

## Duality group actions on fermions

---

Tony Pantev<sup>a</sup> and Eric Sharpe<sup>b</sup>

<sup>a</sup>*Department of Mathematics, University of Pennsylvania,  
David Rittenhouse Lab., 209 South 33rd Street, Philadelphia, PA 19104-6395, U.S.A.*

<sup>b</sup>*Department of Physics MC 0435, Virginia Tech,  
850 West Campus Drive, Blacksburg, VA 24061, U.S.A.*

*E-mail:* [tpantev@math.upenn.edu](mailto:tpantev@math.upenn.edu), [ersharpe@vt.edu](mailto:ersharpe@vt.edu)

**ABSTRACT:** In this short paper we look at the action of T-duality and string duality groups on fermions, in maximally-supersymmetric theories and related theories. Briefly, we argue that typical duality groups such as  $SL(2, \mathbb{Z})$  have sign ambiguities in their actions on fermions, and propose that pertinent duality groups be extended by  $\mathbb{Z}_2$ , to groups such as the metaplectic group. Specifically, we look at duality groups arising from mapping class groups of tori in M theory compactifications, T-duality, ten-dimensional type IIB S-duality, and (briefly) four-dimensional  $N = 4$  super Yang-Mills, and in each case, propose that the full duality group is a nontrivial  $\mathbb{Z}_2$  extension of the duality group acting on bosonic degrees of freedom, to more accurately describe possible actions on fermions. We also walk through U-duality groups for toroidal compactifications to nine, eight, and seven dimensions, which enables us to perform cross-consistency tests of these proposals.

**KEYWORDS:** String Duality, Extended Supersymmetry

**ARXIV EPRINT:** [1609.00011](https://arxiv.org/abs/1609.00011)

---

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Review of the metaplectic group</b>	<b>3</b>
<b>3</b>	<b>Mapping class groups of tori</b>	<b>4</b>
3.1	Elliptic curves	4
3.2	Higher-dimensional tori	5
<b>4</b>	<b>T-duality</b>	<b>7</b>
4.1	Action on worldsheet theories	7
4.2	Moduli spaces of SCFTs	8
<b>5</b>	<b>Ten-dimensional IIB S-duality</b>	<b>9</b>
<b>6</b>	<b>Four-dimensional <math>N = 4</math> SYM</b>	<b>10</b>
<b>7</b>	<b>U-duality</b>	<b>12</b>
7.1	Nine dimensions	12
7.2	Eight dimensions	12
7.3	Seven dimensions	14
<b>8</b>	<b>Conclusions</b>	<b>14</b>

---

## 1 Introduction

In this paper we re-examine duality groups in high-dimensional string theories with maximal supersymmetry, taking a close look at duality group actions<sup>1</sup> on fermions in low-energy effective field theories. Historically, such duality group actions have been primarily discussed only on bosonic degrees of freedom. Briefly, we argue that in many cases, fermions are not uniquely defined under the duality groups as they are typically described, because of square-root-type branch cut ambiguities, and so propose that those duality groups be slightly enlarged.

---

<sup>1</sup>To be clear, although duality transformations should define a map of gauge-invariant local operators, their action on fundamental fields might not be so simply defined. For one example, in four-dimensional  $N = 1$  Seiberg duality, mesons are composite operators on one side and fundamental fields on the other. However, in maximally-supersymmetric theories, actions on fields in low-energy effective actions are typically well-defined, and it is for this reason that one can e.g. describe the R-R and NS-NS two-form fields in ten-dimensional IIB supergravity as transforming in a doublet of the S-duality group. For simplicity, in this note we will largely focus on maximally-supersymmetric theories and related cases.

As one prototypical example, in ten-dimensional type IIB string theory, we argue that under the S-duality group  $SL(2, \mathbb{Z})$ , the transformation of the fermions is not uniquely defined due to a sign ambiguity, and so propose that  $SL(2, \mathbb{Z})$  should be replaced by a  $\mathbb{Z}_2$  extension. In particular, from the form of the duality group action, we propose that  $SL(2, \mathbb{Z})$  should be replaced by a particular nontrivial  $\mathbb{Z}_2$  central extension known as the metaplectic group and denoted  $Mp(2, \mathbb{Z})$ .

We examine several different duality groups — mapping class groups of tori arising in M theory compactifications, T-dualities, ten-dimensional IIB S-duality — proposing such extensions in each case, as well as corresponding U-duality groups, which we use to perform cross-consistency tests of these proposed extensions.

We begin in section 2 by briefly reviewing the metaplectic group  $Mp(2, \mathbb{Z})$ , which will be the most commonly appearing proposed duality group. We review its relation to  $SL(2, \mathbb{Z})$  and properties of this and related groups.

In section 3, we go on to discuss the linear diffeomorphism mapping class groups of tori, that play a crucial role in duality symmetries of toroidal compactifications of M theory. The mapping class group for an  $n$ -torus is ordinarily given as  $SL(n, \mathbb{Z})$ , but we find that this group has an ambiguous action on fermions, and so we propose that in general it be replaced by a  $\mathbb{Z}_2$  extension which we denote  $\widetilde{SL}(n, \mathbb{Z})$ .

In section 4 we turn to perturbative T-duality groups of toroidally-compactified string theories. There, we argue that the worldsheet fermions themselves are well-defined under target-space T-dualities; the Ramond sector vacua, however, are only well-defined under  $\mathbb{Z}_2$  extensions of the ordinary duality groups, just as in the discussion of mapping class groups of tori. We discuss moduli spaces of SCFTs, and argue for similar reasons that, for example, the moduli space of elliptic curves SCFTs is best described as a quotient of the upper half plane by  $Mp(2, \mathbb{Z})$  instead of  $SL(2, \mathbb{Z})$  or  $PSL(2, \mathbb{Z})$ .

In section 5, we turn to S-duality in ten-dimensional type IIB string theory. Here, no tori are manifest; nevertheless, we shall argue that the ten-dimensional fermions are also slightly ambiguous under the action of the S-duality group  $SL(2, \mathbb{Z})$ , due to square-root branch cut ambiguities, and so for a well-defined action, we propose to promote the S-duality group to the  $\mathbb{Z}_2$ -extension  $Mp(2, \mathbb{Z})$ .

In section 6, we briefly turn to four-dimensional  $N = 4$  super Yang-Mills, and discuss how the  $Mp(2, \mathbb{Z})$  action in ten-dimensional type IIB appears to imply an analogous extension in S-duality in the four-dimensional  $N = 4$  theory. That said, for most of this paper, in order to be able to speak meaningfully about duality group actions on fields rather than theories, we focus on maximally-supersymmetric theories in high dimensions.

Finally, in section 7, we put the ideas of the preceding sections together to consider U-duality groups of toroidally-compactified M theory in nine, eight, and seven dimensions. We see that the various occurrences of the metaplectic group and analogous  $\mathbb{Z}_2$  extensions check one another. For example, in nine dimensions, in order for the U-duality group to be consistent, both the mapping class group  $SL(2, \mathbb{Z})$  of  $T^2$  as well as the S-duality group of ten-dimensional type IIB must be extended to  $Mp(2, \mathbb{Z})$ , which is what we have proposed in previous sections.

One of the motivations for this work is to understand the physical significance of a result in [1]. Specifically, it was argued there that the moduli stack of elliptic curves for which the Bagger-Witten line bundle is well-defined is the stacky quotient

$$[(\text{upper half plane}) / Mp(2, \mathbb{Z})],$$

for  $Mp(2, \mathbb{Z})$  the metaplectic group extending  $SL(2, \mathbb{Z})$  by  $\mathbb{Z}_2$ , rather than a quotient by  $SL(2, \mathbb{Z})$  or  $PSL(2, \mathbb{Z})$ . One could naturally ask whether the appearance of the metaplectic group was merely an obscure mathematical quirk of the Bagger-Witten line bundle in that circumstance, or reflected the true T-dualities of the theory. We propose in this paper that the latter is the case.

The proposal of section 5, that the fermions of ten-dimensional IIB supergravity transform under  $Mp(2, \mathbb{Z})$ , has been independently obtained by D. Morrison [2].

## 2 Review of the metaplectic group

We will frequently encounter the metaplectic group  $Mp(2, \mathbb{Z})$  and its various cousins in this paper, so let us briefly review some pertinent facts. The metaplectic group  $Mp(2, \mathbb{Z})$  is the unique nontrivial central extensive of  $\mathbb{Z}_2$ , and can be described as the group with elements of the form

$$\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \pm\sqrt{c\tau + d} \right),$$

where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z}),$$

and  $\sqrt{c\tau + d}$  is considered as a holomorphic function of  $\tau$  in the upper half plane. The multiplication is defined as

$$(A, f(\cdot))(B, g(\cdot)) = (AB, f(B(\cdot))g(\cdot)).$$

More generally, there is a metaplectic group  $Mp(2k, \mathbb{Z})$  which is the unique nontrivial  $\mathbb{Z}_2$  central extension of the symplectic group  $Sp(2k, \mathbb{Z})$ . For  $k = 1$ ,  $Sp(2k, \mathbb{Z}) = SL(2, \mathbb{Z})$ , and so this description reduces to the one above.

Metaplectic groups over  $\mathbb{R}$  also define the symplectic analogue of spin structures on oriented Riemannian manifolds (see e.g. [3] and references therein, of which we shall give only a very brief account here). There is a formal definition of a metaplectic structure on a symplectic manifold, which is an equivariant lift of the symplectic frame bundle. Just as in ordinary spin structures, a metaplectic structure exists on a symplectic manifold  $(X, \omega)$  if and only if the second Stiefel-Whitney class of  $M$  vanishes. Although we will not use metaplectic structures per se in this paper, we will often see metaplectic groups and related extensions arise via a need to define spinors in given situations.

### 3 Mapping class groups of tori

In this section, we will argue that under the action of the ‘mapping class group’  $SL(n, \mathbb{Z})$  of a torus, describing the (analogues of) Dehn twists, spinors transform under the action of a  $\mathbb{Z}_2$  extension of  $SL(n, \mathbb{Z})$  (and the spin structures are permuted). We will argue this solely from the torus itself; in section 4.1, we will describe how the same result appears in the worldsheet theory of a sigma model on a torus.

To be clear, let us define more precisely what we mean by the ‘mapping class group’ of a torus. We follow the language and conventions of [5][section 3.4]. If we describe  $T^n$  as a set of real variables  $y^i$  subject to  $y^i \equiv y^i + n^i$  for  $n^i \in \mathbb{Z}$ , then the ‘mapping class group’ to which we refer is the group of linear orientation-preserving diffeomorphisms  $y^i \mapsto w^i_j y^j$ . Clearly the  $(w^i_j) \in GL(n, \mathbb{Z})$ , and to preserve a choice of orientation, we must require  $(w^i_j) \in SL(n, \mathbb{Z})$ .

In this language, we can now begin to gain some intuition for the subtlety that arises when considering fermions. Broadly speaking, the mapping class group is acting by rotations. However, to define an action on fermions, one must lift to a Spin cover. As a result, one should expect intuitively that the mapping class group will have to be replaced<sup>2</sup> by some sort of  $\mathbb{Z}_2$  cover. In this section, that is exactly the conclusion we shall reach. For  $n = 2$ , the  $\mathbb{Z}_2$  cover of  $SL(2, \mathbb{Z})$  will be the metaplectic group  $Mp(2, \mathbb{Z})$ , and for  $n \geq 3$ , it will be a  $\mathbb{Z}_2$  cover of  $SL(n, \mathbb{Z})$  which we will denote  $\widetilde{SL}(n, \mathbb{Z})$ .

#### 3.1 Elliptic curves

Let us begin by considering elliptic curves. Under the action of the group of Dehn twists  $SL(2, \mathbb{Z})$ , if we describe the elliptic curve as

$$H/(\mathbb{Z} + \mathbb{Z}\tau),$$

for  $H$  the upper half plane, then at the same time that

$$\tau \mapsto \frac{a\tau + b}{c\tau + d},$$

for

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z}),$$

a point  $z$  in the plane transforms as<sup>3</sup> [4][section 2.3]

$$z \mapsto (c\tau + d)^{-1} z,$$

---

<sup>2</sup>We would like to thank D. Auroux for a discussion of this matter.

<sup>3</sup>Briefly, the idea is that if we construct a family of elliptic curves with complex structure parameter  $\tau$  as  $(\mathbb{C} \times H)/\mathbb{Z}^2$ , for  $H$  the upper half plane and  $\mathbb{Z}^2$  action given by

$$(m, n) : (z, \tau) \mapsto (z + m\tau + n, \tau),$$

then to be consistent with the  $\mathbb{Z}^2$  action, under  $\tau \mapsto (a\tau + b)/(c\tau + d)$ , we must rescale  $z$ , so that

$$(z, \tau) \cong \left( \frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right),$$

and so the holomorphic top-form (a one-form) transforms as

$$dz \mapsto (c\tau + d)^{-1} dz.$$

In passing, note that although the parameter  $\tau$  is invariant under the center  $\{\pm 1\} \subset \text{SL}(2, \mathbb{Z})$ , the holomorphic top-form  $dz$  is not invariant under the center, and so is only uniquely defined on a (stacky)  $\text{SL}(2, \mathbb{Z})$  quotient of the upper half plane, not a  $\text{PSL}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z})/\{\pm 1\}$  quotient.

Now, a chiral spinor on an elliptic curve is a section of a square root of the canonical bundle, and so should transform in the same fashion as  $\sqrt{dz}$ . Thus, if  $\psi$  is a chiral spinor, then under  $\text{SL}(2, \mathbb{Z})$ ,

$$\psi \mapsto \pm \sqrt{c\tau + d}^{-1} \psi.$$

However, the element of  $\text{SL}(2, \mathbb{Z})$  does not uniquely determine the sign: the group that is really acting is not  $\text{SL}(2, \mathbb{Z})$ , but a  $\mathbb{Z}_2$  extension of  $\text{SL}(2, \mathbb{Z})$ , and in fact it should be clear to the reader that the  $\mathbb{Z}_2$  extension in question is  $Mp(2, \mathbb{Z})$ .

### 3.2 Higher-dimensional tori

So far we have argued that on a single  $T^2$ , the action of the mapping class group  $\text{SL}(2, \mathbb{Z})$  on fermions is ambiguous up to a  $\tau$ -dependent phase, and so the action on fermions is more properly described as an action of  $Mp(2, \mathbb{Z})$ , the unique nontrivial central extension of  $\text{SL}(2, \mathbb{Z})$ .

Now, consider a higher-dimensional torus  $T^n$ . Setting aside fermions, the mapping class group of such a torus is  $\text{SL}(n, \mathbb{Z})$ . Now, for any  $T^2 \subset T^n$ , the same considerations must apply to fermions, and so every  $\text{SL}(2, \mathbb{Z}) \subset \text{SL}(n, \mathbb{Z})$  must be enhanced to  $Mp(2, \mathbb{Z})$ .

Therefore, to describe the action of the mapping class group on fermions, we need a  $\mathbb{Z}_2$  extension of  $\text{SL}(n, \mathbb{Z})$ , such that every  $\text{SL}(2, \mathbb{Z})$  is extended to  $Mp(2, \mathbb{Z})$ .

We will construct this extension, which we shall denote  $\widetilde{\text{SL}}(n, \mathbb{Z})$ , for  $n \geq 3$  as a pullback of the universal cover  $\widetilde{\text{SL}}(n, \mathbb{R})$  of  $\text{SL}(n, \mathbb{R})$ .

First, let us consider the universal cover  $\widetilde{\text{SL}}(n, \mathbb{R})$ . Since the maximal compact subgroup of  $\text{SL}(n, \mathbb{R})$  is  $\text{SO}(n)$ ,

$$\pi_1(\text{SL}(n, \mathbb{R})) = \pi_1(\text{SO}(n)) = \begin{cases} \mathbb{Z}_2 & n \geq 3, \\ \mathbb{Z} & n = 2, \end{cases}$$

thus the universal cover  $\widetilde{\text{SL}}(n, \mathbb{R})$  is a  $\mathbb{Z}_2$  central extension of  $\text{SL}(n, \mathbb{R})$  for  $n \geq 3$ , and a  $\mathbb{Z}$ -fold central extension for  $n = 2$ . We can understand this topologically as follows. As a

---

making the right-hand side well-defined under the  $\mathbb{Z}^2$  quotient. As consistency tests, note that under the map on  $z$ ,

$$\begin{aligned} \tau &\mapsto \frac{\tau}{c\tau + d} = \frac{a\tau + b}{c\tau + d} + (d-1) \left( \frac{a\tau + b}{c\tau + d} \right) - b \cong \frac{a\tau + b}{c\tau + d}, \\ 1 &\mapsto \frac{1}{c\tau + d} = 1 - c \left( \frac{a\tau + b}{c\tau + d} \right) + (a-1) \cong 1. \end{aligned}$$

topological space,  $SL(n, \mathbb{R})$  is homeomorphic to  $SO(n) \times \mathbb{R}^k$  for some  $k$  (ignoring the group structure), so its universal cover is homeomorphic to  $Spin(n) \times \mathbb{R}^k$ . For  $n = 2$ ,  $Spin(2) = \mathbb{R}$ , a  $\mathbb{Z}$ -fold cover, and for  $n \geq 3$ ,  $Spin(n)$  is a double cover of  $SO(n)$ .

In any event, we can now construct our desired group, that extends the action of the mapping class group to fermions. Define  $\widetilde{SL}(n, \mathbb{Z})$  by the pullback square

$$\begin{array}{ccc} \widetilde{SL}(n, \mathbb{R}) & \xrightarrow{p} & SL(n, \mathbb{R}) \\ \uparrow & & \uparrow i \\ \widetilde{SL}(n, \mathbb{Z}) & \longrightarrow & SL(n, \mathbb{Z}), \end{array}$$

where  $p : \widetilde{SL}(n, \mathbb{R}) \rightarrow SL(n, \mathbb{R})$  is the projection map, and  $i : SL(n, \mathbb{Z}) \hookrightarrow SL(n, \mathbb{R})$  is inclusion. In other words, we define

$$\widetilde{SL}(n, \mathbb{Z}) = \left\{ (a, b) \in \widetilde{SL}(n, \mathbb{R}) \times SL(n, \mathbb{Z}) \mid p(a) = i(b) \right\},$$

with product structure

$$(a, b) \cdot (a', b') \equiv (aa', bb'),$$

which is well-defined because both  $p$  and  $i$  are homomorphisms, hence

$$p(aa') = p(a)p(a') = i(b)i(b') = i(bb').$$

For  $n \geq 3$ , we claim that the group  $\widetilde{SL}(n, \mathbb{Z})$  encodes the action of the mapping class group on the fermions. For  $n = 2$ , the desired group is the metaplectic group  $Mp(2, \mathbb{Z})$ , which is quotient of  $\widetilde{SL}(2, \mathbb{Z})$  by  $2\mathbb{Z}$ , a subgroup of the central  $\mathbb{Z} \subset \widetilde{SL}(2, \mathbb{Z})$ .

It remains to check whether the restriction of  $\widetilde{SL}(n, \mathbb{Z})$  to  $p^{-1}(SL(2, \mathbb{Z}))$  for any  $SL(2, \mathbb{Z}) \subset SL(n, \mathbb{Z})$  is a trivial or nontrivial central extension. (Since  $Mp(2, \mathbb{Z})$  is the unique nontrivial central extension of  $SL(2, \mathbb{Z})$  by  $\mathbb{Z}_2$ , showing that  $p^{-1}(SL(2, \mathbb{Z}))$  is a nontrivial central extension for every  $SL(2, \mathbb{Z})$  would imply  $p^{-1}(SL(2, \mathbb{Z})) = Mp(2, \mathbb{Z})$  for every  $SL(2, \mathbb{Z})$ .) Now, every copy of  $SL(2, \mathbb{Z})$  inside  $SL(n, \mathbb{Z})$  comes from taking the integral points of a copy of  $SL(2, \mathbb{R})$  inside  $SL(n, \mathbb{R})$ , so the question reduces to whether for any copy of  $SO(2)$  inside  $SO(n)$ , the  $Spin$  cover of  $SO(n)$  splits when restricted to that  $SO(2)$ .

It can be shown that the  $Spin$  cover of  $SO(n)$  does not split when restricted to any  $SO(2)$ . This follows<sup>4</sup> from the fact that the group homomorphism  $SO(2) \rightarrow SO(n)$  is surjective on the fundamental group, and the correspondence between covering spaces and actions of the fundamental group.

As a result, the restriction of  $\widetilde{SL}(n, \mathbb{Z})$  to  $p^{-1}(SL(2, \mathbb{Z}))$  for any  $SL(2, \mathbb{Z}) \subset SL(n, \mathbb{Z})$  is a nontrivial central extension, hence

$$p^{-1}(SL(2, \mathbb{Z})) = Mp(2, \mathbb{Z}),$$

and so  $\widetilde{SL}(n, \mathbb{Z})$  for  $n \geq 3$  is the desired group describing the mapping class group action on fermions.

---

<sup>4</sup>One of the authors (E.S.) would like to thank J. Francis for outlining this argument.

## 4 T-duality

### 4.1 Action on worldsheet theories

In this subsection, we will consider the action on worldsheet fermions from two different sources: the action on the worldsheet theory of T-duality on a target elliptic curve, and the action of Dehn twists when the worldsheet itself is an elliptic curve.

Let us first consider a (2,2) supersymmetric sigma model with target space  $T^2$ . Now, as is well-known, the set of T-dualities is larger than merely a  $\mathbb{Z}_2$  for each circle in  $T^2$ , as for example, those T-dualities can be combined with Dehn twists to form a larger symmetry group. Across both complex and Kähler structures on  $T^2$ , the T-duality group acting on the CFT is well-known to be  $O(2, 2; \mathbb{Z})$ .

Now, the part of the T-duality group  $O(2, 2; \mathbb{Z})$  that preserves GSO projections is

$$SO(2, 2; \mathbb{Z}) \cong (SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})) / \mathbb{Z}_2,$$

where one  $SL(2, \mathbb{Z})$  acts on complex structures and the other on Kähler structures. As anticipated above, the largest part of the T-duality group is therefore two  $SL(2, \mathbb{Z})$  factors of Dehn twists, for complex and Kähler structures separately.

Let us consider the  $SL(2, \mathbb{Z})$  action on the complex structure of the target-space  $T^2$ . The worldsheet fermions transform in the pullback of the tangent bundle on the target space, hence (up to dualizations and complex conjugations), they transform as  $dz$ , not  $\sqrt{dz}$ , just as the worldsheet bosons (consistent with worldsheet supersymmetry). As a result, it is  $SL(2, \mathbb{Z})$ , not  $Mp(2, \mathbb{Z})$ , that acts on worldsheet fermions themselves under T-duality on the target space.

The Ramond sector vacua, on the other hand, are a different matter. The Ramond sector vacua of a physical theory transform in principle as sections of the pullback of the canonical bundle on the target space, i.e. as  $\sqrt{dz}$ . (When the target is Calabi-Yau, this can be subtle to see, but is much more manifest in other cases, see for example [6] for a recent discussion of Fock vacua transforming as sections of bundles.) Thus, in principle, the Ramond sector vacua should pick up factors of  $\pm\sqrt{c\tau + d}^{-1}$  under the action of T-duality on the target space, and in particular transform under  $Mp(2, \mathbb{Z})$  rather than  $SL(2, \mathbb{Z})$ . (Similarly, though less relevantly for our purposes, target-space spin structures are also encoded in the R sector vacua, in signs under target-space periodicities.) Since the  $\mathbb{Z}_2$  extension acts only on the vacua, it should commute with the GSO projection, and so our analysis should have no effect on physical states.

In passing, we should note that the transformation of the Ramond sector vacua under target-space Dehn twists is the worldsheet realization of the target-space mapping class group action on fermions discussed in section 3.

T-duality for higher-dimensional tori can be treated in a very similar fashion. Under  $SL(n, \mathbb{Z})$  factors of Dehn twists, the worldsheet fermions themselves will be well-defined, but the Ramond sector vacua will pick up ambiguous signs (in addition to permutations of the spin structures). For the same reasons as in our discussions of mapping class groups of tori, the Ramond sector vacua are well-defined under the action of the  $\mathbb{Z}_2$  extension  $\widehat{SL}(n, \mathbb{Z})$ .



Now, let us ask a different question: how do the worldsheet fermions transform under Dehn twists of the worldsheet itself? Of course, Dehn twists will permute the worldsheet spin structures, but let us consider possible additional phase factors picked up the fields themselves. In principle, the worldsheet fermions in a physical (untwisted) theory with worldsheet  $\Sigma$  are sections of  $\sqrt{K_\Sigma}$ , hence transform as  $\sqrt{dz}$  for  $z$  a coordinate on the worldsheet. As a result, it is natural to propose that the worldsheet fermions transform under  $Mp(2, \mathbb{Z})$  under the action of worldsheet Dehn twists.

## 4.2 Moduli spaces of SCFTs

Let us consider first moduli spaces of SCFTs constructed as sigma models on elliptic curves. As we just argued in the preceding subsection, although the worldsheet fermions themselves are single-valued under the action of  $SL(2, \mathbb{Z})$  on the target space, the R sector vacua are not. In addition to the expected permutations of spin structures, the R sector vacua also necessarily pick up phases, which due to a square root branch cut are ambiguous under  $SL(2, \mathbb{Z})$ . As a result, it is natural to propose that the physically-relevant moduli space of complex structures, half of the moduli space of SCFTs for elliptic curves, should be of the form

$$[(\text{upper half plane}) / Mp(2, \mathbb{Z})],$$

and so the full moduli space, taking into account both complex and Kähler structures, is of the form

$$[(\text{upper half plane}) / Mp(2, \mathbb{Z})] \times [(\text{upper half plane}) / Mp(2, \mathbb{Z})].$$

(We have only discussed the role of the metaplectic group on the space of complex structures, but by mirror symmetry, analogous considerations must also apply to the Kähler structures.)

This sheds new light on a result in [1]. There, it was argued that the Bagger-Witten line bundle over a moduli space of SCFTs for elliptic curves was only well-defined over the stack above, an  $Mp(2, \mathbb{Z})$  quotient of the upper half plane rather than  $SL(2, \mathbb{Z})$  or  $PSL(2, \mathbb{Z})$ . Now, the Bagger-Witten line bundle over any moduli space of SCFTs is a (possibly fractional) line bundle of spectral flow operators, that play an essential role in target-space supersymmetry. Here, we have proposed that the  $Mp(2, \mathbb{Z})$  quotient (as opposed to an  $SL(2, \mathbb{Z})$  or  $PSL(2, \mathbb{Z})$  quotient) is necessary in order to make the R sector vacua well-defined.

Analogous results hold in higher dimensions. For simplicity, let us consider the space of complex structures on a torus  $X$  that is a complex projective manifold, i.e. an abelian variety, and has complex dimension  $g$ , say. For a fixed choice of polarization (Kähler form) on  $X$ , the complex structures preserving the polarization are parametrized by a quotient

$$A_g = H_g / Sp(2g, \mathbb{Z}),$$

where  $H_g$  is the higher-dimensional Siegel upper half plane (meaning, symmetric  $g \times g$  complex matrices with positive-definite imaginary part), and  $Sp(2g, \mathbb{Z})$  is the symplectic group of integral  $2g \times 2g$  matrices preserving the polarization — the ordinary symplectic

group  $\mathrm{Sp}(2g, \mathbb{Z})$  for nondegenerate cases (more properly, principal polarizations). (See e.g. [7][section 2.6] for more information.) Here also, for much the same reasons, the spin structures are parametrized by the stacky quotient

$$[H_g/Mp(2g, \mathbb{Z})],$$

where  $Mp(2g, \mathbb{Z})$  is the metaplectic group extending  $\mathrm{Sp}(2g, \mathbb{Z})$  by  $\mathbb{Z}_2$ .

In general, for an  $n$ -dimensional torus  $T^n$ , the group of all T-dualities is  $O(n, n; \mathbb{Z})$ ; see e.g. [8] for a thorough discussion, hence ordinarily the complete moduli space of tori is described as a quotient by  $O(n, n; \mathbb{Z})$ . We will not try to give a thorough description of the precise metaplectic replacement here, but we will note that elements of  $O(n, n; \mathbb{Z})$  whose determinant is different from one also modify the worldsheet GSO projection (see e.g. [8–10], [11][section 3.2]), for example exchanging type IIA and IIB, so to give a thorough description of the moduli space of SCFTs with target an  $n$ -dimensional torus  $T^n$  will presumably involve a  $\mathbb{Z}_2$  extension of  $\mathrm{SO}(n, n; \mathbb{Z})$  (as well as a means of following spin structures).

## 5 Ten-dimensional IIB S-duality

In this section, we will describe the action of S-duality on fermions in ten-dimensional IIB strings, and we will propose that  $\mathrm{SL}(2, \mathbb{Z})$  should in principle be replaced by the metaplectic group  $Mp(2, \mathbb{Z})$ . (The result of this section has been independently obtained by D. Morrison [2].)

Classically, recall [12–14] that type IIB supergravity in ten dimensions has an  $\mathrm{SU}(1, 1; \mathbb{C}) \cong \mathrm{SL}(2, \mathbb{R})$  symmetry (modulo finite factors) extending a local  $\mathrm{U}(1)$  symmetry. In that theory, the fermions only transform under the local  $\mathrm{U}(1)$  (see e.g. [12][equ'n (7c)]). After gauge-fixing, the local transformation is determined so as to preserve the gauge, and so an  $\mathrm{SL}(2, \mathbb{R})$  transformation determines an action on the fermions [15]. (See also [16][section 13.5] for a review.) Furthermore, in the quantum theory, the continuous  $\mathrm{SL}(2, \mathbb{R})$  symmetry is replaced by a local  $\mathrm{SL}(2, \mathbb{Z})$ , and it is in this fashion that we can understand that the fermions transform under the action of S-duality.

In the conventions of [15], if we pick gauge-fixing condition  $\hat{\phi} = 0$ , then under the  $\mathrm{U}(1)$  subgroup of  $\mathrm{SL}(2, \mathbb{R})$ , the (complex) gravitino  $\psi_\mu$  transforms as [15][equ'n (2.11)]

$$\psi_\mu \mapsto \exp(i\Sigma/2)\psi_\mu,$$

and the (complex) dilatino  $\lambda$  transforms as [15][equ'n (2.11)]

$$\lambda \mapsto \exp(-3i\Sigma/2)\lambda,$$

where for

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R}),$$

the  $\mathrm{U}(1)$  rotation is defined by

$$\exp(i\Sigma) = \left( \frac{c\bar{\tau} + d}{c\tau + d} \right)^{1/2}.$$

Put more simply, and restricting to the local discrete gauge symmetry  $SL(2, \mathbb{Z})$ ,

$$\begin{aligned} \psi_\mu &\mapsto \left(\frac{c\bar{\tau} + d}{c\tau + d}\right)^{1/4} \psi_\mu, \\ \lambda &\mapsto \left(\frac{c\bar{\tau} + d}{c\tau + d}\right)^{-3/4} \lambda. \end{aligned}$$

These transformations were used in e.g. [17] to argue that the coefficients of certain higher-dimension operators should behave as nonholomorphic analogues of modular forms.

Now, as written, the transformations above appear to be invariant under the center of  $SL(2, \mathbb{Z})$ . However, there is a potential subtlety present in the one-quarter-root branch cuts. If we perform a field redefinition, we can construct new fermions with cleaner transformation laws, which is what we will describe next.

Following [18], one can define a modular form of weight  $(m^+, m_-)$  to be a real analytic function  $F$  on the upper half plane  $H$  such that

$$F\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{m^+} (c\bar{\tau} + d)^{m_-} F(\tau).$$

For example,  $\text{Im } \tau$  is such a modular form, of weights  $(-1, -1)$ . Given any such  $F$ , one can transform it to a modular form  $F'$  of weight  $(m^+ - m^-, 0)$  defined by

$$F'(\tau) \equiv (\text{Im } \tau)^{m^-} F.$$

In the present case,  $\psi_\mu$  transforms like a modular form of weights  $(-1/4, +1/4)$  and  $\lambda$  a modular form of weights  $(+3/4, -3/4)$ , so we define

$$\psi'_\mu \equiv (\text{Im } \tau)^{1/4} \psi_\mu, \quad \lambda' \equiv (\text{Im } \tau)^{-3/4} \lambda.$$

Then, under the action of  $SL(2, \mathbb{Z})$ , these new fields transform as

$$\begin{aligned} \psi'_\mu &\mapsto \pm(c\tau + d)^{-1/2} \psi'_\mu, \\ \lambda' &\mapsto \pm(c\tau + d)^{+3/2} \lambda'. \end{aligned}$$

Furthermore, we are free to redefine the fermions as we wish —  $\psi'_\mu, \lambda'$  are no more or less physical than  $\psi_\mu, \lambda$ , so our field redefinition has merely made more manifest a subtle symmetry of the theory.

The transformations above are only defined up to signs — the usual ambiguity in square roots. As a result, the action of  $SL(2, \mathbb{Z})$  is ambiguous, as an element of  $SL(2, \mathbb{Z})$  does not uniquely determine a choice of sign. The group that really is acting is some two-fold cover of  $SL(2, \mathbb{Z})$ . Given the form of the transformations above, it is natural to propose that the full duality group is the metaplectic group  $Mp(2, \mathbb{Z})$ .

## 6 Four-dimensional $N = 4$ SYM

Four-dimensional  $N = 4$  super-Yang-Mills can be directly connected to ten-dimensional IIB string theory by virtue of the AdS/CFT correspondence. In particular, in [19] it was

observed that as the components of the  $N = 4$  supercurrent multiplet couple to the fields of the ten-dimensional IIB supergravity, the transformation properties of the ten-dimensional fields under  $SL(2, \mathbb{Z})$  (or, as we have observed here,  $Mp(2, \mathbb{Z})$ ) imply transformation laws for the four-dimensional multiplets.

For example, as discussed in section 5, if the ten-dimensional dilatino  $\lambda$  has weights  $(+3/4, -3/4)$ , then the ‘dual’ four-dimensional multiplet denoted  $\Lambda_\alpha^i$  in [19] ( $\alpha \in \{1, 2\}$ ,  $i \in \{1, \dots, 4\}$ ) transforms similarly under  $SL(2, \mathbb{Z})$ . We argued in section 5 that a field redefinition of the ten-dimensional theory makes the sign ambiguity and extension to an  $Mp(2, \mathbb{Z})$  action more manifest; similar remarks should apply here.

Given the role of  $Mp(2, \mathbb{Z})$  we have discussed in this section, one might ask what happens after compactification of the four-dimensional theory on a curve. Such compactifications were discussed historically in [20, 21], and more recently in e.g. [27]. For example, these papers argued that the  $SL(2, \mathbb{Z})$  of the four-dimensional  $N = 4$  theory becomes T-duality in the two-dimensional theory. Briefly, if the fermions of the four-dimensional theory transform under the  $\mathbb{Z}_2$  extension  $Mp(2, \mathbb{Z})$  of  $SL(2, \mathbb{Z})$ , then the same must be true of the fermions in the two-dimensional theory, which is consistent with observations in section 4 regarding T-duality and fermions.

In passing, for completeness we should also mention that  $SL(2, \mathbb{Z})$  actions on fermions in four-dimensional  $N = 2$   $U(1)$  gauge theory are discussed in [22]. In the conventions of that reference,  $SL(2, \mathbb{Z})$  acts honestly on the fermions, but the zero modes and partition function pick up factors which could have square-root sign ambiguities. Specifically, if the fermions into modes of R-charge  $R = 1$ , denoted  $\alpha$ , corresponding to a pair of positive-chirality gluinos, and conjugate fields of  $R = -1$ , denoted  $\bar{\alpha}$  and of negative chirality, then under  $\tau \mapsto -1/\tau$ ,

$$\alpha \mapsto \alpha_D \equiv \tau\alpha, \quad \bar{\alpha} \mapsto \bar{\alpha}_D \equiv \bar{\tau}\bar{\alpha}.$$

The zero modes are more nearly relevant for our purposes. The normalized integration measure for any fermi zero mode  $\beta$  is of the form [22][equ’n (3.15)]

$$\frac{d\beta}{\sqrt{\text{Im } \tau}},$$

so that under  $\tau \mapsto -1/\tau \equiv \tau_D$ , [22][equ’n (3.16)]

$$\begin{aligned} \frac{d\alpha}{\sqrt{\text{Im } \tau}} &\mapsto \frac{d\alpha_D}{\sqrt{\text{Im } \tau_D}} = \sqrt{\frac{\bar{\tau}}{\tau}} \frac{d\alpha}{\sqrt{\text{Im } \tau}}, \\ \frac{d\bar{\alpha}}{\sqrt{\text{Im } \tau}} &\mapsto \frac{d\bar{\alpha}_D}{\sqrt{\text{Im } \tau_D}} = \sqrt{\frac{\tau}{\bar{\tau}}} \frac{d\bar{\alpha}}{\sqrt{\text{Im } \tau}} \end{aligned}$$

As a result, since the number of  $\alpha$  zero modes minus the number of  $\bar{\alpha}$  zero modes is  $-(\chi + \sigma)/2$ , the fermion measure picks up a factor of [22][equ’n (3.17)]

$$\left(\frac{\bar{\tau}}{\tau}\right)^{-(\chi+\sigma)/4}$$

under  $\tau \mapsto -1/\tau$ . However, this is not the only part of the path integral measure that transforms. In particular, the gauge measure also picks up a factor of [22][equ’n (3.18)]

$$\tau^{+(\chi-\sigma)/4} \bar{\tau}^{+(\chi+\sigma)/4}$$

under  $\tau \mapsto -1/\tau$ . (The integration measure of  $a$ ,  $a_D$  is invariant.) As a result, altogether the path integral for the  $N = 2$  U(1) super Yang-Mills theory picks up a factor of [22][equ'n (3.19)]

$$\tau^{-\chi/2}$$

under  $\tau \mapsto -1/\tau$ . We leave a more detailed examination of four-dimensional  $N = 2$  theories and the role of the metaplectic group for future work.

## 7 U-duality

In this section we will discuss U-dualities appearing in toroidally-compactified theories, which will not only allow us to display how U-duality groups are modified when one takes into account fermions, but also perform cross-checks of our proposals. Briefly, in each dimension we will propose a  $\mathbb{Z}_2$  extension of the ordinary U-duality group, reflecting the fact that taking fermions into account should only generate sign ambiguities and hence we expect only an overall  $\mathbb{Z}_2$  extension, rather than an extension by a larger finite group.

### 7.1 Nine dimensions

U-duality groups of nine-dimensional theories were discussed in e.g. [5, 23]. Briefly, a nine-dimensional theory can be obtained as either M theory on  $T^2$  or, equivalently, type II on  $S^1$ . As M theory on  $T^2$ , for the reasons discussed in section 3, this theory has an  $Mp(2, \mathbb{Z})$  duality when one takes into account fermions. If we think about this as type IIB on  $S^1$ , then as discussed in section 5, the ten-dimensional type IIB theory has an  $Mp(2, \mathbb{Z})$  symmetry, which coincides with the M theory U-duality group. In addition, there is also the ordinary T-duality on  $S^1$ , but as this exchanges IIA and IIB (albeit shifting the dilaton in the process), it does not contribute to the duality group of IIB per se.

In any event, regardless of how we construct the nine-dimensional theory, we see that when taking into account the fermions, in our proposal it has an  $Mp(2, \mathbb{Z})$  symmetry, slightly enlarging what was previously described as an  $SL(2, \mathbb{Z})$  symmetry. We also see that the results of section 3 and section 5 constrain one another: consistency of the nine-dimensional theory requires the two duality groups obtained independently in those sections to match, as indeed they do.

### 7.2 Eight dimensions

In this section we shall discuss the eight-dimensional theory which can be obtained alternatively as a compactification of M theory on  $T^3$ , or of type IIA/B on  $T^2$ .

In the past, omitting fermions, it was said that the U-duality group is  $SL(3, \mathbb{Z}) \times SL(2, \mathbb{Z})$ . The  $SL(3, \mathbb{Z})$  factor arose from the mapping class group of the  $T^3$  in the M theory compactification, and as explained in e.g. [11][section 4.3], the other factor arises from T-duality of the type II compactification on  $T^2$ .

Omitting fermions, the (GSO-preserving) T-duality group acting on the SCFT is

$$SO(2, 2; \mathbb{Z}) \cong (SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})) / \mathbb{Z}_2.$$

However, in the spacetime theory, we cannot<sup>5</sup> quotient out the  $\mathbb{Z}_2$ , as it acts nontrivially on the RR fields. In fact, to better understand this statement, let us summarize the actions of various  $\text{SL}(2, \mathbb{Z})$ 's appearing. The ten-dimensional  $\text{SL}(2, \mathbb{Z})_S$  of IIB (omitting fermions) acts as [24–26]

$$\begin{aligned} \tau &\mapsto \frac{a\tau + b}{c\tau + d}, \\ \begin{bmatrix} B_N \\ B_R \end{bmatrix} &\mapsto \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \begin{bmatrix} B_N \\ B_R \end{bmatrix}, \\ B_R + \tau B_N &\mapsto \frac{B_R + \tau B_N}{c\tau + d}, \end{aligned}$$

where  $\tau$  is the complexified ten-dimensional string coupling,  $B_N$  is the ten-dimensional NS-NS  $B$  field, and  $B_R$  is the R-R  $B$  field. Part of the T-duality group of the IIB compactification on  $T^2$ , which we shall denote  $\text{SL}(2, \mathbb{Z})_T$ , acts as (e.g. [24])

$$\begin{aligned} \tilde{T} &\mapsto \frac{a\tilde{T} + b}{c\tilde{T} + d}, \\ \rho &\mapsto \frac{\rho}{c\rho + d}, \end{aligned}$$

for  $\tilde{T} = B_N + iV_2$  the  $T^2$  Kähler modulus, with  $B_N$  the NS-NS two-form on  $T^2$  and  $V_2$  the volume of the  $T^2$ , and  $\rho = -B_R + i\tau_1 V_2$ . The other half of the T-duality group we denote  $\text{SL}(2, \mathbb{Z})_U$ , and acts on the complex structure modulus  $U$  of  $T^2$  as (e.g. [24])

$$U \mapsto \frac{aU + b}{cU + d}.$$

From the transformation law of  $\rho$  under  $\text{SL}(2, \mathbb{Z})_T$ , we see that the RR 2-form  $B_R$  picks up a sign under the  $\mathbb{Z}_2$  center of  $\text{SL}(2, \mathbb{Z})_T$ , and so in forming the string duality group, we must lift  $\text{SO}(2, 2; \mathbb{Z})$  to a double cover, namely  $\text{SL}(2, \mathbb{Z})_T \times \text{SL}(2, \mathbb{Z})_U$ .

The  $\text{SL}(2, \mathbb{Z})_S$ ,  $\text{SL}(2, \mathbb{Z})_T$  are combined as a pair of  $2 \times 2$  blocks inside a  $3 \times 3$  matrix to form the  $\text{SL}(3, \mathbb{Z})$  factor in the U-duality group, which altogether is  $\text{SL}(3, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z})_U$ .

We claim that, when fermions are taken into account, the U-duality group in this theory is  $(\widetilde{\text{SL}}(3, \mathbb{Z}) \times \text{Mp}(2, \mathbb{Z}))/\mathbb{Z}_2$ . The  $\widetilde{\text{SL}}(3, \mathbb{Z})$  factor arises from the mapping class group of  $T^3$  in the M theory compactification, as in section 3.

In any event, there would appear to be two natural possibilities for the U-duality group in eight dimensions: either  $\widetilde{\text{SL}}(3, \mathbb{Z}) \times \text{Mp}(2, \mathbb{Z})$ , or  $(\widetilde{\text{SL}}(3, \mathbb{Z}) \times \text{Mp}(2, \mathbb{Z}))/\mathbb{Z}_2$ , where the  $\mathbb{Z}_2$  quotient acts on the two central  $\mathbb{Z}_2$  extension factors. As we are looking for a  $\mathbb{Z}_2$  extension of the ordinary U-duality group rather than a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  extension, we propose that the correct U-duality group in eight dimensions is

$$\frac{\widetilde{\text{SL}}(3, \mathbb{Z}) \times \text{Mp}(2, \mathbb{Z})}{\mathbb{Z}_2}.$$

We shall also see that this group arises in the decompactification limit from seven dimensions in the next section.

---

<sup>5</sup>We would like to thank E. Witten for this observation.

### 7.3 Seven dimensions

In this section, we will discuss the U-duality group of the theory which can alternatively be described as M theory compactified on  $T^4$ , or as type II on  $T^3$ .

In the past, omitting fermions, it was said that the U-duality group is  $SL(5, \mathbb{Z})$ . The mapping class group of the  $T^4$  appearing in the M theory compactification, namely  $SL(4, \mathbb{Z})$ , appears as a subgroup of  $SL(5, \mathbb{Z})$  (embedded in the obvious way, as a  $4 \times 4$  block inside  $5 \times 5$  matrices [5][section 3.4]).

Similarly (see e.g. [11][section 4.3]), the (GSO-preserving and RR-compatible) T-duality group  $SL(4, \mathbb{Z})$  of the type II compactification does not commute with the mapping class group of the M theory compactification, and the two of them together generate  $SL(5, \mathbb{Z})$ .

In the present case, taking into account fermions, our proposal is that the contribution to the U-duality group from the mapping class group arising from M theory on  $T^4$  is the  $\mathbb{Z}_2$  extension  $\widetilde{SL}(4, \mathbb{Z})$  of  $SL(4, \mathbb{Z})$ , as discussed in section 3. Similarly, we expect (though have not carefully checked) that the relevant T-duality group is a (different)  $\widetilde{SL}(4, \mathbb{Z})$ . They should combine into a  $\mathbb{Z}_2$  extension of  $SL(5, \mathbb{Z})$ , and the natural possibility is  $\widetilde{SL}(5, \mathbb{Z})$ , which we conjecture to be the case.

Formally repeating the arguments in [5][section 3.4], the U-duality group of the eight-dimensional decompactification limit should be the subgroup

$$\frac{\widetilde{SL}(3, \mathbb{Z}) \times Mp(2, \mathbb{Z})}{\mathbb{Z}_2},$$

a  $\mathbb{Z}_2$  extension of  $SL(2, \mathbb{Z}) \times SL(3, \mathbb{Z})$ , consistent with our results in section 7.2.

## 8 Conclusions

In this short paper we have proposed that duality group actions in high-dimensional theories with maximal supersymmetry should be slightly enlarged, by nontrivial  $\mathbb{Z}_2$  extensions, to correctly describe duality group actions on fermions. We have argued this separately for mapping class groups of tori in M theory compactifications, T-duality groups, ten-dimensional IIB S-duality, and briefly four-dimensional  $N = 4$  theories, and checked the proposals against one another by exploring U-duality groups in dimensions nine, eight, and seven.

We have only considered U-duality groups of high-dimensional string compactifications. It would be interesting, albeit more technically complex, to extend to lower-dimensional cases and cases with less supersymmetry.

One possible application of such an extension would be to try to identify the Bagger-Witten bundle over moduli spaces of K3 superconformal field theories. One of the original motivations for this work, after all, was to understand whether the metaplectic group appearing in the description of the stringy moduli stack of elliptic curves given in [1] was merely a formal quirk or reflected physical dualities. An understanding of U-duality groups in six-dimensional compactifications of string theory could be used to give analogous information concerning moduli stacks of K3 superconformal field theories.

Another natural question concerns spinors and Bagger-Witten line bundles on special Kähler moduli spaces. As briefly outlined in section 2, the analogue of a spin structure on a symplectic manifold is a metaplectic structure, defining a bundle whose structure group is a metaplectic group. This suggests that a detailed investigation of spinors and Bagger-Witten line bundles on special Kähler manifolds will reveal that the metaplectic group plays an important role there as well. We leave such considerations for future work.

In passing, it is also tempting to wonder whether considerations such as those in this paper for  $N = 4$  theories in four dimensions are relevant to metaplectic geometric Langlands theory, as in [27, 28].

## Acknowledgments

We would like to thank L. Anderson, P. Aspinwall, D. Auroux, J. Distler, J. Francis, J. Gray, I. Melnikov, D. Morrison, B. Pioline, S. Sethi, P. West, and E. Witten for useful conversations. This work was performed in part at the Aspen Center for Physics, which is supported by National Science Foundation grant PHY-1066293. T.P. was partially supported by NSF grants DMS-1302242 and DMS-1601438. E.S. was partially supported by NSF grant PHY-1417410.

**Open Access.** This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

## References

- [1] W. Gu and E. Sharpe, *Bagger-Witten line bundles on moduli spaces of elliptic curves*, [arXiv:1606.07078](https://arxiv.org/abs/1606.07078) [[INSPIRE](#)].
- [2] D. Morrison, private communication.
- [3] M. Forger and H. Hess, *Universal metaplectic structures and geometric quantization*, *Commun. Math. Phys.* **64** (1979) 269 [[INSPIRE](#)].
- [4] R. Hain, *Lectures on moduli spaces of elliptic curves*, in *Transformation groups and moduli spaces of curves*, L. Ji and S.-T. Yau eds., *Adv. Lect. Math.* **16**, International Press, U.S.A. (2011), pg. 95 [[arXiv:0812.1803](https://arxiv.org/abs/0812.1803)].
- [5] E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys.* **B 443** (1995) 85 [[hep-th/9503124](https://arxiv.org/abs/hep-th/9503124)] [[INSPIRE](#)].
- [6] E. Sharpe, *A few Ricci-flat stacks as phases of exotic GLSM's*, *Phys. Lett.* **B 726** (2013) 390 [[arXiv:1306.5440](https://arxiv.org/abs/1306.5440)] [[INSPIRE](#)].
- [7] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley & Sons, New York U.S.A. (1978).
- [8] A. Giveon, N. Malkin and E. Rabinovici, *On discrete symmetries and fundamental domains of target space*, *Phys. Lett.* **B 238** (1990) 57 [[INSPIRE](#)].
- [9] M. Dine, P.Y. Huet and N. Seiberg, *Large and small radius in string theory*, *Nucl. Phys.* **B 322** (1989) 301 [[INSPIRE](#)].



- [10] J. Dai, R.G. Leigh and J. Polchinski, *New connections between string theories*, *Mod. Phys. Lett. A* **4** (1989) 2073 [INSPIRE].
- [11] N.A. Obers and B. Pioline, *U duality and M-theory*, *Phys. Rept.* **318** (1999) 113 [[hep-th/9809039](#)] [INSPIRE].
- [12] J.H. Schwarz and P.C. West, *Symmetries and transformations of chiral  $N = 2$   $D = 10$  supergravity*, *Phys. Lett. B* **126** (1983) 301 [INSPIRE].
- [13] P.S. Howe and P.C. West, *The complete  $N = 2$ ,  $D = 10$  supergravity*, *Nucl. Phys. B* **238** (1984) 181 [INSPIRE].
- [14] J.H. Schwarz, *Covariant field equations of chiral  $N = 2$   $D = 10$  supergravity*, *Nucl. Phys. B* **226** (1983) 269 [INSPIRE].
- [15] M.R. Gaberdiel and M.B. Green, *An  $SL(2, Z)$  anomaly in IIB supergravity and its F-theory interpretation*, *JHEP* **11** (1998) 026 [[hep-th/9810153](#)] [INSPIRE].
- [16] P. West, *An introduction to strings and branes*, Cambridge University Press, Cambridge U.K. (2012).
- [17] M.B. Green and S. Sethi, *Supersymmetry constraints on type IIB supergravity*, *Phys. Rev. D* **59** (1999) 046006 [[hep-th/9808061](#)] [INSPIRE].
- [18] R.E. Borcherds, *Automorphic forms with singularities on Grassmannians*, *Invent. Math.* **132** (1998) 491 [[alg-geom/9609022](#)] [INSPIRE].
- [19] A. Basu, M.B. Green and S. Sethi, *A curious truncation of  $N = 4$  Yang-Mills*, *Phys. Rev. Lett.* **93** (2004) 261601 [[hep-th/0406267](#)] [INSPIRE].
- [20] J.A. Harvey, G.W. Moore and A. Strominger, *Reducing  $S$  duality to  $T$  duality*, *Phys. Rev. D* **52** (1995) 7161 [[hep-th/9501022](#)] [INSPIRE].
- [21] M. Bershadsky, A. Johansen, V. Sadov and C. Vafa, *Topological reduction of 4D SYM to 2D  $\sigma$ -models*, *Nucl. Phys. B* **448** (1995) 166 [[hep-th/9501096](#)] [INSPIRE].
- [22] E. Witten, *On  $S$  duality in Abelian gauge theory*, *Selecta Math.* **1** (1995) 383 [[hep-th/9505186](#)] [INSPIRE].
- [23] P.S. Aspinwall, *Some relationships between dualities in string theory*, *Nucl. Phys. Proc. Suppl.* **46** (1996) 30 [[hep-th/9508154](#)] [INSPIRE].
- [24] A. Basu, *The  $D^4$   $R^4$  term in type IIB string theory on  $T^2$  and  $U$ -duality*, *Phys. Rev. D* **77** (2008) 106003 [[arXiv:0708.2950](#)] [INSPIRE].
- [25] E. Kiritsis and B. Pioline, *On  $R^4$  threshold corrections in IIB string theory and  $(p, q)$  string instantons*, *Nucl. Phys. B* **508** (1997) 509 [[hep-th/9707018](#)] [INSPIRE].
- [26] J.H. Schwarz, *An  $SL(2, Z)$  multiplet of type IIB superstrings*, *Phys. Lett. B* **360** (1995) 13 [*Erratum* *ibid.* **364** (1995) 252] [[hep-th/9508143](#)] [INSPIRE].
- [27] A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric Langlands program*, *Commun. Num. Theor. Phys.* **1** (2007) 1 [[hep-th/0604151](#)] [INSPIRE].
- [28] D. Gaiatsgory and S. Lysenko, *Parameters and duality for the metaplectic geometric Langlands theory*, [arXiv:1608.00284](#).