

More Toda-like $(0,2)$ mirrors

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ABSTRACT: In this paper, we extend our previous work to construct $(0,2)$ Toda-like mirrors to $A/2$ -twisted theories on more general spaces, as part of a program of understanding $(0,2)$ mirror symmetry. Specifically, we propose $(0,2)$ mirrors to GLSMs on toric del Pezzo surfaces and Hirzebruch surfaces with deformations of the tangent bundle. We check the results by comparing correlation functions, global symmetries, as well as geometric blowdowns with the corresponding $(0,2)$ Toda-like mirrors. We also briefly discuss Grassmannian manifolds.

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1 Introduction

Mirror symmetry has historically been of great interest to both physicists and mathematicians. In heterotic string compactifications, there is a natural generalization, known as (0,2) mirror symmetry, see e.g. [1–5]. For mathematics, (0,2) mirror symmetry yields quantum sheaf cohomology, a generalization of quantum cohomology in (2,2) theories, see e.g. [5–18].

A perturbative heterotic compactification is defined by a worldsheet theory with (0,2) supersymmetry. A (0, 2) nonlinear sigma model is defined by a pair (X, \mathcal{E}) , with X a Kähler manifold and $\mathcal{E} \rightarrow X$ a holomorphic vector bundle, satisfying Green-Schwarz anomaly cancellation

$$\text{ch}_2(\mathcal{E}) = \text{ch}_2(TX).$$

In cases in which X is Calabi-Yau, so that the nonlinear sigma model above flows to a SCFT, (0,2) mirror symmetry states that there is a dual pair (X', \mathcal{E}') which gives rise to the same SCFT. If X is Fano, then the (0,2) mirror will be a (0,2) Landau-Ginzburg model.

In this paper, we will focus on topological twists of these theories. In (0,2) theories, broadly speaking, two topological twists exist, now known as the A/2 and B/2 twists. In the case of the nonlinear sigma models above, the A/2 twist will exist when $\det \mathcal{E}^* \cong K_X$, and the B/2 twist will exist when $\det \mathcal{E} \cong K_X$. Clearly, both the Green-Schwarz condition and the conditions for the twists will be satisfied when one takes $\mathcal{E} = TX$, in which case, the A/2 theory becomes the ordinary A model topological field theory, and the B/2 theory becomes the ordinary B model topological field theory.

Quantum sheaf cohomology emerges as the OPE algebra of the A/2-twisted theory, forming a precise (0, 2) analogue of ordinary quantum cohomology. To be specific, recall that the ordinary quantum cohomology ring is a ring of local operators defined in the A twist of a nonlinear sigma model on X as BRST-closed states of the form

$$b_{i_1 \dots i_p \bar{i}_1 \dots \bar{i}_q} \chi^{i_1} \dots \chi^{i_p} \chi^{\bar{i}_1} \dots \chi^{\bar{i}_q}.$$

These BRST-closed states can be identified with closed differential forms on the target space X , elements of $H^q(X, \Omega^p) = H^{p,q}(X)$. Similarly, the quantum sheaf cohomology ring is a ring of local operators defined in the A/2 twist of a nonlinear sigma model on X as right-BRST-closed states of the form

$$b_{a_1 \dots a_p \bar{a}_1 \dots \bar{a}_q} \lambda_-^{a_1} \dots \lambda_-^{a_p} \psi_+^{\bar{a}_1} \dots \psi_+^{\bar{a}_q}.$$

These right-BRST-closed states can be identified with $\bar{\partial}$ -closed bundle-valued differential forms, elements of $H^q(X, \wedge^p \mathcal{E}^*)$.

Ordinary mirror symmetry exchanges A twists with B twists, which means an A twisted nonlinear sigma model is equivalent to a B twisted nonlinear sigma model on the mirror Calabi-Yau manifold [19]. Similarly, (0, 2) mirror symmetry exchanges A/2 twists with B/2 twists, meaning the A/2 twisted nonlinear sigma model on (X, \mathcal{E}) is equivalent to the B/2 twisted nonlinear sigma model on the mirror (X', \mathcal{E}') .

A version of mirror symmetry also exists for Fano spaces (see e.g. [20, 21] for a few early references). An A twisted (respectively A/2 twisted) nonlinear sigma model on a Fano manifold X is equivalent to a B twist (respectively B/2 twist) of a (0, 2) Landau-Ginzburg model. The (2, 2) version of this duality is well studied and is given by Toda duals to Fano manifolds [22]. Comparatively, little is known about (0, 2) analogues. Our previous work [23] constructed (0, 2) Toda-like mirrors to products of projective spaces, generalizing the only example previously in the literature [3]. The goal of this paper is to extend the construction of (0,2) Landau-Ginzburg mirrors to more interesting geometries,

as deformations of (2,2) Landau-Ginzburg mirrors, to help pave the way for a more systematic understanding of (0,2) mirror symmetry. Other recent work on two-dimensional (0,2) theories from different directions includes e.g. [24–33].

In this paper, we extend our previous work [23] and explore (0, 2) Toda-like mirrors to $A/2$ -twisted theories on more spaces. Our previous work studied the (0, 2) Toda-like mirror to $A/2$ model on $\mathbb{P}^n \times \mathbb{P}^m$. In this paper, we will construct ansatzes for (0,2) mirrors to toric del Pezzo surfaces and Hirzebruch¹ surfaces, as part of an on-going program to understand (0,2) mirror symmetry. These ansatzes will be tested in several different ways:

- First, each case reduces to an ordinary (2,2) mirror along the (2,2) locus.
- We check that the fields in the Landau-Ginzburg vacua obey the quantum sheaf cohomology relations of their $A/2$ -model partners.
- We check in each case that all genus zero correlation functions of the proposed $B/2$ -twisted Landau-Ginzburg mirror match those of the original $A/2$ -twisted (0,2) theory.
- Amongst the toric del Pezzo mirrors, we check that our proposed mirrors are related by blowdowns as dictated by² geometry.
- As an implicit check, we also give a proposal for (0,2) mirrors of Hirzebruch surfaces³ of arbitrary degree, which not only correctly captures the genus zero correlation functions, but also includes as special cases our previous proposed mirror for $\mathbb{P}^1 \times \mathbb{P}^1$ [23] and for the del Pezzo dP_1 above, thereby demonstrating that the $\mathbb{P}^1 \times \mathbb{P}^1$ and dP_1 mirrors are indeed elements of a sequence of mirrors, as one would expect.

There is another subtlety we shall encounter in the form of the J functions defining the (0,2) superpotential. Specifically, they will sometimes have poles away from the origin. Now, ordinary (2,2) mirrors to projective spaces and Fano varieties will often have superpotential terms proportional to $1/X^n$ for $n > 0$, but it is understood that those Landau-Ginzburg models are defined over algebraic tori of the form $(\mathbb{C}^\times)^k$, so that the target space does not include places where $X = 0$, and hence the theory never encounters a divergent superpotential. By contrast, in this paper we will encounter some examples which have poles at points which are not disallowed. As a result, we interpret these theories in a low-energy effective theory sense — so long as no vacua are located at those poles, we can understand the theory in a neighborhood of the vacua, which excludes the poles. (Similar remarks have been applied to understand GLSMs for generalized Calabi-Yau complete intersections [35–37].) Of course, this also means that these theories are not UV-complete, but we will leave searches for UV-complete descriptions for other work.

¹More precisely, as we will explain later, gauged linear sigma models (GLSMs) [34] for Hirzebruch surfaces. Most Hirzebruch surfaces are not Fano, and so the UV limits of their GLSMs are not Hirzebruch surfaces but rather different geometries, so in principle we are actually describing mirrors to those different surfaces.

²As by definition the del Pezzos are Fano, the UV GLSM phases correspond to the naive geometries.

³For degree greater than one, Hirzebruch surfaces are not Fano; nevertheless, one expects their sigma models to have isolated vacua in the IR, hence a Toda-type mirror is expected.

We begin in section 2 by reviewing the construction of [22] of (2,2) mirrors (Toda duals) to Fano spaces realized in GLSMs, as well as previous results of [23] on (0,2) mirrors to products of projective spaces with deformations of the tangent bundle, which form the heart of both the proposed del Pezzo and Hirzebruch (0,2) mirrors. In section 3 we turn our attention to toric del Pezzo surfaces, giving proposed mirrors to toric del Pezzo surfaces with tangent bundle deformations, checking that correlation functions match as well as that mirrors to blowdowns are related in the fashion one would expect. In section 4 we turn to Hirzebruch surfaces, and give a proposal that generalizes our results for $\mathbb{F}_1 = dP_1$ and $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, and also satisfies consistency tests. Finally in section 5 we briefly discuss a possible (0,2) analogue of the proposed Grassmannian mirror in [22][appendix A]. An appendix discusses quantum cohomology of dP_1 , which figures into its mirror.

2 Review

2.1 Review of (2,2) Toda dual theories

Consider a (2,2) supersymmetric abelian GLSM, with gauge group $U(1)^k$ and n chiral superfields Φ_i . Let Q_i^a denote the charge of the i th chiral superfield under the a th factor in the gauge group. Following [22], the mirror⁴ of an A-twisted theory of this form is a Landau-Ginzburg model with a superpotential of the form

$$W = \sum_{a=1}^k \Sigma_a \left(\sum_{i=1}^n Q_i^a Y_i - r_a \right) + \sum_{i=1}^n \exp(Y_i), \tag{2.1}$$

where the Y_i are twisted chiral superfields in one-to-one correspondence with chiral superfields in the original theory. We integrate out the Σ_a 's to recover the usual form.

It will also be useful to track R-symmetries, as a consistency test on our proposals. Recall that a two-dimensional $\mathcal{N} = (2, 2)$ theory has classical left-moving $U(1)_L$ and a right-moving $U(1)_R$ R-symmetries,

$$\begin{aligned} U(1)_R : \quad \theta^+ &\mapsto e^{-i\kappa}\theta^+, \\ U(1)_L : \quad \theta^- &\mapsto e^{-i\kappa}\theta^-. \end{aligned}$$

Denoting the generators of the R-symmetry $U(1)_L \times U(1)_R$ as J_L and J_R respectively, then one can combine them to get the vector R-symmetry $U(1)_V$ and axial R-symmetry $U(1)_A$ with generators

$$J_V = \frac{1}{2}(J_R + J_L), \quad J_A = \frac{1}{2}(J_R - J_L).$$

Chiral superfields transform under the R-symmetries as follows [22][equ'ns (2.11)-(2.12)],

$$\begin{aligned} R_V \Phi_i(x, \theta^\pm, \bar{\theta}^\pm) &= e^{-i\alpha q_V^i} \Phi_i(x, e^{-i\alpha}\theta^\pm, e^{i\alpha}\bar{\theta}^\pm), \\ R_A \Phi_i(x, \theta^\pm, \bar{\theta}^\pm) &= e^{-i\beta q_A^i} \Phi_i(x, e^{\mp i\beta}\theta^\pm, e^{\pm i\beta}\bar{\theta}^\pm), \end{aligned}$$

⁴If the toric variety is Fano, this will yield the mirror of the Fano phase. Otherwise, it may yield the mirror of a different phase.

where $q_{V,A}^i$ denote the vector and axial R-charges of Φ_i , chosen so that the superpotential has vector charge 2 and axial charge 0. (A twisted superpotential, a function of twisted chiral superfields, has vector charge 0 and axial charge 2.) In components,

$$\begin{aligned}
 R_V : x &\mapsto e^{i\alpha q_V} x, & \psi_{\pm} &\mapsto e^{i\alpha(q_V+1)} \psi_{\pm}, \\
 R_A : x &\mapsto e^{i\beta q_A} x, & \psi_{\pm} &\mapsto e^{i\beta(q_A\pm 1)} \psi_{\pm}.
 \end{aligned}$$

In the quantum theory, the axial R symmetry is typically anomalous.

Now we turn to the mirror theory. We assume the original theory has no superpotential (as we are taking the mirror of a toric variety), so the vector and axial R-charges of the original chiral superfields both vanish. The twisted chiral superfields Y_i transform as⁵ [22][equ'ns (3.29)-(3.30)]

$$R_V Y_i(x, \theta^{\pm}, \bar{\theta}^{\pm}) = Y_i(x, e^{-i\alpha} \theta^{\pm}, e^{i\alpha} \bar{\theta}^{\pm}), \tag{2.2}$$

$$R_A Y_i(x, \theta^{\pm}, \bar{\theta}^{\pm}) = Y_i(x, e^{\mp i\beta} \theta^{\pm}, e^{\pm i\beta} \bar{\theta}^{\pm}) - 2i\beta. \tag{2.3}$$

It is straightforward to see that in the mirror Landau-Ginzburg model defined by (2.1), the vector R-symmetry is unbroken, but the axial R-symmetry is broken classically by the superpotential, corresponding to the fact that in the original theory, the axial R-symmetry is anomalous.

For example, consider the mirror to \mathbb{P}^n , which (after integrating out Σ) is a Landau-Ginzburg theory defined by the (twisted) superpotential

$$\int d\theta^+ d\bar{\theta}^- \widetilde{W} + c.c. = \int d\theta^+ d\bar{\theta}^- \left(\sum_{i=1}^n X_i + \frac{q}{\prod_{i=1}^n X_i} \right) + c.c.,$$

where $X_i = \exp Y_i$ and $q = \exp(-r)$. Here, the last term, $q \prod X_i^{-1}$, classically breaks the axial R symmetry unless

$$\exp(2i\beta(n+1)) = 1,$$

corresponding to the anomaly of the original theory, breaking the original U(1) symmetry to a $\mathbb{Z}_{2(n+1)}$ subgroup.

2.2 (0,2) mirrors to products of projective spaces

Comparatively little is known about analogous (0,2) mirrors to non-Calabi-Yau spaces. As an attempt to rectify this situation, the recent paper [23] constructed and checked ansatzes for (0,2) mirrors to products of projective spaces with deformations of the tangent bundle. In particular, this paper will make use of results for $\mathbb{P}^1 \times \mathbb{P}^1$, which as both a Hirzebruch surface and a toric Fano surface, will be a starting point for several discussions in this paper.

To make this paper self-contained, we briefly review the pertinent results here.

A deformation of the tangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$ is defined as the cokernel \mathcal{E} below:

$$0 \longrightarrow \mathcal{O}^2 \xrightarrow{E} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0,$$

⁵We follow the conventions of [22] in using the same ‘axial,’ ‘vector’ terminology to describe both the original symmetry and its mirror, to assist in tracking the symmetries.

where E is the map

$$E = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix},$$

where x, \tilde{x} are two-component vectors of homogeneous coordinates on either \mathbb{P}^1 factor and A, B, C, D are four constant 2×2 matrices, whose parameters define the deformation.

The proposed mirror [23] is a (0,2) Landau-Ginzburg model defined in superspace by the (0,2) superpotential

$$\int d\theta^+ W = \int d\theta^+ (\Lambda J + \tilde{\Lambda} \tilde{J}),$$

where Λ and $\tilde{\Lambda}$ are Fermi superfields, $X_i = \exp(Y_i)$, J and \tilde{J} are holomorphic functions given here by

$$J = aX_1 + b\frac{X_2^2}{X_1} + \mu X_2 - \frac{q_1}{X_1}, \tag{2.4}$$

$$\tilde{J} = dX_2 + c\frac{X_1^2}{X_2} + \nu X_1 - \frac{q_2}{X_2}, \tag{2.5}$$

and

$$a = \det A, \quad b = \det B, \quad c = \det C, \quad d = \det D,$$

$$\mu = \det(A + B) - \det A - \det B,$$

$$\nu = \det(C + D) - \det C - \det D.$$

For readers not acquainted with (0,2) theories, on the (2,2) locus the J 's become derivatives of the (2,2) superpotential (with respect to the (2,2) chiral multiplets of which the (0,2) Fermi multiplets are half). It is straightforward to check that, indeed, in this case along the (2,2) locus, $J = \partial W / \partial Y_1$, $\tilde{J} = \partial W / \partial Y_2$ for

$$W = X_1 + \frac{q_1}{X_1} + X_2 + \frac{q_2}{X_2},$$

the (2,2) Toda dual to $\mathbb{P}^1 \times \mathbb{P}^1$.

Next, let us consider U(1) symmetries. On the (2,2) locus, we have both⁶ left- and right-moving R-symmetries; in (0,2), we have instead a right-moving R symmetry and a left-moving U(1) symmetry which becomes an R-symmetry on the (2,2) locus. We can combine those two chiral actions in symmetric and antisymmetric combinations to form vector and axial symmetries $U(1)_{V,A}$ which become the vector and axial R-symmetries $R_{V,A}$

⁶In fact, because the target is a product of two spaces, on the (2,2) locus we have additional symmetries obtained by acting nontrivially on fields associated with only a single \mathbb{P}^1 . A generic (0,2) deformation breaks such symmetries; only those (2,2)-locus-symmetries acting symmetrically on both \mathbb{P}^1 factors survive, and so we focus on those here.

on the (2,2) locus. Explicitly, on (0,2) chiral and Fermi multiplets Y_i, Λ^i , respectively:

$$\begin{aligned} \mathrm{U}(1)_V Y_i(x, \theta^+, \bar{\theta}^+) &= Y_i(x, e^{-i\alpha}\theta^+, e^{i\alpha}\bar{\theta}^+), \\ \mathrm{U}(1)_V \Lambda^i(x, \theta^+, \bar{\theta}^+) &= e^{-i\alpha} \Lambda(x, e^{-i\alpha}\theta^+, e^{i\alpha}\bar{\theta}^+), \end{aligned}$$

$$\begin{aligned} \mathrm{U}(1)_A Y_i(x, \theta^+, \bar{\theta}^+) &= Y_i(x, e^{-i\beta}\theta^+, e^{i\beta}\bar{\theta}^+) - 2i\beta, \\ \mathrm{U}(1)_A \Lambda^i(x, \theta^+, \bar{\theta}^+) &= e^{i\beta} \Lambda^i(x, e^{-i\beta}\theta^+, e^{i\beta}\bar{\theta}^+). \end{aligned}$$

It is straightforward to check that the proposed (0,2) mirror above is invariant under $\mathrm{U}(1)_V$, but not under $\mathrm{U}(1)_A$ because of the $q_1/X_1, q_2/X_2$ terms, except for a finite subgroup defined by $\exp(4i\beta) = 1$. This matches results for the $A/2$ model on $\mathbb{P}^1 \times \mathbb{P}^1$, which is invariant under the vector $\mathrm{U}(1)$ but the axial $\mathrm{U}(1)$ is anomalous and so is broken to a finite subgroup. In fact, this matches results for the (2,2) locus of the $A/2$ model — since anomalies are computed by indices, they are invariant under deformations, and so as a matter of principle one should obtain the same results for the (2,2) locus as its (0,2) deformations.

3 Del Pezzo surfaces

In this section, we will discuss mirrors to toric del Pezzo surfaces. We will use the notation dP_k to indicate \mathbb{P}^2 blown up at k points.

3.1 The first del Pezzo surface, dP_1

The first del Pezzo surface we will consider, dP_1 , corresponding to a single blowup of \mathbb{P}^2 , is isomorphic to the first Hirzebruch surface \mathbb{F}_1 . As mirrors to higher del Pezzo surfaces will be constructed on the ‘foundation’ of dP_1 , let us very by describing its (2,2) and (0,2) mirrors. (Appendix A reviews some standard results on quantum cohomology of dP_1 , standard in the math community but perhaps less well-known in the physics community, that are pertinent for the mirror.)

3.1.1 (2,2) and proposed (0,2) mirrors

The del Pezzo surface dP_1 can be described as a toric variety by a fan with edges $(1, 0), (0, 1), (-1, -1), (0, -1)$. A corresponding GLSM is defined by four chiral superfields $\phi_i, i = 1 \dots 4$ charged under the gauge group $\mathrm{U}(1) \times \mathrm{U}(1)$ as follows:

(1,0)	(-1,-1)	(0,1)	(0,-1)
1	1	1	0
0	0	1	1

The quantum cohomology relations⁷ are

$$\begin{aligned} \psi^2(\psi + \tilde{\psi}) &= q_1, \\ (\psi + \tilde{\psi})\tilde{\psi} &= q_2. \end{aligned}$$

⁷Note that example 7.3 in [38] gives the same quantum cohomology ring relations after identifying $\psi \sim f, \tilde{\psi} \sim e, q_1 \sim r, q_1 q_2^{-1} \sim q$.

As reviewed in section 2.1, the (2,2) mirror to a sigma model on $dP_1 = \mathbb{F}_1$ is [22] a Landau-Ginzburg theory with superpotential

$$W = \exp(Y_1) + \exp(Y_2) + \exp(Y_3) + \exp(Y_4),$$

where the fields obey the constraints

$$Y_1 + Y_2 + Y_3 = r_1, \quad Y_3 + Y_4 = r_2.$$

We will describe ansatzes for (0,2) mirrors based on two different solutions of the constraints above.

Our first description of the (2,2) B -twisted mirror to the A -twisted theory is written in terms of Y_1 and Y_3 . Define $X_1 = \exp(Y_1)$ and $X_3 = \exp(Y_3)$, then the mirror can be described as a Landau-Ginzburg model over $(\mathbb{C}^\times)^2$ with superpotential

$$W = X_1 + X_3 + \frac{q_2}{X_3} + \frac{q_1}{X_1 X_3}. \tag{3.1}$$

(This matches the mirror given in [22][equ'n (5.19)].)

An alternative description of the mirror to the same theory is written in terms of Y_1 and Y_4 . Define $X_1 = \exp(Y_1)$ and $X_4 = \exp(Y_4)$, then on the (2,2) locus, the mirror superpotential is

$$W = X_1 + X_4 + \frac{q_1 X_4}{q_2 X_1} + \frac{q_2}{X_4}. \tag{3.2}$$

On the (2,2) locus, this can be related to the previous expression via the field redefinition

$$X_4 = \frac{q_2}{X_3}.$$

(Analogous field redefinitions can be computed to relate the (0,2) mirrors we discuss next, but their expressions for general parameters are both extremely unwieldy and unhelpful, so we omit them from this paper.)

The (0,2) deformations of dP_1 are defined by a pair of 2×2 matrices A, B , and complex numbers $\gamma_1, \gamma_2, \alpha_1, \alpha_2$, that define a deformation \mathcal{E} of the tangent bundle

$$0 \longrightarrow \mathcal{O}^{\oplus 2} \xrightarrow{E} \mathcal{O}(1,0)^{\oplus 2} \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(0,1) \longrightarrow \mathcal{E} \longrightarrow 0,$$

where E is

$$E = \begin{bmatrix} Ax & Bx \\ \gamma_1 s & \gamma_2 s \\ \alpha_1 t & \alpha_2 t \end{bmatrix},$$

with

$$x = \begin{bmatrix} u \\ v \end{bmatrix}.$$

The (2,2) locus is given by the special case

$$A = I, \quad B = 0, \quad \gamma_1 = 1, \quad \gamma_2 = 1, \quad \alpha_1 = 0, \quad \alpha_2 = 1.$$

If we define

$$Q_{(k)} = \det(\psi A + \tilde{\psi} B), \quad Q_{(s)} = \psi \gamma_1 + \tilde{\psi} \gamma_2, \quad Q_{(t)} = \psi \alpha_1 + \tilde{\psi} \alpha_2,$$

then the quantum sheaf cohomology ring relations are given by [5]

$$Q_{(k)} Q_{(s)} = q_1, \quad Q_{(s)} Q_{(t)} = q_2. \tag{3.3}$$

Next, we shall give an ansatz for a ($B/2$ -twisted) $(0,2)$ Landau-Ginzburg theory which is mirror to the $A/2$ model on dP_1 with deformed tangent bundle as above. For readers not familiar with $(0,2)$ Landau-Ginzburg models, the analogue of the superpotential interactions are described in superspace in the form

$$\sum_i \int d\theta \Lambda^i J_i(\Phi),$$

where the J_α are a set of holomorphic functions and Λ^α are Fermi superfields (forming half of a $(2,2)$ chiral superfield). This reduces to a $(2,2)$ superpotential in the special case that $J_i = \partial_i W$ for some holomorphic function W .

Our proposal for the $(0,2)$ Toda-like mirror of the $A/2$ model on $dP_1 = \mathbb{F}_1$ with a deformation of the tangent bundle is defined by

$$J_1 = aX_1 + \mu_{AB}(X_3 - X_1) + b \frac{(X_3 - X_1)^2}{X_1} - \frac{q_1}{X_1(\gamma_1 X_1 + \gamma_2(X_3 - X_1))}, \tag{3.4}$$

$$J_2 = aX_1 + \mu_{AB}(X_3 - X_1) + b \frac{(X_3 - X_1)^2}{X_1} - \frac{q_1}{X_1(\gamma_1 X_1 + \gamma_2(X_3 - X_1))} + X_3^{-1} \left((\gamma_1 X_1 + \gamma_2(X_3 - X_1)) (\alpha_1 X_1 + \alpha_2(X_3 - X_1)) \right) - \frac{q_2}{X_3}. \tag{3.5}$$

(Because the J 's have poles away from origins, we interpret the resulting action in a low-energy effective field theory sense, as discussed in the introduction.)

We have chosen the labels on the J 's to match q 's, but that also means they are slightly inconsistent with bosons on the $(2,2)$ locus. Here, for example, J_2 on the $(2,2)$ locus corresponds to the Y_3 derivative of W .

It is straightforward to check that the J 's above have the correct $(2,2)$ locus, and that they are invariant under the $U(1)_V$ but the $U(1)_A$ symmetry is classically broken in the fashion expected.

Previously we gave two forms for the B -twisted Landau-Ginzburg mirror to dP_1 , on the $(2,2)$ locus. So far, we have given the mirror that reduces on the $(2,2)$ locus to the first form. An expression for a $(0,2)$ mirror that reduces on the $(2,2)$ locus to the second form is

$$J_1 = aX_1 + \mu_{AB}X_4 + b \frac{X_4^2}{X_1} - \frac{q_1}{q_2} \frac{\alpha_1 X_1 + \alpha_2 X_4}{X_1}, \tag{3.6}$$

$$J_2 = \alpha_2 \gamma_2 X_4 + \alpha_1 \gamma_1 \frac{X_1^2}{X_4} + \frac{q_1}{q_2} \frac{(\alpha_1 X_1 + \alpha_2 X_4)(\gamma_1 \alpha_2 + \gamma_2 \alpha_1)}{aX_1 + \mu_{AB}X_4 + bX_4^2 X_1^{-1}} - \frac{q_2}{X_4}. \tag{3.7}$$

As above, we have chosen subscripts on the J 's to match q 's, which means that J_2 on the (2,2) locus corresponds to the Y_4 derivative of W .

As above, it is straightforward to check that the J 's above have the correct (2,2) locus, and that they are invariant under the $U(1)_V$ but the $U(1)_A$ symmetry is classically broken in the fashion expected.

We will check our proposal by arguing that all genus zero $A/2$ model correlation functions will match those of the $B/2$ -twisted mirror Landau-Ginzburg theory given above, using a variation of an argument in [23] which can be adapted to apply to potential (0,2) Landau-Ginzburg model mirrors to any toric variety realized as a GLSM.

Given a $B/2$ Landau-Ginzburg model with a superpotential J_i , the genus zero correlation functions are given by [13]

$$\langle \phi^{i_1}(x_1) \dots \phi^{i_k}(x_k) \rangle = \sum_{J_i(\phi)=0} \phi^{i_1}(x_1) \dots \phi^{i_k}(x_k) \left[\det_{i,j} J_{i,j} \right]^{-1}, \quad (3.8)$$

where the sum is taken over the classical vacua.

From [14], the one-loop effective theory is described by the following J functions in general:

$$\mathcal{J}_a = \ln \left[q_a^{-1} \prod_{\alpha} Q_{(\alpha)}^{q_{(\alpha)}^a} \right],$$

where $Q_{(\alpha)}$ encodes the tangent bundle deformations (as opposed to gauge charges). In the present case of dP_1 , the superpotential is given by

$$\mathcal{J}_1 = \ln [q_1^{-1} \det(A\psi + B\tilde{\psi})(\psi\gamma_1 + \tilde{\psi}\gamma_2)], \quad (3.9)$$

$$\mathcal{J}_2 = \ln [q_2^{-1}(\psi\gamma_1 + \tilde{\psi}\gamma_2)(\psi\alpha_1 + \tilde{\psi}\alpha_2)], \quad (3.10)$$

and the correlation functions are given by

$$\langle f(\psi, \tilde{\psi}) \rangle = \sum_{\mathcal{J}=0} f(\psi, \tilde{\psi}) \left[\det_{a,b} \mathcal{J}_{a,b} \prod_{\alpha} Q_{(\alpha)} \right]^{-1}.$$

Comparing to the formula calculating the correlation functions of Toda dual Landau-Ginzburg model (3.8), in order to claim the correlation functions match, we only need to verify

$$\det |J_{i,j}| = \det_{a,b} |\mathcal{J}_{a,b}| \prod_{\alpha} Q_{(\alpha)} \quad (3.11)$$

on the space of vacua after identifying X_1 with ψ and X_2 with $\tilde{\psi}$. Expanding the right side of above formula, we get

$$\det \begin{bmatrix} \frac{1}{Q_{(s)}^n} \partial_{\psi} (Q_{(k)} Q_{(s)}) & \frac{1}{Q_{(s)}^n} \partial_{\tilde{\psi}} (Q_{(k)} Q_{(s)}) \\ \partial_{\psi} (Q_{(s)} Q_{(t)}) & \partial_{\tilde{\psi}} (Q_{(s)} Q_{(t)}) \end{bmatrix}.$$

One can then easily verify equation (3.11) holds on the space of vacua with $X_1 \sim \psi$ and $X_3 \sim n\psi + \tilde{\psi}$ for both of the presentations of (0,2) mirrors we have given here.

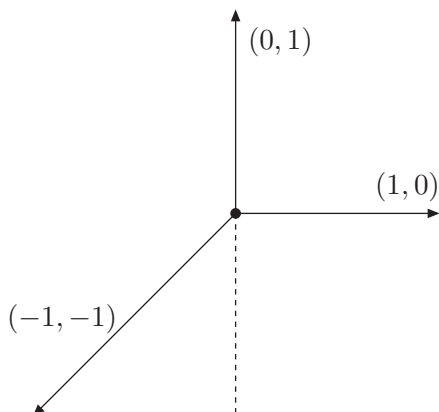


Figure 1. A toric fan of \mathbb{P}^2 can be obtained by removing the edge $(0, -1)$ from the toric fan of dP_1 .

We will use analogous arguments throughout this paper to compare genus zero correlation functions in proposed $(0,2)$ mirrors to the original $A/2$ theories, but for brevity in later sections will only mention the result, not walk through the details of the computation.

So far we have checked that the genus zero correlation functions in this proposed $(0,2)$ mirror to dP_1 match those of the original $A/2$ -twisted theory. In the next section, we will check that there is an analogue of a blowdown in the mirror. In later sections we will describe proposals for $(0,2)$ mirrors to higher del Pezzo surfaces that blow down to this proposal, and we will also describe a family of proposals for $(0,2)$ mirrors to Hirzebruch surfaces that include the proposal of this section for $dP_1 = \mathbb{F}_1$ as well as our earlier proposal for $\mathbb{P}^1 \times \mathbb{P}^1$ as special cases.

3.1.2 Consistency check: mirrors of blowdowns

Geometrically, dP_1 can be blown down to \mathbb{P}^2 , which is visible in the toric fan in figure 1 by removing the edge $(0, -1)$. In the GLSM, although in general Kähler moduli of non-Calabi-Yau manifolds need not correspond to operators in the physical theory, it is nevertheless straightforward to see that there is an analogous limit⁸ in which one recovers \mathbb{P}^2 . The $(2,2)$ mirror of this blowdown is manifest that we only need to take the limit $q_2 \rightarrow 0$ in (3.1), which reduces to the Toda dual superpotential of \mathbb{P}^2 .

In this section we will show that the blowdown limit of the $(0,2)$ mirror of dP_1 with a tangent bundle deformations is also equivalent (as a UV theory) to the mirror of \mathbb{P}^2 . This will provide a consistency test of our proposed $(0,2)$ mirror.

To that end, it will be helpful to first revisit the $(2,2)$ case, albeit in $(0,2)$ language. Recall

$$W = \Lambda^1 J_1 + \Lambda^2 J_2,$$

⁸The same statement will be true of the other blowdown examples considered in this paper — all involving blowups of Fano spaces at smooth points.

where $\Lambda^i, i = 1, 2$ are Fermi superfields, and

$$J_1 = X_1 - \frac{q_1}{X_1 X_3},$$

$$J_2 = X_3 - \frac{q_1}{X_1 X_3}.$$

We can rewrite the (0,2) superpotential as follows,

$$W = \tilde{\Lambda}^1 \tilde{J}_1 + \tilde{\Lambda}^2 \tilde{J}_2,$$

where

$$\tilde{\Lambda}^1 = \Lambda^1 + \Lambda^2, \quad \tilde{\Lambda}^2 = \Lambda^2,$$

and

$$\tilde{J}_1 = J_1 = X_1 - \frac{q_1}{X_1 X_3},$$

$$\tilde{J}_2 = J_2 - J_1 = X_3 - X_1.$$

Then, one can integrate out the Fermi superfield $\tilde{\Lambda}^2$ and obtain a constraint,

$$X_1 = X_3.$$

Plugging the constraint back in, we get

$$W = \tilde{\Lambda}^1 \tilde{J}_1 = \tilde{\Lambda}^1 \left(X_1 - \frac{q_1}{X_1^2} \right). \quad (3.12)$$

Now let us analyze the (0,2) superpotential of dP_1 in the blowdown limit $q_2 \rightarrow 0$,

$$W = \Lambda^1 J_1 + \Lambda^2 J_2,$$

where $\Lambda^i, i = 1, 2$ are Fermi superfields, and

$$J_1 = aX_1 + \mu_{AB}(X_3 - X_1) + b \frac{(X_3 - X_1)^2}{X_1} - \frac{q_1}{X_1(\gamma_1 X_1 + \gamma_2(X_3 - X_1))},$$

$$J_2 = aX_1 + \mu_{AB}(X_3 - X_1) + b \frac{(X_3 - X_1)^2}{X_1} - \frac{q_1}{X_1(\gamma_1 X_1 + \gamma_2(X_3 - X_1))}$$

$$+ X_3^{-1} \left((\gamma_1 X_1 + \gamma_2(X_3 - X_1)) (\alpha_1 X_1 + \alpha_2(X_3 - X_1)) \right).$$

We can rewrite it as

$$W = \tilde{\Lambda}^1 \tilde{J}_1 + \tilde{\Lambda}^2 \tilde{J}_2,$$

where

$$\tilde{\Lambda}^1 = \Lambda^1 + \Lambda^2, \quad \tilde{\Lambda}^2 = \Lambda^2,$$

and

$$\tilde{J}_1 = J_1 = aX_1 + \mu_{AB}(X_3 - X_1) + b \frac{(X_3 - X_1)^2}{X_1} - \frac{q_1}{X_1(\gamma_1 X_1 + \gamma_2(X_3 - X_1))},$$

$$\tilde{J}_2 = J_2 - J_1 = +X_3^{-1} ((\gamma_1 X_1 + \gamma_2(X_3 - X_1)) (\alpha_1 X_1 + \alpha_2(X_3 - X_1))).$$

Then, we integrate out $\tilde{\Lambda}^2$ and obtain the following constraint on X_1, X_3 :

$$\tilde{J}_2 = X_3^{-1} ((\gamma_1 X_1 + \gamma_2 (X_3 - X_1)) (\alpha_1 X_1 + \alpha_2 (X_3 - X_1))) = 0.$$

Notice that $\gamma_1 X_1 + \gamma_2 (X_3 - X_1) \neq 0$ since it is in the denominator of J_1 . Solving the constraint, one obtain the relation

$$X_3 = \frac{\alpha_2 - \alpha_1}{\alpha_2} X_1,$$

where for simplicity we have assumed $\alpha_2 \neq 0$.

Lastly, plugging the above relation back into \tilde{J}_1 , we find

$$W = \tilde{\Lambda}^1 \tilde{J}_1 = \tilde{\Lambda}^1 \left((a - \mu_{AB} \alpha_1 \alpha_2^{-1} + b \alpha_1^2 \alpha_2^{-2}) X_1 - \frac{q_1}{(\gamma_1 - \gamma_2 \alpha_1 \alpha_2^{-1}) X_1^2} \right).$$

(We assume for simplicity that $\gamma_1 \neq \gamma_2 \alpha_1 / \alpha_2$.) One can easily see the above superpotential is equivalent to (3.12) for the mirror to \mathbb{P}^2 , after suitable field redefinitions. Thus, as expected, mirrors and blowdowns commute with one another.

3.2 The second del Pezzo surface, dP_2

3.2.1 Review of the (2,2) mirror

The next del Pezzo surface, dP_2 , is \mathbb{P}^2 blown up at two points, which can be described as a toric variety by a fan with edges $(1, 0), (0, 1), (-1, -1), (0, -1), (-1, 0)$. The gauged linear sigma model has five chiral superfields $\phi_i, i = 1 \dots 5$ which are charged under the gauge group $U(1)^3$ as follows:

(1,0)	(-1,-1)	(0,1)	(0,-1)	(-1,0)
1	1	1	0	0
0	0	1	1	0
1	0	0	0	1

The quantum cohomology relations of the A -twisted theory are

$$\begin{aligned} (\psi_1 + \psi_3)\psi_1(\psi_1 + \psi_2) &= q_1, \\ (\psi_1 + \psi_2)\psi_2 &= q_2, \\ (\psi_1 + \psi_3)\psi_3 &= q_3. \end{aligned}$$

As reviewed in section 2.1, the superpotential of the (2,2) mirror theory is

$$W = \sum_{i=1}^5 \exp(+Y_i)$$

where the Y_i obey constraints

$$Y_1 + Y_2 + Y_3 = r_1, \quad Y_3 + Y_4 = r_2, \quad Y_1 + Y_5 = r_3.$$

One solution is to solve the constraints for Y_4 and Y_5 . Defining $X_4 = \exp(Y_4)$ and $X_5 = \exp(Y_5)$, the superpotential is then

$$W = X_4 + X_5 + \frac{q_3}{X_5} + \frac{q_1}{q_2 q_3} X_4 X_5 + \frac{q_2}{X_4}. \tag{3.13}$$

(The mirror map relates $\psi_2 \sim X_4$ and $X_5 \sim \psi_3$.)

However, we will not use the form of the Toda dual above when building (0,2) deformations. Instead, we will use an alternative form of the Toda dual, which is obtained by retaining an explicit Lagrange multiplier Z , so that one of the constraints naturally embeds into the superpotential,

$$W = X_1 + X_3 + X_5 + \frac{q_1}{X_1 X_3} + \frac{q_2}{X_3} + Z \left(1 - \frac{q_3}{X_1 X_5} \right), \tag{3.14}$$

for $X_i = \exp(Y_i)$. (If we solve the constraint by taking $X_1 = q_3/X_5$, then this form can be related to the previous expression by the holomorphic coordinate transformation $X_4 = q_2/X_3$.) On the space of vacua which is given by,

$$\begin{aligned} X_1 \partial_1 W &= X_1 - \frac{q_1}{X_1 X_3} + Z \frac{q_3}{X_1 X_5} = 0, \\ X_3 \partial_3 W &= X_3 - \frac{q_1}{X_1 X_3} - \frac{q_2}{X_3} = 0, \\ X_5 \partial_5 W &= X_5 + Z \frac{q_3}{X_1 X_5} = 0, \\ \partial_Z W &= 1 - \frac{q_3}{X_1 X_5} = 0, \end{aligned}$$

where

$$\partial_i = \frac{\partial}{\partial X_i}.$$

The quantum cohomology relations are satisfied with the identifications

$$X_1 \sim \psi_1 + \psi_3, \quad X_3 \sim \psi_1 + \psi_2, \quad X_5 \sim \psi_3,$$

It is straightforward to check that the vector R symmetry is preserved, but the axial R symmetry is broken, as expected. Furthermore, it is easy to check that all correlation functions of the alternative description match⁹ those of A -twisted theory for dP_2 . This alternative description turns out to be convenient for constructing ansatzes for (0,2) mirrors.

The choice of constraint embedding in the superpotential should be arbitrary. For example, we could also take the mirror superpotential to be

$$W = X_1 + X_2 + X_3 + \frac{q_2}{X_3} + \frac{q_3}{X_1} + Z \left(1 - \frac{q_1}{X_1 X_2 X_3} \right), \tag{3.15}$$

⁹Technically, keeping an explicit Lagrange multiplier turns out to introduce a sign in correlation functions, which can easily be accounted for.

with vacua,

$$\begin{aligned} X_1 \partial_1 W &= X_1 - \frac{q_3}{X_1} + Z \frac{q_1}{X_1 X_2 X_3} = 0, \\ X_2 \partial_2 W &= X_2 + Z \frac{q_1}{X_1 X_2 X_3} = 0, \\ X_3 \partial_3 W &= X_3 - \frac{q_2}{X_3} + Z \frac{q_1}{X_1 X_2 X_3} = 0, \\ \partial_Z W &= 1 - \frac{q_1}{X_1 X_2 X_3} = 0. \end{aligned}$$

This can be related to the previous form using the holomorphic coordinate transformation $X_2 = (q_1/q_3)(X_5/X_3)$. We will also describe (0,2) deformations of this presentation.

3.2.2 (0,2) deformations and proposed (0,2) mirrors

The (0, 2) deformation of dP_2 is defined by fifteen complex numbers $\alpha_i, \beta_j, \gamma_k, \delta_m, \epsilon_n$, with $i, j, k, m, n = 1, 2, 3$, which define a deformation \mathcal{E} of the tangent bundle as follows:

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}^3 \xrightarrow{E} \mathcal{O}(1, 0, 1) \oplus \mathcal{O}(1, 0, 0) \oplus \mathcal{O}(1, 1, 0) \oplus \mathcal{O}(0, 1, 0) \oplus \mathcal{O}(0, 0, 1) \\ &\longrightarrow \mathcal{E} \longrightarrow 0, \end{aligned}$$

where

$$E = \begin{bmatrix} \alpha_1 s_1 & \alpha_2 s_1 & \alpha_3 s_1 \\ \beta_1 s_2 & \beta_2 s_2 & \beta_3 s_2 \\ \gamma_1 s_3 & \gamma_2 s_3 & \gamma_3 s_3 \\ \delta_1 s_4 & \delta_2 s_4 & \delta_3 s_4 \\ \epsilon_1 s_5 & \epsilon_2 s_5 & \epsilon_3 s_5 \end{bmatrix}.$$

\mathcal{E} reduces to the tangent bundle when

$$\begin{aligned} \alpha_1 &= 1, & \alpha_2 &= 0, & \alpha_3 &= 1, \\ \beta_1 &= 1, & \beta_2 &= \beta_3 = 0, & & \\ \gamma_1 &= \gamma_2 = 1, & \gamma_3 &= 0, & & \\ \delta_1 &= 0, & \delta_2 &= 1, & \delta_3 &= 0, \\ \epsilon_1 &= \epsilon_2 = 0, & \epsilon_3 &= 1. & & \end{aligned}$$

The quantum sheaf cohomology relations are

$$Q_{(1)} Q_{(2)} Q_{(3)} = q_1, \tag{3.16}$$

$$Q_{(3)} Q_{(4)} = q_2, \tag{3.17}$$

$$Q_{(1)} Q_{(5)} = q_3, \tag{3.18}$$

where

$$\begin{aligned} Q_{(1)} &= \sum_{i=1}^3 \alpha_i \psi_i, & Q_{(2)} &= \sum_{i=1}^3 \beta_i \psi_i, & Q_{(3)} &= \sum_{i=1}^3 \gamma_i \psi_i, \\ Q_{(4)} &= \sum_{i=1}^3 \delta_i \psi_i, & Q_{(5)} &= \sum_{i=1}^3 \epsilon_i \psi_i. \end{aligned}$$

We will propose below two (0,2) Toda-like mirrors based on the (2,2) mirrors (3.14) and (3.15) which have a Lagrange multiplier (labelled Z).

Our first (0,2) mirror proposal for dP_2 is defined by the following four holomorphic functions

$$J_1 = -\frac{q_1}{X_1(\gamma \cdot X)} + Z \frac{q_3}{X_1(\epsilon \cdot X)} + \frac{(\alpha \cdot X)(\epsilon \cdot X)}{X_1} + \frac{(\alpha \cdot X)(\beta \cdot X)}{X_1}, \quad (3.19)$$

$$J_3 = -\frac{q_2}{X_3} - \frac{q_1}{X_1(\gamma \cdot X)} + \frac{(\alpha \cdot X)(\beta \cdot X)}{X_1} + \frac{(\gamma \cdot X)(\delta \cdot X)}{X_3}, \quad (3.20)$$

$$J_5 = (\epsilon \cdot X) + Z \frac{q_3}{(\alpha \cdot X)(\epsilon \cdot X)}, \quad (3.21)$$

$$J_Z = \frac{(\epsilon \cdot X)}{X_5} - \frac{q_3}{X_5(\alpha \cdot X)}, \quad (3.22)$$

where on the (2,2) locus, it can be shown that

$$J_1 = \frac{\partial W}{\partial Y_1}, \quad J_3 = \frac{\partial W}{\partial Y_3}, \quad J_5 = \frac{\partial W}{\partial Y_5}, \quad J_Z = \frac{\partial W}{\partial Z},$$

for W defined by (3.14), and

$$\begin{aligned} (\alpha \cdot X) &= \alpha_1(X_1 - X_5) + \alpha_2(X_3 - X_1 + X_5) + \alpha_3 X_5, \\ &\vdots \\ (\epsilon \cdot X) &= \epsilon_1(X_1 - X_5) + \epsilon_2(X_3 - X_1 + X_5) + \epsilon_3 X_5. \end{aligned}$$

Because the J 's have poles away from origins, we interpret the resulting action in a low-energy effective field theory sense, as discussed in the introduction. It is straightforward to check that this proposal has the correct (2,2) locus.

Our second (0,2) mirror ansatz for dP_2 is defined by the following data:

$$J_1 = -\frac{q_3}{X_1} + \frac{(\alpha \cdot X)(\epsilon \cdot X)}{X_1} + (\beta \cdot X) + Z \frac{q_1}{(\alpha \cdot X)(\beta \cdot X)(\gamma \cdot X)}, \quad (3.23)$$

$$J_2 = (\beta \cdot X) + Z \frac{q_1}{(\alpha \cdot X)(\beta \cdot X)(\gamma \cdot X)}, \quad (3.24)$$

$$J_3 = -\frac{q_2}{X_3} + \frac{(\gamma \cdot X)(\delta \cdot X)}{X_3} + (\beta \cdot X) + Z \frac{q_1}{(\alpha \cdot X)(\beta \cdot X)(\gamma \cdot X)}, \quad (3.25)$$

$$J_Z = +\frac{(\beta \cdot X)}{X_2} - \frac{q_1}{X_2(\alpha \cdot X)(\gamma \cdot X)}, \quad (3.26)$$

where

$$\begin{aligned} (\alpha \cdot X) &= \alpha_1 X_2 + \alpha_2(X_3 - X_2) + \alpha_3(X_1 - X_2), \\ &\vdots \\ (\epsilon \cdot X) &= \epsilon_1 X_2 + \epsilon_2(X_3 - X_2) + \epsilon_3(X_1 - X_2). \end{aligned}$$

On the (2,2) locus, the above data reduces to (3.15) (in the sense that each J_i becomes a suitable derivative of W).

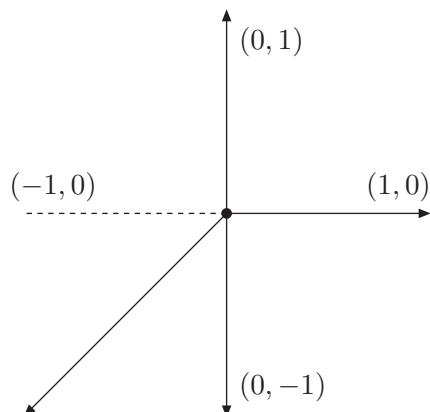


Figure 2. A toric fan for dP_1 can be obtained by removing the edge $(-1,0)$ from the toric fan for dP_2 .

With the identifications,

$$X_1 \sim \psi_1 + \psi_3, \quad X_2 \sim \psi_1, \quad X_3 \sim \psi_1 + \psi_2, \quad X_5 \sim \psi_3,$$

both proposals pass our standard consistency checks: the quantum sheaf cohomology relations are satisfied on the vacua, the $U(1)_V$ symmetry is unbroken but the $U(1)_A$ broken classically, and all correlation functions match those of $A/2$ twisted theory as before.

3.2.3 Consistency check: mirrors of blowdowns to dP_1

The del Pezzo surface dP_2 can be blown down to dP_1 , which one can see from the toric fan by removing the edge $(-1,0)$ in figure 2. (Moreover, essentially because we are discussing blowups of smooth points on Fano varieties, the UV phases of the GLSMs are the geometries described here, so in the cases described here there are no subtleties involving the GLSM giving results for unexpected geometries.)

On the $(2,2)$ locus, the mirror of the blowdown from dP_2 to dP_1 is described in the Toda dual theory (3.15) by taking the limit $q_3 \rightarrow 0$ after integrating out the Lagrange multiplier Z . Next, we will analyze both of the proposed $(0,2)$ mirror theories in section (3.2.2) under the same blowdown limit.

First, to illustrate the method, let us explain how to explicitly follow the blowdown in the $(2,2)$ Toda dual (3.15). That $(2,2)$ superpotential can be rewritten in the $(0,2)$ language as follows,

$$\int d\theta W(\Phi) = \sum_i \int d\theta \Lambda^i J_i(\Phi),$$

where the Λ^i are Fermi superfields and the J_i are derivatives of W , which in the current

case are given by

$$\begin{aligned} J_1 &= X_1 - \frac{q_3}{X_1} + Z \frac{q_1}{X_1 X_2 X_3}, \\ J_2 &= X_2 + Z \frac{q_1}{X_1 X_2 X_3}, \\ J_3 &= X_3 - \frac{q_2}{X_3} + Z \frac{q_1}{X_1 X_2 X_3}, \\ J_Z &= 1 - \frac{q_1}{X_1 X_2 X_3}. \end{aligned}$$

To integrate out the Lagrange multiplier Z , one integrates out the Fermi field Λ^Z corresponding to $J_Z = \partial_Z W$, which implies

$$J_Z = 1 - \frac{q_1}{X_1 X_2 X_3} = 0,$$

or

$$X_2 = \frac{q_1}{X_1 X_3}.$$

As one might expect, the above constraint is the same constraint arising from integrating out the Lagrange multiplier Z in (3.15). Imposing this constraint, the remaining J functions become

$$\begin{aligned} J_1 &= X_1 - \frac{q_3}{X_1} + Z, \\ J_2 &= \frac{q_1}{X_1 X_3} + Z, \\ J_3 &= X_3 + Z - \frac{q_2}{X_3}. \end{aligned}$$

Since we have removed explicit X_2 dependence from the J_i above, we should also integrate out the Fermi field Λ^2 corresponding to $J_2 = -X_2 \partial_{X_2} W$, which implies

$$Z = -\frac{q_1}{X_1 X_3}.$$

Applying the constraint above, one reaches the form

$$\begin{aligned} J_1 &= X_1 - \frac{q_3}{X_1} - \frac{q_1}{X_1 X_3}, \\ J_3 &= X_3 - \frac{q_1}{X_1 X_3} - \frac{q_2}{X_3}. \end{aligned}$$

Finally, taking the limit $q_3 \rightarrow 0$, we see that the J functions above precisely coincide with those for the (2,2) Toda dual of dP_1 presented in (3.1).

Now that we have illustrated the method, let us analyze the mirror of the (0,2) theory (3.19)–(3.22) in the blowdown limit $q_3 \rightarrow 0$. After integrating out the Lagrange multiplier, one obtains the constraints

$$\begin{aligned} J_5 &= (\epsilon \cdot X) + Z \frac{q_3}{(\alpha \cdot X)(\epsilon \cdot X)} = 0, \\ J_Z &= \frac{(\epsilon \cdot X)}{X_5} - \frac{q_3}{X_5(\alpha \cdot X)} = 0, \end{aligned}$$

where

$$\begin{aligned}
 (\alpha \cdot X) &= \alpha_1(X_1 - X_5) + \alpha_2(X_3 - X_1 + X_5) + \alpha_3 X_5, \\
 &\vdots \\
 (\epsilon \cdot X) &= \epsilon_1(X_1 - X_5) + \epsilon_2(X_3 - X_1 + X_5) + \epsilon_3 X_5.
 \end{aligned}$$

In the limit $q_3 \rightarrow 0$, the constraint $J_Z = 0$ implies

$$X_5 = \frac{\epsilon_1 X_1 + \epsilon_2 (X_3 - X_1)}{\epsilon_1 - \epsilon_2 - \epsilon_3}.$$

(For simplicity, we assume $\epsilon_1 - \epsilon_2 - \epsilon_3 \neq 0$.) The mirror blowdown is then given by,

$$\begin{aligned}
 J_1 &= -\frac{q_1}{X_1(\Gamma_1 X_1 + \Gamma_2(X_3 - X_1))} \\
 &\quad + \frac{(A_1 X_1 + A_2(X_3 - X_1))(B_1 X_1 + B_2(X_3 - X_1))}{X_1}, \\
 J_3 &= -\frac{q_1}{X_1(\Gamma_1 X_1 + \Gamma_2(X_3 - X_1))} \\
 &\quad + \frac{(A_1 X_1 + A_2(X_3 - X_1))(B_1 X_1 + B_2(X_3 - X_1))}{X_1} \\
 &\quad + \frac{(\Gamma_1 X_1 + \Gamma_2(X_3 - X_1))(\Delta_1 X_1 + \Delta_2(X_3 - X_1))}{X_3} - \frac{q_2}{X_3},
 \end{aligned}$$

where,

$$\begin{aligned}
 A_1 &= \frac{\epsilon_1(\alpha_2 + \alpha_3) - \alpha_1(\epsilon_2 + \epsilon_3)}{\epsilon_1 - \epsilon_2 - \epsilon_3}, & A_2 &= \frac{\epsilon_2(\alpha_3 - \alpha_1) + \alpha_2(\epsilon_1 - \epsilon_3)}{\epsilon_1 - \epsilon_2 - \epsilon_3}, \\
 B_1 &= \frac{\epsilon_1(\beta_2 + \beta_3) - \beta_1(\epsilon_2 + \epsilon_3)}{\epsilon_1 - \epsilon_2 - \epsilon_3}, & B_2 &= \frac{\epsilon_2(\beta_3 - \beta_1) + \beta_2(\epsilon_1 - \epsilon_3)}{\epsilon_1 - \epsilon_2 - \epsilon_3}, \\
 \Gamma_1 &= \frac{\epsilon_1(\gamma_2 + \gamma_3) - \gamma_1(\epsilon_2 + \epsilon_3)}{\epsilon_1 - \epsilon_2 - \epsilon_3}, & \Gamma_2 &= \frac{\epsilon_2(\gamma_3 - \gamma_1) + \gamma_2(\epsilon_1 - \epsilon_3)}{\epsilon_1 - \epsilon_2 - \epsilon_3}, \\
 \Delta_1 &= \frac{\epsilon_1(\delta_2 + \delta_3) - \delta_1(\epsilon_2 + \epsilon_3)}{\epsilon_1 - \epsilon_2 - \epsilon_3}, & \Delta_2 &= \frac{\epsilon_2(\delta_3 - \delta_1) + \delta_2(\epsilon_1 - \epsilon_3)}{\epsilon_1 - \epsilon_2 - \epsilon_3}.
 \end{aligned}$$

One can see that the resulting superpotential is the same as the superpotential (3.4), (3.5) after adjusting the parameters as follows,

$$\begin{aligned}
 a &= A_1 B_1, & b &= A_2 B_2, & \mu_{AB} &= A_1 B_2 + A_2 B_1, \\
 \gamma_1 &= \Gamma_1, & \gamma_2 &= \Gamma_2, \\
 \alpha_1 &= \Delta_1, & \alpha_2 &= \Delta_2.
 \end{aligned}$$

Thus, as expected, the (0,2) mirror to the blowdown, is the blowdown limit of the mirror. This provides a consistency check on the form of the proposed mirror.

Next, we repeat the analysis for the second form of the (0, 2) mirror (3.23)–(3.26). We first obtain the constraints

$$\begin{aligned}
 J_2 &= (\beta \cdot X) + Z \frac{q_1}{(\alpha \cdot X)(\beta \cdot X)(\gamma \cdot X)} = 0, \\
 J_Z &= \frac{(\beta \cdot X)}{X_2} - \frac{q_1}{X_2(\alpha \cdot X)(\gamma \cdot X)} = 0.
 \end{aligned}$$

In principle, one can use these constraints to eliminate the dependence on X_2 and Z in the remaining J functions. Then, taking the limit $q_3 \rightarrow 0$ one should recover the J functions of the $(0, 2)$ mirror of dP_1 . However, J_Z is effectively a cubic polynomial in X_2 , so directly solving for X_2 in arbitrary $(0, 2)$ deformations is rather complex. For simplicity, we will only consider the blowdown in the second form of the $(0, 2)$ mirror for a special family of deformations, of the form

$$\alpha_1 = 1, \quad \alpha_2 = 0, \quad \alpha_3 = 1, \quad \gamma_1 = 1, \quad \gamma_2 = 1, \quad \gamma_3 = 0,$$

leaving other deformation parameters arbitrary.

Now, for this family of deformations, the constraints become

$$X_2 = (\beta_1 - \beta_2 - \beta_3)^{-1} \left(\frac{q_1}{X_1 X_3} - \beta_3 X_1 - \beta_2 X_3 \right),$$

$$Z = -\frac{q_1}{X_1 X_3} = -\beta \cdot X.$$

Plugging back into the other J functions, we find

$$J'_1 = E J_1,$$

$$= -\frac{q_1}{X_1 X_3} - \frac{\beta_1 \epsilon_3 - \beta_2 \epsilon_3 - \beta_3 \epsilon_1 + \beta_3 \epsilon_2}{\epsilon_1 - \epsilon_2 - \epsilon_3} X_1 - \frac{\beta_1 \epsilon_2 - \beta_3 \epsilon_2 - \beta_2 \epsilon_1 + \beta_2 \epsilon_3}{\epsilon_1 - \epsilon_2 - \epsilon_3} X_3,$$

$$J'_3 = \Delta J_3,$$

$$= -\frac{q'_2}{X_3} - \frac{q_1}{X_1 X_3} - \frac{\beta_1 \delta_3 - \beta_2 \delta_3 - \beta_3 \delta_1 + \beta_3 \delta_2}{\delta_1 - \delta_2 - \delta_3} X_1$$

$$- \frac{\beta_1 \delta_2 - \beta_3 \delta_2 - \beta_2 \delta_1 + \beta_2 \delta_3}{\delta_1 - \delta_2 - \delta_3} X_3,$$

where

$$E = -\frac{\beta_1 - \beta_2 - \beta_3}{\epsilon_1 - \epsilon_2 - \epsilon_3},$$

$$\Delta = -\frac{\beta_1 - \beta_2 - \beta_3}{\delta_1 - \delta_2 - \delta_3},$$

$$q'_2 = -\frac{\beta_1 - \beta_2 - \beta_3}{\delta_1 - \delta_2 - \delta_3} q_2.$$

We assume that

$$\delta_1 - \delta_2 - \delta_3 \neq 0, \quad \epsilon_1 - \epsilon_2 - \epsilon_3 \neq 0.$$

Note that we rescaled J_1 and J_3 : the rescaling parameters E and Δ can always be absorbed in the corresponding Fermi fields. We also rescaled q_2 to match the form of the J functions of dP_1 . As a result, one can see that the J functions reduce to those of dP_1 in

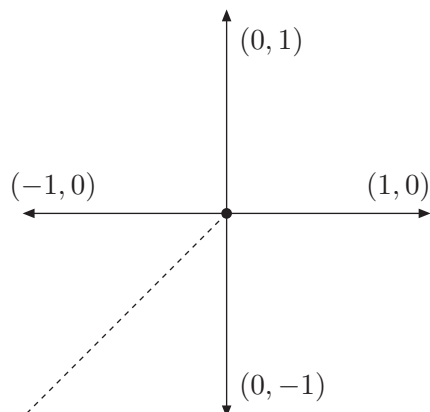


Figure 3. A toric fan for $\mathbb{P}^1 \times \mathbb{P}^1$ can be obtained by removing the edge $(-1, -1)$ from the toric fan for dP_2 .

equations (3.4)–(3.5), with the parameters related as follows:

$$\begin{aligned} \gamma_1 &= \gamma_2 = 1, & b &= 0, \\ a &= -\frac{\beta_1 \epsilon_2 - \beta_2 \epsilon_1 + \beta_1 \epsilon_3 - \beta_3 \epsilon_1}{\epsilon_1 - \epsilon_2 - \epsilon_3}, \\ \mu_{AB} &= -\frac{\beta_1 \epsilon_2 - \beta_3 \epsilon_2 - \beta_2 \epsilon_1 + \beta_2 \epsilon_3}{\epsilon_1 - \epsilon_2 - \epsilon_3}, \\ \alpha_1 &= \frac{\beta_1 \epsilon_3 - \beta_3 \epsilon_1 + \beta_1 \epsilon_2 - \beta_2 \epsilon_1}{\epsilon_1 - \epsilon_2 - \epsilon_3} - \frac{\beta_1 \delta_2 - \beta_3 \delta_1 - \beta_2 \delta_1 + \beta_1 \delta_3}{\delta_1 - \delta_2 - \delta_3}, \\ \alpha_2 &= \frac{\beta_1 \epsilon_2 - \beta_3 \epsilon_2 - \beta_2 \epsilon_1 + \beta_2 \epsilon_3}{\epsilon_1 - \epsilon_2 - \epsilon_3} - \frac{\beta_1 \delta_2 - \beta_3 \delta_2 - \beta_2 \delta_1 + \beta_2 \delta_3}{\delta_1 - \delta_2 - \delta_3}. \end{aligned}$$

3.2.4 Consistency check: mirrors of blowdowns to $\mathbb{P}^1 \times \mathbb{P}^1$

We can also blowdown dP_2 to $\mathbb{P}^1 \times \mathbb{P}^1$, which can be represented in the toric fan we have used previously by removing the edge $(-1, -1)$, as shown in figure 3. (As before, since we are discussing Fano varieties, the geometries described all correspond to UV phases of the GLSMs.)

On the $(2, 2)$ locus, the mirror of the blowdown from dP_2 to $\mathbb{P}^1 \times \mathbb{P}^1$ is described in the mirror theory (3.15) by taking the limit $q_1 \rightarrow 0$ after integrating out the Lagrange multiplier Z . Off the $(2, 2)$ locus, we can follow the same procedure as before, integrating out the Lagrange multiplier in (3.23)–(3.26) and taking the limit $q_1 \rightarrow 0$ to blow down the $(0, 2)$ mirror dual J functions of dP_2 to $\mathbb{P}^1 \times \mathbb{P}^1$. Integrating out the Lagrange multiplier, we obtain the constraints

$$\begin{aligned} J_2 &= (\beta \cdot X) + Z \frac{q_1}{(\alpha \cdot X)(\beta \cdot X)(\gamma \cdot X)} = 0, \\ J_Z &= \frac{(\beta \cdot X)}{X_2} - \frac{q_1}{X_2(\alpha \cdot X)(\gamma \cdot X)} = 0, \end{aligned}$$

where

$$\begin{aligned}
 (\alpha \cdot X) &= \alpha_1 X_2 + \alpha_2 (X_3 - X_2) + \alpha_3 (X_1 - X_2), \\
 &\quad \vdots \\
 (\epsilon \cdot X) &= \epsilon_1 X_2 + \epsilon_2 (X_3 - X_2) + \epsilon_3 (X_1 - X_2).
 \end{aligned}$$

In the limit $q_1 \rightarrow 0$, the only solution of $\{J_Z = 0\}$ for X_2 is

$$X_2 = -\frac{\beta_2 X_3 + \beta_3 X_1}{\beta_1 - \beta_2 - \beta_3}.$$

(For simplicity we assume $\beta_1 - \beta_2 - \beta_3 \neq 0$.) Then, in this limit, the resulting J functions are given by

$$\begin{aligned}
 J_1 &= -\frac{q_3}{X_1} + \frac{(A_1 X_1 + A_2 X_3)(E_1 X_1 + E_2 X_3)}{X_1}, \\
 J_3 &= -\frac{q_2}{X_3} + \frac{(\Gamma_1 X_1 + \Gamma_2 X_3)(\Delta_1 X_1 + \Delta_2 X_3)}{X_3},
 \end{aligned}$$

where

$$\begin{aligned}
 A_1 &= \frac{\beta_3(\alpha_2 - \alpha_1) + \alpha_3(\beta_1 - \beta_2)}{\beta_1 - \beta_2 - \beta_3}, & A_2 &= \frac{\alpha_2(\beta_1 - \beta_3) + \beta_2(\alpha_3 - \alpha_1)}{\beta_1 - \beta_2 - \beta_3}, \\
 E_1 &= \frac{\beta_3(\epsilon_2 - \epsilon_1) + \epsilon_3(\beta_1 - \beta_2)}{\beta_1 - \beta_2 - \beta_3}, & E_2 &= \frac{\epsilon_2(\beta_1 - \beta_3) + \beta_2(\epsilon_3 - \epsilon_1)}{\beta_1 - \beta_2 - \beta_3}, \\
 \Gamma_1 &= \frac{\beta_3(\gamma_2 - \gamma_1) + \gamma_3(\beta_1 - \beta_2)}{\beta_1 - \beta_2 - \beta_3}, & \Gamma_2 &= \frac{\gamma_2(\beta_1 - \beta_3) + \beta_2(\gamma_3 - \gamma_1)}{\beta_1 - \beta_2 - \beta_3}, \\
 \Delta_1 &= \frac{\beta_3(\delta_2 - \delta_1) + \delta_3(\beta_1 - \beta_2)}{\beta_1 - \beta_2 - \beta_3}, & \Delta_2 &= \frac{\delta_2(\beta_1 - \beta_3) + \beta_2(\delta_3 - \delta_1)}{\beta_1 - \beta_2 - \beta_3}.
 \end{aligned}$$

The J 's above are equivalent to (2.4), (2.5) in the mirror to the $A/2$ model on $\mathbb{P}^1 \times \mathbb{P}^1$, if we take

$$\begin{aligned}
 A &= \begin{bmatrix} A_1 & 0 \\ 0 & E_1 \end{bmatrix}, & B &= \begin{bmatrix} A_2 & 0 \\ 0 & E_2 \end{bmatrix}, \\
 C &= \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Delta_1 \end{bmatrix}, & D &= \begin{bmatrix} \Gamma_2 & 0 \\ 0 & \Delta_2 \end{bmatrix}.
 \end{aligned}$$

In principle one could also similarly analyze the mirror of the blowdown $dP_2 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ in the same limit $q_1 \rightarrow 0$ in terms of the J functions (3.19)–(3.22), but we will not do so here.

3.3 The third del Pezzo surface, dP_3

3.3.1 Review of the (2, 2) mirror

In this section, we will consider the last toric del Pezzo dP_3 , which can be described by a fan with edges $(1, 0), (0, 1), (-1, -1), (1, 1), (-1, 0), (0, -1)$. The corresponding GLSM has six

chiral superfields $\phi_i, i = 1, \dots, 6$ which are charged under the gauge group $U(1)^4$ as follows:

(1,0)	(-1,-1)	(0,1)	(0,-1)	(-1,0)	(1, 1)
1	1	1	0	0	0
0	0	1	1	0	0
1	0	0	0	1	0
0	1	0	0	0	1

Following section 2.1, the superpotential of the (2,2) mirror is given by,

$$W = \sum_{i=1}^6 \exp(Y_i) = \sum_{i=1}^6 X_i$$

for $X_i = \exp(Y_i)$, with constraints:

$$Y_1 + Y_2 + Y_3 = r_1, \quad Y_3 + Y_4 = r_2, \quad Y_1 + Y_5 = r_3, \quad Y_2 + Y_6 = r_4.$$

Eliminating X_2 and X_4 via two of the constraints above, and introducing two Lagrange multipliers Z_1 and Z_2 to implement the remaining constraints, the superpotential can be written as

$$W = X_1 + X_3 + X_5 + X_6 + \frac{q_1}{X_1 X_3} + \frac{q_2}{X_3} + Z_1 \left(1 - \frac{q_3}{X_1 X_5}\right) + Z_2 \left(1 - \frac{q_4}{q_1} \frac{X_1 X_3}{X_6}\right). \quad (3.27)$$

The vacua solve the following algebraic equations:

$$\begin{aligned} X_1 \partial_1 W &= X_1 - \frac{q_1}{X_1 X_3} + Z_1 \frac{q_3}{X_1 X_5} - Z_2 \frac{q_4}{q_1} \frac{X_1 X_3}{X_6} = 0, \\ X_3 \partial_3 W &= X_3 - \frac{q_2}{X_3} - \frac{q_1}{X_1 X_3} - Z_2 \frac{q_4}{q_1} \frac{X_1 X_3}{X_6} = 0, \\ X_5 \partial_5 W &= X_5 + Z_1 \frac{q_3}{X_1 X_5} = 0, \\ X_6 \partial_6 W &= X_6 + Z_2 \frac{q_4}{q_1} \frac{X_1 X_3}{X_6} = 0, \\ \partial_{Z_1} W &= 1 - \frac{q_3}{X_1 X_5} = 0, \\ \partial_{Z_2} W &= 1 - \frac{q_4}{q_1} \frac{X_1 X_3}{X_6} = 0. \end{aligned}$$

The quantum cohomology relations are

$$\begin{aligned} (\psi_1 + \psi_3)(\psi_1 + \psi_4)(\psi_1 + \psi_2) &= q_1, \\ (\psi_1 + \psi_2)\psi_2 &= q_2, \\ (\psi_1 + \psi_3)\psi_3 &= q_3, \\ (\psi_1 + \psi_4)\psi_4 &= q_4. \end{aligned}$$

One can check that these quantum cohomology ring relations are satisfied on the space of vacua of the Toda theory after identifying

$$X_1 \sim \psi_1 + \psi_3, \quad X_3 \sim \psi_1 + \psi_2, \quad X_5 \sim \psi_3, \quad X_6 \sim \psi_4.$$

One can also check that all the correlation functions match those of the A -twisted theory on dP_3 .

3.3.2 (0, 2) deformations and proposed (0,2) mirrors

To describe the (0,2) deformation of dP_3 , we will need 24 complex parameters $\alpha_i, \beta_j, \gamma_k, \delta_l, \epsilon_m, \zeta_n, i, j, k, l, m, n = 1 \dots 4$. Those parameters define a deformation \mathcal{E} of the tangent bundle as follows,

$$0 \longrightarrow \mathcal{O}^3 \xrightarrow{E} \mathcal{O}(1, 0, 1, 0) \oplus \mathcal{O}(1, 0, 0, 1) \oplus \mathcal{O}(1, 1, 0, 0) \oplus \mathcal{O}(0, 1, 0, 0) \\ \oplus \mathcal{O}(0, 0, 1, 0) \oplus \mathcal{O}(0, 0, 0, 1) \longrightarrow \mathcal{E} \longrightarrow 0,$$

where E is defined by:

$$E = \begin{bmatrix} \alpha_1 s_1 & \alpha_2 s_2 & \alpha_3 s_3 & \alpha_4 s_4 \\ \beta_1 s_1 & \beta_2 s_2 & \beta_3 s_3 & \beta_4 s_4 \\ \gamma_1 s_1 & \gamma_2 s_2 & \gamma_3 s_3 & \gamma_4 s_4 \\ \delta_1 s_1 & \delta_2 s_2 & \delta_3 s_3 & \delta_4 s_4 \\ \epsilon_1 s_1 & \epsilon_2 s_2 & \epsilon_3 s_3 & \epsilon_4 s_4 \\ \zeta_1 s_2 & \zeta_2 s_2 & \zeta_3 s_3 & \zeta_4 s_4 \end{bmatrix},$$

for s_i the chiral superfields of the GLSM. The (2,2) locus is given by the special case

$$\begin{aligned} \alpha_1 = 1, \alpha_2 = 0, \alpha_3 = 1, \alpha_4 = 0, \\ \beta_1 = 1, \beta_2 = \beta_3 = 0, \beta_4 = 1, \\ \gamma_1 = \gamma_2 = 1, \gamma_3 = \gamma_4 = 0, \\ \delta_1 = 0, \delta_2 = 1, \delta_3 = \delta_4 = 0, \\ \epsilon_1 = \epsilon_2 = 0, \epsilon_3 = 1, \epsilon_4 = 0, \\ \zeta_1 = \zeta_2 = \zeta_3 = 0, \zeta_4 = 1. \end{aligned}$$

If we define:

$$\begin{aligned} Q_{(1)} &= \sum_{i=1}^4 \alpha_i \psi_i, & Q_{(2)} &= \sum_{i=1}^4 \beta_i \psi_i, & Q_{(3)} &= \sum_{i=1}^4 \gamma_i \psi_i, \\ Q_{(4)} &= \sum_{i=1}^4 \delta_i \psi_i, & Q_{(5)} &= \sum_{i=1}^4 \epsilon_i \psi_i, & Q_{(6)} &= \sum_{i=1}^4 \zeta_i \psi_i, \end{aligned}$$

then the quantum sheaf cohomology ring relations are

$$Q_{(1)} Q_{(2)} Q_{(3)} = q_1, \tag{3.28}$$

$$Q_{(3)} Q_{(4)} = q_2, \tag{3.29}$$

$$Q_{(1)} Q_{(5)} = q_3, \tag{3.30}$$

$$Q_{(2)} Q_{(6)} = q_4, \tag{3.31}$$

which reduce to the ordinary quantum cohomology ring relations on the (2, 2) locus.

Our proposal for the (0,2) mirror of the $A/2$ -twisted theory on dP_3 with a deformation of the tangent bundle is defined by the following six J functions:

$$\begin{aligned}
 J_1 &= -\frac{q_1}{X_1(\gamma \cdot X)} + Z_1 \frac{q_3}{X_1(\epsilon \cdot X)} - Z_2 \frac{q_4}{q_1} \frac{(\alpha \cdot X)(\gamma \cdot X)}{(\zeta \cdot X)} + \frac{(\alpha \cdot X)(\epsilon \cdot X)}{X_1} \\
 &\quad - (\zeta \cdot X) + \frac{(\alpha \cdot X)(\beta \cdot X)}{X_1}, \\
 J_3 &= -\frac{q_2}{X_3} - \frac{q_1}{X_1(\gamma \cdot X)} - Z_2 \frac{q_4}{q_1} \frac{(\alpha \cdot X)(\gamma \cdot X)}{(\zeta \cdot X)} + \frac{(\alpha \cdot X)(\beta \cdot X)}{X_1} - (\zeta \cdot X) \\
 &\quad + \frac{(\gamma \cdot X)(\delta \cdot X)}{X_3}, \\
 J_5 &= (\epsilon \cdot X) + Z_1 \frac{q_3}{(\alpha \cdot X)(\epsilon \cdot X)}, \\
 J_6 &= (\zeta \cdot X) + Z_2 \frac{q_4}{q_1} \frac{(\alpha \cdot X)(\gamma \cdot X)}{(\zeta \cdot X)}, \\
 J_{Z1} &= \frac{(\epsilon \cdot X)}{X_5} - \frac{q_3}{X_5(\alpha \cdot X)}, \\
 J_{Z2} &= \frac{(\zeta \cdot X)}{X_6} - \frac{q_4}{q_1} \frac{(\alpha \cdot X)(\gamma \cdot X)}{X_6},
 \end{aligned}$$

where

$$\begin{aligned}
 (\alpha \cdot X) &= \alpha_1(X_1 - X_5) + \alpha_2(-X_1 + X_3 + X_5) + \alpha_3 X_5 + \alpha_4 X_6, \\
 &\quad \vdots \\
 (\zeta \cdot X) &= \zeta_1(X_1 - X_5) + \zeta_2(-X_1 + X_3 + X_5) + \zeta_3 X_5 + \zeta_4 X_6.
 \end{aligned}$$

(Because the J 's have poles away from origins, we interpret the resulting action in a low-energy effective field theory sense, as discussed in the introduction.)

On the (2,2) locus, the above J functions reduce to derivatives of the (2,2) superpotential (3.27) as expected. It is also easy to check that this theory respects the $U(1)_V$ symmetry, but $U(1)_A$ is broken classically. One can also show that on the space of vacua all quantum sheaf cohomology ring relations (3.28)–(3.31) are satisfied after identifying

$$X_1 = \psi_1 + \psi_3, \quad X_3 = \psi_1 + \psi_2, \quad X_5 = \psi_3, \quad X_6 = \psi_4.$$

In all other examples in this paper, we have checked that all of the genus zero correlation functions of the proposed $B/2$ -twisted Landau-Ginzburg mirror match those of the original $A/2$ theory, for all deformations. However, for dP_3 , we have only checked that the genus zero correlation functions match in several families of deformation parameters, described below:

1. families parametrized by $\alpha_i, \beta_i, \gamma_i, \delta_i, \epsilon_i, \zeta_i$, for fixed $i \in \{1, \dots, 4\}$, and other parameters set to their (2,2) locus values,

2.

$$\begin{aligned}\alpha_2 = \alpha_4 = 0, \beta_2 = \beta_3 = 0, \\ \gamma_3 = \gamma_4 = 0, \delta_1 = \delta_3 = \delta_4 = 0, \\ \epsilon_1 = \epsilon_2 = \epsilon_4 = 0, \zeta_1 = \zeta_2 = \zeta_3 = 0,\end{aligned}$$

for a family parametrized by $\alpha_{1,3}, \beta_{1,4}, \gamma_{1,2}, \delta_2, \epsilon_3, \zeta_4$,

3.

$$\begin{aligned}\alpha_1 = 1, \alpha_2 = 0, \alpha_3 = 1, \alpha_4 = 0, \\ \beta_1 = 1, \beta_2 = \beta_3 = 0, \beta_4 = 1,\end{aligned}$$

for a family parametrized by $\gamma_{1-4}, \delta_{1-4}, \epsilon_{1-4}, \zeta_{1-4}$,

4.

$$\begin{aligned}\gamma_1 = \gamma_2 = 1, \gamma_3 = \gamma_4 = 0, \\ \epsilon_1 = \epsilon_2 = 0, \epsilon_3 = 1, \epsilon_4 = 0,\end{aligned}$$

for a family parametrized by $\alpha_{1-4}, \beta_{1-4}, \delta_{1-4}, \zeta_{1-4}$.

For each of the families of deformation parameters above, we have checked that all of the genus zero correlation functions of the proposed $B/2$ -twisted Landau-Ginzburg model match those of the original $A/2$ -twisted theory.

3.3.3 Consistency check: mirrors of blowdowns to dP_2

In this section we will describe the mirror of the blowdown $dP_3 \rightarrow dP_2$, verifying that the blowdown of the mirror is the mirror of the blowdown. This can be represented torically by removing the edge $(1, 1)$ from the toric fan previously discussed for dP_3 , as shown in figure 4. (As before, since all of the varieties in question are Fano, the UV phases of the GLSMs correspond to the geometries described here.)

The $(2,2)$ mirror of this blowdown is given by applying the limit $q_4 \rightarrow 0$ to the superpotential (3.27) after integrate out one of the Lagrange multiplier Z_2 . Following the same procedure for $(0,2)$ mirror dual, one first integrate out the Lagrange multiplier Z_2 and obtain two constraints:

$$\begin{aligned}J_6 = (\zeta \cdot X) + Z_2 \frac{q_4 (\alpha \cdot X)(\gamma \cdot X)}{q_1 (\zeta \cdot X)} = 0, \\ J_{Z_2} = \frac{(\zeta \cdot X)}{X_6} - \frac{q_4 (\alpha \cdot X)(\gamma \cdot X)}{q_1 X_6} = 0.\end{aligned}$$

In the limit $q_4 \rightarrow 0$, the constraints above imply $\zeta \cdot X = 0$ and $Z_2 = 0$, or for $\zeta_4 \neq 0$ (which we assume for simplicity),

$$X_6 = -\zeta_4^{-1} (\zeta_1(X_1 - X_5) + \zeta_2(-X_1 + X_3 + X_5) + \zeta_3 X_5), \quad Z_2 = 0.$$

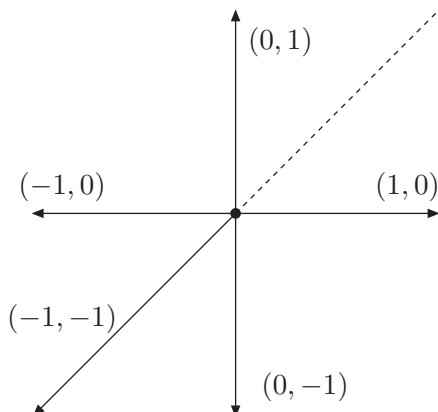


Figure 4. A toric fan for dP_2 can be obtained by removing the edge $(1, 1)$ from the toric fan for dP_3 .

Plugging those constraints into the J s, one gets

$$\begin{aligned}
 J_1 &= -\frac{q_1}{X_1(\gamma \cdot X)} + Z_1 \frac{q_3}{X_1(\epsilon \cdot X)} + \frac{(\alpha \cdot X)(\epsilon \cdot X)}{X_1} + \frac{(\alpha \cdot X)(\beta \cdot X)}{X_1}, \\
 J_3 &= -\frac{q_2}{X_3} - \frac{q_1}{X_1(\gamma \cdot X)} + \frac{(\alpha \cdot X)(\beta \cdot X)}{X_1} + \frac{(\gamma \cdot X)(\delta \cdot X)}{X_3}, \\
 J_5 &= (\epsilon \cdot X) + Z_1 \frac{q_3}{(\alpha \cdot X)(\epsilon \cdot X)}, \\
 J_{Z1} &= \frac{(\epsilon \cdot X)}{X_5} - \frac{q_3}{X_5(\alpha \cdot X)},
 \end{aligned}$$

where

$$\begin{aligned}
 (\alpha \cdot X) &= A_1(X_1 - X_5) + A_2(-X_1 + X_3 + X_5) + A_3X_5, \\
 (\beta \cdot X) &= B_1(X_1 - X_5) + B_2(-X_1 + X_3 + X_5) + B_3X_5, \\
 (\gamma \cdot X) &= G_1(X_1 - X_5) + G_2(-X_1 + X_3 + X_5) + G_3X_5, \\
 (\delta \cdot X) &= D_1(X_1 - X_5) + D_2(-X_1 + X_3 + X_5) + D_3X_5, \\
 (\epsilon \cdot X) &= E_1(X_1 - X_5) + E_2(-X_1 + X_3 + X_5) + E_3X_5,
 \end{aligned}$$

with

$$\begin{aligned}
 A_1 &= \alpha_1 - \alpha_4 \zeta_1 \zeta_4^{-1}, & A_2 &= \alpha_2 - \alpha_4 \zeta_2 \zeta_4^{-1}, & A_3 &= \alpha_3 - \alpha_4 \zeta_3 \zeta_4^{-1}, \\
 B_1 &= \beta_1 - \beta_4 \zeta_1 \zeta_4^{-1}, & B_2 &= \beta_2 - \beta_4 \zeta_2 \zeta_4^{-1}, & B_3 &= \beta_3 - \beta_4 \zeta_3 \zeta_4^{-1}, \\
 G_1 &= \gamma_1 - \gamma_4 \zeta_1 \zeta_4^{-1}, & G_2 &= \gamma_2 - \gamma_4 \zeta_2 \zeta_4^{-1}, & G_3 &= \gamma_3 - \gamma_4 \zeta_3 \zeta_4^{-1}, \\
 D_1 &= \delta_1 - \delta_4 \zeta_1 \zeta_4^{-1}, & D_2 &= \delta_2 - \delta_4 \zeta_2 \zeta_4^{-1}, & D_3 &= \delta_3 - \delta_4 \zeta_3 \zeta_4^{-1}, \\
 E_1 &= \epsilon_1 - \epsilon_4 \zeta_1 \zeta_4^{-1}, & E_2 &= \epsilon_2 - \epsilon_4 \zeta_2 \zeta_4^{-1}, & E_3 &= \epsilon_3 - \epsilon_4 \zeta_3 \zeta_4^{-1}.
 \end{aligned}$$

The resulting J functions are the same as those of dP_2 in (3.19)–(3.22) with parameters

$$\begin{aligned}
 \alpha_1 &= A_1, & \alpha_2 &= A_2, & \alpha_3 &= A_3, \\
 \beta_1 &= B_1, & \beta_2 &= B_2, & \beta_3 &= B_3, \\
 \gamma_1 &= B_1, & \gamma_2 &= B_2, & \gamma_3 &= B_3, \\
 \delta_1 &= D_1, & \delta_2 &= D_2, & \delta_3 &= D_3, \\
 \epsilon_1 &= E_1, & \epsilon_2 &= E_2, & \epsilon_3 &= E_3.
 \end{aligned}$$

4 Hirzebruch surfaces

4.1 Review of the (2,2) mirror

We will first review the construction of Hirzebruch surfaces \mathbb{F}_n and their (2,2) Toda duals along with the ordinary quantum cohomology relations.

Recall a Hirzebruch surface \mathbb{F}_n is a toric variety which can be described by the fan with edges $(1, 0)$, $(0, 1)$, $(0, -1)$, $(-1, -n)$. The corresponding gauged linear sigma model has four chiral superfields $\phi_1, \phi_2, \phi_3, \phi_4$ which are charged under the gauge group $U(1)^2$ as follows

u	v	s	t
1	1	n	0
0	0	1	1

Now, for $n > 1$, Hirzebruch surfaces \mathbb{F}_n are not Fano. This fact manifests itself in the RG flow: en route to the IR, the GLSM for a Hirzebruch surface will enter a different phase, and describe a different geometry. Nevertheless, we expect them to flow in the IR to a discrete set of isolated vacua, and so one can reasonably expect a Toda-type mirror, despite the fact that they are not Fano. On the other hand, for the same reasons, for $n > 2$ the mirrors we describe should not be interpreted as mirrors to Hirzebruch surfaces per se but rather to different phases of the same GLSM, specifically to geometries $\mathbb{P}_{[1,1,n]}^2$ which appear as the UV phases of the same GLSMs. In any event, for simplicity, we will speak loosely of ‘mirrors to Hirzebruch surfaces’ with the understanding that we are speaking of mirrors to GLSMs, and the actual geometries being mirrored are the UV phases, which for $n > 2$ will not in fact be Hirzebruch surfaces.

Following the methods of [22] (with the non-Fano caveat above), the mirror is a Landau-Ginzburg model with superpotential

$$W = \exp(Y_1) + \exp(Y_2) + \exp(Y_3) + \exp(Y_4),$$

with constraints

$$Y_1 + Y_2 + nY_3 = r_1, \quad Y_3 + Y_4 = r_2.$$

We can then use those two constraints to write the result in terms of only Y_1, Y_3 , yielding [22][equ’n (5.19)]

$$W = X_1 + X_3 + \frac{q_2}{X_3} + \frac{q_1}{X_1 X_3^n},$$

where we defined $X_1 = \exp(Y_1)$, $X_3 = \exp(Y_3)$, $q_1 = \exp(r_1)$ and $q_2 = \exp(r_2)$.

As discussed in e.g. [39], the quantum cohomology ring relations are given by

$$\psi^2(n\psi + \tilde{\psi})^n = q_1, \quad \tilde{\psi}(n\psi + \tilde{\psi}) = q_2.$$

To translate to the present case, we use the dictionary

$$\psi \sim X_1, \quad n\psi + \tilde{\psi} \sim X_3, \tag{4.1}$$

so on the space of vacua, we expect that the fields X_1, X_3 of the mirror should obey

$$X_1^2 X_3^n = q_1, \quad (X_3 - nX_1)X_3 = q_2. \tag{4.2}$$

Vacua are computed by taking derivatives of the superpotential with respect to $\ln X_1, \ln X_3$. Doing so one finds that the vacua are defined by

$$X_1 - q_1 X_1^{-1} X_3^{-n} = 0, \tag{4.3}$$

$$X_3 - q_2 X_3^{-1} - nq_1 X_1^{-1} X_3^{-n} = 0. \tag{4.4}$$

After some algebra one can show that these conditions for vacua imply conditions (4.2), as expected.

For geometries previously described, we have given alternative mirrors, and this is no exception. Solving the original constraints for Y_1 and Y_3 , and defining $X_1 = \exp(Y_1)$ and $X_3 = \exp(Y_3)$, we get an alternative expression for the superpotential defining the (2,2) mirror:

$$W = X_1 + q_1 q_2^{-n} X_1^{-1} X_4^n + X_4 + q_2 X_4^{-1}.$$

This expression is related to the one above by the field redefinition

$$X_4 = \frac{q_2}{X_3}.$$

4.2 (0,2) deformations and proposed (0,2) mirrors

In this section, we will give a proposal for the mirror (0,2) Landau-Ginzburg model to an $A/2$ -twisted nonlinear sigma model on a Hirzebruch surface with a deformation of the tangent bundle, which we will check by matching correlation functions.

The (0,2) deformations of a Hirzebruch surface \mathbb{F}_n are defined by a pair of 2×2 matrices A, B , and complex numbers $\gamma_1, \gamma_2, \alpha_1, \alpha_2$, that define a deformation \mathcal{E} of the tangent bundle

$$0 \longrightarrow \mathcal{O}^{\oplus 2} \xrightarrow{E} \mathcal{O}(1,0)^{\oplus 2} \oplus \mathcal{O}(n,1) \oplus \mathcal{O}(0,1) \longrightarrow \mathcal{E} \longrightarrow 0,$$

where E is

$$E = \begin{bmatrix} Ax & Bx \\ \gamma_1 s & \gamma_2 s \\ \alpha_1 t & \alpha_2 t \end{bmatrix},$$

with

$$x = \begin{bmatrix} u \\ v \end{bmatrix}.$$

The (2, 2) locus is given by the special case

$$A = I, \quad B = 0, \quad \gamma_1 = n, \quad \gamma_2 = 1, \quad \alpha_1 = 0, \quad \alpha_2 = 1.$$

If we define

$$Q_{(k)} = \det(\psi A + \tilde{\psi} B), \quad Q_{(s)} = \psi \gamma_1 + \tilde{\psi} \gamma_2, \quad Q_{(t)} = \psi \alpha_1 + \tilde{\psi} \alpha_2,$$

then the quantum sheaf cohomology ring relations are given by [5]

$$Q_{(k)} Q_{(s)}^n = q_1, \quad Q_{(s)} Q_{(t)} = q_2.$$

Our proposal for the (0, 2) Toda mirror of the $A/2$ model on \mathbb{F}_n with a deformation of the tangent bundle is defined by

$$J_1 = aX_1 + \mu_{AB}(X_3 - nX_1) + b \frac{(X_3 - nX_1)^2}{X_1} - q_1 X_1^{-1} (\gamma_1 X_1 + \gamma_2 (X_3 - nX_1))^{-n}, \quad (4.5)$$

$$J_2 = n \left(aX_1 + \mu_{AB}(X_3 - nX_1) + b \frac{(X_3 - nX_1)^2}{X_1} \right) - \frac{nq_1}{X_1 (\gamma_1 X_1 + \gamma_2 (X_3 - nX_1))^n} - \frac{q_2}{X_3} + \frac{(\gamma_1 X_1 + \gamma_2 (X_3 - nX_1)) (\alpha_1 X_1 + \alpha_2 (X_3 - nX_1))}{X_3}. \quad (4.6)$$

(Because the J 's have poles away from origins, we interpret the resulting action in a low-energy effective field theory sense, as discussed in the introduction.) In the expression above, we used the same notation as in our description of the $A/2$ theory, namely

$$a = \det A, \quad b = \det B, \quad \mu_{AB} = \det(A + B) - \det A - \det B.$$

We will check our proposal by arguing that $A/2$ model correlation functions will match those of the $B/2$ -twisted mirror Landau-Ginzburg theory given above. Before doing so, let us first make a few elementary observations. As one might expect, our proposal reduces to the (2, 2) Toda dual when $\mathcal{E} = TX$, corresponding to $A = I, B = 0, \gamma_1 = n, \gamma_2 = 1, \alpha_1 = 0, \alpha_2 = 1$, as in this case each J_i becomes the derivative of the (2,2) superpotential with respect to $Y_1 = \ln X_1$ and $Y_3 = \ln X_3$. As another consistency check, one can show X_1, X_3 satisfy the quantum sheaf cohomology relations on the space of vacua. Specifically, the vacua are defined by $J_1 = 0, J_2 = 0$, which imply

$$\det(A X_1 + B(X_3 - nX_1)) (\gamma_1 X_1 + \gamma_2 (X_3 - nX_1))^n = q_1, \\ (\alpha_1 X_1 + \alpha_2 (X_3 - nX_1)) (\gamma_1 X_1 + \gamma_2 (X_3 - nX_1)) = q_2.$$

With the correspondence (4.1), it is straightforward to show that the quantum sheaf cohomology relations are satisfied on the vacua.

As another consistency check, we observe that this naturally specializes to results obtained in [23] and reviewed in section 2.2 for the mirror to the $A/2$ model on $\mathbb{P}^1 \times \mathbb{P}^1$

with a deformation of the tangent bundle. If we take $n = 0$, then the resulting Hirzebruch surface with tangent bundle deformation corresponds to $\mathbb{P}^1 \times \mathbb{P}^1$ with

$$C = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \alpha_1 \end{bmatrix}, \quad D = \begin{bmatrix} \gamma_2 & 0 \\ 0 & \alpha_2 \end{bmatrix},$$

so that, after simplification,

$$J_1 = aX_1 + \mu_{AB}X_2 + b\frac{X_2^2}{X_1} - \frac{q_1}{X_1}, \quad (4.7)$$

$$J_2 = \alpha_1\gamma_1\frac{X_1^2}{X_2} + (\alpha_1\gamma_2 + \alpha_2\gamma_1)X_1 + \alpha_2\gamma_2X_2 - \frac{q_2}{X_2}. \quad (4.8)$$

One can easily observe that the J functions above (4.7), (4.8) are the same as (4.5), (4.6) after setting $n \mapsto 0$.

Similarly, for the case $n = 1$, the mirror here matches the mirror to $dP_1 = \mathbb{F}_1$ described previously in section 3.1.1.

We have checked that all genus zero correlation functions in this proposed (0,2) mirror match those of the original $A/2$ -twisted theory, following the arguments outlined in section 3.1.1.

So far we have presented a (0,2) mirror proposal that reduces on the (2,2) locus to the first expression for a (2,2) mirror. As we have done for other geometries, we next present a (0,2) mirror proposal that reduces on the (2,2) locus to the second expression for a (2,2) mirror. Specifically, a proposal for a (0,2) mirror of the $A/2$ model on \mathbb{F}_n (with a deformation of the tangent bundle) that reduces on the (2,2) locus to the model above is given by

$$J_1 = \left(aX_1 + \mu_{AB}X_4 + b\frac{X_4^2}{X_1} \right) - \frac{q_1}{q_2^n} \frac{(\alpha_1X_1 + \alpha_2X_4)^n}{X_1}, \quad (4.9)$$

$$J_2 = \left(\alpha_2\gamma_2X_4 + \gamma_1\alpha_1\frac{X_1^2}{X_4} + \frac{q_1}{q_2^n} \frac{(\alpha_1X_1 + \alpha_2X_4)^n(\gamma_1\alpha_2 + \gamma_2\alpha_1)}{aX_1 + \mu_{AB}X_4 + bX_4^2X_1^{-1}} \right) - \frac{q_2}{X_4}. \quad (4.10)$$

In passing, we should mention that an alternative expression which also has matching correlation functions and the correct (2,2) locus can be written which has the same J_1 but a different J_2 given by

$$J_2 = -n \left(aX_1 + \mu_{AB}X_4 + b\frac{X_4^2}{X_1} - \frac{q_1}{q_2^n} \frac{(\alpha_1X_1 + \alpha_2X_4)^n}{X_1} \right) + \left(\alpha_2\gamma_2X_4 + \gamma_1\alpha_1\frac{X_1^2}{X_4} + (\gamma_1\alpha_2 + \gamma_2\alpha_1)X_1 \right) - \frac{q_2}{X_4}. \quad (4.11)$$

Of course, by taking suitable field redefinitions, we can simplify the second J_2 above to write it in the form

$$J_2 = \left(\alpha_2\gamma_2X_4 + \gamma_1\alpha_1\frac{X_1^2}{X_4} + (\gamma_1\alpha_2 + \gamma_2\alpha_1)X_1 \right) - \frac{q_2}{X_4}. \quad (4.12)$$

This third model, with the altered J_2 above, does not reduce to the (2,2) locus expression given previously, but we felt important to point out its existence.

One can check that these alternative proposals also reduce to the (2, 2) mirror, and X_1, X_4 satisfy the quantum sheaf cohomology relations on the space of vacua with identification $X_1 \sim \psi, X_4 \sim \tilde{\psi}$. More importantly, all (genus zero) $A/2$ model correlation functions again match those of the $B/2$ Landau-Ginzburg theory given above. Thus, this is another expression for the mirror.

Setting $n = 0$, we also have a new expression for the $B/2$ mirror Landau-Ginzburg theory to $\mathbb{P}^1 \times \mathbb{P}^1$,

$$J_1 = aX_1 + \mu_{AB}X_4 + b\frac{X_4^2}{X_1} - \frac{q_1}{X_1},$$

$$J_2 = \alpha_2\gamma_2X_4 + \gamma_1\alpha_1\frac{X_1^2}{X_4} + \frac{q_1(\gamma_1\alpha_2 + \gamma_2\alpha_1)}{aX_1 + \mu_{AB}X_4 + bX_4^2X_1^{-1}} - \frac{q_2}{X_4}.$$

On the (2, 2) locus the J functions above reduce to those of the (2, 2) mirror of $\mathbb{P}^1 \times \mathbb{P}^1$ written in (0, 2) language,

$$J_1 = X_1 - \frac{q_1}{X_1},$$

$$J_1 = X_4 - \frac{q_2}{X_4}.$$

All correlation functions of the new mirror theory given above are the same as those given in section (4.2). In both case, all correlation functions match the correlation functions of the same one-loop effective action of the $A/2$ theory on $\mathbb{P}^1 \times \mathbb{P}^1$.

5 Grassmannians

In this section we will propose a (0,2) analogue of the mirror to a Grassmannian proposed in [22][appendix A]. As was remarked to us by one of the authors [40], alternative proposals also exist in the literature, see for example [41–44]. In this paper, we shall only consider (0,2) deformations of the proposal in [22], and will leave (0,2) deformations of other proposals for future work.

5.1 (0,2) deformations

On the (2,2) locus, the Grassmannian $G(k, n)$ is described by a two-dimensional $U(k)$ gauge theory with n chirals in the fundamental representation. We denote these chiral multiplets by $\Phi_\alpha^i, \alpha = 1, \dots, k, i = 1, \dots, n$. These (2,2) chiral multiplets decompose into (0,2) chiral multiplets $\Phi_\alpha^i = (\phi_\alpha^i, \psi_{+\alpha}^i)$ and (0,2) Fermi multiplets $\Lambda_\alpha^i = (\psi_{-\alpha}^i, F_\alpha^i)$, obeying

$$\bar{D}_+\Lambda_\alpha^i = \sigma_\alpha^\beta \Phi_\beta^i. \tag{5.1}$$

For¹⁰ $1 < k < n - 1$, there exist nontrivial (0,2) deformations of this theory, given explicitly as

$$\bar{D}_+\Lambda_\alpha^i = \sigma_\alpha^\beta \Phi_\beta^i + B_j^i(\text{Tr } \sigma)\Phi_\alpha^j,$$

¹⁰In the cases $k = 1, n - 1$, the Grassmannian is a projective space, and its tangent bundle has no deformations.

where B is an $n \times n$ matrix. (As discussed in [16], one can also rotate the first term by an $n \times n$ matrix, but that matrix can be absorbed into field redefinitions, so for simplicity we omit it.) The resulting (0,2) theory describes a deformation \mathcal{E} of the tangent bundle, defined mathematically by the short exact sequence

$$0 \rightarrow \mathcal{S} \otimes \mathcal{S}^* \xrightarrow{*} \mathcal{V} \otimes \mathcal{S}^* \rightarrow \mathcal{E} \rightarrow 0, \tag{5.2}$$

where the map $*$ is given by $\omega_\alpha^\beta \mapsto \omega_\alpha^\beta x_\beta^i + \omega_\beta^\beta B_j^i x_\alpha^j$.

On the Coulomb branch, the one-loop effective J -functions are given by [18]

$$J_a = -\log [q^{-1} \det(M_a)], \tag{5.3}$$

where

$$M_a = \sigma_a I_n + \left(\sum_b \sigma_b \right) B,$$

and I_n is the $n \times n$ identity matrix. The chiral operators are symmetric polynomials in the k fields σ_a [16]. For any such operator \mathcal{O} , the correlation function of the $A/2$ twisted theory is computed by the localization formula [15]

$$\langle \mathcal{O} \rangle = \sum_{J=0} \left(\mathcal{O} \prod_{a \neq b} (\sigma_a - \sigma_b) H^{-1} \right) \tag{5.4}$$

where

$$H = \det_{a,b} \left(\frac{\partial J_a}{\partial \sigma_b} \right) \prod_a \det(M_a).$$

The quantum sheaf cohomology ring of this theory is given by [16]

$$\mathbb{C} [\sigma_{(1)}, \sigma_{(2)}, \dots] / \langle D_{k+1}, D_{k+2}, \dots, R_{(n-k+1)}, \dots, R_{(n-1)}, R_{(n)} + q, R_{(n+1)} + q\sigma_{(1)}, R_{(n+2)} + q\sigma_{(2)}, \dots \rangle, \tag{5.5}$$

where

$$D_m = \det (\sigma_{(1+j-i)})_{1 \leq i, j \leq m},$$

$$R_{(r)} = \sum_{i=0}^{\min(r,n)} I_i \sigma_{(r-i)} \sigma_{(1)}^i,$$

and where I_i are the coefficients of the characteristic polynomial of B , defined by

$$\det(\lambda I + B) = \sum_{i=0}^n I_{n-i} \lambda^i.$$

(For example, $I_0 = 1$, independent of B , $I_1 = \text{tr } B$, $I_n = \det B$.)

5.2 (0,2) mirror

In appendix A of [22], a conjecture was made for the mirror to an A -twisted GLSM for a Grassmannian. The proposed mirror was the “Weyl-group-invariant part” of a Landau-Ginzburg model with the superpotential [22][equ’n (A.1)]:

$$W = \sum_i \Sigma_i (Q_i^\alpha Y_\alpha - t) + \sum_\alpha e^{Y_\alpha},$$

where t is the FI parameter of the original theory, Y_α correspond to the fundamental fields, and Q_i^α are the charges of the fundamental fields with respect to $U(1)^k \subset U(k)$. Taking the Weyl-group-invariant part meant that fundamental fields are to be written in terms of Weyl-group-invariant combinations (not quite the same as orbifolding the theory).

Therefore, in this section, we propose a (0,2) mirror of the model introduced in section 5.1. This means the $B/2$ model of the proposed theory should reproduce the $A/2$ chiral ring and the correlation function (5.4) of the original theory. For this purpose, note that we can rewrite H in (5.4) as

$$H = \det_{a,b} \left(-\frac{\partial \det(M_a)}{\partial \sigma_b} \right).$$

We propose here a (0,2) analogue of the same structure (leaving questions about the correct physical mirror to other work). Specifically, this proposal is built on (0,2) Landau-Ginzburg model with chiral fields $\Sigma_a, a = 1, \dots, k$ and corresponding Fermi fields Λ_a . The J function coupling to Λ_a is

$$J^a = -\det(M_a) + q. \tag{5.6}$$

The constant q is inserted to ensure that the J functions of the two theories have the same zero set. Given an operator defined by a symmetric polynomial in the σ ’s, the $B/2$ correlation function of this dual theory is

$$\langle \mathcal{O} \rangle = \sum_{J=0} (\mathcal{O} H^{-1}).$$

To produce (5.4) we need to define the measure of this mirror theory to be given by

$$\langle \mathcal{O} \rangle = \int [D\Sigma] \prod_{a \neq b} (\Sigma_a - \Sigma_b) \mathcal{O} e^{-S}, \tag{5.7}$$

in effect inserting factors of $\prod_{a \neq b} (\Sigma_a - \Sigma_b)$ in correlation functions, just as in the (2,2) proposal in [22][appendix A]. This clearly is not equivalent to a definition of a new QFT, but rather is merely a (0,2) analogue of the formal structure presented in [22][appendix A].

6 Conclusions

In this paper, we have continued a program of trying to understand (0,2) mirror symmetry by working out proposals for (0,2) mirrors to some more (non-Calabi-Yau) spaces, following up our previous work [23] on (0,2) mirrors to products of projective spaces with tangent

bundle deformations. Specifically, we have given and checked proposals for (0,2) mirrors to toric del Pezzo and Hirzebruch surfaces with tangent bundle deformations, checking not only correlation functions but also e.g. that mirrors to del Pezzos are related by blowdowns in the fashion one would expect.

It remains for the future to find a general construction of (0,2) mirrors, analogous to the general ansatzes in the literature for (2,2) mirrors.

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A Quantum cohomology of dP_1

In this section we briefly outline standard results on the quantum cohomology of dP_1 , following [45][section II.5].

For any $\beta = dH - \alpha E \in H_2(dP_1, \mathbb{Z})$, where H is the pullback of the hyperplane class of \mathbb{P}^2 and E is the class of the exceptional divisor (viewing dP_1 as the blowup of \mathbb{P}^2 at one point), define the Gromov-Witten invariant

$$N_{d,\alpha} = I_{0,n_{d,\alpha},\beta}((pt)^{n_{d,\alpha}})$$

with the expected dimension $n_{d,\alpha} = 3d - 1 - \alpha$. Using results from [45][section II.5], one can compute these $N_{d,\alpha}$ recursively, and thus determine all the Gromov-Witten invariants. For example, $N_{0,-1} = 1, N_{0,-2} = 0, N_{1,2} = 0, N_{1,1} = 1$, from which we see

$$\begin{aligned} I_{0,3,0}(E, E, X) &= E \cdot E = -1, \\ I_{0,3,0,-1}(E, E, E) &= -1, \\ I_{0,3,(1,1)}(E, E, pt) &= N_{1,1} = 1, \end{aligned}$$

and all other three point Gromov-Witten invariants containing two E 's vanish. Thus

$$\begin{aligned} E * E &= \sum_{\beta} [I_{0,3,\beta}(E, E, X)pt + I_{0,3,\beta}(E, E, E)(-E) + I_{0,3,\beta}(E, E, H)H \\ &\quad + I_{0,3,\beta}(E, E, pt)X] q^{\beta}, \\ &= -pt + Eq_1 + Xq_0q_1^{-1}, \end{aligned}$$

where $q_1 = q^E, q_0 = q^H$. If we define $F = H - E$ to be the class of the fiber and $q_2 = q^F$, then the relation can be written as

$$E * E = -pt + Eq_1 + Xq_2.$$

Similarly one can verify all the relations

$$\begin{aligned} E * F &= pt - Eq_1, \\ F * F &= Eq_1, \\ E * pt &= Fq_2, \\ F * pt &= Xq_1q_2, \\ pt * pt &= Hq_1q_2. \end{aligned}$$

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References

- [1] R. Blumenhagen, R. Schimmrigk and A. Wisskirchen, $(0, 2)$ mirror symmetry, *Nucl. Phys. B* **486** (1997) 598 [[hep-th/9609167](#)] [[INSPIRE](#)].
- [2] R. Blumenhagen and S. Sethi, On orbifolds of $(0, 2)$ models, *Nucl. Phys. B* **491** (1997) 263 [[hep-th/9611172](#)] [[INSPIRE](#)].
- [3] A. Adams, A. Basu and S. Sethi, $(0, 2)$ duality, *Adv. Theor. Math. Phys.* **7** (2003) 865 [[hep-th/0309226](#)] [[INSPIRE](#)].
- [4] I.V. Melnikov and M.R. Plesser, A $(0, 2)$ mirror map, *JHEP* **02** (2011) 001 [[arXiv:1003.1303](#)] [[INSPIRE](#)].
- [5] I. Melnikov, S. Sethi and E. Sharpe, Recent developments in $(0, 2)$ mirror symmetry, *SIGMA* **8** (2012) 068 [[arXiv:1209.1134](#)] [[INSPIRE](#)].
- [6] S.H. Katz and E. Sharpe, Notes on certain $(0, 2)$ correlation functions, *Commun. Math. Phys.* **262** (2006) 611 [[hep-th/0406226](#)] [[INSPIRE](#)].
- [7] E. Sharpe, Notes on certain other $(0, 2)$ correlation functions, *Adv. Theor. Math. Phys.* **13** (2009) 33 [[hep-th/0605005](#)] [[INSPIRE](#)].
- [8] A. Adams, J. Distler and M. Ernebjerg, Topological heterotic rings, *Adv. Theor. Math. Phys.* **10** (2006) 657 [[hep-th/0506263](#)] [[INSPIRE](#)].
- [9] J. Guffin, Quantum sheaf cohomology, a précis, *Matemática Contemporânea* **41** (2012) 17 [[arXiv:1101.1305](#)] [[INSPIRE](#)].
- [10] R. Donagi, J. Guffin, S. Katz and E. Sharpe, A mathematical theory of quantum sheaf cohomology, *Asian J. Math.* **18** (2014) 387 [[arXiv:1110.3751](#)] [[INSPIRE](#)].
- [11] R. Donagi, J. Guffin, S. Katz and E. Sharpe, Physical aspects of quantum sheaf cohomology for deformations of tangent bundles of toric varieties, *Adv. Theor. Math. Phys.* **17** (2013) 1255 [[arXiv:1110.3752](#)] [[INSPIRE](#)].
- [12] J. Guffin and S. Katz, Deformed quantum cohomology and $(0, 2)$ mirror symmetry, *JHEP* **08** (2010) 109 [[arXiv:0710.2354](#)] [[INSPIRE](#)].
- [13] I.V. Melnikov and S. Sethi, Half-twisted $(0, 2)$ Landau-Ginzburg models, *JHEP* **03** (2008) 040 [[arXiv:0712.1058](#)] [[INSPIRE](#)].
- [14] J. McOrist and I.V. Melnikov, Summing the instantons in half-twisted linear σ -models, *JHEP* **02** (2009) 026 [[arXiv:0810.0012](#)] [[INSPIRE](#)].

- [15] C. Closset, W. Gu, B. Jia and E. Sharpe, *Localization of twisted $N = (0, 2)$ gauged linear σ -models in two dimensions*, *JHEP* **03** (2016) 070 [[arXiv:1512.08058](#)] [[INSPIRE](#)].
- [16] J. Guo, Z. Lu and E. Sharpe, *Quantum sheaf cohomology on Grassmannians*, *Commun. Math. Phys.* **352** (2017) 135 [[arXiv:1512.08586](#)] [[INSPIRE](#)].
- [17] J. Guo, Z. Lu and E. Sharpe, *Classical sheaf cohomology rings on Grassmannians*, *J. Algebra* **486** (2017) 246 [[arXiv:1605.01410](#)] [[INSPIRE](#)].
- [18] J. McOrist and I.V. Melnikov, *Half-twisted correlators from the Coulomb branch*, *JHEP* **04** (2008) 071 [[arXiv:0712.3272](#)] [[INSPIRE](#)].
- [19] E. Witten, *Mirror manifolds and topological field theory*, in *Mirror symmetry I*, S.-T. Yau ed., U.S.A., (1993), pg. 121 [*AMS/IP Stud. Adv. Math.* **9** (1998) 121] [[hep-th/9112056](#)] [[INSPIRE](#)].
- [20] A. Givental, *Homological geometry and mirror symmetry*, in *Proceedings of the International Congress of Mathematicians (Zürich Switzerland, 1994)*, Birkhäuser, Switzerland, (1995).
- [21] T. Eguchi, K. Hori and S.-K. Yang, *Topological σ -models and large- N matrix integral*, *Int. J. Mod. Phys. A* **10** (1995) 4203 [[hep-th/9503017](#)] [[INSPIRE](#)].
- [22] K. Hori and C. Vafa, *Mirror symmetry*, [hep-th/0002222](#) [[INSPIRE](#)].
- [23] Z. Chen, E. Sharpe and R. Wu, *Toda-like $(0, 2)$ mirrors to products of projective spaces*, *JHEP* **08** (2016) 093 [[arXiv:1603.09634](#)] [[INSPIRE](#)].
- [24] S. Schäfer-Nameki and T. Weigand, *F-theory and 2d $(0, 2)$ theories*, *JHEP* **05** (2016) 059 [[arXiv:1601.02015](#)] [[INSPIRE](#)].
- [25] S. Franco, S. Lee and R.-K. Seong, *Brane brick models and 2d $(0, 2)$ triality*, *JHEP* **05** (2016) 020 [[arXiv:1602.01834](#)] [[INSPIRE](#)].
- [26] F. Apruzzi, F. Hassler, J.J. Heckman and I.V. Melnikov, *UV completions for non-critical strings*, *JHEP* **07** (2016) 045 [[arXiv:1602.04221](#)] [[INSPIRE](#)].
- [27] A. Gadde and P. Putrov, *Exact solutions of $(0, 2)$ Landau-Ginzburg models*, [arXiv:1608.07753](#) [[INSPIRE](#)].
- [28] S. Franco, S. Lee, R.-K. Seong and C. Vafa, *Brane brick models in the mirror*, *JHEP* **02** (2017) 106 [[arXiv:1609.01723](#)] [[INSPIRE](#)].
- [29] S. Franco, S. Lee and R.-K. Seong, *Orbifold reduction and 2d $(0, 2)$ gauge theories*, *JHEP* **03** (2017) 016 [[arXiv:1609.07144](#)] [[INSPIRE](#)].
- [30] F. Apruzzi, F. Hassler, J.J. Heckman and I.V. Melnikov, *From 6d SCFTs to dynamic GLSMs*, [arXiv:1610.00718](#) [[INSPIRE](#)].
- [31] C. Lawrie, S. Schäfer-Nameki and T. Weigand, *The gravitational sector of 2d $(0, 2)$ F-theory vacua*, *JHEP* **05** (2017) 103 [[arXiv:1612.06393](#)] [[INSPIRE](#)].
- [32] S. Franco, D. Ghim, S. Lee and R.-K. Seong, *Elliptic genera of 2d $(0, 2)$ gauge theories from brane brick models*, *JHEP* **06** (2017) 068 [[arXiv:1702.02948](#)] [[INSPIRE](#)].
- [33] M. Dedushenko, S. Gukov and P. Putrov, *Vertex algebras and 4-manifold invariants*, [arXiv:1705.01645](#) [[INSPIRE](#)].
- [34] E. Witten, *Phases of $N = 2$ theories in two-dimensions*, *Nucl. Phys. B* **403** (1993) 159 [*AMS/IP Stud. Adv. Math.* **1** (1996) 143] [[hep-th/9301042](#)] [[INSPIRE](#)].

- [35] L.B. Anderson, F. Apruzzi, X. Gao, J. Gray and S.-J. Lee, *A new construction of Calabi-Yau manifolds: generalized CICYs*, *Nucl. Phys. B* **906** (2016) 441 [[arXiv:1507.03235](#)] [[INSPIRE](#)].
- [36] P. Berglund and T. Hübsch, *A generalized construction of Calabi-Yau models and mirror symmetry*, [arXiv:1611.10300](#) [[INSPIRE](#)].
- [37] L. Anderson, C. Closset, J. Gray and S.-J. Lee, to appear.
- [38] B. Crauder and R. Miranda, *Quantum cohomology of rational surfaces*, [alg-geom/9410028](#).
- [39] D.R. Morrison and M.R. Plesser, *Summing the instantons: quantum cohomology and mirror symmetry in toric varieties*, *Nucl. Phys. B* **440** (1995) 279 [[hep-th/9412236](#)] [[INSPIRE](#)].
- [40] K. Hori, private communication.
- [41] T. Eguchi, K. Hori and C.-S. Xiong, *Gravitational quantum cohomology*, *Int. J. Mod. Phys. A* **12** (1997) 1743 [[hep-th/9605225](#)] [[INSPIRE](#)].
- [42] A. Givental, *Stationary phase integrals, quantum Toda lattices, flag manifolds, and the mirror conjecture*, [alg-geom/9612001](#).
- [43] K. Rietsch, *A mirror symmetric construction for $qH_T^*(G/P)_{(q)}$* , *Adv. Math.* **217** (2008) 2401 [[math.AG/0511124](#)].
- [44] R. Marsh and K. Rietsch, *The B-model connection and mirror symmetry for Grassmannians*, [arXiv:1307.1085](#).
- [45] P. Aluffi ed., *Quantum cohomology at the Mittag-Leffler institute*, Edizioni della Normale, Italy, (1998).