

## D-particles on orientifolds and rational invariants

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**ABSTRACT:** We revisit the D0 bound state problems, of the M/IIA duality, with the Orientifolds. The cases of O4 and O8 have been studied recently, from the perspective of five-dimensional theories, while the case of O0 has been much neglected. The computation we perform for D0-O0 states boils down to the Witten indices for  $\mathcal{N} = 16$   $O(m)$  and  $Sp(n)$  quantum mechanics, where we adapt and extend previous analysis by the authors. The twisted partition function  $\Omega$ , obtained via localization, proves to be *rational*, and we establish a precise relation between  $\Omega$  and the *integral* Witten index  $\mathcal{I}$ , by identifying continuum contributions sector by sector. The resulting Witten index shows surprisingly large numbers of threshold bound states but in a manner consistent with M-theory. We close with an exploration on how the ubiquitous rational invariants of the wall-crossing physics would generalize to theories with Orientifolds.

**KEYWORDS:** D-branes, Differential and Algebraic Geometry, M-Theory, Supersymmetry and Duality

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**1 Introduction**

One of the earliest BPS bound state counting problems in the context of superstring theory is that of multi-D0 threshold bound states. M theory/IIA theory duality anticipates supersymmetric bound states of  $N$  D-particles, for all natural numbers  $N$  [1]. This problem was given a lot of attention since its first inception by Witten, and obviously,  $N = 2$  case, i.e.,  $\mathcal{N} = 16$  SU(2) quantum mechanics, has been dealt with the most rigor [2, 3], while higher  $N$  cases have lead to many new insights over the years.

This problem was given a fresh treatment recently via the localization technique [4, 5]. Previously, computation of twisted partition functions had been performed for  $\mathcal{N} = 16$  SU( $N$ ) theories [6] and some attempts made for other gauge groups [7, 8], but there are often issues with a contour choice in the last stage of such computations. The new derivations

obviate this last uncertainty as they actually derive rigorously what the contour should be. For  $SU(N)$ , one finds in the end the twisted partition function [5]

$$\Omega_{\mathcal{N}=16}^{SU(N)} = 1 + \sum_{p|N; p>1} 1 \cdot \Delta_{\mathcal{N}=16}^{SU(p)}, \tag{1.1}$$

with rational functions  $\Delta$  whose precise form for a general Lie Algebra can be found in eq. (3.10).

This  $\Omega$ , being non-integral, is certainly not the same as the Witten index [9]. Such is usually a symptom of having asymptotic flat directions that cannot be lifted by a parameter tuning. For  $SU(N)$  theory in question, the classical vacua form a cone  $\mathbb{R}^{9(N-1)}/S_N$ , and the plane-wave-like states can also contribute to the relevant path integral. The correct interpretation here is to identify the first term “1” as the index while the rest are attributed to various continuum sectors. In fact, the other “1”’s in the sum are also nothing but the Witten index of the  $SU(N/p)$  subsectors. This interpretation was pioneered in ref. [2], where the nonequivariant version of  $\Omega$  was computed for  $SU(2)$ , and has been generalized to all  $SU(N)$  rather convincingly [6, 10].

Thus one question that has to be resolved if one is to repeat the problem for more complicated spacetime is how to separate the continuum contribution from true Witten index systematically. This does not seem to admit a universal answer, as there are numerous cases where continuum sectors can conspire to contribute a net integral piece to  $\Omega$  [5]. At present, extraction of  $\mathcal{I}$  from  $\Omega$ , when the theory involves gapless asymptotic directions, is more of an art than a science.

Ref. [5], nevertheless, noted how the main feature of  $\mathcal{N} = 16$   $SU(N)$  generalizes straightforwardly to other  $\mathcal{N} = 16$  theories and also to  $\mathcal{N} = 4$  non-primitive quiver theories with bifundamental matters only. The various continuum contributions to  $\Omega_{\mathcal{N}=16}^G$  have been physically understood, identified, and catalogued. Naturally, this opens up the possibility of computing true Witten indices for D-particle binding to Orientifold points. In fact, the results of ref. [5] almost suffice, except for the case of  $O0^-$  orientifold. In this note, we wish to place the last missing piece in the problem and compute Witten indices for all D0-O0 bound states.

Section 2 will give a general discussion on the twisted partition function versus the Witten index, with emphasis on what the localization procedure actually computes. Section 3 will review the recent results for  $\mathcal{N} = 16$  Yang-Mills quantum mechanics, which we will generalize in section 4 to  $O(m)$  gauge groups. This will lead us to the Witten indices that count bound states between D0’s and any one of four types of the orientifold point and to a known M-theory interpretation, adding yet another strong and rather direct confirmation of M/IIA duality. In the final section, we comment on new type of rational expressions we found along the way and propose them as building blocks for the rational invariants suitable for Orientifolded theories.

## 2 Index $\mathcal{I}$ vs. twisted partition function $\Omega$

For supersymmetric quantum theory, one of the useful and accessible quantities that probe the ground state sector is the Witten index [9],

$$\mathcal{I} = \lim_{\beta \rightarrow \infty} \text{tr} \left[ (-1)^{\mathcal{F}} e^{-\beta H} \right] .$$

The chirality operator  $(-1)^{\mathcal{F}}$  can be replaced by any operator that anti-commutes with the supercharges. One often wishes to compute the equivariant version by inserting chemical potentials,  $x$ , associated with global symmetries,  $F$ ,

$$\mathcal{I}(x) \equiv \lim_{\beta \rightarrow \infty} \text{tr} \left[ (-1)^{\mathcal{F}} x^F e^{-\beta H} \right] .$$

Even more useful information emerges if we select out a particular supercharge  $\mathcal{Q}$  which commutes with a linear combination of  $R$ -symmetry generators, call it  $R$ , and one of the  $F$ 's, resulting in a fully equivariant Witten index,

$$\mathcal{I}(\mathbf{y}, x) \equiv \lim_{\beta \rightarrow \infty} \text{tr} \left[ (-1)^{\mathcal{F}} \mathbf{y}^R x^F e^{-\beta \mathcal{Q}^2} \right] . \tag{2.1}$$

However, as is well known, this quantity may not be amenable to straightforward computations.

If the dynamics is compact, i.e., with a fully discrete spectrum,  $\beta$ -dependence can be argued away based on the naive argument that  $\mathcal{I}$  is topological. Under such favorable circumstances, one is motivated to consider instead

$$\Omega(\mathbf{y}, x, \beta) \equiv \text{tr} \left[ (-1)^{\mathcal{F}} \mathbf{y}^R x^F e^{-\beta \mathcal{Q}^2} \right] , \tag{2.2}$$

and compute the other limit, which tends to reduce the path integral to a local expression,

$$\mathcal{I}_{\text{bulk}}(\mathbf{y}, x) \equiv \lim_{\beta \rightarrow 0} \Omega(\mathbf{y}, x, \beta) , \tag{2.3}$$

with the anticipation that  $\Omega$  is independent of  $\beta$  so that  $\mathcal{I} = \mathcal{I}_{\text{bulk}}$ .

For theories with continuum sectors, however, this naive expectation cannot hold in general;  $\mathcal{I}$  is by definition integral, while  $\Omega$  need not be integral and thus can differ from  $\mathcal{I}$ . If the continuum has a gap,  $E \geq E_{\text{gap}} > 0$ , its contribution is suppressed as

$$e^{-\beta E_{\text{gap}}} ,$$

so we may have an option of scaling  $E_{\text{gap}}$  up first and then taking  $\beta \rightarrow 0$  afterward, leaving behind the integral index  $\mathcal{I}$  only [4, 11].

When the continuum cannot be gapped, or when a gap can be introduced only at the expense of qualitative modification of the asymptotic dynamics, however, we are often in trouble. The resulting bulk part  $\mathcal{I}_{\text{bulk}}$  differs from the genuine index. For such theories, isolating  $\mathcal{I}$  hidden inside  $\mathcal{I}_{\text{bulk}}$  requires a method of computing yet another piece, known as the defect term,

$$-\delta\mathcal{I} \equiv \mathcal{I}_{\text{bulk}} - \mathcal{I} . \tag{2.4}$$

This program depends on particulars of the given problem and, in particular, on the boundary conditions.<sup>1</sup> As far as we know there is no general theory for  $\delta\mathcal{I}$ .

For a large class of gauged dynamics, the localization procedure has been applied successfully to reduce the path integral representation of  $\Omega$  to a formulae involving rank-many contour integrations. For  $\mathcal{N} \geq 2$  gauged quantum mechanics [4] and for  $d = 2$  elliptic genera [15, 16], in particular, reasonably complete and reliable derivations exist. At the end of such computations, one finds that  $\beta$ -dependence is absent. When the dynamics is not compact and  $\Omega$  is expected to be  $\beta$ -dependent, the question is exactly which  $\beta$  limit of  $\Omega$  one has computed.

One key trick here is to scale up the gauge kinetic term by sending  $e^2 \rightarrow 0$ , as the term is often BRST-exact for the spacetime dimension  $D$  less than three. In the absence of other dimensionful parameters of the theory, the only obvious answer to the question we posed above is  $\beta \rightarrow 0$ ; the dimensionless combination of the two is

$$\beta e^{2/(4-D)},$$

so  $e^2 \rightarrow 0$  is equivalent to  $\beta \rightarrow 0$  for  $D \leq 3$ . Another typical dimensionful parameters that could be present are Fayet-Iliopoulos constants  $\zeta$ , but, for a sensible results, one often must take a limit of  $\zeta$  first [4]. This raises a gap  $E_{\text{gap}}$  along certain Coulomb directions to infinity, if not all, so we expect that, again, the  $\beta \rightarrow 0$  limit of  $\Omega$  is computed effectively at the end of the localization procedure. After all, one finds a local expression, at the end of such processes, involving zero mode integrals only, which is impossible at the other limit of  $\beta \rightarrow \infty$ .

As such, we will define for this note,

$$\Omega(\mathbf{y}, x) \equiv \Omega(\mathbf{y}, x, \beta) \Big|_{\text{localization}}, \tag{2.5}$$

whereby, according to the above scaling argument, we may identify

$$\mathcal{I}_{\text{bulk}} = \Omega(\mathbf{y}, x). \tag{2.6}$$

We will call this quantity the twisted partition function, although, strictly speaking, the true twisted partition function  $\Omega(\mathbf{y}, x, \beta)$  may yet differ from  $\Omega(\mathbf{y}, x)$ . This brings us to a general statement

$$\mathcal{I}(\mathbf{y}, x) = \Omega(\mathbf{y}, x) + \delta\mathcal{I}(\mathbf{y}, x). \tag{2.7}$$

Even after a successful localization computation of  $\Omega(\mathbf{y}, x)$ , one is often left with an even more difficult task of identifying the continuum contribution,  $-\delta\mathcal{I}$ , inside  $\Omega$  if one wishes to compute  $\mathcal{I}$ .

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<sup>1</sup>A canonical example is the supersymmetric nonlinear sigma models onto a manifold with boundary. If one adopt the so-called APS boundary condition,  $\delta\mathcal{I}$  is then computed by the eta-invariant, leading to the Atiyah-Patodi-Singer index theorem [12–14]. This boundary condition, however, does not in general translate to  $L^2$  condition on the physical space.

There appears to be no single universal relationship between  $\mathcal{I}$  and  $\Omega$ , but surprisingly, as delineated in ref. [5], there exists classes of  $d = 1$  supersymmetric gauged linear sigma models for which this problem may be dealt with honestly. One such is adjoint-only Yang-Mills quantum mechanics, and another is  $\mathcal{N} = 4$  nonprimitive quiver theories with compact classical Higgs vacuum moduli space. In the next section, we recall this phenomenon for  $\mathcal{N} = 4, 8, 16$  pure Yang-Mills quantum mechanics with connected simple group  $G$ .

### 3 Rational $\Omega_{\mathcal{N}}^G$ and integral $\mathcal{I}_{\mathcal{N}}^G$

For gauged linear sigma model with at least two supersymmetries, the localization procedure gives a Jeffrey-Kirwan residue formulae [4],

$$\Omega(\mathbf{y}, x) = \frac{1}{|W|} \text{JK-Res}_{\eta} \frac{g(t)}{\prod_s t_s} d^r t, \quad (3.1)$$

where  $(t_1, \dots, t_r)$  parameterize the  $r$  bosonic zero modes living in  $(\mathbb{C}^*)^r$ , that usually scan the Cartan directions but can be further restricted in topologically nontrivial holonomy sectors. The determinant  $g(t)$  is due to massive modes in the background of  $t$ 's. In this note, we use  $\mathcal{N} = 4$  notations for supermultiplets, and as such,  $g(t)$  takes the general form,

$$g(t) = \left( \frac{1}{\mathbf{y} - \mathbf{y}^{-1}} \right)^r \prod_{\alpha} \frac{t^{-\alpha/2} - t^{\alpha/2}}{t^{\alpha/2} \mathbf{y}^{-1} - t^{-\alpha/2} \mathbf{y}} \times \prod_i \frac{t^{-Q_i/2} x^{-F_i/2} \mathbf{y}^{-(R_i/2-1)} - t^{Q_i/2} x^{F_i/2} \mathbf{y}^{R_i/2-1}}{t^{Q_i/2} x^{F_i/2} \mathbf{y}^{R_i/2} - t^{-Q_i/2} x^{-F_i/2} \mathbf{y}^{-R_i/2}}. \quad (3.2)$$

Here,  $\alpha$  runs over the roots of the gauge group and  $i$  labels the individual chiral multiplets, with the gauge charge  $Q_i$  and the flavor charge  $F_i$  under the Cartans of the gauge group and of the flavor group, respectively. Finally,  $W$  is the Weyl group of the gauge group and  $\eta$  is a choice of  $r$  auxiliary parameters. For detailed definition of the JK residue [17], the condition on the auxiliary parameters  $\eta$ , and the derivation of the above formula, we refer the reader to the section 4 of ref. [4]. We will refer to this general procedure as HKY.

For pure  $\mathcal{N} = 4$  theories, the computation admits the R-charge chemical potential  $\mathbf{y}$  only. For  $\mathcal{N} = 8, 16$ , we have additional adjoint chirals, and the assignment of global charges needs a little bit of thought. For  $\mathcal{N} = 8$ , one more chemical potential  $x$  can be turned on, associated with the natural  $U(1)$  rotation of the chiral field, and  $R = 0$  is assigned to the adjoint chiral. No superpotential is possible under such assignments. For  $\mathcal{N} = 16$ , with three adjoint chirals, a trilinear superpotential term is needed, so at most two flavor chemical potentials are allowed, say,  $x$  and  $\tilde{x}$  associated with  $F$  and  $\tilde{F}$ . We can for example assign  $R = (2, 0, 0)$ ,  $F = (2, -1, -1)$ , and  $\tilde{F} = (0, 1, -1)$  that allow only trilinear superpotential as required by  $\mathcal{N} = 16$ . In actual  $\mathcal{N} = 16$  formula below  $x^F$  should be understood as the product,  $x^F \tilde{x}^{\tilde{F}}$ , over the two flavor chemical potentials.

One thing special about the pure gauge theories is that we are instructed to ignore the poles located at the boundary of the zero mode space  $(\mathbb{C}^*)^r$  [5]. This is a property which holds generally for theories with the total matter content in a real representation under the gauge group.

Although (3.1) and (3.2) are seemingly phrased in terms of the Lie algebra data only, the global properties of the gauge group  $G$  should matter in general. The main computation of this note, in section 4, addresses precisely such a difference,  $\text{SO}(m)$  vs.  $\text{O}(m)$ . However,  $\Omega^G = \Omega^{G/\Gamma}$  must hold if the field content is invariant under  $\Gamma$ , as no local physics would know about such  $\Gamma$ 's. How is this reflected in (3.1)? Note that the integrations in (3.1) implicitly assume  $2\pi i$  period for each  $\log(t_s)$ , independent of one another, and are accompanied by a naive inverse volume  $1/(2\pi)^r$ . This potential mistreatment of the Cartan torus is automatically corrected by the degeneracies of poles in  $g(t)$ , which are, in turn, determined by the field content and hence by the largest compatible  $\Gamma$ . This way, for example,  $\Omega^G = \Omega^{G/Z_G}$  with the center  $Z_G$  is naturally built into (3.1) for pure Yang-Mills [18]. See ref. [18] for a more complete discussion and related subtleties.

### 3.1 $\mathcal{N} = 4, 8$

This gives us an unambiguous procedure of computing the twisted partition functions  $\Omega_{\mathcal{N}}^G$  for all possible  $G$  and  $\mathcal{N}$ . There are some further computational issues, such as how to deal with the degenerate poles, which complicates the task but still allows us to go forward. We will not give too much details here and instead refer the readers to ref. [5] for pure Yang-Mills cases, and to ref. [4] for general gauged quantum mechanics.

It turns out that, after a long and arduous computer-assisted computation of JK residues, the twisted partition functions for pure  $\mathcal{N} = 4, 8$   $G$ -gauged quantum mechanics, can be organized into purely algebraic quantities. For  $\mathcal{N} = 4$ , one finds

$$\Omega_{\mathcal{N}=4}^G(\mathbf{y}) = \frac{1}{|W_G|} \sum'_w \frac{1}{\det(\mathbf{y}^{-1} - \mathbf{y} \cdot w)} . \tag{3.3}$$

The sum is only over the elliptic Weyl elements and  $|W_G|$  is the cardinality of the Weyl group itself. An elliptic Weyl element  $w$  is defined by absence of eigenvalue 1; in other words, in the canonical  $r$ -dimensional representation of the Weyl group on the weight lattice,

$$\det(1 - w) \neq 0 .$$

Some simple examples with  $\mathcal{N} = 4$  are

$$\begin{aligned} \Omega_{\mathcal{N}=4}^{\text{SU}(N)}(\mathbf{y}) &= \frac{1}{N} \cdot \frac{1}{\mathbf{y}^{-N+1} + \mathbf{y}^{-N+3} + \dots + \mathbf{y}^{N-3} + \mathbf{y}^{N-1}} , \\ \Omega_{\mathcal{N}=4}^{\text{SO}(4)}(\mathbf{y}) &= \frac{1}{4} \cdot \frac{1}{(\mathbf{y}^{-1} + \mathbf{y})^2} , \\ \Omega_{\mathcal{N}=4}^{\text{SO}(5)}(\mathbf{y}) = \Omega_{\mathcal{N}=4}^{\text{Sp}(2)}(\mathbf{y}) &= \frac{1}{8} \cdot \left[ \frac{2}{\mathbf{y}^{-2} + \mathbf{y}^2} + \frac{1}{(\mathbf{y}^{-1} + \mathbf{y})^2} \right] , \\ \Omega_{\mathcal{N}=4}^{\text{SO}(7)}(\mathbf{y}) = \Omega_{\mathcal{N}=4}^{\text{Sp}(3)}(\mathbf{y}) &= \frac{1}{48} \cdot \left[ \frac{8}{\mathbf{y}^{-3} + \mathbf{y}^3} + \frac{6}{(\mathbf{y}^{-2} + \mathbf{y}^2)(\mathbf{y}^{-1} + \mathbf{y})} + \frac{1}{(\mathbf{y}^{-1} + \mathbf{y})^3} \right] , \end{aligned}$$

where each term can be associated with a sum over conjugacy classes of the same cyclic decompositions.

For pure  $\mathcal{N} = 8$   $G$ -gauged quantum mechanics, obtained by adding to the  $\mathcal{N} = 4$  theory an adjoint chiral, we can include a flavor chemical potential  $x$  of the adjoint after assigning a unit flavor charge without loss of generality. With  $R = 0$  for the adjoint chiral, we also have the universal formula,

$$\Omega_{\mathcal{N}=8}^G(\mathbf{y}, x) = \frac{1}{|W_G|} \sum_w' \frac{1}{\det(\mathbf{y}^{-1} - \mathbf{y} \cdot w)} \cdot \frac{\det(\mathbf{y}^{-1}x^{1/2} - \mathbf{y}x^{-1/2} \cdot w)}{\det(x^{1/2} - x^{-1/2} \cdot w)}, \quad (3.4)$$

where again the sum is over the elliptic Weyl elements of  $G$ . For example we have

$$\Omega_{\mathcal{N}=8}^{\text{SO}(4)}(\mathbf{y}, x) = \frac{1}{4} \cdot \frac{1}{(\mathbf{y}^{-1} + \mathbf{y})^2} \cdot \frac{(\mathbf{y}^{-1}x^{1/2} + \mathbf{y}x^{-1/2})^2}{(x^{1/2} + x^{-1/2})^2},$$

and the pattern generalizes to higher rank cases in an obvious manner.

The reason why the result can be repackaged into such a simple algebraic formulae has been explained both for nonequivariant form [2, 10, 19, 20] and for equivariant form [5]. Consider  $-\delta\mathcal{I}$ . This part of  $\Omega$  has to arise from the continuum and, because of this, depends only on the asymptotic dynamics. The latter becomes a nonlinear sigma model on an orbifold

$$\mathcal{O}(G)_{\mathcal{N}=4,8} = \mathbb{R}^{3r}/W \text{ or } \mathbb{R}^{5r}/W, \quad (3.5)$$

so that the  $\delta\mathcal{I}$  of the two theories must agree with each other. On the other hand, we expect no quantum mechanical bound state localized at the orbifold point, so

$$\left(\mathcal{I}_{\mathcal{N}=4,8}^{\mathcal{O}(G)}\right)_{\text{bulk}} + \delta\mathcal{I}_{\mathcal{N}=4,8}^{\mathcal{O}(G)} = 0$$

which implies [2]

$$-\delta\mathcal{I}_{\mathcal{N}=4,8}^G = \left(\mathcal{I}_{\mathcal{N}=4,8}^{\mathcal{O}(G)}\right)_{\text{bulk}}. \quad (3.6)$$

The right hand side of (3.6) has been evaluated using the Heat Kernel regularization, when  $\mathbf{y} = 1$  and  $x = 1$ , for  $\text{SU}(2)$  case in ref. [2], and more generally in refs. [10, 19], with the result

$$\frac{1}{|W|} \sum_w' \frac{1}{\det(1 - w)}. \quad (3.7)$$

What we described above in (3.3) and in (3.4), individually confirmed by direct localization computation, are the equivariant uplifts of this expression for  $\mathcal{N} = 4, 8$  respectively.

With this, the origin of  $\Omega_{\mathcal{N}=4,8}^G$  is abundantly clear. They come entirely from the asymptotic continuum states spanned by the free Cartan dynamics, modulo the orbifolding by the Weyl group; the path-integral-computed  $\Omega_{\mathcal{N}=4,8}^G$  has no room for a contribution from threshold bound states. Therefore, the true enumerative part  $\mathcal{I}$  inside  $\Omega$  has to be null,

$$\mathcal{I}_{\mathcal{N}=4}^G = 0 = \mathcal{I}_{\mathcal{N}=8}^G, \quad (3.8)$$



for any simple group  $G$ . Recall that, for classical groups  $G$ ,  $\mathcal{N} = 4, 8$  pure Yang-Mills quantum mechanics has no bound state, as can be argued based on D2/D3-branes multiply-wrapped on  $S^2$  and  $S^3$  in K3 and Calabi-Yau three-fold, possibly together with Orientifold planes, and the Witten index of these theories must vanish. This physical expectation dovetails with the above structure nicely.

The same principle generalizes to  $\mathcal{N} = 16$  cases. However, their asymptotic dynamics will no longer be captured by analog of  $\mathcal{O}(G)$  alone; the presence of threshold bound states implies that the continuum sectors  $\Omega_{\mathcal{N}=16}^G$  will no longer be that simple. There could be additional sectors involving partial bound states tensored with continuum of remaining asymptotic directions. We turn to this next.

### 3.2 On $\mathcal{N} = 16$ continuum sectors

The same kind of continuum sectors as the above  $\mathcal{N} = 4, 8$  examples should exist for  $\mathcal{N} = 16$ , with the asymptotic dynamics of the form,

$$\mathcal{O}(G)_{\mathcal{N}=16} = \mathbb{R}^{9r} / W_G, \tag{3.9}$$

and we can easily guess the contribution to  $\Omega_{\mathcal{N}=16}^G$  from this sector to take the form,

$$\Delta_{\mathcal{N}=16}^G \equiv \frac{1}{|W_G|} \sum_w \frac{1}{\det(\mathbf{y}^{-1} - \mathbf{y} \cdot w)} \cdot \prod_{a=1}^3 \frac{\det(x^{F_a/2} \mathbf{y}^{R_a/2-1} - x^{-F_a/2} \mathbf{y}^{1-R_a/2} \cdot w)}{\det(x^{F_a/2} \mathbf{y}^{R_a/2} - x^{-F_a/2} \mathbf{y}^{-R_a/2} \cdot w)}, \tag{3.10}$$

as a straightforward generalization of  $\mathcal{N} = 4, 8$  expressions. Here,  $a$  labels the three adjoint chirals. Indeed, as we will see below, each  $\Omega_{\mathcal{N}=16}^G$ , computed via localization, is seen to have an additive piece of this type.

The difference for  $\mathcal{N} = 16$  is, however, that threshold bound states are expected in general. For all  $SU(N)$ , e.g., a single threshold bound state must exist for M-theory/IIA theory duality to hold. Since such states can also occur for subgroups of  $G$  as well and since they can explore the remaining asymptotic directions, a far more complex network of continuum sectors exist. Generally a product of subgroups

$$\otimes_A G_A < G$$

correspond to a collection of one-particle-like states, each labeled by  $A$ . When this subgroup equals the Cartan subgroup of  $G$ , the corresponding continuum sector contributes the universal  $\Delta_{\mathcal{N}=16}^G$  to  $\Omega_{\mathcal{N}=16}^G$ . When at least one of  $G_A$  is a simple group, the corresponding partial bound state(s) can contribute a new fractional piece to  $\Omega_{\mathcal{N}=16}^G$ . The relevant continuum sector is the asymptotic Coulombic directions where the “particles” forming the bound state associated with  $G_A$  moves together. In other words, the asymptotic Coulombic directions are parameterized by a subalgebra

$$h[\otimes_A G_A]$$

of the Cartan of  $G$ , where  $\otimes_A G_A$  is the centralizer of  $h[\otimes_A G_A]$ .

Then, the argument leading to (3.6) can be adapted to this slightly more involved case; a continuum contribution from this sector would be associated with a subgroup

$$W'$$

of  $W_G$  that leaves  $h[\otimes_A G_A]$  invariant yet act faithfully. Contribution to  $\Omega$  would arise from generalized elliptic Weyl elements of  $W'$ ,

$$\det(1 - w') \Big|_{h[\otimes_A G_A]} \neq 0,$$

where the determinant is now taken in the smaller representation over  $h[\otimes_A G_A]$ . In a slight abuse of notation, it turns out that the continuum contribution from  $W'$  to  $\Omega_{\mathcal{N}=16}^G$  can be expressed as a product of the form,

$$\prod_I \Delta_{\mathcal{N}=16}^{H_I}$$

where  $\Delta_{\mathcal{N}=16}^{H_I}$  are defined for some subgroups  $H_I$  of  $G$  in the same manner as (3.10). Each  $H_I$  is a simple subgroup of  $G$  whose Weyl group is a subgroup factor of  $W'$ .

### 3.3 $\mathcal{N} = 16$

$\Omega_{\mathcal{N}=16}^G$  can also be directly computed using the HKY procedure [4]. One then searches for a unique decomposition as sum over such continuum pieces as

$$\Omega_{\mathcal{N}=16}^G = \mathcal{I}_{\mathcal{N}=16}^G + \sum_{\otimes G_A < G} n_{\{G_A\}}^G \prod_I \Delta_{\mathcal{N}=16}^{H_I}, \tag{3.11}$$

with nonnegative integral factor,  $n_{\{G_A\}}^G$ . Furthermore, there should be a term

$$1 \cdot \Delta_{\mathcal{N}=16}^G$$

on the right hand side, with the coefficient 1, representing the sector with no partial bound state whatsoever.

Ref. [5] showed that this is indeed the case, even though such a pattern is hardly visible at the stage of JK-residue computations. For  $SU(N)$ , the result takes a particularly simple form,

$$\Omega_{\mathcal{N}=16}^{SU(N)} = 1 + \sum_{p|N; p \neq 1} 1 \cdot \Delta_{\mathcal{N}=16}^{SU(p)}. \tag{3.12}$$

The rational contributions come from the continuum directions,  $h[\otimes_A G_A]$ , parameterized as

$$\text{diag}(v_1, \dots, v_1; v_2, \dots, v_2; \dots; v_p, \dots, v_p)$$

with each eigenvalue repeated  $(N/p)$ -times, and  $\sum_A v_A = 0$ . In this sector,  $p$  number of partial  $SU(N/p)$  bound states form, continuum states of which contribute  $\Delta_{\mathcal{N}=16}^{SU(p)}$ ; the relevant Weyl subgroup is the permutation group that shuffles  $v$ 's, so can be naturally labeled as  $H = SU(p)$ . In the end, this implies

$$\mathcal{I}_{\mathcal{N}=16}^{SU(N)} = 1 \tag{3.13}$$

for all  $N$ . The nonequivariant limit of the same decomposition

$$\Omega_{\mathcal{N}=16}^{\text{SU}(N)} \Big|_{\mathbf{y} \rightarrow 1; x \rightarrow 1} = \Omega_{\mathcal{N}=16}^{\text{SU}(N)} \Big|_{\mathbf{y} \rightarrow 1} = 1 + \sum_{p|N; p \neq 1} \frac{1}{p^2},$$

has been computed and understood early on [2, 6, 10] along this line of reasoning.

The authors have also computed twisted partition functions for more general simple groups, up to rank 4, and decomposed the resulting  $\Omega_{\mathcal{N}}^G$ 's in this manner [5]. See appendix A.1 for the results. The main lesson is again that we can read off the true Witten index  $\mathcal{I}$  from such a decomposition of each  $\Omega$ ; all the rational pieces have to be part of  $-\delta\mathcal{I}$ , sector by sector. The only integral part, the first terms on the right hand sides, may be interpreted as the Witten index, giving us

$$\begin{aligned} \mathcal{I}_{\mathcal{N}=16}^{\text{SO}(4)} &= \left(\mathcal{I}_{\mathcal{N}=16}^{\text{SU}(2)}\right)^2 = 1, \\ \mathcal{I}_{\mathcal{N}=16}^{\text{SO}(5)} &= \mathcal{I}_{\mathcal{N}=16}^{\text{Sp}(2)} = 1, \\ \mathcal{I}_{\mathcal{N}=16}^{\text{SO}(6)} &= \mathcal{I}_{\mathcal{N}=16}^{\text{SU}(4)} = 1, \\ \mathcal{I}_{\mathcal{N}=16}^{\text{SO}(7)} &= 1, \\ \mathcal{I}_{\mathcal{N}=16}^{\text{Sp}(3)} &= 2, \\ \mathcal{I}_{\mathcal{N}=16}^{\text{SO}(8)} &= 2, \\ \mathcal{I}_{\mathcal{N}=16}^{\text{SO}(9)} &= 2, \\ \mathcal{I}_{\mathcal{N}=16}^{\text{Sp}(4)} &= 2, \end{aligned} \tag{3.14}$$

as well as  $\mathcal{I}_{\mathcal{N}=16}^{G_2} = 2$ . In the next section, we will adopt and extend some of these results for D-particles on an Orientifold point.

#### 4 D0-O0<sup>-</sup>

Let us come to the main problem of this note. Just as the Witten index for  $\mathcal{N} = 16$   $\text{SU}(N)$  theory confirms existence of M-theory circle, hidden in IIA theory, one may ask what this M-theory circle will predict in the presence of IIA orientifold planes. For O8 and O4, D-particle states bound to the orientifold planes require additional D-branes: eight D8's for O8, since otherwise M-theory lift does not exist [21], and more than one D4's for O4. See refs. [22, 23] for recent computations of twisted partition functions in the presence of O4/O8 orientifolds. This leaves O0, namely Orientifold points. While it is, a priori, unclear why there should be D-particles trapped at O0, our computation of nontrivial Witten indices for  $\mathcal{N} = 16$   $\text{SO}$  and  $\text{Sp}$  theories suggests that there should be such states after all. An orientifold projection  $\mathbb{R}^9/\mathbb{Z}_2$  can give either  $\text{Sp}(n)$  or  $\text{O}(m)$  gauge groups. For O0<sup>+</sup>'s, the  $\text{Sp}$  computation above suffices. For O0<sup>-</sup>'s, however, one must supplement  $\text{SO}(m)$  computation by taking into account  $Z_2 = \text{O}(m)/\text{SO}(m)$ . In this section, we generalize  $\text{SO}(m)$  to  $\text{O}(m)$  theories, for D-particles bound to O0<sup>-</sup>'s.

Physically, the difference between the two is whether we demand the physical states be invariant under the gauge-parity operation, which we call  $\mathcal{P}$ , in addition to the local Gauss constraint. So if a twisted partition function for  $\text{SO}(m)$  theory has the form,

$$\text{tr} \left[ (-1)^{\mathcal{F}} \dots e^{-\beta H} \right],$$

its  $\text{O}(m)$  counterpart must have the operator insertion,

$$\text{tr} \left[ (-1)^{\mathcal{F}} \dots e^{-\beta H} \cdot \frac{1 + \mathcal{P}}{2} \right],$$

where  $\mathcal{P}$  is the parity operator in  $\text{O}(m)/\text{SO}(m)$ . In the end, the twisted partition function of an  $\text{O}(m)$  theory is the average of two terms,

$$\Omega_{\mathcal{N}}^{\text{O}(m)}(\mathbf{y}, x) = \frac{\Omega_{\mathcal{N}}^{\text{O}^+(m)}(\mathbf{y}, x) + \Omega_{\mathcal{N}}^{\text{O}^-(m)}(\mathbf{y}, x)}{2}. \tag{4.1}$$

The first term  $\Omega_{\mathcal{N}}^{\text{O}^+(m)}(\mathbf{y}, x) = \Omega_{\mathcal{N}}^{\text{SO}(m)}(\mathbf{y}, x)$  has already been computed, while the second term needs to be computed with the insertion of  $\mathcal{P}$  as

$$\Omega_{\mathcal{N}}^{\text{O}^-(m)}(\mathbf{y}, x) \equiv \text{tr} \left[ (-1)^{\mathcal{F}} \mathbf{y}^R x^F e^{-\beta \mathcal{Q}^2 \mathcal{P}} \right] \Big|_{\text{localization}}. \tag{4.2}$$

#### 4.1 $\text{O}(2N)$

First, we turn to  $\text{O}(2N)$  for  $2N \geq 4$ . For  $\Omega_{\mathcal{N}=4,8,16}^{\text{O}^-(2N)}$ , we made an explicit JK-residue evaluation as in the previous section. The insertion of  $\mathcal{P}$  can be represented by a  $Z_2$  holonomy along the Euclidean time circle,

$$\text{diag}_{2N \times 2N}(1, 1, \dots, 1, -1), \tag{4.3}$$

whereby the zero mode space shrinks by one dimension, so  $r = N - 1$  for  $\text{O}^-(2N)$ . The reduced zero modes,  $t_{1,2,\dots,N-1} = e^{2\pi i u_{1,2,\dots,N-1}}$ , parameterize  $\text{O}^-(2N)$  holonomy as

$$\begin{pmatrix} e^{2\pi i \sigma_2 u_1} & 0 & \dots & 0 & 0 \\ 0 & e^{2\pi i \sigma_2 u_2} & \dots & 0 & 0 \\ \dots & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & e^{2\pi i \sigma_2 u_{N-1}} & 0 \\ 0 & 0 & 0 & 0 & \sigma_3 \end{pmatrix},$$

which sets  $t_N = 1$  in  $g(t)$ . The  $N$ -th Cartan elements in all multiplets become massive, instead, and now contribute factors with the signs flipped, e.g., one of the  $N$  overall  $\mathbf{y} - \mathbf{y}^{-1}$  factors in the denominator for the Cartan is flipped to  $\mathbf{y} + \mathbf{y}^{-1}$ . See appendix B. However, we must caution against viewing this as a spontaneous symmetry breaking of the dynamics. Consider very long (Euclidean) time  $\beta$ . The ‘‘symmetry breaking effect’’ becomes diluted arbitrarily, as the size of the time-like gauge field scales with  $1/\beta$ . Moreover, at each time slice, this  $A_0$  can be gauged away, locally, and thus will not alter the dynamics. It is only when we are instructed to perform the trace, this  $\mathcal{P}$  makes a difference.

Finally, one needs to be careful about the usual division by the Weyl group when computing  $O^-(2N)$  contributions. Recall that the Weyl group of  $O^+(2N) = SO(2N)$  is

$$W_{SO(2N)} = S_N \times (\mathbb{Z}_2)^{N-1}, \tag{4.4}$$

with the latter factor representing the even number of sign flips. For the  $O^-(2N)$  sector of the path integral, the  $N$ -th zero mode is turned off and hence, the nontrivial permutation reduces to  $S_{N-1}$  while the effective number of sign-flips remains the same. We thus need to divide by

$$|S_{N-1} \times (\mathbb{Z}_2)^{N-1}| = 2^{N-1} \cdot (N-1)!, \tag{4.5}$$

instead of dividing by  $|W_{SO(2N)}| = 2^{N-1} \cdot N!$ . We warn the readers not to confuse these groups with the Weyl group of  $O(2N)$

$$W_{O(2N)} = S_N \times (\mathbb{Z}_2)^N, \tag{4.6}$$

which will enter the continuum interpretation of the rational pieces below. Just as in  $O^+(2N) = SO(2N)$ , the results for the twisted partition function for  $O(2N)$  can be organized physically, in terms of plane-wave-like states that explore the classical vacua. These plane waves will see all  $N$  Cartan directions as flat, even though in the localization computations one must regard the  $N$ -th as massive. This means that the continuum contributions to  $\Omega^{O(2N)}$  will take a similar form as those to  $\Omega^{SO(2N)}$  with  $W_{O(2N)}$  replacing  $W_{SO(2N)}$ . However,  $W_{O(2N)}$  itself does not enter the residue computation of  $\Omega_{\mathcal{N}}^{O^-(2N)}$  directly.

#### 4.1.1 $\mathcal{N} = 4, 8$

As in section 3, we present  $\mathcal{N} = 4, 8$  results first, and motivate how  $\mathcal{N} = 16$  continuum sectors should look like. This will enable us to decompose uniquely  $\mathcal{N} = 16$  results into the integral part and the rational parts, in much the same way as  $\Omega_{\mathcal{N}=16}^{O^+(2N)=SO(2N),s}$  were decomposed. Having computed  $\Omega_{\mathcal{N}=4,8}^{O^-(2N)}$  by a direct path integral evaluation, we again find the results can be all organized into the following simple expressions,

$$\Omega_{\mathcal{N}=4}^{O^-(2N)}(\mathbf{y}) = \frac{1}{|W_{SO(2N)}|} \sum_{\tilde{w}} \frac{1}{\det(\mathbf{y}^{-1} - \mathbf{y} \cdot \tilde{w}P)}. \tag{4.7}$$

The sum is now over the Weyl elements of  $SO(2N)$  such that

$$\det(1 - \tilde{w}P) \neq 0,$$

where  $P$  inside the determinant

$$P = \text{diag}_{N \times N}(1, 1, \dots, 1, -1)$$

is the representation of  $\mathcal{P}$  on the weight lattice of  $\text{SO}(2N)$ . In this note, we will call these  $\tilde{w}$ 's the *twisted* Elliptic Weyl elements.<sup>2</sup>

Why this happens is fairly clear in view of the heuristic arguments in section 3. The origin of  $\Omega_{\mathcal{N}=4,8}^G$  was understood as a result of the orbifolding of the asymptotic Cartan dynamics by the Weyl action, or equivalently via the insertion of the Weyl projection operator in the Hilbert space trace for  $\mathcal{O}(G)$ ,

$$\frac{1}{|W|} \sum_{\sigma \in W} \sigma .$$

Only the elliptic Weyl elements  $w$  with  $\det(1 - w) \neq 0$  contribute to  $\Omega$ , and produce

$$\frac{1}{|W|} \sum'_w \frac{1}{\det(\mathbf{y}^{-1} - \mathbf{y} \cdot w)} .$$

For  $\text{O}^-$ 's, the operator  $P$  multiplies on the right, so the only difference is that the Weyl projection for  $\text{O}^-(2N)$  is now shifted to

$$\frac{1}{|W_{\text{SO}(2N)}|} \sum_{\sigma \in W} \sigma P .$$

This leads to the modified sum (4.7), where  $w$  is replaced by  $\tilde{w} \cdot P$ . See appendix A for more details on Elliptic Weyl elements and *twisted* Elliptic Weyl elements.

Although we computed  $\text{O}^\pm(2N)$  sector contributions separately, the total partition function

$$\Omega_{\mathcal{N}=4}^{\text{O}(2N)}(\mathbf{y}) = \frac{1}{2} \left( \Omega_{\mathcal{N}=4}^{\text{SO}(2N)}(\mathbf{y}) + \Omega_{\mathcal{N}=4}^{\text{O}^-(2N)}(\mathbf{y}) \right) \tag{4.9}$$

can be more succinctly written as  $\Omega_{\mathcal{N}=4}^{\text{O}(2N)}(\mathbf{y}) = \Xi_{\mathcal{N}=4}^{(N)}$  with

$$\Xi_{\mathcal{N}=4}^{(N)} \equiv \frac{1}{|W^{(N)}|} \sum'_w \frac{1}{\det(\mathbf{y}^{-1} - \mathbf{y} \cdot w)} , \tag{4.10}$$

where the sum is now over elliptic Weyl elements of  $\text{O}(2N)$  and, likewise,  $W^{(N)} = W_{\text{O}(2N)}$ . This follows from the fact that  $P$  is a Weyl element of  $\text{O}(2N)$  which generates  $W_{\text{O}(2N)}/W_{\text{SO}(2N)}$ . The universal role played by elliptic Weyl elements is evident here again.

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<sup>2</sup>As an illustration, we list the first few for  $\Omega_{\mathcal{N}=4}^{\text{O}^-(2N)}(\mathbf{y})$ ,

$$\begin{aligned} \Omega_{\mathcal{N}=4}^{\text{O}^-(4)}(\mathbf{y}) &= \frac{1}{2} \cdot \frac{1}{\mathbf{y}^{-2} + \mathbf{y}^2} , \\ \Omega_{\mathcal{N}=4}^{\text{O}^-(6)}(\mathbf{y}) &= \frac{1}{24} \left[ \frac{8}{\mathbf{y}^{-3} + \mathbf{y}^3} + \frac{1}{(\mathbf{y}^{-1} + \mathbf{y})^3} \right] , \\ \Omega_{\mathcal{N}=4}^{\text{O}^-(8)}(\mathbf{y}) &= \frac{1}{16} \left[ \frac{4}{\mathbf{y}^{-4} + \mathbf{y}^4} + \frac{1}{(\mathbf{y}^{-2} + \mathbf{y}^2)(\mathbf{y}^{-1} + \mathbf{y})^2} \right] . \end{aligned} \tag{4.8}$$

As in the previous section,  $\mathcal{N} = 8$  is a straightforward extension of this, with additional factors from the single adjoint chiral multiplet,

$$\Omega_{\mathcal{N}=8}^{\text{O}^-(2N)}(\mathbf{y}, x) = \frac{1}{|W_{\text{SO}(2N)}|} \sum_{\tilde{w}} \frac{1}{\det(\mathbf{y}^{-1} - \mathbf{y} \cdot \tilde{w}P)} \cdot \frac{\det(\mathbf{y}^{-1}x^{1/2} - \mathbf{y}x^{-1/2} \cdot \tilde{w}P)}{\det(x^{1/2} - x^{-1/2} \cdot \tilde{w}P)}, \quad (4.11)$$

the simplest of which is

$$\Omega_{\mathcal{N}=8}^{\text{O}^-(4)}(\mathbf{y}, x) = \frac{1}{2} \cdot \frac{1}{\mathbf{y}^{-2} + \mathbf{y}^2} \cdot \frac{\mathbf{y}^{-2}x + \mathbf{y}^2x^{-1}}{x + x^{-1}}. \quad (4.12)$$

Again, we can write the total partition function as

$$\Omega_{\mathcal{N}=8}^{\text{O}(2N)}(\mathbf{y}, x) = \Xi_{\mathcal{N}=8}^{(N)} \equiv \frac{1}{|W^{(N)}|} \sum_w \frac{1}{\det(\mathbf{y}^{-1} - \mathbf{y} \cdot w)} \cdot \frac{\det(\mathbf{y}^{-1}x^{1/2} - \mathbf{y}x^{-1/2} \cdot w)}{\det(x^{1/2} - x^{-1/2} \cdot w)}, \quad (4.13)$$

where the sum is over elliptic Weyl elements of  $\text{O}(2N)$ .

#### 4.1.2 $\mathcal{N} = 16$

After computing  $\Omega_{\mathcal{N}=16}^{\text{O}^-(2N)}$ , we again wish to decompose it into the integral part and other rational parts from various continuum sectors. Our findings for  $\mathcal{N} = 4, 8$  imply that there are new types of continuum contributions that can enter  $\Omega_{\mathcal{N}=16}^{\text{O}^-(2N)}$ , of the form

$$\Delta_{\mathcal{N}=16}^{\text{O}^-(2N)} \equiv \frac{1}{|W_{\text{SO}(2N)}|} \sum_{\tilde{w}} \frac{1}{\det(\mathbf{y}^{-1} - \mathbf{y} \cdot \tilde{w}P)} \cdot \prod_{a=1}^3 \frac{\det(x^{F_a/2} \mathbf{y}^{R_a/2-1} - x^{-F_a/2} \mathbf{y}^{1-R_a/2} \cdot \tilde{w}P)}{\det(x^{F_a/2} \mathbf{y}^{R_a/2} - x^{-F_a/2} \mathbf{y}^{-R_a/2} \cdot \tilde{w}P)}, \quad (4.14)$$

where the sum is over the *twisted* elliptic Weyl elements of  $\text{SO}(2N)$ . For  $\Omega_{\mathcal{N}=16}^{\text{O}^-(2N)}$ , we can also have continuum contributions constructed from,

$$\Delta_{\mathcal{N}=16}^{\text{O}^-(2r+1)} = \Delta_{\mathcal{N}=16}^{\text{SO}(2r+1)}, \quad \text{for } r < N. \quad (4.15)$$

The reason for the equality is explained in next subsection.

Upon direct computations of the twisted partition functions, the analogs of (3.12) and (A.1) are found for  $\text{O}^-(2N)$  as follows

$$\begin{aligned} \Omega_{\mathcal{N}=16}^{\text{O}^-(4)} &= 1 + \Delta_{\mathcal{N}=16}^{\text{O}^-(4)}, \\ \Omega_{\mathcal{N}=16}^{\text{O}^-(6)} &= 1 + 3\Delta_{\mathcal{N}=16}^{\text{O}^-(3)} + 2\Delta_{\mathcal{N}=16}^{\text{SO}(3)} \cdot \Delta_{\mathcal{N}=16}^{\text{O}^-(3)} + \Delta_{\mathcal{N}=16}^{\text{O}^-(6)}, \\ \Omega_{\mathcal{N}=16}^{\text{O}^-(8)} &= 2 + 2\Delta_{\mathcal{N}=16}^{\text{O}^-(3)} + \Delta_{\mathcal{N}=16}^{\text{O}^-(5)} + \Delta_{\mathcal{N}=16}^{\text{SO}(3)} \cdot \Delta_{\mathcal{N}=16}^{\text{O}^-(4)} + \Delta_{\mathcal{N}=16}^{\text{O}^-(8)}. \end{aligned} \quad (4.16)$$

Note that the decomposition is unique.<sup>3</sup> The fact that each term on the right hand side has only one of the latter type factor is also reasonable, as at most one subgroup  $H$  would see the projection operator  $\mathcal{P}$ .

<sup>3</sup>Up to the accidental identity,  $\Delta_{\mathcal{N}=16}^{\text{O}^-(2)} = 2\Delta_{\mathcal{N}=16}^{\text{O}^\pm(3)}$ . See the subsection 4.3.

As with  $\mathcal{N} = 4, 8$ , the full partition function of  $O(2N)$  gauge theory can also be expressed in terms of the elliptic Weyl sums,

$$\Xi_{\mathcal{N}=16}^{(N)} \equiv \frac{1}{|W(N)|} \sum_w \frac{1}{\det(\mathbf{y}^{-1} - \mathbf{y} \cdot w)} \cdot \prod_{a=1}^3 \frac{\det(x^{F_a/2} \mathbf{y}^{R_a/2-1} - x^{-F_a/2} \mathbf{y}^{1-R_a/2} \cdot w)}{\det(x^{F_a/2} \mathbf{y}^{R_a/2} - x^{-F_a/2} \mathbf{y}^{-R_a/2} \cdot w)}, \quad (4.17)$$

as follows

$$\begin{aligned} \Omega_{\mathcal{N}=16}^{O(4)} &= 1 + \Xi_{\mathcal{N}=16}^{(1)} + \Xi_{\mathcal{N}=16}^{(2)}, \\ \Omega_{\mathcal{N}=16}^{O(6)} &= 1 + 2\Xi_{\mathcal{N}=16}^{(1)} + \Xi_{\mathcal{N}=16}^{(1)} \cdot \Xi_{\mathcal{N}=16}^{(1)} + \Xi_{\mathcal{N}=16}^{(3)}, \\ \Omega_{\mathcal{N}=16}^{O(8)} &= 2 + 3\Xi_{\mathcal{N}=16}^{(1)} + \Xi_{\mathcal{N}=16}^{(1)} \cdot \Xi_{\mathcal{N}=16}^{(1)} + 2\Xi_{\mathcal{N}=16}^{(2)} + \Xi_{\mathcal{N}=16}^{(2)} \cdot \Xi_{\mathcal{N}=16}^{(1)} + \Xi_{\mathcal{N}=16}^{(4)}. \end{aligned}$$

The partition functions of  $SO(2N)$  theories do not equal those of  $O(2N)$  theories,

$$\Omega_{\mathcal{N}=16}^{O(2N)} \neq \Omega_{\mathcal{N}=16}^{SO(2N)}, \quad (4.18)$$

yet we observe that the integral pieces that enumerate threshold bound states do agree between  $O(2N)$  and  $SO(2N)$ ,

$$\mathcal{I}_{\mathcal{N}=16}^{O(2N)} = \mathcal{I}_{\mathcal{N}=16}^{SO(2N)}. \quad (4.19)$$

Explicit computations have shown this latter identity for up to rank 4, and we believe this holds for all  $N$ .<sup>4</sup>

## 4.2 $O(2N + 1)$

One can similarly compute  $\Omega_{\mathcal{N}}^{O^-(2N+1)}$  for  $N \geq 1$  via HKY procedure, but in the end finds  $\Omega_{\mathcal{N}}^{O^-(2N+1)} = \Omega_{\mathcal{N}}^{O^+(2N+1)}$ . Perhaps the simplest way to understand this is to use a different form of  $\mathcal{P}$ ,

$$\text{diag}_{(2N+1) \times (2N+1)}(-1, -1, \dots, -1). \quad (4.20)$$

On representations with an even number of vector-like indices, such as the adjoint representation or symmetric 2-tensors, the action of  $\mathcal{P}$  is trivial. Neither the determinants nor the zero modes are affected by  $\mathcal{P}$ , so we find

$$\Omega_{\mathcal{N}}^{O(2N+1)} = \Omega_{\mathcal{N}}^{SO(2N+1)}, \quad (4.21)$$

for all  $N$  and all  $\mathcal{N} = 4, 8, 16$ . Consistent with this is the fact that the *twisted* elliptic Weyl elements  $\tilde{w}$  are in fact ordinary elliptic Weyl elements for the case of  $O(2N + 1)$ . This, from the trivial action of  $\mathcal{P}$  on the Cartan of  $SO(2N + 1)$ , implies that the decomposition into continuum sectors are also intact under the projection, leading us from (4.21) to

$$\mathcal{I}_{\mathcal{N}}^{O(2N+1)} = \mathcal{I}_{\mathcal{N}}^{SO(2N+1)}. \quad (4.22)$$

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<sup>4</sup>See section 5 for related discussions.



### 4.3 O(2) and O(1)

Let us close with two exceptional cases of O(2) and O(1). In the  $O^+(2) = SO(2)$  sector, the twisted partition function vanishes

$$\Omega_{\mathcal{N}=4,8,16}^{O^+(2)} = 0, \tag{4.23}$$

as all fields are charge-neutral and the determinant  $g(t)$  is independent of the gauge variable  $t$ ; the relevant JK-residue sum has to vanish identically, since we are supposed to pick up residue only from physical poles for these pure Yang-Mills quantum mechanics [5].

For the  $O^-(2)$  sector, however,  $t$  no longer appears as a zero mode, so there is no final residue integral to perform. The localization merely reduces to a product of determinants,

$$\begin{aligned} \Omega_{\mathcal{N}=4}^{O^-(2)} &= \frac{1}{\mathbf{y}^{-1} + \mathbf{y}} = 2\mathfrak{E}_{\mathcal{N}=4}^{(1)} \\ \Omega_{\mathcal{N}=8}^{O^-(2)} &= \frac{1}{\mathbf{y}^{-1} + \mathbf{y}} \cdot \frac{x^{1/2}\mathbf{y}^{-1} + x^{-1/2}\mathbf{y}}{x^{1/2} + x^{-1/2}} = 2\mathfrak{E}_{\mathcal{N}=8}^{(1)} \\ \Omega_{\mathcal{N}=16}^{O^-(2)} &= \frac{1}{\mathbf{y}^{-1} + \mathbf{y}} \cdot \prod_{a=1}^3 \frac{x^{F_a/2}\mathbf{y}^{R_a/2-1} + x^{-F_a/2}\mathbf{y}^{1-R_a/2}}{x^{F_a/2}\mathbf{y}^{R_a/2} + x^{-F_a/2}\mathbf{y}^{-R_a/2}} = 2\mathfrak{E}_{\mathcal{N}=16}^{(1)} \end{aligned} \tag{4.24}$$

and, in view of (4.23),

$$\Omega_{\mathcal{N}}^{O(2)} = \mathfrak{E}_{\mathcal{N}}^{(1)} \tag{4.25}$$

for each  $\mathcal{N} = 4, 8, 16$ . Since  $\mathfrak{E}$ 's are inherently of continuum contributions, this implies that not only for  $\mathcal{N} = 4, 8$  but also for  $\mathcal{N} = 16$ ,  $\Omega_{\mathcal{N}}^{O(2)}$ , the integral index vanishes,

$$\mathcal{I}_{\mathcal{N}=16}^{O(2)} = 0 = \mathcal{I}_{\mathcal{N}=16}^{SO(2)}. \tag{4.26}$$

Finally, O(1) means a single D0 trapped in O0. As such, even though the theory is empty literally, it still makes sense to assign,

$$\mathcal{I}_{\mathcal{N}=16}^{O(1)} = 1, \tag{4.27}$$

as the counting of a IIA quantum state. This, together with higher rank computations above, completes O( $m$ ) cases. This result may look a little odd in that, of all orientifold theories, the O(2) theory proves to be the only case with null Witten index. In the next section, we will explain this from a simple and elegant M-theory reasoning.

## 5 Witten index and M-theory on $\mathbb{R}^9/\mathbb{Z}_2$

Combining results of the previous two sections, and with help of some foresight [24], we end up with the following, rather compelling expressions as the generating functions,

$$1 + \sum_{n \geq 1} z^{2n} \mathcal{I}_{\mathcal{N}=16}^{Sp(n)} = \prod_{k=2,4,6,\dots} (1 + z^k), \tag{5.1}$$

$$1 + \sum_{m \geq 1} z^m \mathcal{I}_{\mathcal{N}=16}^{O(m)} = \prod_{k=1,3,5,\dots} (1 + z^k). \tag{5.2}$$

The two generating functions count the number of partitions of  $2n$  and  $m$  into, respectively, distinct even natural numbers and distinct odd natural numbers. Our path-integral computation confirmed this formulae up to  $2n = 8$  and  $m = 9$ , that is, up to nine D-particles in the covering space. Recall that  $O(2)$  is the only Orientifold theory with no bound states,  $\mathcal{I}_{\mathcal{N}=16}^{O(2)} = 0$ . We find the manner in which (5.2) realizes this  $m = 2$  result, quite compelling and elegant:  $m = 2$  is the only positive integer that cannot be expressed as a sum of distinct odd natural numbers.

A further evidence in favor of these generating functions can be found in ref. [19], which counted classical isolated vacua of mass-deformed theories instead. The mass deformation is easiest to see when  $\mathcal{N} = 16$  theory is viewed as  $\mathcal{N} = 4$  with three adjoint chirals and a particular trilinear superpotential  $\mathcal{W}$ . Adding a quadratic mass term to  $\mathcal{W}$ , one finds certain “distinguished” classical vacua which are cataloged by  $su(2)$  embedding, with trivial centralizers so that the solution is isolated. Kac and Smilga proposed the counting of such special subsets of classical vacua equals the true Witten index of the undeformed theory. Interestingly, this drastic approach had previously produced the desired results of  $\mathcal{I}_{\mathcal{N}=16}^{SU(N)} = 1$  [25].

Extending this to  $SO$  and  $Sp$  groups, Kac and Smilga found numbers which can be seen to be consistent with the generating functions as above. Since  $SO(m)$  theories and  $O(m)$  theories are different, one further needs to check  $\mathcal{I}^{SO(m)} = \mathcal{I}^{O(m)}$  for all  $m$ , but this equality follows easily: the classical vacua for the mass-deformed  $SO(m)$  theory can be thought of as a triplet of  $m \times m$  matrices forming a  $su(2)$  representation [19]. The defining representation of  $SO(m)$  is real, so only integral spins can enter, while the absence of centralizer demands these spins be distinct. Each partition of  $m$  into distinct odd natural numbers,

$$m = \sum k_s ; \quad k_s + 1 \in 2\mathbb{Z}_+, \quad k_s \neq k_{s'} \text{ if } s \neq s'$$

then gives a solution where the three adjoints are block-diagonal with  $k_s \times k_s$  blocks. The action of  $\mathcal{P}$  on such solutions is trivial, up to possible shift along  $SO(m)$  orbits, regardless of even or odd  $m$ , for the same reason as  $\mathcal{P}$  acts trivially on  $SO(2N + 1)$  pure Yang-Mills theories.

It has been observed by Hanany et al. [24] that spectrum of type (5.1) and (5.2) have a simple explanation in M-theory. For this, we must first go back to the story of M-theory on  $\mathbb{T}^{4p+1}/\mathbb{Z}_2$  originally due to Dasgupta and Mukhi [26].  $p = 0$  is the well-known Horava-Witten [27], while  $p = 1$  is relevant for  $D$ -type (2,0) theories and anomaly inflow thereof [28–30]. The lesser-known case of  $p = 2$  was also discussed, however, where the authors noted that the net anomaly after the projection can be canceled by a single chiral fermion supported at each fixed point. As first proposed in ref. [24], this implies certain spectrum of D-particle states at the Orientifold point  $\mathbb{R}^9/\mathbb{Z}_2$ . Upon a further  $S^1$  compactification, the fixed point will become a IIA orientifold point, and at this point the chiral fermion will generate infinite towers of harmonic oscillators, with either integral or half-integral KK momenta, depending on a choice of the spin structure.

With the anti-periodic spin structure, we have fermionic harmonic oscillators  $b_{k/2}, b_{k/2}^\dagger$  with odd  $k$ 's. The Hilbert space built out of these, with positive KK momenta  $k/2$  has the partition function of the second type above, i.e., (5.2). An even number of oscillators corresponds to  $O(2N)$  cases of  $O0^-$  while an odd number of oscillators corresponds to  $O(2N+1)$  cases of  $\widetilde{O0^-}$ . With the periodic boundary conditions, we have  $b_{k/2}, b_{k/2}^\dagger$  with even  $k$ 's, instead, so this would lead to partition function of the first type, i.e., (5.1). With periodic spin structure, the zero mode  $b_0, b_0^\dagger$  also appear, meaning that there are actually two towers, built on either the vacuum  $|0\rangle$  or on  $b_0^\dagger|0\rangle$ . It looks reasonable that we associated these two towers with  $O0^+$  and  $\widetilde{O0^+}$ , respectively.<sup>5</sup> The correspondence is complete once we recall that  $2n$  and  $m$  are the D-particle charges in the covering space and must be divided by 2. These four towers also explain neatly the four possible types of  $O0$ 's.

There are a few noteworthy facts. First, apart from the anti-D0 towers due to oscillators with negative  $k$ 's, there are additional states with positive and negative  $k$  oscillators mixed. These correspond to mixture of D0 and anti-D0 from the standard M/IIA duality, and a pair annihilation must occur to reduce them to collection of either D0 and anti-D0 only. The relevant coupling involves the closed string multiplet in the bulk, as the energy must be radiated away to transverse space. With nothing that prevents the necessary couplings, the above four towers we reproduced from D0-O0 perspective are the only stable states from these free fermions.

Second, each of these stable states is, for any such collection of  $k$ 's of the same sign, a single quantum state rather than a supermultiplet. Although this may sound strange given the extensive supersymmetry, there is really no contradiction as these states are strictly one-dimensional. Supersymmetry does not always imply an on-shell supermultiplet for quantum mechanical degrees of freedom. Recall that the usual D0 problem in the flat IIA case is governed by  $U(N) = U(1) \times SU(N)$ , and  $U(1)$  is responsible for  $\mathbb{R}^9$  center of mass degrees of freedom and the BPS multiplet structure of 256. In the orientifold analog, this  $U(1)$  is projected out, which is consistent with the fact that  $O0$  breaks the spatial translational invariance completely.

Finally, the number of states at a given large D-particle quantum number  $k$  seems to grow pretty fast with  $k$ . For example, the number of threshold bound states in  $Sp(n)$  case equals to the number of distinct partitions of  $n$ , with the known asymptotic formula [32],

$$\frac{1}{4 \cdot 3^{1/4} \cdot n^{3/4}} \exp\left(\pi\sqrt{n/3}\right) + \dots \quad (5.3)$$

This exponential growth is a straightforward consequence of the single chiral fermion along the M-theory circle at the origin of the IIA theory. Whether this has other physical consequences remains to be explored.

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<sup>5</sup>For D0-O0 quantum mechanics and indices thereof, the distinction is clearly lost. The distinction between  $Dp\text{-}Op^+$  and  $Dp\text{-}\widetilde{Op}^+$  survives the field theory limit for  $p = 4$ , only thanks to  $\pi_4(Sp(n)) = \mathbb{Z}_2$ , while already for  $p = 3$  the two field theories are continuously connected via a  $2\pi$  shift of the  $\theta$ -angle [31].

## 6 Toward rational invariants for orientifolds

For  $\mathcal{N} = 4$  quiver theories based on  $U(N)$ -type gauge groups,<sup>6</sup> it has been observed that there is a universal relationship between  $\Omega$ 's and  $\mathcal{I}$ 's of the form,

$$\Omega_\Gamma(\mathbf{y}) = \sum_{N|\Gamma} \frac{1}{N} \cdot \frac{\mathbf{y}^{-1} - \mathbf{y}}{\mathbf{y}^{-N} - \mathbf{y}^N} \cdot \mathcal{I}_{\Gamma/N}(\mathbf{y}^N) \tag{6.1}$$

where the sum is over possible divisor  $N$  of the quiver  $\Gamma$  [5], in the sense that  $\Gamma/N$  is the same quiver except the rank vector is divided by  $N$ . Not only is this structure evident in the final answers but also in the computational middle steps as well, and is thus quite ubiquitous in counting problems in the wall-crossing [20, 33, 34]. The object of type (6.1), prior to being identified as the twisted partition functions [5], was also known as the rational invariants for the obvious reason. Note that the universal factor

$$\frac{1}{N} \cdot \frac{\mathbf{y}^{-1} - \mathbf{y}}{\mathbf{y}^{-N} - \mathbf{y}^N} \tag{6.2}$$

in this expression coincides with  $\Omega_{\mathcal{N}=4}^{\text{SU}(N)}$ , and carries the continuum contribution from a plane-wave sector of  $N$ -identical 1-particle-like states. This is because the continuum sector in question resides in the Coulomb branch, and, as such, any other  $\mathcal{N} = 4$   $U(N)$  type quiver theory with Coulombic flat directions can receive the same type of contributions. Universality of this begs for the question whether there is an analog of this rational structure for D-brane theories with Orientifolds.

Indeed, one of the most tantalizing outcome is the ‘‘orientifolded’’ version of (6.2)

$$\Xi_{\mathcal{N}}^{(N)}$$

precisely defined in (4.10), (4.13), and (4.17), as building blocks for  $\Omega_{\mathcal{N}}^G$  for orthogonal and symplectic groups. These functions  $\Xi_{\mathcal{N}}^{(N)}$  appear universally for these theories, simply because  $O(2N)$ ,  $O(2N + 1)$ , and  $Sp(N)$  share a common Weyl group;

$$W_{O(2N)} = W_{O(2N+1)} = W_{Sp(N)} = W^{(N)} \equiv S_N \times (\mathbb{Z}_2)^N .$$

One difference of  $\Xi_{\mathcal{N}=4}^{(N)}$  from the above  $U(N)$  version (6.2) is that  $\Xi_{\mathcal{N}}^{(N)}$  has increasing large number of linearly independent terms, due to large number of contributing conjugacy classes. Another complication is that, as we saw in various  $\mathcal{N} = 16$  Orientifolded theories, the continuum sectors are no longer constrained to sectors with identical partial bound states.

We note here that at least the first issue has a simple and elegant solution;  $\Xi_{\mathcal{N}}^{(N)}$ , even though they look individually quite complicated, can be all constructed from a single function  $\Xi_{\mathcal{N}}^{(1)}$ . Introducing

$$\chi_{\mathcal{N}}^{(n)}(\mathbf{y}, \dots) \equiv \Xi_{\mathcal{N}}^{(1)}(\mathbf{y}^n, \dots) \tag{6.3}$$

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<sup>6</sup>This has been extensively tested in the class of quivers where 1-cycles and of 2-cycles are absent, meaning absence of adjoint chirals and of complex conjugate pairs.

where the ellipsis on the left hand side denotes other possible equivariant parameters, while the one on the right hand side denotes the same parameters raised to the  $n$ -th power,  $\Xi_{\mathcal{N}}^{(N)}$  can be seen to be sums of products of  $\chi_{\mathcal{N}}^{(n)}$  with contributing  $n$ 's sum to  $N$ . One then finds the generating functions,

$$1 + \sum_{N=1}^{\infty} q^N \cdot \Xi_{\mathcal{N}}^{(N)} = \text{Exp} \left( \sum_{k=1}^{\infty} \frac{q^k}{k} \chi_{\mathcal{N}}^{(k)} \right) = \text{P.E.} \left[ q \cdot \chi_{\mathcal{N}}^{(1)} \right] \tag{6.4}$$

for all  $\mathcal{N}$ , where P.E. is the Plethystic Exponential [35]. We expect that these quantities, term by term in  $q$ -expansion, should play a role similar to (6.2), now for Orientifolded quiver theories.

We are not aware of a general answer to the second complication, yet. Trivial examples, in this sense, are  $\mathcal{N} = 4, 8$  Orientifold theories, partition functions of which can be paraphrased as

$$1 + \sum_{N=1}^{\infty} q^N \Omega_{\mathcal{N}=4}^{G_N}(\mathbf{y}) = \text{P.E.} \left[ \frac{q}{2(\mathbf{y}^{-1} + \mathbf{y})} \right], \tag{6.5}$$

and

$$1 + \sum_{N=1}^{\infty} q^N \Omega_{\mathcal{N}=8}^{G_N}(\mathbf{y}, x) = \text{P.E.} \left[ \frac{q}{2(\mathbf{y}^{-1} + \mathbf{y})} \cdot \frac{x^{1/2} \mathbf{y}^{-1} + x^{-1/2} \mathbf{y}}{x^{1/2} + x^{-1/2}} \right], \tag{6.6}$$

common for  $G_N = \text{O}(2N)$ ,  $\text{O}(2N + 1)$ , or  $\text{Sp}(N)$ . But the analog of (6.1) for general Orientifolded quiver theories, which may have nontrivial ground states, is yet another matter. Even for  $\mathcal{N} = 16$  theories computed in this note, we are yet to find a closed form of generating functions, inclusive of all ranks. We wish to come back to the problem of finding generic Orientifold version of the rational invariants in near future.

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### A Elliptic Weyl elements and rational invariants

An elliptic element  $w$  of Weyl group  $W$  is defined by absence of eigenvalue 1 in the canonical representation of  $W$  on the weight lattice.

For  $\text{SU}(N)$ , the Weyl group  $S_N$  is a little special because the rank is actually  $N - 1$ . The only elliptic Weyl's are the fully cyclic ones, say,  $(123 \cdots N)$  and all of these belong to a single conjugacy class. For  $\text{SO}(2N)$ ,  $\text{SO}(2N + 1)$ , and  $\text{Sp}(N)$  groups, the Weyl groups

are  $S_N$  semi-direct-product with  $(Z_2)^{N-1}$ ,  $(Z_2)^N$ , and  $(Z_2)^N$ , respectively. The elements can be therefore represented as follows

$$\sigma = (abc\dot{d}\dots)(klm\dot{n}\dots)\dots$$

where dots above a number indicate a sign flip. For example  $(12\dot{3})$  represents the element,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} .$$

In this form, the above  $(Z_2)^{N-1}$  for  $SO(2N)$  means that the total number of sign flip has to be even. Since the determinant factorizes upon the above decomposition of  $w$ , this should be true for each cyclic component. It is fairly easy to see that this requires each cyclic component of  $w$  to have an odd number of sign flips.

Let us list the conjugacy classes of elliptic Weyl elements for classical groups, for some low rank cases, from which the pattern should be quite obvious,

- $SU(N)$ 

$$(123\dots N)$$
- $SO(4)$ 

$$(\dot{1})(\dot{2})$$
- $SO(5)$  and  $Sp(2)$ 

$$(1\dot{2}), (\dot{1})(\dot{2})$$
- $SO(6)$ 

$$(1\dot{2})(\dot{3})$$
- $SO(7)$  and  $Sp(3)$ 

$$(\dot{1}\dot{2}\dot{3}), (12\dot{3}), (1\dot{2})(\dot{3}), (\dot{1})(\dot{2})(\dot{3})$$
- $SO(8)$ 

$$(\dot{1}\dot{2}\dot{3})(\dot{4}), (12\dot{3})(\dot{4}), (1\dot{2})(3\dot{4}), (\dot{1})(\dot{2})(\dot{3})(\dot{4})$$
- $SO(9)$  and  $Sp(4)$ 

$$(1\dot{2}\dot{3}\dot{4}), (123\dot{4}), (\dot{1}\dot{2}\dot{3})(\dot{4}), (12\dot{3})(\dot{4}), (1\dot{2})(3\dot{4}), (1\dot{2})(\dot{3})(\dot{4}), (\dot{1})(\dot{2})(\dot{3})(\dot{4})$$

We may classify the twisted elliptic Weyl elements,  $\tilde{w}$ , for  $O(m)$ 's, similarly. We take this to be defined by absence of eigenvalue 1 in  $\tilde{w} \cdot P$  where  $\tilde{w}$  is an element of  $W_{SO(m)}$ . One immediate fact is that the underlying action of  $\mathcal{P}$  is trivial on the root lattice of  $SO(2N+1)$ , so for  $SO(2N+1)$ , the elliptic Weyl elements coincide with the twisted elliptic Weyl elements. This is, in retrospect, another reason behind why  $\Omega^{O^-(2N+1)} = \Omega^{O^+(2N+1)}$  and hence  $\Omega^{O(2N+1)} = \Omega^{SO(2N+1)}$ . For  $O(2N)$ , however,  $\mathcal{P}$  flips an odd number of Cartan's,

Using the same notation as above, we can then classify the conjugacy classes of  $\tilde{w} \cdot P$  as follows,

- $O^-(4)$ 

$$(1\dot{2})$$
- $O^-(6)$ 

$$(1\dot{2}\dot{3}), (12\dot{3}), (\dot{1})(\dot{2})(\dot{3})$$
- $O^-(8)$ 

$$(1\dot{2}\dot{3}\dot{4}), (123\dot{4}), (\dot{1})(\dot{2})(\dot{3})(\dot{4}),$$
- $O^-(10)$ 

$$(1\dot{2}\dot{3}\dot{4}\dot{5}), (12\dot{3}\dot{4}\dot{5}), (1234\dot{5}),$$

$$(\dot{1}\dot{2}\dot{3})(\dot{4})(\dot{5}), (1\dot{2}\dot{3})(\dot{4})(\dot{5}), (\dot{1}\dot{2})(\dot{3}\dot{4})(\dot{5}), (\dot{1})(\dot{2})(\dot{3})(\dot{4})(\dot{5})$$

Note that  $P$  is in fact nothing but the generator of  $W_{O(2N)}/W_{SO(2N)} = Z_2$ . Therefore, one can also think of  $\tilde{w} \cdot P$  as elliptic Weyl elements of  $O(2N)$  which are not in  $W_{SO(2N)}$ . In particular, this means that  $W_{O(2N)} = W_{O(2N+1)} = W_{Sp(N)}$  and the the respective elliptic Weyl elements also coincide.

### A.1 $\Omega_{\mathcal{N}=16}^G$ with simple and connected $G$

We list results for twisted partition functions with  $\mathcal{N} = 16$ , from ref. [5];

$$\begin{aligned}
 \Omega_{\mathcal{N}=16}^{SO(4)} &= 1 + 2\Delta_{\mathcal{N}=16}^{SO(3)} + \Delta_{\mathcal{N}=16}^{SO(4)}, & (A.1) \\
 \Omega_{\mathcal{N}=16}^{SO(5)} &= 1 + 2\Delta_{\mathcal{N}=16}^{SO(3)=Sp(1)} + \Delta_{\mathcal{N}=16}^{SO(5)=Sp(2)} = \Omega_{\mathcal{N}=16}^{Sp(2)}, \\
 \Omega_{\mathcal{N}=16}^{G_2} &= 2 + 2\Delta_{\mathcal{N}=16}^{SU(2)} + \Delta_{\mathcal{N}=16}^{G_2}, \\
 \Omega_{\mathcal{N}=16}^{SO(6)} &= 1 + \Delta_{\mathcal{N}=16}^{SO(3)} + \Delta_{\mathcal{N}=16}^{SO(6)}, \\
 \Omega_{\mathcal{N}=16}^{SO(7)} &= 1 + 3\Delta_{\mathcal{N}=16}^{SO(3)} + \left(\Delta_{\mathcal{N}=16}^{SO(3)}\right)^2 + \Delta_{\mathcal{N}=16}^{SO(5)} + \Delta_{\mathcal{N}=16}^{SO(7)}, \\
 \Omega_{\mathcal{N}=16}^{Sp(3)} &= 2 + 3\Delta_{\mathcal{N}=16}^{Sp(1)} + \left(\Delta_{\mathcal{N}=16}^{Sp(1)}\right)^2 + \Delta_{\mathcal{N}=16}^{Sp(2)} + \Delta_{\mathcal{N}=16}^{Sp(3)}, \\
 \Omega_{\mathcal{N}=16}^{SO(8)} &= 2 + 4\Delta_{\mathcal{N}=16}^{SO(3)} + 2\left(\Delta_{\mathcal{N}=16}^{SO(3)}\right)^2 + \left(\Delta_{\mathcal{N}=16}^{SO(3)}\right)^3 + 3\Delta_{\mathcal{N}=16}^{SO(5)} + \Delta_{\mathcal{N}=16}^{SO(8)}, \\
 \Omega_{\mathcal{N}=16}^{SO(9)} &= 2 + 4\Delta_{\mathcal{N}=16}^{SO(3)} + 2\left(\Delta_{\mathcal{N}=16}^{SO(3)}\right)^2 + 2\Delta_{\mathcal{N}=16}^{SO(5)} + \Delta_{\mathcal{N}=16}^{SO(3)} \cdot \Delta_{\mathcal{N}=16}^{SO(5)} + \Delta_{\mathcal{N}=16}^{SO(7)} + \Delta_{\mathcal{N}=16}^{SO(9)}, \\
 \Omega_{\mathcal{N}=16}^{Sp(4)} &= 2 + 5\Delta_{\mathcal{N}=16}^{Sp(1)} + 2\left(\Delta_{\mathcal{N}=16}^{Sp(1)}\right)^2 + 2\Delta_{\mathcal{N}=16}^{Sp(2)} + \Delta_{\mathcal{N}=16}^{Sp(1)} \cdot \Delta_{\mathcal{N}=16}^{Sp(2)} + \Delta_{\mathcal{N}=16}^{Sp(3)} + \Delta_{\mathcal{N}=16}^{Sp(4)},
 \end{aligned}$$

where  $\Delta$ 's are defined in (3.10). As with  $SU(N)$  case in (3.12), these decompositions are unique.

## A.2 Common building blocks for orthogonal and symplectic groups

Since the Weyl groups of  $O(2N)$ ,  $O(2N+1)$ , and  $Sp(N)$  coincide, the quantities defined in (4.10), (4.13), and (4.17) are common to all three classes of the gauge groups. These can be classified by the rank alone, without reference to the type of orientifolding projection, suggesting universal building blocks for continuum contributions. Here we list a few low rank examples of  $\Xi_{\mathcal{N}=4}^{(N)}(\mathbf{y})$  of (4.10);

- rank 1

$$\frac{1}{2} \frac{1}{\mathbf{y}^{-1} + \mathbf{y}} \quad (\text{A.2})$$

- rank 2

$$\frac{1}{8} \left[ \frac{2}{\mathbf{y}^{-2} + \mathbf{y}^2} + \frac{1}{(\mathbf{y}^{-1} + \mathbf{y})^2} \right] \quad (\text{A.3})$$

- rank 3

$$\frac{1}{48} \left[ \frac{8}{\mathbf{y}^{-3} + \mathbf{y}^3} + \frac{6}{(\mathbf{y}^{-2} + \mathbf{y}^2)(\mathbf{y}^{-1} + \mathbf{y})} + \frac{1}{(\mathbf{y}^{-1} + \mathbf{y})^3} \right] \quad (\text{A.4})$$

- rank 4

$$\frac{1}{384} \left[ \frac{48}{\mathbf{y}^{-4} + \mathbf{y}^4} + \frac{32}{(\mathbf{y}^{-3} + \mathbf{y}^3)(\mathbf{y}^{-1} + \mathbf{y})} + \frac{12}{(\mathbf{y}^{-2} + \mathbf{y}^2)^2} + \frac{12}{(\mathbf{y}^{-2} + \mathbf{y}^2)(\mathbf{y}^{-1} + \mathbf{y})^2} + \frac{1}{(\mathbf{y}^{-1} + \mathbf{y})^4} \right] \quad (\text{A.5})$$

- rank 5

$$\frac{1}{3840} \left[ \frac{384}{\mathbf{y}^{-5} + \mathbf{y}^5} + \frac{240}{(\mathbf{y}^{-4} + \mathbf{y}^4)(\mathbf{y}^{-1} + \mathbf{y})} + \frac{160}{(\mathbf{y}^{-3} + \mathbf{y}^3)(\mathbf{y}^{-2} + \mathbf{y}^2)} + \frac{80}{(\mathbf{y}^{-3} + \mathbf{y}^3)(\mathbf{y}^{-1} + \mathbf{y})^2} + \frac{60}{(\mathbf{y}^{-2} + \mathbf{y}^2)^2(\mathbf{y}^{-1} + \mathbf{y})} + \frac{20}{(\mathbf{y}^{-2} + \mathbf{y}^2)(\mathbf{y}^{-1} + \mathbf{y})^3} + \frac{1}{(\mathbf{y}^{-1} + \mathbf{y})^5} \right] \quad (\text{A.6})$$

Elevating these to building blocks of  $\mathcal{N} = 8, 16$  orientifolded theories is a matter of attaching chiral field contributions to each linearly-independent rational pieces, as in (4.13) and in (4.17).  $\Omega_{\mathcal{N}=4,8}$  and  $\Delta_{\mathcal{N}=16}$ 's are related simply to these as

$$\Xi_{\mathcal{N}=4,8}^{(N)} = \Omega_{\mathcal{N}=4,8}^{O(2N)} = \Omega_{\mathcal{N}=4,8}^{O(2N+1)} = \Omega_{\mathcal{N}=4,8}^{SO(2N+1)} = \Omega_{\mathcal{N}=4,8}^{Sp(N)}, \quad (\text{A.7})$$

and

$$\Xi_{\mathcal{N}=16}^{(N)} = \Delta_{\mathcal{N}=16}^{O(2N)} = \Delta_{\mathcal{N}=16}^{O(2N+1)} = \Delta_{\mathcal{N}=16}^{SO(2N+1)} = \Delta_{\mathcal{N}=16}^{Sp(N)}. \quad (\text{A.8})$$



### A.3 $\Omega_{\mathcal{N}=16}^G$ for D-particles on an orientifold point

Although there is a universal form (4.17) of continuum contributions to  $\mathcal{N} = 16$  theories with an Orientifold point, the actual partition functions and the indices differ among  $O(2N)$ ,  $O(2N + 1)$ , and  $Sp(N)$  groups. Here we list all three series, for comparison, although  $O(2N + 1)$  and  $Sp(N)$  cases were already shown in section A.1 in a different notation;

$$\Omega_{\mathcal{N}=16}^{O(2)} = 0 + \Xi_{\mathcal{N}=16}^{(1)}, \quad (\text{A.9})$$

$$\Omega_{\mathcal{N}=16}^{O(4)} = 1 + \Xi_{\mathcal{N}=16}^{(1)} + \Xi_{\mathcal{N}=16}^{(2)},$$

$$\Omega_{\mathcal{N}=16}^{O(6)} = 1 + 2\Xi_{\mathcal{N}=16}^{(1)} + \left(\Xi_{\mathcal{N}=16}^{(1)}\right)^2 + \Xi_{\mathcal{N}=16}^{(3)},$$

$$\Omega_{\mathcal{N}=16}^{O(8)} = 2 + 3\Xi_{\mathcal{N}=16}^{(1)} + \left(\Xi_{\mathcal{N}=16}^{(1)}\right)^2 + 2\Xi_{\mathcal{N}=16}^{(2)} + \Xi_{\mathcal{N}=16}^{(1)} \cdot \Xi_{\mathcal{N}=16}^{(2)} + \Xi_{\mathcal{N}=16}^{(4)},$$

$$\Omega_{\mathcal{N}=16}^{O(3)} = 1 + \Xi_{\mathcal{N}=16}^{(1)}, \quad (\text{A.10})$$

$$\Omega_{\mathcal{N}=16}^{O(5)} = 1 + 2\Xi_{\mathcal{N}=16}^{(1)} + \Xi_{\mathcal{N}=16}^{(2)},$$

$$\Omega_{\mathcal{N}=16}^{O(7)} = 1 + 3\Xi_{\mathcal{N}=16}^{(1)} + \left(\Xi_{\mathcal{N}=16}^{(1)}\right)^2 + \Xi_{\mathcal{N}=16}^{(2)} + \Xi_{\mathcal{N}=16}^{(3)},$$

$$\Omega_{\mathcal{N}=16}^{O(9)} = 2 + 4\Xi_{\mathcal{N}=16}^{(1)} + 2\left(\Xi_{\mathcal{N}=16}^{(1)}\right)^2 + 2\Xi_{\mathcal{N}=16}^{(2)} + \Xi_{\mathcal{N}=16}^{(1)} \cdot \Xi_{\mathcal{N}=16}^{(2)} + \Xi_{\mathcal{N}=16}^{(3)} + \Xi_{\mathcal{N}=16}^{(4)},$$

$$\Omega_{\mathcal{N}=16}^{Sp(1)} = 1 + \Xi_{\mathcal{N}=16}^{(1)}, \quad (\text{A.11})$$

$$\Omega_{\mathcal{N}=16}^{Sp(2)} = 1 + 2\Xi_{\mathcal{N}=16}^{(1)} + \Xi_{\mathcal{N}=16}^{(2)},$$

$$\Omega_{\mathcal{N}=16}^{Sp(3)} = 2 + 3\Xi_{\mathcal{N}=16}^{(1)} + \left(\Xi_{\mathcal{N}=16}^{(1)}\right)^2 + \Xi_{\mathcal{N}=16}^{(2)} + \Xi_{\mathcal{N}=16}^{(3)},$$

$$\Omega_{\mathcal{N}=16}^{Sp(4)} = 2 + 5\Xi_{\mathcal{N}=16}^{(1)} + 2\left(\Xi_{\mathcal{N}=16}^{(1)}\right)^2 + 2\Xi_{\mathcal{N}=16}^{(2)} + \Xi_{\mathcal{N}=16}^{(1)} \cdot \Xi_{\mathcal{N}=16}^{(2)} + \Xi_{\mathcal{N}=16}^{(3)} + \Xi_{\mathcal{N}=16}^{(4)}.$$

## B Integrand for the $O^-(2N)$

The determinant  $g_{O^-(2N)}(t)$  that appears in the localization formula (3.1) for the twisted partition function of the  $O^-(2N)$  pure Yang-Mills theory can be obtained by modifying the following  $O^+(2N)$  counterpart,

$$\begin{aligned} g_{O^+(2N)}(t) &= \left(\frac{1}{\mathbf{y} - \mathbf{y}^{-1}}\right)^N \cdot \prod_a \left(\frac{x^{-F_a/2} \mathbf{y}^{-(R_a/2-1)} - x^{F_a/2} \mathbf{y}^{R_a/2-1}}{x^{F_a/2} \mathbf{y}^{R_a/2} - x^{-F_a/2} \mathbf{y}^{-R_a/2}}\right)^N \\ &\quad \times \prod_\alpha \frac{t^{-\alpha/2} - t^{\alpha/2}}{t^{\alpha/2} \mathbf{y}^{-1} - t^{-\alpha/2} \mathbf{y}} \cdot \prod_a \prod_\alpha \frac{t^{-\alpha/2} x^{-F_a/2} \mathbf{y}^{-(R_a/2-1)} - t^{\alpha/2} x^{F_a/2} \mathbf{y}^{R_a/2-1}}{t^{\alpha/2} x^{F_a/2} \mathbf{y}^{R_a/2} - t^{-\alpha/2} x^{-F_a/2} \mathbf{y}^{-R_a/2}} \\ &= \left(\frac{1}{\mathbf{y} - \mathbf{y}^{-1}}\right)^N \cdot \prod_a \left(\frac{\mathbf{y}^{-(R_a/2-1)} - x^{F_a} \mathbf{y}^{R_a/2-1}}{x^{F_a} \mathbf{y}^{R_a/2} - \mathbf{y}^{-R_a/2}}\right)^N \\ &\quad \times \prod_\alpha \frac{1 - t^\alpha}{t^\alpha \mathbf{y}^{-1} - \mathbf{y}} \cdot \prod_a \prod_\alpha \frac{\mathbf{y}^{-(R_a/2-1)} - t^\alpha x^{F_a} \mathbf{y}^{R_a/2-1}}{t^\alpha x^{F_a} \mathbf{y}^{R_a/2} - \mathbf{y}^{-R_a/2}}, \end{aligned} \quad (\text{B.1})$$

so that the parity action is appropriately taken into account. Here,  $\alpha$ 's are the roots of  $\text{SO}(2N)$  and  $a$ 's label the 0, 1, and 3 adjoint chiral multiplets for  $\mathcal{N} = 4, 8,$  and 16 theories, respectively. With the parity represented as in eq. (4.3),

$$\text{diag}_{2N \times 2N}(1, 1, \dots, 1, -1), \quad (\text{B.2})$$

the  $N$ -th zero mode is frozen to  $t_N = 1$  and some of the one-loop determinants relevant to the  $N$ -th Cartan  $\text{U}(1)$  undergo appropriate sign flips as described in the paragraph including eq. (4.3). The determinant  $g_{\text{O}^-(2N)}(t)$  is then a function of the  $N - 1$  zero modes,  $t = \{t_1, \dots, t_{N-1}\}$ , and can be written as

$$\begin{aligned} g_{\text{O}^-(2N)}(t) &= g_{\text{O}^+(2N-2)}(t) \cdot \frac{1}{\mathbf{y} + \mathbf{y}^{-1}} \cdot \prod_a \frac{\mathbf{y}^{-(Ra/2-1)} + x^{Fa} \mathbf{y}^{Ra/2-1}}{x^{Fa} \mathbf{y}^{Ra/2} + \mathbf{y}^{-Ra/2}} \\ &\times \prod_{i=1}^{N-1} \frac{1 - t_i}{t_i \mathbf{y}^{-1} - \mathbf{y}} \frac{1 + t_i}{t_i \mathbf{y}^{-1} + \mathbf{y}} \cdot \prod_{i=1}^{N-1} \frac{1 - t_i^{-1}}{t_i^{-1} \mathbf{y}^{-1} - \mathbf{y}} \frac{1 + t_i^{-1}}{t_i^{-1} \mathbf{y}^{-1} + \mathbf{y}} \\ &\times \prod_a \prod_{i=1}^{N-1} \frac{\mathbf{y}^{-(Ra/2-1)} - t_i x^{Fa} \mathbf{y}^{Ra/2-1}}{t_i x^{Fa} \mathbf{y}^{Ra/2} - \mathbf{y}^{-Ra/2}} \frac{\mathbf{y}^{-(Ra/2-1)} + t_i x^{Fa} \mathbf{y}^{Ra/2-1}}{t_i x^{Fa} \mathbf{y}^{Ra/2} + \mathbf{y}^{-Ra/2}} \\ &\times \prod_a \prod_{i=1}^{N-1} \frac{\mathbf{y}^{-(Ra/2-1)} - t_i^{-1} x^{Fa} \mathbf{y}^{Ra/2-1}}{t_i^{-1} x^{Fa} \mathbf{y}^{Ra/2} - \mathbf{y}^{-Ra/2}} \frac{\mathbf{y}^{-(Ra/2-1)} + t_i^{-1} x^{Fa} \mathbf{y}^{Ra/2-1}}{t_i^{-1} x^{Fa} \mathbf{y}^{Ra/2} + \mathbf{y}^{-Ra/2}}, \end{aligned} \quad (\text{B.3})$$

where the expression for  $g_{\text{O}^+(2N-2)}(t)$  can be read from eq. (B.1).

For an illustration, we list below the determinants for the  $\text{O}^-(4)$  theories with  $\mathcal{N} = 4, 8,$  and 16:

$$\begin{aligned} g_{\text{O}^-(4)}^{\mathcal{N}=4}(t_1) &= \frac{1}{\mathbf{y} - \mathbf{y}^{-1}} \cdot \frac{1}{\mathbf{y} + \mathbf{y}^{-1}} \\ &\times \frac{1 - t_1}{t_1 \mathbf{y}^{-1} - \mathbf{y}} \cdot \frac{1 + t_1}{t_1 \mathbf{y}^{-1} + \mathbf{y}} \cdot \frac{1 - t_1^{-1}}{t_1^{-1} \mathbf{y}^{-1} - \mathbf{y}} \cdot \frac{1 + t_1^{-1}}{t_1^{-1} \mathbf{y}^{-1} + \mathbf{y}}, \\ g_{\text{O}^-(4)}^{\mathcal{N}=8}(t_1) &= \frac{1}{\mathbf{y} - \mathbf{y}^{-1}} \cdot \frac{1}{\mathbf{y} + \mathbf{y}^{-1}} \cdot \frac{\mathbf{y} - x \mathbf{y}^{-1}}{x - 1} \cdot \frac{\mathbf{y} + x \mathbf{y}^{-1}}{x + 1} \\ &\times \frac{1 - t_1}{t_1 \mathbf{y}^{-1} - \mathbf{y}} \cdot \frac{1 + t_1}{t_1 \mathbf{y}^{-1} + \mathbf{y}} \cdot \frac{1 - t_1^{-1}}{t_1^{-1} \mathbf{y}^{-1} - \mathbf{y}} \cdot \frac{1 + t_1^{-1}}{t_1^{-1} \mathbf{y}^{-1} + \mathbf{y}} \\ &\times \frac{\mathbf{y} - t_1 x \mathbf{y}^{-1}}{t_1 x - 1} \cdot \frac{\mathbf{y} + t_1 x \mathbf{y}^{-1}}{t_1 x + 1} \cdot \frac{\mathbf{y} - t_1^{-1} x \mathbf{y}^{-1}}{t_1^{-1} x - 1} \cdot \frac{\mathbf{y} + t_1^{-1} x \mathbf{y}^{-1}}{t_1^{-1} x + 1}, \\ g_{\text{O}^-(4)}^{\mathcal{N}=16}(t_1) &= \frac{1}{\mathbf{y} - \mathbf{y}^{-1}} \cdot \frac{1}{\mathbf{y} + \mathbf{y}^{-1}} \cdot \frac{1 - x^2}{x^2 \mathbf{y} - \mathbf{y}^{-1}} \cdot \frac{1 + x^2}{x^2 \mathbf{y} + \mathbf{y}^{-1}} \\ &\times \frac{\mathbf{y} - x^{-1} \tilde{x} \mathbf{y}^{-1}}{x^{-1} \tilde{x} - 1} \cdot \frac{\mathbf{y} + x^{-1} \tilde{x} \mathbf{y}^{-1}}{x^{-1} \tilde{x} + 1} \cdot \frac{\mathbf{y} - x^{-1} \tilde{x}^{-1} \mathbf{y}^{-1}}{x^{-1} \tilde{x}^{-1} - 1} \cdot \frac{\mathbf{y} + x^{-1} \tilde{x}^{-1} \mathbf{y}^{-1}}{x^{-1} \tilde{x}^{-1} + 1} \\ &\times \frac{1 - t_1}{t_1 \mathbf{y}^{-1} - \mathbf{y}} \cdot \frac{1 + t_1}{t_1 \mathbf{y}^{-1} + \mathbf{y}} \cdot \frac{1 - t_1^{-1}}{t_1^{-1} \mathbf{y}^{-1} - \mathbf{y}} \cdot \frac{1 + t_1^{-1}}{t_1^{-1} \mathbf{y}^{-1} + \mathbf{y}} \\ &\times \frac{1 - t_1 x^2}{t_1 x^2 \mathbf{y} - \mathbf{y}^{-1}} \cdot \frac{1 + t_1 x^2}{t_1 x^2 \mathbf{y} + \mathbf{y}^{-1}} \cdot \frac{1 - t_1^{-1} x^2}{t_1^{-1} x^2 \mathbf{y} - \mathbf{y}^{-1}} \cdot \frac{1 + t_1^{-1} x^2}{t_1^{-1} x^2 \mathbf{y} + \mathbf{y}^{-1}} \end{aligned}$$

$$\begin{aligned} & \times \frac{\mathbf{y} - t_1 x^{-1} \tilde{x} \mathbf{y}^{-1}}{t_1 x^{-1} \tilde{x} - 1} \cdot \frac{\mathbf{y} + t_1 x^{-1} \tilde{x} \mathbf{y}^{-1}}{t_1 x^{-1} \tilde{x} + 1} \cdot \frac{\mathbf{y} - t_1^{-1} x^{-1} \tilde{x} \mathbf{y}^{-1}}{t_1^{-1} x^{-1} \tilde{x} - 1} \cdot \frac{\mathbf{y} + t_1^{-1} x^{-1} \tilde{x} \mathbf{y}^{-1}}{t_1^{-1} x^{-1} \tilde{x} + 1} \\ & \times \frac{\mathbf{y} - t_1 x^{-1} \tilde{x}^{-1} \mathbf{y}^{-1}}{t_1 x^{-1} \tilde{x}^{-1} - 1} \cdot \frac{\mathbf{y} + t_1 x^{-1} \tilde{x}^{-1} \mathbf{y}^{-1}}{t_1 x^{-1} \tilde{x}^{-1} + 1} \cdot \frac{\mathbf{y} - t_1^{-1} x^{-1} \tilde{x}^{-1} \mathbf{y}^{-1}}{t_1^{-1} x^{-1} \tilde{x}^{-1} - 1} \cdot \frac{\mathbf{y} + t_1^{-1} x^{-1} \tilde{x}^{-1} \mathbf{y}^{-1}}{t_1^{-1} x^{-1} \tilde{x}^{-1} + 1}, \end{aligned}$$

where R-charges and flavor charges have been assigned as  $R = 0$  and  $F = 1$  to the adjoint chiral multiplet of the  $\mathcal{N} = 8$  theory and as  $R = (2, 0, 0)$ ,  $F = (2, -1, -1)$  and  $\tilde{F} = (0, 1, -1)$  to the three adjoint chirals of the  $\mathcal{N} = 16$  theory.

As a final remark, the determinant formula (B.3) has the following subtlety in sign. It is natural to expect that the massive Cartan factors in the first line of eq. (B.3) each come with an additional minus sign,<sup>7</sup> just like they do in the  $O^+$  theory,

$$\frac{\mathbf{y}^{-(R_a/2-1)} - x^{F_a} \mathbf{y}^{R_a/2-1}}{x^{F_a} \mathbf{y}^{R_a/2} - \mathbf{y}^{-R_a/2}} = - \frac{x^{F_a/2} \mathbf{y}^{R_a/2-1} - x^{-F_a/2} \mathbf{y}^{-(R_a/2-1)}}{x^{F_a/2} \mathbf{y}^{R_a/2} - x^{-F_a/2} \mathbf{y}^{-R_a/2}}. \quad (\text{B.4})$$

If true, the formula would have an incorrect overall sign for  $\mathcal{N} = 8$  and 16 cases as there exist one and three such massive Cartan factors, respectively. However, we propose that they do not come with an expected minus sign and eq. (B.3) is correct as it is. For a consistency check, let us consider  $\mathcal{N} = 4 O^-(2N)$  theory with an adjoint chiral multiplet, to which  $R = 1$  and  $F = 0$  are assigned. Since this theory admits a mass term for the chiral field, it should flow to pure  $\mathcal{N} = 4 O^-(2N)$  theory and hence, the twisted partition functions of the two theories must agree, with the same overall sign. We have indeed confirmed this for  $N = 2$  and 3 based on the one-loop determinants (B.3).

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<sup>7</sup>Similar argument applies to all the flipped factors in the other lines of eq. (B.3), although the total number of such factors is always even so that they may never affect the final result.

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