STRUCTURES AND PROPERTIES OF REPEATED MEASUREMENT DESIGNS

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(ABSTRACT)

In this study the structure and properties of repeated measurement (RM) designs are investigated from different points of view, such as (i) balancedness or partial balancedness, (ii) construction versus estimation, (iii) underlying linear models, (iv) factorial treatment structure.

In studying balanced repeated measurement designs for the first order residual effects model it becomes apparent that one has to distinguish between balancedness with resepct to construction and balancedness with respect to estimation. These two concepts do not necessarily imply each other as they do, for example, for the balanced incomplete block design. Such designs are referred to as BRM1 and BRM1E designs, respectively. It is shown that they are imbedded in a much larger class of RM designs. This class is based on generalized partially balanced incomplete block designs and hence referred to as GPBRM1 designs.

The properties of GPBRM1 designs can be investigated by means of association matrices. For the construction of these designs the concept of asymmetrically repeated differences is introduced as a

generalization of symmetrically repeated differences used for constructing certain PBIB designs.

Another generalization of RM designs concerns the underlying linear model. In particular, the situation is considered where in addition to first order residual effects the model also contains second order residual effects. This leads to BRM2 and BRM2E designs. Extension to k^{th} order residual effect models are mentioned briefly.

Modifications of existing RM designs can be achieved if the treatments have a factorial structure and if certain, usually higher order, interactions can be considered negligible. In particular, it is shown how this can lead to a substantial reduction in the number of periods and/or subjects for a RM design.

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DEDICATION

To my father and mother, Dr. Zie-chung and Mrs. Sharn-jun Shing, and my wife Chie-mey and the rest of my family for their love and support.

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I. INTRODUCTION

A repeated measurement design, also referred to as crossover, changeover, carryover, switchover, switchback, reversal, before-after or multiple time series, with n subjects, t treatments and p periods, is an experimental design in which n subjects (experimental units) can be used repeatedly by exposing them to a sequence of t different or identical treatments in p periods. The effect of the treatment being applied is called the direct effect, and the effect of the previous treatment is called the residual effect (or carryover effect). If a residual effect is incurred after i periods, then we call it the i^{th} order residual effect.

The simplest such design is that of two subjects and two treatments, A and B, in two periods. We can apply treatment A to subject 1 in period 1 and treatment B in period 2. For subject 2, we reverse the order of application, that is, we apply treatment B to subject 2 in period 1 and treatment A in period 2. In this case, we say that we apply sequence 1 (treatment A first, then treatment B) to subject 1 and apply sequence 2 (treatment B first, then treatment A) to subject 2. Since there are two treatments applied to each subject, this is a repeated measurement design with $t = 2$, $p = 2$ and $n = 2$.

Repeated measurement designs are often used, for example, in pharmacology, in animal science, and in agriculture, as illustrated by the following examples:

(1) In drug manufacturing the question is whether a new drug is better than an old drug. The observation y_{ij} is the cumulative

amount of the drug in the blood stream of the j^{th} person in the i^{th} period.

(2) In animal science (e.g. in experiments involving the feeding of dairy cows), we are interested in finding a particular ration for the best milk yield during a lactation period. The treatments are the rations and the observation y_{ij} is the yield of milk in pounds for the j^{th} cow in the i^{th} period.

(3) In agriculture (e.g. crop rotation trials), we are interested in the residual fertility effect of the soil from the last harvest period. The treatments are the fertilizers and the observation y_{ij} is the amount of harvest on the j^{th} plot in the i^{th} year.

There are several reasons for using this design in practice:

(1) Budget limitations: either because subjects are expensive or because we have limited funds, we need to apply more than one treatment to each subject.

(2) Time limitations: sometimes the experiment needs to train subjects over a long period of time. Because we have limited time to conduct our experiment, we need to apply more treatments to each subject.

(3) Source limitations: even if we have enough time and money to conduct our experiment, we may have a limited source of subjects, so we must use the subjects repeatedly.

(4) The effects of the treatments do not have a serious damaging effect on the subjects, so the subjects can be used repeatedly for successive experiments.

(5) Sequence effects: the objective of an experiment, such as the drug, nutrition or learning experiment, is to find out the effect of different sequences.

(6) Residual effects: we are actually interested in the residual effects, as in the agriculture experiment.

(7) The variation between the subjects is much larger than the variation within the individual subjects.

Since each observation consists of cumulative effects, involving both direct and residual effects, the analysis of the repeated measurement design will be more complicated than that of the design without residual effects. There are three ways to make the analysis simple: (1) by inserting a washout period (called a rest period) between the treatment periods; (2) by adding a preperiod; (3) by using balanced repeated measurement designs.

1.1 Washout Period

The washout period consists of a period of no treatment. For example:

		subjects	
		1	2
	1	A	B
periods	2	-	-
	3	B	A

In this arrangement period 2 is a washout period. In this way the residual effects can be separated from the direct effects by using the washout period above, or they can be ignored completely if we are interested in the direct effects only.

The advantage of this method is that we can obtain the estimates of the direct and/or the residual effects more precisely than those of any other repeated measurement designs. The disadvantage is that we need more periods resulting in an excessively long experiment.

1.2 Extra Period

In repeated measurement designs Lucas (1957) and Cochran and Cox (1957) have pointed out the following: (1) residual effects are less precisely determined than direct effects; and (2) there is a positive covariance between direct and residual effects.

These results lead to certain problems if residual effects are just as important as direct effects. To remove the nonorthogonality between direct effects and residual effects, Lucas (1957) suggested using an extra period for designs with $t = p$, while Patterson and Lucas (1959) suggested using an extra period for designs with $t > p$

For example, we can add a preperiod to the simplest design by using the same treatments in the preperiod (period 0) as in the first period without taking any observation in the preperiod:

		subjects	
		1	2
periods	0	A	B
	1	A	B
	2	B	A

Then we can see that all the direct effects occur together exactly once with all the residual effects; that is, they are orthogonal. Thus we can get all the direct effects and all the residual effects with equal precision; also we can estimate the residual effects more precisely than from designs without the preperiod.

1.3 Balanced Repeated Measurement Designs

Since the residual effects and the treatment effects are non-orthogonal, the residual effects should be adjusted for the treatment effects. Naturally we expect that each treatment is preceded by every other treatment equally often so that the residual effects can be estimated easily by balancing out the treatment effects which are applied in the previous period. These designs are the balanced repeated measurement designs from the combinatorial point of view. We will discuss this in more detail in the next chapter. At this point we provide the following definition for a specific kind of repeated measurement designs, called RM(t,n,p) designs, where no treatment is applied to the same subject more than once.

Definition 1.3.1

A RM (t,n,p) design is a repeated measurement design, based on t distinct treatments applied to n subjects in p periods.

1.4 Purpose of Investigation

The purpose of this thesis is to examine the structures and properties of repeated measurement designs. Using association matrices for generalized PBIB designs we generalize a class of partially balanced repeated measurement designs and thereby introduce and characterize a larger class of repeated measurement in Chapter III. The principles of analysis for first residual effect RM designs are given in Chapter IV and are applied to generalized partially balanced repeated measurement (GPBRM) designs in Chapter V. A method of constructing these GPBRM designs under the existence of first order residual effects is given in Chapter VI. Further generalizations are discussed in Chapter VII.

In Chapter VIII we consider the special case of RM designs with factorial treatment structures, in particular 2^n factorials. We investigate the possible structures of such designs when higher order interactions can be ignored. Of particular interest is the reduction in the number of periods and/or subjects.

II. LITERATURE REVIEW

In this chapter we will discuss the development of the balanced repeated measurement and the partially balanced repeated measurement designs under the existence of the first and possibly second order residual effects. And we will also review the definitions of the balanced repeated measurement designs as used by different authors.

2.1 Designs for First Order Residual Effects

2.1.1. BRM1 Designs

The most commonly used RM (t,n,p) designs are balanced designs for first order residual effects given by Williams (1949) and Hedayat and Afsarinejad (1975). The combinatorial properties of such designs are given in the following:

Definition 2.1.1 (Williams 1949, Hedayat and Afsarinejad 1975)

A RM (t,n,p) design is said to be balanced, or a BRM1 (t,n,p) design, with respect to sets of direct and first order residual effects, if (1) each treatment occurs λ_1 times in each period; (2) each treatment is preceded by every other treatment λ_2 times.

Williams (1949, 1950) called these designs balanced residual effects designs. He constructed a series of these designs by using one or two cyclic Latin Squares for $t = p$. For example for $t = 6$, we have a Latin Square:

		subjects					
		1	2	3	4	5	6
periods	1	1	2	3	4	5	6
	2	6	1	2	3	4	5
	3	2	3	4	5	6	1
	4	5	6	1	2	3	4
	5	3	4	5	6	1	2
	6	4	5	6	1	2	3

This is a BRM1 ($t = 6$, $n = 6$, $p = 6$) design, where $\lambda_1 = 1$ and $\lambda_2 = 1$.

For $t = 3$, we have a BRM1 ($t = 3$, $n = 6$, $p = 3$) design, with $\lambda_1 = 2$, $\lambda_2 = 2$, consisting of two Latin Squares:

		subjects					
		1	2	3	4	5	6
periods	1	1	2	3	3	1	2
	2	3	1	2	1	2	3
	3	2	3	1	2	3	1

For reference in later chapters we shall discuss briefly Williams' method of constructing these designs.

For t treatments we construct the first subject with t periods by entering successive number starting from integer 1 in every other period from period 1 to period t and reversing the direction once period t has been reached. The remaining $p - 1$ subjects can be constructed cyclically from the first subject.

For example, for a 2^3 factorial design, a RM (2^3, 8, 8) design can be constructed by Williams' method as follows:

periods \ subjects	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	8	1	2	3	4	5	6	7
3	2	3	4	5	6	7	8	1
4	7	8	1	2	3	4	5	6
5	3	4	5	6	7	8	1	2
6	6	7	8	1	2	3	4	5
7	4	5	6	7	8	1	2	3
8	5	6	7	8	1	2	3	4

If we associate those eight numbers with the eight factorial treatments, e.g., 1 with (1), 2 with a, 3 with b, 4 with c, 5 with ab, 6 with ac, 7 with bc, 8 with abc, then we have the following design:

periods \ subjects	1	2	3	4	5	6	7	8
1	(1)	a	b	c	ab	ac	bc	abc
2	abc	(1)	a	b	c	ab	ac	bc
3	a	b	c	ab	ac	bc	abc	(1)
4	bc	abc	(1)	a	b	c	ab	ac
5	b	c	ab	ac	bc	abc	(1)	a
6	ac	bc	abc	(1)	a	b	c	ab
7	c	ab	ac	bc	abc	(1)	a	b
8	ab	ac	bc	abc	(1)	a	b	c

The properties of Williams' BRM1 designs are as follows:

(1) all direct effects are estimated with equal precision;

(2) all residual effects are estimated with equal precision.

The advantages of Williams' BRM1 designs over unbalanced RM designs are these: (1) the efficiency is increased (i.e. we can get more precise estimates of direct and residual effects than any unbalanced RM design); (2) the design and analysis are simpler than those for other designs.

However, when $p < t$, we cannot use Williams' designs. Patterson (1950, 1951, 1952) used incomplete Latin Squares (called Generalized Youden Squares) to construct BRM1 designs for such a situation.

For example, the RM ($t = 4$, $n = 12$, $p = 3$) design

		subjects											
		1	2	3	4	5	6	7	8	9	10	11	12
	1	1	2	3	4	1	2	3	4	1	2	3	4
periods	2	2	1	4	3	3	4	1	2	4	3	2	1
	3	3	4	1	2	4	3	2	1	2	1	4	3

is a BRM1 design consisting of three 3×4 incomplete Latin Squares.

2.1.2 BRMP Designs

Patterson (1950, 1951, 1952) also gave a definition for a BRM1 design, which we shall refer to as a BRMP design.

Definition 2.1.2.1 (Patterson 1952)

A BRMP (t,n,p) design is a BRM1 (t,n,p) design which satisfies the following conditions:

(1) the design is a BIB with subjects used as blocks;

(2) deleting the last period, the design is still a BIB;

(3) for every pair of treatments (A, B), the number of subjects in which B occurs when A is in the last period is the same as the

number of subjects in which A occurs when B is in the last period.

From definition 2.1.2.1, a BRMP design is a BRM1 design, but not vice versa. For example, the BRM1 ($t=7$, $n=21$, $p=3$) design with $\lambda_1 = 2$, $\lambda_2 = 1$ is a BRMP design:

	subjects																				
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
periods 1	1	2	3	4	5	6	7	1	2	3	4	5	6	7	1	2	3	4	5	6	7
periods 2	6	7	1	2	3	4	5	4	5	6	7	1	2	3	7	1	2	3	4	5	6
periods 3	7	1	2	3	4	5	6	6	7	1	2	3	4	5	4	5	6	7	1	2	3

However, the following BRM1 ($t=5$, $n=10$, $p=3$) design with $\lambda_1 = 2$, $\lambda_2 = 1$ is not a BRMP design because for the treatment pair (2,3), the number of subjects in which treatment 3 occurs when treatment 2 is in the last period is 0, on the other hand, the number of subjects in which treatment 2 occurs when treatment 3 is in the last period is 1:

	subjects									
	1	2	3	4	5	6	7	8	9	10
periods 1	1	3	5	4	2	4	3	2	5	1
periods 2	2	4	1	5	3	3	2	1	4	5
periods 3	4	1	4	2	5	1	5	3	2	3

2.1.3 PBRMP Designs

In practice, we may have situations where the number of subjects is not large enough for a BRM1 design. Although we may not be able then to estimate all the direct effects or all the residual effects with the same precision, we can try to estimate some direct effects or some residual effects more precisely than others. For this reason, using subjects as blocks, Patterson and Lucas (1962) developed partially balanced repeated measurement designs which we shall refer to as PBRMP designs.

Definition 2.1.3.1 (Patterson and Lucas 1962)

A PBRMP design is a RM design which satisfies the following conditions:

(1) each treatment occurs λ_1 times in each period;

(2) for any given treatment, the remaining treatments can be divided into two sets of associates; the first associates immediately follow the treatment m_1 times, and the second associates immediately follow the treatment m_2 times;

(3) the design is a PBIB(2) with the same associate classes as in (2);

(4) deleting the last period, the design is still a PBIB(2) with the same associate classes as in (2);

(5) for every pair of treatments (A, B), the number of subjects in which B occurs when A is in the last period is the same as the number of subjects in which A occurs when B is in the last period.

Patterson and Lucas (1962) used some Latin Squares to construct these designs in which each Latin Square is a BRM1 design and the whole design is a PBIB(2) design.

For example, the following RM ($t = 6$, $n = 12$, $p = 4$) design, which has a PBIB(2) structure, is a PBRMP design:

		subjects											
		1	2	3	4	5	6	7	8	9	10	11	12
periods	1	1	4	2	5	2	5	3	6	3	6	1	4
	2	4	5	1	2	5	6	2	3	6	4	3	1
	3	2	1	5	4	3	2	6	5	1	3	4	6
	4	5	2	4	1	6	3	5	2	4	1	6	3

where $\lambda_1 = 2$, $m_1 = 1$ and $m_2 = 2$.

The above PBIB(2) design has the following association scheme:

treatment	1st associate μ_1=4	2nd associate μ_2=8
1	2, 3, 5, 6	4
2	1, 3, 4, 6	5
3	1, 2, 4, 5	6
4	2, 3, 5, 6	1
5	1, 3, 4, 6	2
6	1, 2, 4, 5	3

where, e.g., treatment 1 and treatment 2 appear together in $\mu_1 = 4$ subjects whereas treatment 1 and treatment 4 appear together in $\mu_2 = 8$ subjects. And deleting the last period, the resulting design is still a PBIB(2) design with $\mu_1 = 2$ and $\mu_2 = 4$.

Davis and Hall (1969) constructed BRM1 and partially balanced repeated measurement designs cyclically, using PBIB's with more than two associate classes in the PBRMP's definition. These designs are therefore called cyclic RM designs. For example, the RM ($t = 7$, $n = 21$, $p = 3$) design is a BRM1 design and a BRMP design constructed by three initial subjects (1 6 7), (1 4 6) and (1 7 4) and then developed cyclically:

		subjects																				
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
	1	1	2	3	4	5	6	7	1	2	3	4	5	6	7	1	2	3	4	5	6	7
periods	2	6	7	1	2	3	4	5	4	5	6	7	1	2	3	7	1	2	3	4	5	6
	3	7	1	2	3	4	5	6	6	7	1	2	3	4	5	4	5	6	7	1	2	3

This is a BIB design with $\lambda_1 = 3$ and $\lambda_2 = 1$, where each treatment occurs with every other treatment on three subjects.

The following RM $(t = 6,\ n = 12,\ p = 4)$ design is a partially balanced RM design constructed by developing two intial subjects (1 2 4 3) and (1 4 2 5) cyclically:

		subjects											
		1	2	3	4	5	6	7	8	9	10	11	12
	1	1	2	3	4	5	6	1	2	3	4	5	6
periods	2	2	3	4	5	6	1	4	5	6	1	2	3
	3	4	5	6	1	2	3	2	3	4	5	6	1
	4	3	4	5	6	1	2	5	6	1	2	3	4

The above design is a PBIB(3) design with $\lambda_1 = 2$ and the following association scheme:

treatment	1st associate μ_1=6	2nd associate μ_2=5	3rd associate μ_3=4
1	4	2, 6	3, 5
2	5	3, 1	4, 6
3	6	2, 4	5, 1
4	1	3, 5	2, 6
5	2	4, 6	3, 1
6	3	5, 1	4, 2

where, e.g., treatment 1 and treatment 4 appear together in six subjects, treatment 1 and treatment 2 appear together in five subjects, and treatment 1 and treatment 3 appear together in four subjects.

2.1.4 PBRM(m) Designs

Blaisdell and Raghavarao (1980) formulated a partially balanced RM design based on an m-associate class PBIB design and provided some series of triangular and rectangular designs.

Definition 2.1.4.1 (Blaisdell and Raghavarao 1980)

A RM (t,n,p) design is said to be partially balanced, or a PBRM (t,n,p;m) deisgn, based on an m-associate class PBIB design with t treatments in p periods and n subjects, such that the following properties hold:

(1) Every treatment occurs λ_1 times in each period;

(2) Every pair of treatments (A, B) occurs together in μ_i subjects if A and B are i^{th} associates;

(3) Deleting the last period of the design, every pair of treatments (A, B) occurs together in ν_i subjects if A and B are i^{th} associates;

(4) Every ordered pair of treatments (A, B) occurs together in successive periods in ρ_i subjects if A and B are i^{th} associates;

(5) For every pair of treatments (A, B), the number of subjects in which B occurs when A is in the last period is the same as the number of subjects in which A occurs when B is in the last period (Condition C_1).

For example, the RM ($t = 9$, $n = 18$, $p = 4$) design given below is a PBRM(m=3) design, based on a rectangular association scheme with parameters $\lambda_1 = 2$, $\mu_1 = 6$, $\mu_2 = 2$, $\mu_3 = 2$, $\nu_1 = 6$, $\nu_2 = 0$, $\nu_3 = 0$, $\rho_1 = 2$, $\rho_2 = 1$, $\rho_3 = 0$ and, it satisfies condition C_1 since, e.g., for treatment pair (1, 2), the number of subjects in which treatment 2 occurs when treatment 1 is in the last period is zero and the number of subjects in which treatment 1 occurs when treatment 2 is in the last period is zero.

		subjects																	
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
periods	1	1	2	3	3	1	2	4	5	6	6	4	5	7	8	9	9	7	8
	2	2	3	1	2	3	1	5	6	4	5	6	4	8	9	7	8	9	7
	3	3	1	2	1	2	3	6	4	5	4	5	6	9	7	8	7	8	9
	4	6	4	5	7	8	9	3	1	2	7	8	9	3	1	2	4	5	6

The above PBIB(3) design has the following association scheme:

treatment	1st associate μ_1=6	2nd associate μ_2=2	3rd associate μ_3=2
1	2, 3	4, 7	5, 6, 8, 9
2	1, 3	5, 8	4, 6, 7, 9
3	2, 1	6, 9	4, 5, 7, 8
4	5, 6	1, 7	2, 3, 8, 9
5	4, 6	2, 8	1, 3, 7, 9
6	4, 5	3, 9	1, 2, 7, 8
7	8, 9	4, 1	2, 3, 5, 6
8	7, 9	5, 2	1, 3, 4, 6
9	7, 8	6, 3	1, 2, 4, 5

The properties of PBRM(m) designs are as follows: (1) we can estimate the direct effects with at most m different precisions;

(2) we can also estimate the residual effects with at most m different precisions.

2.2 Second Order Residual Effects

Williams (1949, 1950) also gave a definition for a RM design which is balanced for both first and second order residual effects, which we shall refer to as BRMW2 designs.

Definition 2.2.1 (Williams 1949, 1950)

A RM (t,n,p) design is said to be balanced, or a BRMW2 (t,n,p) design, with respect to sets of direct, first and second order residual effects, if (1) each treatment occurs λ_1 times in each period; and (2) each treatment is preceded by each ordered pair of other treatments λ_3 times.

Williams used a set of $t-1$ mutually orthogonal Latin Squares to construct BRMW2 designs, where t is a prime or a power of a prime.

For instance, the following RM ($t=4$, $n=12$, $p=4$) design, which is constructed from three Latin Squares, is a BRMW2 design, where $\lambda_1 = 3$ and $\lambda_3 = 1$:

		subjects											
		1	2	3	4	5	6	7	8	9	10	11	12
periods	1	4	3	2	1	4	3	2	1	4	3	2	1
	2	1	2	3	4	3	4	1	2	2	1	4	3
	3	2	1	4	3	1	2	3	4	3	4	1	2
	4	3	4	1	2	2	1	4	3	1	2	3	4

(2.1)

III. GENERALIZED PARTIALLY BALANCED REPEATED MEASUREMENT DESIGNS FOR FIRST ORDER RESIDUAL EFFECTS

In general, with incomplete block designs we would like to choose the most efficient design, that is, a balanced one, because the notion of balancedness implies that comparisons among direct effects are estimated with the same precision, as are differences among residual effects. With respect to direct effects (and assuming no residual effects) this is certainly the case for BIB designs. For the BRM1 design as defined earlier this is, however, not necessarily true: Combinatorial balance does not necessarily imply variance balance as described above. In fact, if we consider the following three properties:

P_1: there are M_1 different variances among all the comparisons of direct effects;

P_2: there are M_2 different variances among all the comparisons of first order residual effects;

P_3: there are M_3 different covariances among all the direct and first order residual effects,

then from the estimation (i.e. inferential) point of view, a BRM1 design can have P_1, P_2 and P_3, where M_1, M_2 and M_3 are not all equal to 1.

For example, the following RM ($t = 9$, $n = 18$, $p = 5$) design is a BRM1 design constructed from a rectangular association scheme. It has the properties P_1, P_2 and P_3 with $M_1 = M_2 = M_3 = 3$:

		subjects																	
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
	1	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9
	2	7	8	4	9	5	6	2	9	3	6	3	5	1	8	1	7	2	4
periods	3	2	3	3	1	2	1	7	6	8	9	4	7	6	4	9	5	5	8
	4	8	7	9	4	6	5	9	2	6	3	5	3	8	1	7	1	4	2
	5	5	4	8	6	7	9	5	1	4	1	8	2	9	3	6	2	7	3

The cyclic BRM1 designs also have the same properties. For instance, the RM ($t = 5$, $n = 10$, $p = 3$) design below (from Hedayat and Afsarinejad 1975) is a cyclic BRM1 design with two initial subjects (1 2 4) and (4 3 1):

		subjects									
		1	2	3	4	5	6	7	8	9	10
	1	1	2	3	4	5	4	5	1	2	3
periods	2	2	3	4	5	1	3	4	5	1	2
	3	4	5	1	2	3	1	2	3	4	5

It has P_1, P_2 and P_3 with $M_1 = M_2 = M_3 = 2$. BRMP designs, which have P_1, P_2 and P_3 with $M_1 = M_2 = M_3 = 1$, are the most efficient designs.

In practice if we need to use a design which has the properties P_1 and P_2 where $M_1 > 1$ and $M_2 > 1$, we can use a PBRM(m) design which has P_1, P_2 and P_3, with $M_1 = M_2 = M_3 = m > 1$.

However, we may want to use designs for which M_1, M_2 and M_3 are not all equal. For example, a design with $M_1 = 1$ and M_2, M_3 arbitrary may be an attractive design, if it exists. This cannot be achieved with PBRM(m) designs; instead one may have to generalize the definition of a PBRM(m) design.

There do exist RM designs which are more general than those defined earlier. For example, the RM $(t=4,\ n=8,\ p=4)$ design given below has the properties P_1, P_2 and P_3, where $M_1 = M_2 = M_3 = 2$, but does not satisfy the condition (4) in definition 2.1.4.1 of a PBRM $(t=4,\ n=8,\ p=4;\ m=2)$ design:

		subjects							
		1	2	3	4	5	6	7	8
periods	1	1	1	2	2	3	3	4	4
	2	3	4	3	4	1	2	1	2
	3	2	2	1	1	4	4	3	3
	4	4	3	4	3	2	1	2	1

3.1 PBRM1 Designs

To generalize PBRM(m) designs for the first order residual effects, we give the following definition for partially balanced repeated measurement designs:

Defintion 3.1.1

A PBRM1 (t,n,p) design is a repeated measurement design, which is partially balanced for the first order residual effects, such that (1) it is a RM (t,n,p) design; (2) every treatment occurs λ_1 times in each period; (3) it is a PBIB design with subjects as blocks; (4) by deleting the last period, it is still a PBIB design; (5) for each treatment the remaining treatments can be divided into L associate classes, the first associates immediately follow the treatment M_1 times, ..., the L^{th} associates immediately follow the treatment M_L times.

For example, the following RM ($t = 3$, $n = 3$, $p = 3$) design has P_1, P_2 and P_3 with $M_1 = M_2 = 1$ and $M_3 = 2$ but it is not a PBRM(m) design because the design is a PBIB(m=1) design with subjects as blocks where all treatments are in one associate, however, concerning the order of treatments, for each treatment the remaining treatments can be divided into two associate classes, the first associates immediately follow the treatment zero times and the second associates immediately follow the treatment twice:

		subjects 1	2	3
	1	1	2	3
periods	2	2	3	1
	3	3	1	2

However, since for a PBRM1 design, M_1, M_2 and M_3 could be different, the above design is a PBRM1 design for the first order residual effects.

3.2 GPBRM1 Designs

We now propose a definition for a generalized partially balanced repeated measurement design (GPBRM1 design) assuming the existence of first order residual effects only.

To motivate this definition one can visualize conceptually that at any given period (except the first) each subject is exposed to two "treatments", the "direct treatment" and the "residual treatment" which produce the combined direct and residual effects as assumed in the linear model.

Definition 3.2.1

A GPBRM1 (t,n,p) design is a partially balanced repeated measurement design if the following conditions are met:

(1) it is a RM (t,n,p) design;

(2) every treatment occurs λ_1 times in each period;

(3) there exist 2t treatments (i.e. direct and first-order residual treatments) divided into two groups of t elements each (first group treatments and second group treatments);

(4) it has the following generalized PBIB design (GPBIB) properties: (Shah, 1959)

(a) any i^{th}-group treatment and j^{th}-group treatment are either $ij{:}0^{th}$, $ij{:}1^{th}$, ... or $ij{:}m_{ij}{}^{th}$ associates, where i, j = 1, 2, (every i^{th}-group treatment is the $ii{:}0^{th}$ associate of itself and of no other treatment); $ij{:}t^{th}$ associates are the same as $ji{:}t^{th}$ associates;

(b) each i^{th}-group treatment has exactly $n_{ij:t}$ $ij{:}t^{th}$ associates, where $j = 1, 2$, $t = 0, 1, \ldots, m_{ij}$;

(c) given any two treatments which are $ij{:}t^{th}$ associates, the number of treatments common to the $i_1j_1{:}t_1{}^{th}$ associates of the first and the $i_2j_2{:}t_2{}^{th}$ of the second plus the number of treatments common to the $i_1j_1{:}t_1{}^{th}$ associates of the second and the $i_2j_2{:}t_2{}^{th}$ of the first is $2\,P_{ij:t}\,(i_1j_1{:}t_1,\ i_2j_2{:}t_2)$. And it is also independent of the pair of treatments with which we start;

(d) two treatments which are $ij{:}t^{th}$ associates occur together in exactly $\lambda_{ij:t}$ subjects;

(e) two treatments which are $ij{:}t^{th}$ associates precede each other $\mu_{ij:t}$ times, where $i \neq j$ and $m_{ij} > 1$.

It follows from this definition that when the association schemes for 11, 22 and 12 associate classes are the same as those of a PBIB(m) under condition C_1, a GPBRM1 design will reduce to a PBRM(m) design. (See Definition 2.1.4.1.)

For example, the following RM ($t = 6$, $n = 18$, $p = 3$) design is a GPBRM design which has P_1, P_2 and P_3, with $M_1 = 3$, $M_2 = 4$ and $M_3 = 5$, and is constructed from a GPBIB design:

		subjects																	
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
	1	1	3	5	1	3	5	1	2	4	2	2	3	4	4	5	6	6	6
periods	2	2	4	6	2	4	6	6	3	5	1	1	2	3	3	4	5	5	1
	3	4	2	2	6	6	4	3	6	2	3	5	5	1	5	1	1	3	4

(3.1)

The above design is a PBIB design with subjects as blocks and the following association scheme:

1st group treatment	11:1 associates	11:2 associates
1	2, 4, 6	3, 5
2	1, 3, 5	4, 6
3	2, 4, 6	1, 5
4	1, 3, 5	2, 6
5	2, 4, 6	1, 3
6	1, 3, 5	2, 4

It can be seen from the actual design that any two treatments that are 11:1 associates occur together four times in the same subject; however, treatments that are 11:2 associates occur together only three times in the same subject.

On the other hand, deleting the last period of (3.1), the design results in a PBIB design with subjects as blocks and the following association scheme for the second group treatments:

2nd group treatment	22:1 associates	22:2 associates	22:3 associates
1	2	6	3, 4, 5
2	1	3	4, 5, 6
3	4	2	1, 5, 6
4	3	5	1, 2, 6
5	6	4	1, 2, 3
6	5	1	2, 3, 4

It follows from (3.1) that any two second group treatments that are 22:1 associates occur together four times in the same subject and any two second group treatments that are 22:2 associates occur together twice in the same subject. However, any two second group treatments that are 22:3 associates do not occur together in the same subject.

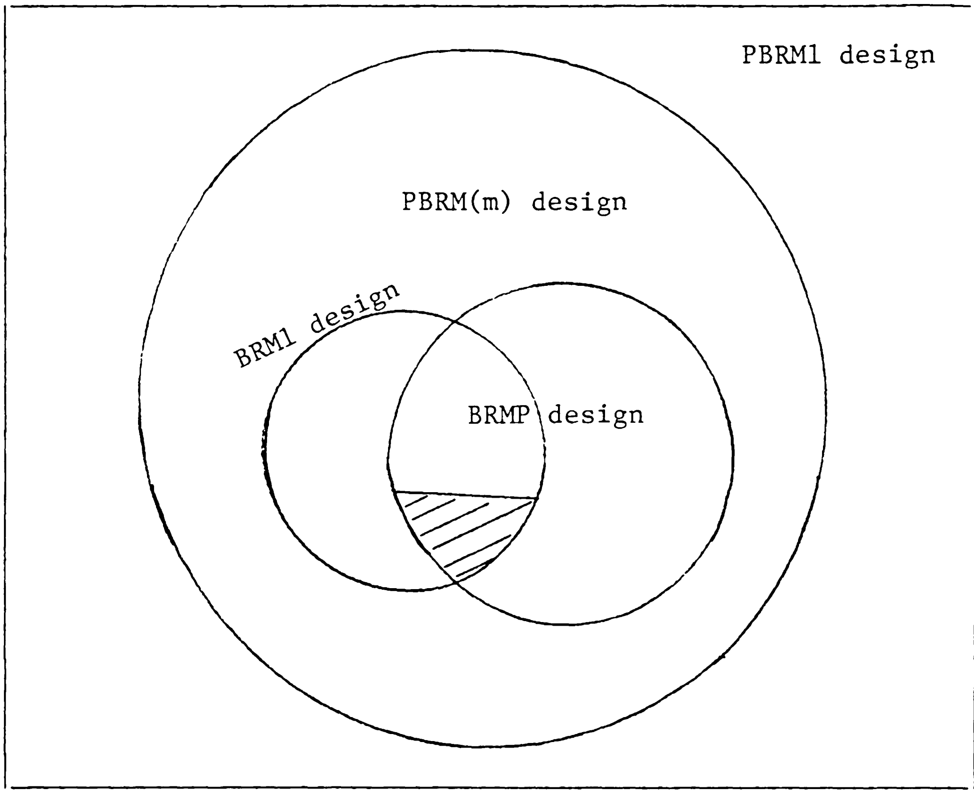
In (3.1) we can also see that the association scheme for first and second group treatments is as follows:

1st group treatment	second group treatments 12:1 associates	12:2 associates	12:3 associates	12:4 associates	12:5 associates
1	2	6	3	4	5
2	1	3	6	5	4
3	4	2	5	6	1
4	3	5	2	1	6
5	6	4	1	2	3
6	5	1	4	3	2

So far, we have given the definition for GPBRM1 designs. In the next two chapters, we will analyze the properties and the structures of GPBRM designs according to Definition 3.2.1.

In summary, we can see clearly that under condition C_1, a GPBRM1 design is a PBRM1 design and a PBRM1 design is also a PBRM(m) design. Meanwhile, a PBRM(m) design is a BRMP design. Since BRM1 designs are defined from the combinatorial point of view, a BRMP design then is also a BRM1 design. However, a BRM1 design is not a PBRM(m) design and vice versa in general. On the other hand, a BRMP design is a Williams' design. Their relationship can be shown in the following graph:

GPBRM1 design



where the set of BRMP designs is the intersection of the set of BRM1 designs and the set of PBRM(m) designs, and the shaded area is the set of Williams' designs.

In PBRM(m) designs if we estimate some direct effects more precisely than other direct effects, then the corresponding residual effects can also be estimated more precisely. However, GPBRM1 designs do not have this property.

For example, in the following GPBRM1 design,

		subjects																	
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
	1	1	3	5	1	3	5	1	2	4	2	2	3	4	4	5	6	6	6
periods	2	2	4	6	2	4	6	6	3	5	1	1	2	3	3	4	5	5	1
	3	4	2	2	6	6	4	3	6	2	3	5	5	1	5	1	1	3	4

the variances of $\hat{\tau}_1 - \hat{\tau}_2$ and $\hat{\tau}_1 - \hat{\tau}_4$ are

$$\mathrm{Var}(\hat{\tau}_1-\hat{\tau}_2) = 0.3664\sigma^2$$

and

$$\mathrm{Var}(\hat{\tau}_1-\hat{\tau}_4) = 0.3664\sigma^2 \quad .$$

However, the variances of $\hat{\alpha}_1 - \hat{\alpha}_2$ and $\hat{\alpha}_1 - \hat{\alpha}_4$ are not the same; it can be shown that

$$\mathrm{Var}(\hat{\alpha}_1-\hat{\alpha}_2) = 0.5102\sigma^2$$

and

$$\mathrm{Var}(\hat{\alpha}_1-\hat{\alpha}_4) = 0.7902\sigma^2 \ .$$

IV. ANALYSIS AND PROPERTIES OF THE FIRST ORDER RESIDUAL EFFECT MODEL

In this chapter we shall give the analysis and explore the structure of GPBRM1 designs under a simple model.

4.1 Analysis

An appropriate linear model for the observations $y_{ijk\ell}$ from a RM (t,n,p) design is as follows:

$$y_{ijk\ell} = \mu + \tau_i + s_j + \pi_k + \alpha_\ell + \varepsilon_{ijk\ell} \tag{4.1}$$

where

τ_i: direct effect, $i = 1, \ldots, t$

s_j: subject effect, $j = 1, \ldots, n$

π_k: period effect, $k = 1, \ldots, p$

α_ℓ: first order residual effect, $\ell = 1, \ldots, t$

are fixed effects, and $\varepsilon_{ijk\ell} \sim (0,\sigma^2)$ independently distributed. In matrix notation we can rewrite (4.1) as

$$\underline{Y} = X\underline{\beta} + \underline{\varepsilon} \tag{4.2}$$

where

$$\underline{\beta} = [\tau_1,\ldots,\tau_t,\alpha_1,\ldots,\alpha_t,\pi_1,\ldots,\pi_p,s_1,\ldots,s_n]' \quad .$$

The normal equations are

$$X'X\hat{\underline{\beta}} = X'\underline{Y}$$

where X'X and $X'\underline{y}$ can be partitioned as follows:

$$X'X = \begin{bmatrix} D & S & T & N \\ S' & D^* & T^* & N^* \\ T' & T^{*\prime} & nI_p & J_{p\times n} \\ N' & N^{*\prime} & J_{n\times p} & pI_n \end{bmatrix} \tag{4.3}$$

and

$$X'\underline{Y} = [T_1,\ldots,T_t,R_1,\ldots,R_t,P_1,\ldots,P_p,B_1,\ldots,B_n]'$$

where

T_i is the i^{th} direct effect total

R_ℓ is the ℓ^{th} residual effect total

P_k is the k^{th} period effect total

B_j is the j^{th} subject effect total

and

$$D = \begin{bmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_t \end{bmatrix}$$

where γ_i denotes the number of times that treatment i occurs in the design. Likewise,

$$D^* = \begin{bmatrix} \gamma_1^* & & 0 \\ & \ddots & \\ 0 & & \gamma_t^* \end{bmatrix}$$

where γ_i^* is the number of times that the residual effect of treatment i occurs in the design.

Also $S = (s_{ij})$ is the direct-residual effects incidence matrix where s_{ij} is the number of times treatment i and residual effect j occur in the design. And $N \equiv (n_{ij})$ is the direct-subject effect incidence matrix where n_{ij} is the number of times treatment i occurs in the j^{th} subject, $T \equiv (t_{ij})$ is the direct-period effects incidence matrix where t_{ij} is the number of times treatment i occurs in the j^{th} period. Finally, $N^* \equiv (n_{ij}^*)$ is the residual-subject effects incidence matrix where n_{ij}^* represents the number of times residual effect i occurs in the j^{th} subject and $T^* \equiv (t_{ij}^*)$ is the residual-period effects incidence matrix where t_{ij}^* represents the number of times residual effect i occurs in the j^{th} period.

The reduced normal equations for direct and residual effects can be written as

$$C \begin{bmatrix} \hat{\underline{\tau}} \\ \hat{\underline{\alpha}} \end{bmatrix} = Q_{(1)}$$

where

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

is the C-matrix adjusted for subjects and periods, i.e.,

$$C = \begin{bmatrix} D & S \\ S' & D^* \end{bmatrix} - \begin{bmatrix} T & N \\ T^* & N^* \end{bmatrix} \begin{bmatrix} nI_p & J_{p\times n} \\ J_{n\times p} & pI_n \end{bmatrix}^{-} \begin{bmatrix} T' & T^{*\prime} \\ N' & N^{*\prime} \end{bmatrix} . \quad (4.4)$$

Using the following general result (e.g. Rao, 1965)

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = |A_{11}|\,|A_{22} - A_{21}A_{11}^{-1}A_{12}|$$

we have

$$\begin{vmatrix} nI_p & J_{p\times n} \\ J_{n\times p} & pI_n \end{vmatrix} = |nI_p|\,|pI_n - J_{n\times p}(nI_p)^{-1}J_{p\times n}|$$

$$= n^p\,|pI_n - \frac{p}{n} J_n|$$

$$= n^p\, p^n\, |I_n - \frac{1}{n} J_n|$$

$$= 0$$

which implies that

$$\text{rank} \begin{bmatrix} nI_p & J_{p\times n} \\ J_{n\times p} & pI_n \end{bmatrix} \le n + p - 1 \quad .$$

Using a result from Rohde (1965), a generalized inverse of a matrix

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z'_{12} & Z_{22} \end{bmatrix}$$

is given by

$$Z = \begin{bmatrix} Z_{11}^{-} + Z_{11}^{-}Z_{12}Z_{1}^{-}Z'_{12}Z_{11}^{-} & -Z_{11}^{-}Z_{12}Z_{1}^{-} \\ -Z_{1}^{-}Z'_{12}Z_{11}^{-} & Z_{1}^{-} \end{bmatrix}$$

where

$$Z_1 = Z_{22} - Z'_{12}Z_{11}^{-}Z_{12} \quad .$$

Hence

$$\begin{bmatrix} nI_p & J_{p\times n} \\ J_{n\times p} & pI_n \end{bmatrix}^{-} = \begin{bmatrix} \frac{1}{n} I_p & 0 \\ 0 & \frac{1}{p} I_n - \frac{1}{np} J_n \end{bmatrix} \quad .$$

It follows from (4.4) that

$$C_{11} = D - \frac{1}{n} TT' - \frac{1}{p} NN' + \frac{1}{np} NJ_nN'$$

$$C_{12} = S - \frac{1}{n} TT^{*\prime} - \frac{1}{p} NN^{*\prime} + \frac{1}{np} NJ_nN^{*\prime} \tag{4.5}$$

$$C_{22} = D^* - \frac{1}{n} T^*T^{*'} - \frac{1}{p} N^*N^{*'} + \frac{1}{np} N^*J_n N^{*'}$$

$$C_{21} = C_{12}' \tag{4.5}$$

and

$$Q_{(1)} = \begin{bmatrix} \underset{\sim}{T}_0 \\ \underset{\sim}{R} \end{bmatrix} - \begin{bmatrix} T & N \\ T^* & N^* \end{bmatrix} \begin{bmatrix} nI_p & J_{p\times n} \\ J_{n\times p} & pI_n \end{bmatrix}^- \begin{bmatrix} \underset{\sim}{P} \\ \underset{\sim}{B} \end{bmatrix}$$

$$= \begin{bmatrix} \underset{\sim}{T}_0 - \frac{1}{p} N\underset{\sim}{B} - \frac{1}{n} T\underset{\sim}{P} + \frac{1}{np} TJ_p\underset{\sim}{P} \\ \underset{\sim}{R} - \frac{1}{p} N^*\underset{\sim}{B} - \frac{1}{n} T^*\underset{\sim}{P} + \frac{1}{np} T^*J_p\underset{\sim}{P} \end{bmatrix}$$

where $\underset{\sim}{T}_0 = [T_1,\ldots,T_t]'$ is the treatment total vector in which T_i is the treatment i total, $\underset{\sim}{B} = [B_1,\ldots,B_n]'$ is the subject total vector where B_i is the i^{th} subject total, $\underset{\sim}{R} = [R_1,\ldots,R_t]'$ is the residual total vector where R_i is the i^{th} residual effect total, $\underset{\sim}{P} = [P_1,\ldots,P_p]'$ is the period effect total vector where P_i is the i^{th} period total.

From (4.4) we have

$$\begin{bmatrix} \hat{\underline{\tau}} \\ \hat{\underline{\alpha}} \end{bmatrix} = C^- Q_{(1)} \tag{4.6}$$

where

$$C^{-} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^{-}$$

$$= \begin{bmatrix} (C_{11}-C_{12}C_{22}^{-}C_{21})^{-} & -C_{11}^{-}C_{12}(C_{22}-C_{21}C_{11}^{-}C_{12})^{-} \\ -C_{22}^{-}C_{21}(C_{11}-C_{12}C_{22}^{-}C_{21})^{-} & (C_{22}-C_{21}C_{11}^{-}C_{12})^{-} \end{bmatrix} .$$

Therefore, the variance-covariance matrix of direct effects, $\Sigma_{\underline{\hat{\tau}}}$, the variance-covariance matrix of residual effects, $\Sigma_{\underline{\hat{\alpha}}}$, and the variance-covariance matrix of direct-residual effects, $\Sigma_{\underline{\hat{\tau}},\underline{\hat{\alpha}}}$, are given as follows:

$$\Sigma_{\underline{\hat{\tau}}} = (C_{11}-C_{12}C_{22}^{-}C_{21})^{-}\sigma^2$$

$$\Sigma_{\underline{\hat{\alpha}}} = (C_{22}-C_{21}C_{11}^{-}C_{12})^{-}\sigma^2 \qquad (4.7)$$

$$\Sigma_{\underline{\hat{\tau}},\underline{\hat{\alpha}}} = -C_{11}^{-}C_{12}(C_{22}-C_{21}C_{11}^{-}C_{12})^{-}\sigma^2$$

4.1.1 BRM1 Designs

For the special case of a BRM1 design, we have the following:

$$D = \lambda_1 p I_t$$

$$S = -\lambda_2 I_t + \lambda_2 J_t$$

$$T = \lambda_1 J_{t\times p}$$

$$D^* = \lambda_1(p-1)I_t$$

$$T^* = \lambda_1\begin{bmatrix} 0 & \cdot & \\ \vdots & \vdots & J_{t\times(p-1)} \\ 0 & \cdot & \end{bmatrix} \quad .$$

Then

$$TT' = p\lambda_1^2 J_t$$

$$TT^{*\prime} = \lambda_1^2(p-1)J_t$$

$$T^*T^{*\prime} = \lambda_1^2(p-1)J_t$$

$$NJ_nN' = \lambda_1^2 p^2 J_t$$

$$NJ_nN^{*\prime} = \lambda_1^2 p(p-1)J_t$$

$$N^*J_nN^{*\prime} = \lambda_1^2(p-1)^2 J_t$$

and (4.5) yields

$$\begin{aligned} C_{11} &= \lambda_1 p I_t - \frac{1}{p}\,NN' \\ C_{12} &= -\lambda_2 I_t + \lambda_2 J_t - \frac{1}{p}\,NN^{*\prime} \\ C_{22} &= \lambda_1(p-1)I_t - \frac{\lambda_1(p-1)}{pt}\,J_t - \frac{1}{p}\,N^*N^{*\prime} \\ C_{21} &= C_{12}' \quad . \end{aligned} \tag{4.8}$$

4.1.2 GPBRM1 Designs

For a GPBRM1 design, we have

$$D = \lambda_1 p I_t$$

$$T = \lambda_1 J_{t\times p}$$

$$D^* = \lambda_1 (p-1) I_t$$

$$T^* = \lambda_1 \begin{bmatrix} 0 & \vdots & \\ \vdots & \vdots & J_{t\times(p-1)} \\ 0 & \vdots & \end{bmatrix} .$$

Then

$$TT' = p\lambda_1^2 J_t$$

$$TT^{*\prime} = \lambda_1^2 (p-1) J_t$$

$$T^* T^{*\prime} = \lambda_1^2 (p-1) J_t$$

$$N J_n N' = \lambda_1^2 p^2 J_t$$

$$N J_n N^{*\prime} = \lambda_1^2 p(p-1) J_t$$

$$N^* J_n N^{*\prime} = \lambda_1^2 (p-1)^2 J_t$$

and from (4.5) we obtain

$$
\begin{aligned}
C_{11} &= \lambda_1 p I_t - \frac{1}{p} NN' \\
C_{12} &= S - \frac{1}{p} NN^{*\prime} \\
C_{22} &= \lambda_1 (p-1) I_t - \frac{\lambda_1 (p-1)}{pt} J_t - \frac{1}{p} N^* N^{*\prime} \\
C_{21} &= C_{12}' \quad .
\end{aligned}
\tag{4.9}
$$

4.2 Properties

From Chapter II we know that the condition C_1 is that the number of subjects receiving treatment i' when treatment i is in the last period is equal to the number of subjects receiving treatment i when treatment i' is in the last period. Under C_1, we have the following Lemma:

Lemma 4.2.1

The condition C_1 implies that

$$NN^{*\prime} = \frac{1}{2}(N^*N^{*\prime} + NN') - \frac{1}{2}\lambda_1 I_t \quad .$$

Proof:

Using the notation of (4.3), suppose that n_{ij} is the number of times that treatment i occurs in subject j and n^*_{ij} is the number of times that residual effect i occurs in subject j, and n_{ijk} is the number of times that treatment i occurs in subject j and period k, then

$$N = (n_{ij})$$

and

$$N^* = (n^*_{ij})$$

or

$$N = (\sum_{k=1}^{p} n_{ijk})_{t\times n}$$

and

$$N^* = (\sum_{k=2}^{p} n_{ij(k-1)})_{t\times n} \quad .$$

Thus

$$NN^{*\prime} \equiv (d_{ii'}) = (\sum_{j,k'} n_{ijk'} \sum_{k=2}^{p} n_{i'j(k-1)})$$

$$NN' \equiv (b_{ii'}) = (\sum_{j,k'} n_{ijk'} \sum_{k} n_{i'jk})$$

$$N^*N^{*\prime} \equiv (s_{ii'}) = (\sum_{i=2}^{p} \sum_{j} n_{ij(k-1)}(\sum_{k'=2}^{p} n_{i'j(k'-1)})) \quad .$$

Since

$$d_{ii'} = \sum_{j,k'} n_{ijk'} \sum_{k=2}^{p} n_{i'j(k-1)}$$

$$= \sum_j (\sum_{k'=2}^{p} n_{ij(k'-1)} + n_{ijp})(\sum_{k=2}^{p} n_{i'j(k-1)})$$

$$= \sum_j [(\sum_{k'=2}^{p} n_{ij(k'-1)})(\sum_{k=2}^{p} n_{i'j(k-1)} + n_{ijp}(\sum_{k=2}^{p} n_{i'j(k-1)})]$$

$$= \sum_j [\sum_{k'=2}^{p} n_{ij(k'-1)}][\sum_{k=2}^{p} n_{i'j(k-1)}] + \sum_j n_{ijp}(\sum_{k=2}^{p} n_{i'j(k-1)})$$

$$= s_{ii'} + \sum_j n_{ijp}(\sum_{k=2}^{p} n_{i'j(k-1)}) \quad , \qquad (4.11)$$

and

$$b_{ii'} = \sum_{j,k'} n_{ijk'} \sum_k n_{i'jk}$$

$$= \sum_j (\sum_{k'=2}^{p} n_{i'j(k'-1)} + n_{i'jp})(\sum_k n_{ijk})$$

$$= \sum_j [(\sum_{k'=2}^{p} n_{i'j(k'-1)})\sum_k n_{ijk} + n_{i'jp}(\sum_k n_{ijk})]$$

$$= \sum_j [\sum_{k'=2}^{p} n_{i'j(k'-1)}]\sum_k n_{ijk} + \sum_j n_{i'jp}(\sum_k n_{ijk})$$

$$= d_{ii'} + \sum_j n_{i'jp}(\sum_k n_{ijk})$$

$$= d_{ii'} + \sum_j n_{i'jp}\left(\sum_{k=2}^{p} n_{ij(k-1)} + n_{ijp}\right)$$

$$= d_{ii'} + \sum_j n_{i'jp}\left(\sum_{k=2}^{p} n_{ij(k-1)}\right) + \sum_j n_{ijp}n_{i'jp} \qquad (4.12)$$

then C_1 implies that

$$\sum_j n_{ijp}\left(\sum_{k=2}^{p} n_{i'j(k-1)}\right) = \sum_j n_{i'jp}\left(\sum_{k=2}^{p} n_{ij(k-1)}\right) . \qquad (4.13)$$

If we define

$$LHS = \sum_j n_{ijp}\left(\sum_{k=2}^{p} n_{i'j(k-1)}\right)$$

and

$$RHS = \sum_j n_{i'jp}\left(\sum_{k=2}^{p} n_{ij(k-1)}\right) ,$$

it follows then from (4.11) and (4.13) that

$$LHS = d_{ii'} - s_{ii'} , \qquad (4.14)$$

and from (4.12) and (4.13), that

$$RHS = b_{ii'} - d_{ii'} - \sum_j n_{ijp}n_{i'jp} . \qquad (4.15)$$

From (4.13), (4.14) and (4.15) we obtain

$$d_{ii'} - s_{ii'} = b_{ii'} - d_{ii'} - \sum_j n_{ijp} n_{i'jp}$$

or

$$d_{ii'} = \frac{1}{2}(s_{ii'} + b_{ii'}) - \frac{1}{2} \sum_j n_{ijp} n_{i'jp} \quad .$$

Since

$$\sum_j n_{ijp} n_{i'jp} = \lambda_1 \delta_{ii'} \quad ,$$

it follows that

$$d_{ii'} = \frac{1}{2}(s_{ii'} + b_{ii'}) - \frac{1}{2} \lambda_1 \delta_{ii'}$$

i.e.,

$$NN^{*\prime} = \frac{1}{2}(N^*N^{*\prime} + NN') - \frac{1}{2} \lambda_1 I_t \quad .$$

Note that the condition C_1 ensures that $NN^{*\prime}$ is a symmetric matrix. In general, if we have the results that C_{11} and C_{22} are nonnegative in (4.4), then it follows from (4.6) that the estimators of the direct and residual effects can be found accordingly, as we shall prove in the following lemma:

Lemma 4.2.2

C_{11} and C_{22} as given in (4.5) are nonnegative definite.

Proof:

Obviously, $C_{11} = \lambda_1 pI_t - \frac{1}{p} NN'$ is nonnegative definite since it is the C-matrix for the incomplete block design. But since

$$C_{11} = (\lambda_1 pI_t - \frac{p\lambda_1^2}{n} J_t) + (\frac{p\lambda_1^2}{n} J_t - \frac{1}{p} NN')$$

$$= (\lambda_1 pI_t - \frac{p\lambda_1^2}{n} J_t) + (\frac{NJ_nN'}{np} - \frac{1}{p} NN')$$

$$= (\lambda_1 pI_t - \frac{p\lambda_1^2}{n} J_t) - \frac{1}{p} N(I_n - \frac{J_n}{n})N' \quad , \tag{4.16}$$

and since C_{11} is nonnegative, the right hand side of (4.16) is nonnegative. Also,

$$C_{22} = \lambda_1(p-1)I_t - \frac{\lambda_1(p-1)}{pt} J_t - \frac{1}{p} N^*N^{*'}$$

$$= [\lambda_1(p-1)I_t - \frac{\lambda_1^2(p-1)}{n} J_t] - \frac{1}{p} [N^*N^{*'} - \frac{\lambda_1^2(p-1)^2}{n} J_t]$$

$$= [\lambda_1(p-1)I_t - \frac{\lambda_1^2(p-1)}{n} J_t] - \frac{1}{p} N^*(I_n - \frac{J_n}{n})N^{*'}$$

$$= [\lambda_1(p-1)I_t - \frac{\lambda_1^2(p-1)}{n} J_t] - \frac{1}{p-1} N^*(I_n - \frac{J_n}{n})N^{*'}$$

$$+ N^*[(\frac{1}{p-1} - \frac{1}{p})(I_n - \frac{J_n}{n})]N^{*'} \quad .$$

From (4.16) and the argument given before, for a RM (t,n,p-1) design, we have that

$$[\lambda_1(p-1)I_t - \frac{\lambda_1^2(p-1)}{n} J_t] - \frac{1}{p-1} N^*(I_n - \frac{J_n}{n})N^{*'}$$

is nonnegative definite. And $I_n - \frac{J_n}{n}$ is nonnegative definite, so is

$$(\frac{1}{p-1} - \frac{1}{p})(I_n - \frac{J_n}{n}) \quad .$$

Therefore, $N^*[(\frac{1}{p-1} - \frac{1}{p})(I_n - \frac{J_n}{n})]N^{*'}$ is nonnegative definite. This implies that C_{22} is also nonnegative definite.

From Lemma 4.2.2, the nonnegative definiteness of C_{11} and C_{22} guarantees the forms of C_{11}^- and C_{22}^- according to Shah (1959). Therefore we can find the explicit forms for the estimators and variances and covariances of all the direct and residual effects. These forms will be discussed in the next chapter.

V. APPLICATION OF ASSOCIATION MATRICES UNDER THE FIRST ORDER RESIDUAL MODEL

In the previous chapter, we have analyzed GPBRM1 designs and also listed their properties. In this chapter, we shall use the association schemes for GPBRM1 designs to examine their properties and structures more deeply. In this way we can detect whether the direct effects and the residual effects are estimable, and can find their estimators if they are estimable. To begin with, we introduce the concept of association matrices which have been used in connection with PBIB designs.

5.1 Using the Association Matrices

Let

$$b_{\alpha\beta}^{ij:v} = \begin{cases} 1 & \text{if treatments } \alpha \text{ and } \beta \text{ are } ij{:}v \text{ th associates} \\ 0 & \text{otherwise} \end{cases}$$

where α and β are elements from the first and/or second group treatments. Then we define the (ij:v)th association matrix, $B_{ij:v}$, as

$$B_{ij:v} = (b_{\alpha\beta}^{ij:v}) = \begin{bmatrix} b_{11}^{ij:v} & \cdots & b_{1t}^{ij:v} \\ \cdots & \cdots & \cdots \\ b_{t1}^{ij:v} & \cdots & b_{tt}^{ij:v} \end{bmatrix} .$$

From Definition 3.2.1, $n_{ij:v}$ is the total number of treatments in the ij:v th associate class. Using the definition of $B_{ij:v}$, it follows then that every row total or column total of $B_{ij:v}$ is equal to $n_{ij:v}$ for any v. Obviously, we also have the following results:

$$B_{ij:0} = I_t$$

and (5.1)

$$\sum_{v=0}^{m_{ij}} B_{ij}:v = J_t \quad , \quad \text{for all } (ij) \quad .$$

Using the definition of $B_{ij:v}$, the GPBIB properties of GPBRM1 designs in condition (4) of Definition 3.2.1 can be written mathematically (Shah, 1959) in terms of the association matrices as follows:

$$B_{i_1j_1:v_1} B_{i_2j_2:v_2} + B_{i_2j_2:v_2} B_{i_1j_1:v_1}$$
$$= 2 \sum_{v=0}^{m_{ij}} P_{ij:v} (i_1j_1:v_1, i_2j_2:v_2) B_{ij:v} \quad . \qquad (5.2)$$

Since the parameters of GPBRM1 designs are related to the association matrices in (5.1) and (5.2), the parameters in Definition 3.2.1 are related in the following Lemma:

Lemma 5.1

For any GPBRM1 design, the following relationships hold among the parameters t, n, p, λ_1, $n_{ij:v}$, $P_{ij:v}$, $\lambda_{ij:v}$ and $\mu_{ij:v}$:

(5.1.1) $n = \lambda_1 t$

(5.1.2) $\sum_{v=0}^{m_{ij}} n_{ij:v} = t$ where $i,j = 1,2$

(5.1.3) $\sum_{\ell=0}^{m_{jk}} P_{ij:v}\,(ik:t_1,\ jk:\ell) = n_{ik:t_1}$

and

$\sum_{t_1=0}^{m_{ik}} P_{ij:v}\,(ik:t_1,\ jk:\ell) = n_{jk:\ell}$ where $i,j,k = 1,2$

(5.1.4) $n_{ij:v}\, P_{ij:v}\,(ik:\ell,\ jk:t_1) = n_{ik:\ell}\, P_{ik:\ell}\,(ij:v,\ jk:t_1)$

where $i \neq j$ and $i \neq k$ and $i,j,k = 1,2$

(5.1.5) $\sum_{v=0}^{m_{ii}} n_{ii:v}\, \lambda_{ii:v} = \lambda_1 (p+1-i)^2$ where $i = 1,2$

(5.1.6) $\sum_{v=0}^{m_{12}} n_{12:v}\, \mu_{12:v} = \lambda_1 (p-1)$

(5.1.7) $\sum_{v=0}^{m_{12}} n_{12:v}\, \lambda_{12:v} = \lambda_1 p(p-1)$

(5.1.8) $\sum_{v=0}^{m_{12}} n_{12:v}\, \mu_{12:v} = \sum_{v=0}^{m_{21}} n_{21:v}\, \mu_{21:v}$

and

$\sum_{v=0}^{m_{12}} n_{12:v}\, \lambda_{12:v} = \sum_{v=0}^{m_{21}} n_{21:v}\, \lambda_{21:v}$

Proof:

(5.1.1): trivial

(5.1.2): trivial

(5.1.3): Using (5.1), we obtain

$$\left(\sum_{\ell=0}^{m_{jk}} B_{jk:\ell}\right) B_{ik:t_1} = J_t B_{ik:t_1} = n_{ik:t_1} J_t$$

$$= n_{ik:t_1} \sum_{v=0}^{m_{ij}} B_{ij:v} \quad .$$

Hence,

$$\sum_{\ell=0}^{m_{jk}} (B_{jk:\ell} B_{ik:t_1} + B_{ik:t_1} B_{jk:\ell})$$

$$= n_{ik:t_1} \sum_{v=0}^{m_{ij}} B_{ij:v} + n_{ik:t_1} \sum_{v=0}^{m_{ij}} B_{ij:v}$$

$$= 2\, n_{ik:t_1} \sum_{v=0}^{m_{ij}} B_{ij:v} \quad . \tag{5.3}$$

If we define

$$\text{LHS} = \sum_{\ell=0}^{m_{jk}} (B_{jk:\ell} B_{ik:t_1} + B_{ik:t_1} B_{jk:\ell})$$

and

$$\text{RHS} = 2\, n_{ik:t_1} \sum_{v=0}^{m_{ij}} B_{ij:v}$$

then it follows from (5.2) that

$$\text{LHS} = \sum_{\ell=0}^{m_{jk}} \left[\sum_{v=0}^{m_{ij}} 2\, P_{ij:v}\,(ik:t_1,\ jk:\ell)\; B_{ij:v} \right]$$

$$= \sum_{v=0}^{m_{ij}} \left[\sum_{\ell=0}^{m_{jk}} 2\, P_{ij:v}\,(ik:t_1,\ jk:\ell)\; B_{ij:v} \right] \quad . \qquad (5.4)$$

Therefore, from (5.3) and (5.4) we obtain

$$\sum_{\ell=0}^{m_{jk}} 2\, P_{ij:v}\,(ik:t_1,\ jk:\ell) = 2\, n_{ik:t_1} \quad .$$

On the other hand,

$$\left(\sum_{t_1=0}^{m_{ik}} B_{ik:t_1} \right) B_{jk:\ell} = J_t\, B_{jk:\ell} = n_{jk:\ell}\, J_t$$

$$= n_{jk:\ell} \sum_{v=0}^{m_{ij}} B_{ij:v}$$

implies

$$\sum_{t_1=0}^{m_{ik}} \left(B_{jk:\ell}\, B_{ik:t_1} + B_{ik:t_1}\, B_{jk:\ell} \right)$$

$$= n_{jk:\ell} \sum_{v=0}^{m_{ij}} B_{ij:v} + n_{jk:\ell} \sum_{v=0}^{m_{ij}} B_{ij:v}$$

$$= 2\, n_{jk:\ell} \sum_{v=0}^{m_{ij}} B_{ij:v} \quad . \qquad (5.4a)$$

Since from (5.2)

$$\sum_{t_1=0}^{m_{ik}} (B_{jk:\ell}\, B_{ik:t_1} + B_{ik:t_1}\, B_{jk:\ell})$$

$$= \sum_{t_1=0}^{m_{ik}} [\sum_{v=0}^{m_{ij}} 2\, P_{ij:v}\, (ik:t_1,\ jk:\ell)\, B_{ij:v}]$$

$$= \sum_{v=0}^{m_{ij}} [\sum_{t_1=0}^{m_{ik}} 2\, P_{ij:v}\, (ik:t_1,\ jk:\ell)\, B_{ij:v}] \quad ,$$

(5.4a) implies that

$$\sum_{t_1=0}^{m_{ik}} P_{ij:v}\, (ik:t_1,\ jk:\ell) = n_{jk:\ell} \quad .$$

(5.1.4): by Shah (1959)

(5.1.5): Since from Definition 3.2.1 $\lambda_{ii:v}$ is defined as the number of times any two i^{th} group treatments occur together in the same subject, then $\sum_{v=0}^{m_{ii}} n_{ii:v}\, \lambda_{ii:v}$ represents the number of times any i^{th} group treatment occurs together with all the other i^{th} group treatments in the same subject.

Let R_i be the number of subjects in which any i^{th} group treatment occurs. Since any i^{th} group treatment occurs in only p-i+1 periods and since it occurs λ_1 times in each period, then we obtain

$$R_i = \lambda_1(p-i+1) \quad . \tag{5.5}$$

On the other hand,

$$\sum_{v=0}^{m_{ii}} n_{ii:v} \lambda_{ii:v} = R_i(p-i+1) \quad . \tag{5.6}$$

It follows from (5.5) and (5.6) that

$$\sum_{v=0}^{m_{ii}} n_{ii:v} \lambda_{ii:v} = \lambda_1(p-i+1)^2 \quad .$$

(5.1.6): From Definition 3.2.1, $\mu_{12:v}$ is defined as the number of times any second group treatment immediately precedes a first group treatment, those two treatments being 12:v th associates. Then an argument similar to that for (5.1.5) shows that $\sum_{w=1}^{t} [\sum_{v=0}^{m_{12}} n_{12:v} \mu_{12:v}]$ represents the number of times all the second group treatments immediately precede all the first group treatments. This is equal to n(p-1) where n is the total number of subjects.

Since $\sum_{v=0}^{m_{12}} n_{12:v} \mu_{12:v}$ is independent of the number of treatments, we obtain

$$\sum_{v=0}^{m_{12}} n_{12:v} \mu_{12:v} = \frac{n(p-1)}{t} = \lambda_1(p-1) \quad .$$

(5.1.7): This follows an argument similar to that used for (5.1.5). $\sum_{v=0}^{m_{12}} n_{12:t} \lambda_{12:t}$ represents the number of times any first group treatment occurs together with all the second group treatments in the same subject. Since for any first group treatment there are p-1 second group treatments occurring togehter in the same subject, and since any first group treatment occurs $\lambda_1 p$ times in a GPBRM1 design, then

$$\sum_{v=0}^{m_{12}} n_{12:v}\, \lambda_{12:v} = \lambda_1 p(p-1).$$

(5.1.8): obvious.

5.2 Solving the Normal Equations

Having introduced the concept of the association matrices, we apply it to the analysis of GPBRM1 designs described in Chapter IV. And after solving the normal equations, we can find a rule to determine whether a GPBRM1 design is connected. For a connected GPBRM1 design we give the estimators for direct and residual effects, variances of the estimators and the efficiency of the design.

To solve the normal equations given in (4.4) and (4.9), we first express NN', NN*' and N*N*' as given in (4.9) in terms of the association matrices.

For a GPBRM1 design, we call NN', NN*' and N*N*' concordance matrices associated with the PBIB(m_{11}), PBIB(m_{12}) and PBIB(m_{22}) and based on the 11, 12 and 22 association schemes, respectively. Then

$$NN' = \sum_{v=0}^{m_{11}} \lambda_{11:v} B_{11:v}$$

$$NN^{*\prime} = \sum_{v=0}^{m_{12}} \lambda_{12:v} B_{12:v} \tag{5.7}$$

$$N^{*}N^{*\prime} = \sum_{v=0}^{m_{22}} \lambda_{22:v} B_{22:v}$$

where, e.g., the ij^{th} element of NN' represents the number of times treatment i occurs together with treatment j in the same subject. This is equal to $\lambda_{11:w}$ for a specific w, which is also the ij^{th} element of $\sum_{v=0}^{m_{11}} \lambda_{11:v} B_{11:v}$.

After substituting (5.7) in (4.9), every element c_{ij} in the C matrix of (4.4) can be expressed as a linear combination of the association matrices in the following way:

$$C_{11} = (\lambda_1 p - \frac{\lambda_{11:0}}{p}) B_{11:0} - \sum_{v=1}^{m_{11}} \frac{\lambda_{11:v}}{p} B_{11:v}$$

$$C_{12} = \sum_{v=0}^{m_{12}} (\mu_{12:v} - \frac{\lambda_{12:v}}{p}) B_{12:v}$$

where $\mu_{12:0}$ is defined to be zero

$$C_{22} = [\lambda_1 (p-1) - \frac{\lambda_1 (p-1)}{pt} - \frac{\lambda_{22:0}}{p}] B_{22:0}$$

$$- \sum_{v=1}^{m_{22}} \frac{1}{p} [\frac{\lambda_1 (p-1)}{t} + \lambda_{22:v}] B_{22:v}$$

$$C_{21} = C'_{12} \quad . \tag{5.8}$$

Also, (4.4) can be rewritten in a simplified form using (5.8) as follows:

$$C^* \begin{bmatrix} \hat{\underline{\tau}} \\ \hat{\underline{\alpha}} \end{bmatrix} = Q_{(1)}$$

where

$$C^* \equiv \begin{bmatrix} C^*_{11} & C^*_{12} \\ C^*_{21} & C^*_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{v=0}^{m_{11}} \lambda^*_{11:v} B_{11:v} & \sum_{v=0}^{m_{12}} \lambda^*_{12:v} B_{12:v} \\ \sum_{v=0}^{m_{21}} \lambda^*_{21:v} B_{21:v} & \sum_{v=0}^{m_{22}} \lambda^*_{22:v} B_{22:v} \end{bmatrix}$$

with

$$\lambda^*_{11:0} = \lambda_1 p - \frac{\lambda_{11:0}}{p}$$

$$\lambda^*_{11:v} = - \frac{\lambda_{11:v}}{p}$$

$$\text{for } v = 1, \ldots, m_{11}$$

$$\lambda^*_{22:0} = \lambda_1(p-1) - \frac{\lambda_1(p-1)}{pt} - \frac{\lambda_{22:0}}{p}$$

$$\lambda^*_{22:v} = - \frac{1}{p} \left[\frac{\lambda_1(p-1)}{t} + \lambda_{22:v}\right]$$

$$\text{for } v = 1, \ldots, m_{22}$$

$$\lambda^*_{12:0} = \mu_{12:0} - \frac{\lambda_{12:0}}{p}$$

$$\lambda^*_{12:v} = \mu_{12:v} - \frac{\lambda_{12:v}}{p}$$

for $v = 1, \ldots, m_{12}$ and $\mu_{12:0} = 0$

$$\lambda^*_{21:0} = \mu_{21:0} - \frac{\lambda_{21:0}}{p}$$

$$\lambda^*_{21:v} = \mu_{21:v} - \frac{\lambda_{21:v}}{p}$$

for $v = 1, \ldots, m_{21}$ and $\mu_{21:0} = 0$. (5.12)

Using the new parameters $\lambda^*_{ij:v}$ and Lemma 5.1, a GPBRM1 design has the following properties:

Lemma 5.2

With the $\lambda^*_{ij:v}$ as defined in (5.12), the following relationships hold for any GPBRM1 design:

$$\sum_{v=0}^{m_{ij}} n_{ij:v} \lambda^*_{ij:v} = 0$$

for $i, j = 1, 2$.

Proof:

From (5.12) we obtain

$$\sum_{v=0}^{m_{11}} n_{11:v}\,\lambda^*_{11:v}$$

$$= \lambda_1 p - \frac{\lambda_{11:0}}{p} - \frac{1}{p}\sum_{v=1}^{m_{11}} n_{11:v}\,\lambda_{11:v}$$

$$= \lambda_1 p - \frac{1}{p}\sum_{v=0}^{m_{11}} n_{11:v}\,\lambda_{11:v} \quad .$$

It follows from Lemma 5.1 that

$$\sum_{v=0}^{m_{11}} n_{11:v}\,\lambda^*_{11:v} = \lambda_1 p - \frac{1}{p}\,(\lambda_1 p^2) = 0 \quad .$$

Similarly, from (5.12) we have

$$\sum_{v=0}^{m_{22}} n_{22:v}\,\lambda^*_{22:v}$$

$$= \lambda_1(p-1) - \frac{\lambda_1(p-1)}{pt} - \frac{\lambda_{22:0}}{p} - \frac{1}{p}\sum_{v=1}^{m_{22}} n_{22:v}\left[\frac{\lambda_1(p-1)}{t} + \lambda_{22:v}\right]$$

$$= \lambda_1(p-1) - \frac{\lambda_1(p-1)}{pt}\sum_{v=0}^{m_{22}} n_{22:v} - \frac{1}{p}\sum_{v=0}^{m_{22}} n_{22:v}\,\lambda_{22:v} \quad .$$

Since

$$\sum_{v=0}^{m_{22}} n_{22:v} = t \ ,$$

Lemma 5.1 implies that

$$\sum_{v=0}^{m_{22}} n_{22:v}\, \lambda^*_{22:v}$$

$$= \lambda_1(p-1) - \frac{\lambda_1(p-1)}{pt}\, t - \frac{1}{p}\, \lambda_1(p-1)^2 = 0 \quad .$$

Also from (5.12) and Lemma 5.1, we obtain

$$\sum_{v=0}^{m_{12}} n_{12:v}\, \lambda^*_{12:v}$$

$$= \mu_{12:0} - \frac{\lambda_{12:0}}{p} + \sum_{v=1}^{m_{12}} (\mu_{12:v} - \frac{\lambda_{12:v}}{p}) n_{12:v}$$

$$= \sum_{v=0}^{m_{12}} n_{12:v}\, \mu_{12:v} - \frac{1}{p} \sum_{v=0}^{m_{12}} n_{12:v}\, \lambda_{12:v}$$

$$= \sum_{v=0}^{m_{12}} n_{12:v}\, \mu_{12:v} - \frac{1}{p} \sum_{v=0}^{m_{12}} n_{12:v}\, \lambda_{12:v}$$

$$= \lambda_1(p-1) - \frac{1}{p}\, \lambda_1 p(p-1) = 0 \quad .$$

Similarly,

$$\sum_{v=0}^{m_{21}} n_{21:v}\, \lambda^*_{21:v} = 0 \quad .$$

Using the definition of C* in (5.12), Lemma 5.2 implies that the sums of rows or columns in C^*_{11}, C^*_{12}, C^*_{21} and C^*_{22} of C* are equal to zero. Thus, Lemma 5.2 implies that for any GPBRM1 design,

$$\text{rank}(C^*) \leq 2t - 2 \; . \tag{5.12a}$$

The rank of C* of any GPBRM1 design depends on the actual construction. In order to find the estimators of direct and residual effects for a GPBRM1 design we first require that it be connected.

Definition 5.2.1

A GPBRM1 design is said to be connected if all linear contrasts $\tau_i - \tau_{i'}$ and $\alpha_i - \alpha_{i'}$ $(i, i' = 1, \ldots, t)$ are estimable.

From (5.12a) and Definition 5.2.1, we can check the connectedness of a GPBRM1 design in terms of the rank of C* according to the following theorem:

Theorem 5.2.1

A necessary condition for a GPBRM1 design to be connected is that $\text{rank}(C^*) = 2(t-1)$.

Proof:

If a GPBRM1 design is connected, then Definition 5.2.1 implies that there are $2(t-1)$ linearly independent contrasts among the $\hat{\tau}_i$'s and $\hat{\alpha}_i$'s. Thus $\text{rank}(C^*) \geq 2(t-1)$.

From (5.12a), we know that $\text{rank}(C^*) \leq 2(t-1)$.

Therefore $\text{rank}(C^*) = 2(t-1)$.

If a GPBRM1 design is connected, then by solving the normal equations (5.12) and using the association matrices, we can derive estimators for direct and residual effects as follows.

Rewrite (5.12) as

$$C^* \begin{bmatrix} \underline{\hat{\tau}} \\ \\ \underline{\hat{\alpha}} \end{bmatrix} = \begin{bmatrix} Q_1 \\ \\ Q_2 \end{bmatrix} \tag{5.13}$$

or

$$C^*_{11}\ \underline{\hat{\tau}} + C^*_{12}\ \underline{\hat{\alpha}} = Q_1 \tag{5.14}$$

$$C^*_{21}\ \underline{\hat{\tau}} + C_{22}\ \underline{\hat{\alpha}} = Q_2 \quad . \tag{5.15}$$

From (5.14) we obtain

$$C^*_{11}\ \underline{\hat{\tau}} = Q_1 - C^*_{12}\ \underline{\hat{\alpha}} \quad .$$

Using the association matrices we have

$$\left(\sum_{v=0}^{m_{11}} \lambda^*_{11:v}\ B_{11:v}\right)\underline{\hat{\tau}} = Q_1 - \left(\sum_{v=0}^{m_{12}} \lambda^*_{12:v}\ B_{12:v}\right)\underline{\hat{\alpha}} \ . \tag{5.16}$$

To express $\underline{\hat{\tau}}$ in terms of $\underline{\hat{\alpha}}$, we must invert $\sum_{v=0}^{m_{11}} \lambda^*_{11:v}\ B_{11:v}$ or C^*_{11}. For a connected GPBRM1 design we know that

$$\text{rank}(C^*_{11}) = t - 1 \quad .$$

Then Lemma 4.2.2 guarantees that C_{11} and C_{22} are nonnegative definite. From Shah (1959), (5.16) yields

$$\underline{\hat{\tau}} = A_{11}\left[Q_1 - \left(\sum_{v=0}^{m_{12}} \lambda^*_{12:v}\ B_{12:v}\right)\underline{\hat{\alpha}}\right] \tag{5.17}$$

where

$$A_{11} = \sum_{v=0}^{m_{11}} d_{11:v} \, B_{11:v}$$

and

$$\sum_{i=0}^{m_{11}} \sum_{j=0}^{m_{11}} \lambda^*_{11:i} \, P_{11:k}(11:i,\ 11:j) d_{11:j}$$

$$= \begin{cases} 1 - \frac{1}{t} & \text{if } k = 0 \\ \\ -\frac{1}{t} & \text{if } k = 1, \ldots, m_{11} \end{cases} \tag{5.17a}$$

with constraint

$$\sum_{j=0}^{m_{11}} n_{11:j} \, d_{11:j} = 0 \quad .$$

For simplicity (omitting the 11 subscripts), we can write (5.17a) as

$$\sum_{i=0}^{m_{11}} \sum_{j=0}^{m_{11}} \lambda^*_i \, P_k^{i,j} \, d_j$$

$$= \begin{cases} 1 - \frac{1}{t} & \text{if } k = 0 \\ \\ -\frac{1}{t} & \text{if } k = 1, \ldots, m_{11} \end{cases} \tag{5.17b}$$

with constraint

$$\sum_{j=0}^{m_{11}} n_j d_j = 0 \quad .$$

Alternatively, in terms of matrices, we have

$$\underline{\lambda}_{11}^{*'} P^{11,11} \underline{d}_{11}$$

$$= \begin{cases} 1 - \frac{1}{t} & \text{if } k = 0 \\ - \frac{1}{t} & \text{if } k = 1, \ldots, m_{11} \end{cases} \tag{5.17c}$$

with

$$\underline{n}_{11}' \underline{d}_{11} = 0$$

where

$$\underline{\lambda}_{11}^{*'} = (\lambda_{11:0}^*, \ldots, \lambda_{11:m_{11}}^*)$$

$$\underline{n}_{11}' = (n_{11:0}, \ldots, n_{11:m_{11}})$$

and

$$\underline{d}_{11}' = (d_{11:0}, \ldots, d_{11:m_{11}}) \quad .$$

Substituting (5.17c) in (5.17), we obtain

$$\underline{\hat{\tau}} = Q_1 \sum_{v=0}^{m_{11}} d_v \, B_{11:v} - \sum_{s=0}^{m_{12}} (\underline{d}'_{11} \, P^{11,12} \, \underline{\lambda}^*_{12}) B_{12:s} \, \underline{\hat{\alpha}} \, . \quad (5.19)$$

Since $\underline{\hat{\tau}}$ has been expressed in terms of $\underline{\hat{\alpha}}$, we can solve for $\underline{\hat{\alpha}}$ by inserting (5.19) into (5.15) and obtain

$$[\sum_{s'=0}^{m_{22}} (\lambda^*_{22:s'} - \underline{\lambda}^{*\prime}_{21} \, P^{21,12} \, e_{12}) B_{22:s'}]\underline{\hat{\alpha}}$$

$$= Q_2 - C^*_{21} \, Q_1 \sum_{v=0}^{m_{11}} d_{11:v} \, B_{11:v} \quad (5.21)$$

where e_{12} is defined as $d'_{11} \, P^{11,12} \, \lambda^*_{12}$. If we define

$$_{\underline{\alpha}}\lambda^*_s = \lambda^*_{22:s} - \lambda^{*\prime}_{21} \, P^{21,12} \, e_{12}$$

and

$$B_{\underline{\alpha}} = \sum_{s=0}^{m_{22}} {}_{\underline{\alpha}}\lambda^*_{22:s} \, B_{22:s} \quad ,$$

then

$$B_{\underline{\alpha}} \, \underline{\hat{\alpha}} = Q_2 - C^*_{21} \, Q_1 \sum_{v=0}^{m_{11}} d_{11:v} \, B_{11:v}$$

or

$$B_{\underline{\alpha}} \, \underline{\hat{\alpha}} = Q_{\underline{\alpha}} \quad (5.23)$$

where

$$Q_{\underline{\alpha}} = Q_2 - C^*_{21}\, Q_1 \sum_{v=0}^{m_{11}} d_{11:v}\, B_{11:v} \quad .$$

We now check the rank of $B_{\underline{\alpha}}$ in (5.23). Following Lemma 5.2 and the definition of $_{\underline{\alpha}}\lambda^*$, we have

$$n'_{22}\; {}_{\underline{\alpha}}\lambda^* = \sum_{v=0}^{m_{22}} n_{22:v}\lambda^*_{22:v} - C \tag{5.24}$$

where

$$C = \sum_{s',t'',s,t,t'} n_{s'}\, d_t\, P_s^{t,t'}\, \lambda^*_{t'}\, P_{s'}^{s,t''}\, \lambda^*_{t''}$$

$$= \sum_{s',t'',s,t,t'} d_t\, P_s^{t,t'}\, \lambda^*_{t'}\, \lambda^*_{t''}\, n_{s'}\, P_{s'}^{s,t''} \quad . \tag{5.25}$$

From Lemma 5.1, we know that

$$n_{s'}\, P_{s'}^{s,t''} = n_s\, P_s^{s',t''}$$

and hence (5.25) becomes

$$C = \sum_{t'',s,t,t'} n_s\, d_t\, P_s^{t,t'}\, \lambda^*_{t'}\, \lambda^*_{t''} \sum_{s'} P_s^{s',t''} \quad . \tag{5.26}$$

Since, from Lemma 5.1,

$$\sum_{s'} P_s^{s',t''} = n_{t''} \quad ,$$

it follows from (5.26) that

$$C = \sum_{t'',s,t,t'} d_t P_s^{t,t'} \lambda^*_{t'} \lambda^*_{t''} n_s n_{t''} . \tag{5.27}$$

To eliminate the $P_s^{t,t'}$ term from (5.27), we use the same development as in (5.25) to obtain

$$C = \sum_{t'',t,t'} d_t n_{t''} \lambda^*_{t'} \lambda^*_{t''} \sum_s n_s P_s^{t,t'}$$

$$= \sum_{t'',t,t'} d_t n_{t''} \lambda^*_{t'} \lambda^*_{t''} \sum_s n_t P_t^{s,t'}$$

$$= \sum_{t'',t,t'} d_t n_{t''} \lambda^*_{t'} \lambda^*_{t''} n_t n_{t'}$$

$$= (\sum_t n_t d_t)(\sum_{t'} n_{t'} \lambda^*_{t'})(\sum_{t''} n_{t''} \lambda^*_{t''}) .$$

Therefore by Lemma 5.2, we see that

$$C = 0 .$$

Then using Lemma 5.2 and the fact that C = 0, (5.24) implies

$$\underline{n}'_{22} \, {}_{\underline{\alpha}}\lambda^* = 0,$$

that is, the sum of the elements in any row or any column of $B_{\underline{\alpha}}$ is equal to zero.

Thus,

$$\text{rank}(B_{\underline{\alpha}}) \leq t - 1 . \tag{5.29}$$

Concerning the estimability of linear functions of residual effects we now state

Theorem 5.2.2

All linear contrasts of the form $\alpha_i - \alpha_{i'}$ are estimable if and only if rank $(B_{\underline{\alpha}}) = t - 1$.

Proof:

Assume that rank $(B_{\underline{\alpha}}) = t - 1$. For any linear function of residual effects

$$\sum_{i=1}^{t} a_i \alpha_i \quad ,$$

since the sums of column elements of $B_{\underline{\alpha}}$ is equal to zero, from (5.23) this means that $\sum_{i=1}^{t} a_i = 0$. Since rank $(B_{\underline{\alpha}}) = t - 1$, we know that there are $t - 1$ linear independent estimable functions for α_i's. Hence $\sum_{i=1}^{t} a_i \alpha_i$ is estimable. Therefore all linear contrasts of the form $\alpha_i - \alpha_{i'}$ are estimable.

Conversely, if all $\alpha_i - \alpha_{i'}$ are estimable, then trivially rank $(B_{\underline{\alpha}}) \geq t - 1$. However, from (5.29) rank $(B_{\underline{\alpha}}) \leq t - 1$, so rank $(B_{\underline{\alpha}}) = t - 1$.

Similarly, by interchanging the roles of $\underline{\alpha}$ and $\underline{\tau}$ in solving (5.14) and (5.15) (i.e., by interchanging subscripts 1 and 2 in (5.23)), we obtain

$$\sum_{s=0}^{m_{11}} \lambda^*_{\underline{\tau} s} B_s \hat{\underline{\tau}} = Q_{\underline{\tau}}$$

where

$$\underline{\tau}\lambda^*_s = \lambda^*_{11:s} - \lambda^{*\prime}_{12}\, P_s^{21,12}\, h$$

$$h' \equiv (h_{21:0}, h_{21:1}, \ldots, h_{21:m_{12}})$$

$$= (b'\, P^{22,21}\, \lambda^*_{21})'$$

$$b' \equiv (b_{22:0}, \ldots, b_{22:m_{22}})$$

and also

$$\sum_{i=0}^{m_{22}} \sum_{j=0}^{m_{22}} \lambda^*_{22:i}\, P_t^{i,j}\, b_{22:j}$$

$$= \begin{cases} 1 - \frac{1}{t} & \text{for } k = 0 \\ \\ -\frac{1}{t} & \text{for } k = 1, \ldots, m_{22} \end{cases}$$

with

$$\sum_{j=0}^{m_{22}} n_{22:j}\, b_{22:j} = 0$$

and

$$Q_{\underline{\tau}} = Q_1 - Q_2 \sum_{s=0}^{m_{21}} h_{21:s}\, B_{21:s} \quad .$$

Define

$$\underline{B}_{\tau} \equiv \sum_{s=0}^{m_{11}} \underline{\lambda}^{*}_{\tau s} B_s$$

then

$$\underline{B}_{\tau} \hat{\underline{\tau}} = \underline{Q}_{\tau}$$

where the sum of the elements of any row or any column of $\underline{B}_{\tau}$ is equal to zero; i.e., rank $(\underline{B}_{\tau}) \leq t - 1$.

Thus, we have the following theorem:

Theorem 5.2.3

All linear contrast of the form $\hat{\tau}_i - \hat{\tau}_{i'}$ are estimable iff rank $(\underline{B}_{\tau}) = t - 1$.

Combining Theorems 5.2.2 and 5.2.3, we find

Theorem 5.2.4

A GPBRM1 design is connected iff rank $(\underline{B}_{\alpha}) = t - 1$ and rank $(\underline{B}_{\tau})$ $= t - 1$.

Proof:

If a GPBRM1 design is connected, then it implies that all $\alpha_i - \alpha_{i'}$ and all $\tau_i - \tau_{i'}$ are estimable. Then by Theorem 5.2.2, this implies rank $(\underline{B}_{\alpha}) = t - 1$. Similarly, by Theorem 5.2.3 we have rank $(\underline{B}_{\tau}) = t - 1$.

Conversely, if rank $(B_{\underline{\alpha}}) = t - 1$, then by Theorem 5.2.2 all $\alpha_i - \alpha_{i'}$ are estimable. And since rank $(B_{\underline{\tau}}) = t - 1$ it implies (by Theorem 5.2.3) that all $\tau_i - \tau_{i'}$ are estimable.

We shall obtain the variances for the estimators obtained in the connected GPBRM1 design. We have

$$B_{\underline{\alpha}} \, \underline{\hat{\alpha}} = Q_{\underline{\alpha}}$$

where

$$B_{\underline{\alpha}} = \sum_{s=0}^{m_{22}} \lambda^*_{22:s} \, B_{22:s} \quad .$$

From Shah (1959)

$$\underline{\hat{\alpha}} = A_{\underline{\alpha}} \, Q_{\underline{\alpha}}$$

where

$$A_{\underline{\alpha}} = \sum_{v=0}^{m_{22}} a_{22:v} \, B_{22:v} \tag{5.30a}$$

and

$$A_{\underline{\alpha}} \, B_{\underline{\alpha}} = B_{\underline{\alpha}} \, A_{\underline{\alpha}} = I_t - \frac{1}{t} J_t$$

with

$$\sum_{s=0}^{m_{22}} \sum_{t=0}^{m_{22}} {}_{\underline{\alpha}}\lambda^*_s \, P_u^{s,V} \, a_{22:v} = \delta_{u0} - \frac{1}{t} \tag{5.30}$$

.

where u = 0, 1, ..., m_{22} and δ_{u0} denotes the Kronecker delta function.

Since

$$\text{rank}(B_{\underline{\alpha}}) = t - 1$$

with

$$\sum_{i=o}^{m_{22}} n_i^* \, {}_{\underline{\alpha}}\lambda_i^* = 0 \ ,$$

it follows that the equations in (5.30) are not independent.

Using the constraint

$$\sum_{v=0}^{m_{22}} n_{22:v} \, a_{22:v} = 0$$

we obtain

$$a_{22:0} = - \sum_{v=1}^{m_{22}} n_{22:v} \, a_{22:v}$$

and

$$\sum_{s=0}^{m_{22}} {}_{\underline{\hat{\alpha}}}\lambda_s^* [\sum_{v=1}^{m_{22}} P_{22:u}(22:s,22:v) a_{22:v} - P_{22:u}(22:s,22:0) \sum_{v=1}^{m_{22}} n_{22:v} \, a_{22:v}]$$

$$= \delta_{u0} - \frac{1}{t}$$

or

$$\sum_{s=0}^{m_{22}} \sum_{v=1}^{m_{22}} \underline{\hat{\alpha}}\lambda_s^* \, a_{22:v}[P_{22:u}(22:s,22:v) - P_{22:u}(22:s,22:0)n_{22:v}]$$

$$= \delta_{u0} - \frac{1}{t} \quad .$$

Since $P_{22:u}(22:s,22:0) = \delta_{us}$, the set of independent equations are

$$\sum_{s=0}^{m_{22}} \sum_{v=1}^{m_{22}} \underline{\alpha}\lambda_s^* \, a_{22:v}[P_{22:u}(22:s,22:v) - \delta_{us} \, n_{22:v}]$$

$$= -\frac{1}{t} \qquad \text{for } u = 1, \ldots, m_{22} \quad .$$

Solving for the $a_{22:v}$'s, we get $A_{\underline{\alpha}}$ and since

$$\underline{\hat{\alpha}} = A_{\underline{\alpha}} \, Q_{\underline{\alpha}} \tag{5.31}$$

it follows that

$$\mathrm{Var}(\hat{\alpha}_i - \hat{\alpha}_{i'}) = 2(a_{22:0} - a_{22:v})\sigma^2$$

if i and i' are 22:v associates. The average variance of $\hat{\alpha}_i - \hat{\alpha}_{i'}$ is

$$\overline{\mathrm{Var}}(\hat{\alpha}_i - \hat{\alpha}_{i'})$$

$$= \sum_{v=1}^{m_{22}} n_{22:v} \frac{2(a_{22:0} - a_{22:v})}{t-1} \sigma^2$$

$$= \frac{2\sigma^2}{t-1} \sum_{v=1}^{m_{22}} n_{22:v} \, (a_{22:0} - a_{22:v}) \quad .$$

Since

$$a_{22:0} = - \sum_{t=1}^{m_{22}} n_{22:v}\, a_{22:v} \quad ,$$

then

$$\overline{\mathrm{Var}}(\hat{\alpha}_i - \hat{\alpha}_{i'})$$

$$= \frac{2\sigma^2}{t-1}\, a_{22:0}\left(\sum_{v=1}^{m_{22}} n_{22:v} + 1\right)$$

$$= \frac{2\sigma^2 t}{t-1}\, a_{22:0} \quad .$$

Similarly, for direct effects, we have

$$\hat{\underline{\tau}} = Q_1 \sum_{v=0}^{m_{11}} d_{11:v}\, B_{11:v} - \sum_{s=0}^{m_{12}} (d'_{11}\, P_s^{11,12}\, \lambda^*_{12}) B_{12:s}\, \hat{\underline{\alpha}}$$

$$= Q_1 \sum_{v=0}^{m_{11}} d_{11:v}\, B_{11:v} - \sum_{s=0}^{m_{12}} (d'_{11}\, P_s^{11,12}\, \lambda^*_{12}) B_{12:s}\, A_{\underline{\alpha}}\, Q_{\underline{\alpha}}$$

$$= Q_1 \sum_{v=0}^{m_{11}} d_{11:v}\, B_{11:v} - \sum_{s=0}^{m_{12}} d'_{11}\, P_s^{11,12}\, \lambda^*_{12}\, B_{12:s}\, A_{\underline{\alpha}}$$

$$\times \left[Q_2 - Q_1 \sum_{s'=0}^{m_{12}} \lambda^{*\prime}_{21}\, P_s^{21,11}\, d_{11}\, B_{12:s'}\right]$$

$$= Q_1[\sum_{v=0}^{m_{11}} d_{11:v} B_{11:v} + \sum_{s=0}^{m_{12}} d_{11}' P_s^{11,12} \lambda_{12}^* B_{12:s} (\sum_{s'=0}^{m_{22}} a_{22:s'} B_{22:s'})$$

$$\times (\sum_{s''=0}^{m_{12}} \lambda_{21}^{*\prime} P_{s''}^{21,11} d_{11} B_{12:s''}) - Q_2(\sum_{s=0}^{m_{12}} d_{11}' P_s^{11,12} \lambda_{12}^* B_{12:s} \underline{A_\alpha})$$

$$= Q_1[\sum_{v=0}^{m_{11}} d_{11:v} B_{11:v} + \sum_{t'=0}^{m_{12}} \sum_{s=0}^{m_{12}} (d_{11}' P_s^{11,12} \lambda_{12}^*)' P_{t'}^{12,22} a_{22:t'} B_{12:t'}$$

$$\times (\sum_{s''=0}^{m_{12}} \lambda_{21}^{*\prime} P_{s''}^{21,11} d_{11} B_{12:s''})] - Q_2(\sum_{s=0}^{m_{12}} e_{12:s} B_{12:s} \underline{A_\alpha}) \; .$$

If we define

$$w_{21} \equiv \lambda_{21}^{*\prime} P_{12}^{21,11} d_{11}$$

$$e_{12} \equiv d_{11}' P_{12}^{11,12} \lambda_{12}^*$$

$$\ell_{12} \equiv e_{12}' P_{12}^{12,22} a_{22} \quad ,$$

then

$$\underline{\hat{\tau}} = Q_1[\sum_{t=0}^{m_{11}} d_{11:t} B_{11:t} + \sum_{v=0}^{m_{11}} (\ell_{12}' P_v^{12,21} w_{21}) B_{11:t}]$$

$$- Q_2(\sum_{s=0}^{m_{12}} e_{12:s} B_{12:s} \underline{A_\alpha}) \quad .$$

Let

$$g_{11:i} = d_{11:i} + \ell' P_{11:i}^{12,21} w_{21} \qquad i = 0, 1, \ldots, m_{11}.$$

Then

$$\underline{\hat{\tau}} = Q_1 = \sum_{i=0}^{m} g_{11:i} B_{11:i} - Q_2(\sum_{s=0}^{m_{11}} e_{12:s} B_{12:s} A_{\underline{\alpha}})$$

and hence

$$\mathrm{Var}(\hat{\tau}_i - \hat{\tau}_{i'}) = 2(g_{11:0} - g_{11:v})\sigma^2$$

if i and i' are 11:v associates. Furthermore, the average variance is

$$\overline{\mathrm{Var}}(\hat{\tau}_i - \hat{\tau}_{i'})$$

$$= \frac{\sum_{v=1}^{m_{11}} n_{11:v} \, 2(g_{11:0} - g_{11:v})\sigma^2}{t-1}$$

$$= \frac{2\sigma^2}{t-1} \sum_{v=1}^{m_{11}} n_{11:v}(g_{11:0} - g_{11:v})$$

and since

$$\sum_{v=1}^{m_{11}} g_{11:v} n_{11:v} = -a_{11:0} \quad ,$$

we obtain

$$\overline{\mathrm{Var}}(\hat{\tau}_i - \hat{\tau}_{i'}) = \frac{2\sigma^2 t}{t-1}\, g_{11:0} \quad .$$

Now, to calculate the efficiency of the comparison between the i^{th} and j^{th} direct effects in a GPBRM1 design relative to a complete randomized design, we know

$$E_{\tau_i\tau_j} = \frac{2\sigma^2/\gamma_1}{\mathrm{Var}(\hat{\tau}_i-\hat{\tau}_j)}$$

where γ_1 is the number of times each treatment occur in the complete randomized design. If treatments i and j are $11{:}v^{\ th}$ associates, then

$$E_{\tau_i\tau_j} = \frac{2\sigma^2}{\lambda_1 p}\ \frac{1}{2\sigma^2(g_{11:0}-g_{11:v})}$$

$$= \frac{1}{\lambda_1 p(g_{11:0}-g_{11:v})} \quad .$$

The overall efficiency can be defined as follows:

$$E^0_{\tau_i\tau_j} = \frac{2\sigma^2/\gamma_1}{\overline{\mathrm{Var}}(\hat{\tau}_i-\hat{\tau}_j)}$$

$$= \frac{2\sigma^2}{\lambda_1 p}\ \frac{t-1}{2\sigma^2 t\ g_{11:0}}$$

$$= \frac{t-1}{\lambda_1 p\, t\, g_{11:0}} \ .$$

Similarly, the efficiency of comparison between the i^{th} and j^{th} residual effects relative to a completely randomized design is

$$E_{\alpha_i\alpha_j} = \frac{2\sigma^2}{\lambda_1(p-1)} \; \frac{1}{2\sigma^2(a_{22:0}-a_{22:v})}$$

$$= \frac{1}{\lambda_1(p-1)(a_{22:0}-a_{22:v})}$$

where α_i and α_j are 22:v th associates, and the overall efficiency is

$$E^0_{\alpha_i\alpha_j} = \frac{2\sigma^2}{\lambda_1(p-1)} \; \frac{t-1}{2\sigma^2 \, t \, a_{22:0}}$$

$$= \frac{t-1}{\lambda_1(p-1) \, t \, a_{22:0}} \quad .$$

VI. CONSTRUCTION OF GPBRM1 DESIGNS

So far we have discussed the structure of GPBRM1 designs. Now we consider the problem of constructing such designs. The proposed method gives designs that are always GPBIB designs, and also cyclic designs. However, before discussing the method of construction, we introduce the concept of a modified difference set which augments the definition of a difference set, a concept used frequently for the construction of incomplete block designs.

6.1 Modified Difference Set

To begin, we establish several definitions, made concrete by examples.

Definition 6.1.1

Let $G = \{0, 1, \ldots, t-1\}$ be an Abelian group under addition. Let A be a subset of G with k elements such that the $k(k-1)$ differences (mod t) between members of A comprise all non-zero elements of G $\eta_1, \ldots, \eta_\gamma$ times where $\eta_1 > \eta_2 > \ldots > \eta_\gamma$ $(1 \le \gamma \le t-1)$. Then we call A a modified difference set of size k with γ *associates*.

In Definition 6.1.1 we mean that if we count the number of occurrences of each non-zero difference and find m different values, then $\eta_1, \ldots, \eta_m$ are those values arranged from the largest to the smallest.

Example 6.1.1

Suppose G = {0, 1, 2, 3} (mod 4). If A = {0, 1}, then all the differences between members of A (mod 4) are given as follows:

$$0 - 0 = 0$$
$$0 - 1 = 3$$
$$1 - 0 = 1$$
$$1 - 1 = 0$$

The non-zero elements 1 and 3 occur once, 2 not at all. Therefor $\eta_1 = 1$ and $\eta_2 = 0$, i.e., A is a modified difference set of size 2 with 2 associates.

Consider now a finite additive group G containing u elements, say G = {0, 1, 2, ..., u-1}. To each element of G let there correspond q symbols, the symbols associated with i being denoted by $i_1, i_2, \ldots, i_q$, where i = 0, 1, ..., u-1. Symbols with the subscript j are said to belong to the j^{th} class, i.e., $0_j, 1_j, \ldots, (u-1)_j$ belong to the j^{th} class. We consider differences of the form $v_i - w_j$, which is called a difference of type (i, j), and its value is given by $v - w = d$, say, where $d \in G$. When i = j the differences are called pure, when $i \neq j$ the differences are called mixed.

Then using pure differences and mixed differences of a modified difference set A, we have the following definition:

Definition 6.1.2

Let $A = \{I_{10}, \ldots, I_{s0}\}$ be a collection of s sets satisfying the following conditions:

(1) Each set contains k elements with $k = n_{1\ell} + \ldots + n_{q\ell}$ ($\ell = 1, 2, \ldots, s$), where $n_{j\ell}$ denotes the number of elements of the j^{th} class in set ℓ ($j = 1, 2, \ldots, q$);

(2) Among the $\sum_{\ell=1}^{s} n_{i\ell}(n_{i\ell}-1)$ pure differences of type (i, i) arising from the s sets, the non-zero element j of G is repeated ξ_j times ($j = 1, 2, \ldots, u-1$), which is the same for each $i = 1, 2, \ldots, q$;

(3) Among the $\sum_{\ell=1}^{s} n_{i\ell}n_{j\ell}$ mixed differences of type (i, j) arising from the s sets, every element v of G is repeated ξ'_v times ($v = 0, 1, 2, \ldots, u-1$), which is the same for each (i, j), $i, j = 1, 2, \ldots, q$, $i \neq j$;

(4) Suppose among the u pairs $(0, \xi'_0)$, (ξ_1, ξ'_1), $\ldots$, (ξ_{u-1}, ξ'_{u-1}) there are m for which at least one of its members is greater than zero, i.e., there occurs m out of the u possible values $0, 1, 2, \ldots, u-1$ for pure and mixed differences. The values are denoted by $\eta_1, \eta_2, \ldots, \eta_m$ and arranged in decreasing order of magnitude;

Then the differences in A are said to be asymmetrically repeated in G with m associates.

Example 6.1.2

Suppose $G = \{0, 1, 2\}$ (mod 3). If $A = \{I_{10}, I_{20}\}$, where

$$I_{10} = (0_1, 0_2, 1_1)$$

and

$$I_{20} = (1_2, 2_1, 2_2)$$

then $u = 3$, $s = 2$ and $q = 2$ according to Definition 6.1.2. The pure differences of type (1, 1) are:

$$0_1 - 1_1 = 2, \qquad 1_1 - 0_1 = 1 .$$

The pure differences of type (2, 2) are:

$$1_2 - 2_2 = 2, \qquad 2_2 - 1_2 = 1 .$$

Hence, $\xi_1 = 1$, and $\xi_2 = 1$. And the mixed differences of type (1, 2) are:

$$0_1 - 0_2 = 0, \quad 1_1 - 0_2 = 1, \quad 2_1 - 1_2 = 1, \quad 2_1 - 2_2 = 0 .$$

Hence,

$$\xi'_0 = 2, \quad \xi'_1 = 2, \quad \text{and} \quad \xi'_2 = 0 .$$

Combining ξ_1, ξ_2 and ξ'_0, ξ'_1, ξ'_2, we find that there are three different values, and if we order these values, we have

$$\eta_1 = 2, \quad \eta_2 = 1 \quad \text{and} \quad \eta_3 = 0 .$$

Therefore, the differences in A are asymmetrically repeated in G with three associates.

Using the concept of asymmetrically repeated differences with m associates, we can construct a PBIB design, which will be used to construct GPBRM1 designs. First of all, we can construct a design such that any two treatments occur together in either η_1 or η_2, ..., or η_m blocks if we have an asymmetrically repeated difference set A with m associates.

Lemma 6.1.1

Suppose a collection of s sets $A = \{I_{10}, \ldots, I_{s0}\}$ satisfies the following conditions:

(1) Among the ks elements occurring in the s sets (blocks), exactly u elements belong to the j^{th} class ($j = 1, 2, \ldots, q$); i.e., $ks = qu$;

(2) The differences in A are asymmetrically repeated with m associates and the numbers of occurrences are $\eta_1, \eta_2, \ldots, \eta_m$;

If each set in A is cyclically developed (mod u) obtaining $I_{\ell\theta}$ by adding $\theta \varepsilon G$ (mod u) to each element in $I_{\ell 0}$ ($\ell = 1, 2, \ldots, s$) and retaining the class number, then from the resulting us sets $I_{\ell\theta}$ ($\ell = 1, 2, \ldots, s$; $\theta \varepsilon G$), any two elements occur together in either η_1 or η_2 or ... η_m blocks.

Proof:

We denote an element in $I_{\ell 0}$ by $V_{(i)}^{(\ell)}$ where $V \varepsilon G$, $i = 1, 2, \ldots, q$, and $\ell = 1, 2, \ldots, s$.

Since $V \varepsilon G$ (mod u), it follows that by cyclically developing $I_{\ell 0}$ (mod u), the number of blocks $b = us$. Within these us blocks, there are usk elements and since there are qu elements in sets $\{I_{\ell 0}\}$ ($\ell = 1, 2, \ldots, s$), hence each element repeats itself $\gamma = \frac{usk}{qu}$ times in sets $\{I_{\ell\theta}\}$ ($\ell = 1, 2, \ldots, s$; $\theta \varepsilon G$).

To show that any two elements occur together in η_1 or η_2 or ... η_m blocks, consider, without loss of generality, any two elements $V_{(i)}$ and $W_{(j)}$, where $V, W \varepsilon G$, and either $i = j$ or $i \neq j$.

If $i = j$ and suppose that for $V'_{(i)}$ and $W'_{(i)} \varepsilon I_{\ell 0}$ ($\ell = 1, 2, \ldots, s$), we have

$$
\begin{aligned}
V_{(i)} &= V'_{(i)} + \theta \\
W_{(i)} &= W'_{(i)} + \theta
\end{aligned}
\tag{6.1}
$$

i.e., $V_{(i)}$ and $W_{(i)}$ occur together in $I_{\ell\theta}$ ($\theta \varepsilon G$).

From (6.1) we have

$$
V_{(i)} - V'_{(i)} = W_{(i)} - W'_{(i)} \tag{6.2}
$$

Because $V'_{(i)} - W'_{(i)} = C$ is of type (i, i) with $C = \{1, 2, \ldots, u-1\}$ it follows from (6.2), $V'_{(i)} - W'_{(i)} = C$ with $C \varepsilon \{1, 2, \ldots, u-1\}$. Since A is an asymmetrically repeated difference set, it follows that for each $V'_{(i)} - W'_{(i)} = C$, there exist ξ_c such differences where ξ_c is defined in Definition 6.1.2. Hence, there exist exactly ξ_c solutions to (6.1) as equations in θ.

On the other hand, if $i \neq j$ and suppose that $V''_{(i)}$ and $W''_{(j)}$ ε $I_{\ell 0}$ ($\ell = 1, 2, \ldots, s$), we have, similar to (6.1),

$$
\begin{aligned}
V_{(i)} &= V''_{(i)} + \theta' \\
W_{(j)} &= W''_{(j)} + \theta' \quad .
\end{aligned}
\tag{6.2a}
$$

From (6.2a), because $V''_{(i)} - W''_{(j)} = d$ is of type (i, j), where $d \varepsilon \{1, 2, \ldots, u-1\}$, there exist ξ'_d such differences. Hence, there exist exactly ξ'_d solutions to (6.2a) as equations in θ.

It follows then that there are η_1 or η_2 or ... η_m different values, as defined in Definition 6.1.2, which satisfy either (6.1) or (6.2a). Therefore for any two elements $V_{(i)}$ and $W_{(j)}$, they occur together in η_1 or η_2 or ... or η_m blocks.

In case that the condition (2) in Lemma 6.1.1 forms m associate classes and there are u elements in each $I_{\ell 0}$ ($\ell = 1, 2, \ldots, s$), then it implies from Lemma 6.1.1 that the resulting us sets $I_{\ell\theta}$ ($\ell = 1, 2, \ldots, s$; $\theta \varepsilon G$) form a PBIB design with parameters ($t = uq$, $b = us$, $k = u$, $r = \frac{us}{q}$; $\eta_1, \ldots, \eta_m$). That is true, in particular, for Example 6.1.2, which is shown in the following example:

Example 6.1.3

In Example 6.1.2, we can see $k = 3$, $s = 2$, $q = 2$ and $u = 3$. Then by Lemma 6.1.1, we obtain

$$r = 3, \ t = 6, \ b = 6$$

Thus the resulting design obtained from cyclical development of A (mod 3) keeping the subscript of each element unchanged is

	blocks					
	1	2	3	4	5	6
	0_1	1_1	2_1	1_2	2_2	0_2
	0_2	1_2	2_2	2_1	0_1	1_1
	1_1	2_1	0_1	2_2	0_2	1_2

(6.3)

where block 1 and 4 are initial blocks.

By inspection, the association scheme for the design (6.3) is as follows:

treatment	1st associate class	2nd associate class	3rd associate class
0_1	1_2	1_1, 2_1	0_2, 2_2
1_1	2_2	2_1, 0_1	1_2, 0_2
2_1	0_2	0_1, 1_1	2_2, 1_2
0_2	2_1	1_2, 2_2	0_1, 1_1
1_2	0_1	2_2, 0_2	1_1, 2_1
2_2	1_1	0_2, 1_2	2_1, 0_1

where any two treatments which are in the second associate class occur together in the same block once; and any two treatments which are in the third associate class occur together in the same block twice; however, any two treatments which are in the first associate class do not occur together in the same block.

Then the design (6.3) can be checked to be a PBIB ($t = 6$, $b = 6$, $k = 3$, $r = 3$; $\eta_1 = 2$, $\eta_2 = 1$, $\eta_3 = 0$) design. In order to construct an asymmetrically repeated difference set, we can use the following method.

Lemma 6.1.2

Suppose $t = u \times 2$, where $u > 2$ and u is odd, and $G = \{0, 1, \ldots, u-1\}$ (mod u) is an Abelian group under addition. Also suppose that $A = \{I_{10}, I_{20}\}$, where

$$I_{10} = \{(\tfrac{u-1}{2})_1, (\tfrac{u-3}{2})_2, (\tfrac{u-3}{2})_1, \ldots, 0_2, 0_1\}$$

and

$$I_{20} = \{(\tfrac{u-1}{2})_2, (\tfrac{u+1}{2})_1, (\tfrac{u+1}{2})_2, \ldots, (u-1)_1, (u-1)_2\} .$$

If we use A as the set of initial blocks and develop it cyclically (mod u), then the differences in A are asymmetrically repeated.

Proof:

The differences in A of type (1, 1) are:

there are	$u-2$	1's	occurring,
	$u-4$	2's	occurring,
	$\vdots$		
	1	$\frac{u-1}{2}$'s	occurring,
	1	$\frac{u+1}{2}$'s	occurring,
	$\vdots$		
	$u-4$	$u-2$'s	occurring,
	$u-2$	$u-1$'s	occurring.

The same results hold for differences of type (2, 2).

The differences in A of type (1, 2) are:

there are u-1 0's occurring,

u-1 1's occurring,

u-3 2's occurring,

u-5 3's occurring,

⋮

u-5 u-2's occurring,

u-3 u-1's occurring.

Similarly for (2, 1) type. Combined the above results, we have

$$\eta_1 = u-1, \quad \eta_2 = u-2, \quad \eta_3 = u-3, \quad \ldots, \quad \eta_u = 0$$

Therefore, A is an asymmetrically repeated difference set.

Actually the design obtained from Lemma 6.1.2 has the following association scheme: e.g.,

treatment	1st associate class η_1=u-1	2nd associate class η_2=u-2	3rd associate class η_3=u-3
0_1	0_2, $(u-1)_2$	1_1, $(u-1)_1$	1_2, $(u-2)_2$

	4th associate class $_4$=u-4	. . .
	2_1, $(u-2)_1$	. . .

This is a cyclic (mod u) association scheme.

Using Lemma 6.1.2 and Lemma 6.1.1, we can construct a PBIB design. This design is also a GPBRM1 design, which will be shown in the next section.

6.2 Construction of GPBRM1 Designs

The following theorem gives a method of constructing GPBRM1 designs.

Theorem 6.2.1

Suppose $t = u \times 2$, where u is odd and $u > 2$, and $G = \{0, 1, \ldots, u-1\}$ (mod u), A is the set defined in Lemma 6.1.2 and A* is the set resulting from deleting the last element of each $I_{\ell 0}$ in A ($\ell = 1, 2, \ldots, s$). Then the resulting design A_1 generated by A cyclically (mod u) is a GPBRM1 ($t = 2u$, $n = 2u$, $p = u$) design.

Proof:

Clearly, each treatment occurs the same number of times in each period of A_1.

(i) To show that A_1 is a PBIB (m_1) design, consider, e.g., the following association scheme:

treatment	1st associate class	2nd associate class	...	(u-1)th associate class
0_1	1_1	2_1	...	$(u-1)_1$

	u^{th} associate class	(u+1)th associate class	...	(2u-1)th associate class
	0_2	1_2	...	$(u-1)_2$

where the association scheme is cyclic (mod u), then the association matrices can be written as

$$B_{11:1} = \left[\begin{array}{c:c} \begin{matrix} 0 & 1 & 0 & . & 0 \\ 0 & 0 & 1 & . & . \\ \vdots & . & . & . & . \\ 1 & 0 & 0 & . & 0 \end{matrix} & 0 \\ \hdashline 0 & \begin{matrix} 0 & 0 & . & 1 \\ 1 & 0 & . & 0 \\ \vdots & . & . & . \\ 0 & 0 & . & 0 \end{matrix} \end{array}\right]$$

$$B_{11:2} = \left[\begin{array}{c:c} \begin{matrix} 0 & 0 & 1 & . & . & . & 0 \\ 0 & 0 & 0 & 1 & \ldots & & 0 \\ & . & . & . & & & \\ 0 & 1 & 0 & . & . & . & 0 \end{matrix} & 0 \\ \hdashline 0 & \begin{matrix} 0 & 0 & . & 0 \\ 0 & 0 & . & 1 \\ 1 & 0 & . & 0 \\ \vdots & 1 & . & \vdots \\ & \vdots & & \\ 0 & 0 & . & 0 \end{matrix} \end{array}\right]$$

$$\vdots$$

$$B_{11:u-1} = \left[\begin{array}{c|c} \begin{matrix} 0 & . & . & . & . & 1 \\ 1 & . & . & . & . & 0 \\ . & . & . & . & . & . \\ 0 & . & . & . & . & 0 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} 0 & 1 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & \vdots \\ . & . & . & . & 1 \\ 1 & 0 & . & . & 0 \end{matrix} \end{array}\right]$$

$$B_{11:u} = \left[\begin{array}{c|c} 0 & \begin{matrix} 1 & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & 1 \end{matrix} \\ \hline \begin{matrix} 1 & . & . & . & . & 0 \\ . & . & . & . & . & . \\ 0 & . & . & . & . & 1 \end{matrix} & 0 \end{array}\right]$$

$$B_{11:u+1} = \left[\begin{array}{c|c} 0 & \begin{matrix} 0 & 1 & . & . & . & 0 \\ 0 & 0 & 1 & \ldots & & 0 \\ . & . & . & . & . & . \\ 1 & 0 & . & . & . & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 & . & 1 \\ 1 & 0 & . & 0 \\ . & 1 & . & . \\ \vdots & \vdots & & \vdots \\ 0 & 0 & . & 0 \end{matrix} & 0 \end{array}\right]$$

$$\vdots$$

$$B_{11:2u-1} = \left[\begin{array}{cccc|ccccc} & & & & 0 & . & . & . & 1 \\ & & & & 1 & 0 & . & . & 0 \\ & & 0 & & & & & & \\ & & & & . & . & . & . & . \\ & & & & 0 & . & . & . & 0 \\ \hline 0 & 1 & . & 0 & & & & & \\ . & . & . & \vdots & & & & & \\ . & . & . & & & & 0 & & \\ . & . & . & 1 & & & & & \\ 1 & 0 & . & 0 & & & & & \end{array}\right]$$

where rows and columns of the association matrices both represent $0_1, 1_1, \ldots, (u-1)_1, 0_2, 1_2, \ldots, (u-1)_2$. Since this is a cyclic association scheme, it can be easily shown that

$$B_{11:i}\, B_{11:j} = \sum_{k=0}^{2u-1} P_k^{i,j}\, B_{11:k}$$

where $i, j = 1, \ldots, u-1$.

That is, A_1 is a PBIB (2u-1) design (by Shah 1959). Similarly, A* generates a PBIB (2u-1) design. Since a GPBIB design exists iff

$$B_{i_1j_1:t_1}\, B_{i_2j_2:t_2} + B_{i_2j_2:t_2}\, B_{i_1j_1:t_1}$$

$$= 2 \sum P_{ij:t}\, (i_1j_1:t_1,\ i_2j_2:t_2)\, B_{ij:t} \qquad (6.4)$$

The above association scheme can be used also for (12) associate classes. Therefore, (6.4) is satisfied. That is, A_1 is a GPBRM1 design.

Example 6.2.1

Given G = {0, 1, 2, 3} (mod 4), $t = 4 \times 2$ where $u = 4$, and $A = \{I_{10}, I_{20}\}$, where

$$I_{10} = \{1_2, 1_1, 0_2, 0_1\}$$

and

$$I_{20} = \{2_1, 2_2, 3_1, 3_2\} \ .$$

Then A generates the following design cyclically (mod 4):

		subjects							
		1	2	3	4	5	6	7	8
periods	1	1_2	2_2	3_2	0_2	2_1	3_1	0_1	1_1
	2	1_1	2_1	3_1	0_1	2_2	3_2	0_2	1_2
	3	0_2	1_2	2_2	3_2	3_1	0_1	1_1	2_1
	4	0_1	1_1	2_1	3_1	3_2	0_2	1_2	2_2

If we relabel 0_1, 1_1, 2_1, 3_1, 0_2, 1_2, 2_2 and 3_2 as 1, 2, 3, 4, 5, 6, 7 and 8, respectively, then the resulting design, by Theorem 6.2.1, is a GPBRM1 design.

		subjects							
		1	2	3	4	5	6	7	8
periods	1	6	7	8	5	3	4	1	2
	2	2	3	4	1	7	8	5	6
	3	5	6	7	8	4	1	2	7
	4	1	2	3	4	8	5	6	8

So far we have discussed the GPBRM designs for the first residual effects, we will use the same procedures discussed in Chapters IV and V to find the properties and the structures of GPBRM designs for second residual effects.

VII. ANALYSIS AND PROPERTIES OF THE SECOND ORDER RESIDUAL MODEL

So far we have discussed the structure of RM designs with first order residual effects. When both first and second order residual effects exist in the model, we call it a second order residual model. To discuss the properties and structure of suitable RM designs we use a method similar to that in Chapter IV and V. This leads to definitions of BRM1 and BRM2 designs.

7.1 Analysis

The second order residual model can be written as follows:

$$y_{ijk\ell m} = \mu + \tau_i + s_j + \pi_k + \alpha_\ell + \gamma_m + \varepsilon_{ijk\ell m} \tag{7.1}$$

where:

τ_i: direct effect $i = 1, \ldots, t$

s_j: subject effect $j = 1, \ldots, n$

π_k: period effect $k = 1, \ldots, p$

α_ℓ: first order residual effect, $\ell = 1, \ldots, t$

γ_m: second order residual effect, $m = 1, \ldots, t$

are fixed effects, and $\varepsilon_{ijk\ell m} \sim (0, \sigma^2)$ independently distributed.

In matrix notation we can rewrite (7.1) as

$$\underline{Y} = X\underline{\beta} + \underline{\varepsilon} \tag{7.2}$$

where

$$\underline{\beta} = [\tau_1,\ldots,\ \tau_t,\ \alpha_1,\ldots,\ \alpha_t,\ \gamma_1,\ldots,\ \gamma_t,\ \pi_1,\ldots,\ \pi_p,\ s_1,\ldots,s_n]' .$$

The normal equations are

$$x'x\hat{\underline{\beta}} = x'y$$

where x'x and x'y can be partitioned as follows:

$$x'x = \begin{bmatrix} D & S_1 & S_2 & T & N \\ S_1' & D_1^* & S_{12} & T_1^* & N_1^* \\ S_2' & S_{12}' & D_2^* & T_2^* & N_2^* \\ T' & T_1^{*\prime} & T_2^{*\prime} & nI_p & J_{p\times n} \\ N' & N_1^{*\prime} & N_2^{*\prime} & J_{n\times p} & pI_n \end{bmatrix} \tag{7.3}$$

and

$$x'y = [T_{01},\ldots,\ T_{0t},\ R_1,\ldots,\ R_t,\ \Gamma_1,\ldots,\ \Gamma_t,\ P_1,\ldots,\ P_p,\ B_1.,,,\ B_n]'$$

where

T_{0i} is the ith direct effect total

R_ℓ is the ℓth first order residual effect total

Γ_m is the mth second order residual effect total

P_k is the kth period total

B_j is the jth subject total

and

$D = (d_{ii'})$ is the direct-direct effects incidence matrix, where $d_{ii'}$ is the number of times treatment i and treatment i' appear in the same subject;

$S_1 = ({}_1S_{ij})$ is the direct-first order residual effects incidence matrix, where ${}_1S_{ij}$ is the number of times treatment i immediately precedes treatment j;

$S_2 = ({}_2S_{ij})$ is the direct-second order residual effects incidence matrix, where ${}_2S_{ij}$ is the number of times treatment i precedes treatment j by two periods;

$T = (t_{ij})$ is the direct-period incidence matrix, where t_{ij} is the number of times treatment i occurs in period j;

$N = (n_{ij})$ is the direct-subject incidence matrix, where n_{ij} is the number of times treatment i occurs in subject j;

$D_k^* = ({}_kd_{ii'}^*)$ is the kth order residual-kth order residual effects incidence matrix, where $d_{ii'}^*$ is the number of times the kth order residual effect i occurs together with the kth order residual effect i' in the same subject, k = 1, 2;

$S_{12} = ({}_{12}S_{ij})$ is the first order residual-second order residual effects incidence matrix, where ${}_{12}S_{ij}$ is the number of times the first order residual effect i immediately precedes the second order residual effect j;

$T_k^* = ({}_k t_{ij}^*)$ is the kth order residual effect-period incidence matrix, where ${}_k t_{ij}^*$ represents the number of times the kth order residual effect i occurs in jth period, k = 1, 2,;

$N_k^* = ({}_k n_{ij}^*)$ is the kth order residual effect-subject incidence matrix, where ${}_k n_{ij}^*$ is the number of times kth order residual effect i occurs in jth subject.

Then the reduced normal equation for direct and residual effects can be written as

$$C\begin{bmatrix} \hat{\underline{\tau}} \\ \hat{\underline{\alpha}} \\ \hat{\underline{\gamma}} \end{bmatrix} = Q_{(2)} \tag{7.4}$$

where

$$Q_{(2)} = \begin{bmatrix} \underset{\sim}{T}_0 \\ \underset{\sim}{R}_1 \\ \underset{\sim}{R}_2 \end{bmatrix} - \begin{bmatrix} T & N \\ T_1^* & N_1^* \\ T_2^* & N_2^* \end{bmatrix} \begin{bmatrix} nI_p & J_{p\times n} \\ J_{n\times p} & pI_n \end{bmatrix} \begin{bmatrix} \underset{\sim}{P} \\ \underset{\sim}{B} \end{bmatrix}$$

$$= \begin{bmatrix} \underset{\sim}{T}_0 - \frac{1}{n} T\underset{\sim}{P} - \frac{1}{p} N\underset{\sim}{B} + \frac{1}{np} NJ_n\underset{\sim}{B} \\ \underset{\sim}{R}_1 - \frac{1}{n} T_1^*\underset{\sim}{P} - \frac{1}{p} N_1^*\underset{\sim}{B} + \frac{1}{np} N_1^*J_n\underset{\sim}{B} \\ \underset{\sim}{R}_2 - \frac{1}{n} T_2^*\underset{\sim}{P} - \frac{1}{p} N_2^*\underset{\sim}{B} + \frac{1}{np} N_2^*J_n\underset{\sim}{B} \end{bmatrix}$$

and $\underset{\sim}{T}_0$, $\underset{\sim}{R}_1$, $\underset{\sim}{B}$, $\underset{\sim}{P}$ are defined in (4.4) and $\underset{\sim}{R}_2 = [R_{21}, \ldots, R_{2t}]'$, where R_{2i} is the ith second order residual effect total.

and where

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

is the c-matrix adjusted for subjects and periods, i.e.,

$$C = \begin{bmatrix} D & S_1 & S_2 \\ S_1' & D_1^* & S_{12} \\ S_2' & S_{12}' & D_2^* \end{bmatrix}$$

$$- \begin{bmatrix} T & N \\ T_1^* & N_1^* \\ T_2^* & N_2^* \end{bmatrix} \begin{bmatrix} nI_p & J_{p\times n} \\ J_{n\times p} & pI_n \end{bmatrix}^{-} \begin{bmatrix} T' & T_1^{*\prime} & T_2^{*\prime} \\ N' & N_1^{*\prime} & N_2^{*\prime} \end{bmatrix} . \tag{7.5}$$

Since

$$\begin{bmatrix} nI_p & J_{p\times n} \\ J_{n\times p} & pI_n \end{bmatrix}^{-} = \begin{bmatrix} \frac{1}{n} I_p & 0 \\ 0 & \frac{1}{p} I_n - \frac{1}{np} J_n \end{bmatrix} ,$$

we obtain from (7.5)

$$C_{11} = D - \frac{1}{n} TT' - \frac{1}{p} NN' + \frac{1}{np} NJ_n N'$$

$$C_{12} = S_1 - \frac{1}{n} TT_1^{*\prime} - \frac{1}{p} NN_1^{*\prime} + \frac{1}{np} NJ_n N_1^{*\prime}$$

$$C_{13} = S_2 - \frac{1}{n} TT_2^{*\prime} - \frac{1}{p} NN_2^{*\prime} + \frac{1}{np} NJ_n N_2^{*\prime}$$

$$C_{22} = D_1^* - \frac{1}{n} T_1^* T_1^{*\prime} - \frac{1}{p} N_1^* N_1^{*\prime} + \frac{1}{np} N_1^* J_n N_1^{*\prime}$$

$$C_{23} = S_{12} - \frac{1}{n} T_1^* T_2^{*'} - \frac{1}{p} N_1^* N_2^{*'} + \frac{1}{np} N_1^* J_n N_2^{*'}$$

$$C_{33} = D_2^* - \frac{1}{n} T_2^* T_2^{*'} - \frac{1}{p} N_2^* N_2^{*'} + \frac{1}{np} N_2^* J_n N_2^{*'} ,$$

and

$$C_{ji} = C_{ij}', \quad \text{where } i,\ j = 1,\ 2,\ 3 \text{ and } i \neq j\ . \tag{7.6}$$

A generalized partially balanced repeated measurement design for the second order residual model, GPBRM2 design, can be defined by extending Definition 3.2.1 using i, j = 1, 2, 3. For a GPBRM2 design, the partitioned matrices of x'x in (7.3) are as follows:

$$D = \lambda_1 p I_t$$

$$T = \lambda_1 J_{t\times p}$$

$$D_1{}^* = \lambda_1 (p-1) I_t$$

$$T_1{}^* = \lambda_1 [0 : J_{tx(p-1)}]$$

$$T_2{}^* = \lambda_1 [0 : J_{tx(p-2)}]$$

$$D_2{}^* = \lambda_1 (p-2) I_t\ .$$

Hence,

$$TT' = p\lambda_1{}^2\, J_t$$

$$TT_1{}^{*'} = \lambda_1{}^2 (p-1) J_t$$

$$T_1{}^* T_1{}^{*'} = \lambda_1{}^2 (p-1) J_t$$

$$TT_2{}^{*'} = \lambda_1{}^2(p-2)J_t$$

$$T_1{}^*T_2{}^{*'} = \lambda_1{}^2(p-2)J_t$$

$$T_2{}^*T_2{}^{*'} = \lambda_1{}^2(p-2)J_t \ .$$

Since

$$N = \begin{bmatrix} \sum_{k=1}^{p} n_{11k} & \cdots\cdots & \sum_{k=1}^{p} n_{1nk} \\ \cdots & \cdots\cdots & \cdots \\ \sum_{k=1}^{p} n_{t1k} & \cdots\cdots & \sum_{k=1}^{p} n_{tnk} \end{bmatrix}$$

$$N_1{}^* = \begin{bmatrix} \sum_{k=2}^{p} n_{11(k-1)} & \cdots\cdots & \sum_{k=2}^{p} n_{1n(k-1)} \\ \cdots & \cdots\cdots & \cdots \\ \sum_{k=2}^{p} n_{t1(k-1)} & \cdots\cdots & \sum_{k=2}^{p} n_{tn(k-1)} \end{bmatrix}$$

$$N_2{}^* = \begin{bmatrix} \sum_{k=3}^{p} n_{11(k-2)} & \cdots\cdots & \sum_{k=3}^{p} n_{1n(k-2)} \\ \cdots & \cdots\cdots & \cdots \\ \sum_{k=3}^{p} n_{t1(k-2)} & \cdots\cdots & \sum_{k=3}^{p} n_{tn(k-2)} \end{bmatrix}$$

where n_{ijk} is the number of times treatment i occurs in subject j and period k, we have

$$NJ_nN' = \lambda_1 pJ_tN' = \lambda_1{}^2p^2J_t$$

$$NJ_nN_1{}^{*'} = \lambda_1 pJ_tN_1{}^{*'} = \lambda_1{}^2p(p-1)J_t$$

$$N_1{*}J_nN_1{*}' = \lambda_1^{\,2}(p-1)^2 J_t$$

$$NJ_nN_2{*}' = \lambda_1^{\,2}p(p-2)J_t$$

$$N_1{*}J_nN_2{*}' = \lambda_1^{\,2}(p-1)(p-2)J_t$$

$$N_2{*}J_nN_2{*}' = \lambda_1^{\,2}(p-2)^2 J_t \quad .$$

It follows then that

$$C_{11} = \lambda_1 pJ_t - \frac{1}{p}\,NN'$$

$$C_{12} = S_1 - \frac{1}{p}\,NN_1{*}'$$

$$C_{13} = S_2 - \frac{1}{p}\,NN_2{*}'$$

$$C_{22} = \lambda_1(p-1)I_t - \frac{\lambda_1^{\,2}(p-1)}{np}\,J_t - \frac{1}{p}\,N_1{*}N_1{*}'$$

$$C_{23} = S_{12} - \frac{\lambda_1^{\,2}(p-2)}{np}\,J_t - \frac{1}{p}\,N_1{*}N_2{*}'$$

$$C_{33} = \lambda_1(p-2)I_t - \frac{2\lambda_1^{\,2}(p-2)}{np}\,J_t - \frac{1}{p}\,N_2{*}N_2{*}' \quad ,$$

and

$$C_{ji} = C_{ij}' \quad \text{for } i,\ j = 1,\ 2,\ 3 \text{ and } i \neq j \ . \tag{7.7}$$

To examine the properties and structures of GPBRM2 designs we use the association matrices and follow a similar procedure as in Section (5.2).

For a GPBRM2 design, let NN', $N_1{*}N_1{*}'$, $N_2{*}N_2{*}'$, $NN_1{*}'$, $NN_2{*}'$ and $N_1{*}N_2{*}'$ are the PBIB (m_{11}), PBIB (m_{22}), PBIB (m_{33}), PBIB (m_{12}), PBIB (m_{13}) and PBIB (m_{23}) concordance matrices, respectively. Then, similar to (5.7), we have

$$NN' = \sum_{v=0}^{m_{11}} \lambda_{11:v} B_{11:v}$$

$$N_1{*}N_1{*}' = \sum_{v=0}^{m_{22}} \lambda_{22:v} B_{22:v}$$

$$N_2{*}N_2{*}' = \sum_{v=0}^{m_{33}} \lambda_{33:v} B_{33:v}$$

$$NN_1{*}' = \sum_{v=0}^{m_{12}} \lambda_{12:v} B_{12:v}$$

$$NN_2{*}' = \sum_{v=0}^{m_{13}} \lambda_{13:v} B_{13:v}$$

$$N_1{*}N_2{*}' = \sum_{v=0}^{m_{23}} \lambda_{23:v} B_{23:v}$$

$$C_{11} = (\lambda_1 p - \frac{\lambda_{11:0}}{p}) B_{11:0} + \sum_{v=1}^{m_{11}} (- \frac{\lambda_{11:t}}{p}) B_{11:v}$$

$$C_{12} = \sum_{v=0}^{m_{12}} (\mu_{12:t} - \frac{\lambda_{12:t}}{p}) B_{12:v}$$

$$C_{13} = \sum_{v=0}^{m_{13}} (\lambda_{13:v} - \frac{\lambda_{13:v}}{p}) B_{13:v}$$

$$C_{22} = [\lambda_1 (p-1) - \frac{\lambda_1 (p-1)}{pt} - \frac{\lambda_{22:0}}{p}] B_{22:0}$$

$$+ \sum_{v=1}^{m_{22}} [- \frac{\lambda_1 (p-1)}{pt} - \frac{\lambda_{22:v}}{p}] B_{22:v}$$

$$C_{23} = [\mu_{23:0} - \frac{\lambda_1(p-2)}{pt} - \frac{\lambda_{23:0}}{p}]\ B_{23:0}$$

$$+ \sum_{v=1}^{m_{23}} [\mu_{23:v} - \frac{\lambda_1(p-2)}{pt} - \frac{\lambda_{23:v}}{p}]\ B_{23:v}$$

$$C_{33} = [\lambda_1(p-2) - \frac{2\lambda_1(p-2)}{pt} - \frac{\lambda_{33:0}}{p}]\ B_{33:0}$$

$$+ \sum_{v=1}^{m_{33}} [- \frac{2\lambda_1(p-2)}{pt} - \frac{\lambda_{33:v}}{p}]\ B_{33:v}$$

$$C_{21} = C_{12}'$$

$$C_{31} = C_{13}'$$

and

$$C_{32} = C_{23}' \ . \tag{7.8}$$

And then (7.4) and (7.8) can be expressed in the following form:

$$C^* \begin{bmatrix} \hat{\underset{\sim}{t}} \\ \hat{\underset{\sim}{\alpha}} \\ \hat{\underset{\sim}{\gamma}} \end{bmatrix} = Q_{(2)} \tag{7.9}$$

where

$$\dot{C}^* = \begin{bmatrix} \sum_{v=0}^{m_{11}} \lambda^*_{11:v} B_{11:v} & \sum_{v=0}^{m_{12}} \lambda^*_{12:v} B_{12:v} & \sum_{v=0}^{m_{13}} \lambda^*_{13:v} B_{13:v} \\ \sum_{v=0}^{m_{21}} \lambda^*_{21:v} B_{21:v} & \sum_{v=0}^{m_{22}} \lambda^*_{22:v} B_{22:v} & \sum_{v=0}^{m_{23}} \lambda^*_{23:v} B_{23:v} \\ \sum_{v=0}^{m_{31}} \lambda^*_{31:v} B_{31:v} & \sum_{v=0}^{m_{32}} \lambda^*_{32:v} B_{32:v} & \sum_{v=0}^{m_{33}} \lambda^*_{33:v} B_{33:v} \end{bmatrix}$$

and where,

$$\lambda^*_{11:0} = \lambda_1 p - \frac{\lambda_{11:0}}{p}$$

$$\lambda^*_{11:v} = -\frac{\lambda_{11:v}}{p} \qquad , \; v=1,2,\ldots,m_{11}$$

$$\lambda^*_{12:0} = \mu_{12:0} - \frac{\lambda_{12:0}}{p}$$

$$\lambda^*_{12:v} = \mu_{12:v} - \frac{\lambda_{12:v}}{p} \; , \; v=1,2,\ldots,m_{12}$$

$$\lambda^*_{13:0} = \mu_{13:0} - \frac{\lambda_{13:0}}{p}$$

$$\lambda^*_{13:v} = \mu_{13:v} - \frac{\lambda_{13:v}}{p} \; , \; v=1,2,\ldots,m_{13}$$

$$\lambda^*_{21:0} = \mu_{21:0} - \frac{\lambda_{21:0}}{p}$$

$$\lambda^*_{21:v} = \mu_{21:v} - \frac{\lambda_{21:v}}{p} \; , \; v=1,2,\ldots,m_{21}$$

$$\lambda^*_{22:0} = \lambda_1(p-1) - \frac{\lambda_1(p-1)}{pt} - \frac{\lambda_{22:0}}{p}$$

$$\lambda^*_{22:v} = - \frac{\lambda_1(p-1)}{pt} - \frac{\lambda_{22:v}}{p} \quad , \; v=1,2,\ldots,m_{22}$$

$$\lambda^*_{23:0} = \mu_{23:0} - \frac{\lambda_1(p-2)}{pt} - \frac{\lambda_{23:0}}{p}$$

$$\lambda^*_{23:v} = \mu_{23:v} - \frac{\lambda_1(p-2)}{pt} - \frac{\lambda_{23:v}}{p} \quad , \; v=1,\ldots,m_{23}$$

$$\lambda^*_{31:0} = \mu_{31:0} - \frac{\lambda_{31:0}}{p}$$

$$\lambda^*_{31:v} = \mu_{31:v} - \frac{\lambda_{31:v}}{p} \quad , \; v=1,\ldots,m_{31}$$

$$\lambda^*_{32:0} = \mu_{32:0} - \frac{\lambda_1(p-2)}{pt} - \frac{\lambda_{32:0}}{p}$$

$$\lambda^*_{32:v} = \mu_{32:v} - \frac{\lambda_1(p-2)}{pt} - \frac{\lambda_{32:v}}{p} \quad , \; v=1,\ldots,m_{32}$$

$$\lambda^*_{33:0} = \lambda_1(p-2) - \frac{2\lambda_1(p-2)}{pt} - \frac{\lambda_{33:0}}{p}$$

$$\lambda^*_{33:v} = - \frac{2\lambda_1(p-2)}{pt} - \frac{\lambda_{33:v}}{p} \quad , \; v=1,\ldots,m_{33} \; . \qquad (7.10)$$

It follows from (7.9) that the estimators for $\hat{\underset{\sim}{\tau}}$, $\hat{\underset{\sim}{\alpha}}$ and $\hat{\underset{\sim}{\gamma}}$ are given by

$$\begin{bmatrix} \hat{\underset{\sim}{\tau}} \\ \hat{\underset{\sim}{\alpha}} \\ \hat{\underset{\sim}{\gamma}} \end{bmatrix} = C^- \, Q_{(2)} \qquad (7.11)$$

where

$$C^{-} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^{-} .$$

The variance-covariance matrix $\Sigma_{\hat{\underset{\sim}{t}},\hat{\underset{\sim}{\alpha}},\hat{\underset{\sim}{\gamma}}}$ between $\hat{\underset{\sim}{t}}$, $\hat{\underset{\sim}{\alpha}}$ and $\hat{\underset{\sim}{\gamma}}$ is

$$\Sigma_{\hat{\underset{\sim}{t}},\hat{\underset{\sim}{\alpha}},\hat{\underset{\sim}{\gamma}}} = C^{-}\sigma^2 \tag{7.12}$$

and

$$\Sigma_{\hat{\underset{\sim}{t}},\hat{\underset{\sim}{\alpha}},\hat{\underset{\sim}{\gamma}}} = \begin{bmatrix} \Sigma_{\hat{\underset{\sim}{t}}} & \Sigma_{\hat{\underset{\sim}{t}};\hat{\underset{\sim}{\alpha}},\hat{\underset{\sim}{\gamma}}} \\ \Sigma'_{\hat{\underset{\sim}{t}};\hat{\underset{\sim}{\alpha}},\hat{\underset{\sim}{\gamma}}} & \Sigma_{\hat{\underset{\sim}{\alpha}},\hat{\underset{\sim}{\gamma}}} \end{bmatrix}$$

where $\Sigma_{\hat{\underset{\sim}{t}}}$, $\Sigma_{\hat{\underset{\sim}{t}};\hat{\underset{\sim}{\alpha}},\hat{\underset{\sim}{\gamma}}}$ and $\Sigma_{\hat{\underset{\sim}{\alpha}},\hat{\underset{\sim}{\gamma}}}$ are the variance-covariance matrix for $\hat{t}$, the covariance matrix for $\hat{t}$ and $\hat{\underset{\sim}{\alpha}}$, $\hat{\underset{\sim}{\gamma}}$, the covariance matrix for $\hat{\underset{\sim}{\alpha}}$ and $\hat{\underset{\sim}{\gamma}}$, respectively.

Then we obtain, from (7.12), that

$$\frac{\Sigma_{\hat{\underset{\sim}{t}}}}{\sigma^2} = \left(C_{11} - [C_{12} \quad C_{13}] \begin{bmatrix} C_{22} & C_{23} \\ C_{32} & C_{33} \end{bmatrix}^{-} \begin{bmatrix} C_{21} \\ C_{31} \end{bmatrix} \right)^{-}$$

$$\begin{aligned} &= [C_{11} - C_{12}(C_{22} - C_{23}C_{33}^{-}C_{32})^{-}C_{21} \\ &\quad + C_{13}(C_{33} - C_{32}C_{22}^{-}C_{23})^{-}C_{32}C_{22}^{-}C_{21} \\ &\quad + C_{12}C_{22}^{-}C_{23}(C_{33} - C_{32}C_{22}^{-}C_{23})^{-}C_{31} \\ &\quad - C_{13}(C_{33} - C_{32}C_{22}^{-}C_{23})^{-}C_{31}]^{-} \end{aligned} \tag{7.13}$$

$$\frac{\Sigma_{\hat{\alpha},\hat{\gamma}}}{\sigma^2} = \left(\begin{bmatrix} C_{22} & C_{23} \\ C_{32} & C_{33} \end{bmatrix} - \begin{bmatrix} C_{21}C_{11}^{-}C_{12} & C_{21}C_{11}^{-}C_{13} \\ C_{31}C_{11}^{-}C_{12} & C_{31}C_{11}^{-}C_{13} \end{bmatrix} \right)^{-}$$

$$= \begin{bmatrix} C_{22} - C_{21}C_{11}^{-}C_{12} & C_{23} - C_{21}C_{11}^{-}C_{13} \\ C_{32} - C_{31}C_{11}^{-}C_{12} & C_{33} - C_{31}C_{11}^{-}C_{13} \end{bmatrix}^{-}$$

or

$$\frac{\Sigma_{\hat{\alpha},\hat{\gamma}}}{\sigma^2} \equiv \begin{bmatrix} F_2 & F_{23} \\ F_{23}' & F_3 \end{bmatrix},$$

where

$$F_2 = [C_{22} - C_{21}C_{11}^{-}C_{12} - (C_{23} - C_{21}C_{11}^{-}C_{13})$$
$$(C_{33} - C_{31}C_{11}^{-}C_{13})^{-}(C_{32} - C_{31}C_{11}^{-}C_{12})]^{-}$$

$$F_3 = [C_{33} - C_{31}C_{11}^{-}C_{13} - (C_{32} - C_{31}C_{11}^{-}C_{12})$$
$$(C_{22} - C_{21}C_{11}^{-}C_{12})^{-}(C_{23} - C_{21}C_{11}^{-}C_{13})]^{-}$$

and

$$F_{23} = -(C_{22} - C_{21}C_{11}^{-}C_{12})^{-}(C_{23} - C_{21}C_{11}^{-}C_{13})F_3 \ . \qquad (7.14)$$

Therefore,

$$\frac{\Sigma_{\tilde{\hat{\tau}},\tilde{\hat{\alpha}},\tilde{\hat{\gamma}}}}{\sigma^2} = - C_{11}^{-}[C_{12} \quad C_{13}] \begin{bmatrix} F_2 & F_{23} \\ F_{23}' & F_3 \end{bmatrix}$$

$$= [- C_{11}^{-}C_{12}F_2 - C_{11}^{-}C_{13}F_{23}^{-} \quad - C_{11}^{-}C_{12}F_{23} -$$

$$C_{11}^{-}C_{13}F_3] \quad . \tag{7.15}$$

In summary, (7.13), (7.14) and (7.15) can be rewritten as follows:

$$\frac{\Sigma_{\hat{\underset{\sim}{\tau}}}}{\sigma^2} = [C_{11} - C_{12}(C_{22} - C_{23}C_{33}^{-}C_{32})^{-}C_{21} + C_{13}(C_{33} - C_{32}C_{22}^{-}C_{23})^{-}C_{32}C_{22}^{-}C_{21}$$

$$+ C_{12}C_{22}^{-}C_{23}(C_{33} - C_{32}C_{22}^{-}C_{23})^{-}C_{31} - C_{13}(C_{33} - C_{32}C_{22}^{-}C_{23})^{-}C_{31}]^{-}$$

$$\frac{\Sigma_{\hat{\underset{\sim}{\tau}},\hat{\underset{\sim}{\alpha}}}}{\sigma^2} = - C_{11}^{-}C_{12}F_2 - C_{11}^{-}C_{13}F_{23}'$$

$$\frac{\Sigma_{\hat{\underset{\sim}{\tau}},\hat{\underset{\sim}{\alpha}}}}{\sigma^2} = - C_{11}^{-}C_{12}F_{23} - C_{11}^{-}C_{13}F_3$$

$$\frac{\Sigma_{\hat{\underset{\sim}{\alpha}}}}{\sigma^2} = F_2$$

$$\frac{\Sigma_{\hat{\underset{\sim}{\alpha}},\hat{\underset{\sim}{\gamma}}}}{\sigma^2} = F_{23}$$

$$\frac{\Sigma_{\hat{\underset{\sim}{\gamma}}}}{\sigma^2} = F_3 \tag{7.16}$$

where F_2, F_3 and F_{23} are given in (7.14).

7.2 Properties

Similar to Lemma 5.1, we have the following:

Lemma 7.2.1

For any GPBRM2 design, the parameters p, λ_1, $n_{ij:v}$, $\lambda_{ij:v}$ and $\mu_{ij:v}$ satisfy the following relationships:

$$\sum_{v=0}^{m_{11}} n_{11:v}\ \lambda_{11:V} = \lambda_1 p^2$$

$$\sum_{v=0}^{m_{12}} n_{12:v}\ \lambda_{12:v} = \sum_{v=0}^{m_{21}} n_{21:v}\ \lambda_{21:v} = \lambda_1 p(p-1)$$

$$\sum_{v=0}^{m_{13}} n_{13:v}\ \lambda_{13:v} = \sum_{v=0}^{m_{31}} n_{31:v}\ \lambda_{31:v} = \lambda_1 p(p-2)$$

$$\sum_{v=0}^{m_{22}} n_{22:v}\ \lambda_{22:v} = \lambda_1 (p-1)^2$$

$$\sum_{v=0}^{m_{23}} n_{23:v}\ \lambda_{23:v} = \sum_{v=0}^{m_{32}} n_{32:v}\ \lambda_{32:v} = \lambda_1 (p-1)(p-2)$$

$$\sum_{v=0}^{m_{33}} n_{33:v}\ \lambda_{33:v} = \lambda_1 (p-2)^2$$

$$\sum_{v=0}^{m_{12}} n_{12:v}\ \mu_{12:v} = \sum_{v=0}^{m_{21}} n_{21:v}\ \mu_{21:v} = \lambda_1 (p-1)$$

$$\sum_{v=0}^{m_{13}} n_{13:v}\ \mu_{13:v} = \sum_{v=0}^{m_{31}} n_{31:v}\ \mu_{31:v} = \lambda_1 (p-2)$$

$$\sum_{v=0}^{m_{23}} n_{23:v}\, \mu_{23:v} = \sum_{v=0}^{m_{32}} n_{32:v}\, \mu_{32:v} = \lambda_1(p-2) \ .$$

The proof of Lemma 7.2.1 is similar to that of Lemma 5.1. A lemma similar to Lemma 5.2 is the following:

Lemma 7.2.2

The $\lambda^*_{ij:v}$'s in (7.10) are restricted by the following constraints:

$$\sum_{v=0}^{m_{ij}} n_{ij:v}\, \lambda^*_{ij:v} = 0$$

where i, j = 1, 2, 3.

Proof:

It follows from Lemma 7.2.1 that

$$\sum_{v=0}^{m_{11}} n_{11:v}\, \lambda^*_{11:v}$$

$$= \lambda_1 p - \frac{\lambda_{11:0}}{p} - \frac{1}{p} \sum_{v=1}^{m_{11}} n_{11:v}\, \lambda_{11:v}$$

$$= \lambda_1 p - \frac{1}{p} \sum_{v=0}^{m_{11}} n_{11:v}\, \lambda_{11:v}$$

$$= \lambda_1 p - \frac{1}{p}\,(\lambda_1 p^2) = 0 \quad .$$

Similarly,

$$\sum_{v=0}^{m_{12}} n_{12:v}\, \lambda^*_{12:v}$$

$$= \mu_{12:0} - \frac{\lambda_{12:0}}{p} + \sum_{v=1}^{m_{12}} n_{12:v}(\mu_{12:v} - \frac{\lambda_{12:v}}{p})$$

$$= \sum_{v=0}^{m_{12}} n_{12:v}\, \mu_{12:v} - \frac{1}{p} \sum_{v=0}^{m_{12}} n_{12:v}\, \lambda_{12:v}$$

$$= \lambda_1(p-1) - \frac{1}{p}\, \lambda_1 p(p-1) = 0 \ ,$$

$$\sum_{v=0}^{m_{13}} n_{13:v}\, \lambda^*_{13:v}$$

$$= \mu_{13:0} - \frac{\lambda_{13:0}}{p} + \sum_{v=1}^{m_{13}} n_{13:v}(\mu_{13:v} - \frac{\lambda_{13:v}}{p})$$

$$= \sum_{v=0}^{m_{13}} n_{13:v}\, \mu_{13:v} - \frac{1}{p} \sum_{v=0}^{m_{13}} n_{13:v}\, \lambda_{13:v}$$

$$= \lambda_1(p-2) - \frac{1}{p}\, \lambda_1 p(p-2) = 0 \ ,$$

$$\sum_{v=0}^{m_{22}} n_{22:v}\, \lambda^*_{22:v}$$

$$= \lambda_1(p-1) - \frac{\lambda_1(p-1)}{pt} - \frac{\lambda_{22:0}}{p} - \frac{\lambda_1(p-1)}{pt} \sum_{v=1}^{m_{22}} n_{22:v}$$

$$- \frac{1}{p} \sum_{v=1}^{m_{22}} n_{22:v}\, \lambda_{22:v}$$

$$= \lambda_1(p-1) - \frac{\lambda_1(p-1)}{pt} \sum_{v=0}^{m_{22}} n_{22:v} - \frac{1}{p} \sum_{v=0}^{m_{22}} n_{22:v}\, \lambda_{22:v}$$

$$= \lambda_1(p-1) - \frac{\lambda_1(p-1)}{p} - \frac{1}{p}\, \lambda_1(p-1)^2 = 0 \ ,$$

$$\sum_{v=0}^{m_{23}} n_{23:v}\, \lambda^{*}_{23:v}$$

$$= \mu_{23:0} - \frac{\lambda_1(p-2)}{pt} - \sum_{v=1}^{m_{23}} n_{23:v}\, \mu_{23:v} - \frac{\lambda_1(p-2)}{pt} \sum_{v=1}^{m_{23}} n_{23:v}$$

$$- \frac{1}{p} \sum_{v=1}^{m_{23}} n_{23:v}\, \lambda_{23:v}$$

$$= \sum_{v=0}^{m_{23}} n_{23:v}\, \mu_{23:v} - \frac{\lambda_1(p-2)}{pt} \sum_{v=0}^{m_{23}} n_{23:v} - \frac{1}{p} \sum_{v=0}^{m_{23}} n_{23:v}\, \lambda_{23:v}$$

$$= \lambda_1(p-2) - \frac{\lambda_1(p-2)}{p} - \frac{1}{p}\, \lambda_1(p-1)(p-2)$$

$$= \lambda_1(p-2) - \lambda_1(p-2) = 0 \ ,$$

$$\sum_{v=0}^{m_{33}} n_{33:v}\, \lambda^{*}_{33:v}$$

$$= \lambda_1(p-2) - \frac{2\lambda_1(p-2)}{pt} - \frac{\lambda_{33:0}}{p} - \frac{2\lambda_1(p-2)}{pt} \sum_{v=1}^{m_{33}} n_{33:v}$$

$$- \frac{\sum_{v=1}^{m_{33}} n_{33:v}\, \lambda_{33:v}}{p}$$

$$= \lambda_1(p-2) - \frac{2\lambda_1(p-2)}{p} - \frac{1}{p} \sum_{v=0}^{m_{33}} n_{33:v}\, \lambda_{33:v}$$

$$= \lambda_1(p-2) - \frac{2\lambda_1(p-2)}{p} - \frac{1}{p}\, \lambda_1(p-2)^2$$

$$= \lambda_1(p-2) - \lambda_1(p-2) = 0 \ .$$

Then similar to Lemma 4.2.2, we have

Lemma 7.2.3

C_{11}, C_{22}, and C_{33} as given in (7.4) are nonnegative definite.

Hence from Lemma 7.2.3 C_{11}^{-}, C_{22}^{-}, and C_{33}^{-} can be expressed in terms of P-matrices similar to (5.17c). And if $\hat{\underset{\sim}{\tau}}$, $\hat{\underset{\sim}{\alpha}}$ and $\hat{\underset{\sim}{\gamma}}$ are estimable, then they can also be expressed in terms of P-matrices. From Lemma 7.2.2 the sum of columns or rows of C_{ij}^{*} of C^{*} in (7.9) is equal to zero for all i, j = 1, 2, 3. Therefore we have

$$\text{rank}(C^{*}) \leq 3t - 3 . \tag{7.17}$$

In order to find the estimators of direct, first order residual and second order residual effects for a GPBRM2 design, we first require it to be connected which we define as follows:

Definition 7.2.1

A GPBRM2 design is said to be connected if all linear contrasts $\tau_i - \tau_{i'}$, $\alpha_i - \alpha_{i'}$ and $\gamma_i - \gamma_{i'}$ are estimable where i, i' = 1, ... , t.

From (7.17) and Definition 7.2.1, we can find a disconnected GPBRM2 design in terms of the rank of C^{*} according to the following theorem:

Theorem 7.2.1

A necessary condition for a GPBRM2 design to be connected is that $\text{rank}(C^{*}) = 3t - 3$.

The proof of Theorem 7.2.1 is similar to that of Theorem 5.2.1.

Following an argument similar to that in Lemma 4.2.1, the concordance matrices NN_1*', NN_2*' and N_1*N_2*' in (7.6) will be determined by the concordance matrices NN', N_1*N_1*' and N_2*N_2*' subject to one of the conditions C_1, C_2, and C_3 to be explained later. This property will be used to define a balanced repeated measurement design when second order residual effects exist, i.e., a BRM2 design.

Recall that condition C_1 is satisfied if the number of subjects receiving treatment i' when treatment i is in the last period is equal to the number of subjects receiving treatment i when treatment i' is in the last period. Now we introduce two more conditions in the following way:

Condition C_2 is satisfied if, after deleting the last period of a RM design, the number of subjects receiving treatment i' when treatment i is in the last period of the resulting design is equal to the number of subjects receiving treatment i when treatment i' is in the last period of the resulting design.

Condition C_3 is satisfied if after deleting the last two periods of a RM design, in the resulting design, the number of subjects receiving treatment i' when treatment i is in the last period is equal to the number of subjects receiving treatment i when treatment i' is in the last period.

Using C_1, C_2, and C_3, we can now prove the following lemma:

Lemma 7.2.4

(1) Under condition C_1, NN_1*' can be represented in terms of

NN' and $N_1^*N_1^{*'}$ by the following relation:

$$NN_1^* = \tfrac{1}{2}(NN' + N_1^*N_1^{*'}) - \tfrac{1}{2}\lambda_1 I_t$$

(2) Under condition C_2, $N_1^*N_2^{*'}$ can be represented in terms of $N_1^*N_1^{*'}$ and $N_2^*N_2^{*'}$ by the following relation:

$$N_1^*N_2^{*'} = \tfrac{1}{2}(N_1^*N_1^{*'} + N_2^*N_2^{*'}) - \frac{\lambda_1}{2} I_t$$

(3) Letting $NN_2^{*'} = (a_{ii'})$, $NN' = (b_{ii'})$ and $N_2^*N_2^{*'} = (t_{ii'})$, we have under conditions C_1, C_2, and C_3 that

$$a_{ii'} = \tfrac{1}{2}(t_{ii'} + b_{ii'}) - \lambda_1\delta_{ii'} - \sum_j n_{i'jp}\, n_{ij(p-1)} \,.$$

Proof:

We can see that (1) is the same as Lemma 4.2.1.

To prove (2), let

$$N = \Big(\sum_{k=1}^{p} n_{ijk}\Big)_{t\times n}$$

$$N_1^* = \Big(\sum_{k=2}^{p} n_{ij(k-1)}\Big)_{t\times n}$$

$$N_2^* = \Big(\sum_{k=3}^{p} n_{ij(k-2)}\Big)_{t\times n}$$

where n_{ijk} represents the number of times treatment i occurs in subject j and period k.

Then

$$NN_1^{*'} = \Big(\sum_{j,k'} n_{ijk'} \sum_{k=2}^{p} n_{i'j(k-1)}\Big)$$

$$NN' = (\sum_{j,k'} n_{ijk'} \sum_{k=1}^{p} n_{i'jk})$$

$$NN_2*' = (\sum_{j,k'} n_{ijk'} \sum_{k=3}^{p} n_{i'j(k-2)})$$

$$N_1*N_1*' = (\sum_{k=2}^{p} \sum_{j} n_{ij(k-1)} (\sum_{k'=2}^{p} n_{i'j(k'-1)}))$$

$$N_1*N_2*' = (\sum_{k=2}^{p} \sum_{j} n_{ij(k-1)} (\sum_{k'=3}^{p} n_{i'j(k'-2)}))$$

$$N_2*N_2*' = (\sum_{k=3}^{p} \sum_{j} n_{ij(k-2)} (\sum_{k'=3}^{p} n_{i'j(k'-2)})) .$$

If we write $NN_1*' \equiv (d_{ii'})$, $NN' \equiv (b_{ii'})$, $NN_2*' \equiv (a_{ii'})$, $N_1*N_1*' \equiv (s_{ii'})$, $N_1*N_2*' \equiv (u_{ii'})$, and $N_2*N_2*' \equiv (t_{ii'})$, then

$$u_{ii'} = \sum_{j} (\sum_{k'=3}^{p} n_{ij(k'-2)} + n_{ij(p-1)})(\sum_{k=3}^{p} n_{i'j(k-2)})$$

$$= \sum_{j} [(\sum_{k'=3}^{p} n_{ij(k'-2)})(\sum_{k=3}^{p} n_{i'j(k-2)})$$

$$+ n_{ij(p-1)} \sum_{k=3}^{p} n_{i'j(k-2)}]$$

$$= t_{ii'} + \sum_{j} n_{ij(p-1)} \sum_{k=3}^{p} n_{i'j(k-2)} \qquad (7.18)$$

and

$$s_{ii'} = \sum_{j} (\sum_{k'=3}^{p} n_{i'j(k'-2)} + n_{i'j(p-1)})(\sum_{k=2}^{p} n_{ij(k-1)})$$

$$= \sum_j [(\sum_{k'=3}^{p} n_{i'j(k'-2)})(\sum_{k=2}^{p} n_{ij(k-1)}) + n_{i'j(p-1)} \sum_{k=2}^{p} n_{ij(k-1)}]$$

$$= u_{ii'} + \sum_j n_{i'j(p-1)} (\sum_{k=2}^{p} n_{ij(k-1)})$$

$$= u_{ii'} + \sum_j n_{i'j(p-1)} (\sum_{k=3}^{p} n_{ij(k-2)} + n_{ij(p-1)})$$

$$= u_{ii'} + \sum_j n_{i'j(p-1)} (\sum_{k=3}^{p} n_{ij(k-2)}) + \sum_j n_{i'j(p-1)} n_{ij(p-1)} . \tag{7.19}$$

From (7.18) we have

$$\sum_j n_{ij(p-1)} \sum_{k=3}^{p} n_{i'j(k-2)} = u_{ii'} - t_{ii'} . \tag{7.20}$$

On the other hand, from (7.19)

$$\sum_j n_{i'j(p-1)} \sum_{k=3}^{p} n_{ij(k-2)}$$

$$= s_{ii'} - u_{ii'} - \sum_j n_{i'j(p-1)} n_{ij(p-1)} . \tag{7.21}$$

It follows from (7.20) and (7.21) that under C_2,

$$u_{ii'} - t_{ii'} = s_{ii'} - u_{ii'} - \sum_j n_{i'j(p-1)} n_{ij(p-1)} . \tag{7.22}$$

Since every treatment occurs in each period λ_1 times for a GPBRM2 design, we have

$$\sum_j n_{ij(p-1)} n_{i'j(p-1)} = \lambda_1 \delta_{ii'} . \tag{7.23}$$

Substituting (7.23) into (7.22) we obtain

$$u_{ii'} - t_{ii'} = s_{ii'} - u_{ii'} - \lambda_1 \delta_{ii'}$$

or

$$u_{ii'} = \tfrac{1}{2}(t_{ii'} + s_{ii'}) - \frac{\lambda_1}{2} \delta_{ii'}$$

i.e.,

$$N_1{}^*N_2{}^{*'} = \tfrac{1}{2}(N_1{}^*N_1{}^{*'} + N_2{}^*N_2{}^{*'}) - \frac{\lambda_1}{2} I_t$$

To show that (3) holds, we write

$$a_{ii'} = \sum_j \left(\sum_{k'=2}^{p} n_{ij(k'-1)} + n_{ijp} \right) \left(\sum_{k=3}^{p} n_{i'j(k-2)} \right)$$

$$= \sum_j \left[\left(\sum_{k'=2}^{p} n_{ij(k'-1)} \right) \left(\sum_{k=3}^{p} n_{i'j(k-2)} \right) \right.$$

$$\left. + n_{ijp} \left(\sum_{k=3}^{p} n_{i'j(k-2)} \right) \right]$$

$$= u_{ii'} + \sum_j n_{ijp} \left(\sum_{k=3}^{p} n_{i'j(k-2)} \right) . \qquad (7.24)$$

Substituting (7.18) into (7.24) we obtain

$$a_{ii'} = t_{ii'} + \sum_j n_{ij(p-1)} \sum_{k=3}^{p} n_{i'j(k-2)}$$

$$+ \sum_j n_{ijp} \left(\sum_{k=3}^{p} n_{i'j(k-2)} \right) . \qquad (7.25)$$

Similarly,

$$b_{ii'} = \sum_{j,k'} n_{ijk'} \sum_k n_{i'jk}$$

$$= \sum_j [\sum_{k'} n_{ijk'} (\sum_{k=2}^{p} n_{i'j(k-1)} + n_{i'jp})]$$

$$= \sum_j [\sum_{k'} n_{ijk'} \sum_{k=2}^{p} n_{i'j(k-1)} + n_{i'jp} \sum_{k'} n_{ijk'}]$$

$$= \sum_j \sum_{k'} n_{ijk'} \sum_{k=2}^{p} n_{i'j(k-1)} + \sum_j n_{i'jp} \sum_{k'} n_{ijk'}$$

$$= d_{ii'} + \sum_j n_{i'jp} (\sum_{k'=2}^{p} n_{ij(k'-1)} + n_{ijp})$$

$$= d_{ii'} + \sum_j n_{ijp} n_{i'jp} + \sum_j n_{i'jp} \sum_{k'=2}^{p} n_{ij(k'-1)}$$

$$= d_{ii'} + \sum_j n_{ijp} n_{i'jp} + \sum_j n_{i'jp}[\sum_{k'=3}^{p} n_{ij(k'-2)} + n_{ij(p-1)}]$$

$$= d_{ii'} + \lambda_1 \delta_{ii'} + \sum_j n_{i'jp} \sum_{k'=3}^{p} n_{ij(k'-2)}$$

$$+ \sum_j n_{i'jp} n_{ij(p-1)}$$

or

$$\sum_j n_{i'jp} \sum_{k'=2}^{p} n_{ij(k'-2)}$$

$$= b_{ii'} - d_{ii'} - \lambda_1 \delta_{ii'} - \sum_j n_{i'jp} n_{ij(p-1)} \ . \qquad (7.26)$$

Under C_3, (7.26) becomes

$$\sum_j n_{ijp}\left(\sum_{k=3}^{p} n_{i'j(k-2)}\right)$$

$$= b_{ii'} - d_{ii'} - \lambda_1\delta_{ii'} - \sum_j n_{i'jp}n_{ij(p-1)} \; . \tag{7.27}$$

Substituting (7.21), (7.23), (7.27) into (7.25), we have

$$a_{ii'} = t_{ii'} + (s_{ii'} - u_{ii'} - \lambda_1\delta_{ii'})$$

$$+ b_{ii'} - d_{ii'} - \lambda_1\delta_{ii'} - \sum_j n_{i'jp}n_{ij(p-1)} \; . \tag{7.28}$$

Since (1) implies that

$$d_{ii'} = \tfrac{1}{2}(s_{ii'} + b_{ii'}) - \frac{\lambda_1}{2}\,\delta_{ii'} \tag{7.29}$$

and (2) implies that

$$u_{ii'} = \tfrac{1}{2}(t_{ii'} + s_{ii'}) - \frac{\lambda_1}{2}\,\delta_{ii'} \; , \tag{7.30}$$

we obtain, after substituting (7.29) and (7.30) into (7.28),

$$a_{ii'} = t_{ii'} + s_{ii'} - \tfrac{1}{2}(t_{ii'} + s_{ii'}) + \frac{\lambda_1}{2}\,\delta_{ii'}$$

$$- \lambda_1\delta_{ii'} + b_{ii'} - \tfrac{1}{2}(s_{ii'} + b_{ii'}) + \frac{\lambda_1}{2}\,\delta_{ii'}$$

$$- \lambda_1\delta_{ii'} - \sum_j n_{i'jp}n_{ij(p-1)}$$

$$= \tfrac{1}{2}(t_{ii'} + b_{ii'}) - \lambda_1\delta_{ii'} - \sum_j n_{i'jp}n_{ij(p-1)} \; .$$

In the next section we will discuss a special case of GPBRM2 designs, which are the balance repeated measurement designs under the second order residual effects model from a combinatorial point of view (BRM2 designs) and from an estimation point of view (BRM2E designs.)

7.3 Balanced Repeated Measurement Designs

7.3.1 BRM2 Designs (Combinatorial)

It is similar to Definition 2.1.1 for BRM1 designs that we can define a RM design as a balanced RM design for the existence of second order residual effects if its direct effects are balanced with respect to first and second order residual effects and also its first order residual effects are balanced with respect to second order residual effects. This is defined in the combinatorial point of view. We can formalize the definition of BRM2 designs in the following:

Definition 7.3.1.1

A RM(t, n, p) design is said to be balanced, or a BRM2 (t, n, p) design, with respect to sets of direct, the first and the second order residual effects if

(1) each treatment occurs λ_1 times in each period;

(2) each treatment is preceded by each other treatment λ_2 times;

(3) ignoring the last period, each treatment is preceded by each other treatment ζ_3 times;

(4) each treatment is preceded by each other treatment by two periods λ_4 times.

We can see from Definition 2.2.1 that in a BRMW2 design each treatment is preceded by each ordered pair of other treatments the same number of times. Obviously, this means that the condition (2) of Definition 2.2.1 satisfies conditions (2), (3), and (4) of Definition 7.3.1.1. Therefore, we know that a BRMW2 design is also a BRM2 design. That is, the set of BRMW2 designs is a subset of the set of BRM2 designs. For example, the design (2.1) is a BRM2 design with $\lambda_1 = 3$, $\lambda_2 = 3$, $\zeta_3 = 2$, and $\lambda_4 = 2$.

7.3.2 BRM2E Designs (Estimation)

From (7.4) and (7.5) we know that in order to have only one variance for direct effects or residual effects, C_{ij} (i, j = 1, 2, 3) should be in aI + bJ forms, where a and b are constants. This means that S_1, S_2, S_{12}, NN', NN_1*', NN_2*', N_1*N_1*', N_1*N_2*' and N_2*N_2*' should be in an aI + bJ form. However, from Definition 7.3.1.1 a BRM2 design can guarantee S_1, S_2 and S_{12} to be in an aI + bJ form. To achieve that NN', $N_1*'N_1*'$ and N_2*N_2*' are all in aI + bJ forms, we need to put more restrictions on our BRM2 designs. If the design is a BIB design, then NN' is in an aI + bJ form. Also by deleting the last period of the design, if the resulting design is a BIB design, then N_1*N_1*' is in an aI + bJ form. Similarly, by deleting the last two periods, if the resulting design is a BIB design, then N_2*N_2*' is also in an aI + bJ form.

Since Lemma 7.2.3 implies that under condition C_1, NN_1*' can be expressed in terms of NN', and under condition C_2, N_1*N_2*' can be expressed in terms of N_1*N_1*' and N_2*N_2*', NN_1*' and N_1*N_2*' can be in aI + bJ forms under conditions C_1 and C_2.

We can also see that if the term $\sum_j n_{i'jp}n_{ij(p-1)}$ in (3) of Lemma 7.2.3 is a constant for all treatment pairs (i, i') where i ≠ i', then we can express NN_2*' in an aI + bJ form. Now, we can introduce the following condition C_4 to achieve this.

Condition C_4 represents that the number of subjects receiving treatment i' in the last second period, when treatment i is in the last period is the same as the number of subjects receiving treatment i in the last second period, when treatment i' is in the last period, this is a constant for all i ≠ i'.

In summary, we can define a balanced repeated measurement design under second order residual effects in the estimating sense.

Definition 7.3.2.1

A RM(t, n, p) design is said to be balanced, or a BRM2E (t, n, p) design, with respect to sets of direct, first order and second order residual effects, if

(1) it is a BRM2 design;

(2) the design is a BIB design by using subjects as blocks;

(3) deleting the last period, the resulting design is a BIB design;

(4) deleting the last two periods, the resulting design is a BIB design;

(5) it satisfies conditions C_1, C_2, C_3 and C_4.

For example, the design (2.1) is also a BRM2E design.

So far we have derived the properties of GPBRM1 and GPBRM2 designs. We can follow the same procedure as before to extend GPBRM designs to kth order residual model in the next section.

7.4 GPBRMk designs

To generalize the definition of GPBRM1 and GPBRM2 designs, we propose a definition for a generalized partially balanced repeated measurement design under the existence of kth order residual effects.

7.4.1 Definition

Definition 7.4.1.1

A GPBRMk(t, n, p) design is a partially balanced repeated measurement design if:

(1) it is a RM(t, n, p) design;

(2) every treatment occurs λ_1 times in each period;

(3) there exist $(k + 1)t$ effects (including direct, first order to kth order residual effects) which are divided into $k + 1$ groups of t treatments each, where, for convenience and purposes of description, we refer to the elements in these groups as Lth group treatments, i.e.,

L = 1: actual treatments

L = 2: "treatments" giving rise to the first order residual effects

L = 3: "treatments" giving rise to the second order residual effects;

.
.
.
.
.

L = k + 1: "treatments" giving rise to the kth order residual effects;

(4) it has the following generalized PBIB design properties:

(a) Any ith group treatment and jth group treatment are either ij : 0th, ij : 1th, ... , or ij : M_{ij}th associates, where i, j = 1, ... , k + 1 (every ith group treatment is the ii : 0th associate of itself and of no other treatment); ij : Vth associates are the same as ji : Vth associates;

(b) Each ith group treatment has exactly $N_{ij:v}$ ij : Vth associates, where j = 1, ... , k + 1 and V = 0, 1, ... , M_{ij};

(c) Given any two treatments which are ij : Vth associates, the number of treatments common to the i_1j_1 : t_1th associates of the first and the i_2j_2 : t_2th of the second, plus the number of treatments common to the i_1j_1 : t_1th associates of the second and the i_2j_2 : t_2th of the first, is $2P_{ij:v}(i_1j_1: t_1, i_2j_2 : t_2)$. And it is independent of the pair of treatments with which we start.

(d) Two treatments which are ij : Vth associates occur together in exactly $\lambda_{ij:v}$ subjects;

(e) Two treatments which are ij:Vth associates precede each other $\mu_{ij:v}$ times, where i $\neq$ j and M_{ij} > 1.

7.4.2 Analysis and Properties

We will follow the same procedure as that in Chapter IV and V to extend some of those results to GPBRMk designs. Since the new results look similar to the old ones, we only list the results without proving them.

An appropriate linear model for GPBRMk designs is in the following:

$$y_{ijm\ell_1\ldots\ell_k} = \mu + \tau_i + s_j + \pi_m + \alpha_{\ell_1}^{(1)} + \ldots + \alpha_{\ell_k}^{(k)} + \varepsilon_{ijm\ell_1\ldots\ell_k}$$

where

τ_i = direct effect, i = 1, ... , t

s_j = subject effect, j = 1, ... , n

π_m = period effect, m = 1, ... , p

$\alpha_{\ell_1}^{(1)}$ = first order residual effect, ℓ_1 = 1, ... , t

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$\alpha_{\ell_k}^{(k)}$ = kth order residual effect, ℓ_k = 1, ... , t are fixed effects, and $\varepsilon_{ijk\ell_1\ldots\ell_k} \sim (0, \sigma^2)$ are independently distributed.

We can write the model in matrix form as

$$Y = X\underline{\beta} + \underline{\varepsilon}$$

where

$$\underline{\beta}' = [\tau_1, \ldots, \tau_t, \alpha_1^{(1)}, \ldots, \alpha_t^{(1)}, \ldots, \alpha_1^{(k)}, \ldots, \alpha_t^{(k)},$$

$$\pi_1, \ldots, \pi_p, s_1, \ldots, s_n] .$$

Then the normal equations are

$$x'x\hat{\underline{\beta}} = x'\underline{y}$$

where

$$x'x = \begin{bmatrix} D & S_1 & \cdots\cdots & S_k & T & N \\ S_1' & D_1^* & \cdots\cdots & S_{1k} & T_1^* & N_1^* \\ \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots \\ S_k' & S_{1k}' & \cdots\cdots & D_k^* & T_k^* & N_k^* \\ T' & T_1^{*\prime} & \cdots\cdots & T_k^{*\prime} & nI_p & J_{p\times n} \\ N' & N_1^{*\prime} & \cdots\cdots & N_k^{*\prime} & J_{n\times p} & pI_n \end{bmatrix}$$

and

D is direct-direct effect incidence matrix,

S_i is direct-ith order residual effect incidence matrix, $i = 1, \ldots, k$,

T is direct-period effect incidence matrix,

N is direct-subject effect incidence matrix,

D_i^* is ith order residual - ith order residual effect incidence matrix, $i = 1, \ldots, k$,

T_i^* is ith order residual-period effect incidence matrix,

N_i^* is ith order residual-subject effect incidence matrix.

Then the normal equations can be adjusted to the following reduced normal equation in terms of direct effects and residual effects as

$$c\begin{bmatrix} \underline{\hat{t}} \\ \underline{\hat{\alpha}}_{\ell_1} \\ \cdot \\ \cdot \\ \cdot \\ \underline{\hat{\alpha}}_{\ell_k} \end{bmatrix} = Q_{(k)}$$

where C is the matrix adjusted for subjects and periods, and C can be defined by the following symbols:

$$C \equiv \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1(k+1)} \\ \cdot & \cdot & \cdots & \cdot \\ C_{(k+1)1} & C_{(k+1)2} & \cdots & C_{(k+1)(k+1)} \end{bmatrix}$$

and

$$C = \begin{bmatrix} D & S_1 & \cdots & S_k \\ S_1' & D_1^* & & S_{1k} \\ \cdot & \cdot & \cdots & \cdot \\ S_k' & S_{1k}' & \cdots & D_k^* \end{bmatrix}$$

$$- \begin{bmatrix} T & N \\ T_1^* & N_1^* \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ T_k^* & N_k^* \end{bmatrix} \begin{bmatrix} nI_p & J_{p\times n} \\ & \\ J_{n\times p} & pI_n \end{bmatrix}^{-} \begin{bmatrix} T' & T_1^{*\prime} & \cdots & T_k^{*\prime} \\ & & & \\ N' & N_1^{*\prime} & \cdots & N_k^{*\prime} \end{bmatrix} .$$

Since

$$\begin{bmatrix} nI_p & J_{p\times n} \\ J_{n\times p} & pI_n \end{bmatrix}^- = \begin{bmatrix} \frac{1}{n} I_p & 0 \\ 0 & \frac{1}{p} I_n - \frac{1}{np} J_n \end{bmatrix} ,$$

C_{ij} in C can be represented as follows:

$$C_{11} = D - \frac{1}{n} TT' - \frac{1}{p} NN' + \frac{1}{np} NJ_nN'$$

$$C_{12} = S_1 - \frac{1}{n} TT_1{*}' - \frac{1}{p} NN_1{*}' + \frac{1}{np} NJ_nN_1{*}'$$

$$\vdots$$

$$C_{1(k+1)} = S_k - \frac{1}{n} TT_k{*}' - \frac{1}{p} NN_t{*}' + \frac{1}{np} NJ_nN_k{*}'$$

$$C_{22} = D_1{*} - \frac{1}{n} T_1{*}T_1{*}' - \frac{1}{p} N_1{*}N_1{*}' + \frac{1}{np} N_1{*}J_nN_1{*}'$$

$$\vdots$$

$$C_{2(k+1)} = S_{1k} - \frac{1}{n} T_1{*}T_k{*}' - \frac{1}{p} N_1{*}N_k{*}' + \frac{1}{np} N_1J_nN_k{*}'$$

$$\vdots$$

$$C_{(k+1)(k+1)} = D_k{*} - \frac{1}{n} T_k{*}T_k{*}' + \frac{1}{p} N_k{*}N_k{*}' + \frac{1}{np} N_k{*}J_nN_k{*}'$$

and

$$C_{ji} = C_{ij}', \quad \text{where } i, j = 1, \ldots, k+1 \text{ and } i \neq j .$$

Also

$$Q_{(k)} = \begin{bmatrix} \underset{\sim}{T}_0 \\ \underset{\sim}{R}_1 \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{R}_k \end{bmatrix} - \begin{bmatrix} T & N \\ T_1^* & N_1^* \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ T_k^* & N_k^* \end{bmatrix} \begin{bmatrix} nI_p & J_{n\times p} \\ & \\ J_{n\times p} & pI_n \end{bmatrix}^- \begin{bmatrix} \underset{\sim}{P} \\ \\ \underset{\sim}{B} \end{bmatrix}$$

$$= \begin{bmatrix} \underset{\sim}{T}_0 - \frac{1}{n} T\underset{\sim}{P} - \frac{1}{p} N\underset{\sim}{B} + \frac{1}{np} NJ_n\underset{\sim}{B} \\ \underset{\sim}{R}_1 - \frac{1}{n} T_1^*\underset{\sim}{P} - \frac{1}{p} N_1^*\underset{\sim}{B} + \frac{1}{np} N_1^*J_n\underset{\sim}{B} \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{R}_k - \frac{1}{n} T_k^*\underset{\sim}{P} - \frac{1}{p} N_k^*\underset{\sim}{B} + \frac{1}{np} N_k^*J_n\underset{\sim}{B} \end{bmatrix},$$

where

$\underset{\sim}{T}_0 = [T_{01}, \ldots, T_{0t}]'$ is the 'direct effect total' vector,

$\underset{\sim}{R}_1 = [R_{11}, \ldots, R_{1t}]'$ is the 'first order residual effect total' vector,

$\underset{\sim}{R}_k = [R_{k1}, \ldots, R_{kt}]'$ is the 'kth order residual effect total' vector,

$\underset{\sim}{B} = [B, \ldots, B_n]'$ is the 'subject total' vector,

and

$\underset{\sim}{P} = [P_1, \ldots, P_p]'$ is the 'period total' vector.

For a GPBRMk design, the terms in C_{ij} can be simplified by the following relations:

Since

$$D = \lambda_1 p I_t$$

$$T = \lambda_1 J_{t\times p}$$

$$T_1* = \lambda_1 [0 : J_{t\times(p-1)}]$$

$$T_2* = \lambda_1 [0 : J_{t\times(p-2)}]$$

$$\vdots$$

$$T_k* = \lambda_1 [0 : J_{t\times(p-k)}]$$

$$D_1* = \lambda_1 (p-1) I_t$$

$$D_2* = \lambda_1 (p-2) J_t$$

$$\vdots$$

$$D_k* = \lambda_1 (p-k) I_t ,$$

we have

$$TT' = P\lambda_1^{\;2}\; J_t$$

$$TT_1{}^{*'} = \lambda_1^{\;2}(p-1)J_t$$

$$\vdots$$

$$TT_k{}^{*'} = \lambda_1^{\;2}(p-k)^{J}t$$

$$T_1{}^{*}T_1{}^{*'} = \lambda_1^{\;2}(p-1)J_t$$

$$T_1{}^{*}T_2{}^{*'} = \lambda_1^{\;2}(p-2)J_t$$

$$\vdots$$

$$T_1{}^{*}T_k{}^{*'} = \lambda_1^{\;2}(p-k)J_t$$

$$\vdots$$

$$T_k{}^{*}T_k{}^{*'} = \lambda_1^{\;2}(p-k)J_t$$

$$NJ_nN' = \lambda_1^{\;2}\; p^2\; J_n$$

$$NJ_nN_1{}^{*'} = \lambda_1^{\;2}\; p(p-1)J_t$$

$$\vdots$$

$$NJ_nN_k{}^{*'} = \lambda_1^{\;2}\; p(p-k)J_t$$

$$N_1{}^*J_nN_1{}^{*'} = \lambda_1{}^2(p-1)^2\ J_t$$

$$\vdots$$

$$N_1{}^*J_nN_k{}^{*'} = \lambda_1{}^2(p-1)(p-k)J_t$$

$$\vdots$$

$$N_k{}^*J_nN_k{}^{*'} = \lambda_1{}^2(p-k)^2\ J_t\quad .$$

Then for any GPBRMk design,

$$C_{11} = \lambda_1 pI_t - \frac{1}{p}\ NN'$$

$$C_{12} = S_1 - \frac{1}{p}\ NN_1{}^{*'}$$

$$\vdots$$

$$C_{1(k+1)} = S_k - \frac{1}{p}\ NN_k{}^{*'}$$

$$C_{22} = \lambda_1(p-1)I_t - \frac{\lambda_1{}^2(p-1)}{np}\ J_t - \frac{1}{p}\ N_1{}^*N_1{}^{*'}$$

$$\vdots$$

$$C_{2(k+1)} = S_{1k} - \frac{\lambda_1{}^2(p-k)}{np}\ J_t - \frac{1}{p}\ N_1{}^*N_k{}^{*'}$$

$$\vdots$$

$$C_{(k+1)(k+1)} = \lambda_1(p-k)I_t - \frac{k\lambda_1^{\,2}(p-k)}{np} J_t - \frac{1}{p} N_k{*}N_k{*}'$$

and

$$C_{ji} = C_{ij}' \quad \text{for } i,\ j = 1, \ldots, (k+1) \text{ and } i \neq j \ .$$

Now, using association matrices, we can proceed along steps like those in Chapter V in the following:

Suppose that

$$NN' = \sum_{v=0}^{m_{11}} \lambda_{11:v}\, B_{11:v}$$

$$NN_j{*}' = \sum_{v=0}^{m_{1(j+1)}} \lambda_{1(j+1):v}\, B_{1(j+1):v}$$

$$N_iN_j{*}' = \sum_{v=0}^{m_{(i+1)(j+1)}} \lambda_{(i+1)(j+1):v}\, B_{(i+1)(j+1):v}$$

$$S_i = \sum_{v=0}^{m_{1(i+1)}} \mu_{1(i+1):v}\, B_{1(i+1):v}$$

$$\text{where } \mu_{1(i+1):0} = 0$$

and

$$S_{ij} = \sum_{v=0}^{m_{(i+1)(j+1)}} \mu_{(i+1)(j+1):v}\, B_{(i+1)(j+1):v}$$

$$\text{where } \mu_{(i+1)(j+1):0} = 0$$

for all i, j.

Then C_{ij} can be expressed by $B_{ij:v}$'s and the reduced normal equation can be expressed as

$$C^* \begin{bmatrix} \underline{\hat{\tau}} \\ \underline{\hat{\alpha}}_{\ell_1} \\ \cdot \\ \cdot \\ \cdot \\ \underline{\hat{\alpha}}_{\ell_k} \end{bmatrix} = Q_{(k)}$$

where

$$C^* = \begin{bmatrix} \sum_{v=0}^{m_{11}} \lambda^*_{11:v} B_{11:v} & \cdots & \sum_{v=0}^{m_{1(k+1)}} \lambda^*_{1(k+1):v} B_{1(k+1):v} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \sum_{v=0}^{m_{(k+1)1}} \lambda^*_{(k+1)1:v} B_{(k+1)1:v} & \cdots & \sum_{v=0}^{m_{(k+1)(k+1)}} \lambda^*_{(k+1)(k+1):v} \\ & & \times B_{(k+1)(k+1):v} \end{bmatrix} .$$

If all effects are estimable, then

$$\begin{bmatrix} \hat{\tau} \\ \hat{\alpha}_{\ell_1} \\ \cdot \\ \cdot \\ \cdot \\ \hat{\alpha}_{\ell_k} \end{bmatrix} = C^{*-} Q_{(k)}$$

and

$$\Sigma_{\hat{\tau},\hat{\alpha}_{\ell_1},\ldots,\hat{\alpha}_{\ell_k}} = C^{*-}\sigma^2 .$$

The properties for GPBRMk designs are the generalizations in Chapter V. We will list some without proof.

Lemma 7.4.2.1

For any GPBRMk design, the relationship among its parameters can be written as follows:

(1) $n = \lambda_1 t$

(2) $\sum_{v=0}^{m_{ij}} n_{ij:v} = t$

(3) $\sum_{\ell=0}^{m_{jk}} P_{ij:v}(ik : t_1,\ jk : \ell) = n_{ik} : t_1$

and

$$\sum_{t_1=0}^{m_{ik}} P_{ij:v}(ik : t_1,\ jk : \ell) = n_{jk:\ell}$$

(4) $n_{ij:v}\, P_{ij:v}(ik : \ell,\ jk : t_1) = n_{ik:\ell}\, P_{ik:\ell}(ij : v,\ jk : t_1)$

where $i \neq j$ and $i \neq k$

(5) $\sum_{v=0}^{m_{ii}} n_{ii:v}\, \lambda_{ii:v} = \lambda_1 (p+1-i)^2$

$$(6)\quad \sum_{v=0}^{m_{ij}} n_{ij:v}\,\lambda_{ij:v} = \lambda_1(p-j+1) \quad \text{where } i<j$$

$$(7)\quad \sum_{v=0}^{m_{ij}} n_{ij:v}\,\lambda_{ij:v} = \lambda_1(p-i+1)(p-j+1) \quad \text{where } i<j$$

$$(8)\quad \sum_{v=0}^{m_{ij}} n_{ij:v}\,\mu_{ij:v} = \sum_{v=0}^{m_{ji}} n_{ji:v}\,\mu_{ji:v}$$

and

$$\sum_{v=0}^{m_{ij}} n_{ij:v}\,\lambda_{ij:v} = \sum_{v=0}^{m_{ji}} n_{ji:v}\,\lambda_{ji:v}\,.$$

Lemma 7.4.2.2

The $\lambda^*_{ij;v}$'s are restricted by the following relations:

$$\sum_{v=0}^{m_{ij}} n_{ij:v}\,\lambda^*_{ij:v} = 0 \quad \text{where } i,\ j = 1, \ldots, (k+1)$$

By Lemma 7.4.2.2, we know that the sum of any row or column of C* matrix is zero, i.e., rank(C*) $\leq$ (k+1)(t-1).

To express C^- in terms of P-matrices, we have

Lemma 7.4.2.3

C_{ij} are nonnegative definite, i, j = 1, ..., k+1. To define a connected GPBRMk design, we have the following definition in terms of the estimable direct effects and residual effects.

Definition 7.4.2.1

A GPBRMk design is said to be connected if all linear contrasts

$\tau_j - \tau_{i'}$ and $\alpha_{\ell_{im}}^{(i)} - \alpha_{\ell_{in}}^{(i)}$ are estimable, where $i = 1, \ldots, k$ and $i', j, m, n = 1, \ldots, t$. Then by checking the rank of C^* matrix, we can determine whether a GPBRMk design is connected in the following theorem:

Theorem 7.4.2.1

A GPBRMk design is connected iff

$$\text{rank}(C^*) = (k+1)(t-1) .$$

Since the results for GPBRMk designs are similar to those for GPBRM1 designs, we will not progress further. In the next chapter we will deal with RM designs when treatments have a factorial structure.

VIII. FACTORIAL STRUCTURE

In the preceding chapters we have discussed general classes of RM designs involving t treatments. It is quite obvious that these treatments can be anything; in particular if $t = p^n$ they can have a factorial structure, i.e., n factors with p levels each. In this situation the designs discussed earlier are suitable to estimate all main effects and interactions in such a p^n factorial.

A natural question then is whether one can reduce the size of the design, i.e., either reduce the number of subjects, or the number of periods or the total number of observations, if certain interactions are assumed to be negligible such that all remaining direct main effects and interactions remain estimable. We shall investigate this question for $t = 2^n$ and provide suitable designs when some higher order interactions are assumed to be negligible.

8.1 2^n Designs When the Highest Order Interaction is Negligible

Using the familiar notation for 2^n factorial experiments (e.g. Kempthorne, 1952) we have, for example, for a 2^2 design, the following effects:

$$\text{Mean} = M = \tfrac{1}{4}(a + 1)(b + 1) = \tfrac{1}{4}(ab + a + b + (1))$$

Direct Effects:

$$A = \tfrac{1}{2}(a - 1)(b + 1) = \tfrac{1}{2}(ab + a - b - (1))$$

$$B = \tfrac{1}{2}(a + 1)(b - 1) = \tfrac{1}{2}(ab - a + b - (1))$$

$$AB = \tfrac{1}{2}(a - 1)(b - 1) = \tfrac{1}{2}(ab - a - b + (1))$$

In addition, we may define for the residual effects:

$$A_r = \tfrac{1}{2}(ab_r + a_r - b_r - (1)_r)$$

$$B_r = \tfrac{1}{2}(ab_r - a_r + b_r - (1)_r)$$

$$AB_r = \tfrac{1}{2}(ab_r - a_r - b_r + (1)_r)$$

where $(1)_r$, a_r, b_r, ab_r are the residual effects of (1), a, b, ab, respectively. We note here that A_r, B_r, and AB_r are merely linear combinations of residual effects and do not have any real meaning.

A suitable design to estimate all the above seven effects, the subject effects and the period effects is the following RM(2^2, 4, 4) design obtained according to Williams' method.

		subjects			
		1	2	3	4
periods	1	(1)	a	b	ab
	2	ab	(1)	a	b
	3	a	b	ab	(1)
	4	b	ab	(1)	a

Likewise, in a 2^3 factorial we have the following design using Williams' method:

		subjects							
		1	2	3	4	5	6	7	8
periods	1	(1)	a	b	c	ab	ac	bc	abc
	2	abc	(1)	a	b	c	ab	ac	bc
	3	a	b	c	ab	ac	bc	abc	(1)
	4	bc	abc	(1)	a	b	c	ab	ac
	5	b	c	ab	ac	bc	abc	(1)	a
	6	ac	bc	abc	(1)	a	b	c	ab
	7	c	ab	ac	bc	abc	(1)	a	b
	8	ab	ac	bc	abc	(1)	a	b	c

(8.1)

Since in design (8.1) we can estimate all the individual parameters (1), a, b, c, ab, ac, bc, and abc, we can also estimate all the linear combinations of (1), a, b, c, ab, ac, bc, and abc. Therefore, all the effects A, B, C, AB, AC, BC, ABC and A_r, B_r, C_r, AB_r, AC_r, BC_r, ABC_r can be estimated. However, if ABC is negligible, then the RM(2^3, 8, 8) design in (8.1) can be reduced in order to estimate all the effects except ABC. Confounding ABC with subjects leads to the following RM(2^3, 8, 4) design.

		subjects							
		1	2	3	4	5	6	7	8
periods	1	(1)	ac	bc	ab	abc	b	a	c
	2	ab	(1)	ac	bc	c	abc	b	a
	3	ac	bc	ab	(1)	b	a	c	abc
	4	bc	ab	(1)	ac	a	c	abc	b

(8.2)

To describe the construction of this design and to facilitate the discussion about the general case, it is convenient to denote the treatment combinations by ordered triples (x_1, x_2, x_3), where x_1 represents the level of treatment factor A, x_2 the level of treatment factor B, and x_3 the level of treatment factor C, with $x_i = 0,1$ ($i = 1, 2, 3$).

Using this notation we start by constructing two initial subjects as follows:

The treatment combinations for the first subject satisfying $x_1 + x_2 + x_3 = 0 \pmod 2$ are applied in any order, e.g.,

periods	
1	(0,0,0)
2	(1,1,0)
3	(1,0,1)
4	(0,1,1)

We shall refer to this subject as the first initial subject.

For another subject which we shall refer to as the second initial subject, we apply the treatments satisfying $x_1 + x_2 + x_3 = 1 \pmod 2$, e.g.,

periods	
1	(1,1,1)
2	(0,0,1)
3	(0,1,0)
4	(1,0,0)

Each initial subject is then developed into a Latin Square according to Williams' method as described in Section (2.1.1). These two Latin Squares are then combined and we obtain the following RM(2^3, 8, 4) design:

		subjects							
		1	2	3	4	5	6	7	8
periods	1	(0,0,0)	(1,0,1)	(0,1,1)	(1,1,0)	(1,1,1)	(0,1,0)	(1,0,0)	(0,0,1)
	2	(1,1,0)	(0,0,0)	(1,0,1)	(0,1,1)	(0,0,1)	(1,1,1)	(0,1,0)	(1,0,0)
	3	(1,0,1)	(0,1,1)	(1,1,0)	(0,0,0)	(0,1,0)	(1,0,0)	(0,0,1)	(1,1,1)
	4	(0,1,1)	(1,1,0)	(0,0,0)	(1,0,1)	(1,0,0)	(0,0,1)	(1,1,1)	(0,1,0)

(8.3)

In general we have n factors, A_1, A_2, ... , A_n say, where A_i has levels $x_i = 0$ and $x_i = 1$. If we can assume the n-factor interaction $A_1A_2 \ldots A_n$ to be negligible we can construct a suitable RM(2^n, 2^n, 2^{n-1}) design as stated in Theorem 9.1.1. Before discussing it, however, we first introduce the following notation.

For the treatment combination $x' = (x_1, x_2, \ldots, x_n)$, with $x_i = 0, 1$, let

$$a(x) = a_1^{x_1} a_2^{x_2} \ldots a_n^{x_n}$$

denote the true response, where we set $a_i^0 = 1$ and $a_i^1 = a_i$ ($i = 1, 2, \ldots, n$). For a partition $\alpha' = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i = 0, 1$ we write the direct effect as

$$E^{\alpha} = A_1^{\alpha_1} A_2^{\alpha_2} \ldots A_n^{\alpha_n}$$

and we set $A_i^0 = 1$ and $A_i^1 = A_i$, with $A_1^0 A_2^0 \ldots A_n^0 = M$, the grand mean. In terms of treatment effects we then write

$$E^{\alpha} = \frac{1}{2^{n-1}} \prod_{i=1}^{n} [a_i + (-1)^{\alpha_i}]$$

and

$$M = \frac{1}{2^n} \prod_{i=1}^{n} (a_i + 1) .$$

Similarly, for residual effects we define

$$E_r^{\alpha} = \frac{1}{2^{n-1}} \prod_{i=1}^{n} [a_{i_r} + (-1)^{\alpha_i}] .$$

We now define for j = 0, 1:

T_j^{α} = total of all a(x) satisfying $\sum_i \alpha_i x_i = j \pmod 2$,

E_j^{α} = mean of all a(x) satisfying $\sum_i \alpha_i x_i = j \pmod 2$.

Similarly, if $a_r(x) = a_{1_r}^{x_1} a_{2_r}^{x_2} \ldots a_{n_r}^{x_n}$ is the residual effect of a(x), then

R_j^{α} = total of all $a_r(x)$ satisfying $\sum_i \alpha_i x_i = j \pmod 2$

and

$(E_j^{\alpha})_r$ = mean of all $a_r(x)$ satisfying $\sum_i \alpha_i x_i = j \pmod 2$.

Letting

$$E_d^{\alpha} = \hat{E}_1^{\alpha} - \hat{E}_0^{\alpha} \qquad \text{if } \sum_i \alpha_i \text{ is odd}$$

$$= \hat{E}_0^{\alpha} - \hat{E}_1^{\alpha} \qquad \text{if } \sum_i \alpha_i \text{ is even}$$

where $\hat{E}_i^{\alpha}$ denotes the observed E_i^{α} we have

$$E_d^{\alpha} = (-1)^{\Sigma\alpha_i} (\hat{E}_0^{\alpha} - \hat{E}_1^{\alpha}) \quad . \tag{8.5}$$

Similarly,

$$(E_d^{\alpha})_r = (-1)^{\Sigma\alpha_i} [(\hat{E}_0^{\alpha})_r - (\hat{E}_1^{\alpha})_r] \quad ,$$

where $(\hat{E}_i^{\alpha})_r$ is the observed $(E_i^{\alpha})_r$.

If there is no ambiguity we shall, in what follows, use the same notation for true and observed responses.

We define further

$(S_i^{\alpha})_{LP}$ = sum of subject total for all subjects whose treatment x in the last period satisfies $\sum_j \alpha_j x_j = i \pmod 2$ for $i = 0, 1$, and

$$S_{LP}^{\alpha} = (S_1^{\alpha})_{LP} - (S_0^{\alpha})_{LP} \qquad \text{if } \sum_i \alpha_i \text{ is odd}$$

$$= (S_0^{\alpha})_{LP} - (S_1^{\alpha})_{LP} \qquad \text{if } \sum_i \alpha_i \text{ is even}$$

or

$$S_{LP}^{\alpha} = (-1)^{\Sigma\alpha_i} [(S_0)_{LP}^{\alpha} - (S_1)_{LP}^{\alpha}] \quad .$$

Theorem 8.1.1

For a 2^n factorial with $A_1 A_2 \ldots A_n$ negligible, where $n \geq 3$, there exists a RM(2^n, 2^n, 2^{n-1}) design such that all effects and interactions are estimable. Furthermore, for any effects E^{α} and E^{α}_r with $\alpha' \neq (1, \ldots, 1)$,

$$\hat{E}^{\alpha} = \frac{2^{n-1} - 1}{2^{n-1}(2^{n-1} - 1) - 2}$$

$$\left[(2^{n-1} - \frac{1}{2^{n-1} - 1}) E^{\alpha}_d + (E_d^{\alpha})_r + \frac{1}{(2^{n-1})^2 (2^{n-1} - 1)} S^{\alpha}_{LP} \right] ,$$

and

$$\hat{E}^{\alpha}_r = \frac{2^{n-1}(2^{n-1} - 1)}{2^{n-1}(2^{n-1} - 1) - 2}$$

$$\left[\frac{1}{2^{n-1} - 1} E^{\alpha}_d + (E^{\alpha}_d)_r + \frac{1}{(2^{n-1})^2 (2^{n-1} - 1)} S^{\alpha}_{LP} \right]$$

and, for $\alpha' = (1, 1, \ldots, 1)$,

$$\hat{E}^{\alpha}_r = 2^{n-1}(E^{\alpha}_d)_r - \frac{1}{2^{n-1}} S^{\alpha}_{LP} .$$

Proof:

(1) Construction

First we assign the set of treatment combinations satisfying $x_1 + x_2 + \ldots + x_n = 0 \pmod 2$ to the first initial subject in arbitrary order and assign to the second initial subject the set of treatment combinations satisfying $x_1 + x_2 + \ldots + x_n = 1 \pmod 2$. Then we construct one Latin Square from each initial subject using Williams' method. Finally, we combine these two Latin Squares to obtain a $RM(2^n, 2^n, 2^{n-1})$ design.

(2) Estimation

From the construction of the design in part (1), it follows that each a(x) occurs 2^{n-1} times and each $a_r(x)$ occurs $2^{n-1} - 1$ times in the design. Hence we have

$$E_j^\alpha = \frac{1}{(2^{n-1})^2} T_j^\alpha$$

$$(E_j^\alpha)_r = \frac{1}{2^{n-1}(2^{n-1} - 1)} R_j^\alpha \tag{8.6}$$

where j = 0, 1.

Then from (8.5) and (8.6), we obtain

$$E_d^\alpha = \frac{(-1)^{\Sigma\alpha_i}}{2^{n-1}} \frac{1}{2^{n-1}} (T_0^\alpha - T_1^\alpha)$$

$$= \frac{(-1)^{\Sigma\alpha_i}}{2^{n-1}} \frac{1}{2^{n-1}} \left(\sum_x T_{a_0^\alpha(x)} - \sum_x T_{a_1^\alpha(x)}\right), \tag{8.7}$$

where $T_{a_j^\alpha(x)}$ is the total of all observations associated with $a_j^\alpha(x)$ (j = 0, 1), and

$a_j^\alpha(x)$ = any a(x) which satisfies $\sum_i \alpha_i x_i = j \pmod 2$,

$a_{j_r}^\alpha(x)$ = the residual effect of $a_j^\alpha(x)$ (j = 0, 1).

Since a given $a_i^\alpha(x)$ $(i = 0, 1)$ occurs 2^{n-1} times in the design and exactly once in the last period and since $\sum_{i,\alpha} \hat{a}^\alpha_{i_r}(x) = 0$ (a condition imposed to solve the N.E.) we obtain from model (4.1) and the normal equations (4.2),

$$T_{a_i^\alpha}(x) = 2^{n-1}\, \hat{a}_i^\alpha(x) - \hat{a}^\alpha_{i_r}(x) + \Sigma\, \hat{S}_j \tag{8.8}$$

where the summation in (8.8) is over all subjects with $a_i^\alpha(x)$ $(i = 0, 1)$.

Since there is the same number of $a_0^\alpha(x)$ and $a_1^\alpha(x)$ associated with each subject, it follows from (8.7) and (8.8) that

$$E_d^\alpha = \frac{(-1)^{\Sigma\alpha_i}}{(2^{n-1})^2}\{2^{n-1}[\Sigma\,\hat{a}_0^\alpha(x) - \Sigma\,\hat{a}_1^\alpha(x)] - [\Sigma\,\hat{a}^\alpha_{0_r}(x) - \Sigma\,\hat{a}^\alpha_{1_r}(x)]\}$$

$$= \frac{(-1)^{\Sigma\alpha_i}}{2^{n-1}}[\Sigma\,\hat{a}_0^\alpha(x) - \Sigma\,\hat{a}_1^\alpha(x)] - \frac{1}{2^{n-1}}\{\frac{(-1)^{\Sigma\alpha_i}}{2^{n-1}}[\Sigma\,\hat{a}^\alpha_{0_r}(x) - \Sigma\,\hat{a}^\alpha_{1_r}(x)]\}$$

$$= \hat{E}^\alpha - \frac{1}{2^{n-1}}\hat{E}^\alpha_r \quad . \tag{8.9}$$

Further, from (8.5) and (8.6) we have

$$(E_d^\alpha)_r = (-1)^{\Sigma\alpha_i}[(E_0^\alpha)_r - (E_1^\alpha)_r]$$

$$= (-1)^{\Sigma\alpha_i}\frac{1}{2^{n-1}}\frac{1}{2^{n-1}-1}(R_0^\alpha - R_1^\alpha)$$

$$= (-1)^{\Sigma\alpha_i}\frac{1}{2^{n-1}}\frac{1}{2^{n-1}-1}(\Sigma\, T_{a^\alpha_{0_r}}(x) - \Sigma\, T_{a^\alpha_{1_r}}(x)) \tag{8.11}$$

We now define

$$S^{\alpha} = S_1^{\alpha} - S_0^{\alpha} \quad \text{for } \sum_i \alpha_i \text{ odd}$$

$$= S_0^{\alpha} - S_1^{\alpha} \quad \text{for } \sum_i \alpha_i \text{ even}$$

or

$$S^{\alpha} = (-1)^{\Sigma\alpha_i} (S_0^{\alpha} - S_1^{\alpha}) \tag{8.12}$$

where S_i^{α} = sum of subject effects for those subjects for which in the last period a(x) satisfies $\sum_j \alpha_j x_j = i \pmod 2$ $(i = 0, 1)$.

It follows then from model (4.1) and the normal equations (4.2) that

$$T_{a^{\alpha}_{i_r}}(x) = (2^{n-1} - 1)\hat{a}^{\alpha}_{i_r}(x) - \hat{a}^{\alpha}_i(x) - \hat{s}^{\alpha}_i$$

$$(i = 0, 1) \quad , \tag{8.13}$$

since a given $a^{\alpha}_{i_r}(x)$ occurs $2^{n-1} - 1$ times in the whole design and exactly once in the last period, and since $\sum_{i,\alpha} \hat{a}^{\alpha}_i(x) = 0$.

Using (8.12) and (8.13), (8.11) can be written as

$$(E_d^{\alpha})_r = \hat{E}_r^{\alpha} - \frac{1}{2^{n-1} - 1} \hat{E}^{\alpha} - \frac{(-1)^{\Sigma\alpha_i}}{2^{n-1}(2^{n-1} - 1)} (\hat{S}_0^{\alpha} - \hat{S}_1^{\alpha})$$

$$= \hat{E}_r^{\alpha} - \frac{1}{2^{n-1} - 1} \hat{E}^{\alpha} - \frac{\hat{S}^{\alpha}}{2^{n-1}(2^{n-1} - 1)} \quad . \tag{8.14}$$

In (8.14) $\hat{S}^{\alpha}$ can be obtained as follows:

Since there are 2^{n-1} periods and since a given $a^{\alpha}_{i_r}(x)$ occurs exactly once in the last period and $\sum_{i,\alpha} \hat{a}^{\alpha}_{i}(x) = 0$, it follows from (4.1) and (4.2) that

$$S^{\alpha}_{LP} = 2^{n-1}S^{\hat{\alpha}} - 2^{n-1}\hat{E}^{\alpha}_{r}$$

or

$$\hat{S}^{\alpha} = \frac{1}{2^{n-1}} S^{\alpha}_{LP} + \hat{E}^{\alpha}_{r} \quad . \tag{8.15}$$

Substituting (8.15) into (8.14), we have

$$(E^{\alpha}_{d})_{r} + \frac{1}{(2^{n-1})^{2}(2^{n-1}-1)} S^{\alpha}_{LP} = 1 - \frac{1}{2^{n-1}(2^{n-1}-1)} \hat{E}^{\alpha}_{r} - \frac{1}{2^{n-1}-1} \hat{E}^{\alpha} \; . \tag{8.16}$$

From (8.9) and (8.16) we obtain

$$\hat{E}^{\alpha}_{r} = \frac{1}{1 - \dfrac{2}{2^{n-1}(2^{n-1}-1)}} \left[\frac{1}{2^{n-1}-1} E^{\alpha}_{d} + (E^{\alpha}_{d})_{r} + \frac{1}{(2^{n-1})^{2}(2^{n-1}-1)} S^{\alpha}_{LP} \right] ,$$

and

$$\hat{E}^{\alpha} = \frac{1}{2^{n-1} - \dfrac{2}{2^{n-1}-1}}$$

$$\left[(2^{n-1} - \frac{1}{2^{n-1}-1}) E^{\alpha}_{d} + (E^{\alpha}_{d})_{r} + \frac{1}{(2^{n-1})^{2}(2^{n-1}-1)} S^{\alpha}_{LP} \right] .$$

We now consider $\hat{E}_r^\alpha$ for $\alpha' = (1, 1, \ldots, 1)$. It follows from the construction that all $a_0^\alpha(x)$ are in the first 2^{n-1} subjects and all $a_1^\alpha(x)$ are in the remaining 2^{n-1} subjects. Thus $\hat{E}_r^\alpha$ involves the comparison of the observations in those two sets of subjects. It follows then, similar to (8.14), that

$$(E_d^\alpha)_r = \hat{E}_r^\alpha - \frac{\hat{S}^\alpha}{2^{n-1}(2^{n-1}-1)} + \frac{2^{n-1}\ \hat{S}^\alpha}{2^{n-1}(2^{n-1}-1)}$$

$$= \hat{E}_r^\alpha + \frac{\hat{S}^\alpha}{2^{n-1}} \quad , \tag{8.16a}$$

and similar to (8.15) we have

$$S_{LP}^\alpha = 2^{n-1}\hat{S}^\alpha + 2^{n-1}(2^{n-1} - 1)\hat{E}_r^\alpha$$

or

$$\hat{S}^\alpha = \frac{1}{2^{n-1}} S_{LP}^\alpha - (2^{n-1} - 1)\hat{E}_r^\alpha \quad . \tag{8.17}$$

Substituting (8.17) into (8.16a), we obtain that for $\alpha = (1, 1, \ldots, 1)'$

$$\hat{E}_r^\alpha = 2^{n-1}(E_d^\alpha)_r - \frac{1}{2^{n-1}} S_{LP}^\alpha \quad .$$

We shall illustrate this result by an example. For simplicity, we will use the symbol y_{ij} to denote the observation in the ith period and for the jth subject:

From design (8.2) or (8.3) we find, e.g., for $\alpha = (1, 0, 0)'$,

$$T_0^{\alpha} = y_{11} + y_{22} + y_{34} + y_{43} + y_{16} + y_{27} + y_{35} + y_{48}$$
$$+ y_{18} + y_{25} + y_{37} + y_{46} + y_{13} + y_{24} + y_{32} + y_{41} ,$$

$$T_1^{\alpha} = y_{17} + y_{28} + y_{36} + y_{45} + y_{14} + y_{21} + y_{33} + y_{42}$$
$$+ y_{12} + y_{23} + y_{31} + y_{44} + y_{15} + y_{26} + y_{38} + y_{47} ,$$

$$R_0^{\alpha} = y_{21} + y_{32} + y_{44} + y_{26} + y_{37} + y_{45}$$
$$+ y_{28} + y_{35} + y_{47} + y_{23} + y_{34} + y_{42} ,$$

$$R_1^{\alpha} = y_{27} + y_{38} + y_{46} + y_{24} + y_{31} + y_{43}$$
$$+ y_{22} + y_{33} + y_{41} + y_{25} + y_{36} + y_{48} ,$$

$$E_d^{\alpha} = \frac{1}{16} [-y_{11} + y_{12} - y_{13} + y_{14} + y_{15} - y_{16} + y_{17} - y_{18}$$
$$+ y_{21} - y_{22} + y_{23} - y_{24} - y_{25} + y_{26} - y_{27} + y_{28}$$
$$+ y_{31} - y_{32} + y_{33} - y_{34} - y_{35} + y_{36} - y_{37} + y_{38}$$
$$- y_{41} + y_{42} - y_{43} + y_{44} + y_{45} - y_{46} + y_{47} - y_{48}] ,$$

and

$$(E_d^{\alpha})_r = \frac{1}{12} [-y_{21} + y_{22} - y_{23} + y_{24} + y_{25} - y_{26} + y_{27} - y_{28}$$
$$+ y_{31} - y_{32} + y_{33} - y_{34} - y_{35} + y_{36} - y_{37} + y_{38}$$
$$+ y_{41} - y_{42} + y_{43} - y_{44} - y_{45} + y_{46} - y_{47} + y_{48}] .$$

Since

$$S^{\alpha}_{0_{LP}} = y_{11} + y_{21} + y_{31} + y_{41} + y_{13} + y_{23} + y_{33} + y_{43}$$
$$+ y_{16} + y_{26} + y_{36} + y_{46} + y_{18} + y_{28} + y_{38} + y_{48}$$

and

$$S^{\alpha}_{1_{LP}} = y_{12} + y_{22} + y_{32} + y_{42} + y_{14} + y_{24} + y_{34} + y_{44}$$
$$+ y_{15} + y_{25} + y_{35} + y_{45} + y_{17} + y_{27} + y_{37} + y_{47} ,$$

we have

$$S^{\alpha}_{LP} = -y_{11} + y_{12} - y_{21} + y_{22} - y_{31} + y_{32} - y_{41} + y_{42}$$
$$- y_{13} + y_{14} - y_{23} + y_{24} - y_{33} + y_{34} - y_{43} + y_{44}$$
$$+ y_{15} - y_{16} + y_{25} - y_{26} + y_{35} - y_{36} + y_{45} - y_{46}$$
$$+ y_{17} - y_{18} + y_{27} - y_{28} + y_{37} - y_{38} + y_{47} - y_{48} .$$

Then by Theorem 8.1.1, the least squares estimators are

$$\hat{A} = \hat{E}^{\alpha} = \frac{11}{10} E^{\alpha}_{d} + \frac{3}{10} (E^{\alpha}_{d})_{r} + \frac{1}{160} S^{\alpha}_{LP}$$
$$= \frac{1}{160} [12(-y_{11} + y_{12} - y_{13} + y_{14} + y_{15} - y_{16} + y_{17} - y_{18})$$
$$+ 7(y_{21} - y_{22} + y_{23} - y_{24} - y_{25} + y_{26} - y_{27} + y_{28})$$
$$+ 15(y_{31} - y_{32} + y_{33} - y_{34} - y_{35} + y_{36} - y_{37} + y_{38})$$
$$- 7(-y_{41} + y_{42} - y_{43} + y_{44} + y_{45} - y_{46} + y_{47} - y_{48})]$$

$$\hat{A}_r = \hat{E}_r^{\alpha} = \frac{2}{5} E_d^{\alpha} + \frac{6}{5} (E_d^{\alpha})_r + \frac{1}{40} S_{LP}^{\alpha}$$

$$= \frac{1}{40} [2(-y_{11} + y_{12} - y_{13} + y_{14} + y_{15} - y_{16} + y_{17} - y_{18})$$

$$- 4(y_{21} - y_{22} + y_{23} - y_{24} - y_{25} + y_{26} - y_{27} + y_{28})$$

$$+ 4(y_{31} - y_{32} + y_{33} - y_{34} - y_{35} + y_{36} - y_{37} + y_{38})$$

$$- 2(-y_{41} + y_{42} - y_{43} + y_{44} + y_{45} - y_{46} + y_{47} - y_{48})] .$$

We can proceed similarly for the other estimators.

Variance of Estimators

Before considering the general case we derive first the variances for $\hat{A}$ and $\hat{A}_r$ from the previous example.

To find the variances of the estimators $\hat{A}$ and $\hat{A}_r$, we have, for $\alpha' = (1, 0, 0)$, from the discussion of the preceding example,

$$\frac{\mathrm{Var}(\hat{A})}{\sigma^2} = \left(\frac{11}{10}\right)^2 \frac{\mathrm{Var}(E_d^{\alpha})}{\sigma^2} + \left(\frac{3}{10}\right)^2 \frac{\mathrm{Var}((E_d^{\alpha})_r)}{\sigma^2}$$

$$+ \left(\frac{1}{160}\right)^2 \frac{\mathrm{Var}(S_{LP}^{\alpha})}{\sigma^2}$$

$$+ 2 \times \frac{11}{10} \times \frac{3}{10} \frac{\mathrm{Cov}(E_d^{\alpha}, (E_d^{\alpha})_r)}{\sigma^2}$$

$$+ 2 \times \frac{11}{10} \times \frac{1}{160} \frac{\mathrm{Cov}(E_d^{\alpha}, S_{LP}^{\alpha})}{\sigma^2}$$

$$+ 2 \times \frac{3}{10} \times \frac{1}{160} \frac{\mathrm{Cov}((E_d^{\alpha})_r, S_{LP}^{\alpha})}{\sigma^2} . \tag{8.18}$$

Since from model (4.1),

$$\frac{\mathrm{Var}(E_d^{\alpha})}{\sigma^2} = \frac{1}{(16)^2}\,(8 \times 4) = \frac{1}{8}$$

$$\frac{\mathrm{Var}((E_d^{\alpha})_r)}{\sigma^2} = \frac{1}{(16)^2}\,(8 \times 3) = \frac{3}{32}$$

and

$$S_{LP}^{\alpha} = B_2 + B_4 + B_5 + B_7 - B_1 - B_3 - B_6 - B_8$$

$$\frac{\mathrm{Var}(S_{LP}^{\alpha})}{\sigma^2} = 8 \times 4 = 32$$

$$\frac{\mathrm{Cov}(E_d^{\alpha},\ (E_d^{\alpha})_r)}{\sigma^2} = \frac{1}{(16)^2}\,(-2^3) = -\,\frac{1}{32}$$

$$\frac{\mathrm{Cov}(E_d^{\alpha},\ S_{LP}^{\alpha})}{\sigma^2} = 0$$

$$\frac{\mathrm{Cov}((E_d^{\alpha})_r,\ B_2 + B_4 + B_5 + B_7)}{\sigma^2} = \frac{1}{16}\,(-4 \times 1) = -\,\frac{1}{4}$$

$$\frac{\mathrm{Cov}((E_d^{\alpha})_r,\ B_1 + B_3 + B_6 + B_8)}{\sigma^2} = \frac{1}{16}\,(4 \times 1) = \frac{1}{4}$$

$$\frac{\mathrm{Cov}((E_d^{\alpha})_r,\ S_{LP}^{\alpha})}{\sigma^2} = -\,\frac{1}{2}\ .$$

From (8.18) we have

$$\frac{\mathrm{Var}(\hat{A})}{\sigma^2} = \frac{\mathrm{Var}(\hat{E}^{\alpha})}{\sigma^2} = 0.1375$$

Further,

$$\mathrm{Var}(\hat{A}_r) = \mathrm{Var}(\hat{E}^{\alpha}_r)$$

$$= (\tfrac{6}{5})^2[(\tfrac{1}{3})^2 \mathrm{Var}(E^{\alpha}_d) + \mathrm{Var}(E^{\alpha}_{d_r}) + (\tfrac{1}{48})^2 \mathrm{Var}(S^{\alpha}_{LP})$$

$$+ \tfrac{1}{3} \mathrm{Cov}(E^{\alpha}_d, (E^{\alpha}_d)_r) + \tfrac{1}{144} \mathrm{Cov}(E^{\alpha}_d, S^{\alpha}_{LP})$$

$$+ \tfrac{1}{48} \mathrm{Cov}((E^{\alpha}_d)_r, S^{\alpha}_{LP})]$$

and hence

$$\frac{\mathrm{Var}(\hat{A})_r}{\sigma^2} = 0.20 .$$

Likewise, for any effect E^{α} except ABC and ABC_r, we have

$$\frac{\mathrm{Var}(\hat{E}^{\alpha})}{\sigma^2} = 0.1375$$

and

$$\frac{\mathrm{Var}(\hat{E}^{\alpha})_r}{\sigma^2} = 0.20 .$$

To find $\mathrm{Var}(A\hat{B}C_r)$, recall first that

$$A\hat{B}C_r = \hat{E}^{\alpha}_r = 4(E^{\alpha}_d)_r - \tfrac{1}{4} S^{\alpha}_{LP} .$$

Hence,

$$\mathrm{Var}(A\hat{B}C_r)$$

$$= 16 \mathrm{Var}(E^{\alpha}_d)_r + \frac{1}{16} \mathrm{Var}(S^{\alpha}_{LP}) - 2 \mathrm{Cov}((E^{\alpha}_d)_r, S^{\alpha}_{LP}) . \qquad (8.19)$$

Since

$$\frac{\mathrm{Var}(E_d^{\alpha})_r}{\sigma^2} = \frac{1}{(16)^2}(8 \times 3) = \frac{3}{32}$$

$$\frac{\mathrm{Var}(S_{LP}^{\alpha})}{\sigma^2} = 32$$

and

$$\frac{\mathrm{Cov}((E_d^{\alpha})_r,\ B_5 + B_6 + B_7 + B_8)}{\sigma^2} = \frac{\mathrm{Cov}((E_d^{\alpha})_r,\ B_1 + B_2 + B_3 + B_4)}{\sigma^2}$$

$$= \frac{1}{16}(3 \times 4) = \frac{3}{4}\ ,$$

(8.19) becomes

$$\frac{\mathrm{Var}(\widehat{ABC}_r)}{\sigma^2} = \frac{2}{3}$$

In general, for a 2^n design constructed according to Theorem 8.1.1, the variances of the least squares estimators can be found as follows:

For $\alpha \neq (1, 1, \ldots, 1)'$ we have from Theorem 8.1.1 that

$$\mathrm{Var}(\hat{E}^{\alpha}) = \left[\frac{2^{n-1}-1}{2^{n-1}(2^{n-1}-1)-2}\right]^2$$

$$\left\{(2^{n-1} - \frac{1}{2^{n-1}-1})^2\ \mathrm{Var}(E_d^{\alpha}) + \mathrm{Var}(E_d^{\alpha})_r + \frac{1}{(2^{n-1})^4(2^{n-1}-1)^2}\ \mathrm{Var}(S_{LP}^{\alpha})\right.$$

$$+ 2\left(2^{n-1} - \frac{1}{2^{n-1}-1}\right)\mathrm{Cov}(E_d^{\alpha}, (E_d^{\alpha})_r)$$

$$+ 2 \cdot \frac{2^{n-1} - \dfrac{1}{2^{n-1}-1}}{(2^{n-1})^2(2^{n-1}-1)} \mathrm{Cov}(E_d^{\alpha}, S_{LP}^{\alpha})$$

$$+ \frac{2}{(2^{n-1})^2(2^{n-1}-1)} \mathrm{Cov}((E_d^{\alpha})_r, S_{LP}^{\alpha})\Big\} . \tag{8.20}$$

Obviously,

$$\frac{\mathrm{Var}(E_d^{\alpha})}{\sigma^2} = \left[\frac{1}{(2^{n-1})^2}\right]^2 (2^n \times 2^{n-1}) = \frac{2}{(2^{n-1})^2}$$

$$\frac{\mathrm{Var}(E_d^{\alpha})_r}{\sigma^2} = \left[\frac{1}{2^{n-1}(2^{n-1}-1)}\right]^2 2^n \cdot (2^{n-1} - 1)$$

$$= \frac{2}{2^{n-1}(2^{n-1}-1)}$$

$$\frac{\mathrm{Var}(S_{LP}^{\alpha})}{\sigma^2} = 2^n \times 2^{n-1} = 2^{2n-1} . \tag{8.21}$$

To evaluate $\mathrm{Cov}(E_d^{\alpha}, (E_d^{\alpha})_r)$, we define the following quantities:

S_{A_0} = the set of all treatments x with $a_0^{\alpha}(x)$,

S_{A_1} = the set of all treatments x with $a_1^{\alpha}(x)$,

$y_{i_1 j_1; a_1(x)}$ = the observation in the i_1th period and for the j_1th subject with which a specific treatment combination $a_1(x)$ is associated.

If $y_{i_1 j_1; a_1(x)}$ enters into a E_d^{α}, then $y_{i_1+1, j_1; a_2(x)}$ enters into $(E_d^{\alpha})_r$. Then

$$\frac{\mathrm{Cov}(y_{i_1 j_1; a_1(x)}\ ,\ y_{i_1+1, j_1; a_2(x)})}{\sigma^2}$$

$$\begin{aligned}
&= 1 \quad \text{if } a_1(x) \in S_{A_1} \text{ and } a_2(x) \in S_{A_1} \\
&\qquad \text{or } a_1(x) \in S_{A_0} \text{ and } a_2(x) \in S_{A_0} \\
&= -1 \quad \text{if } a_1(x) \in S_{A_1} \text{ and } a_2(x) \in S_{A_0} \\
&\qquad \text{or } a_1(x) \in S_{A_0} \text{ and } a_2(x) \in S_{A_1} \\
&= 0 \quad \text{otherwise}
\end{aligned}$$

where $1 \leq i_1 \leq 2^{n-1} - 1$, $1 \leq j_1 \leq 2^n$.

All $a(x)$ in the first 2^{n-1} subjects of the design constructed in Theorem 9.1.1, are either in S_{A_0} or S_{A_1} and each of those $a(x)$ is preceded by every other $a(x)$ in S_{A_0} or S_{A_1}, respectively. There are an equal number of $a(x)$ in S_{A_0} and S_{A_1}.

Now consider a fixed $a_1(x)$ in the first 2^{n-1} subjects. The sum of the covariances of any $y_{i_1 j_1; a_1(x)}$ with the observation following it, say $y_{i_1+1, j; a_2(x)}$ is given by

$$\sum_{i_1=1}^{2^{n-1}-1} \sum_{j_1=1}^{2^{n-1}} \frac{\mathrm{cov}(y_{i_1 j_1;a_1(x)}, y_{i_1+1,j_1;a_2(x)})}{\sigma^2}$$

$$= -1 \ .$$

Similarly, for any $a_1(x)$ in the 2nd set of 2^{n-1} subjects, we have

$$\sum_{i_1=1}^{2^{n-1}-1} \sum_{j_1=2^{n-1}+1}^{2^n} \frac{\mathrm{Cov}(y_{i_1 j_1;a_1(x)}, y_{i_1+1,j_1;a_2(x)})}{\sigma^2}$$

$$= -1 \ . \tag{8.22}$$

Therefore, since we have 2^n $a(x)$ in the design,

$$\frac{\mathrm{Cov}(E_d^\alpha, (E_d^\alpha)_r)}{\sigma^2} = \left(\frac{1}{2^{n-1}}\right)^3 \left(\frac{1}{2^{n-1}-1}\right)$$

$$\sum_{i_1=1}^{2^{n-1}-1} \sum_{j_1=1}^{2^n} \frac{\mathrm{Cov}(y_{i_1 j_1;a_1(x)}, y_{i_1+1,j_1;a_2(x)})}{\sigma^2}$$

$$= \frac{1}{(2^{n-1})^3} \ \frac{1}{2^{n-1}-1} \ (-2^n)$$

$$= - \frac{2}{(2^{n-1})^2 (2^{n-1}-1)} \ . \tag{8.23}$$

Obviously,

$$\frac{\mathrm{Cov}(E_d^\alpha, S_{LP}^\alpha)}{\sigma^2} = 0 \tag{8.24}$$

$$\mathrm{Cov}((E_d^\alpha)_r,\ S_{LP}^\alpha)$$

$$= (-1)^{\Sigma\alpha_i}\left[\mathrm{Cov}((E_d^\alpha)_r,\ (S_0^\alpha)_{LP}) - \mathrm{Cov}((E_d^\alpha)_r,\ (S_1^\alpha)_{LP}\right]$$

To find $\mathrm{Cov}((E_d^\alpha)_r,\ (S_0^\alpha)_{LP})$, consider the subjects with $a_0^\alpha(x)$ in the last period. Since in those subjects the number of a(x) in S_{A_0} is the same as in S_{A_1}, we have, for every such subject S,

$$\frac{\mathrm{Cov}_s((E_d^\alpha)_r,\ (S_0^\alpha)_{LP})}{\sigma^2} = \frac{1}{2^{n-1}(2^{n-1}-1)}\,(-1)^{\Sigma\alpha_i+1} \ .$$

Since there are 2^{n-1} such s in each Latin Square of the design, we have

$$\frac{\mathrm{Cov}((E_d^\alpha)_r,\ (S_0^\alpha)_{LP})}{\sigma^2} = \frac{1}{2^{n-1}(2^{n-1}-1)}\,(-1)^{\Sigma\alpha_i+1}\ 2^{n-1} \ .$$

Similarly,

$$\frac{\mathrm{Cov}((E_d^\alpha)_r,(S_1^\alpha)_{LP})}{\sigma^2} = \frac{(-1)^{\Sigma\alpha_i}}{2^{n-1}(2^{n-1}-1)}\,\frac{2^n}{2} \ . \tag{8.25}$$

From (8.25) we obtain

$$\frac{\mathrm{Cov}((E_d^\alpha)_r,\ S_{LP}^\alpha)}{\sigma^2} = \frac{-1}{2^{n-1}(2^{n-1}-1)}\,\frac{2^n}{2}\cdot 2$$

$$= \frac{-2}{2^{n-1}-1} \ . \tag{8.26}$$

Then using (8.21), (8.23), (8.24) and (8.26), (8.20) becomes

$$\frac{\mathrm{Var}(E^{\alpha})}{\sigma^2} = \frac{2[2^{n-1}(2^{n-1}-1) - 1]^2 - 2 \cdot 2^{n-1}(2^{n-1}-1) + 2}{\{2^{n-1}[2^{n-1}(2^{n-1}-1)-2]\}^2} . \tag{8.26a}$$

Likewise, by using Theorem 8.1.1, we can evaluate the variance of $\hat{E}_r^{\alpha}$ as follows:

$$\mathrm{Var}(\hat{E}_r^{\alpha})$$

$$= \left[\frac{2^{n-1}(2^{n-1}-1)}{2^{n-1}(2^{n-1}-1)-2}\right]^2$$

$$\Bigg\{\frac{1}{(2^{n-1}-1)^2}\mathrm{Var}(E_d^{\alpha}) + \mathrm{Var}(E_d^{\alpha})_r + \frac{1}{(2^{n-1})^4(2^{n-1}-1)^2}\mathrm{Var}(S_{LP}^{\alpha})$$

$$+ \frac{2}{2^{n-1}-1}\mathrm{Cov}(E_d^{\alpha}, (E_d^{\alpha})_r) + \frac{2}{(2^{n-1})^2(2^{n-1}-1)^2}\mathrm{Cov}(E_d^{\alpha}, S_{LP}^{\alpha})$$

$$+ \frac{2}{(2^{n-1})^2(2^{n-1}-1)}\mathrm{Cov}((E_d^{\alpha})_r, S_{LP}^{\alpha})\Bigg\} . \tag{8.27}$$

Using (8.21), (8.23), (8.24) and (8.26), (8.27) becomes

$$\frac{\mathrm{Var}(E_r^{\alpha})}{\sigma^2} = \frac{2}{2^{n-1}(2^{n-1}-1)-2} . \tag{8.28}$$

To calculate the variance of $\hat{E}_r^{\alpha}$ for $\alpha' = (1, 1, \ldots, 1)$, from Theorem 8.1.1, we have

$$\frac{\mathrm{Var}(\hat{E}_r^{\alpha})}{\sigma^2}$$

$$= 2^{2n-2}\ \mathrm{Var}(E_d^{\alpha})_r + \frac{1}{2^{2n-2}}\ \mathrm{Var}(S_{LP}^{\alpha})$$

$$- 2\ \mathrm{Cov}((E_d^{\alpha})_r,\ S_{LP}^{\alpha})\ . \tag{8.29}$$

Obviously, since there are 2^{n-1} -1 's residual effects of the treatment combinations a(x) in each subject,

$$\frac{\mathrm{Var}(E_d^{\alpha})_r}{\sigma^2} = \left(\frac{1}{2^{n-1}-1}\right)^2\left(\frac{1}{2^{n-1}}\right)^2 \left[2^n(2^{n-1} - 1)\right]$$

$$= \frac{2}{2^{n-1}(2^{n-1}-1)}\ . \tag{8.30}$$

Also we have 2^{n-1} subjects to which the treatment combination x, satisfying $\Sigma x_i = j$ ($j = 0, 1$), are assigned.

Hence

$$\frac{\mathrm{Cov}((E_d^{\alpha})_r,\ S_{LP}^{\alpha})}{\sigma^2} = 1 - (-1) = 2\ . \tag{8.31}$$

Similar to (8.21), we have

$$\frac{\mathrm{Var}(S_{LP}^{\alpha})}{\sigma^2} = 2^{2n-1}\ . \tag{8.32}$$

Using (8.30), (8.31) and (8.32), (8.29) becomes

$$\frac{\mathrm{Var}(\hat{E}_r^{\alpha})}{\sigma^2} = \frac{2}{2^{n-1}-1} \quad . \tag{8.32a}$$

We summarize these results in (8.26a), (8.28) and (8.32a) in the following:

Theorem 8.1.2

The variances of the estimators in Theorem 8.1.1 are given by the following:

For $\alpha' \neq (1, \ldots, 1)$,

$$\mathrm{Var}(\hat{E}^{\alpha}) = \frac{2\{[2^{n-1}(2^{n-1}-1)-1]^2 - 2^{n-1}(2^{n-1}-1)+1\}}{\{2^{n-1}[2^{n-1}(2^{n-1}-1)-2]\}^2}\,\sigma^2$$

$$\mathrm{Var}(\hat{E}_r^{\alpha}) = \frac{2}{2^{n-1}(2^{n-1}-1)-2}\,\sigma^2$$

and for $\alpha' = (1, \ldots, 1)$,

$$\mathrm{Var}(\hat{E}_r^{\alpha}) = \frac{2}{2^{n-1}-1}\,\sigma^2 \ .$$

Remark: From Theorem 8.1.2 we can see that for a 2^n factorial with a negligible highest order interaction all the direct effects are estimated equally precisely and all the residual effects are also estimated equally precisely. Also the residual effects except the residual effect of the highest order interaction, are estimated less precisely than the direct effects and they are estimated more precisely than the residual effect of the highest order interaction.

In practice, if the highest order interaction E^{α} ($\alpha' = (1, 1, \ldots, 1)$) in a 2^n design is negligible, then we can even use designs which have fewer subjects than is required by Theorem 8.1.1, and still be able to estimate all other effects. This can be achieved by deleting one replicate of the 2^n treatment combinations, e.g., deleting every last subject from every Latin Square constructed in Theorem 8.1.1.

For example, we can delete the subjects 4 and 8 in the design (8.2) and obtain the following:

		subjects					
		1	2	3	4	5	6
periods	1	(1)	ac	bc	abc	b	a
	2	ab	(1)	ac	c	abc	b
	3	ac	bc	ab	b	a	c
	4	bc	ab	(1)	a	c	abc

where all the direct and residual effects are estimable except the highest order interaction ABC.

8.2 2^n Designs When Several Interactions are Negligible

Suppose that in a 2^4 design the effects AB, CD and ABCD are all negligible rather than only ABCD. We may then expect to use an even smaller design than the one from Theorem 8.1.1 with all direct and residual effects estimable except AB, CD and ABCD. In fact, the following design is an appropriate design for this purpose:

		subjects							
		1	2	3	4	5	6	7	8
periods	1	(1)	cd	abcd	ab	ac	bc	bd	ad
	2	ab	(1)	cd	abcd	ad	ac	bc	bd
	3	cd	abcd	ab	(1)	bc	bd	ad	ac
	4	abcd	ab	(1)	cd	bd	ad	ac	bc

		subjects							
		9	10	11	12	13	14	15	16
periods	1	c	abc	abd	d	a	acd	bcd	b
	2	d	c	abc	abd	b	a	acd	bcd
	3	abc	abd	d	c	acd	bcd	b	a
	4	abd	d	c	abc	bcd	b	a	acd

(8.33)

To construct the design in (8.33), we confound AB, CD and ABCD (as the generalized interaction of AB and CD) with subjects assigning 2^2 treatment combinations, in arbitrary order, to each of four initial subjects; for example:

(1)	ac	c	a
ab	ad	d	b
cd	bc	abc	acd
abcd	bd	abd	bcd

(8.34)

Then, similar to Section 8.1, we use each initial subject in (8.34) to construct a Latin Square using Williams' method yielding the design (8.33).

In general, we can state the following:

Theorem 8.2.1

For a 2^n factorial ($n \geq 4$) with two effects E^{α} and E^{β} and their generalized interaction negligible, there exists a RM(2^n, 2^n, 2^{n-2}) design such that all other direct and residual effects and interactions are estimable. Furthermore, for any effect E^{α_1} and $E_r^{\alpha_1}$ ($\alpha_1 \neq \alpha$, β, $\alpha + \beta$)

$$\hat{E}^{\alpha_1} = \frac{2^{n-2}-1}{2^{n-2}(2^{n-2}-1)-2}$$

$$\left[(2^{n-2} - \frac{1}{2^{n-2}-1})E_d^{\alpha_1} + (E_d^{\alpha_1})_r + \frac{1}{2^{n-1}\cdot 2^{n-2}(2^{n-2}-1)} S_{LP}^{\alpha_1}\right] ,$$

and

$$\hat{E}_r^{\alpha_1} = \frac{2^{n-2}(2^{n-2}-1)}{2^{n-2}(2^{n-2}-1)-2}$$

$$\left[\frac{1}{2^{n-2}-1} E_d^{\alpha_1} + (E_d^{\alpha_1})_r + \frac{1}{2^{n-1}\cdot 2^{n-2}(2^{n-2}-1)} S_{LP}^{\alpha_1}\right] .$$

For $\alpha_2 = \alpha$, β, $\alpha + \beta$,

$$\hat{E}_r^{\alpha_2} = 2^{n-2}(E_d^{\alpha_2})_r - \frac{1}{2^{n-1}} S_{LP}^{\alpha_2} .$$

Proof:

(1) Construction

First, we assign 2^{n-2} treatment combinations to each of four initial subjects in arbitrary order by confounding E^{α} and E^{β}, and also their generalized interaction $E^{\alpha+\beta}$, with subjects. That is, we assign each set of 2^{n-2} treatment combinations x satisfying $\Sigma d_{\ell} x_{\ell} = i$ and $\Sigma \beta_{\ell} x_{\ell} = j \pmod 2$ $(i, j = 0, 1)$ to one of four initial subjects in arbitrary order. Each initial subject is then developed into a Latin Square using Williams' method (see Section 2.1.1). Finally, we combine these four Latin Squares yielding a $RM(2^n, 2^n, 2^{n-2})$ design.

(2) Estimation

A proof which is similar to that of Theorem 8.1.1 will apply.

For $\alpha_1 \neq \alpha$, β, or $\alpha + \beta$, since, from the construction of the design in part (1), there are $2^{n-2} a(x)$ and $2^{n-2}-1$ $a_r(x)$ in the design, we have

$$E_j^{\alpha_1} = \frac{1}{2^{n-1} \cdot 2^{n-2}} T_j^{\alpha_1}$$

and

$$(E_j^{\alpha_1})_r = \frac{1}{2^{n-1}(2^{n-2}-1)} R_j^{\alpha_1}$$

where $j = 0, 1$.

Then, similar to (8.9), we have

$$E_d^{\alpha_1} = \hat{E}^{\alpha_1} - \frac{1}{2^{n-2}} \hat{E}_r^{\alpha_1} . \tag{8.39}$$

Also, similar to (8.16), we have

$$(E_d^{\alpha})_r + \frac{1}{2^{n-1} \cdot 2^{n-2}(2^{n-2}-1)} S_{LP}^{\alpha_1}$$

$$= \left[1 - \frac{1}{2^{n-2}(2^{n-2}-1)} \hat{E}_r^{\alpha_1} - \frac{1}{2^{n-2}-1} \hat{E}^{\alpha_1}\right] . \tag{8.46}$$

From (8.39) and (8.46) we obtain

$$\hat{E}_r^{\alpha_1} = \frac{2^{n-2}(2^{n-2}-1)}{2^{n-2}(2^{n-2}-1)-2}$$

$$\left[\frac{1}{2^{n-2}-1} E_d^{\alpha_1} + (E_d^{\alpha_1})_r + \frac{1}{2^{n-1} \cdot 2^{n-2}(2^{n-2}-1)} S_{LP}^{\alpha_1}\right]$$

and

$$\hat{E}^{\alpha_1} = \frac{2^{n-2}-1}{2^{n-2}(2^{n-2}-1)-2}$$

$$\left[(2^{n-2} - \frac{1}{2^{n-2}-1}) E_d^{\alpha_1} + (E_d^{\alpha_1})_r + \frac{1}{2^{n-1} \cdot 2^{n-2}(2^{n-2}-1)} S_{LP}^{\alpha_1}\right] .$$

Similar to (8.16a) and (8.17) we have, for $\alpha_2 = \alpha$, β, or $\alpha + \beta$,

$$(E_d^{\alpha_2})_r + \frac{1}{2^{n-1} \cdot 2^{n-2}} S_{LP}^{\alpha_2}$$

$$= \hat{E}_r^{\alpha_2} - \frac{2^{n-2}-1}{2^{n-2}} \hat{E}_r^{\alpha_2}$$

$$= \frac{1}{2^{n-2}} \hat{E}_r^{\alpha_2} \quad .$$

Therefore,

$$\hat{E}_r^{\alpha_2} = 2^{n-2}(E_d^{\alpha_2})_r - \frac{1}{2^{n-1}} S_{LP}^{\alpha_2} \quad .$$

A result about the variances of the estimators obtained in Theorem 8.2.1, which is similar to Theorem 8.1.2, will be given as follows:

Theorem 8.2.2

The variances of the estimators in Theorem 8.2.1 are given by the following:

For $\alpha_1 \neq \alpha$, β, or $\alpha + \beta$,

$$\mathrm{Var}(\hat{E}^{\alpha_1}) = \frac{2^{n-2}(2^{n-2}-1)-1}{(2^{n-2})^2[2^{n-2}(2^{n-2}-1)-2]} \sigma^2 ,$$

$$\mathrm{Var}(\hat{E}_r^{\alpha_1}) = \frac{1}{2^{n-2}(2^{n-2}-1)-2} \sigma^2 ,$$

and for $\alpha_2 = \alpha$, β, or $\alpha + \beta$,

$$\mathrm{Var}(\hat{E}_r^{\alpha_2}) = \frac{1}{2^{n-2}-1}\,\sigma^2 \ .$$

Proof: This proof is similar to the one in Theorem 8.1.2.

Given $\hat{E}^{\alpha_1}$ in Theorem 8.2.1, where $\alpha_1 \neq \alpha$, β, or $\alpha + \beta$,

$$\mathrm{Var}(\hat{E}^{\alpha_1})$$

$$= \left[\frac{2^{n-2}-1}{2^{n-2}(2^{n-2}-1)-2}\right]$$

$$\{(2^{n-2} - \frac{1}{2^{n-2}-1})^2\,\mathrm{Var}(E_d^{\alpha_1}) + \mathrm{Var}(E_d^{\alpha_1})_r$$

$$+ \frac{1}{(2^{n-1}2^{n-2})^2(2^{n-2}-1)^2}\,\mathrm{Var}(S_{LP}^{\alpha_1})$$

$$+ 2(2^{n-2} - \frac{1}{2^{n-2}-1})\,\mathrm{Cov}(E_d^{\alpha_1}, (E_d^{\alpha_1})_r)$$

$$+ 2\,\frac{2^{n-2} - \frac{1}{2^{n-2}-1}}{2^{n-1}\cdot 2^{n-2}(2^{n-2}-1)}\,\mathrm{Cov}(E_d^{\alpha_1}, S_{LP}^{\alpha_1})$$

$$+ \frac{2}{2^{n-1}2^{n-2}(2^{n-2}-1)}\,\mathrm{Cov}((E_d^{\alpha_1})_r, S_{LP}^{\alpha_1})\} \ . \tag{8.47}$$

Since

$$\frac{\mathrm{Var}(E_d^{\alpha_1})}{\sigma^2} = \left(\frac{1}{2^{n-1}}\right)^4 (2^{n-2} \times 2^n)$$

$$\frac{\mathrm{Var}(E_d^{\alpha_1})_r}{\sigma^2} = \left[\frac{1}{2^{n-1}(2^{n-2}-1)}\right]^2 [2^n \times (2^{n-2}-1)]$$

$$\frac{\mathrm{Var}(S_{LP}^{\alpha_1})}{\sigma^2} = 2^n \times 2^{n-2}$$

$$\frac{\mathrm{Cov}(E_d^{\alpha_1}, (E_d^{\alpha_1})_r)}{\sigma^2} = \left(\frac{1}{2^{n-1}}\right)^2 \frac{1}{2^{n-2}} \quad \frac{1}{(2^{n-2}-1)} (-2^n)$$

$$\frac{\mathrm{Cov}(E_d^{\alpha_1}, S_{LP}^{\alpha_1})}{\sigma^2} = 0 ,$$

and

$$\frac{\mathrm{Cov}((E_d^{\alpha_1})_r, S_{LP}^{\alpha_1})}{\sigma^2} = - \frac{2}{2^{n-2}-1} , \tag{8.48}$$

(8.47) becomes

$$\frac{\mathrm{Var}(\hat{E}^{\alpha_1})}{\sigma^2} = \frac{2^{n-2}(2^{n-2}-1)-1}{(2^{n-2})^2[2^{n-2}(2^{n-2}-1)-2]} .$$

And

$$\begin{aligned}
\mathrm{Var}(\hat{E}_r^{\alpha_1}) \\
&= \left[\frac{2^{n-2}(2^{n-2}-1)}{2^{n-2}(2^{n-2}-1)-2}\right]^2 \\
&\left\{\frac{1}{(2^{n-2}-1)^2}\,\mathrm{Var}(E_d^{\alpha_1}) + \mathrm{Var}(E_d^{\alpha_1})_r\right. \\
&+ \frac{1}{(2^{n-1})^2(2^{n-2})^2(2^{n-2}-1)^2}\,\mathrm{Var}(S_{LP}^{\alpha_1}) \\
&+ \frac{2}{2^{n-2}-1}\,\mathrm{Cov}(E_d^{\alpha_1},\ (E_d^{\alpha_1})_r) \\
&+ \frac{2}{2^{n-1}\cdot 2^{n-2}(2^{n-2}-1)^2}\,\mathrm{Cov}(E_d^{\alpha_1},\ S_{LP}^{\alpha_1}) \\
&\left.+ \frac{2}{2^{n-1}\cdot 2^{n-2}(2^{n-2}-1)}\,\mathrm{Cov}((E_d^{\alpha_1})_r,\ S_{LP}^{\alpha_1})\right\} \ . \qquad (8.49)
\end{aligned}$$

It follows from (8.48) that (8.49) becomes

$$\frac{\mathrm{Var}(\hat{E}_r^{\alpha_1})}{\sigma^2} = \frac{1}{2^{n-2}(2^{n-2}-1)-2} \ .$$

To calculate $\mathrm{Var}(\hat{E}^{\alpha_2})_r$ for $\alpha_2 = \alpha,\ \beta$, or $\alpha + \beta$, since

$$\mathrm{Var}(\hat{E}_r^{\alpha_2})$$

$$= (2^{n-2})^2 \ \mathrm{Var}(E_d^{\alpha_2})_r + \frac{1}{(2^{n-1})^2} \ \mathrm{Var}(S_{LP}^{\alpha_2})$$

$$- \frac{2}{2} \ \mathrm{Cov}((E_d^{\alpha_2})_r, \ S_{LP}^{\alpha_2}) \ , \tag{8.50}$$

use the procedures similar to (8.30) - (8.32),

$$\mathrm{Var}(E_d^{\alpha_2})_r = \left(\frac{1}{2^{n-1}}\right)^2 \left(\frac{1}{2^{n-2}-1}\right)^2 \ [2^n(2^{n-2}-1)]$$

$$\frac{\mathrm{Cov}((E_d^{\alpha_2})_r, \ S_{LP}^{\alpha_2})}{\sigma^2} = 2 \ ,$$

(8.50) becomes

$$\frac{\mathrm{Var}(\hat{E}_r^{\alpha_2})}{\sigma^2} = \frac{1}{2^{n-2}-1} \ .$$

More generally, if there are several high order interactions which are negligible in a 2^n design, we have the following theorem, which extends the results of Theorems 8.1.1 and 8.2.1.

Theorem 8.2.3

For a 2^n factorial with $m = 2^q - 1$ effects $E^{\beta_1}, \ldots, E^{\beta_m}$ and their generalized interactions are negligible, where $n \geq q + 2$, there exists a $RM(2^n, 2^n, 2^{n-q})$ design such that all non-negligible effects and interactions are estimable. Furthermore, for any effect E^{α_1} and $E_r^{\alpha_1}$, where $\alpha_1 \neq \beta_1, \ldots, \beta_m$ or $\sum_{i=1}^{m} c_i\beta_i$ $(c_i = 0, 1)$

$$\hat{E}^{\alpha_1} = \frac{2^{n-q}-1}{2^{n-q}(2^{n-q}-1)-2}$$

$$\left[(2^{n-q} - \frac{1}{2^{n-q}-1})E_d^{\alpha_1} + (E_d^{\alpha_1})_r + \frac{1}{2^{n-1}\cdot 2^{n-q}(2^{n-q}-1)} S_{LP}^{\alpha_1} \right],$$

and

$$\hat{E}_r^{\alpha_1} = \frac{2^{n-q}(2^{n-q}-1)}{2^{n-q}(2^{n-q}-1)-2}$$

$$\left[\frac{1}{2^{n-q}-1} E_d^{\alpha_1} + (E_d^{\alpha_1})_r + \frac{1}{2^{n-1}\cdot 2^{n-q}(2^{n-q}-1)} S_{LP}^{\alpha_1} \right].$$

For all $\alpha_2 = \sum_{i=1}^{m} c_i\beta_i$ $(c_i = 0, 1)$

$$\hat{E}_r^{\alpha_2} = 2^{n-q}(E_d^{\alpha_2})_r - \frac{1}{2^{n-1}} S_{LP}^{\alpha_2} .$$

Theorem 8.2.4

The variances of the estimators in Theorem 8.2.3 are given by the following:

For $\alpha_1 \neq \beta_1, \ldots, \beta_m$ or $\sum_{i=1}^{m} c_i\beta_i$ $(c_i = 0,1)$,

$$\mathrm{Var}(\hat{E}^{\alpha_1}) = \frac{2^{n-q}(2^{n-q}-1)-1}{(2^{n-q})^2[2^{n-q}(2^{n-q}-1)-2]}\,\sigma^2 ,$$

$$\mathrm{Var}(\hat{E}_r^{\alpha_1}) = \frac{1}{2^{n-q}(2^{n-q}-1)-2}\,\sigma^2 ,$$

and for all $\alpha_2 = \sum_{i=1}^{m} c_i\beta_i$ $(c_i = 0,1)$

$$\mathrm{Var}(\hat{E}_r^{\alpha_2}) = \frac{1}{2^{n-q}-1}\,\sigma^2$$

where $n \geq q + 2$.

The proofs of Theorem 8.2.3 and 8.2.4 are similar to those of Theorem 8.2.1 and 8.2.2, respectively.

The design given in Theorem 8.2.3 can be reduced further by deleting one replicate of the 2^n treatment combinations. For example, we can delete subjects 4, 8, 12, and 16 from the design in (8.33) and get the following:

		subjects					
		1	2	3	4	5	6
periods	1	(1)	cd	abcd	ac	bc	bd
	2	ab	(1)	cd	ad	ac	bc
	3	cd	abcd	ab	bc	bd	ad
	4	abcd	ab	(1)	bd	ad	ac

		subjects						
		7	8	9	10	11	12	
periods	1	c	abc	abd	a	acd	bcd	
	2	d	c	abc	b	a	acd	
	3	abc	abd	d	acd	bcd	b	
	4	abd	d	c	bcd	b	a	(8.51)

All the direct and residual effects in (8.51) except AB, CD, and ABCD are estimable and those forty-two effects can be estimated by the forty-eight observations in (8.51).

IX. CONCLUSION

In this study we have investigated the structure and properties of repeated measurement designs from different points of view, such as

(i) balancedness or partial balancedness

(ii) construction versus estimation

(iii) underlying linear models

(iv) factorial treatment structure.

In studying balanced repeated measurement designs for the first order residual effects model it becomes apparent that one really has to distinguish between balancedness with respect to construction and balancedness with respect to estimation. These two concepts do not necessarily imply each other as they do, for example, for the balanced incomplete block design. We refer to these designs as BRM1 and BRM1E designs respectively. We show that these designs are imbedded in a much larger class of RM designs. This class is based on generalized partially balanced incomplete block designs and hence we refer to these designs as GPBRM1 designs.

The properties of GPBRM1 designs can be investigated by means of association matrices. For the construction of these designs we introduce the concept of asymmetrically repeated differences, a generalization of symmetrically repeated differences used for constructing certain PBIB designs.

Another generalization of RM designs discussed in this dissertation is with respect to the underlying linear model. In particular, we consider the situation where in addition to first order residual effects the model also contains second order residual effects. This leads to BRM2 and BRM2E designs. Extension to kth order residual effect models are mentioned briefly.

Modifications of existing RM designs can be achieved if the treatments have a factorial structure and if certain, usually higher order, interactions can be considered negligible. In particular, we discuss how this can lead to a substantial reduction in the number of periods for a RM design. This is of great practical interest.

The following topics may be of interest for future study:

Finding a method to construct generalized partially balanced repeated measurement designs under a kth order residual effect model and even a nonadditive model.

Having given the definitions of BRM1, BRM1E, BRM2 and BRM2E designs, we could extend these defintions to BRMk and BRMkE designs.

We have been concerned with the characterization and properties of RM designs in general. Of importance then is the actual construction of connected BRM1 or BRM1E designs and even BRM2 or BRM2E designs.

For a 2^n factorial with several negligible interactions, one may want to reduce the number of subjects instead of the number of periods or both the number of subjects and the number of periods.

We could investigate the same problem for a 3^n and even a p^n factorial treatments.

To reduce even more the number of subjects and/or periods, we may try to use fractional factorials.

Finally, extensions to asymmetrical factorial, complete or fractionated, are of great practical as well as theoretical importance.

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APPENDIX

A RM(t = 3, n = 3, p = 3) design is given as follows:

		subjects		
		1	2	3
	1	1	2	3
periods	2	2	3	1
	3	3	1	2

We show that this is a GPBRM1 design with $\lambda_1 = 1$. As before, we define $\lambda_{11:t}$, $\lambda_{22:t}$, $\lambda_{12:t}$ and $\mu_{12:t}$ as follows:

$\lambda_{11:t}$ is the number of times treatment i and treatment i' occur on the same subject, and they are the 11:t th associates;

$\lambda_{22:t}$ is the number of times residual effect i and residual effect i' occur on the same subject, and they are the 22:t th associates;

$\lambda_{12:t}$ is the number of times direct effect i and residual effect i' occur on the same subject;

$\lambda_{21:t}$ is the number of times residual effect i and direct effect i' occur on the same subject;

$\mu_{12:t}$ is the number of times that treatment i' precedes treatment i on the same subject;

$\mu_{21:t}$ is the number of times that treatment i precedes treatment i' on the same subject.

Then we have the following association scheme:

11:0	11:1
1	2,3
2	1,3
3	1,2

with

$$n_{11:0} = 1, \; n_{11:1} = 2$$

$$\lambda_{11:0} = 3, \; \lambda_{11:1} = 3$$

22:0	22:1
1	2,3
2	1,3
3	1,2

with

$$n_{22:0} = 1, \; n_{22:1} = 2$$

$$\lambda_{22:0} = 2, \; \lambda_{22:1} = 1$$

12:0	12:1	12:2
1	2	3
2	3	1
3	1	2

with

$$n_{12:0} = 1,\ n_{12:1} = 1,\ n_{12:2} = 1$$

$$\mu_{12:0} = 0,\ \mu_{12:1} = 0,\ \mu_{12:2} = 2$$

$$\lambda_{12:0} = 2,\ \lambda_{12:1} = 2,\ \lambda_{12:2} = 2$$

21:0	21:1	21:2
1	2	3
2	3	1
3	1	2

with

$$n_{21:0} = 1,\ n_{21:1} = 1,\ n_{21:2} = 1$$

$$\mu_{21:0} = 0,\ \mu_{21:1} = 2,\ \mu_{21:2} = 0$$

$$\lambda_{21:0} = 2,\ \lambda_{21:1} = 2,\ \lambda_{21:2} = 2$$

We see from the design given that $\lambda_1 = 1$, $p = 3$ and $t = 3$. To verify Lemma 5.1 we note that

$$\sum_{v=0}^{1} n_{11:v}\,\lambda_{11:v} = 9 = \lambda_1 p^2$$

$$\sum_{v=0}^{1} n_{22:v}\,\lambda_{22:v} = 4 = \lambda_1 (p-1)^2$$

$$\sum_{v=0}^{2} n_{12:v}\,\lambda_{12:v} = 6 = \lambda_1 p(p-1)$$

$$\sum_{v=0}^{2} n_{21:v}\,\lambda_{21:v} = 6 = \lambda_1 p(p-1)$$

$$\sum_{v=0}^{2} n_{12:v}\,\mu_{12:v} = 2 = \lambda_1 (p-1)$$

$$\sum_{v=0}^{2} n_{21:v}\,\mu_{21:v} = 2 = \lambda_1 (p-1)$$

$$\sum_{t=0}^{m_{ij}} n_{ij:t} = 3 = t \quad \text{for } i,\ j = 1,\ 2\ .$$

Now, for our example, the association matrices and the P matrices are as follows:

$$B_{11:0} = I_3$$

$$B_{11:1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$B_{11:0}^{2} = B_{11:0}$$

$$B_{11:0}\, B_{11:1} = B_{11:1}$$

$$B_{11:1}^{2} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = 2\, B_{11:0} + B_{11:1}$$

$$P_0^{11,11} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$P_1^{11,11} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad .$$

Similarly,

$$B_{22:0} = I_3$$

$$B_{22:1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and

$$B_{22:0}^2 = B_{22:0}$$

$$B_{22:0}\ B_{22:1} = B_{22:1}$$

$$B_{22:1}^2 = 2\ B_{22:0} + B_{22:1}$$

$$P_0^{22,22} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$P_1^{22,22} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} .$$

Finally,

$$B_{12:0} = I_3$$

$$B_{12:1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$B_{12:2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$B_{11:0}\ B_{12:0} = B_{12:0}$$

$$B_{11:0}\, B_{12:1} = B_{12:1}$$

$$B_{11:0}\, B_{12:2} = B_{12:2}$$

$$B_{11:1}\, B_{12:0} = B_{12:1} + B_{12:2}$$

$$B_{11:1}\, B_{12:1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = B_{12:0} + B_{12:2}$$

$$B_{11:1}\, B_{12:2} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = B_{12:0} + B_{12:1}$$

and

$$P_0^{11,12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$P_1^{11,12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$P_2^{11,12} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Also,

$$P_0^{22,21} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$P_1^{22,21} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$P_2^{22,21} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} .$$

For ease of computation it is convenient to artificially introduce the same association scheme for the 11, 22, 12 associate classes, i.e.,

$$\lambda'_{ij} \equiv (\lambda_{ij:0}, \lambda_{ij:1}, \dots, \lambda_{ij:t}) ,$$

$$n^{*\prime}_{ij} \equiv (n^*_{ij:0}, n^*_{ij:1}, \dots, n^*_{ij:t})$$

we have

$$\lambda'_{11} = (3 \quad 3 \quad 3)$$

$$\mu'_{12} = (0 \quad 0 \quad 2)$$

$$\mu'_{21} = (0 \quad 2 \quad 0)$$

$$\lambda'_{22} = (2 \quad 1 \quad 1)$$

and

$$n^{*'}_{11} = (1 \quad 1 \quad 1)$$

$$n^{*'}_{22} = (1 \quad 1 \quad 1)$$

$$n^{*'}_{12} = (1 \quad 1 \quad 1)$$

$$n^{*'}_{21} = (1 \quad 1 \quad 1)$$

$p = 3$, $\lambda_1 = 1$.

$$\therefore \quad \lambda^{*}_{11:0} = 2$$

$$\lambda^{*}_{11:1} = -1 \qquad\qquad \lambda^{*'}_{11} = (2 \quad -1 \quad -1)$$

$$\lambda^{*}_{11:2} = -1$$

$$\lambda^{*}_{22:0} = \frac{10}{9}$$

$$\lambda^{*}_{22:1} = -\frac{5}{9} \qquad\qquad \lambda^{*'}_{22} = \left(\frac{10}{9} \quad -\frac{5}{9} \quad -\frac{5}{9}\right)$$

$$\lambda^{*}_{22:2} = -\frac{5}{9}$$

$$\lambda^{*}_{12:0} = -\frac{2}{3}$$

$$\lambda^{*}_{12:1} = -\frac{2}{3} \qquad\qquad \lambda^{*'}_{12} = \left(-\frac{2}{3} \quad -\frac{2}{3} \quad \frac{4}{3}\right)$$

$$\lambda^{*}_{12:2} = \frac{4}{3} \qquad .$$

Similarly, $\lambda^{*'}_{21} = \left(-\frac{2}{3} \quad \frac{4}{3} \quad -\frac{2}{3}\right)$. Therefore,

$$C^{*} = \begin{pmatrix} 2\,B_{12:0} - B_{12:1} - B_{12:2} & -\frac{2}{3}\,B_{12:0} - \frac{2}{3}\,B_{12:1} + \frac{4}{3}\,B_{12:2} \\ -\frac{2}{3}\,B_{12:0} + \frac{4}{3}\,B_{12:1} - \frac{2}{3}\,B_{12:2} & \frac{10}{9}\,B_{12\cdot 0} - \frac{5}{9}\,B_{12:1} - \frac{5}{9}\,B_{12:2} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & -1 & -0.67 & -0.67 & 1.33 \\ -1 & 2 & -1 & 1.33 & -0.67 & -0.67 \\ -1 & -1 & 2 & -0.67 & 1.33 & -0.67 \\ -0.67 & 1.33 & -0.67 & 1.11 & -0.56 & -0.56 \\ -0.67 & -0.67 & 1.33 & -0.56 & 1.11 & -0.56 \\ 1.33 & -0.67 & -0.67 & -0.56 & -0.56 & 1.11 \end{pmatrix}$$

and rank(C*) = 4 = 2t - 2.

$\therefore$ the design is connected.

To find the estimators for $\underline{\tau}$ and $\underline{\alpha}$, since $\lambda'_{11}\ P_0^{11,11}\ d_{11} = \frac{2}{3}$ and $\lambda'_{11}\ P_1^{11,11}\ d_{11} = -\frac{1}{3}$ with constraint $n'_{11}\ d_{11} = 0$. As a result, we have

$$(2 \quad -1)\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} d_{11:0} \\ d_{11:1} \end{pmatrix} = \frac{2}{3}$$

$$(2 \quad -1)\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} d_{11:0} \\ d_{11:1} \end{pmatrix} = -\frac{1}{3}$$

with

$$(1 \quad 2)\begin{pmatrix} d_{11:0} \\ d_{11:1} \end{pmatrix} = 0$$

$$\therefore\ d'_{11} = (\tfrac{2}{9} \quad -\tfrac{1}{9}) \ .$$

Since

$$d'_{11}\ P_0^{11,12}\ \lambda_{12} = -\frac{2}{9}$$

$$d'_{11}\ P_1^{11,12}\ \lambda_{12} = -\frac{2}{9}$$

$$d'_{11}\ P_2^{11,12}\ \lambda_{12} = \frac{4}{9} \ ,$$

hence,

$$e' \equiv (d'_{11}\ P^{11,12}\ \lambda^*_{12})'$$

$$= (-\tfrac{2}{9} \quad -\tfrac{2}{9} \quad \tfrac{4}{9}) \ .$$

Also since

$$\lambda'_{21}\ P_0^{21,12}\ e = \frac{8}{9}$$

$$\lambda'_{21}\ P_1^{21,12}\ e = -\ \frac{4}{9}$$

$$\lambda'_{21}\ P_2^{21,12}\ e = -\ \frac{4}{9} \ .$$

hence

$$f' \equiv (\lambda'_{21}\ P^{21,12}\ e)'$$

$$= (\tfrac{8}{9} \quad -\tfrac{4}{9} \quad -\tfrac{4}{9}) \ .$$

We know

$$\hat{\alpha}^{\lambda^{*'}} \equiv (\lambda^*_{22} - f)'$$

$$= (\tfrac{2}{9} \quad -\tfrac{1}{9} \quad -\tfrac{1}{9}) \ .$$

Therefore,

$$B_\alpha = \frac{2}{9}\ B_{12:0} - \frac{1}{9}\ B_{12:1} - \frac{1}{9}\ B_{12:2}$$

$$= \begin{pmatrix} \frac{2}{9} & -\frac{1}{9} & -\frac{1}{9} \\ -\frac{1}{9} & \frac{2}{9} & -\frac{1}{9} \\ -\frac{1}{9} & -\frac{1}{9} & \frac{2}{9} \end{pmatrix}$$

$\mathrm{rank}(B_{\hat{\alpha}}) = 2 = t - 1$.

On the other hand,

$$\left(\frac{2}{9} B_{22:0} - \frac{1}{9} B_{22:1}\right)\hat{\alpha} = Q_\alpha$$

and

$$\hat{\alpha} = A_\alpha Q_\alpha$$

where

$$A_\alpha = \sum_{t=0}^{1} a_{22:t} B_{22:t}$$

and

$$\begin{cases} \hat{\alpha}^{\lambda'_s} P_0^{22,22} a = \frac{2}{3} \\ \hat{\alpha}^{\lambda'_s} P^{22,22} a = -\frac{1}{3} \end{cases} \qquad \text{constraint } n'_{22} a = 0 \quad .$$

Therefore,

$$\left(\frac{2}{9} \quad -\frac{1}{9}\right)\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a_{22:0} \\ a_{22:1} \end{pmatrix} = \frac{2}{3}$$

$$\left(\frac{2}{9} \quad -\frac{1}{9}\right)\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a_{22:0} \\ a_{22:1} \end{pmatrix} = -\frac{1}{3}$$

with

$$(1 \quad 2)\begin{pmatrix} a_{22:0} \\ a_{22:1} \end{pmatrix} = 0$$

$\therefore\ a_{22:1} = -1,\ a_{22:0} = 2$

$\therefore\ a' = (2 \quad -1)\,.$

Also

$$A_\alpha = 2\,B_{22:0} - B_{22:1}$$

$$= \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} .$$

Similarly,

$$\left\{\begin{array}{l} \lambda_{22}'\ P_0^{22,22}\ b_{22} = \frac{2}{3} \\ \\ \lambda_{22}'\ P_1^{22,22}\ b_{22} = -\frac{1}{3} \end{array}\right. \qquad \text{constraint } n_{22}'\ b_{22} = 0$$

$$(\tfrac{10}{9} \quad -\tfrac{5}{9}) \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} b_{22:0} \\ b_{22:1} \end{pmatrix} = \frac{2}{3}$$

$$(\tfrac{10}{9} \quad -\tfrac{5}{9}) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} b_{22:0} \\ b_{22:1} \end{pmatrix} = -\frac{1}{3}$$

with

$$(1 \quad 2) \begin{pmatrix} b_{22:0} \\ b_{22:1} \end{pmatrix} = 0$$

$$\therefore\ b'_{22} = (\tfrac{2}{5} \quad -\tfrac{1}{5}) .$$

Since

$$b'_{22}\ P_0^{22,21}\ \lambda_{21} = -\frac{2}{5}$$

$$b'_{22}\ P_1^{22,21}\ \lambda_{21} = \frac{4}{5}$$

$$b'_{22}\ P_2^{22,21}\ \lambda_{21} = -\frac{2}{5} ,$$

hence,

$$h' \equiv (b'_{22}\ P^{22,21}\ \lambda_{21})' = (-\tfrac{2}{5} \quad \tfrac{4}{5} \quad -\tfrac{2}{5}) .$$

Also since

$$\lambda'_{12}\ P_0^{12,21}\ h = \frac{8}{5}$$

$$\lambda'_{12}\ P_1^{12,21}\ h = -\frac{4}{5}$$

$$\lambda'_{12}\ P_2^{12,21}\ h = -\frac{4}{5} ,$$

we have

$$\begin{aligned}\hat{\tau}^{\lambda*} &= \lambda_{11}^{*'} - (\lambda'_{12}\ P^{12,21}\ h)' \\ &= (\tfrac{2}{5} \quad -\tfrac{1}{5} \quad -\tfrac{1}{5}) .\end{aligned}$$

Therefore,

$$B_{\hat{\tau}} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

$\mathrm{rank}(B_\tau) = 2 = 5 - 1.$

Since

$$\begin{cases} \hat{\tau}^{\lambda'} \; P_0^{11,11} \; g = \frac{2}{3} \\ \hat{\tau}^{\lambda} \; P_1^{11,11} \; g = -\frac{1}{3} \end{cases} \qquad \text{with } n'_{11} \; g = 0 \ ,$$

hence

$$(\tfrac{2}{5} \quad -\tfrac{1}{5}) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} g_{11:0} \\ g_{11:1} \end{pmatrix} = -\frac{1}{3}$$

with

$$(1 \quad 2) \begin{pmatrix} g_{11:0} \\ g_{11:1} \end{pmatrix} = 0$$

i.e.,

$$g_{11:0} = \frac{10}{9}, \ g_{11:1} = -\frac{5}{9} \ .$$

Therefore

$$\hat{\tau} = A_\tau \; Q_\tau$$

where

$$A_\tau = \begin{pmatrix} \frac{10}{9} & -\frac{5}{9} & -\frac{5}{9} \\ -\frac{5}{9} & \frac{10}{9} & -\frac{5}{9} \\ -\frac{5}{9} & -\frac{5}{9} & \frac{10}{9} \end{pmatrix} .$$

To calculate the average variances of $\underline{\alpha}$ and $\underline{\tau}$, since

$$\hat{\alpha} = A_\alpha Q_\alpha$$

$$\mathrm{Var}(\hat{\alpha}_i - \hat{\alpha}_i') = 2(a_{22:0} - a_{22:t})\sigma^2$$

where

$$a_{22:0} = 2, \; a_{22:1} = -1 ,$$

hence,

$$\mathrm{Var}(\hat{\alpha}_i - \hat{\alpha}_i') = 6\sigma^2$$

if i, i' are 22:1 associates. Therefore,

$$\overline{\mathrm{Var}}(\hat{\alpha}_i - \hat{\alpha}_i') = \frac{2\sigma^2 t}{t-1} a_0 = \frac{2\sigma^2(3)}{2} (2) = 6\sigma^2 .$$

Since

$$g_{11:i} = d_{11:i} + \ell' \, P_i^{21,12} \, w_{21}$$

$$d_{11'} = (\tfrac{2}{9} \quad - \tfrac{1}{9})$$

$$\ell' = (-\tfrac{2}{3} \quad -\tfrac{2}{3} \quad \tfrac{4}{3})$$

$$e' = (-\tfrac{2}{9} \quad -\tfrac{2}{9} \quad \tfrac{4}{9})$$

and

$$w_{21:0} = \lambda_{21}' \, P_0^{21,11} \, d_{11} = -\tfrac{2}{9}$$

$$w_{21:1} = \lambda_{21}' \, P_1^{21,11} \, d_{11} = \tfrac{4}{9}$$

$$w_{21:2} = \lambda_{21}' \, P_2^{21,11} \, d_{11} = -\tfrac{2}{9}$$

so

$$w_{21}' = (-\tfrac{2}{9} \quad \tfrac{4}{9} \quad -\tfrac{2}{9})$$

and

$$\ell' \, P_0^{21,12} \, w_{21} = \frac{8}{9}$$

$$\ell' \, P_1^{21,12} \, w_{21} = -\frac{4}{9}$$

$$\ell' \, P_2^{21,12} \, w_{21} = -\frac{4}{9}$$

$$g_{11}' = d_{11}' + \ell' \, P^{21,12} \, w_{21} \quad .$$

Therefore,

$$g_{11}' = (\frac{10}{9} \quad -\frac{5}{9} \quad -\frac{5}{9})$$

and

$$\mathrm{Var}(\hat{\tau}_i - \hat{\tau}_i') = 2(\frac{10}{9} + \frac{5}{9})\sigma^2$$

$$= \frac{10}{3}\,\sigma^2$$

if i, i' are 11:1 associates, and then

$$\mathrm{Var}(\hat{\tau}_i - \hat{\tau}_i') = \frac{2\sigma^2 t \; g_{11:0}}{t-1} = \frac{10}{3}\,\sigma^2 \quad .$$

We have the same results as those from A_τ directly.

The vita has been removed from the scanned document