p}\|A\|_q + 1)^p \; \|\dot{\varphi}\|_p^p + 2^p \int_{t_1^N}^{0} |\dot{\varphi}(\theta)|^p d\theta.$$

Also, we have that

$$\|[D^N \varphi^N] - \dot{\varphi}\|_p = (\int_{-r}^{0} ([D^N \varphi^N](\theta) - \dot{\varphi}(\theta)|^p d\theta)^{1/p}$$

$$\leqslant (2^p (\frac{r}{N})^p [(\frac{5}{2})^p + 1] \|\ddot{\varphi}\|_p^p$$

$$+ 2^p ([\frac{r}{N})^{1/p} \|A\|_q + 1]^p + 1) \; \|\dot{\varphi}\|_p^p)^{1/p}.$$

Substituting (3.2.11) and (3.2.12) into (3.2.10) we obtain

$$\|\mathcal{a}^N P^N(\eta,\varphi) - \mathcal{a}(\eta,\varphi)\| \leqslant (K\cdot \tfrac{r}{N}\|\dot{\varphi}\|_p^p + 2^p (\tfrac{r}{N})^p [(\tfrac{5}{2})^p + 1] \|\ddot{\varphi}\|_p^p$$

$$+ 2^p ([(\tfrac{r}{N})^{1/p} \|B\|_p + 1]^{p+1}) \|\dot{\varphi}\|_p^p)^{1/p}$$

$$\leqslant \tilde{K} \|\varphi\|_{W^{2,p}},$$

where the constant $\tilde{K}$ is defined by

$$\tilde{K}^p = \max\ ((K + 2^p([r^{1/p}\|A\|_q + 1]^{p+1})),\ 2^p r^p [(\tfrac{5}{2})^p + 1]). \qquad \blacksquare$$

<u>Lemma</u> <u>3.2.6</u>: There exists a function $m(\cdot,(\eta,\varphi)) \in L_{1,loc}(0,\infty)$ such that if $(\eta,\varphi) \in \mathcal{D}(\mathcal{a}^2)$, then

$$\|\mathcal{a}^N P^N S(t)(\eta,\varphi)\| \leqslant m(t,(\eta,\varphi)) \text{ on } 0 \leqslant t < \infty.$$

<u>Proof</u>: The semigroup $S(t)$ (generated by $\mathcal{a}$) restricted to $\mathcal{D}(\mathcal{a}^2)$ is a C_0-semigroup on $\mathcal{D}(\mathcal{a}^2)$ equipped with the norm

$$\|\eta,\varphi\|_2 = \|(\eta,\varphi)\| + \|\mathcal{a}(\eta,\varphi)\| + \|\mathcal{a}^2(\eta,\varphi)\|$$

(see e.g. [19], [30]). We have that if $(\eta,\varphi) \in \mathcal{D}(\mathcal{A}^2)$, then

$$\|S(t)(\eta,\varphi)\|_2 \leqslant \bar{M}e^{wt}\|(\eta,\varphi)\|_2, \quad t \geqslant 0.$$

Furthermore, $\mathcal{A}^2(\eta,\varphi) = (L\dot{\varphi},\ddot{\varphi})$, and hence it follows that

$$\|(\eta,\varphi)\|_2 = \|(\eta,\varphi)\| + \|A(\eta,\varphi)\| + \|A^2(\eta,\varphi)\|$$

$$= \|(\eta,\varphi)\| + \|(L\varphi,\dot{\varphi})\| + \|(L\dot{\varphi},\ddot{\varphi})\|$$

$$\geqslant \|\varphi\|_p + \|\dot{\varphi}\|_p + \|\ddot{\varphi}\|_p \geqslant \|\varphi\|_{W^{2,p}}$$

This yields the estimate

$$\|\mathcal{A}^N P^N S(t)(\eta,\varphi)\| \leqslant \|\mathcal{A}S(t)(\eta,\varphi)\|$$

$$+ \|\mathcal{A}^N P^N S(t)(\eta,\varphi) - \mathcal{A}S(t)(\eta,\varphi)\|$$

$$\leqslant \|S(t)\| \, \|\mathcal{A}(\eta,\varphi)\| + \tilde{K}\|S(t)(\eta,\varphi)\|_{W^{2,p}}$$

$$\leqslant \bar{M}e^{wt}\|\mathcal{A}(\eta,\varphi)\| + \tilde{K}\bar{M}e^{wt}\|(\eta,\varphi)\|_{W^{2,p}}$$

$$\leqslant \bar{M}(1 + \tilde{K})e^{\omega t}\|(\eta,\varphi)\|_2 = m(t;(\eta,\varphi))$$

which finishes the proof. ■

We are in a position now to prove the main results of Section 3.2.

<u>THEOREM</u> <u>3.2.7</u>: Let $D\in\mathcal{B}(C;\mathbb{R}^n)$, $L\in\mathcal{B}(W^{1,p};\mathbb{R}^n)$, have the representations (3.2.1) (D is atomic at s = 0), (3.2.2), respectively and $\mathcal{Q}$, defined by (2.1.3)-(2.1.4), be the infinitesimal generator of a C_o-semigroup $(S(t))_{t\geqslant 0}$. If the sequence $(\mathbb{Z}^N, P^N, \mathcal{Q}^N)$ is the "averaging projection" scheme, then for all $z_0\in\mathbb{Z}$

$$\lim_{N\to\infty} e^{\mathcal{Q}^N t}P^N z_0 = S(t)z_0$$

uniformly for t in bounded intervals.

<u>PROOF</u>: We show that the sequence $(\mathbb{Z}^N, P^N, \mathcal{Q}^N)$ satisfies H1)-H3) in Lemma 3.1.1. In particular:

1) H1) follows by Lemma 3.2.3.
2) Lemma 3.2.1 and Lemma 3.2.2 imply H2)* and since H2) and H2)* are equivalent, H2) follows.
3) If $D = \mathcal{D}(\mathcal{Q})$ in H3), then Lemma 3.2.4 yields H3)i and the function $m(\cdot,z)$ defined in Lemma 3.2.6,

satisfies H3)ii.

The proof is complete. ∎

<u>THEOREM</u> <u>3.2.8:</u> Lemma 3.1.4 is valid for the sequence $(S^N(t))_{t\geqslant 0}$, $(N = 1,2,\ldots)$ of C_o-semigroups, defined on $\mathbb{Z}$ by

$$S^N(t)z_0 = e^{\mathcal{Q}^N t}P^N z_0 + z_0 - P^N z_0, \quad t\geqslant 0, \quad z_0 \in \mathbb{Z},$$

where $(\mathbb{Z}^N, P^N, \mathcal{Q}^N)$, $N = 1,2,\ldots$ is the "averaging projection" scheme.

<u>PROOF:</u> The uniform boundedness of the sequence (P^N) and Theorem 3.2.7 imply that there exist constants $M\geqslant 1$, $\omega\in\mathbb{R}$ such that $\|S^N(t)\|\leqslant Me^{\omega t}$, $N = 1,2,\ldots$ and

$$\lim_{N\to\infty} S^N(t)P^N z_0 = S(t)z_0$$

uniformly for t in bounded intervals. Therefore, the sequence $(S^N(t))_{t\geqslant 0}$ satisfies the conditions of Lemm 3.1.4 and the proof is complete. ∎

We conclude the section with the following observation.

<u>REMARK</u> <u>3.2.9:</u> The equivalence between generalized solutions to the NFDE and mild solutions to the corresponding abstract Cauchy problem implies that for $(\eta,\varphi)\in\mathbb{R}^n\times L_p$ the "averaging projection" approximation scheme gives an approximation of $(y(t),x_t)$ with respect to the product space norm. Assuming that $(\eta,\varphi)\in\mathcal{D}(\mathcal{A})$ we get an approximation of (Dx_t,x_t) with respect to the norm in $\mathbb{Z} = \mathbb{R}^n\times L_p$. It is not known at this time if one can get direct approximation of the solution in general without assuming additional smoothness conditions on the initial data. ∎

3.3 Concluding remarks and future work

We have considered general necessary and sufficient conditions for the well-posedness of linear functional differential equations of neutral type (NFDE) on the product spaces $\mathbb{R}^n \times Lp$. This research is motivated by certain NFDE-systems ("aero problem") which have been used to model the electric motions of a two-dimensional airfoil in unsteady flows. Previous sufficient conditions do not guarantee the well-posedness of these systems because the difference operator is non-atomic at s=0.

We developed new sets of necessary and sufficient conditions and in particular, we established the well-posedness of the above mentioned non-atomic problems on the Banach spaces $\mathbb{R}^n \times Lp$, $1 \leqslant p < 2$. Employing the equivalence between generalized solutions of the NFDE and mild solutions of the "corresponding" abstract Cauchy-problem we considered the approximation problem for the abstract Cauchy-problem in the Banach space $\mathbb{R}^n \times Lp$ $(1 < p < \infty)$. Using a general approximation result we established the convergence of the "averaging projection" approximation scheme in the Banach space situation.

Elsewhere we have also considered the approximation problem for certain non-atomic problems and constructed a convergent approximation scheme for the "aero problem".

Since this scheme uses heavily the special structure of this NFDE and is not strongly related to the general scheme presented in section 3.1, we have decided not to include it here. We conclude this paper by listing some related problems we intend to study in the near future:

1, Developing stable numerical schemes for the simulation (control and identification) of non-atomic NFDE-systems, and conduct numerical experiments in order to test the theoretical convergence of these schemes.

2, Extend theoretical and computational results to include non-atomic NFDE systems with infinite delay.

3, Refine the necessary conditions of Section 2.2 and eventually find the "largest" class of NFDEs which generate C_o-semigroups on the product spaces $\mathbb{R}^n$xLp.

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