MULTIVARIATE SEQUENTIAL PROCEDURES FOR TESTING MEANS

by

James Edward Jackson A.B., M.A.

Thesis submitted to the Graduate Faculty of the
Virginia Polytechnic Institute
in candidacy for the degree of

DOCTOR OF PHILOSOPHY
in
STATISTICS

August, 1959
Blacksburg, Virginia

TABLE OF CONTENTS

Chapter		Page
I.	INTRODUCTION	1
II.	REVIEW OF THE LITERATURE	4
	2.1 Sequential analysis	4
	2.2 Distribution of means from a multivariate normal distribution	9
	2.3 Generalized hypergeometric functions	11
III.	PROCEDURES FOR THE SEQUENTIAL χ^2- AND T^2-TEST	14
	3.1 Sequential χ^2-test	14
	3.1.1 Introduction	14
	3.1.2 Test procedure	15
	3.1.3 The case where $p = 1$	17
	3.1.4 The case where $\lambda^2 \neq 0$ under H_0	18
	3.2 Sequential T^2-test	18
	3.2.1 Introduction	18
	3.2.2 Test procedure	19
	3.2.3 The case where $p = 1$	21
	3.2.4 The case where $\lambda^2 \neq 0$ under H_0	21
	3.3 The two-sample case	22
IV.	EXISTENCE THEOREMS	24
	4.1 A theorem of Cox and the problem of composite hypotheses	24
	4.2 Fulfillment of the conditions of Cox's Theorem	28

Chapter Page

4.2.1 The sequential χ^2-test 28

4.2.2 The sequential T^2-test 35

4.3 Termination proofs. 40

4.3.1 Proof that the sequential χ^2-test terminates with probability one. . . 40

4.3.2 Proof that the sequential T^2-test terminates with probability one. . . 45

4.3.3 Behavior of the bounds for the sequential χ^2- and T^2-tests. 46

V. PROPERTIES AND PROBLEMS OF SEQUENTIAL TESTINGS . 50

5.1 Introductory statements 50

5.2 The ASN function. 50

5.3 Generalized χ^2- and T^2-statistics 56

5.4 The problem of tolerances 62

VI. COMPUTATIONAL PROCEDURES 67

6.1 Introduction. 67

6.2 Inadequacy of the methods of interpolation and iteration 68

6.3 The Newton-Raphson method 70

6.4 Use of tables to facilitate sequential χ^2- and T^2-tests. 72

VII. NUMERICAL EXAMPLE. 76

7.1 Introduction. 76

Chapter | Page

	7.2 Case I: Σ unknown.	78
	7.3 Case II: Σ known	84
VIII.	SUMMARY.	88
IX.	ACKNOWLEDGMENTS.	92
X.	BIBLIOGRAPHY	93
XI.	APPENDICES	98
	Appendix A Auxiliary proofs for Section 4.3.1 .	98
	Appendix B Tables to facilitate the sequential χ^2-test.	101
	Appendix C Tables to facilitate the sequential T^2-test.	117
	Appendix D A short table of the hypergeometric function ${}_0F_1(c;x)$.	133
XII.	VITA	139

I. INTRODUCTION

The motivation for this project stems from the acceptance sampling programs used in the evaluation of production lots of ballistic missiles such as the Honest John and the Nike. These particular missiles are operational and have been on a production basis for some time. They are currently produced in lots and subjected to the ordiary acceptance sampling schemes used in quality control. These sampling systems will be discussed in more detail in Chapter VII but it will suffice now to say that the manufacture and testing of these missiles is very expensive. Consequently, judgments on lots should be made with as little inspection as possible consistent with the prescribed risks of accepting poor lots and rejecting good ones. This suggests <u>sequential</u> sampling which has come into widespread use in the past few years and, in fact, some types of missiles are now inspected in that manner.

There are several important parameters to be inspected on each round in missile production; a few such characteristics are action time, thrust or impulse, and some measure of chamber pressure. These variables are inter-related and hence the problem is a <u>multivariate</u> one. In present day operations, separate sequential plans must be set up for each parameter. It is, therefore, possible to get conflicting answers about the quality of a lot, sampling may terminate for one characteristic before another and there is no appreciation of the true sampling risks involved in the overall program. It is obvious that a sequential multivariate technique should be used. We have discussed our problem in relation to the field of ballistic missiles but the same problem exists any place where sequential sampling is required and more than one parameter is under consideration.

The purpose of this project, then, is to derive some multivariate sequential inspection schemes for the characteristic averages both for the case where the population variance-covariance matrix (hereafter referred to as the covariance matrix) is known or assumed to be known (a typical quality control situation) and where it must be estimated from the sample. When the covariance matrix is known, we develop a sequential χ^2-test; when the covariance matrix is estimated from the sample, we develop a sequential T^2-test.

A review of the pertinent literature concerning sequential analysis, the χ^2- and T^2-distributions, and some hypergeometric functions that will be required is given in Chapter II. One of these functions, the hypergeometric function ${}_0F_1(c;x)$ is tabulated in Appendix D. The sequential X^2- and T^2-tests are derived in Chapter III and the related mathematical proofs dealing with the existence and termination of these procedures is given in Chapter IV and Appendix A. Chapter V contains some general discussions on topics associated with applications of our methods such as the expected sample size required for our procedures, the extension of these procedures to some other multivariate situations, and the problem of relating tolerances to the specifications of the procedures.

Some tables are required to facilitate use of our sequential tests. The tables are given in Appendices B and C and a discussion of their construction is given in Chapter VI. Chapter VII contains a numerical example taken from the field of ballistic missiles. Chapter VIII is the conclusion and also contains mention of a number of problems related to the multivariate sequential testing of means which are as yet no completely solved and which may suggest directions for further research.

II REVIEW OF THE LITERATURE

2.1 Sequential analysis.

The notion of sequential analysis goes back at least as far as DeMoivre (Fieller, 1931) but, in the modern sense, it has its origin during World War II with Wald and the Statistical Research Group at Columbia University (Wald, 1943; Freeman, 1944) and simultaneously in England with such people as Stockman (1944) and Barnard (1946) working with the British Ministry of Supply. A historical account of the development of this technique is given in Wald's book Sequential Analysis (Wald, 1947) which, to date, is the only book devoted exclusively to this subject. In both countries, the immediate requirements of this technique were the same: to cut down on the amount of inspection necessary in the acceptance sampling of military supplies without losing the protection afforded by the standard acceptance sampling techniques. The general philosophy motivating this research was that, if a production lot was of extremely good or extremely poor quality, one should be able to diagnose this more readily than if the lot contained borderline material. In this way very good or very bad lots could be accepted or rejected on the basis of very limited sampling while more extensive sampling could be used where most necessary on lots of borderline quality.

Some work had previously been done along this line by Dodge and Romig (1944) with their double sampling techniques. Although these plans allegedly gave the lot "another chance", in actuality they were designed to cut down on the amount of inspection required for good or poor lots by allowing a decision after a sub-set of the total sample had been inspected. Dodge and Romig also considered the multiple sampling problem in 1927 although their work was never very well publicized and it was not until Bartky's work (1943) was published that any practical system of multiple sampling was available to quality control technicians. This was contemporary with the work of Wald and, since then, very little work has been done on formal multiple sampling plans. The reason for this is that sequential techniques are essentially item-by-item sampling techniques. These are, comparatively speaking, quite easy to derive and can be modified in many cases to yield approximations to multiple sampling schemes. A great deal has been done along this line for the establishment of multiple sampling plans for proportion defective (Freeman et al. 1948, Jackson 1948).

In Wald's sequential analysis procedures, use is made of the <u>probability ratio test</u>. In this procedure, both the null and alternative hypotheses are stated more or less specifically and risks α and β are attached to errors associated with incorrect acceptance-rejection decisions. For example, in a one-parameter situation with parameter θ,

H_0: $\theta = \theta_0$

H_1: $\theta = \theta_1$ (rather than $\theta > \theta_0$, say).

This implies that lots with $\theta \leq \theta_0$ are acceptable, lots with $\theta \geq \theta_1$ are unacceptable, and acceptance is a matter of indifference if $\theta_0 < \theta < \theta_1$. The probability ratio is formed by taking the likelihood function $L(x; \theta)$ with the region in the parameter space restricted as specified by H_1 and dividing this by the same function with the parameter space restricted as specified by H_0. If the probability ratio based on n observations is denoted by p_{1n}/p_{0n}, then

$p_{1n}/p_{0n} = L(x;\theta_1)/L(x;\theta_0)$. The sequential procedure is as follows: After each observation form p_{1n}/p_{0n};

a. If $p_{1n}/p_{0n} \leq \beta/(1-\alpha)$, accept H_0;

b. If $p_{1n}/p_{0n} \geq (1-\beta)/\alpha$, accept H_1;

c. If $\beta/(1-\alpha) < p_{1n}/p_{0n} < (1-\beta)/\alpha$, continue sampling.

Provided the sequential process terminates, the risks of accepting $H_1(\theta \geq \theta_1)$ when H_0 is true and of accepting $H_0(\theta \leq \theta_0)$ when H_1 is true are approximately equal to α and β respectively. Experience has shown that, except for borderline cases, the sequential probability ratio test will require considerably less sampling on the average than corresponding fixed-sample procedures.

Most of the examples that Wald dealt with were sequential tests of simple hypotheses. If the hypotheses were composite, Wald showed that the probability ratio test could be derived using the method of weight functions. This method can prove to be quite cumbersone in addition to the fact that no method is known of constructing weight functions that is optimum in the sense that the risks of accepting the wrong hypothesis are less than or equal to α and β respectively. With the exception of some work done by Hoel (1954, 1955), very little use has been made of this method in the last ten years and at least one of the earlier results, Wald's sequential t-test, has since been shown to be slightly in error.

The reason for the lack of enthusiasm for the method of weight functions is the fact that an alternative method, the method of frequency functions has become much more popular and has, after several years of conjecture, received sound mathematical backing. This method was first proposed by Goldberg (Wallis, 1947) in making a sequential test of the proportion defective of a normal population using the sample mean and variance. Since the fixed-sample solution to this problem involves the non-central t-distribution under H_1, Goldberg conjectured that the probability ratio would merely be the ratio of the non-central t-distribution under H_1 to the non-central t-distribution under H_0. Nandi (1948) proposed this solution to a whole class of tests of composite hypotheses and stated the conditions under which this could be done but no proofs were given. Later Goldberg's probability ratio was used for the sequential t-test itself, the work being carried on independently by Rushton (1950, 1952) and Arnold (Nat. Bur. Stds., 1951). Finally Barnard (1952, 1953) and Cox (1952), independently established the conditions under which the frequency function of a test statistic might be used in a sequential probability ratio test and still guarantee approximately the risks α and β.

Wald showed that the sequential probability ratio test would terminate with probability unity and also established approximate formulae for construction the OC (Operating Characteristic) and ASN (Average Sample Number or expected sample size) functions. However, his proofs and derivations assumed that the observations were independent. In the frequency function method of testing composite hypotheses, the "observations" are now the successive values of the test-statistic and hence the observations may no longer be independent. When this is the case, Wald's work regarding termination, OC functions, or ASN functions no longer holds. Termination proofs must be carried out for each test and there is no known method for constructing the OC or ASN function although there are some conjectural approaches to the latter (Ray, 1956).

The method of frequency functions is used in deriving the sequential tests discussed in Chapter III. The existence and termination proofs associated with these tests are given in Chapter IV.

Of historical interest, it should be mentioned that Rao (1950) has proposed a sequential procedure for testing the hypotheses:

H_0: $\theta = \theta_0$

H_1: $\theta > \theta_0$, say,

but except for the most simple cases, the operations involved appear to be too difficult to handle under present conditions.

Jackson (1959) has compiled a very complete bibliography of all work done in the field of sequential analysis through May 1959.

2.2 Distributions of means from a multivariate normal distribution.

The sequential tests discussed deal with the distributions of the means from a multivariate normal population for both the case where the population covariance matrix in known and for the case where it is not known and must be estimated from the sample. In the former case, the χ^2-distribution is used; in the latter case, the T^2-distribution.

The central χ^2-distribution was derived by Helmert (1875). The non-central χ^2-distribution was derived by Fisher (1928) when he also derived the non-central F-distribution and the non-central distribution of the correlation coefficient. The central T^2-distribution was derived by Hotelling (1931) in which he showed that T^2 itself had an F-distribution. Similarly, Hsu (1938) and Bose and Roy (1938) independently showed that the non-central T^2 had a non-central F-distribution.

With the exception of Tang's tables (1938) of the non-central F-distribution, very little work was done with either the non-central χ^2- or F-distributions for 20 years after their discovery. However, Patnaik (1949) investigated many properties of these two distributions and showed that both distributions could be approximated by their respective central distributions with modified arguments and degrees of freedom. Several people have since prepared various tables and nomographs for the non-central F; among them Evelyn Fix (1949), Pearson and Hartley (1951) and Fox (1956).

2.3 Generalized hypergeometric functions.

The family of generalized hypergeometric functions (Erdelyi, et al. 1953) ${}_pF_q(a_1, a_2, \ldots a_p;\ c_1,\ c_2, \ldots c_q;\ x)$ are solutions of the generalized differential equation:

$$\{\delta(\delta+c_1-1)(\delta+c_2-1)\ldots(\delta+c_q-1) -x(\delta+a_1)(\delta+a_2)\ldots(\delta+a_p)\}\,u = 0, \tag{2.3.1}$$

where δ denotes the operator $\delta = x\frac{d}{dx}$, $a_1, \ldots, a_p$ and $c_1, \ldots, c_q$ are constants, x is the independent variable and u is the dependent variable. These functions are generally used in solving physical problems. The most familiar of these functions is:

$$ {}_2F_1(a_1, a_2, c; x) = 1 + \frac{a_1 a_2 x}{c} + \frac{a_1(a_1+1)a_2(a_2+1)x^2}{c(c+1)\ 2!} + \frac{a_1(a_1+1)(a_1+2)a_2(a_2+1)(a_2+2)x^3}{c(c+1)(c+2)\ 3!} + \ldots \tag{2.3.2}$$

which, in addition to being a solution of the differential equation:

$$x(1-x)\frac{d^2u}{dx^2} + [c-(a_1+a_2+1)x]\frac{du}{dx} - a_1 a_2 u = 0, \tag{2.3.3}$$

is also involved in the non-central distribution of the simple and multiple correlation coefficients. This is sometimes known as Gauss's series. Second of importance is the Confluent Hypergeometric function:

$$(2.3.4)\qquad {}_1F_1(a,c;x) = 1 + \frac{ax}{c} + \frac{a(a+1)x^2}{c(c+1)2!} + \frac{a(a+1)(a+2)x^3}{c(c+1)(c+2)3!} + \cdots$$

when, in addition to being a solution of the differential equation:

$$(2.3.5)\qquad x\frac{d^2u}{dx^2} + (c-x)\frac{du}{dx} - au = 0,$$

is also involved in the non-central F-distribution. This is sometimes called Kummer's series and is closely related to Whittaker's M-function.

Less prominent so far is the function:

$$(2.3.6)\qquad {}_0F_1(c;x) = 1 + \frac{x}{c} + \frac{x^2}{c(c+1)2!} + \frac{x^3}{c(c+1)(c+2)3!} + \cdots$$

which, in addition to being a solution of the differential equation:

$$(2.3.7)\qquad x\frac{d^2u}{dx^2} + c\frac{du}{dx} - u = 0,$$

is also involved in the non-central χ^2-distribution. As stated in Section 2.2, the derivation of all three of these non-central distributions was first shown in a single paper by Fisher, although, in that case, all of the work was in terms of correlation coefficients and the Kummer ${}_pF_q$ notation was not generally employed (nor has it been generally employed off the European continent until the last decade). The generalized hypergeometric series will converge for all finite x if $p \leq q$ which is satisfied by the non-central

F- and X^2-distribution and for $|x| < 1$ if $p = q + 1$ which is satisfied for the non-central distribution of the correlation coefficients. These series diverge for $X \neq 0$ if $p > q + 1$.

This thesis will make extensive use of the functions ${}_0F_1(c;x)$ and ${}_1F_1(a,c;x)$. The latter function has been discussed in great detail by Tricomi (Erdelyi et al.,1953; Tricomi,1954) and Buchholz (1953). There are several tables of ${}_1F_1(a,c;x)$ available but the most extensive are those of Nath(1951) and of Rushton and Lang (Rushton 1954, Rushton and Lang 1954) who have tabulated the function for $c = .5(.5)4.5$, a ranging from .5 to 49.5 and x ranging from .2 to 200. Another table has been prepared by the National Bureau of Standards (1949). Considerable work has been done on the asymptotic behavior of ${}_1F_1(a,c;x)$ and, although most of this has been covered in the books by Tricomi and Buchholz, the pioneering work of Perron (1921) and a recent publication by Erdelyi and Swanson (1957) should also be mentioned.

Less is known about the function ${}_0F_1(c;x)$. However, as shown in Appendix D, where this function is tabulated, ${}_0F_1(c;x)$ can be transformed into a Bessel function, a Whittaker function, or a confluent hypergeometric function and the behavior of these functions is well known.

III. PROCEDURES FOR THE SEQUENTIAL χ^2-TEST AND T^2-TEST

3.1 Sequential χ^2-test.

3.1.1 Introduction

Consider a p-variate population which has a multivariate normal distribution with unknown means $\mu_1, \ldots, \mu_p$ and known population covariance matrix Σ. It is required to test that the true means of the p variates, $\underline{\mu}$, are equal to some hypothetical or standard values $\underline{\mu}_0$. However, rather than test the hypothesis $\underline{\mu} = \underline{\mu}_0$ against some alternative $\underline{\mu} = \underline{\mu}_1$, we will work with the quadratic form $(\underline{\mu}-\underline{\mu}_0)\Sigma^{-1}(\underline{\mu}-\underline{\mu}_0)'$. The hypotheses become:

$$H_0: \quad (\underline{\mu} - \underline{\mu}_0)\ \Sigma^{-1}\ (\underline{\mu} - \underline{\mu}_0)' = 0,$$

$$H_1: \quad (\underline{\mu} - \underline{\mu}_0)\ \Sigma^{-1}\ (\underline{\mu} - \underline{\mu}_0)' = \lambda^2.$$

H_0 is equivalent to the hypothesis that $\underline{\mu} = \underline{\mu}_0$ but H_1 now represents the surface of a p-dimensional ellipsoid rather than a single designated point. For most hypotheses that are to be tested sequentially, the procedure generally followed is to reduce the problem, essentially, to a univariate problem involving simple hypotheses as will be discussed in Chapter IV.

In the present problem, the p-dimensional situation involving $\underline{\mu}$ and $\underline{\mu}_0$ is reduced to a univariate situation involving λ^2.

3.1.2 Test Procedure

As will be shown in Chapter IV, the probability ratio for n observations is the ration of the non-central χ^2-distribution with p degrees of freedom and non-centrality parameter $n\lambda^2$ to the central χ^2-distribution with p degrees of freedom. Thus:

$$\frac{p_{1n}}{p_{0n}} = \frac{\dfrac{e^{-\frac{1}{2}(\chi_n^2+n\lambda^2)}}{2^{\frac{p}{2}}} \sum\limits_{j=0}^{\infty} \dfrac{(\chi_n^2)^{\frac{p}{2}+j-1}\,(n\lambda^2)^j}{\Gamma(\frac{p}{2}+j)2^{2j}\;j!}}{\dfrac{(\chi_n^2)^{\frac{p}{2}-1}\,e^{-\frac{\chi_n^2}{2}}}{2^{\frac{p}{2}}\Gamma(\frac{p}{2})}}$$

$$= \Gamma(\tfrac{p}{2})\,e^{\frac{-n\lambda^2}{2}} \sum_{j=0}^{\infty} \frac{\left(\frac{n\lambda^2\chi_n^2}{4}\right)^j}{\Gamma(\frac{p}{2}+j)\,j!}$$

$$= e^{\frac{-n\lambda^2}{2}}\left[1+\left(\frac{n\lambda^2\chi_n^2}{4}\right)/(\tfrac{p}{2}) + \left(\frac{n\lambda^2\chi_n^2}{4}\right)^2/(\tfrac{p}{2})(\tfrac{p}{2}+1)2! + \ldots\right]$$

$$(3.1.2.1) \quad = e^{\frac{-n\lambda^2}{2}}\;{}_0F_1(p/2,\; n\lambda^2\chi_n^2/4)$$

This in turn can be written (Erdelyi et al., 1953) as:

$$(3.1.2.2) \quad \frac{p_{1n}}{p_{0n}} = e^{\frac{-n\lambda^2}{2}}\Gamma(\tfrac{p}{2})\left(\frac{n\lambda^2\chi_n^2}{4}\right)^{(\frac{2-p}{4})} I_{(\frac{p-2}{2})}\left(\sqrt{n\lambda^2\chi_n^2}\right)$$

$$(3.1.2.3) = e^{\frac{-n\lambda^2}{2} - \sqrt{n\lambda^2\chi_n^2}} {}_1F_1[\tfrac{(p-1)}{2}, p-1; 2\sqrt{n\lambda^2\chi_n^2}]$$

$$(3.1.2.4) = e^{\frac{-n\lambda^2}{2}} (2\sqrt{n\lambda^2\chi_n^2})^{\frac{-(p-1)}{2}} M_{0,\frac{(p-2)}{2}} (2\sqrt{n\lambda^2\chi_n^2})$$

where $I_\nu(x)$ is a modified Bessel function of the first kind, ${}_1F_1(a,c;x)$ is a confluent hypergeometric function and $M_{\mu,k}(x)$ is a Whittaker function. In actual use of the sequential χ^2-test, form (3.1.2.1) is used but in the study of the properties of the procedure, form (3.1.2.3) is sometimes more convenient since the sequential T^2-test may also be expressed in terms of the confluent hypergeometric function.

The sequential χ^2-test procedure is the following: take a sequence of n vector observations $(n = 1,2,\dots)$, compute ${}_n\bar{\underline{x}}$, the vector of sample means based on n observations, compute $\chi_n^2 = n({}_n\bar{\underline{x}} - \underline{\mu}_0)\Sigma^{-1}({}_n\bar{\underline{x}} - \underline{\mu}_0)'$ and evaluate (3.1.2.1).

a. If $p_{1n}/p_{0n} \leq \beta/(1-\alpha)$, accept H_0;

b. If $p_{1n}/p_{0n} \geq (1-\beta)/\beta$, accept H_1;

c. If $\beta/(1-\alpha) < p_{1n}/p_{0n} < (1-\beta)/\alpha$, take another vector observation and repeat the procedure. As long as p_{1n}/p_{0n} remains between $\beta/(1-\alpha)$ and $(1-\beta)\alpha$, continue sampling but as soon as it overruns either limit, take the appropriate action. The limits, since they involve only α and β, are constant for the entire procedure. However, the computation of p_{1n}/p_{0n} is quite tedious apart from the computation of χ^2_n itself. Furthermore, the use of the ${}_0F_1(c;x)$ tables would involve considerable Langrangian interpolation. Putting the probability ratio into the form involving either the confluent hypergeometric function or the Bessel function would only add to the difficulty since χ^2_n would appear twice in the expression for p_{1n}/p_{0n}. It is better to prepare tables, similar to those of the National Bureau of Standards (1951) for the univariate t-test in which the equations $p_{1n}/p_{0n} = \beta/(1-\alpha)$ and $p_{1n}/p_{0n} = (1-\beta)/\alpha$ with α, β, λ^2, and n given, are solved for boundary values of χ^2_n. This is done in Chapter VI.

3.1.3 The case where p = 1

For the case where p = 1,

$$\frac{p_{1n}}{p_{0n}} = e^{\frac{-n\lambda^2}{2}} [1 + (\frac{n\lambda^2\chi^2_n}{4})/\tfrac{1}{2} + (\frac{n\lambda^2\chi^2_n}{4})^2/(\tfrac{1}{2})(\tfrac{3}{2})2! + \ldots]$$

$$= e^{\frac{-n\lambda^2}{2}} [1 + \frac{n\lambda^2\chi^2_n}{2!} + \frac{(n\lambda^2\chi^2_n)^2}{4!} + \frac{(n\lambda^2\chi^2_n)^3}{6!} + \ldots]$$

$$(3.1.3.1) = e^{\frac{-n\lambda^2}{2}} \cosh \sqrt{n\lambda^2\chi_n^2} = e^{\frac{-n\lambda^2}{2}} \cosh [\lambda \sum_{j=1}^{n} (x_j-\mu_0)/\sigma].$$

This is the same as formula 9:5, p. 155, in Wald's book, the univariate sequential test for the mean with σ known.

3.1.4 The case where $\lambda^2 \neq 0$ under H_0

The case may conceivably arise where the stated value for λ^2 under H_0 may be different from zero. In that case,

$$H_0: \quad (\underline{\mu} - \underline{\mu}_0)\Sigma^{-1}(\underline{\mu} - \underline{\mu}_0)' = \lambda_0^2$$

$$H_1: \quad (\underline{\mu} - \underline{\mu}_0)\Sigma^{-1}(\underline{\mu} - \underline{\mu}_0)' = \lambda_1^2; \quad (\lambda_1^2 > \lambda_0^2).$$

The probability ratio then becomes the ratio of the non-central χ^2-distribution with non-centrality parameter $n\lambda_1^2$ to the non-central χ^2-distribution with non-centrality $n\lambda_0^2$.

$$(3.1.4.1) \quad \frac{p_{1n}}{p_{0n}} = e^{-n(\lambda_1^2 - \lambda_0^2)/2} \, \frac{{}_0F_1(p/2;n\lambda_1^2\chi_n^2/4)}{{}_0F_1(p/2;n\lambda_0^2\chi_n^2/4)} \, .$$

The operational procedure is the same as in Section 3.1.2. However, no tables are yet available to facilitate this test. Instead two ${}_0F_1(c;x)$ functions would have to be evaluated each time.

3.2 Sequential T^2-test.

3.2.1 Introduction

The sequential procedure followed when the population covariance matrix Σ is not known is quite similar to that discussed in Section 3.1. The null and alternative hypotheses are:

$$H_0: \quad (\underline{\mu} - \underline{\mu}_0)\Sigma^{-1} (\underline{\mu} - \underline{\mu}_0)' = 0$$

$$H_1: \quad (\underline{\mu} - \underline{\mu}_0)\Sigma^{-1} (\underline{\mu} - \underline{\mu}_0)' = \lambda^2,$$

the same as for the case where Σ is known. Again these are composite hypotheses involving $\mu_1, \ldots \mu_p$, $\mu_{10}, \ldots \mu_{p0}$ and Σ but now Σ is not known. As will be shown in Chapter IV, this can be reduced to a univariate situation involving T_n^2 and $n\lambda^2$ only.

3.2.2 Test procedure

As will be shown in Chapter IV, the probability ratio for n observations $(n > p)$ is the ratio of the non-central T3-distribution with non-centrality parameter $n\lambda^2$ to the central T^2-distribution. T_n^2 is distributed like $(n-1)pF/(n-p)$ with $\nu_1 = p$ and $\nu_2 = n-p$ degrees of freedom. Substituting, we have:

$$\frac{p_{1n}}{p_{0n}} = \frac{\sum\limits_{j=0}^{\infty} e^{\frac{-n\lambda^2}{2}} \left(\frac{n\lambda^2}{2}\right)^j \left(\frac{p}{n-p}\right)^{\frac{p}{2}+j} \left[\frac{(n-p)T_n^2}{(n-1)p}\right]^{\frac{p}{2}-1+j} \left[1 + \frac{T_n^2}{n-1}\right]^{-\frac{n}{2}-j}}{\left(\frac{p}{n-p}\right)^{\frac{p}{2}} \left[\frac{(n-p)T_n^2}{(n-1)p}\right]^{\frac{p}{2}-1} \left[1 + \frac{T_n^2}{n-1}\right]^{-\frac{n}{2}}} \cdot \frac{B(\frac{p}{2}, \frac{n-p}{2})}{j!B(\frac{p}{2} + j, \frac{n-p}{2})}$$

$$(3.2.2.1) = e^{\frac{-n\lambda^2}{2}} \frac{\Gamma(\frac{p}{2})}{\Gamma(\frac{n}{2})} \sum_{j=0}^{\infty} \frac{1}{j!} \left[\frac{n\lambda^2 T_n^2}{2(n-1+T_n^2)} \right]^j \frac{\Gamma(\frac{n}{2} + j)}{\Gamma(\frac{p}{2} + j)}$$

$$= e^{\frac{-n\lambda^2}{2}} \left\{ 1 + \frac{(\frac{n}{2})}{(\frac{p}{2})} \left[\frac{n\lambda^2 T_n^2}{2(n-1+T_n^2)} \right] + \frac{(\frac{n}{2})(\frac{n}{2} + 1)}{2!(\frac{p}{2})(\frac{p}{2} + 1)} \left[\frac{n\lambda^2 T_n^2}{2(n-1+T_n^2)} \right]^2 + \cdots \right\}$$

$$(3.2.2.2) = e^{\frac{-n\lambda^2}{2}} {}_1F_1 \left[\frac{n}{2}, \frac{p}{2} ; \frac{n\lambda^2 T_n^2}{2(n-1+T_n^2)} \right] .$$

The sequential T^2-test procedure is the following: take a sequence of n vector observations ($n = p + 1, p + 2, \ldots$), compute ${}_n\bar{x}$ and S_n, the covariance matrix based on n observations, compute $T_n^2 = n({}_n\bar{x} - \mu_0)S_n^{-1}({}_n\bar{x} - \mu_0)$ and evaluate (3.2.2.2).

a. If $p_{1n}/p_{0n} \leq \beta/(1-\alpha)$, accept H_0;

b. If $p_{1n}/p_{0n} \geq (1-\beta)/\alpha$, accept H_1;

c. If $\beta/(1-\alpha) < p_{1n}/p_{0n} < (1-\beta)/\alpha$, take another vector observation and repeat the procedure. As in the case of the χ^2-test, the limits are constant for the entire procedure since they involve only α and β. Again, the main problem is in the computation of the confluent hypergeometric function. Although tables of ${}_1F_1(a,c;x)$ are available, much laborious interpolation would be required. Again, it is better to prepare tables in which the equations $p_{1n}/p_{0n} = \beta/(1-\alpha)$ and $p_{1n}/p_{0n} = (1-\beta)/\alpha$, with α, β, λ^2 and n given, are solved for T_n^2. This will be done in Chapter VI.

3.2.3 The case where $p = 1$

When $p = 1, \lambda^2 = (\mu - \mu_0)^2/\sigma^2$ and $T_n^2 = t^2$,

$$(3.2.3.1)\quad p_{1n}/p_{0n} = e^{-n\lambda^2/2}\,{}_1F_1[n/2,\ 1/2,\ n\lambda^2t^2/2(n-1+t^2)]$$

which is the same as equation 5 in Rushton's 1952 paper.

3.2.4 The case where $\lambda^2 \neq 0$ under H_0

As in Section 3.1.4 for the sequential χ^2-test, the case may arise where λ^2 is different from zero under H_0. Then, in a similar manner the probability ratio becomes the ratio of two non-central T^2-distributions.

(3.2.4.1)

$$p_{1n}/p_{0n} = e^{-n(\lambda_1^2-\lambda_0^2)/2}\ \frac{{}_1F_1\left[\frac{n}{2},\ \frac{p}{2};\ \frac{n\lambda_1^2T_n^2}{2(n-1+T_n^2)}\right]}{{}_1F_1\left[\frac{n}{2},\ \frac{p}{2};\ \frac{n\lambda_0^2T_n^2}{2(n-1+T_n^2)}\right]}.$$

The operational procedure is the same as in Section 3.2.2. However, no tables are yet available to facilitate this test. Instead, two ${}_1F_1(a,c;x)$ functions would have to be evaluated each time.

3.3 The two-sample case.

The sequential techniques discussed in this chapter can also be used to test the hypothesis that two samples are drawn from populations with the same means or from populations whose means differ by prescribed amounts. Let Population 1 have means $\mu_1^{(1)},\ldots,\mu_p^{(1)}$ with covariance matrix Σ_{11} and let Population 2 have means $\mu_1^{(2)},\ldots,\mu_p^{(2)}$ and covariance matrix Σ_{22}. Let the cross-covariances be represented by Σ_{12}. Therefore, the covariance matrix of $x_1^{(1)},\ldots,x_p^{(1)}$; $x_1^{(2)},\ldots,x_p^{(2)}$ is

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{bmatrix} .$$

Let $y_i = x_i^{(1)} - x_i^{(2)} - \delta_i$ (δ_i is a constant, quite often zero) for $i = 1,2,\ldots,p$. The covariance matrix of $y_1,\ldots,y_p$ is $\Sigma_y = \Sigma_{11} + \Sigma_{22} - \Sigma_{12} - \Sigma_{12}'$ (Anderson 1958). Now let $\underline{\eta} = \underline{\mu}_1 - \mu_2 - \underline{\delta}$.

Again, rather than test $\underline{\eta} = 0$ against $\underline{\eta} = \underline{\eta}_1$ (say), we will use the quadratic form $\underline{\eta}\Sigma_y\underline{\eta}'$. The hypotheses become:

H_0: $\underline{\eta}\Sigma_y^{-1}\underline{\eta}' = 0$

H_1: $\underline{\eta}\Sigma_y^{-1}\underline{\eta}' = \lambda^2$.

If the covariance matrices are known, the test statistic is $\chi_n^2 = n\,{}_n\underline{\bar{y}}\Sigma_{yn}^{-1}\,{}_n\underline{\bar{y}}'$ and the probability ratio test is

(3.3.1) $\frac{p_{1n}}{p_{0n}} = e^{-n\lambda^2/2} \; {}_0F_1(p/2;\; n\lambda^2\chi_n^2/4)$ as in (3.1.2.1).

If $\underline{\eta}\Sigma_y^{-1}\underline{\eta}'$ is other than zero under H_0, then the procedures discussed in Section 3.1.4 may be used.

To test the same hypotheses in the case where the population covariance matrices are not known, the problem may be considerably simpler than that of the non-sequential two-sample situation. The observations $\underline{x}_1$ and $\underline{x}_2$ are paired in the sequential case and can be treated as paired observations from the start so the procedure is in terms of $\underline{y} = \underline{x}_1 - \underline{x}_2$ (or $\underline{x}_1 - \underline{x}_2 - \underline{\delta}$). This removes the problem of unequal covariance matrices between the two populations. S_{yn} can be computed directly from the individual observations on $\underline{y}$ and the test statistic becomes $T_n^2 = n\,{}_n\underline{\bar{y}}S_{yn}^{-1}\,{}_n\underline{\bar{y}}'$. The probability ratio is

(3.3.2) $\frac{p_{1n}}{p_{0n}} = e^{-n\lambda^2/2} \; {}_1F_1[n/2, p/2;\; n\lambda^2T_n^2/2(n-1+T_n^2)]$

as in (3.2.2.2). If $\underline{\eta}\Sigma_y^{-1}\underline{\eta}'$ is not equal to zero under H_0, the procedure discussed in Section 3.2.4 may be used.

IV. EXISTENCE THEOREMS

4.1 A theorem of Cox and the problem of composite hypotheses.

In the description of the sequential χ^2- and T^2-tests, we have delayed the matter of relating to properties of these tests for consideration in this chapter. A theorem of D. R. Cox (1952) is particularly useful in this work.

Cox's Theorem is quoted as follows:

"Let $\underline{x} = [x_1,\ldots,x_n]$ be random variables whose probability density function (p.d.f.) depends on unknown parameters $\theta_1,\ldots,\theta_p$. The x_i themselves may be vectors. Suppose that

(i) $t_1,\ldots,t_p$ are a functionally independent jointly sufficient set of estimators for $\theta_1,\ldots,\theta_p$;

(ii) the distribution of t_1 involves θ_1 but not $\theta_2,\ldots,\theta_p$;

(iii) $u_1,\ldots,u_m$ are functions of $\underline{x}$ functionally independent of each other and of $t_1,\ldots,t_p$;

(iv) there exists a set S of transformations of $\underline{x} = [x_1,\ldots,x_n]$ into $\underline{x}^* = [x_1^*,\ldots,x_n^*]$ such that

(a) t_1, $u_1,\ldots,u_m$ are unchanged by all transformations in S;

(b) the transformation of $t_2,\ldots,t_p$ into $t_2^*,\ldots,t_p^*$ defined by each transformation in S is one-to-one;

(c) if $T_2,\ldots,T_p$ and $T_2^*,\ldots,T_p^*$ are two sets of values of $t_2,\ldots,t_p$ each having non-zero probability density under at least one of the distributions of $\underline{x}$, then there exists a transformation in S such that if $t_2 = T_2,\ldots,t_p = T_p$ then $t_2^* = T_2^*,\ldots,t_p^* = T_p^*$.

Then the joint p.d.f. of $t_1,u_1,\ldots,u_m$ factorizes into

$$g(t_1|\theta_1)\,\ell\,(u_1,\ldots,u_m,t_1),$$

where g is the p.d.f. of t_1 and ℓ does not involve θ_1." The proof of this theorem is given in Cox's paper. Cox defines the term <u>functional independence</u> as follows:

"When it is stated that certain functions of $x_1,\ldots,x_n$ denoted by $t_1,\ldots,t_p$, $u_1,\ldots,u_m$ are functionally independent, it is meant that there is a transformation from $x_1,\ldots,x_n$ to a set of new variables including $t_1,\ldots,t_p$, $u_1,\ldots,u_m$, and that the Jacobian of the transformation is different from zero (except possibly for a set of values of total probability zero)."

The application of this theorem to sequential analysis is as follows: when confronted with a composite hypothesis to be tested sequentially, one can use either Wald's method of weight functions or this present method of frequency functions. In this latter case, a transformation is generally made on the original parameters $\phi_1,\ldots,\phi_k$ to another set

$\theta_1,\ldots,\theta_k$ with estimators $t_1,\ldots,t_k$ such that the distribution of t_1 does not involve any parameter except θ_1. This reduces the problem from that of a composite hypothesis involving $\phi_1,\ldots,\phi_k$ to a simple hypotheses involving θ_1 alone. The test of significance of a simple hypothesis is then carried out for this single parameter. Wald has shown that a probability ratio test of the form

$$\frac{p_{1n}(x_1,\ldots,x_n)}{p_{0n}(x_1,\ldots,x_n)}$$

can be carried out even when the x_i's themselves are not independent although the usual properties asscciated with the termination proof, OC function and ASN function may no longer hold. In a sequential test of significance associated with the parameter θ_1, successive values of the estimator t_1 based on $1,2,\ldots,n$ observations (i.e. $u_1,\ldots,u_{n-1},t_1$) are treated as "observations" themselves and the problem is to derive a probability ratio test based on these "observations". If it can be shown that all of the conditions of Cox's theorem are satisfied, then $f(x_1,\ldots,x_n|\phi_1,\ldots,\phi_k)$ the likelihood function for a sample of size n can be transformed and reduced into $p_n(u_1,\ldots,u_{n-1},t_1|\theta_1)$ which in turn factorizes into $g(t_1|\theta_1)\ \ell\ (u_1,\ldots,u_{n-1},t_1)$. If θ_{10} and θ_{11} represent the values of θ_1 under the null and alternative hypotheses respectively, then

$$(4.1.1) \quad \frac{p_{1n}}{p_{0n}} = \frac{g(t_1|\theta_{11})\ \ell\ (u_1,\ldots,u_{n-1},t_1)}{g(t_1|\theta_{10})\ \ell\ (u_1,\ldots,u_{n-1},t_1)} = \frac{g(t_1|\theta_{11})}{g(t_1|\theta_{10})}\ .$$

The probability ratio then becomes merely the ratio of the distributions of the statistic t_1 under the alternative and null hypotheses respectively. The sequential test is then defined, consistent with our procedures defined in Chapter III, as follows:

a. Accept H_0 if $p_{1n}/p_{0n} \leq \beta/(1-\alpha)$;

b. Accept H_1 if $p_{1n}/p_{0n} \geq (1-\beta)/\alpha$;

c. Continue sampling if $\beta/(1-\alpha) < p_{1n}/p_{0n} < (1-\beta)/\alpha$.

Provided that the probability is one that the test terminates, the probabilities of error under the null and alternative hypotheses are approximately α and β respectively [Wald, 1947, p. 43].

Specifically, in this problem, it will be shown that for the sequential χ^2-test, the probability ratio is the ratio of the non-central χ^2-distribution under H_1 to the central χ^2-distribution under H_0 and that the probability ratio for the sequential T^2-test is the ratio of the non-central T^2-distribution under H_1 to the central T^2-distribution under H_0. These proofs will be given in Section 4.2 and the termination proofs will be given in Section 4.3.

As stated above, the usual properties associated with the OC function and the ASN function no longer hold since the "observations" are no longer independent. For this reason, no expression can be given for the OC function other than that it is approximately equal to α when $\theta_1 = \theta_{10}$ and $1-\beta$ when $\theta_1 = \theta_{11}$. The only method known so far to approximate to this function is the Monte Carlo approach. Similarly, no expression can be given for the ASN function either but some "conjectural" approximations are given for $\theta_1 = \theta_{10}$ and $\theta_1 = \theta_{11}$ in Chapter V. These seem to be in line with the savings in numbers of observations generally realized in sequential procedures and agree fairly well with the limited amount of Monte Carlo work done so far.

4.2 <u>Fulfillment of the conditions of Cox's Theorem</u>.

4.2.1 The sequential χ^2-test

We now show that Cox's Theorem applies and that the conditions of the theorem hold for the sequential χ^2-test. The parameters initially involved are $\mu_1,\ldots,\mu_p$, the population means. These are transformed, as we shall see below, to $n\lambda^2$, $\alpha_1,\ldots,\alpha_{p-1}$ corresponding to $\theta_1,\ldots,\theta_p$ of the theorem. Estimators χ_n^2, $a_{1n},\ldots,a_{p-1,n}$ are defined corresponding to $t_1,\ldots,t_p$ of the theorem.

<u>Condition (i)</u>. χ_n^2, $a_{1n},\ldots,a_{p-1n}$ are functionally independent jointly sufficient estimators for

$n\lambda^2$, $\alpha_1,\ldots,\alpha_{p-1}$, where the a_{in}'s and α_i's are defined below.

Proof: From Anderson, Section 3.3.3, ${}_n\bar{x}_1,\ldots,{}_n\bar{x}_p$ are a jointly sufficient set of estimators for $\mu_1,\ldots,\mu_p$ when Σ is known and they are obviously functionally independent. Then $f(x_{11},\ldots,x_{pn}|\mu_1,\ldots,\mu_p)$ can be written as $H({}_n\bar{x}_1,\ldots,{}_n\bar{x}_p|\mu_1,\ldots,\mu_p)m(x_{11},\ldots,x_{pn})$. Now apply the transformation $\underline{y} = ({}_n\underline{\bar{x}} - \underline{\mu}_0)G$ where $GG' = \Sigma^{-1}$ and $\underline{\mu}_0$ is a vector of constants. Then $\underline{y}$ has a multivariate normal distribution with mean $\underline{\eta} = (\underline{\mu} - \underline{\mu}_0)G$ and covariance matrix I. The Jacobian of the transformation is $|G|^{-1}$. Apply to $\underline{y}$ and $\underline{\eta}$ the following polar transfromations:

$$(4.2.1.1) \qquad \underline{y}' = \begin{bmatrix} \chi_n \cos a_{1n} \\ \chi_n \sin a_{1n} \cos a_{2n} \\ \vdots \\ \chi_n \sin a_{1n} \sin a_{2n},\ldots,\cos a_{p-1,n} \\ \chi_n \sin a_{1n} \sin a_{2n},\ldots,\sin a_{p-1,n} \end{bmatrix},$$

$$\underline{\eta}' = \begin{bmatrix} \sqrt{n}\,\lambda \cos \alpha_1 \\ \sqrt{n}\,\lambda \sin \alpha_1 \cos \alpha_2 \\ \vdots \\ \sqrt{n}\,\lambda \sin \alpha_1 \sin \alpha_2, \ldots, \cos \alpha_{p-1} \\ \sqrt{n}\,\lambda \sin \alpha_1 \sin \alpha_2, \ldots, \sin \alpha_{p-1} \end{bmatrix}$$

Restricting χ_n and λ to positive values, then $0 \leq a_{in} \leq \pi$ and $0 \leq \alpha_i \leq \pi$ for $i = 1,2,\ldots, p-2$, and $0 \leq a_{p-1,n} \leq 2\pi$ and $0 \leq \alpha_{p-1} \leq 2\pi$. The Jacobian of the transformation on the variates is $\chi_n^p \sin^{p-1} a_{1n} \sin^{p-2} a_{2n}, \ldots, \sin a_{p-2,n}$. The quadratic form $n({}_n\underline{\bar{x}}-\underline{\mu}_0)\Sigma^{-1}({}_n\underline{\bar{x}}-\underline{\mu}_0)'$ has been transformed into $n\underline{y}\underline{y}'$ and in turn into χ_n^2 while the corresponding quadratic form $n(\underline{\mu}-\underline{\mu}_0)\Sigma^{-1}(\underline{\mu}-\underline{\mu}_0)'$ has been transformed into $n\underline{\eta}\underline{\eta}'$ and in turn into $n\lambda^2$. Since the transformations ${}_n\bar{x}_1,\ldots,{}_n\bar{x}_p$ to χ_n^2, $a_{1n},\ldots,a_{p-1,n}$ do not involve $\mu_1,\ldots,\mu_p$, λ^2 or $\alpha_1,\ldots,\alpha_{p-1}$, $f(x_{11},\ldots,x_{pn})$ can, using non-singular transformations, now be written as

$$h(\chi_n^2,\ a_{1n},\ldots,a_{p-1,n} \mid n\lambda^2,\ \alpha_1,\ldots,\alpha_{p-1})m(x_{11},\ldots,x_{pn}),$$

the form necessary for sufficiency. Since ${}_n\bar{x}_1,\ldots,{}_n\bar{x}_p$ are functionally independent and the transformation to χ_n^2, $a_{1n},\ldots,a_{p-1,n}$ is non-singular, χ_n^2, $a_{1n},\ldots,a_{p-1,n}$ are also functionally independent. Condition (i) is fulfilled.

Condition (ii). The marginal distribution of χ_n^2 involves only $n\lambda^2$. This was first shown by Fisher (1928).

Condition (iii). $\chi_1^2,\ldots,\chi_{n-1}^2$ are functions of $x_{11},\ldots,x_{pn}$, functionally independent of each other and of χ_n^2, $a_{1n},\ldots,a_{p-1,n}$.

Proof: consider the matrix of observations on the variables $\dot{x}_i = x_i - \mu_{i0}$, $i = 1,2,\ldots,p$:

$$\dot{X}' = \begin{bmatrix} \dot{x}_{11} & \cdots & \dot{x}_{1n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \dot{x}_{p1} & \cdots & \dot{x}_{pn} \end{bmatrix} .$$

Transform $\dot{X}$ to new variables $Y = \dot{X}G$ where, again, $GG' = \Sigma^{-1}$. The Jacobian of this transformation is $|G|^{-n}$. Now transform Y to new variables $Z = TY$ where:

$$T = \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \cdot & \cdot & 0 & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{\sqrt{n-1}} & \frac{1}{\sqrt{n-1}} & \frac{1}{\sqrt{n-1}} & \cdot & \cdot & \frac{1}{\sqrt{n-1}} & 0 \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdot & \cdot & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} \end{bmatrix}$$

with the Jacobian equal to $(n!)^{p/2}$. Now the polar transformation used in (4.2.1.1) is generalized as follows:

$$z_{1j} = \chi_j \cos a_{1j}$$
$$z_{2j} = \chi_j \sin a_{1j} \cos a_{2j}$$
$$\vdots$$
$$z_{pj} = \chi_j \sin a_{1j} \sin a_{2j}, \ldots, \sin a_{p-1,j}$$

for $j = 1, 2, \ldots, n$. The Jacobian of this transformation is:

$$\prod_{j=1}^{n} \chi^p \sin_j^{p-1} a_{1j} \sin^{p-2} a_{2j}, \ldots, \sin a_{p-2,j}.$$

As a result of these transformations, $\Sigma z_{i1}^2 = \chi_1^2, \ldots, \Sigma z_{in}^2 = \chi_n^2$ and has been transformed into $\chi_1^2, \ldots, \chi_{n-1}^2$, $\chi_n^2, a_{11}, \ldots, a_{1,n-1}, \ldots, a_{p-1,1}, \ldots, a_{p-1,n-1}, a_{1n}, \ldots, a_{p-1,n}$. Therefore $x_{11}, \ldots, x_{pn}$ has been transformed into $\chi_n^2, a_{1n}, \ldots, a_{p-1,n}, \chi_1^2, \ldots, \chi_{n-1}^2$ and some other angles which are of no further interest in this problem. The Jacobian of the combined transformations is non-singular since all three of the individual Jacobians are non-singular. Therefore, $\chi_1^2, \ldots, \chi_{n-1}^2$ are functions of $x_{11}, \ldots, x_{pn}$ functionally independent of each other and of $\chi_n^2, a_{1n}, \ldots, a_{p-1,n}$. Condition (iii) has been fulfilled.

Condition (iv). There exists a set of transformations S of $\dot{X}$ into $\dot{X}^*$ such that:

(a) $\chi_1^2, \chi_2^2, \ldots, \chi_{n-1}^2, \chi_n^2$ are unchanged by all transformations in S.

(b) The transformation of $a_{1n}, \ldots, a_{pn}$ into $a_{1n}^*, \ldots, a_{pn}^*$ defined by each transformation in S is one-to-one.

(c) If $A_1, \ldots, A_{p-1}$ and $A_1^*, \ldots, A_{p-1}^*$ are two sets of values of $a_{1n}, \ldots, a_{p-1,n}$ each having non-zero probability density under at least one of the distributions of X, then there exists a transformation in S such that, if $a_{1n}=A_1, \ldots, a_{p-1,n}=A_{p-1}$, then $a_{1n}^*=A_1^*, \ldots, a_{p-1,n}^*=A_{p-1}^*$.

Proof: Consider the transformation $\dot{X}^*G = \dot{X}GB$ where G is defined as before and B is a p x p orthogonal matrix. (This amounts to making a transformation on Y rather than on $\dot{X}$ but this is permissible since G is known in advance and would be common to both transformations.) Now consider the transformation:

$$Z^* = T\dot{X}^*G = T\dot{X}GB = ZB.$$

Since B is orthogonal, the sums of squares of elements in each row of Z^* will be the same as the corresponding quantity for each row of Z. Therefore $\chi_1^2, \chi_2^2, \ldots, \chi_n^2$ are all unchanged by the transformation. Condition (iv a) is satisfied.

For the purposes of parts (b) and (c) of Condition (iv), only the last row of Z and of Z^*, denoted by $\underline{z}_n$ and $\underline{z}_n^*$

respectively, need be considered. The transformation $\underline{z}_n^* = \underline{z}_n B$ is one-to-one since B is an orthogonal matrix and has a unique inverse. Since $\underline{z}_n$ and $\underline{z}_n^*$ differ and χ_n^2 is unchanged, the difference must appear in the angles. Hence the transformation from $a_{1n},\ldots,a_{p-1,n}$ to $a_{1n}^*,\ldots,a_{p-1,n}^*$ is one-to-one and condition (iv b) is satisfied.

If $A_1,\ldots,A_{p-1}$ and $A_1^*,\ldots,A_{p-1}^*$ are two sets of values of $a_{1n},\ldots,a_{p-1,n}$ suitably restricted between 0 and π or 0 and 2π as the case may be, then $\underline{z}_n$ and $\underline{z}_n^*$ may be evaluated except for a scalar quantity χ_n which is the same in both cases. Given two sets of values $\underline{z}_n$ and $\underline{z}_n^*$, they are related through $\underline{z}_n^* = \underline{z}_n B$. This relation defines p-1 independent equations on the elements of B. There are also $p(p+1)/2$ additional equations on the elements of B imposed through the requirement that B is orthogonal. The solution for the p^2 elements of B is not unique (except for $p = 2$) but matrices B satisfying the requirements may be found and this is sufficient to satisfy Condition (iv c).

All of the conditions of the theorem have been fulfilled. Hence the joint p.d.f. of $\chi_1^2,\ldots,\chi_n^2$ factorized into $g(\chi_n^2|n\lambda^2)\ \ell\ (\chi_1^2,\ldots,\chi_n^2)$ and p_{1n}/p_{0n} can be written as

$$g(\chi_n^2|n\lambda_1^2)/g(\chi_n^2|n\lambda_0^2),$$

the form of Section 3.1.4. For the situation discussed in Section 3.1.1, $\lambda_1^2 = \lambda^2$ and $\lambda_0^2 = 0$. Then

(4.2.1.2) $p_{1n}/p_{0n} = g(\chi_n^2|n\lambda^2)/g(\chi_n^2|0)$

which is the ratio of the non-central χ^2-distribution with non-centrality parameter $n\lambda^2$ to the central χ^2-distribution and gives us (3.1.2.1) as a final result.

4.2.2 The sequential T^2-test

We now show that the conditions of Cox's Theorem also hold for the sequential T^2-test. The parameters initially involved are $\mu_1,\ldots,\mu_p$, σ_{11}, $\sigma_{12},\ldots,\sigma_{pp}$, the population means, variances and covariances. The means are transformed, as in the χ^2 case, into $n\lambda^2,\alpha_1,\ldots,\alpha_{p-1}$ and these together with σ_{11}, $\sigma_{12},\ldots,\sigma_{pp}$ correspond to the θ's in the theorem. Estimators $T_n^2, a_{1n},\ldots,a_{p-1,n}, {}_ns_{11}, {}_ns_{12},\ldots, {}_ns_{pp}$ are defined corresponding to the t's in the theorem.

<u>Condition (i)</u>. $T_n^2, a_{1n},\ldots,a_{p-1,n}, {}_ns_{11}, {}_ns_{12},\ldots, {}_ns_{pp}$ are functionally independent jointly sufficient estimators for $n\lambda^2$, $\alpha_1,\ldots,\alpha_{p-1},\sigma_{11},\sigma_{12},\ldots,\sigma_{pp}$ where the a_{in}'s and α_i's are defined in a similar manner to those discussed in Section 4.2.1.

Proof: Again from Anderson, Section 3.3.3, ${}_n\bar{x}_1,\ldots {}_n\bar{x}_p$, ${}_ns_{11}, {}_ns_{12},\ldots, {}_ns_{pp}$ are a jointly sufficient set of estimators for $\mu_1,\ldots,\mu_p$, $\sigma_{11},\sigma_{12},\ldots,\sigma_{pp}$. It is well known that these estimators are functionally independent. In a manner similar to that used in Section 4.2.1, the ${}_n\bar{x}_i$'s can be transformed into T_n^2 and $a_{1n},\dots,a_{p-1,n}$ and the μ_i's can be transformed into $n\lambda^2$ and $\alpha_1,\dots,\alpha_{p-1}$ with $S_n = [{}_n s_{ij}]$ and Σ being unchanged by the transformations. Therefore condition (i) is easily fulfilled.

Condition (ii). The marginal distribution of T_n^2 involves only $n\lambda^2$. This was proved independently by Hsu (1938) and Bose and Roy (1938).

Condition (iii). $T_{p+1}^2,\dots,T_{n-1}^2$ are functions of $x_{11},\dots,x_{pn}$ functionally independent of each other and of $T_n^2, a_{1n},\dots,a_{p-1,n},\ {}_n s_{11}, {}_n s_{12},\dots,{}_n s_{pp}$.

Proof: Again, consider the matrix of observations on the variables $\dot{x}_i = x_i - \mu_{i0}$, $i = 1,2,\dots,p$:

$$\dot{X}' = \begin{bmatrix} \dot{x}_{11} & \cdots & \dot{x}_{1n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \dot{x}_{p1} & \cdots & \dot{x}_{pn} \end{bmatrix}, \quad n > p + 1.$$

The procedure here is similar to that employed in satisfying condition (iii) for the χ^2-case except that the T^2-statistics are functions of the sample variances and covariances as well. Let ${}_j\underline{\bar{X}}, S_j$ and T_j^2 denote, respectively, the vector of sample means, the sample covariance matrix, and T^2 based on the first j observations of $\dot{X}$. T^2_{p+1} is a function of the first p+1 rows of $\dot{X}$. Each successive $T^2_j = j\,{}_j\bar{\dot{x}}S_j^{-1}{}_j\bar{\dot{x}}'$ is a function of one more row of $\dot{X}$ and hence $T^2_{p+1},\dots,T^2_{n-1}$ are functionally independent with T^2_{n-1} being a function of the first n-1 rows of $\dot{X}$. However, $T^2_n, a_{1n},\dots,a_{p-1,n}, {}_ns_{11}, {}_ns_{12},\dots,{}_ns_{pp}$ all depend on the last row of $\dot{X}$ and hence are functionally independent of $T^2_{p+1},\dots,T^2_{n-1}$. Condition (iii) has been fulfilled.

<u>Condition (iv)</u>. There exists a set S of transformations of X into X* such that:

(a) $T^2_{p+1},\dots,T^2_n$ are unchanged by all transformations in S.

(b) The transformation of $a_{1n},\dots,a_{p-1,n}, {}_ns_{11}, {}_ns_{12},\dots,{}_ns_{pp}$ into $a^*_{1n},\dots,a^*_{p-1,n}, {}_ns^*_{11}, {}_ns^*_{12},\dots,{}_ns^*_{pp}$ defined by each transformation in S is one-to-one.

(c) If two sets $a_{1n},\dots,a_{p-1,n}, {}_ns_{11}, {}_ns_{12},\dots,{}_ns_{pp}$ and $a^*_{1n},\dots,a^*_{p-1,n}, {}_ns^*_{11}, {}_ns^*_{12},\dots,{}_ns^*_{pp}$ are given, each having non-zero probability density under at least one of the distributions of X, then there exists a transformation in S which will transform one set into the other.

Proof: Let B be an orthogonal p x p matrix, C be a nonsingular triangular p x p matrix and finally let E be a triangular p x p matrix such that $S_n^{-1} = EE'$.

Let the class of transformations be:

(4.2.2.1) $\dot{X}^* = \dot{X}EBC'$ (i.e. $\dot{X} = \dot{X}^*C'^{-1}B'E^{-1}$).

The covariance matrix of the new variables is:

$$\begin{aligned} S_n^* &= CB'E'S_nEBC' \\ &= CB'BC', \qquad \text{since } S_n = E'^{-1}E^{-1}, \\ &= CC', \qquad \text{since } B' = B^{-1}. \end{aligned}$$

The transformation applies also to the averages, viz:

(4.2.2.2) ${}_n\bar{\dot{\underline{x}}}^* = {}_n\bar{\dot{\underline{x}}}EBC'$ or ${}_n\bar{\dot{\underline{x}}} = {}_n\bar{\dot{\underline{x}}}^*C'^{-1}B'E^{-1}$.

Now let

$$\underline{z}' = E'{}_n\bar{\underline{x}}' = \begin{bmatrix} T_n \cos a_{1n} \\ T_n \sin a_{1n} \cos a_{2n} \\ \cdot \\ \cdot \\ \cdot \\ T_n \sin a_{1n} \sin a_{2n}, \ldots, \sin a_{p-1,n} \end{bmatrix}$$

so that $\underline{zz}' = T_n^2$. Similarly, let

$$\underline{z}^{*\prime} = C^{-1}{}_n\bar{\underline{x}}^{*\prime} = \begin{bmatrix} T_n^* \cos a_{1n}^* \\ T_n^* \sin a_{1n}^* \cos a_{2n}^* \\ \cdot \\ \cdot \\ \cdot \\ T_n^* \sin a_{1n}^* \sin a_{2n}^*, \ldots, \sin a_{p-1,n}^* \end{bmatrix}$$

since $S_n^* = CC'$. Now by (4.2.2.2), $\underline{z}^* = n\,\underline{\bar{x}}EBC'C'^{-1} = \underline{z}B$. Therefore, $T_n^{2*} = \underline{z}^*\underline{z}^{*\prime} = \underline{z}BB'\underline{z}' = \underline{z}\underline{z}' = T_n^2$ since B is orthogonal, so T_n^2 is unchanged by this transformation.

For every $p < j < n$, $T_j^2 = j\,{}_j\underline{\bar{x}}S_j^{-1}\,{}_j\underline{\bar{x}}'$. Now, ${}_j\underline{\bar{x}}^* = {}_j\underline{\bar{x}}EBC'$ and $S_j^* = CB'E'S_jEBC'$. Therefore

$$\begin{aligned} T_j^{2*} &= j\,{}_j\underline{\bar{x}}^*S_j^{*-1}\,{}_j\underline{\bar{x}}^{*\prime} \\ &= j\,{}_j\underline{\bar{x}}EBC'C'^{-1}B'E^{-1}S_j^{-1}E'^{-1}BC^{-1}CB'E'\,{}_j\underline{\bar{x}}' \\ &= j\,{}_j\underline{\bar{x}}S_j^{-1}\,{}_j\underline{\bar{x}}'. \end{aligned}$$

Hence T_j^2 is invariant under this class of transformations for all $p < j \leq n$. Condition (iv a) is fulfilled.

Given B, C, and E, the transformations of S_n into S_n^* and $\underline{z}$ into $\underline{z}^*$ are one-to-one since all of these matrices have unique inverses and the transformations are linear. The transformation $\underline{z}^* = \underline{z}B$ actually transforms $T_n^2, a_{1n}, \ldots, a_{p-1,n}$ into $T_n^{2*}, a_{1n}^*, \ldots, a_{p-1,n}^*$ and it has already been established that $T_n^{2*} = T_n^2$. Then the transformation really involves only the angles as noted in the similar proof for the χ^2-test. This transformation is one-to-one and condition (iv b) is fulfilled.

E is uniquely determined by the relationship $S_n^{-1} = EE'$ and C is uniquely determined by the relationship $S_n^* = CC'$. There exists an orthogonal matrix B such that $\underline{z}^* = \underline{z}B$. Therefore, once $S_n, a_{1n}, \ldots, a_{p-1,n}, S_n^*, a_{1n}^*, \ldots, a_{p-1,n}^*$ are given, E, C, and B, the components of the transformation in S can be determined. Condition (iv c) is fulfilled.

All of the conditions of the theorem have now been fulfilled. Hence the joint p.d.f. of $T^2_{p+1},\dots,T^2_n$ factorizes into $g(T^2_n|n\lambda^2)\,\ell\,(T^2_{p+1},\dots,T^2_n)$ and p_{1n}/p_{0n} can be written as

$$g(T^2_n|n\lambda^2_1)/g(T^2_n|n\lambda^2_0),$$

the form in Section 3.2.4. For the situation discussed in Section 3.2.1, $\lambda^2_1 = \lambda^2$ and $\lambda^2_0 = 0$. Therefore,

(4.2.2.1) $p_{1n}/p_{0n} = g(T^2_n|n\lambda^2)/g(T^2_n|0)$

which is the ratio of the non-central T^2-distribution with non-centrality parameter $n\lambda^2$ to the central T^2-distribution and gives us (3.2.2.2) as a final result.

4.3 Termination proofs.

4.3.1 Proof that the sequential χ^2-test terminates with probability one

In Section 3.1.2, we showed that the sequential χ^2-test, based on n observations, was of the form:

a. If $p_{1n}/p_{0n} \leq \beta/(1-\alpha)$, accept H_0;

b. If $p_{1n}/p_{0n} \geq (1-\beta)/\alpha$, accept H_1;

c. If $\beta/(1-\alpha) \leq p_{1n}/p_{0n} \leq (1-\beta)/\alpha$, continue sampling.

We had

(4.3.1.1) $p_{1n}/p_{0n} = e^{-n\lambda^2/2}\,{}_0F_1(p/2;n\lambda^2\chi^2/4)$

where λ^2 is the value of $(\underline{\mu} - \underline{\mu}_0)\Sigma^{-1}(\underline{\mu} - \underline{\mu}_0)'$ specified by the alternative hypothesis. It was shown in Section 4.2.1 that $f(\chi_1^2,\ldots,\chi_n^2)$ could be decomposed in such a way that p_{1n}/p_{0n} can be written in the form (4.3.1). For the risks α and β to hold approximately, it is still necessary to show that the sequential χ^2-test terminates with probability one.

We assert that

$$P\left\{\text{Sample size} > n\right\} \leq P\left\{\underline{\chi_n^2} < \chi_n^2 < \overline{\chi_n^2}\right\}$$

where $\underline{\chi_n^2}$ and $\overline{\chi_n^2}$ are the boundary values of χ_n^2 corresponding to $p_{1n}/p_{0n} = \beta/(1-\alpha)$ and $p_{1n}/p_{0n} = (1-\beta)/\alpha$ respectively. Now

$$P\left\{\underline{\chi_n^2} < \chi_n^2 < \overline{\chi_n^2}\right\} = \int_0^{\overline{\chi_n^2}} h(\chi_n^2)d\chi_n^2 - \int_0^{\underline{\chi_n^2}} h(\chi_n^2)d\chi_n^2$$

where $h(\chi_n^2)$ denotes the non-central χ^2-distribution with p degrees of freedom and non-centrality parameter $n\lambda^2$. We will now show that as $n \to \infty$, $\underline{\chi_n^2} \to \overline{\chi_n^2} \to n\lambda^2/4$ where λ^2 is specified under H_1. Hence, in the limit, $P\left\{\underline{\chi_n^2} < \chi_n^2 < \overline{\chi_n^2}\right\} = 0$ and therefore the sequential process must terminate with probability one.

To show that $\underline{\chi_n^2}$ and $\overline{\chi_n^2}$ approach a common limit, we employ an argument similar to that used by Ray (1957) for the sequential analysis of variance. First, we let $\chi_n^2 = nU^2$.

Then, using the confluent hypergeometric function form of the probability ratio, we can write equation (3.1.2.3) as

(4.3.1.2)

$$p_{1n}/p_{0n} = e^{-n\lambda^2/2 \; -\sqrt{n^2\lambda^2U^2}} \; {}_1F_1[(p-1)/2, p-1;\; 2\sqrt{n^2\lambda^2U^2}]$$

$$= f_n(U^2), \quad 0 < U^2 < \infty$$

and

(4.3.1.3)

$$\begin{aligned} g_n(U^2) &= \ln f_n(U^2) \\ &= -n\lambda^2/2 - \sqrt{n^2\lambda^2U^2} \\ &\quad + \ln {}_1F_1[(p-1)/2, p-1;\; 2\sqrt{n^2\lambda^2U^2}]. \end{aligned}$$

Certain properties of $g_n(U^2)$ must be investigated.

We must show that:

(i) $g_n'(U^2) > 0$ for $U^2 > 0$ and $\lim\limits_{n \to \infty} g_n'(U^2) = \infty$.

(4.3.1.4)

$$g_n'(U^2) = \frac{1}{2}\sqrt{\frac{n^2\lambda^2}{U^2}} \left\{ \frac{{}_1F_1[(p+1)/2, p; 2\sqrt{n^2\lambda^2U^2}]}{{}_1F_1[(p-1)/2, p-1; 2\sqrt{n^2\lambda^2U^2}]} - 1 \right\} .$$

By Appendix A, it can be seen that the ratio

(4.3.1.5)

$$G = \frac{{}_1F_1[(p+1)/2, p;\; 2\sqrt{n^2\lambda^2U^2}]}{{}_1F_1[(p-1)/2, p-1; 2\sqrt{n^2\lambda^2U^2}]}$$

is strictly increasing in U^2, is not less than unity, is greater than unity for $U^2 > 0$, and approaches 2 in the limit for $U^2 > 0$. Hence the quantity G-1 is always positive for $U^2 > 0$ and approaches unity in the limit. Therefore $g_n'(U^2)$ is always positive for $U^2 > 0$ and will tend to infinity as $n \to \infty$.

We must also show that:

(ii) $g_n(U^2) = n[-n\lambda^2/2 - \sqrt{\lambda^2 U^2} + O(1/n)]$.

By Erdelyi, et al. (1953)

(4.3.1.6) $${}_1F_1(a,c;x) = \frac{\Gamma(c)}{\Gamma(a)} e^x x^{a-c}[1 + O(|x|^{-1})]$$

or in the present case:

$$\begin{aligned}&{}_1F_1[(p-1)/2, p-1; 2\sqrt{n^2\lambda^2U^2}] \\ &= \frac{\Gamma(p-1)}{\Gamma[(p-1)/2]} e^{2\sqrt{n^2\lambda^2U^2}} [2\sqrt{n^2\lambda^2U^2}]^{-(p-1)/2} \\ &\quad \cdot [1 + O(1/2\sqrt{n^2\lambda^2U^2})].\end{aligned}$$

Then,

$$\begin{aligned}g_n(U^2) &= \ln f_n(U^2) \\ &= -n\lambda^2/2 - \sqrt{n^2\lambda^2U^2} + \ln\left\{[\Gamma(p-1)]/\Gamma[(p-1)/2]\right\} \\ &\quad -[(p-1)/2][\ln 2 + \ln n + (1/2)\ln\lambda^2 + (1/2)\ln U^2] \\ &\quad +\ln[1 + O(1/2\sqrt{n^2\lambda^2U^2})]\end{aligned}$$

(4.3.1.7) $$= n[(-\lambda^2/2) - \sqrt{\lambda^2U^2} + O(1/n)]$$

since $(1/n)\ln n = O(1/n)$.

Since $0 < f_n(U^2) < \infty$, then $-\infty < g_n(U^2) < \infty$. The function $y = g_n(U^2)$ comprises a family of curves starting at $y = -n\lambda^2/2$ for $U^2 = 0$ and increasing strictly to ∞ as $U^2 \to \infty$. Hence $g_n(U^2)$ has one root U^2_{0n}. This root is determined by setting $g_n(U^2) = 0$ or

$$-\lambda^2/2 - \sqrt{\lambda^2U^2} + O(1/n) = 0$$

from whence

(4.3.1.8) $U_{0n}^2 = \lambda^2/4 + O(1/n)$.

Then $\lim_{n \to \infty} U_{0n}^2 = U_0^2 = \lambda^2/4$.

The horizontal line $y = \ln[\beta/(1-\alpha)]$ is the lower boundary line for the sequential process and the intersections of the family of curves with this line determine the values $\underline{U_n^2}$ and in turn $\underline{\chi_n^2}$. To solve, we set

$$g_n(U^2) = \ln[\beta/(1-\alpha)]$$

or

$$-\lambda^2/2 - \sqrt{\lambda^2 U^2} + O(1/n) = (1/n)\ln[\beta/(1-\alpha)]$$

and

$$\underline{U_n^2} = \lambda^2/4 + O(1/n)$$

with

$$\lim_{n \to \infty} \underline{U_n^2} = U_0^2 = \lambda^2/4.$$

Similarly, the horizontal line $y = \ln[(1-\beta)/\alpha]$ is the upper boundary line for the sequential process and the intersections of the family of curves with this line determine the values of $\overline{U_n^2}$ and in turn $\overline{\chi_n^2}$. Again, we solve

$$-\lambda^2/2 - \sqrt{\lambda^2 U^2} + O(1/n) = (1/n)\ln[(1-\beta)/\alpha]$$

and

$$\overline{U_n^2} = \lambda^2/4 + O(1/n)$$

with

$$\lim_{n \to \infty} \overline{U_n^2} = U_0^2 = \lambda^2/4.$$

But $\underline{\chi}_n^2 = n\underline{U}_n^2$ and $\overline{\chi_n^2} = n\overline{U_n^2}$. Hence, $\underline{\chi}_n^2$, $\overline{\chi_n^2} \to n\lambda^2/4$ and the termination proof is complete.

4.3.2 Proof that the sequential T^2-test terminates with probability one

The argument here will be similar to that employed in the case of the sequential χ^2-test. For the sequential T^2-test, the probability ratio is

(4.3.2.1)

$$p_{1n}/p_{0n} = e^{-n\lambda^2/2}\,{}_1F_1[n/2, p/2; n\lambda^2 T_n^2/2(n-1+T_n^2)].$$

$\underline{T}_n^2$ and $\overline{T_n^2}$ are boundary values of T_n^2 corresponding to $p_{1n}/p_{0n} = \beta/(1-\alpha)$ and $p_{1n}/p_{0n} = (1-\beta)/\alpha$ respectively.

Again we assert that

$$P\left\{\text{Sample size} > n\right\} \leq P\left\{\underline{T}_n^2 < T_n^2 < \overline{T_n^2}\right\}$$

$$= \int_0^{\overline{T_n^2}} h(T_n^2)\,dT_n^2 - \int_0^{\underline{T}_n^2} h(T_n^2)\,dT_n^2$$

where $h(T_n^2)$ denotes the non-central T^2-distribution with p and n-p degrees of freedom and non-centrality parameter $n\lambda^2$. We will again show that, as $n \to \infty$ $\underline{T}_n^2 \to \overline{T_n^2} \to n\lambda^2/4$ where λ^2 is specified under H_1. Hence in the limit, $P\left\{\underline{T}_n^2 < T_n^2 < \overline{T_n^2}\right\} = 0$ and therefore the sequential procedure must terminate with probability one.

To show that $\underline{T}_n^2$ and $\overline{T_n^2}$ approach a common limit, we expand (4.3.2.1) which can be written as:

(4.3.2.2)

$$\frac{p_{1n}}{p_{0n}} = \frac{\Gamma(p/2)}{\Gamma(n/2)} e^{-n\lambda^2/2} \sum_{j=0}^{\infty} \frac{\left[\frac{n\lambda^2 T_n^2}{2(n-1)}\right]^j \left[1 + \frac{T_n^2}{n-1}\right]^{-j} \Gamma(\frac{n}{2} + j)}{j!\,\Gamma(p/2 + j)} .$$

As n grows large, $[1 + (T^2)/(n-1)] \to 1$, for fixed T^2, $\Gamma[(n/2) + j]/\Gamma(n/2) = (n/2)^j$ using Stirling's formula and $n\lambda^2T^2/[2(n-1)] \to \lambda^2T^2/2$. Then, as $n \to \infty$,

$$\frac{p_{1n}}{p_{0n}} \to \Gamma(p/2)e^{-n\lambda^2/2} \sum_{j=0}^{\infty} \frac{(n\lambda^2T_n^2/4)^j}{j!\,\Gamma(p/2+j)} ,$$

(4.3.2.3)

$$= e^{-n\lambda^2/2} \; {}_0F_1(p/2; n\lambda^2T_n^2/4),$$

(4.3.2.4)

$$= e^{-n\lambda^2/2 - \sqrt{n\lambda^2T_n^2}} \; {}_1F_1[(p-1)/2, p-1; 2\sqrt{n\lambda^2T_n^2}].$$

This is in the same form as equation (4.3.1.2). Hence, using the same arguments as those employed in Section 4.3.1, it can be shown that $\underline{T_n^2}$ and $\overline{T_n^2}$ both approach a common limit $n\lambda^2/4$ as before and hence the sequential T^2-test terminates with probability one.

4.3.3 Behavior of the bounds for the sequential χ^2- and T^2-tests

The proofs of the theorems discussed in Sections 4.3.1 and 4.3.2 are illustrated to some extent in Figure 4.3.3.

Using the χ^2-test as an example, we plot $g_n(U^2)$ as a function of U^2 for several values of n when $p = 3$ and $\lambda^2 = 1.0$. In this case, $U_0^2 = .25$ and it can be seen that these curves are approaching the line $U^2 = .25$ asymptotically.

$\underline{U_n^2}$ and $\overline{U_n^2}$ are determined by the intersection of these curves with the lines $y = \ln[\beta/(1-\alpha)]$ and $y = \ln[(1-\beta)/\alpha]$. The values of $\overline{U_n^2}$ approach U_0^2 from above. In general, the values of $\underline{U_n^2}$ will begin near zero, pass above U_0^2 and then approach U_0^2 asymptotically from above. It is difficult to show graphically that this is so but Table 4.3.3 will give somenumerical evidence of this behavior. Table 4.3.3 gives the values of $\underline{U_n^2}$ and $\overline{U_n^2}$ for some selected values of n when $\lambda^2 = 1.0$ and p takes the values 3 and 9. It can be seen in both cases that $\overline{U_n^2}$ decreases steadily as n increases and that it approaches .25 in both case although the convergence is very slow. $\underline{U_n^2}$ on the other hand, increases essentially from zero to a point above .25 and then decreases to that value. The convergence again is slow. The maximum value of $\underline{U_n^2}$ is larger as p grows smaller. For $p = 9$, the maximum occurs at $n = 16$ but for $p = 3$, the maximum occurs at $n = 50$. Table 4.3.3 also gives $\underline{V_n^2} = \underline{T_n^2}/n$ and $\overline{V_n^2} = \overline{T_n^2}/n$ for the same values of n, p and λ^2. The behavior of these functions is generally the same as that of $\underline{U_n^2}$ and $\overline{U_n^2}$ but the changes are more pronounced.

BEHAVIOR OF THE FUNCTION

$g_n(U^2)$ for $p = 3$, $\lambda^2 = 1.0$, $\alpha = \beta = .05$

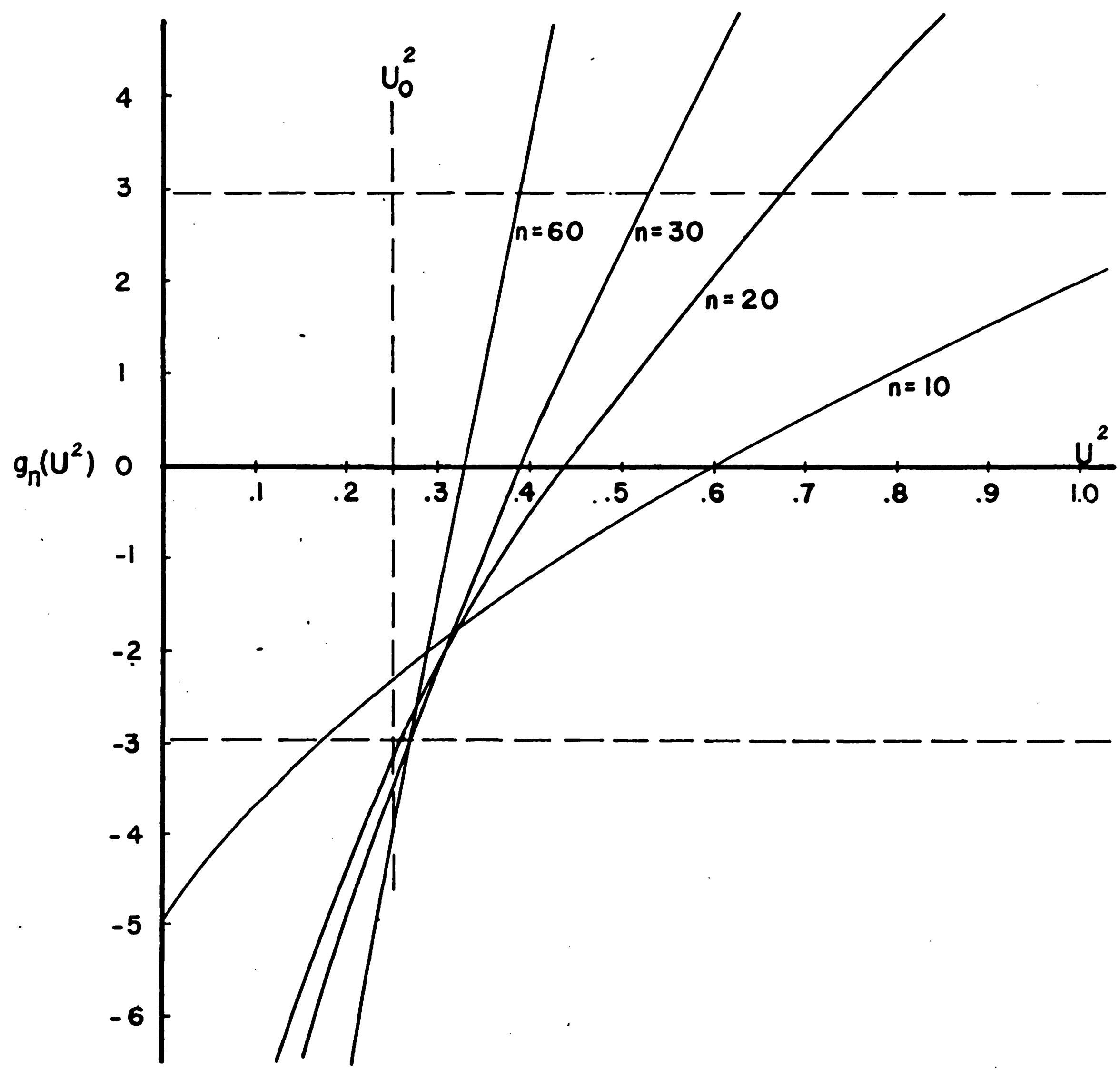

Figure 4.3.3

Table 4.3.3

Behavior of $\underline{U}_n^2$, $\overline{U_n^2}$, $\underline{V}_n^2$, and $\overline{V_n^2}$

for $\lambda^2 = 1.0$ and p = 3, 9.

		p = 3		
n	$\underline{U}_n^2$	$\overline{U_n^2}$	$\underline{V}_n^2$	$\overline{V_n^2}$
6	.0094	2.158	.0078	-
10	.1746	1.219	.1654	4.440
15	.2321	.8413	.2283	1.401
20	.2528	.6755	.2519	.9245
25	.2621	.5828	.2625	.7300
30	.2666	.5240	.2677	.6243
35	.2688	.4831	.2704	.5574
40	.2700	.4532	.2718	.5112
45	.2704	.4304	.2722	.4776
50	.2706	.4123	.2724	.4518
55	.2704	.3977	.2722	.4314
60	.2702	.3856	.2720	.4149

		p = 9		
n	$\underline{U}_n^2$	$\overline{U_n^2}$	$\underline{V}_n^2$	$\overline{V_n^2}$
6	.0279	4.422	-	-
10	.4362	2.340	.5418	-
15	.5031	1.519	.6497	-
20	.4988	1.161	.6285	3.958
25	.4828	.9628	.5916	2.062
30	.4657	.8377	.5577	1.462
35	.4500	.7511	.5289	1.166
40	.4362	.6878	.5050	.9898
45	.4244	.6396	.4849	.8722
50	.4139	.6013	.4678	.7882
55	.4047	.5703	.4533	.7251
60	.3965	.5447	.4407	.6759

V. PROPERTIES AND PROBLEMS OF SEQUENTIAL TESTING

5.1 Introductory statements.

This chapter is devoted to the discussion of several types of miscellaneous topics related to multivariate sequential testing of means. No complete solutions are given to the problems raised but rather, a general statement of each problem is made along with a discussion of various attempts at its solution. The topics to be discussed are:
(1) Methods of estimating the expected sample size of a sequential test of a composite hypothesis, (2) Generalized χ^2- and T^2-statistics, and (3) The problem of tolerances.

5.2 The ASN function.

The primary purpose for using sequential analysis is that, except for borderline cases, the average amount of inspection (denoted by ASN for Average Sample Number) is less than would be required for the comparable fixed-sample-size test. However, the experimenter or quality control engineer may want to know how large such expected savings will be and for what values of θ, the parameter under consideration, if any, will the expected sample size be larger than the fixed-sample-size test.

For the case of sequential tests of simple hypotheses, Wald (1947) has established procedures for determining the approximate expected value of the sample size required to make a decision. In some cases, such as for binomial sampling, the ASN function can be fairly well determined over the whole range of the unknown parameter. However, this procedure is based on the assumption that the successive observations are independent. In the case of sequential tests of composite hypotheses where successive values of the estimator of θ are themselves considered as "observations", this assumption of independence is not valid and therefore, Wald's procedures can no longer be used.

To date, no general expression has been found for the ASN function when successive observations are not independent. One possible solution is to rephrase the problem in such a way that the successive observations are independent. N. L. Johnson (1954) has done this in sequential testing of variance components by sampling additional groups rather than additional items within groups. In the present situation, this technique is possible but would require too many observations to be worthwhile.

A second approach is to assume that all but one of the parameters are known, reducing the problem to a simple hypothesis concerning the one remaining unknown parameter, and to use Wald's formula as a lower bound. Rushton (1950, 1952) has suggested this for the univariate sequential t-test by assuming after, say, thirty observations have been taken, that the estimate of the variance thus far obtained is close enough to the true value and then applying the Wald formula for the test of the mean when the variance is known to obtain a lower bound for the ASN function. Rushton's method would not work in the present situation, even if the population covariance matrix were known (the case of the χ^2-test), since the resultant hypothesis is still composite.

A third approach is the Monte Carlo technique. This has been done for the univariate sequential t-test and will be discussed later in connection with our work.

Still a fourth method is the use of what is known as Bhate's Conjecture (Ray, 1956). Bhate's work is unpublished and is summarized by Ray who used it in estimating some expected sample sizes for the sequential analysis of variance. In essence, Bhate appears to have generalized Wald's original approximation on the assumption that lack of independence would not affect the final results greatly. He works with the logarithm of the probability ratio and conjectures that

(5.2.1) $E[\ln(p_{1n}/p_{0n})|n=E(n)]\approx(1-\alpha)\ln[\beta/(1-\alpha)]+\alpha\ln[(1-\beta)/\alpha]$

when H_0 is the true and

(5.2.2) $E[\ln(p_{1n}/p_{0n})|n=E(n)]\approx\beta\ln[\beta/(1-\alpha)]+(1-\beta)\ln[(1-\beta)/\alpha]$

when H_1 is true.

For the χ^2-case,

$$E[\ln(p_{1n}/p_{0n})] = E[-n\lambda^2/2 + \ln {}_0F_1(p/2;x)]$$

where $x = n\lambda^2\chi^2/4$. We expand $\ln{}_0F_1(p/2;x)$ in the powers of $x - E(x)$. Then,

$$E[\ln(p_{1n}/p_{0n})] = -n\lambda^2/2 + \ln{}_0F_1[p/2;E(x)]$$

$$+ (2/p)[x-E(x)]\frac{{}_0F_1[(p+2)/2;E(x)]}{{}_0F_1[(p/2);E(x)]}\left(\frac{dx}{d\chi_n^2}\right)_{x = [E(x)]} + \ldots \text{ etc.}$$

Using just the first term of the series expansion as a first approximation, we have,

(5.2.3) $\quad E[\ln(p_{1n}/p_{0n})] = -n\lambda^2/2 + \ln{}_0F_1[(p/2);E(x)].$

Under H_0, $E(x) = (n\lambda^2/4)E(\chi_n^2) = np\lambda^2/4$ since $E(\chi_n^2) = p$ under H_0. Under H_1, $E(\chi_n^2) = p + n\lambda^2$ so $E(x) = n\lambda^2(p+n\lambda^2)/4$ (Patnaik, 1949). After taking the antilogarithms of these expressions, the solutions for the ASN are obtained by solving for n the equations:

(5.2.4) $\quad e^{-n\lambda^2/2}{}_0F_1(p/2;np\lambda^2/4) = [\beta/(1-\alpha)]^{(1-\alpha)}[(1-\beta)/\alpha]^{\alpha}$

when H_0 is true and

(5.2.5)

$$e^{-n\lambda^2/2}{}_0F_1[p/2;n\lambda^2(p+n\lambda^2)/4] = [\beta/(1-\alpha)]^{\beta}[(1-\beta)/\alpha]^{(1-\beta)}$$

when H_1 is true.

Following the same procedure for T^2 as a first approximation,

(5.2.6) $\quad E[\ln(p_{1n}/p_{0n})] \approx -n\lambda^2/2 + \ln{}_1F_1[n/2,p/2;E(x)]$

where $x = (n\lambda^2T_n^2)/[2(n-1+T^2)]$. Since $T_n^2 = (n-1)pF/(n-p)$, $x = (n\lambda^2pF)/[2(n-p+pF)]$ where F has $\nu_1 = p$ and $\nu_2 = n-p$

degrees of freedom. Therefore x can be written as $(n\lambda^2\nu_1F)/[2(\nu_2+\nu_1F)]$. The expected value of $\nu_1F/(\nu_2+\nu_1F)$ under H_0 is $\nu_1/(\nu_1+\nu_2)$ or p/n in the present case so that $E(x) = p\lambda^2/2$ (Wishart, 1932). Under H_1, again by Wishart,

$$\begin{aligned} E[\nu_1F/(\nu_2+\nu_1F)] &= E(y, \text{ say}) \\ &= 1 - [\nu_2/(\nu_1+\nu_2)]\,{}_1F_1[1,(\nu_1+\nu_2+2)/2;y] \\ &= 1 - e^{-y}[\nu_2/(\nu_1+\nu_2)]\,{}_1F_1[(\nu_1+\nu_2)/2,(\nu_1+\nu_2+2)/2;-y] \end{aligned}$$

the latter form based on Kummer's Formula. In the present situation,

$$\begin{aligned} E(x) &= (n\lambda^2/2)E[\nu_1F/(\nu_2+\nu_1F)] \\ &= (n\lambda^2/2)\left\{1-e^{-n\lambda^2/2}[(n-p)/n]\,{}_1F_1[n/2,(n+2)/2;n\lambda^2/2]\right\}. \end{aligned}$$

Again, after taking antilogarithms of these expressions, the solutions for the ASN are obtained by solving for n, the equations:

$$(5.2.7) \qquad e^{-n\lambda^2/2}\,{}_1F_1(n/2,p/2;p\lambda^2/2) = [\beta/(1-\alpha)]^{(1-\alpha)}[(1-\beta)/\alpha]^{\alpha}$$

when H_0 is true and

(5.2.8)

$$e^{-n\lambda^2/2}$$

$$\cdot\,{}_1F_1\left\{n/2,p/2;n\lambda^2/2(1-e^{-n\lambda^2/2}[(n-p)/n]\,{}_1F_1[n/2,(n+2)/2;n\lambda^2/2])\right\}$$

$$= [\beta/(1-\alpha)]^{\beta}[(1-\beta)/\alpha]^{(1-\beta)}$$

when H_1 is true.

Table 5.1 gives the conjectural expected sample sizes for both the χ^2-and T^2-tests for $p = 3, 8$; $\lambda^2 = .5, 1.0, 2.0$;

$\alpha = \beta = .05$ along with the corresponding sample size required by the fixed-sample-size test of the same strength.

Table 5.1

Conjectural Expected Sample Sizes, $\alpha = \beta = .05$

χ^2-test				
p	λ^2	under H_0	under H_1	Fixed Sample
3	.5	27	17	36
3	1.0	13	9	18
3	2.0	6	4	9
8	.5	32	25	48
8	1.0	16	12	24
8	2.0	7	6	12

T^2-test				
p	λ^2	under H_0	under H_1	Fixed Sample
3	.5	27	24	37
3	1.0	14	15	21
3	2.0	6	10	13
8	.5	37	36	55
8	1.0	21	23	29
8	2.0	13+	17	20

It is to be noted that for small values of λ^2, larger sample sizes are required to reach a decision under H_0 than H_1 and that the situation reverses itself as λ^2 increases.

The only check on these conjectural values would be the use of the Monte Carlo approach. Unfortunately, only one such study is currently available and that is for the case $p = 1$. This study was performed by K. J. Arnold (Natl. Bur. Stds. 1951) for the case $p = 1$, $\lambda^2 = 1.0$ and $\alpha = \beta = .05$. In that study 500 sets of observations were sampled for the cases $\lambda^2 = 0$ and $\lambda^2 = 1.0$. The average sample size to reach a decision under H_0 was 10.00 while the conjectural value was 10.7. Conversely, the average sample size to reach a decision under H_1 was 11.2 while the conjectural value was 9.7. Both of these differences are significant based on the variability of the 500 results in each case but, at least for the case $p = 1$, the conjectural values appear to be of the right order of magnitude. The fixed sample size required for this same test is $n = 15$. It is interesting to note that the actual α and β values from this Monte Carlo study were .044 and .034 respectively, somewhat different from the intended values.

It might be pointed out that Ray used this example too, considering the t-test as a special case of a one-way classification, but, in his work, a rounding error occurs which makes his conjoctured value appear to be closer to Arnold's result than it really is.

5.3 Generalized X^2- and T^2-statistics.

In addition to the X^2- and T^2-statistics already discussed, there are two others in each case which deserve mention and complete the family of χ^2- and T^2-statistics (Hotelling, 1947). The χ^2-statistic considered in Chapter III is actually χ^2_M in Hotelling's terminology and is a measure of the departure of the vector of sample means from the standard values relative to preassigned values of the variances and covariances which, in this section, will be denoted by Σ_0. (In this section, the subscript n will be assumed for χ^2, T^2, and x unless otherwise noted.) As has already been stated, $n(\underline{\bar{x}} - \underline{\mu}_0)\Sigma_0^{-1}(\underline{\bar{x}} - \underline{\mu}_0)'$ is distributed like non-central χ^2 with p degrees of freedom. Let

$$(\underline{\mu} - \underline{\mu}_0)\Sigma_0^{-1}(\underline{\mu} - \underline{\mu}_0)' = \lambda^2_M$$

in our new notation. The parameter of non-centrality is now $n\lambda^2_M$. As before, it will be assumed that the hypotheses are:

$$H_0\text{:}\quad \lambda^2_M = 0,$$

$$H_1\text{:}\quad \lambda^2_M = \lambda^2_{M_1}.$$

In addition, there is a second statistic:

$$\chi^2_D = (n-1)\mathrm{Tr}S\Sigma_0^{-1} \quad (5.3.1)$$

This is a measure of the variation of the sample observations about their means and χ^2_D is essentially a test of whether or not this variation is significantly greater than the preassigned variances and covariances. The hypotheses to be tested are:

$$H_0: \ \Sigma = \Sigma_0 \ (\text{or } \lambda_D^2 = \mathrm{Tr}\Sigma\Sigma_0^{-1} = p),$$

$$H_1: \ \Sigma > \Sigma_0 \ (\text{or } \lambda_D^2 = \lambda_{D_1}^2 > p).$$

χ_D^2 is distributed like χ^2 with $(n-1)p$ degrees of freedom and non-centrality parameter $(n-1)\lambda_D^2$. It can be seen that χ_M^2 and χ_D^2 are used as multivariate extensions of univariate operations with $\bar{x}$ and s or the range.

The sum of χ_M^2 and χ_D^2 is a measure of the overall variation of the sample from standard. This statistic is denoted by χ_0^2, distributed like χ^2 with np degrees of freedom and non-centrality parameter $n\lambda_0^2$. This quantity can be determined either by the relationship

$$(5.3.2) \qquad \chi_0^2 = \chi_M^2 + \chi_D^2$$

or

$$(5.3.3) \qquad \chi_0^2 = \sum_{i=1}^{n} \chi_i^2$$

where $\chi_i^2 = (\underline{x}_i - \underline{\mu}_0)\Sigma_0^{-1}(\underline{x}_i - \underline{\mu}_0)'$, $i = 1, \ldots, n$ observations. In this latter form, χ_D^2 can be obtained by subtraction if χ_M^2 and χ_0^2 are both known and the sample covariance matrix S need not be computed at all. Since $\chi_0^2 = \chi_M^2 + \chi_D^2$, it follows that, in general, $n\lambda_0^2 = n\lambda_M^2 + (n-1)\lambda_D^2$. Therefore, the logical hypotheses to be tested would be:

$$H_0: \ n\lambda_0^2 = (n-1)p$$

$$H_1: \ n\lambda_0^2 = n\lambda_{M_1}^2 + (n-1)\lambda_{D_1}^2 = n\lambda_{0_1}^2 .$$

In the case of the χ^2-statistics, sequential tests can be applied not only to χ_M^2 but also to χ_D^2 and χ_0^2 as well. That is, not only can the mean of a lot be tested sequentially, but at the same time a check can be made on the variability of the lot as well. The test procedures are identical, the only difference being in the computation of the test statistic, the ratio p_{1n}/p_{0n} and the preparation of additional tables. These extra tables would be required since both λ_D^2 and λ_0^2 are different from zero under the null hypothesis. The test statistics have already been described above. The probability ratios are as follows:

χ_M^2:

$$(5.3.4)\quad p_{1n}/p_{0n} = e^{-n\lambda^2_{M_1}/2}\ {}_1F_1(p/2;n\lambda^2_{M_1}(\chi^2_M/4),$$

χ_D^2:

$$(5.3.5)\quad p_{1n}/p_{0n} = e^{-(n-1)(\lambda^2_{D_1}-p)/2}\ \frac{{}_0F_1[(n-1)p/2;(n-1)\lambda^2_{D_1}\chi^2_D/4]}{{}_0F_1[(n-1)p/2;(n-1)p\chi^2_D/4]},$$

χ_0^2:

$$(5.3.6)\quad p_{1n}/p_{0n} = e^{-n(\lambda^2_{0_1}-p)/2}\ \frac{{}_0F_1[np/2;n\lambda^2_{0_1}\chi^2_0/4]}{{}_0F_1[np/2;np\chi^2_0/4]}.$$

If this family of sequential tests were used, the experimenter or inspector could tell after each item tested or inspected:

(1) Whether or not the sample means differed significantly from standard,

(2) Whether or not the variation about the sample means was greater than the preassigned Σ_0, and

(3) Whether or not the overall variability of the sample thus far is larger than should have been expected.

All of these sequential processes terminate with probability unity; the proof in Section 4.3.1, suitably modified, should suffice for all of these.

The same method as employed in Section 5.2, (Bhate's conjecture) may be used to obtain the conjectural Average Sample Numbers for these procedures as follows:

χ^2_M: These are given in Section 5.2, equations (5.2.4) and (5.2.5).

χ^2_D:

$$(5.3.7)\quad e^{-(n-1)(\lambda^2_{D_1}-p)/2}\ \frac{{}_0F_1[(n-1)p/2;(n-1)^2p\lambda^2_{D_1}/4]}{{}_0F_1[(n-1)p/2;(n-1)^2p^2/4]} = [\beta/(1-\alpha)]^{(1-\alpha)}[(1-\beta)/\alpha]^{\alpha}$$

for H_0, and

$$(5.3.8)\quad e^{-(n-1)(\lambda^2_{D_1}-p)/2}\ \frac{{}_0F_1[(n-1)p/2;(n-1)^2\lambda^2_{D_1}(p+\lambda^2_{D_1})/4]}{{}_0F_1[(n-1)p/2;(n-1)^2p(p+\lambda^2_{D_1})/4]} = [\beta/(1-\alpha)]^{\beta}[(1-\beta)/\alpha]^{(1-\beta)}$$

for H_1.

χ^2_O:

$$(5.3.9)\quad e^{-n(\lambda^2_{O_1}-p)/2}\,\frac{{}_0F_1[np/2;n^2p\lambda^2_{O_1}/4]}{{}_0F_1[np/2;(np)^2/4]} = [\beta/(1-\alpha)]^{(1-\alpha)}[(1-\beta)/\alpha]^{\alpha}$$

for H_0 and

$$(5.3.10)\quad e^{-n(\lambda^2_{O_1}-p)/2}\,\frac{{}_0F_1[np/2;n^2\lambda^2_{O_1}(p+\lambda^2_{O_1})/4]}{{}_0F_1[np/2;n^2p(p+\lambda^2_{O_1})/4]} = [\beta/(1-\alpha)]^{\beta}[(1-\beta)/\alpha]^{(1-\beta)}$$

for H_1.

Given α, β, p, $\lambda^2_{D_1}$, and $\lambda^2_{O_1}$, these equations can be solved for n as before.

Although the T^2-statistics generalize themselves in the same way as the χ^2-statistics for the fixed sample situation, they do not lend themselves to sequential tests. The T^2_M statistic is the same one described in Chapter III. However, there are no counterparts for χ^2_D or χ^2_O. T^2_D generally represents the variability in a subgroup of an experiment compared to, say, the average subgroup variability of the experiment. This situation does not exist in the sequential procedures under discussion since the variability of the total sample is not known in advance. It could conceivably be used to check the variability of the latter part of a sequence of observations against the first part of the sequence but this is not very likely unless the expected sample size is rather large. Somewhat more conceivable is a T_M^2-test run in parallel with a χ_D^2-test where the test on variances is based on previous experience but where the test on means is based only on the variability of the sample itself. Examples of this situation will be mentioned in Section 5.4. However, for routine acceptance sampling these cases must be considered somewhat pathological. T_M^2 and T_0^2 are useful statistics in the multivariate analysis of variance and could conceivably be used in some sequential multivariate schemes when more is known about the forms of their distributions. Sequential tests for the roots of determinantal equations might also prove useful although the computational procedures in carrying out a sequential test of this type would be considerably more involved.

5.4 <u>The problem of tolerances</u>.

Another problem in connection with sequential multivariate tests for means is the relationship between the tolerances established on a product and the specifications of the null and alternative hypotheses. Practically all of the material in this section is also applicable to fixed sample tests in the multivariate case as well; very little seems to have been written on this problem. In acceptance sampling procedures, the values of the parameters under consideration, corresponding to each hypothesis, are usually related to standards and tolerances previously established. The standards are no problem, either in specification or operation, since they are easily specified and can be substituted directly for the $\underline{\mu}_0$'s in the expression $(\underline{x}-\underline{\mu}_0)\Sigma^{-1}(\underline{x}-\underline{\mu}_0)'$. However, the incorporation of the tolerances into this expression may be an entirely different matter. It may be quite possible to set up tolerances on each individual variable but the additional problem of specifying the correlations between variables has received scant attention so far. The most common practices are either to specify the tolerances for each variable and assume that the variables are independent (so that Σ_0 is a diagonal matrix) or use <u>natural tolerances</u>, that is, tolerances based on the variability of the production process. Most difficult of all, usually, is the specification of λ^2, although in the years to come, as more use is made of multivariate analysis in industry, engineering personnel should become more familiar with this concept.

The establishment of tolerances in industry is usually a joint effort involving more than one department and may include such widely separated functions as sales, production design, production control and quality control. The opinions expressed by representatives of these various groups may vary considerably in the establishment of a system of tolerances. The production interests, for example, may recommend natural tolerances while the sales departments may argue that in certain instances, natural tolerances are too wide and would permit the shipping of material that would be unacceptable to the customer. A great deal has been said and written about this problem for the univariate situation; much less has been done for the multivariate case. Since this is a highly controversial subject for the univariate case, it would be even more so when more than one variable is under consideration.

If natural tolerances were employed, the establishment of sequential procedures would be a relatively simple matter since Σ could be estimated from prior knowledge of the process. If the process is a well established one, Σ might be assumed to be known and the sequential χ^2-test could be employed. If this were not the case and there was only a limited amount of information available concerning the process variabilities, then this information could be used to estimate Σ in the expression for λ^2 but the sample covariance matrix would be used for a sequential procedure and a sequential T^2-test would be in order. This is a situation in which the use of parallel T_M^2- and χ_D^2-tests, mentioned in the previous section, might be used.

If the tolerances are tighter than the process is capable of maintaining, their use will perhaps result in strained relations between production and sales personnel and eventually either the sales department will have to make less stringent demands or improvements will have to be made to the production process or both. The problem of tolerances that are too tight, of course, is an old one. If corrective action is to be taken on the basis of the levels of the rejected lots, unnecessary and harmful adjustments may be made to the process. A multivariate analogue of the work of Jackson, Freund, and Howe (1959) regarding errors of process adjustments would be desirable in order to investigate the effects of such an operation.

If the stated tolerances are wider than the process variability, one could continue to use the measure of process variability in the test procedure. However, if it seems that this might reject a considerable amount of acceptable material, the use of some multivariate analogue of sequential acceptance sampling for variables schemes (Wallis, 1947) might be more appropriate. A third alternative would be to increase λ^2. If the correlations between variables were high, this procedure might also result in acceptable material being rejected. There is no one solution to this problem. Specified correlations may be important in one case and not in another. Each situation has to be treated on its own merits.

There remains the problem of specifying λ^2, α, and β. This is essentially the reverse of the ordinary confidence limits problem in that n is a function of λ^2, α, and β rather than λ^2 being a function of n, α, and β. The smaller λ^2, α, and β are, the larger will be the expected sample size and with small λ^2 or β (or large α), there is also the possibility of rejecting good product. On the other hand, if λ^2 and β are large, (or α small) there is the reverse problem of accepting material of poor quality. At present, the answer in specifying λ^2, α, and β seems to be to couple intuition with experience. This is currently true however of all acceptance sampling schemes, be they univariate or multivariate, fixed-sample or sequential, so the situation for these sequential multivariate techniques is no better and, except for a few isolated cases, no worse than anywhere else in the field of acceptance sampling. The choice of these values for a particular problem will be discussed in Chapter VII.

VI. COMPUTATIONAL PROCEDURES

6.1 Introduction.

As described in Chapters II and III, a sequential test of significance consists of computing a probability ratio p_{1n}/p_{0n} and comparing this ratio with the constants $\beta/(1-\alpha)$ and $(1-\beta)/\alpha$. However, for the sequential χ^2- and T^2-tests the calculation of p_{1n}/p_{0n} involves evaluating functions of the form ${}_0F_1(c;x)$ or ${}_1F_1(a,c;x)$ after each observation.

Consider the χ^2-test. It is better, once and for all, to solve

$$(6.1.1) \qquad e^{-n\lambda^2/2}\,{}_0F_1(p/2;n\lambda^2\chi_n^2/4) = \beta/(1-\alpha)$$

and

$$(6.1.2) \qquad e^{-n\lambda^2/2}\,{}_0F_1(p/2;n\lambda^2\chi_n^2/4) = (1-\beta)/\alpha$$

for χ_n^2, which yield respectively $\underline{\chi_n^2}$ and $\overline{\chi_n^2}$ for various values of n, p, λ^2, α, and β. The sequential χ^2-test procedure would then be: from a sample of size n, obtain χ_n^2;

a. if $\chi_n^2 \leq \underline{\chi_n^2}$, accept H_0;

b. if $\chi_n^2 \geq \overline{\chi_n^2}$, accept H_1; and

c. if $\underline{\chi_n^2} < \chi_n^2 < \overline{\chi_n^2}$, continue sampling.

The only computation required for each sampled observation vector would be the computation of χ_n^2. Values of $\underline{\chi_n^2}$ and $\overline{\chi_n^2}$ can be obtained from tables. We discuss these tables in in Section 6.4.

Similar procedures may be carried out for the sequential T^2-test and lead to values $\underline{T}^2_n$ and $\overline{T}^2_n$ used in the same way as $\underline{\chi}^2_n$ and $\overline{\chi}^2_n$.

6.2 Inadequacy of the methods of interpolation and iteration.

To find, say, $\underline{\chi}^2_n$, one must solve (6.1.1) for χ^2_n. This involves solving for x in the expression

$$_0F_1(p/2;x) = e^{n\lambda^2/2}\beta/(1-\alpha)$$

when n, p, λ^2, α, and β are given. This in turn would involve Lagrangian interpolation in our tables of $_0F_1(c;x)$ since the values of the argument x are not evenly spaced. These interpolation procedures can become laboricus and in some instances, the results are not very accurate. The same problem arises in finding $\underline{T}^2_n$ and $\overline{T}^2_n$ since we are involved with the same sort of interpolation problems in using the tables of Nath, Rushton, and Lang for the function $_1F_1(a,c;x)$

An iterative technique is much more desirable since it can be more readily programmed for a computer and the results can be obtained to the degree of accuracy desired. The so-called method of iteration (Nielsen, 1956) is used when an equation $f(x) = 0$ can be expressed in the form $x = \phi(x)$. A trial value x_0 of x is substituted in the function $\phi(x)$ and the result is the first approximation to the true value of x, say $x^{(1)}$. This in turn is substituted in $\phi(x)$ to get $x^{(2)}$ and so on. If the iteration procedure converges, the sequence of values $x^{(1)}$, $x^{(2)}$,..., etc. will converge to the true value of x. The method of iteration will converge provided $|\phi'(x)| < 1$ in the neighborhood of the desired root of the equation.

The confluent hypergeometric form of p_{1n}/p_{0n} of the χ^2-test can be placed in this form. Rewrite equations (6.1.1) and (6.1.2) as

$$e^{-n\lambda^2/2-\sqrt{n\lambda^2\chi_n^2}}\ {}_1F_1[(p-1)/2, p-1; 2\sqrt{n\lambda^2\chi_n^2}] = R$$

where R is a general symbol representing either risk constant. Letting $x = 2\sqrt{n\lambda^2\chi_n^2}$, $a = (p-1)/2$ and taking logarithms of both sides, we have

$$\ln R = -n\lambda^2/2 - x/2 + \ln {}_1F_1(a,2a;x)$$

or

$$x = -2 \ln R - n\lambda^2 + 2 \ln {}_1F_1(a,2a;x) = \phi(x).$$

which is the required form for the method of iteration. Now

$$\phi'(x) = {}_1F_1(a+1,2a+1:x)/{}_1F_1(a,2a;x).$$

This is a special case of the function G discussed in Appendix A and it can easily be seen that $|\phi'(x)| > 1$ for all positive x and therefore the method of iteration will not converge for the χ^2-test.

A similar situation occurs for the sequential T^2-test. Starting with

$$e^{-n\lambda^2/2}\,{}_1F_1(n/2,p/2;x) = R$$

where $x = n\lambda^2T_n^2/2(n-1+T_n^2)$ and R, again, represents either risk constant, we multiply both sides of the equation by x and divide both sides by R so that

$$\phi(x) = (x/R)e^{-n\lambda^2/2}\,{}_1F_1(n/2,p/2;x) = x.$$

This procedure will converge only when

$$|\phi'(x)| = \frac{1}{R}e^{-n\lambda^2/2} \cdot \left\{{}_1F_1(n/2,p/2;x)+(nx/p)\,{}_1F_1[(n+2)/2,(p+2)/2;x]\right\} < 1.$$

For the sequential T^2-test, $\phi'(x) < 1$ for some combinations of n, p, λ^2, R, and x but not for others.

It appears, then, that the method of iteration will result in only partial success.

6.3 The Newton-Raphson method.

Another iterative procedure is the Newton-Raphson method (Nielsen, 1956). In this procedure, the solution to the equation $f(x) = 0$ is obtained by the following method of successive approximations: if x_k is the k^{th} approximation and x_{k-1} is the (k-1)th approximation to the true value x, then

$$x_k = x_{k-1} - f(x_{k-1})/f'(x_{k-1})$$

providing $f'(x)$ is not equal to or close to zero in the neighborhood of the desired root of the equation. Again, representing $\beta/(1-\alpha)$ or $(1-\beta)/\alpha$ by the general symbol R, we have for the χ^2-test:

(6.3.1)

$$x_k = x_{k-1} - \frac{[e^{-n\lambda^2}\,{}_0F_1(p/2;x_{k-1}) - R]}{\left\{(2/p)e^{-n\lambda^2/2}\,{}_0F_1[(p+2)/2;x_{k-1}]\right\}}$$

where $x = n\lambda^2\chi_n^2/4$. For the T^2-test:

(6.3.2)

$$x_k = x_{k-1} - \frac{[e^{-n\lambda^2/2}\,{}_1F_1(n/2,p/2;x_{k-1}) - R]}{\left\{(n/p)e^{-n\lambda^2/2}\,{}_1F_1[(n+2)/2,(p+2)/2,x_{k-1}]\right\}}$$

where $x = n\lambda^2T_n^2/2(n-1+T_n^2)$. The behavior of the first derivative in each case is such that the processes converge fairly rapidly over the range of values n, p, and λ^2 considered.

Since the Newton-Raphson method would apply for both the χ^2- and T^2-tests over the desired range of values of n and λ^2, equations (6.3.1) and (6.3.2) were employed to construct tables. These computations were performed on an IBM 650 computer. For a given test with λ^2, R, and p specified, a trial value of x was made for the lowest value of n for which a decision could be reached. The computational technique was such that when a final value of x was obtained for a given value of n, the computer would automatically "step up" to n+1 and use the final approximation for x corresponding to n as the starting approximation for x corresponding to n+1. In general four or five iterations were necessary for five-digit accuracy.

6.4 Use of tables to facilitate sequential χ^2- and T^2-tests.

Appendix B contains tables designed to facilitate application of the sequential χ^2-test. For $\alpha = \beta = .05$, these tables give $\underline{\chi}^2_n$ and $\overline{\chi}^2_n$ to four digits, for various values of n when $\lambda^2 = 0.5$, 1.0, and 2.0 and $p = 2,3,\ldots,9$. (The case $p = 1$ is not included as it is already covered in Wald's book.) Since the expected sample size decreases with increasing λ^2, n runs from 1 to 60 for $\lambda^2 = 0.5$, from 1 to 45 for $\lambda^2 = 1.0$, and from 1 to 30 for $\lambda^2 = 2.0$. While it is possible to accept H_1 after the first observation regardless of the value of λ^2, H_0 can be accepted only after at least $\frac{6}{\lambda^2}$ observations have been made since the solutions for $\underline{\chi}^2_n$ result in negative values of $\underline{\chi}^2_n$ up to that point. These tables are shown for $p = 3$ in Figure 6.4.

The corresponding tables for the sequential T^2-test for the same values of α, β, λ^2, p, and the same ranges for n are given in Appendix C. [The case $p = 1$ is not included since it has already been tabulated in a slightly different form by the National Bureau of Standards (1951).] In the case of the sequential T^2-test, neither hypothesis can

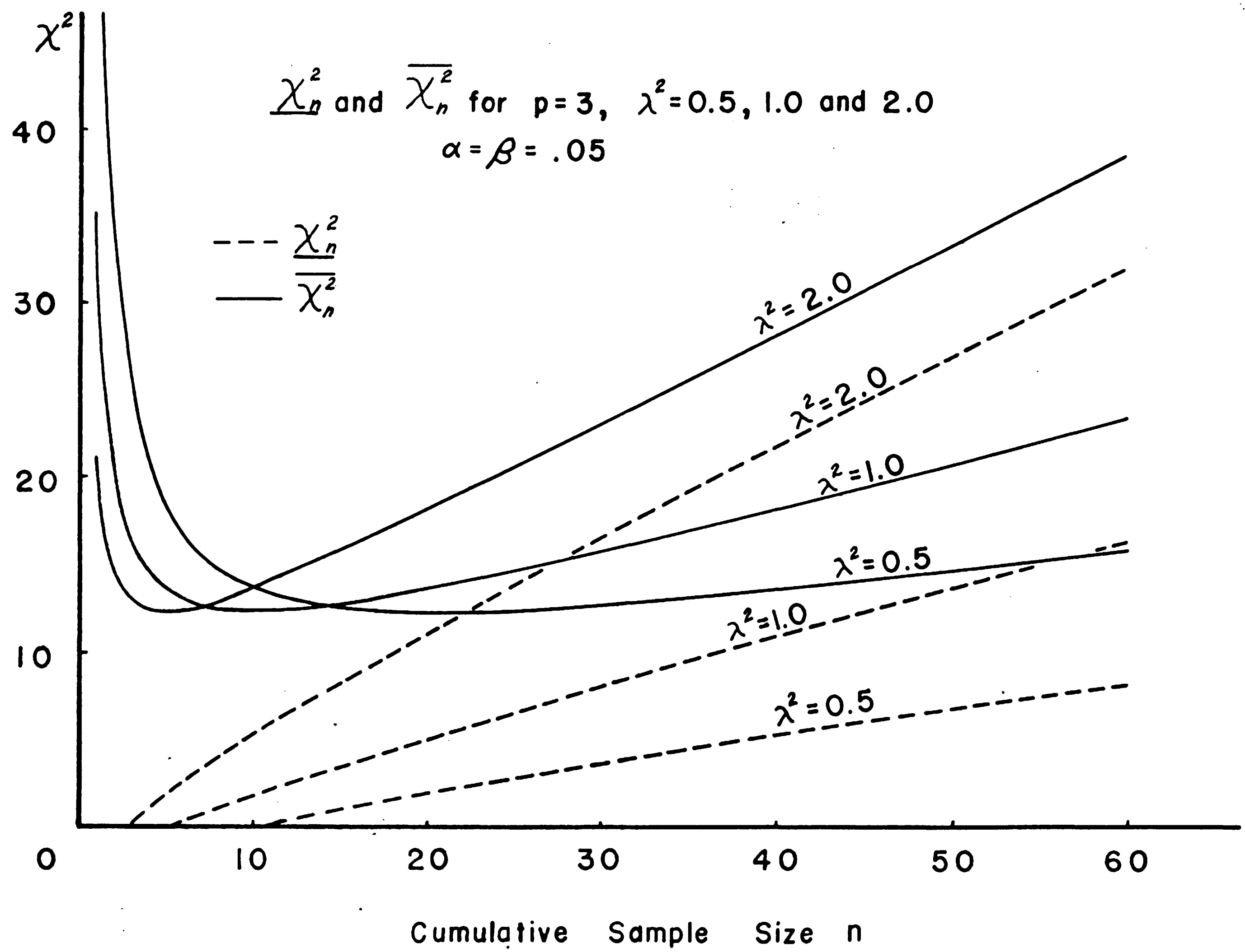
$\underline{\chi}^2_n$ and $\overline{\chi^2_n}$ for p = 3, λ^2 = 0.5, 1.0 and 2.0
$\alpha = \beta = .05$
$\underline{\chi}^2_n$
$\overline{\chi^2_n}$
χ^2
40
30
20
10
0
10
20
30
40
50
60
$\lambda^2 = 2.0$
$\lambda^2 = 2.0$
$\lambda^2 = 1.0$
$\lambda^2 = 0.5$
$\lambda^2 = 1.0$
$\lambda^2 = 0.5$
Cumulative Sample Size n

Figure 6.4

be accepted until at least p+1 observations have been made. In addition, the solutions for both $\underline{T^2_n}$ and $\overline{T^2_n}$ are negative for small values of n. $\underline{T^2_n}$ becomes positive when $n > \frac{6}{\lambda^2}$ as in the case of $\underline{\chi^2_n}$. Therefore, H_0 cannot be accepted until a maximum of p+1 and $\frac{6}{\lambda^2}$ observations have been made. There is no corresponding rule for $\overline{T^2_n}$ but the solution yields positive values beginning around the (p+6)th observation when $\lambda^2 = 1.0$. If $\lambda^2 = 0.5$, usually two or three additional observations will be required and if $\lambda^2 = 2.0$, one or two less observations will be required in order to accept H_1.

Let $\underline{\chi^2_n}(\lambda^2)$ denote the value of $\underline{\chi^2_n}$ given λ^2 and define a similar relationship for $\overline{\chi^2_n}(\lambda^2)$. It is interesting to note that for given α, β, and p, the relations:

$$(6.4.1)\qquad \begin{aligned} \underline{\chi^2_n}(\lambda^2) &= \underline{\chi^2_{n/d}}(d\lambda^2), \\ \overline{\chi^2_n}(\lambda^2) &= \overline{\chi^2_{n/d}}(d\lambda^2), \end{aligned}$$

hold for all positive n, λ^2, and d. This means that tables can be prepared for any multiple of $\lambda^2 = 0.5$, say, using only the values of $\underline{\chi^2_n}$ and $\overline{\chi^2_n}$ for $\lambda^2 = 0.5$ although these values could not be obtained, necessarily, for every value of n. Nevertheless, this relationship is useful for getting approximate values of $\underline{\chi^2_n}$ and $\overline{\chi^2_n}$ for values of λ^2 other than 0.5, 1.0, and 2.0. For example, $\underline{\chi^2_2}(.5) = \underline{\chi^2_1}(1)$, $\underline{\chi^2_4}(.5) = \underline{\chi^2_2}(1)$, $\underline{\chi^2_6}(.5) = \underline{\chi^2_3}(1), \ldots$; d in this case is equal to 2. The same relationships hold for $\overline{\chi^2_n}$. No similar relationship exists for the T^2-test.

To obtain approximate valus of $\underline{\chi^2_n}$, $\overline{\chi^2_n}$, $\underline{T^2_n}$, and $\overline{T^2_n}$ not covered by our tables when no computer is available, one can still resort to the existing tables of ${}_0F_1(c;x)$ and ${}_1F_1(a,c;x)$ using interpolation.

VII. NUMERICAL EXAMPLE

7.1 Introduction.

In this chapter, we illustrate the procedures discussed in Chapter III by way of an example drawn from the area of ballistic missile testing. This testing can be carried on in two ways: (1) flight or operational testing where the missile is actually put into flight and (2) static testing where the propulsion unit is fastened down to prevent its flight during burning and measures of characteristics such as thrust and pressure are obtained by placing strain gages on the nose of the missile. This latter type of testing is used in the acceptance sampling of such missiles as the Honest John and the Nike. Our problem specifically deals with the inspection of the solid propellant boosters for the Nike-Ajax and the Nike-Hercules.

Three characteristics will be considered:

(1) Action time. This is a measure of the total elapsed time during which the propellant in the booster is burning (neglecting after-burning).

(2) Total impulse. This is a measure of the total force exerted by the booster during burning.

(3) Maximum pressure. This is a measure of the maximum pressure development inside the chamber of the booster during burning.

These boosters are manufactured in regular production lots which are passed or rejected on the basis of sampling inspection. Current inspection procedures require nine rounds to be tested in order to pass judgment on the quality of the lot. In order to illustrate both the sequential X^2- and T^2-tests, some liberties will be taken with the assumptions related to the inspection program for these missiles. We will consider two cases: Case I will consist of an example where only the standards and tolerances are given and only scant information is available concerning the actual variability of the production process as regards action time, total impulse, and maximum pressure. Case II will assume that sufficient sampling inspection experience has been obtained in order for the population covariance matrix for these three variables to be known. The other liberty we shall take is in the establishment of upper and lower tolerances for all variables when, in fact, only one-sided tolerances exist in some instances. Our procedures at present will only accomodate two-sided tolerances. Extension of these methods to the variables inspection techniques as discussed in Section 5.4 should enable us to handle the single tolerance problem also.

It should be emphasized that although the data and, to some extent, the tolerances used in this chapter are from an actual production situation, the methods employed to establish λ^2 under H_1 in no way reflect the actual acceptance sampling policy employed in the Nike program.

Our variables will be coded as follows:

x_1 = (Action time - 3.000) seconds x 10^3.

x_2 = (Total impulse - 143000) pounds-seconds x 10^{-1}.

x_3 = (Maximum pressure - 1000) PSI.

In terms of these coded variables, our standards and tolerances for individual rounds are defined as follows:

Action time: $x_1 = 100 \pm 120$,

Total impulse: $x_2 = 200 \pm 300$,

and

Maximum pressure: $x_3 = 50 \pm 200$.

7.2 Case I: Σ unknown.

In this first example, it will be assumed that a sequential sampling plan is required which will guarantee with risk $\beta = .05$ that no lot shall be accepted which contains more than 2.5% defective rounds and at the same time guarantee with risk $\alpha = .05$ that no lot shall be rejected when the true means of all three characteristics are on standard. This implies the following hypotheses:

H_0: Proportion defective $\leq .025$;

H_1: Proportion defective $\geq .025$.

It will further be assumed that only a meager amount of information is available concerning the variability of these characteristics. From this information, it is inferred that the individual tolerances constitute limits of $\pm 3\sigma$ for individual observations about their standards and that there is no evidence that the variables are correlated. (The assumption of independence regarding tolerances is not always valid. In some types of missiles, total impulse and action time must be negatively correlated to insure a fixed range for their flight.)

Considering each characteristic separately, the requirement that 97.5% of the lot must be within tolerances implies that the true lot mean for that characteristic cannot be closer than 2.24σ to either tolerance limit; conversely, the true mean must be within $.76\sigma$ of the standard since the tolerances were assumed to be $\pm 3\sigma$ limits. Therefore, the lot means for each characteristic should be in the intervals:

(7.2.1) $\quad x_1$: 100 ± 30.4,

$\qquad x_2$: 200 ± 76.0,

and

$\qquad x_3$: 50 ± 50.7.

This would imply that the non-centrality parameter under H_1 for a given characteristic would be $(.76)^2$ and for a three-variable problem, $\lambda^2 = 3(.76)^2 = 1.73$ under H_1 where now:

(7.2.1)

$$\lambda^2 = [\mu_1-100 \;\; \mu_2-200 \;\; \mu_3-50] \begin{bmatrix} 1600 & 0 & 0 \\ 0 & 10000 & 0 \\ 0 & 0 & 4489 \end{bmatrix}^{-1} \begin{bmatrix} \mu_1-100 \\ \mu_2-200 \\ \mu_3-50 \end{bmatrix} .$$

The closest tabular value of λ^2 in our tables is 2.0 and this is probably quite adequate considering the crudeness of the determination of λ^2 in the first place. Our hypotheses have now been restated as:

H_0: $\lambda^2 = 0$,

H_1: $\lambda^2 = 2.0$.

Since the covariance matrix is unknown, we now employ the sequential T^2-test using the tables in Appendix C for $p = 3$ and $\lambda^2 = 2.0$. Acceptance of a lot is not permissible until at least four rounds have been fired. The individual coded observations, the deviations of their corresponding cumulative averages from standard, and the resultant values of T_n^2 are shown in Table 7.2.1. From either Appendix C or Figure 7.2, where the successive values of T_n^2 are plotted, it can be seen that the sequential process has terminated on the firing of the ninth round with the acceptance of the null hypothesis that the lot is of acceptable quality. Table 7.2.2 shows the sample variances and covariances used in the calculation of T_n^2 in Table 7.2.1.

Sequential Inspection

of Ballistic Missiles

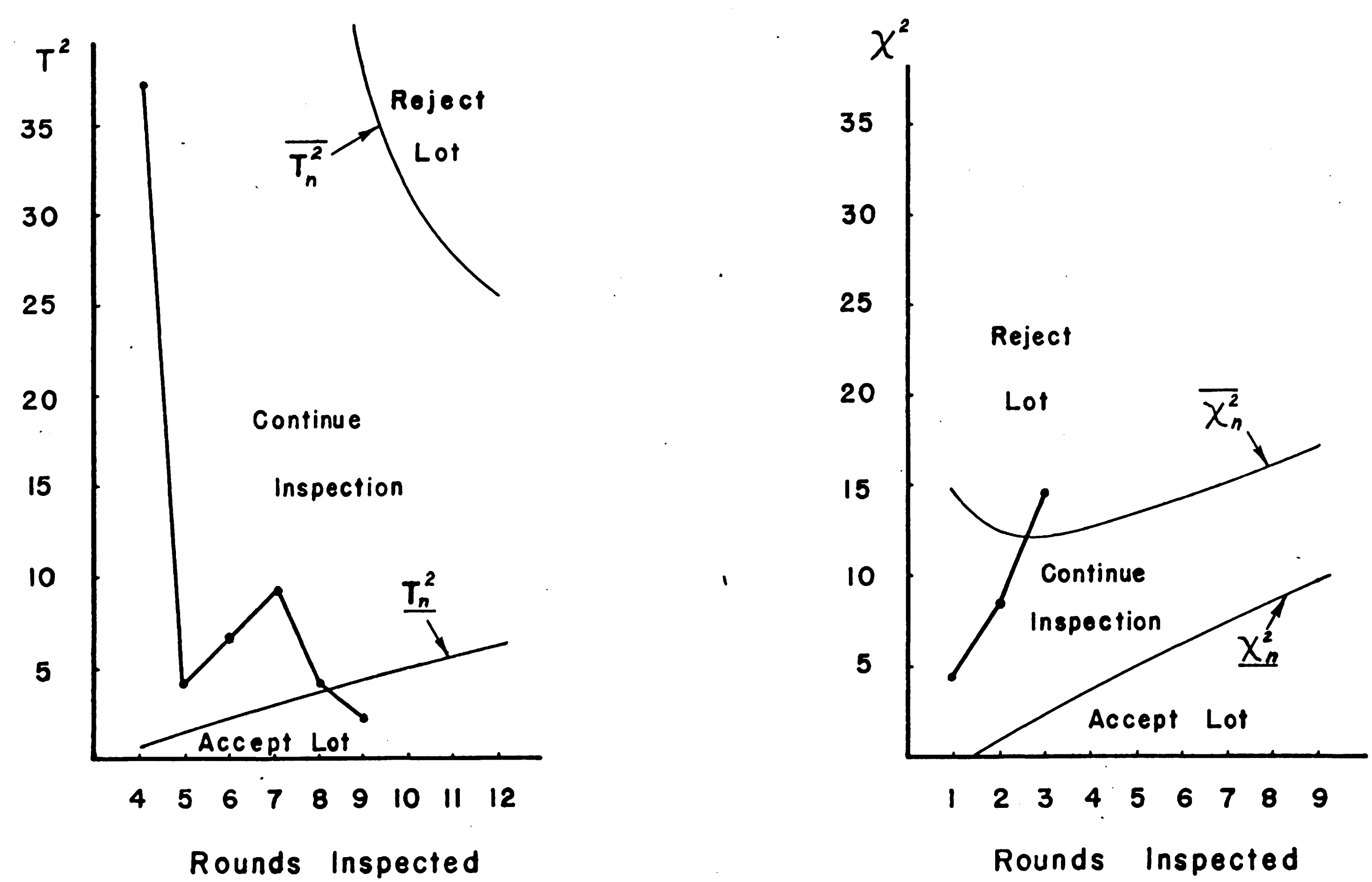

Figure 7.2

Table 7.2.1

Illustration of the sequential T^2-test

Round Number	Observations x_1	x_2	x_3	$n\bar{x}_1-\mu_{10}$	$n\bar{x}_2-\mu_{20}$	$n\bar{x}_3-\mu_{30}$	T_n^2
1	98	252	68				
2	120	77	72				
3	113	277	90				
4	61	60	79	-2.0	-33.5	27.2	37.362
5	78	39	2	-6.0	-59.0	12.2	4.1340
6	115	263	108	-2.5	-38.7	19.8	6.7542
7	103	167	76	-1.7	-37.9	20.9	9.7384
8	126	167	-36	1.8	-37.2	7.4	4.0904
9	82	215	52	- .4	-31.4	6.8	2.4210

Table 7.2.2

Variances and covariances used in Table 7.2.1

Sample Size n	s_{11}	s_{12}	s_{13}	s_{22}	s_{23}	s_{33}
4	693	1210	4	12958	301	93
5	600	1417	304	12970	2114	1202
6	553	1561	403	12856	2647	1311
7	465	1305	341	10718	2211	1098
8	495	1136	-77	9190	1830	2365
9	476	879	-56	8345	1570	2072

This is only one method of determining λ^2 for this situation and consists, essentially, of circumscribing an ellipsoid around the rectangular solid bounded by

$$\mu_{i0} \pm .76\sigma_i \qquad (i = 1,2,3, \text{ variables})$$

where σ_i is the "assumed" standard deviation for the ith variable. This procedure will result in accepting slightly more material than intended under the original statement of the problem. An alternative procedure would be to inscribe an ellipsoid inside the rectangular solid with a resultant smaller λ^2 under H_1 and by contrast, this procedure would be conservative in nature. This indicates that the optimum solution of this type lies somewhere between the two solutions indicated.

Another alternative that can be used when nothing is known about the variability of the characteristics is an extension of the variables inspection technique (Wallis, 1947). In this procedure, two values of proportion defectives are stated. One of these, p_1, represents an acceptable lot and the other, p_2, represents an unacceptable lot. The hypotheses are:

$$H_0: \; p = p_1,$$

$$H_1: \; p = p_2 > p_1.$$

In the univariate case, once the tolerances are given and p_1, p_2, α, and β specified, a sequential scheme can be set up utilizing the mean and standard deviation of the sample. No similar technique is yet available in the multivariate case for specifying λ^2. The form of the test would actually require non-zero values of λ^2 for both hypotheses and we would then employ equation (3.2.4.1) to carry out our procedure.

Since the present non-sequential procedure mentioned in Section 7.1 involves testing nine rounds, another alternative could be to use the confidence limits for the mean of nine rounds in determining λ^2. In the present situation, the univariate 95% confidence limits for averages of nine is .77s and for three independent variables an estimate of λ^2 would be $3(.77)^2 = 1.78$ which, by coincidence, is nearly the same value of λ^2 as that used in this example.

We see, then, that there are many ways of specifying λ^2 under H_1 and the choice of methods will depend on the individual situation.

7.3 Case II: Σ known.

We will now assume that in the time which has elapsed since the acceptance of the lot discussed in Section 7.2, sufficient information has been gathered so that the population covariance matrix can be assumed to be known. From the data of approximately 100 rounds, the covariance matrix was estimated to be:

(7.3.1)
$$\Sigma = \begin{bmatrix} 870 & -400 & -200 \\ -400 & 7075 & 1535 \\ -200 & 1535 & 1300 \end{bmatrix}$$

so that the standard deviations are $\sigma_1 = 29.5$, $\sigma_2 = 84.1$, and $\sigma_3 = 36.1$. It appears that the tolerances imposed on the process are greater than $\pm 3\sigma$ as assumed in Case I. This suggests several possibilities. One possibility would be to use the natural tolerances of the process and let λ^2 remain at 2.0. This procedure is often employed when the acceptance sampling program is also used to control the process but, in the case of these Nike Boosters, too much material of acceptable quality would be rejected and, considering the cost factor, this would not be a recommended procedure. A second possibility would be to allow for the fact that the process is well contained within the tolerances and repeat the procedure used in Case I to obtain a new rectangular solid based on the relationships between the tolerances and the new variances and then circumscribe an ellipsoid about this solid using the known covariance matrix. In this particular instance, the resultant value of λ^2 would be quite large and since some of the variables are correlated, the ellipsoid is now rotated and parts of it may actually be out of tolerance. A third and possible compromise procedure would be to assume that the tolerances imposed on the true lot mean in Case I are adequate but that the true variances should be taken into account in getting a new value of λ^2 under H_1. This amounts to circumscribing a rotated ellipse about this original rectangular solid. Since the individual means were to be within the limits given by (7.2.1), λ^2 can now be determined by

(7.3.2)

$$\lambda^2 = [30.4\ 76.0\ 50.7]\begin{bmatrix} 870 & -400 & -200 \\ -400 & 7075 & 1535 \\ -200 & 1535 & 1300 \end{bmatrix}^{-1}\begin{bmatrix} 30.4 \\ 76.0 \\ 50.7 \end{bmatrix} = 3.90.$$

We will now employ a sequential χ^2-test using the covariance matrix given by (7.3.1) and $\lambda^2 = 4.0$ under H_1.

No tables have been prepared for this sequential χ^2-test but using the relationships discussed in Section 6.4, tables for $\lambda^2 = 4.0$, p = 3 can be obtained by taking every other row of the χ^2 tables in Appendix B for $\lambda^2 = 2.0$, p = 3 starting with n = 2. A short table of $\underline{\chi_n^2}$ and $\overline{\chi_n^2}$ for $\lambda^2 = 4.0$ and p = 3 is given in Table 7.3.1. Table 7.3.2 shows the individual observations, deviations of cumulative averages from standard and χ_n^2 for this new lot. The action time measurements (x_1) are considerably longer than in Case I and from either Table 7.3.1 or Figure 7.2, it can be seen that the sequential process has terminated on the firing of the third round with the rejection of the lot

Table 7.3.1

A short table of $\underline{\chi}^2_n$ and $\overline{\chi^2_n}$

for $\lambda^2 = 4.0$, $p = 3$, $\alpha = \beta = .05$

n	$\underline{\chi}^2_n$	$\overline{\chi^2_n}$
1		14.73
2	.9607	12.33
3	2.469	12.27
4	3.806	12.78
5	5.057	13.51
6	6.257	14.35
7	7.423	15.25
8	8.565	16.19
9	9.689	17.15
10	10.80	18.13

Table 7.3.2

Illustration of the sequential χ^2-test

Round Number	Observations x_1	x_2	x_3	$n\bar{x}_1-\mu_{10}$	$n\bar{x}_2-\mu_{20}$	$n\bar{x}_3-\mu_{30}$	χ^2_n
1	151	272	70	51.0	72.0	20.0	4.4896
2	159	215	46	55.0	43.5	8.0	8.4468
3	178	157	48	62.7	14.7	4.7	14.797

VIII. SUMMARY

Procedures have been derived for sequentially testing the hypothesis:

$$H_0: \quad (\underline{\mu}-\underline{\mu}_0)\Sigma^{-1}(\underline{\mu}-\underline{\mu}_0)' = \lambda_0^2 \text{ (usually zero)}$$

against the alternative:

$$H_1: \quad (\underline{\mu}-\underline{\mu}_0)\Sigma^{-1}(\underline{\mu}-\underline{\mu}_0)' = \lambda_1^2$$

both for the case where Σ is known (the sequential χ^2-test) and where Σ must be estimated from the sample (the sequential T^2-test). Similar procedures are given to test the hypothesis:

$$H_0: \quad (\underline{\mu}_1-\underline{\mu}_2-\underline{\delta})\Sigma^{-1}(\underline{\mu}_1-\underline{\mu}_2-\underline{\delta})' = \lambda_0^2 \text{ (usually zero)}$$

against the alternative:

$$H_1: \quad (\underline{\mu}_1-\underline{\mu}_2-\underline{\delta})\Sigma^{-1}(\underline{\mu}_1-\underline{\mu}_2-\underline{\delta})' = \lambda_1^2.$$

It is shown that these sequential procedures all exist in the sense that the risks of accepting H_0 when H_1 is true and of accepting H_1 when H_0 is true are α and β respectively and that these sequential procedures terminate with probability unity. Some of these situations have been generalized to give simultaneous tests on the means and covariance matrix of a sample.

No expressions yet exist for the OC or ASN functions although some conjectured values have been determined for the latter and suggest, in comparison with their corresponding fixed-sample tests, substantial reductions in the sample sizes required when either H_0 or H_1 is true.

The general problem of tolerances is discussed and then some of these procedures are demonstrated with a numerical example drawn from the field of ballistic missiles.

Tables to facilitate both the sequential χ^2- and T^2-tests are given for $p = 2, 3, \ldots, 9$; $\lambda^2 = 0.5, 1.0, 2.0$; $\alpha = \beta = .05$ for n ranging from the minimum value necessary to reach a decision to 30, 45, and 60 for $\lambda^2 = 0.5, 1.0,$ and 2.0 respectively. Finally a discussion is given for the hypergeometric function ${}_0F_1(c;x)$ and a table given of this function for $c = .5(.5)5.0$; and $x = .1(.1)1(1)10(10)100(50)1000$.

Since the topic of sequential multivariate tests is fairly new, a whole host of problems for future research can be suggested. These include:

(1) Determination of truncated or restricted schemes for composite hypotheses similar to the work of Wald (1947) or Armitage (1957).

(2) Determination of multiple sampling schemes for χ^2- and T^2-tests. This would overcome some of the scheduling and computing problems associated with sequential sampling of a high volume, low cost product. Until such time as this can be done, Wald's approximate method can be used. This

would consist of using $\underline{\chi}_n^2$ and $\overline{\chi_n^2}$ or $\underline{T}_n^2$ and $\overline{T_n^2}$ with n corresponding to the total sample size at that stage. The main disadvantage of this technique is that the ASN values are somewhat higher of course as it is with all multiple sampling plans. However, this is partially compensated at least by α and β being reduced although the extent of this reduction is unknown.

(3) Determination of explicit (or even approximate) OC and ASN functions when the hypotheses are composite.

(4) A Monte Carlo evaluation of the sequential χ^2- and T^2-tests to determine empirical ASN values to compare with those obtained using Bhate's conjecture and also to determine OC values in order to study the power of these tests and to see just how close the stated α and β are to the true risks.

(5) More actual experience with these techniques, particularly in specifying the non-centrality parameter λ^2 under H_1.

(6) An extension of the tables in Appendices B and C as well as the development of some methods for getting rapid approximations for these values. It appears that larger values of λ^2 are desirable for increasing p. Larger values of n are also desirable for some of the existing tables. An

extension of these tables could also be carried out to cover the sequential χ_O^2- and χ_D^2-tests.

IX ACKNOWLEDGMENTS

The author expresses his sincere appreciation to:

Dr. Ralph A. Bradley for his aid in formulating the problems considered in this thesis and for his guidance in the solution of these problems,

Dr. Boyd Harshbarger for his assistance, encouragement and counsel,

Dr. R. J. Freund and his staff for assistance rendered in the numerous computations required on the IBM 650,

Dr. L. McFadden, Dr. H. A. David and Dr. R. E. Bargmann for their many helpful suggestions,

The administration of the Radford Arsenal and, in particular, Mr. R. C. Rhodes of the Hercules Powder Company who originally proposed the basic problem considered in this thesis and furnished the data used in the numerical example,

Mrs. Patricia Vandevender and Mrs. Sue Ann Moore for their careful typing of this manuscript,

Mr. Roger Lineberry for his preparation of the illustration, and finally to

his wife, Suzanne, and children, Jimmy and Janice for several years of patience and understanding.

This study was supported, in part, by the Office of Naval Research, Department of the Navy, Contract No. NONR-2352(01), Task Order NR 042-019.

X. BIBLIOGRAPHY

Anderson, T. W. (1958). An Introduction to Multivariate Statistical Analysis. John Wiley and Sons, Inc., New York.

Armitage, P. (1957). Restricted sequential procedures. Biometrika, 44, 9-26.

Barnard, G. A. (1946). Sequential tests in industrial statistics (with discussion). J. Royal Stat. Soc. Ser. B., 9, 1-26.

Barnard, G. A. (1952). The frequency justification of certain sequential tests. Biometrika, 39, 144-150.

Barnard, G. A. (1953). The frequency justification of sequential tests-addendum. Biometrika, 40, 468-469.

Bartky, W. (1943). Multiple sampling with constant probability. Ann. Math. Stat., 14, 363-377.

Bose, R. C. and Roy, S. N. (1938). The exact distribution of the studentized D^2-statistic. Sankhyā, 4, 19-38.

Buchholz, H. (1953). Die Konfluente Hypergeometrische Funktion mit besonder Berucksichtigung ihrer Andwendungen. Ergebnisse der Angewanten Mathematik Bd. 2 Springer-Verlag, Berlin.

Cox, D. R. (1952). Sequential tests for composite hypotheses. Proceedings of the Cambridge Philosophical Society, 48, 290-299.

Dodge, H. F. and Romig, H. G. (1944). Sampling Inspection Tables, John Wiley and Sons, Inc., New York.

Erdelyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G. (1953). Higher Transcendental Functions. McGraw-Hill Book Co., Inc., New York.

Erdelyi, A. and Swanson, C. A. (1957). Asymptotic forms of Whittaker's confluent hypergeometric functions. Memoirs of the American Mathematical Society, No. 25.

Fieller, E. C. (1931). The duration of play. Biometrika, 22, 375-404.

Fisher, R. A. (1928). The general sampling distribution of the multiple correlation coefficient. Proc. Roy. Soc. London, Ser. A., 121, 654-673.

Fix, Evelyn, (1949). Tables of non-central χ^2. University of California Publications in Statistics, 1, No. 2, 15-19.

Fox, M. (1956). Charts of the power of the F-test. Ann. Math. Stat., 27, 484-497.

Freeman, H. A. (1944). Sequential Analysis of Statistical Data: Applications. Statistical Research Group, Columbia University.

Freeman, H. A., Friedman, M., Mosteller, F., and Wallis, W. A., Editors, (1948). Sampling Inspection, McGraw-Hill Book Co., Inc., New York.

Helmert, F. R. (1875). Uber die Berechnung des wahrscheinlichen Fehlers aus einer endlichen Anzahl wahrer Beobachtungsfehler. Zeit. fur Math. und Phys. 21, 192.

Hoel, P. G. (1954). On a property of the sequential t-test. Skandinavisk Aktuarietidskrift, 37, 19-22.

Hoel, P. G. (1955). On a sequential test for the general linear hypothesis. Ann. Math. Stat., 26, 136-139.

Hotelling, H. (1931). The generalization of Student's ratio. Ann. Math. Stat., 2, 360-378.

Hotelling, H. (1947). Multivariate quality control. Techniques of Statistical Analysis, Edited by Eisenhart, C., Hastay, M. W. and Wallis, W. A., McGraw-Hill Book Co., Inc., New York.

Hsu, P. L. (1938). Notes on Hotellings's generalized T^2. Ann. Math. Stat., 9, 231-243.

Jackson, J. E. (1948). Evaluation of Acceptance Sampling Plans in Statistical Quality Control. Master's Thesis, University of North Carolina.

Jackson, J. E. (1959). Bibliography on Sequential Analysis. The development of Statistical Methods for Experimental Designs in Quality Control and Surveillance Testing, Tech. Report No. 6, Virginia Polytechnic Inst.

Jackson, J. E., Freund, R. A., and Howe, W. G. (1959). Errors associated with process adjestments. Virginia J. of Science, 10, 3-26.

Johnson, N. L. (1954). Sequential procedures in certain component of variance problems. Ann. Math. Stat., 25, 357-366.

Nandi, H. K. (1948). Use of well-known statistics in sequential analysis. Sankhya, 8, 339-344.

Nath, P. (1951). Confluent hypergeometric function. Sankhyā 11, 155-166.

National Bureau of Standards (1949). Tables of the confluent hypergeometric function F(n/2, 1/2; x) and related functions. Applied Mathematics Series No. 3.

National Bureau of Standards (1951). Tables to facilitate sequential t-tests. Applied Mathematics Series No. 7.

Nielsen, K. L. (1956). Methods in Numerical Analysis. The Macmillan Co., New York.

Patnaik, P. B. (1949). The non-central χ^2- and F-distributions and their applications. Biometrika, 36, 202-232.

Pearson, E. S. and Hartley, H. O. (1951). Charts of the power function for analysis of variance tests, derived from the non-central F-distribution. Biometrika, 38, 112-130.

Perron, O. (1921). Uber das Verhalten einer ausgearteten hypergeometrischen Reihe bei unbegrenzten wachstum eines Parameters. J. fur reine und ang. Math. 151, 63-78.

Rao, C. R. (1950). Sequential tests of null hypotheses. Sankhya, 10, 361-370.

Ray, W. D. (1956). Sequential analysis applied to certain experimental designs in the analysis of variance. Biometrika, 43, 388-403.

Ray, W. D. (1957). A proof that the sequential probability ratio test (S.P.R.T.) of the general linear hypothesis terminates with probability unity. Ann. Math. Stat., 28, 521-523.

Rushton, S. (1950). On a sequential t-test. Biometrika, 37, 326-333.

Rushton, S. (1952). On a two-sided sequential t-test. Biometrika, 39, 302-308.

Rushton, S. (1954). On the confluent hypergeometric function $M(\alpha,\gamma;x)$. Sankhya, 13, 3λ9-376.

Rushton, S. and Lang, E. D. (1954). Tables of the confluent hypergeometric function. Sankhyā, 13, 377-411.

Stockman, C. M. (1944). A method of obtaining an approximation for the operating characteristic of a Wald sequential probability test applied to a binomial distribution. (British) Ministry of Supply Advisory Service on Statistical Method and Quality Control, Tech. Report Q.C./R/19.

Tang, P. C. (1938). The power function of the analysis of variance tests with tables and illustration of their use. Statistical Research Memoirs, 2, 126-157.

Tricomi, F. G. (1954). Funzioni Ipergeometrische Confluenti. Edizioni Cremonese, Rome.

Wald, A. (1943). Sequential Analysis of Statistical Data: Theory. Statistical Research Group, Columbia University.

Wald, A. (1947). Sequential Analysis. John Wiley and Sons, Inc., New York.

Wallis, W. A. (1947). Use of variables in acceptance inspection for percent defective. Techniques of Statistical Analysis, Edited by Eisenhart, C., Hastay, M. W., and Wallis, W. A. McGraw-Hill Book Co., Inc., New York.

Wishart, J. (1932). A note on the distribution of the correlation ratio. Biometrika, 24, 441-456.

XI. APPENDICES

APPENDIX A

AUXILIARY PROOFS FOR SECTION 4.3.1

Section 4.3.1 deals with the proof that a sequential χ^2-test terminates with probability one. In order to present that section in a more concise form, two auxiliary proofs or lemmas have been deferred to this section.

Lemma A.1 The ratio:

$$G = {}_1F_1[(p+1)/2,p;2\sqrt{n^2\lambda^2U^2}]/{}_1F_1[(p-1)/2,p-1;2\sqrt{n^2\lambda^2U^2}]$$

is a strictly increasing function of $2\sqrt{n^2\lambda^2U^2}$.

Proof: Let $a = (p-1)/2$, $c = p-1$, and $x = 2\sqrt{n^2\lambda^2U^2}$. Then G can be rewritten as $G = {}_1F_1(a+1,c+1;x)/{}_1F_1(a,c;x)$ with $0 < a < c$. The derivative of G with respect to x is:

$$\frac{dG}{dx} = \frac{[(a+1)/(c+1)]{}_1F_1(a,c;x){}_1F_1(a+2,c+2;x)-(\frac{a}{c})[{}_1F_1(a+1,c+1;x)]^2}{[{}_1F_1(a,c;x)]^2} .$$

The denominator is positive so only the numerator need be investigated.

Let $A = [(a+1)/(c+1)]{}_1F_1(a,c;x){}_1F_1(a+2,c+2;x)]$ and let $B = (a/c)[{}_1F_1(a+1,c+1;x)]^2$ so that the numerator of dG/dx can be written as A-B. Expanding, we have:

$$A = \frac{a+1}{c+1} + \frac{a+1}{c+1}\left[\frac{a+2}{c+2} + \frac{a}{c}\right] x$$
$$+ \frac{a+1}{c+1}\left[\left(\frac{a+2}{c+2}\right)\left(\frac{a+3}{c+3}\right) + \frac{a(a+1)}{c(c+1)} + \frac{2a(a+2)}{c(c+2)}\right]\frac{x^2}{2!} + \cdots,$$

and

$$B = \frac{a}{c} + \frac{2a(a+1)}{c(c+1)}, x + \frac{2a}{c}\left[\left(\frac{a+1}{c+1}\right)\left(\frac{a+2}{c+2}\right) + \left(\frac{a+1}{c+1}\right)^2\right]\frac{x^2}{2!} + \cdots .$$

When $0 < a < c$, the sequentc $\frac{a}{c}$, $\frac{a+1}{c+1}$, $\frac{a+2}{c+2}$, etc., is strictly increasing concave and as such

$$\frac{a}{c} < \frac{a+1}{c+1}, \quad \frac{2a(a+1)}{c(c+1)} < \frac{a+1}{c+1}\left[\frac{a+2}{c+2} + \frac{a}{c}\right], \text{ etc.}$$

That is, each term of B is less than the corresponding term of A. Hence dG/dx is positive for all $0 < a < c$, $x > 0$ and therefore, G is a strictly increasing function of x. Carrying over to the present problem of the sequential χ^2-test, G is a strictly increasing function of $x = 2\sqrt{n^2\lambda^2U^2}$.

<u>Lemma A.2</u> $1 \leq G \leq 2$.

Proof: If $U^2 = 0$, both the numerator and denominator of G are equal to unity. Since G is a strictly increasing function of $2\sqrt{n^2\lambda^2U^2}$, it must therefore be greater than 1 for all $U^2 > 0$.

By (4.3.1.6)

$$_1F_1(a,c;x) = \frac{\Gamma(c)}{\Gamma(a)} e^x x^{a-c}[1 + O(|x|)^{-1}].$$

Hence,

$$G = \frac{\Gamma(c+1)\Gamma(a)\ e^x x^{(a+1)-(c+1)}[1 + O(|x|^{-1})]}{\Gamma(a+1)\Gamma(c)\ e^x x^{a-c}\ [1 + O(|x|^{-1})]}$$

$$= c/a.$$

In the present case, $a = (p-1)/2$ and $c = p-1$ so that in the limit, $G \to 2$.

APPENDIX B

TABLES TO FACILITATE THE SEQUENTIAL χ^2-TEST

Tables for Sequential χ^2-test

$\alpha = \beta = .05$

$p = 2$ variables

Sample Size n	$\lambda^2 = 0.5$		$\lambda^2 = 1.0$		$\lambda^2 = 2.0$	
	$\underline{\chi}^2_n$	$\overline{\chi}^2_n$	$\underline{\chi}^2_n$	$\overline{\chi}^2_n$	$\underline{\chi}^2_n$	$\overline{\chi}^2_n$
1		47.56		26.59		16.30
2		26.59		16.30		11.57
3		19.69		13.06	.03756	10.34
4		16.30		11.57	.6732	9.999
5		14.33		10.78	1.273	10.00
6		13.06	.03756	10.34	1.857	10.18
7		12.19	.3627	10.11	2.430	10.46
8		11.57	.6732	9.999	2.995	10.79
9		11.11	.9756	9.972	3.553	11.17
10		10.78	1.273	10.00	4.106	11.58
11		10.53	1.566	10.08	4.654	12.00
12	.03756	10.34	1.857	10.18	5.198	12.44
13	.2026	10.21	2.144	10.31	5.739	12.89
14	.3627	10.11	2.430	10.46	6.277	13.36
15	.5193	10.04	2.713	10.62	6.812	13.82
16	.6732	9.999	2.995	10.79	7.345	14.30
17	.8252	9.977	3.275	10.98	7.876	14.77
18	.9756	9.972	3.553	11.17	8.405	15.26
19	1.125	9.982	3.830	11.37	8.933	15.74
20	1.273	10.00	4.106	11.58	9.459	16.23
21	1.420	10.04	4.380	11.79	9.984	16.72
22	1.566	10.08	4.654	12.00	10.51	17.21
23	1.712	10.12	4.926	12.22	11.03	17.70
24	1.857	10.18	5.198	12.44	11.55	18.19
25	2.001	10.24	5.469	12.67	12.07	18.69
26	2.144	10.31	5.739	12.89	12.59	19.19
27	2.287	10.38	6.008	13.12	13.11	19.68
28	2.430	10.46	6.277	13.36	13.63	20.18
29	2.572	10.54	6.545	13.59	14.15	20.68
30	2.713	10.62	6.812	13.82	14.67	21.18

p = 2 variables (concluded)

Sample Size n	$\lambda^2 = 0.5$ $\underline{\chi}^2_n$	$\lambda^2 = 0.5$ $\overline{\chi}^2_n$	$\lambda^2 = 1.0$ $\underline{\chi}^2_n$	$\lambda^2 = 1.0$ $\overline{\chi}^2_n$
31	2.854	10.70	7.079	14.06
32	2.995	10.79	7.345	14.30
33	3.135	10.88	7.611	14.53
34	3.275	10.98	7.876	14.77
35	3.414	11.07	8.141	15.01
36	3.553	11.17	8.405	15.26
37	3.692	11.27	8.669	15.50
38	3.830	11.37	8.933	15.74
39	3.968	11.47	9.196	15.98
40	4.106	11.58	9.459	16.23
41	4.243	11.68	9.722	16.47
42	4.380	11.79	9.984	16.72
43	4.517	11.89	10.25	16.96
44	4.654	12.00	10.51	17.21
45	4.790	12.11	10.77	17.45
46	4.926	12.22		
47	5.062	12.33		
48	5.198	12.44		
49	5.334	12.55		
50	5.469	12.67		
51	5.604	12.78		
52	5.739	12.89		
53	5.874	13.01		
54	6.008	13.12		
55	6.142	13.24		
56	6.277	13.36		
57	6.411	13.47		
58	6.545	13.59		
59	6.678	13.71		
60	6.812	13.82		

Tables for Sequential χ^2-test

$\alpha = \beta = .05$

p = 3 variables

Sample Size n	$\lambda^2 = 0.5$ $\underline{\chi}^2_n$	$\lambda^2 = 0.5$ $\overline{\chi}^2_n$	$\lambda^2 = 1.0$ $\underline{\chi}^2_n$	$\lambda^2 = 1.0$ $\overline{\chi}^2_n$	$\lambda^2 = 2.0$ $\underline{\chi}^2_n$	$\lambda^2 = 2.0$ $\overline{\chi}^2_n$
1		63.00		34.99		21.20
2		34.99		21.20		14.73
3		25.75		16.79	.05618	12.95
4		21.20		14.73	.9607	12.33
5		18.52		13.60	1.746	12.19
6		16.79	.05618	12.95	2.469	12.27
7		15.59	.5295	12.56	3.151	12.48
8		14.73	.9607	12.33	3.806	12.78
9		14.09	1.364	12.22	4.439	13.12
10		13.60	1.746	12.19	5.057	13.51
11		13.23	2.114	12.21	5.662	13.92
12	.05618	12.95	2.469	12.27	6.257	14.35
13	.2994	12.73	2.814	12.36	6.844	14.80
14	.5295	12.56	3.151	12.48	7.423	15.25
15	.7493	12.43	3.481	12.62	7.997	15.72
16	.9607	12.33	3.806	12.78	8.565	16.19
17	1.165	12.27	4.125	12.94	9.129	16.67
18	1.364	12.22	4.439	13.12	9.689	17.15
19	1.557	12.20	4.750	13.31	10.24	17.64
20	1.746	12.19	5.057	13.51	10.80	18.13
21	1.932	12.19	5.361	13.71	11.35	18.62
22	2.114	12.21	5.662	13.92	11.90	19.12
23	2.293	12.23	5.961	14.13	12.44	19.62
24	2.469	12.27	6.257	14.35	12.99	20.12
25	2.643	12.31	6.552	14.57	13.53	20.62
26	2.814	12.36	6.844	14.80	14.07	21.12
27	2.984	12.42	7.134	15.02	14.61	21.62
28	3.151	12.48	7.423	15.25	15.14	22.12
29	3.317	12.55	7.711	15.49	15.68	22.63
30	3.481	12.62	7.997	15.72	16.21	23.13

p = 3 variables (concluded)

Sample Size n	$\lambda^2 = 0.5$ $\underline{\chi}^2_n$	$\lambda^2 = 0.5$ $\overline{\chi}^2_n$	$\lambda^2 = 1.0$ $\underline{\chi}^2_n$	$\lambda^2 = 1.0$ $\overline{\chi}^2_n$
31	3.644	12.70	8.282	15.95
32	3.806	12.78	8.565	16.19
33	3.966	12.86	8.847	16.43
34	4.125	12.94	9.129	16.67
35	4.283	13.03	9.409	16.91
36	4.439	13.12	9.689	17.15
37	4.595	13.22	9.967	17.40
38	4.750	13.31	10.24	17.64
39	4.904	13.41	10.52	17.89
40	5.057	13.51	10.80	18.13
41	5.210	13.61	11.07	18.38
42	5.361	13.71	11.35	18.62
43	5.512	13.82	11.62	18.87
44	5.662	13.92	11.90	19.12
45	5.812	14.03	12.17	19.37
46	5.961	14.13		
47	6.109	14.24		
48	6.257	14.35		
49	6.405	14.46		
50	6.552	14.57		
51	6.698	14.68		
52	6.844	14.80		
53	6.989	14.91		
54	7.134	15.02		
55	7.279	15.14		
56	7.423	15.25		
57	7.567	15.37		
58	7.711	15.49		
59	7.854	15.60		
60	7.997	15.72		

Tables for Sequential χ^2-test

$\alpha = \beta = .05$

$p = 4$ variables

Sample Size n	$\lambda^2 = 0.5$		$\lambda^2 = 1.0$		$\lambda^2 = 2.0$	
	$\underline{\chi}^2_n$	$\overline{\chi}^2_n$	$\underline{\chi}^2_n$	$\overline{\chi}^2_n$	$\underline{\chi}^2_n$	$\overline{\chi}^2_n$
1		77.63		42.93		25.81
2		42.93		25.81		17.69
3		31.47		20.30	.07477	15.38
4		25.81		17.69	1.240	14.51
5		22.47		16.24	2.200	14.21
6		20.30	.07477	15.38	3.050	14.20
7		18.79	.6936	14.84	3.831	14.35
8		17.69	1.240	14.51	4.565	14.60
9		16.87	1.738	14.31	5.266	14.92
10		16.24	2.200	14.21	5.941	15.28
11		15.76	2.635	14.18	6.597	15.68
12	.07477	15.38	3.050	14.20	7.237	16.10
13	.3952	15.07	3.447	14.26	7.864	16.54
14	.6936	14.84	3.831	14.35	8.480	16.99
15	.9743	14.65	4.203	14.47	9.087	17.45
16	1.240	14.51	4.565	14.60	9.686	17.93
17	1.494	14.39	4.919	14.76	10.28	18.40
18	1.738	14.31	5.266	14.92	10.87	18.89
19	1.973	14.25	5.606	15.10	11.45	19.38
20	2.200	14.21	5.941	15.28	12.02	19.87
21	2.421	14.19	6.271	15.48	12.60	20.37
22	2.635	14.18	6.597	15.68	13.17	20.87
23	2.845	14.18	6.919	15.89	13.73	21.37
24	3.050	14.20	7.237	16.10	14.29	21.87
25	3.250	14.23	7.552	16.32	14.85	22.37
26	3.447	14.26	7.864	16.54	15.41	22.88
27	3.640	14.30	8.173	16.76	15.97	23.39
28	3.831	14.35	8.480	16.99	16.52	23.89
29	4.018	14.41	8.784	17.22	17.07	24.40
30	4.203	14.47	9.087	17.45	17.62	24.91

p = 4 variables (concluded)

Sample Size n	$\lambda^2 = 0.5$ $\underline{\chi^2_n}$	$\lambda^2 = 0.5$ $\overline{\chi^2_n}$	$\lambda^2 = 1.0$ $\underline{\chi^2_n}$	$\lambda^2 = 1.0$ $\overline{\chi^2_n}$
31	4.385	14.53	9.387	17.69
32	4.565	14.60	9.686	17.93
33	4.743	14.68	9.983	18.16
34	4.919	14.76	10.28	18.40
35	5.093	14.84	10.57	18.65
36	5.266	14.92	10.87	18.89
37	5.437	15.01	11.16	19.13
38	5.606	15.10	11.45	19.38
39	5.774	15.19	11.74	19.62
40	5.941	15.28	12.02	19.87
41	6.107	15.38	12.31	20.12
42	6.271	15.48	12.60	20.37
43	6.435	15.58	12.88	20.62
44	6.597	15.68	13.17	20.87
45	6.758	15.78	13.45	21.12
46	6.919	15.89		
47	7.078	15.99		
48	7.237	16.10		
49	7.395	16.21		
50	7.552	16.32		
51	7.708	16.43		
52	7.864	16.54		
53	8.019	16.65		
54	8.173	16.76		
55	8.327	16.88		
56	8.480	16.99		
57	8.632	17.11		
58	8.784	17.22		
59	8.936	17.34		
60	9.087	17.45		

Tables for Sequential χ^2-test

$\alpha = \beta = .05$

$p = 5$ variables

Sample Size n	$\lambda^2 = 0.5$ $\underline{\chi}^2_n$	$\overline{\chi}^2_n$	$\lambda^2 = 1.0$ $\underline{\chi}^2_n$	$\overline{\chi}^2_n$	$\lambda^2 = 2.0$ $\underline{\chi}^2_n$	$\overline{\chi}^2_n$
1		91.81		50.62		30.26
2		50.62		30.26		20.54
3		37.00		23.68	.09334	17.70
4		30.26		20.54	1.516	16.58
5		26.28		18.78	2.643	16.14
6		23.68	.09334	17.70	3.613	16.04
7		21.87	.8562	17.02	4.486	16.13
8		20.54	1.516	16.58	5.295	16.34
9		19.55	2.105	16.30	6.058	16.63
10		18.78	2.643	16.14	6.787	16.97
11		18.18	3.143	16.06	7.489	17.35
12	.09334	17.70	3.613	16.04	8.170	17.76
13	.4905	17.32	4.059	16.07	8.834	18.19
14	.8562	17.02	4.486	16.13	9.483	18.63
15	1.196	16.78	4.897	16.22	10.12	19.09
16	1.516	16.58	5.295	16.34	10.75	19.56
17	1.818	16.43	5.682	16.48	11.37	20.04
18	2.105	16.30	6.058	16.63	11.98	20.52
19	2.379	16.21	6.426	16.79	12.58	21.01
20	2.643	16.14	6.787	16.97	13.18	21.51
21	2.897	16.09	7.141	17.15	13.78	22.01
22	3.143	16.06	7.489	17.35	14.36	22.51
23	3.381	16.04	7.832	17.55	14.95	23.01
24	3.613	16.04	8.170	17.76	15.53	23.52
25	3.838	16.05	8.504	17.97	16.11	24.03
26	4.059	16.07	8.834	18.19	16.68	24.54
27	4.275	16.09	9.160	18.41	17.25	25.05
28	4.486	16.13	9.483	18.63	17.82	25.56
29	4.694	16.17	9.804	18.86	18.38	26.07
30	4.897	16.22	10.12	19.09	18.95	26.59

p = 5 variables (concluded)

Sample Size n	$\lambda^2 = 0.5$ $\underline{\chi^2_n}$	$\lambda^2 = 0.5$ $\overline{\chi^2_n}$	$\lambda^2 = 1.0$ $\underline{\chi^2_n}$	$\lambda^2 = 1.0$ $\overline{\chi^2_n}$
31	5.098	16.28	10.44	19.33
32	5.295	16.34	10.75	19.56
33	5.490	16.41	11.06	19.80
34	5.682	16.48	11.37	20.04
35	5.871	16.55	11.67	20.28
36	6.058	16.63	11.98	20.52
37	6.243	16.71	12.28	20.77
38	6.426	16.79	12.58	21.01
39	6.607	16.88	12.88	21.26
40	6.787	16.97	13.18	21.51
41	6.965	17.06	13.48	21.76
42	7.141	17.15	13.78	22.01
43	7.315	17.25	14.07	22.26
44	7.489	17.35	14.36	22.51
45	7.661	17.45	14.66	22.76
46	7.832	17.55		
47	8.001	17.65		
48	8.170	17.76		
49	8.337	17.86		
50	8.504	17.97		
51	8.669	18.08		
52	8.834	18.19		
53	8.997	18.30		
54	9.160	18.41		
55	9.322	18.52		
56	9.483	18.63		
57	9.644	18.75		
58	9.804	18.86		
59	9.963	18.98		
60	10.12	19.09		

Tables for Sequential χ^2-test

$\alpha = \beta = .05$

p = 6 variables

Sample Size n	$\lambda^2 = 0.5$		$\lambda^2 = 1.0$		$\lambda^2 = 2.0$	
	$\underline{\chi}^2_n$	$\overline{\chi}^2_n$	$\underline{\chi}^2_n$	$\overline{\chi}^2_n$	$\underline{\chi}^2_n$	$\overline{\chi}^2_n$
1		105.7		58.14		34.61
2		58.14		34.61		23.32
3		42.40		26.98	.1119	19.97
4		34.61		23.32	1.788	18.60
5		30.00		21.25	3.078	18.01
6		26.98	.1119	19.97	4.164	17.82
7		24.86	1.018	19.14	5.126	17.85
8		23.32	1.788	18.60	6.006	18.01
9		22.15	2.467	18.24	6.828	18.27
10		21.25	3.078	18.01	7.607	18.59
11		20.54	3.641	17.88	8.352	18.95
12	.1119	19.97	4.164	17.82	9.072	19.35
13	.5855	19.51	4.658	17.81	9.770	19.77
14	1.018	19.14	5.126	17.85	10.45	20.21
15	1.417	18.84	5.575	17.92	11.12	20.66
16	1.788	18.60	6.006	18.01	11.77	21.13
17	2.137	18.40	6.423	18.13	12.41	21.61
18	2.467	18.24	6.828	18.27	13.05	22.09
19	2.780	18.11	7.222	18.42	13.67	22.58
20	3.078	18.01	7.607	18.59	14.29	23.08
21	3.365	17.94	7.983	18.77	14.91	23.58
22	3.641	17.88	8.352	18.95	15.51	24.08
23	3.907	17.84	8.715	19.15	16.11	24.58
24	4.164	17.82	9.072	19.35	16.71	25.09
25	4.414	17.81	9.424	19.56	17.30	25.61
26	4.658	17.81	9.770	19.77	17.89	26.12
27	4.895	17.83	10.11	19.99	18.48	26.63
28	5.126	17.85	10.45	20.21	19.06	27.15
29	5.353	17.88	10.79	20.43	19.64	27.67
30	5.575	17.92	11.12	20.66	20.21	28.18

p = 6 variables (concluded)

Sample Size n	$\lambda^2 = 0.5$		$\lambda^2 = 1.0$	
	$\underline{\chi^2_n}$	$\overline{\chi^2_n}$	$\underline{\chi^2_n}$	$\overline{\chi^2_n}$
31	5.793	17.96	11.45	20.90
32	6.006	18.01	11.77	21.13
33	6.217	18.07	12.09	21.37
34	6.423	18.13	12.41	21.61
35	6.627	18.20	12.73	21.85
36	6.828	18.27	13.05	22.09
37	7.026	18.35	13.36	22.33
38	7.222	18.42	13.67	22.58
39	7.416	18.50	13.98	22.83
40	7.607	18.59	14.29	23.08
41	7.796	18.68	14.60	23.32
42	7.983	18.77	14.91	23.58
43	8.169	18.86	15.21	23.83
44	8.352	18.95	15.51	24.08
45	8.535	19.05	15.81	24.33
46	8.715	19.15		
47	8.894	19.25		
48	9.072	19.35		
49	9.248	19.45		
50	9.424	19.56		
51	9.597	19.66		
52	9.770	19.77		
53	9.942	19.88		
54	10.11	19.99		
55	10.28	20.10		
56	10.45	20.21		
57	10.62	20.32		
58	10.79	20.43		
59	10.95	20.55		
60	11.12	20.66		

Tables for Sequential χ^2-test

$\alpha = \beta = .05$

p = 7 variables

Sample Size n	$\lambda^2 = 0.5$ $\underline{\chi^2_n}$	$\lambda^2 = 0.5$ $\overline{\chi^2_n}$	$\lambda^2 = 1.0$ $\underline{\chi^2_n}$	$\lambda^2 = 1.0$ $\overline{\chi^2_n}$	$\lambda^2 = 2.0$ $\underline{\chi^2_n}$	$\lambda^2 = 2.0$ $\overline{\chi^2_n}$
1		119.4		65.55		38.89
2		65.55		38.89		26.04
3		47.72		30.22	.1304	22.19
4		38.89		26.04	2.059	20.57
5		33.66		23.67	3.509	19.84
6		30.22	.1304	22.19	4.708	19.56
7		27.81	1.179	21.22	5.756	19.52
8		26.04	2.059	20.57	6.704	19.65
9		24.71	2.825	20.13	7.582	19.87
10		23.67	3.509	19.84	8.409	20.16
11		22.84	4.132	19.66	9.196	20.51
12	.1304	22.19	4.708	19.56	9.952	20.89
13	.6803	21.65	5.247	19.52	10.68	21.30
14	1.179	21.22	5.756	19.52	11.39	21.74
15	1.636	20.86	6.240	19.57	12.09	22.19
16	2.059	20.57	6.704	19.65	12.77	22.65
17	2.454	20.33	7.151	19.75	13.43	23.12
18	2.825	20.13	7.582	19.87	14.09	23.61
19	3.176	19.97	8.001	20.01	14.73	24.10
20	3.509	19.84	8.409	20.16	15.37	24.59
21	3.828	19.74	8.807	20.33	16.00	25.09
22	4.132	19.66	9.196	20.51	16.63	25.60
23	4.426	19.60	9.577	20.70	17.24	26.11
24	4.708	19.56	9.952	20.89	17.85	26.62
25	4.982	19.53	10.32	21.10	18.46	27.13
26	5.247	19.52	10.68	21.30	19.06	27.65
27	5.505	19.51	11.04	21.52	19.66	28.16
28	5.756	19.52	11.39	21.74	20.26	28.68
29	6.001	19.54	11.74	21.96	20.85	29.20
30	6.240	19.57	12.09	22.19	21.44	29.72

p = 7 variables (concluded)

Sample Size n	$\lambda^2 = 0.5$ $\underline{\chi^2_n}$	$\lambda^2 = 0.5$ $\overline{\chi^2_n}$	$\lambda^2 = 1.0$ $\underline{\chi^2_n}$	$\lambda^2 = 1.0$ $\overline{\chi^2_n}$
31	6.475	19.60	12.43	22.42
32	6.704	19.65	12.77	22.65
33	6.930	19.69	13.10	22.89
34	7.151	19.75	13.43	23.12
35	7.368	19.81	13.76	23.36
36	7.582	19.87	14.09	23.61
37	7.793	19.94	14.41	23.85
38	8.001	20.01	14.73	24.10
39	8.206	20.09	15.05	24.34
40	8.409	20.16	15.37	24.59
41	8.609	20.25	15.69	24.84
42	8.807	20.33	16.00	25.09
43	9.002	20.42	16.31	25.34
44	9.196	20.51	16.63	25.60
45	9.388	20.60	16.93	25.85
46	9.577	20.70		
47	9.766	20.79		
48	9.952	20.89		
49	10.14	20.99		
50	10.32	21.10		
51	10.50	21.20		
52	10.68	21.30		
53	10.86	21.41		
54	11.04	21.52		
55	11.22	21.63		
56	11.39	21.74		
57	11.57	21.85		
58	11.74	21.96		
59	11.92	22.07		
60	12.09	22.19		

Tables for Sequential χ^2-test

$\alpha = \beta = .05$

p = 8 variables

Sample Size n	$\lambda^2 = 0.5$ $\underline{\chi}^2_n$	$\lambda^2 = 0.5$ $\overline{\chi}^2_n$	$\lambda^2 = 1.0$ $\underline{\chi}^2_n$	$\lambda^2 = 1.0$ $\overline{\chi}^2_n$	$\lambda^2 = 2.0$ $\underline{\chi}^2_n$	$\lambda^2 = 2.0$ $\overline{\chi}^2_n$
1		132.9		72.88		43.12
2		72.88		43.12		28.73
3		52.98		33.42	.1490	24.37
4		43.12		28.73	2.329	22.50
5		37.26		26.05	3.937	21.63
6		33.42	.1490	24.37	5.247	21.26
7		30.72	1.340	23.26	6.378	21.16
8		28.73	2.329	22.50	7.392	21.24
9		27.22	3.182	21.99	8.325	21.43
10		26.05	3.937	21.63	9.197	21.71
11		25.12	4.620	21.40	10.02	22.03
12	.1490	24.37	5.247	21.26	10.82	22.40
13	.7750	23.76	5.830	21.19	11.58	22.80
14	1.340	23.26	6.378	21.16	12.32	23.23
15	1.855	22.85	6.897	21.19	13.04	23.67
16	2.329	22.50	7.392	21.24	13.74	24.13
17	2.770	22.22	7.867	21.33	14.43	24.60
18	3.182	21.99	8.325	21.43	15.10	25.08
19	3.570	21.79	8.768	21.56	15.77	25.57
20	3.937	21.63	9.197	21.71	16.42	26.07
21	4.286	21.51	9.616	21.86	17.07	26.57
22	4.620	21.40	10.02	22.03	17.71	27.08
23	4.939	21.32	10.42	22.21	18.34	27.59
24	5.247	21.26	10.82	22.40	18.97	28.10
25	5.543	21.21	11.20	22.60	19.59	28.61
26	5.830	21.19	11.58	22.80	20.21	29.13
27	6.108	21.17	11.95	23.01	20.82	29.65
28	6.378	21.16	12.32	23.23	21.43	30.17
29	6.641	21.17	12.68	23.45	22.03	30.70
30	6.897	21.19	13.04	23.67	22.63	31.22

p = 8 variables (concluded)

Sample Size n	$\lambda^2 = 0.5$ $\underline{\chi^2_n}$	$\overline{\chi^2_n}$	$\lambda^2 = 1.0$ $\underline{\chi^2_n}$	$\overline{\chi^2_n}$
31	7.148	21.21	13.39	23.90
32	7.392	21.24	13.74	24.13
33	7.632	21.28	14.08	24.37
34	7.867	21.33	14.43	24.60
35	8.098	21.38	14.77	24.84
36	8.325	21.43	15.10	25.08
37	8.548	21.50	15.44	25.33
38	8.768	21.56	15.77	25.57
39	8.984	21.63	16.10	25.82
40	9.197	21.71	16.42	26.07
41	9.408	21.78	16.75	26.32
42	9.616	21.86	17.07	26.57
43	9.821	21.95	17.39	26.82
44	10.02	22.03	17.71	27.08
45	10.23	22.12	18.03	27.33
46	10.42	22.21		
47	10.62	22.31		
48	10.82	22.40		
49	11.01	22.50		
50	11.20	22.60		
51	11.39	22.70		
52	11.58	22.80		
53	11.76	22.91		
54	11.95	23.01		
55	12.13	23.12		
56	12.32	23.23		
57	12.50	23.34		
58	12.68	23.45		
59	12.86	23.56		
60	12.04	23.67		

Tables for Sequential χ^2-test

$\alpha = \beta = .05$

p = 9 variables

Sample Size n	$\lambda^2 = 0.5$		$\lambda^2 = 1.0$		$\lambda^2 = 2.0$	
	$\underline{\chi^2_n}$	$\overline{\chi^2_n}$	$\underline{\chi^2_n}$	$\overline{\chi^2_n}$	$\underline{\chi^2_n}$	$\overline{\chi^2_n}$
1		146.3		80.15		47.30
2		80.15		47.30		31.39
3		58.19		36.58	.1675	26.53
4		47.30		31.39	2.598	24.42
5		40.84		28.41	4.362	23.40
6		36.58	.1675	26.53	5.781	22.94
7		33.59	1.500	25.28	6.994	22.78
8		31.39	2.598	24.42	8.073	22.81
9		29.71	3.537	23.82	9.058	22.97
10		28.41	4.362	23.40	9.976	23.22
11		27.37	5.104	23.12	10.84	23.53
12	.1675	26.53	5.781	22.94	11.67	23.88
13	.8695	25.84	6.408	22.83	12.46	24.27
14	1.500	25.28	6.994	22.78	13.22	24.69
15	2.072	24.81	7.547	22.78	13.97	25.13
16	2.598	24.42	8.073	22.81	14.69	25.58
17	3.084	24.09	8.576	22.88	15.40	26.05
18	3.537	23.82	9.058	22.97	16.10	26.53
19	3.961	23.59	9.524	23.09	16.78	27.02
20	4.362	23.40	9.976	23.22	17.45	27.51
21	4.742	23.25	10.41	23.37	18.12	28.01
22	5.104	23.12	10.84	23.53	18.77	28.52
23	5.449	23.02	11.26	23.70	19.42	29.03
24	5.781	22.94	11.67	23.88	20.06	29.55
25	6.100	22.88	12.07	24.07	20.70	30.06
26	6.408	22.83	12.46	24.27	21.32	30.58
27	6.705	22.80	12.84	24.48	21.95	31.11
28	6.994	22.78	13.22	24.69	22.57	31.63
29	7.274	22.77	13.60	24.91	23.18	32.16
30	7.547	22.78	13.97	25.13	23.79	32.68

p = 9 variables (concluded)

Sample Size n	$\lambda^2 = 0.5$: $\underline{\chi^2_n}$	$\lambda^2 = 0.5$: $\overline{\chi^2_n}$	$\lambda^2 = 1.0$: $\underline{\chi^2_n}$	$\lambda^2 = 1.0$: $\overline{\chi^2_n}$
31	7.813	22.79	14.33	25.35
32	8.073	22.81	14.69	25.58
33	8.327	22.84	15.05	25.82
34	8.576	22.88	15.40	26.05
35	8.819	22.92	15.75	26.29
36	9.058	22.97	16.10	26.53
37	9.293	23.03	16.44	26.77
38	9.524	23.09	16.78	27.02
39	9.752	23.15	17.12	27.26
40	9.976	23.22	17.45	27.51
41	10.20	23.29	17.79	27.76
42	10.41	23.37	18.12	28.01
43	10.63	23.45	18.45	28.27
44	10.84	23.53	18.77	28.52
45	11.05	23.61	19.10	28.78
46	11.26	23.70		
47	11.46	23.79		
48	11.67	23.88		
49	11.87	23.98		
50	12.07	24.07		
51	12.26	24.17		
52	12.46	24.27		
53	12.65	24.38		
54	12.84	24.48		
55	13.04	24.58		
56	13.22	24.69		
57	13.41	24.80		
58	13.60	24.91		
59	13.78	25.02		
60	13.97	25.13		

APPENDIX C

TABLES TO FACILITATE THE SEQUENTIAL T^2-TEST

Table for Sequential T^2-test

$\alpha = \beta = .05$

$p = 2$ variables

Sample Size n	$\lambda^2 = 0.5$		$\lambda^2 = 1.0$		$\lambda^2 = 2.0$	
	$\underline{T^2_n}$	$\overline{T^2_n}$	$\underline{T^2_n}$	$\overline{T^2_n}$	$\underline{T^2_n}$	$\overline{T^2_n}$
3					.02512	
4					.5202	
5					1.053	
6			.03135		1.598	72.52
7			.3143	146.3	2.148	31.94
8			.5977	39.00	2.698	23.89
9		122.7	.8812	25.81	3.246	20.69
10		44.20	1.164	20.80	3.792	19.13
11		29.42	1.447	18.24	4.335	18.32
12	.03445	23.22	1.730	16.75	4.876	17.91
13	.1877	19.85	2.011	15.82	5.414	17.75
14	.3386	17.75	2.292	15.21	5.949	17.75
15	.4878	16.34	2.571	14.80	6.483	17.85
16	.6357	15.34	2.850	14.54	7.014	18.03
17	.7826	14.60	3.127	14.38	7.543	18.26
18	.9287	14.04	3.404	14.30	8.071	18.54
19	1.074	13.61	3.680	14.27	8.596	18.85
20	1.219	13.28	3.954	14.28	9.121	19.19
21	1.363	13.03	4.228	14.33	9.644	19.55
22	1.507	12.82	4.501	14.40	10.17	19.93
23	1.651	12.67	4.773	14.50	10.69	20.32
24	1.794	12.55	5.044	14.61	11.21	20.73
25	1.936	12.46	5.315	14.74	11.72	21.14
26	2.078	12.39	5.585	14.89	12.24	21.57
27	2.220	12.35	5.854	15.04	12.76	22.00
28	2.362	12.32	6.123	15.20	13.27	22.43
29	2.503	12.31	6.391	15.38	13.79	22.88
30	2.643	12.31	6.658	15.56	14.30	23.32

p = 2 variables (concluded)

Sample Size n	$\lambda^2 = 0.5$		$\lambda^2 = 1.0$	
	$\underline{T^2_n}$	$\overline{T^2_n}$	$\underline{T^2_n}$	$\overline{T^2_n}$
31	2.783	12.33	6.925	15.74
32	2.923	12.35	7.191	15.93
33	3.063	12.38	7.457	16.13
34	3.202	12.42	7.723	16.33
35	3.341	12.46	7.988	16.53
36	3.480	12.51	8.252	16.74
37	3.618	12.57	8.516	16.95
38	3.756	12.63	8.780	17.16
39	3.894	12.70	9.044	17.38
40	4.032	12.77	9.307	17.60
41	4.169	12.84	9.570	17.82
42	4.306	12.92	9.832	18.04
43	4.443	13.00	10.09	18.26
44	4.580	13.08	10.36	18.49
45	4.716	13.17	10.62	18.72
46	4.852	13.25		
47	4.988	13.34		
48	5.124	13.43		
49	5.259	13.53		
50	5.395	13.62		
51	5.530	13.72		
52	5.665	13.82		
53	5.800	13.91		
54	5.934	14.01		
55	6.069	14.12		
56	6.203	14.22		
57	6.337	14.32		
58	6.471	14.43		
59	6.605	14.53		
60	6.739	14.64		

Tables for Sequential T^2-test

$\alpha = \beta = .05$

$p = 3$ variables

Sample Size n	$\lambda^2 = 0.5$ $\underline{T}^2_n$	$\lambda^2 = 0.5$ $\overline{T}^2_n$	$\lambda^2 = 1.0$ $\underline{T}^2_n$	$\lambda^2 = 1.0$ $\overline{T}^2_n$	$\lambda^2 = 2.0$ $\underline{T}^2_n$	$\lambda^2 = 2.0$ $\overline{T}^2_n$
4					.7831	
5					1.551	
6			.04699		2.295	
7			.4667		3.008	239.7
8			.8755		3.694	56.94
9			1.271	84.54	4.358	37.87
10			1.654	44.40	5.003	30.90
11		115.8	2.026	32.75	5.633	27.47
12	.05160	55.44	2.388	27.32	6.249	25.54
13	.2788	33.87	2.740	24.24	6.855	24.40
14	.4986	31.17	3.086	22.30	7.451	23.72
15	.7117	26.75	3.424	21.01	8.040	23.33
16	.9190	23.91	3.757	20.12	8.621	23.14
17	1.121	21.94	4.084	19.48	9.197	23.09
18	1.319	20.51	4.406	19.03	9.767	23.14
19	1.512	19.44	4.724	18.72	10.33	23.26
20	1.702	18.61	5.038	18.49	10.89	23.45
21	1.888	17.96	5.349	18.35	11.45	23.68
22	2.071	17.45	5.657	18.26	12.01	23.95
23	2.252	17.03	5.961	18.23	12.56	24.24
24	2.430	16.69	6.263	18.22	13.11	24.57
25	2.605	16.42	6.563	18.25	13.65	24.91
26	2.779	16.20	6.861	18.31	14.20	25.27
27	2.951	16.02	7.156	18.39	14.74	25.65
28	3.120	15.87	7.450	18.49	15.28	26.04
29	3.288	15.75	7.742	18.60	15.82	26.44
30	3.455	15.66	8.032	18.73	16.35	26.85

p = 3 variables (concluded)

Sample Size n	$\lambda^2 = 0.5$: $\underline{T^2_n}$	$\lambda^2 = 0.5$: $\overline{T^2_n}$	$\lambda^2 = 1.0$: $\underline{T^2_n}$	$\lambda^2 = 1.0$: $\overline{T^2_n}$
31	3.620	15.59	8.321	18.86
32	3.783	15.54	8.608	19.01
33	3.945	15.51	8.894	19.17
34	4.106	15.49	9.179	19.33
35	4.266	15.48	9.463	19.51
36	4.425	15.48	9.746	19.68
37	4.583	15.49	10.03	19.87
38	4.740	15.51	10.31	20.06
39	4.895	15.54	10.59	20.25
40	5.050	15.58	10.87	20.45
41	5.205	15.62	11.14	20.65
42	5.358	15.66	11.42	20.86
43	5.511	15.72	11.70	21.07
44	5.663	15.77	11.97	21.28
45	5.814	15.83	12.25	21.49
46	5.965	15.90		
47	6.115	15.97		
48	6.264	16.04		
49	6.413	16.12		
50	6.561	16.19		
51	6.709	16.27		
52	6.856	16.36		
53	7.003	16.44		
54	7.150	16.53		
55	7.296	16.62		
56	7.441	16.71		
57	7.586	16.80		
58	7.731	16.90		
59	7.875	16.99		
60	8.019	17.09		

Tables for Sequential T^2-test

$\alpha = \beta = .05$

$p = 4$ variables

Sample Size n	$\lambda^2 = 0.5$ $\underline{T}^2_n$	$\lambda^2 = 0.5$ $\overline{T}^2_n$	$\lambda^2 = 1.0$ $\underline{T}^2_n$	$\lambda^2 = 1.0$ $\overline{T}^2_n$	$\lambda^2 = 2.0$ $\underline{T}^2_n$	$\lambda^2 = 2.0$ $\overline{T}^2_n$
5					2.120	
6			.06270		3.091	
7			.6234		3.983	
8			1.164		4.812	1920.
9			1.678		5.591	91.92
10			2.165	183.1	6.331	55.13
11			2.628	70.39	7.040	42.88
12	.06875		3.071	47.66	7.725	36.97
13	.3703	100.9	3.495	38.03	8.390	33.61
14	.6593	60.77	3.904	32.79	9.039	31.54
15	.9368	45.54	4.300	29.55	9.673	30.21
16	1.204	37.56	4.684	27.39	10.30	29.34
17	1.461	32.68	5.058	25.88	10.91	28.78
18	1.709	29.40	5.423	24.78	11.51	28.44
19	1.950	27.07	5.781	23.97	12.11	28.26
20	2.184	25.33	6.131	23.37	12.70	28.19
21	2.411	24.00	6.476	22.92	13.29	28.21
22	2.633	22.96	6.814	22.58	13.87	28.31
23	2.849	22.12	7.148	22.34	14.44	28.46
24	3.061	21.44	7.478	22.16	15.01	28.66
25	3.268	20.89	7.804	22.04	15.58	28.89
26	3.471	20.43	8.125	21.97	16.15	29.16
27	3.671	20.05	8.444	21.94	16.71	29.46
28	3.867	19.73	8.759	21.94	17.27	29.77
29	4.061	19.47	9.072	21.97	17.82	30.11
30	4.251	19.25	9.382	22.02	18.38	30.46

p = 4 variables (concluded)

Sample Size n	$\lambda^2 = 0.5$ $\underline{T}^2_n$	$\lambda^2 = 0.5$ $\overline{T}^2_n$	$\lambda^2 = 1.0$ $\underline{T}^2_n$	$\lambda^2 = 1.0$ $\overline{T}^2_n$
31	4.439	19.06	9.690	22.09
32	4.624	18.91	9.995	22.18
33	4.807	18.78	10.30	22.28
34	4.988	18.68	10.60	22.40
35	5.167	18.60	10.90	22.52
36	5.344	18.54	11.20	22.66
37	5.520	18.49	11.49	22.81
38	5.693	18.45	11.79	22.96
39	5.866	18.43	12.08	23.13
40	6.036	18.42	12.38	23.30
41	6.206	18.42	12.67	23.47
42	6.374	18.43	12.96	23.65
43	6.540	18.44	13.25	23.84
44	6.706	18.47	13.53	24.03
45	6.871	18.50	13.82	24.22
46	7.034	18.53		
47	7.197	18.58		
48	7.358	18.62		
49	7.519	18.67		
50	7.679	18.73		
51	7.838	18.79		
52	7.996	18.85		
53	8.153	18.92		
54	8.310	18.99		
55	8.466	19.06		
56	8.622	19.14		
57	8.776	19.22		
58	8.931	19.30		
59	9.084	19.38		
60	9.237	19.47		

Tables for Sequential T^2-test

$\alpha = \beta = .05$

$p = 5$ variables

Sample Size n	$\lambda^2 = 0.5$		$\lambda^2 = 1.0$		$\lambda^2 = 2.0$	
	$\underline{T}^2_n$	$\overline{T}^2_n$	$\underline{T}^2_n$	$\overline{T}^2_n$	$\underline{T}^2_n$	$\overline{T}^2_n$
6			.07848		4.035	
7			.7858		5.131	
8			1.468		6.112	
9			2.110		7.007	
10			2.710		7.838	139.2
11			3.271	460.4	8.620	75.91
12	.08594		3.798	106.3	9.365	56.74
13	.4625		4.297	65.98	10.08	47.70
14	.8223	189.7	4.771	50.57	10.77	42.58
15	1.166	91.82	5.224	42.53	11.44	39.38
16	1.493	63.77	5.659	37.65	12.10	37.26
17	1.807	50.52	6.079	34.42	12.74	35.82
18	2.108	42.84	6.485	32.15	13.37	34.83
19	2.397	37.85	6.880	30.50	13.99	34.14
20	2.675	34.36	7.265	29.27	14.61	33.68
21	2.944	31.80	7.641	28.33	15.21	33.38
22	3.204	29.86	8.008	27.61	15.81	33.22
23	3.456	28.33	8.369	27.04	16.40	33.15
24	3.701	27.12	8.724	26.61	16.99	33.16
25	3.939	26.13	9.072	26.27	17.57	33.24
26	4.172	25.32	9.416	26.02	18.15	33.38
27	4.399	24.65	9.755	25.83	18.73	33.55
28	4.622	24.08	10.09	25.69	19.30	33.76
29	4.840	23.60	10.42	25.60	19.87	34.01
30	5.053	23.20	10.75	25.54	20.43	34.28

p = 5 variables (concluded)

Sample Size n	$\lambda^2 = 0.5$ $\underline{T}^2_n$	$\lambda^2 = 0.5$ $\overline{T}^2_n$	$\lambda^2 = 1.0$ $\underline{T}^2_n$	$\lambda^2 = 1.0$ $\overline{T}^2_n$
31	5.263	22.86	11.07	25.52
32	5.469	22.57	11.39	25.53
33	5.672	22.32	11.71	25.56
34	5.872	22.11	12.03	25.61
35	6.069	21.93	12.34	25.67
36	6.263	21.78	12.65	25.76
37	6.455	21.65	12.96	25.86
38	6.645	21.55	13.27	25.97
39	6.832	21.46	13.57	26.09
40	7.017	21.39	13.88	26.22
41	7.201	21.33	14.18	26.36
42	7.382	21.29	14.48	26.51
43	7.562	21.26	14.78	26.67
44	7.741	21.24	15.08	26.83
45	7.917	21.23	15.37	27.00
46	8.093	21.23		
47	8.267	21.24		
48	8.439	21.25		
49	8.611	21.28		
50	8.781	21.30		
51	8.950	21.34		
52	9.118	21.38		
53	9.285	21.42		
54	9.451	21.47		
55	9.617	21.52		
56	9.781	21.58		
57	9.944	21.64		
58	10.11	21.70		
59	10.27	21.77		
60	10.43	21.84		

Tables for Sequential T^2-test

$\alpha = \beta = .05$

p = 6 variables

Sample Size n	$\lambda^2 = 0.5$ $\underline{T}^2_n$	$\lambda^2 = 0.5$ $\overline{T}^2_n$	$\lambda^2 = 1.0$ $\underline{T}^2_n$	$\lambda^2 = 1.0$ $\overline{T}^2_n$	$\lambda^2 = 2.0$ $\underline{T}^2_n$	$\lambda^2 = 2.0$ $\overline{T}^2_n$
7			.9551		6.523	
8			1.792		7.669	
9			2.575		8.680	
10			3.298		9.594	
11			3.965		10.44	201.5
12	.1032		4.584	2741.	11.23	100.4
13	.5558		5.161	155.6	11.98	72.52
14	.9880		5.702	88.14	12.70	59.70
15	1.399	407.9	6.214	65.05	13.40	52.48
16	1.790	136.6	6.701	53.50	14.07	47.95
17	2.162	86.85	7.166	46.63	14.73	44.92
18	2.517	65.99	7.612	42.12	15.38	42.81
19	2.856	54.56	8.043	38.97	16.02	41.30
20	3.180	47.38	8.460	36.67	16.64	40.22
21	3.491	42.47	8.865	34.94	17.26	39.43
22	3.790	38.91	9.259	33.61	17.87	38.87
23	4.079	36.22	9.644	32.58	18.47	38.49
24	4.358	34.14	10.02	31.76	19.07	38.23
25	4.628	32.47	10.39	31.11	19.66	38.09
26	4.890	31.13	10.75	30.59	20.25	38.03
27	5.145	30.02	11.11	30.17	20.83	38.04
28	5.393	29.10	11.46	29.85	21.41	38.11
29	5.635	28.32	11.81	29.59	21.99	38.23
30	5.871	27.66	12.15	29.39	22.56	38.39

p = 6 variables (concluded)

Sample Size n	$\lambda^2 = 0.5$		$\lambda^2 = 1.0$	
	$\underline{T_n^2}$	$\overline{T_n^2}$	$\underline{T_n^2}$	$\overline{T_n^2}$
31	6.102	27.10	12.49	29.25
32	6.329	26.62	12.82	29.14
33	6.551	26.21	13.16	29.07
34	6.769	25.85	13.48	29.03
35	6.984	25.55	13.81	29.02
36	7.195	25.28	14.13	29.04
37	7.402	25.05	14.45	29.07
38	7.607	24.85	14.77	29.12
39	7.809	24.68	15.08	29.19
40	8.008	24.54	15.40	29.27
41	8.204	24.41	15.71	29.37
42	8.399	24.31	16.02	29.47
43	8.591	24.22	16.33	29.59
44	8.781	24.15	16.63	29.72
45	8.969	24.09	16.94	29.85
46	9.156	24.04		
47	9.340	24.01		
48	9.523	23.98		
49	9.704	23.97		
50	9.884	23.96		
51	10.06	23.96		
52	10.24	23.97		
53	10.42	23.99		
54	10.59	24.01		
55	10.76	24.04		
56	10.94	24.07		
57	11.11	24.11		
58	11.28	24.15		
59	11.45	24.20		
60	11.62	24.25		

Tables for Sequential T^2-test

$\alpha = \beta = .05$

p = 7 variables

Sample Size n	$\lambda^2 = 0.5$ $\underline{T^2_n}$	$\lambda^2 = 0.5$ $\overline{T^2_n}$	$\lambda^2 = 1.0$ $\underline{T^2_n}$	$\lambda^2 = 1.0$ $\overline{T^2_n}$	$\lambda^2 = 2.0$ $\underline{T^2_n}$	$\lambda^2 = 2.0$ $\overline{T^2_n}$
8			2.138		9.586	
9			3.077		10.70	
10			3.938		11.69	
11			4.723		12.57	
12	.1204		5.441		13.39	281.9
13	.6502		6.102		14.16	128.8
14	1.157		6.714	223.1	14.89	90.28
15	1.639		7.286	114.6	15.59	73.00
16	2.096	1497.	7.825	81.61	16.27	63.34
17	2.529	203.3	8.334	65.76	16.94	57.27
18	2.940	116.0	8.820	56.52	17.59	53.19
19	3.330	84.33	9.285	50.52	18.22	50.32
20	3.702	68.00	9.732	46.35	18.85	48.23
21	4.057	58.08	10.16	43.31	19.47	46.69
22	4.397	51.43	10.58	41.01	20.08	45.54
23	4.723	46.68	10.99	39.24	20.68	44.68
24	5.036	43.13	11.39	37.85	21.28	44.04
25	5.338	40.38	11.77	36.74	21.88	43.57
26	5.630	38.21	12.15	35.84	22.47	43.24
27	5.913	36.45	12.53	35.12	23.05	43.02
28	6.187	35.00	12.89	34.53	23.63	42.90
29	6.453	33.80	13.25	34.05	24.21	42.85
30	6.712	32.79	13.61	33.66	24.79	42.86

p = 7 variables (concluded)

Sample Size n	$\lambda^2 = 0.5$: $\underline{T}_n^2$	$\lambda^2 = 0.5$: $\overline{T_n^2}$	$\lambda^2 = 1.0$: $\underline{T}_n^2$	$\lambda^2 = 1.0$: $\overline{T_n^2}$
31	6.964	31.93	13.96	33.34
32	7.211	31.19	14.31	33.09
33	7.452	30.55	14.65	32.89
34	7.688	30.01	14.99	32.74
35	7.919	29.53	15.32	32.62
36	8.146	29.11	15.65	32.54
37	8.369	28.75	15.98	32.49
38	8.588	28.43	16.31	32.47
39	8.803	28.16	16.63	32.47
40	9.016	27.91	16.95	32.49
41	9.225	27.70	17.27	32.53
42	9.432	27.52	17.59	32.58
43	9.635	27.36	17.90	32.65
44	9.836	27.22	18.21	32.74
45	10.04	27.10	18.53	32.83
46	10.23	26.99		
47	10.43	26.91		
48	10.62	26.83		
49	10.81	26.77		
50	11.00	26.73		
51	11.19	26.69		
52	11.37	26.66		
53	11.56	26.64		
54	11.74	26.63		
55	11.92	26.63		
56	12.10	26.63		
57	12.28	26.65		
58	12.45	26.66		
59	12.63	26.69		
60	12.81	26.71		

Tables for Sequential T^2-test

$\alpha = \beta = .05$

$p = 8$ variables

Sample Size n	$\lambda^2 = 0.5$ $\underline{T}^2_n$	$\lambda^2 = 0.5$ $\overline{T}^2_n$	$\lambda^2 = 1.0$ $\underline{T}^2_n$	$\lambda^2 = 1.0$ $\overline{T^2}$	$\lambda^2 = 2.0$ $\underline{T}^2_n$	$\lambda^2 = 2.0$ $\overline{T}^2_n$
9			3.626		13.22	
10			4.641		14.24	
11			5.557		15.13	
12	.1378		6.384		15.94	
13	.7459		7.135		16.69	384.0
14	1.329		7.823		17.41	161.2
15	1.885		8.457	315.9	18.09	110.0
16	2.411		9.048	145.9	18.75	87.60
17	2.908		9.601	100.4	19.40	75.15
18	3.378	307.1	10.12	79.36	20.03	67.35
19	3.823	152.9	10.62	67.76	20.66	62.08
20	4.245	106.0	11.10	59.64	21.27	58.35
21	4.646	83.32	11.55	54.30	21.88	55.61
22	5.027	70.02	11.99	50.42	22.48	53.56
23	5.392	61.29	12.42	47.49	23.07	52.00
24	5.741	55.14	12.83	45.22	23.66	50.80
25	6.076	50.59	13.24	43.43	24.25	49.88
26	6.398	47.09	13.63	41.99	24.83	49.17
27	6.708	44.34	14.02	40.83	25.41	48.64
28	7.008	42.11	14.40	39.87	25.99	48.25
29	7.299	40.29	14.77	39.09	26.56	47.96
30	7.580	38.77	15.13	38.44	27.13	47.78

p = 8 variables (concluded)

Sample Size n	$\lambda^2 = 0.5$		$\lambda^2 = 1.0$	
	$\underline{T}^2_n$	$\overline{T}^2_n$	$\underline{T}^2_n$	$\overline{T}^2_n$
31	7.854	37.49	15.49	37.90
32	8.120	36.40	15.85	37.45
33	8.380	35.47	16.20	37.08
34	8.633	34.66	16.54	36.78
35	8.881	33.96	16.88	36.53
36	9.123	33.35	17.22	36.34
37	9.361	32.82	17.56	36.18
38	9.594	32.35	17.89	36.06
39	9.823	31.94	18.22	35.97
40	10.05	31.57	18.55	35.92
41	10.27	31.25	18.87	35.88
42	10.49	30.97	19.20	35.87
43	10.70	30.72	19.52	35.88
44	10.91	30.49	19.83	35.91
45	11.12	30.30	20.15	35.95
46	11.33	30.13		
47	11.53	29.98		
48	11.73	29.84		
49	11.93	29.73		
50	12.13	29.63		
51	12.33	29.55		
52	12.52	29.47		
53	12.71	29.41		
54	12.90	29.37		
55	13.09	29.33		
56	13.27	29.30		
57	13.46	29.28		
58	13.64	29.26		
59	13.83	29.26		
60	14.01	29.26		

Tables for Sequential T^2-test

$\alpha = \beta = .05$

p = 9 variables

Sample Size n	$\lambda^2 = 0.5$		$\lambda^2 = 1.0$		$\lambda^2 = 2.0$	
	$\underline{T}^2_n$	$\overline{T}^2_n$	$\underline{T}^2_n$	$\overline{T}^2_n$	$\underline{T}^2_n$	$\overline{T}^2_n$
10			5.418		17.43	
11			6.482		18.26	
12	.1551		7.430		19.01	
13	.8429		8.280		19.69	
14	1.506		9.046		20.35	512.11
15	2.138		9.745		20.97	197.8
16	2.736		10.39	445.2	21.59	131.8
17	3.301		10.99	182.4	22.19	103.5
18	3.833		11.54	121.4	22.78	87.94
19	4.335	481.8	12.07	94.33	23.36	78.19
20	4.809	199.7	12.57	79.15	23.94	71.60
21	5.258	131.4	13.05	69.49	24.52	66.91
22	5.684	100.7	13.51	62.84	25.09	63.45
23	6.089	83.28	13.95	25.67	25.67	60.83
24	6.474	72.10	14.38	54.38	26.23	58.82
25	6.843	64.33	14.79	51.56	26.80	57.25
26	7.196	58.63	15.20	49.33	27.37	56.02
27	7.535	54.28	15.59	47.53	27.93	55.05
28	7.862	50.87	15.98	46.07	28.49	54.29
29	8.176	48.12	16.36	44.86	29.05	53.70
30	8.481	45.88	16.73	43.86	29.61	53.25

p = 9 variables (concluded)

Sample Size n	$\lambda^2 = 0.5$ $\underline{T^2_n}$	$\lambda^2 = 0.5$ $\overline{T^2_n}$	$\lambda^2 = 1.0$ $\underline{T^2_n}$	$\lambda^2 = 1.0$ $\overline{T^2_n}$
31	8.775	44.01	17.10	43.02
32	9.061	42.43	17.46	42.32
33	9.339	41.09	17.82	41.73
34	9.610	39.94	18.17	41.23
35	9.874	38.95	18.51	40.81
36	10.13	38.09	18.86	40.47
37	10.38	37.33	19.20	40.18
38	10.63	36.67	19.53	39.94
39	10.87	36.08	19.87	39.74
40	11.11	35.56	20.20	39.59
41	11.34	35.11	20.53	39.46
42	11.56	34.70	20.85	39.37
43	11.79	34.34	21.18	39.31
44	12.02	34.01	21.50	39.27
45	12.23	33.72	21.82	39.25
46	12.45	33.47		
47	12.66	33.24		
48	12.87	33.04		
49	13.08	32.85		
50	13.28	32.69		
51	13.49	32.55		
52	13.69	32.43		
53	13.89	32.32		
54	14.08	32.22		
55	14.28	32.14		
56	14.47	32.07		
57	14.66	32.01		
58	14.85	31.97		
59	15.04	31.93		
60	15.23	31.90		

APPENDIX D

A SHORT TABLE OF THE HYPERGEOMETRIC FUNCTION ${}_0F_1(c;x)$

The function

$$(D.1) \qquad {}_0F_1(c;x) = 1 + \frac{x}{c} + \frac{x^2}{c(c+1)2!} + \frac{x^3}{c(c+1)(c+2)3!} + \cdots\cdot$$

is a special case of a generalized hypergeometric function and is a solution of the differential equation:

$$(D.2) \qquad x\frac{d^2u}{dx^2} + c\frac{du}{dx} - u = 0.$$

It is convergent for all finite x. This function is related to many other well known transcendental functions among them:

a-Modified Bessel Function of the first kind.

$${}_0F_1(c;x) = \Gamma(c)x^{\frac{1-c}{2}} I_{c-1}(2\sqrt{x}).$$

b-Confluent hypergeometric function.

$${}_0F_1(c;x) = e^{-2\sqrt{x}}\ {}_1F_1(c-\tfrac{1}{2},\ 2c-1;\ 4\sqrt{x}).$$

c-Whittaker's M-function.

$${}_0F_1(c;x) = (4\sqrt{x})^{\frac{1-c}{2}} M_{0,c-1}\ (4\sqrt{x}).$$

None of these functions are tabulated to the extent that is required for the sequential χ^2-test and it seemed best to make the following short table of the function ${}_0F_1(c;x)$ itself.

The present table gives the value of the function ${}_0F_1(c;x)$ to six significant figures for $x = .1(.1)1(1)10(10)100(50)1000$ and $c = .5(.5)5$. Thus the table has a uniform graduation for c and is divided into four "bands" for values of x. The value in parentheses directly following each tabulated value indicates the power of 10 by which the tabulated value should be multiplied to produce the true result. For example:

$${}_0F_1(4.5;100) = 1.17108(5) = 117108.$$

When interpolating, it is best to work with log ${}_0F_1(c;x)$ rather than the function itself. In general, when interpolating for given values of x within "bands", four-point Lagrangian interpolation will yield four-digit accuracy. For small values of x, linear interpolation (of the logs) will suffice. When interpolating near either end of a band, four-point Newton interpolation formulae will prove quite satisfactory. When interpolating for given values of c, the results will not be as accurate. If ${}_0F_1(c;x)$ is to be obtained when neither c nor x appear in the table, one must resort either to double interpolation or to a numerical solution of the original differential equation itself.

Table of ${}_0F_1(c;x)$

c	x = .1	.2	.3	.4
.5	1.20676(0)	1.42739(0)	1.66245(0)	1.91252(0)
1.0	1.10253(0)	1.21022(0)	1.32326(0)	1.44182(0)
1.5	1.06801(0)	1.13877(0)	1.21235(0)	1.28883(0)
2.0	1.05084(0)	1.10339(0)	1.15769(0)	1.21379(0)
2.5	1.04058(0)	1.08232(0)	1.12526(0)	1.16942(0)
3.0	1.03375(0)	1.06836(0)	1.10383(0)	1.14018(0)
3.5	1.02889(0)	1.05843(0)	1.08862(0)	1.11949(0)
4.0	1.02525(0)	1.05101(0)	1.07729(0)	1.10409(0)
4.5	1.02243(0)	1.04526(0)	1.06851(0)	1.09219(0)
5.0	1.02017(0)	1.04067(0)	1.06152(0)	1.08272(0)

c	x = .5	.6	.7	.8
.5	2.17818(0)	2.46004(0)	2.75873(0)	3.07487(0)
1.0	1.56608(0)	1.69623(0)	1.83246(0)	1.97496(0)
1.5	1.36830(0)	1.45084(0)	1.53653(0)	1.62547(0)
2.0	1.27172(0)	1.33155(0)	1.39330(0)	1.45703(0)
2.5	1.21483(0)	1.26151(0)	1.30950(0)	1.35882(0)
3.0	1.17744(0)	1.21562(0)	1.25473(0)	1.29480(0)
3.5	1.15104(0)	1.18328(0)	1.21623(0)	1.24990(0)
4.0	1.13143(0)	1.15931(0)	1.18774(0)	1.21673(0)
4.5	1.11629(0)	1.14083(0)	1.16582(0)	1.19125(0)
5.0	1.10427(0)	1.12617(0)	1.14844(0)	1.17108(0)

c	x = .9	1	2	3
.5	3.40914(0)	3.76220(0)	8.48897(0)	1.59895(0)
1.0	2.12393(0)	2.27959(0)	4.25235(0)	7.15900(0)
1.5	1.71774(0)	1.81343(0)	2.98041(0)	4.60674(0)
2.0	1.52280(0)	1.59064(0)	2.39483(0)	3.46865(0)
2.5	1.40950(0)	1.46157(0)	2.06571(0)	2.84570(0)
3.0	1.33585(0)	1.37790(0)	1.85752(0)	2.46023(0)
3.5	1.28431(0)	1.31946(0)	1.71505(0)	2.20131(0)
4.0	1.24630(0)	1.27644(0)	1.61195(0)	2.01684(0)
4.5	1.21714(0)	1.24350(0)	1.53412(0)	1.87946(0)
5.0	1.19410(0)	1.21749(0)	1.47343(0)	1.77358(0)

c	x = 4	5	6	7
.5	2.73082(1)	4.37775(1)	6.70801(1)	9.93234(1)
1.0	1.13019(1)	1.70578(1)	2.48921(1)	3.53766(1)
1.5	6.82248(0)	9.78638(0)	1.36911(1)	1.87694(1)
2.0	4.87973(0)	6.70929(0)	9.05436(0)	1.20304(1)
2.5	3.84108(0)	5.09866(0)	6.67362(0)	8.63078(0)
3.0	3.21109(0)	4.13939(0)	5.27926(0)	6.67034(0)
3.5	2.79506(0)	3.51579(0)	4.38596(0)	5.43140(0)
4.0	2.50296(0)	3.08387(0)	3.77510(0)	4.59432(0)
4.5	2.28816(0)	2.77002(0)	3.33618(0)	3.99922(0)
5.0	2.12441(0)	2.53325(0)	3.00832(0)	3.55889(0)

c	x = 8	9	10	20
.5	1.43125(2)	2.01716(2)	2.79056(2)	3.83193(3)
1.0	4.92086(1)	6.72344(1)	9.04759(1)	1.03763(3)
1.5	2.53006(1)	3.36189(1)	4.41223(1)	4.28423(2)
2.0	1.57742(1)	2.04473(1)	2.62399(1)	2.18638(2)
2.5	1.10460(1)	1.40081(1)	1.76200(1)	1.27632(2)
3.0	8.35859(0)	1.03971(1)	1.28472(1)	8.18993(1)
3.5	6.68180(0)	8.17116(0)	9.93836(0)	5.63984(1)
4.0	5.56170(0)	6.70012(0)	8.03558(0)	4.10217(1)
4.5	4.77340(0)	5.67476(0)	6.72144(0)	3.11645(1)
5.0	4.19534(0)	4.92935(0)	5.77396(0)	2.45265(1)

c	x = 30	40	50	60
.5	2.86041(4)	1.55743(5)	6.93141(5)	2.67320(6)
1.0	6.97878(3)	3.53018(4)	1.48419(5)	5.46442(5)
1.5	2.61118(3)	1.23126(4)	4.90124(4)	1.72554(5)
2.0	1.21452(3)	5.35631(3)	2.02334(4)	6.82291(4)
2.5	6.49822(2)	2.68932(3)	9.66192(3)	3.12580(4)
3.0	3.84284(2)	1.49727(3)	5.12743(3)	1.59404(4)
3.5	2.45170(2)	9.02180(2)	2.95129(3)	8.83101(3)
4.0	1.66047(2)	5.78855(2)	1.81271(3)	5.22887(3)
4.5	1.18023(2)	3.90937(2)	1.17436(3)	3.27061(3)
5.0	8.72949(1)	2.75525(2)	7.95532(2)	2.14231(3)

	x			
c	70	80	90	100
.5	9.24925(6)	2.93674(7)	8.69217(7)	2.42582(8)
1.0	1.81805(6)	5.58012(6)	1.60300(7)	4.35583(7)
1.5	5.52748(5)	1.64169(6)	4.58118(6)	1.21291(7)
2.0	2.10702(5)	6.06180(5)	1.64456(6)	4.24550(6)
2.5	9.31768(4)	2.59929(5)	6.86171(5)	1.72840(6)
3.0	4.59241(4)	1.24348(5)	3.19677(5)	7.86255(5)
3.5	2.46199(4)	6.47700(4)	1.62292(5)	3.90027(5)
4.0	1.41238(4)	3.61373(4)	8.83258(4)	2.07554(5)
4.5	8.56961(3)	2.13455(4)	5.09327(4)	1.17108(5)
5.0	5.45149(3)	1.32317(4)	3.08468(4)	6.94441(4)

	x			
c	150	200	250	300
.5	2.17254(10)	9.60888(11)	2.70749(13)	5.53824(14)
1.0	3.52074(9)	1.44809(11)	3.85702(12)	7.53541(13)
1.5	8.86938(8)	3.39725(10)	8.56184(11)	1.59875(13)
2.0	2.81537(8)	1.00569(10)	2.40051(11)	4.28731(12)
2.5	1.04192(8)	3.47593(9)	7.86562(10)	1.34459(12)
3.0	4.31894(7)	1.34752(9)	2.89358(10)	4.73779(11)
3.5	1.95686(7)	5.71811(8)	1.16629(10)	1.83037(11)
4.0	9.53391(6)	2.61280(8)	5.06676(9)	7.62707(10)
4.5	4.93639(6)	1.27055(8)	2.34476(9)	3.38787(10)
5.0	2.69244(6)	6.51744(7)	1.14571(9)	1.59003(10)

	x			
c	350	400	450	500
.5	8.88753(15)	1.17693(17)	1.33206(18)	1.32193(19)
1.0	1.16322(15)	1.48948(16)	1.63659(17)	1.58168(18)
1.5	2.37529(14)	2.94231(15)	3.13970(16)	2.95593(17)
2.0	6.13399(13)	7.35370(14)	7.62351(15)	6.99397(16)
2.5	1.85357(13)	2.15157(14)	2.16777(15)	1.93856(16)
3.0	6.29643(12)	7.07970(13)	6.93493(14)	6.04698(15)
3.5	2.34636(12)	2.55671(13)	2.43577(14)	2.07156(15)
4.0	9.43603(11)	9.96859(12)	9.24002(13)	7.66713(14)
4.5	4.04734(11)	4.14727(12)	3.74149(13)	3.02996(14)
5.0	1.83526(11)	1.82485(12)	1.60291(13)	1.26726(14)

c	x: 550	600	650	700
.5	1.17270(20)	9.43990(20)	6.97799(21)	4.78270(22)
1.0	1.36991(19)	1.07888(20)	7.81632(20)	5.25848(21)
1.5	2.50020(18)	1.92691(19)	1.36849(20)	9.03846(20)
2.0	5.77869(17)	4.35933(18)	3.03560(19)	1.96865(20)
2.5	1.56504(17)	1.15590(18)	7.89362(18)	5.02749(19)
3.0	4.77134(16)	3.45096(17)	2.31162(18)	1.44617(19)
3.5	1.59798(16)	1.13207(17)	7.43976(17)	4.57270(18)
4.0	5.78352(15)	4.01424(16)	2.58871(17)	1.56345(18)
4.5	2.23561(15)	1.52059(16)	9.62452(16)	5.71277(17)
5.0	9.14835(14)	6.09908(15)	3.78969(16)	2.21114(17)

c	x: 750	800	850
.5	3.06378(23)	1.84661(24)	1.05311(25)
1.0	3.31069(22)	1.96334(23)	1.10276(24)
1.5	5.59368(21)	3.26438(22)	1.80606(23)
2.0	1.19780(21)	6.87984(21)	3.74986(22)
2.5	3.00785(20)	1.70059(21)	9.13275(21)
3.0	8.50909(19)	4.73636(20)	2.50650(21)
3.5	2.64645(19)	1.45046(20)	7.56500(20)
4.0	8.90172(18)	4.80465(19)	2.47003(20)
4.5	3.20040(18)	1.70138(19)	8.62261(19)
5.0	1.21903(18)	6.38385(18)	3.18988(19)

c	x: 900	950	1000
.5	5.71003(25)	2.95583(26)	1.46610(27)
1.0	5.89407(24)	3.00997(25)	1.47385(26)
1.5	9.51672(23)	4.79499(24)	2.31811(25)
2.0	1.94825(23)	9.68609(23)	4.62374(24)
2.5	4.67905(22)	2.29569(23)	1.08219(24)
3.0	1.26650(22)	6.13285(22)	2.85523(23)
3.5	3.77034(21)	1.80214(22)	8.28710(22)
4.0	1.21440(21)	5.73019(21)	2.60293(22)
4.5	4.18252(20)	1.94847(21)	8.74405(21)
5.0	1.52675(20)	7.02295(20)	3.11393(21)

The two page vita has been removed from the scanned document. Page 1 of 2

The two page vita has been removed from the scanned document. Page 2 of 2

ABSTRACT

We consider a multivariate situation with means $\mu_1, \ldots, \mu_p$ and covariance matrix Σ. We wish to derive sequential procedures for testing the hypothesis:

$$H_0: \quad (\underline{\mu} - \underline{\mu}_0)\Sigma^{-1}(\underline{\mu} - \underline{\mu}_0)' = \lambda_0^2 \text{ (usually zero)}$$

against the alternative:

$$H_1: \quad (\underline{\mu} - \underline{\mu}_0)\Sigma^{-1}(\underline{\mu} - \underline{\mu}_0)' = \lambda_1^2$$

both for the case where Σ is known (the sequential χ^2-test) and where Σ is unknown and must be estimated from the sample (the sequential T^2-test). These sequential procedures should guarentee that the probability of accepting H_1 when H_0 is true is equal to α and the probability of accepting H_0 when H_1 is true is equal to β.

For the case where Σ is known, $\lambda_0^2 = 0$ and $\lambda_1^2 = \lambda^2$, the test procedure is as follows: for a sample of n observations, form the probability ratio:

$$p_{1n}/p_{0n} = e^{-n\lambda^2/2}\,{}_0F_1(p/2; n\lambda^2\chi_n^2/4)$$

where p denotes the number of variables, ${}_n\underline{\bar{x}}$ denotes the vector of sample means based on n observations, $\chi_n^2 = n({}_n\underline{\bar{x}} - \underline{\mu}_0)\Sigma^{-1}({}_n\underline{\bar{x}} - \underline{\mu}_0)'$ and ${}_0F_1(c;x)$ is a type of generalized hypergeometric function.

a. If $p_{1n}/p_{0n} \leq \beta/(1-\alpha)$, accept H_0;

b. If $p_{1n}/p_{0n} \geq (1-\beta)/\alpha$, accept H_1;

c. If $\beta/(1-\alpha) < p_{1n}/p_{0n} < (1-\beta)/\alpha$, continue sampling.

For the case where Σ is unknown, the procedure is exactly the same except that the probability ratio is now:

$$p_{1n}/p_{0n} = e^{-n\lambda^2/2}\,{}_1F_1[n/2, p/2; n\lambda^2 T_n^2/2(n-1+T_n^2)]$$

where $T_n^2 = n({}_n\underline{\bar{x}} - \underline{\mu}_0)S_n^{-1}({}_n\underline{\bar{x}} - \underline{\mu}_0)'$, S_n denotes the sample covariance matrix based on n observations and ${}_1F_1(a,c;x)$ is a confluent hypergeometric function. Procedures are also given for the case $\lambda_0^2 \neq 0$.

Similar procedures are given to test the hypothesis:

$$H_0: \ (\underline{\mu}_1 - \underline{\mu}_2 - \underline{\delta})\Sigma^{-1}(\underline{\mu}_1 - \underline{\mu}_2 - \underline{\delta})' = \lambda_0^2 \text{ (usually zero)}$$

against the alternative:

$$H_1: \ (\underline{\mu}_1 - \underline{\mu}_2 - \underline{\delta})\Sigma^{-1}(\underline{\mu}_1 - \underline{\mu}_2 - \underline{\delta})' = \lambda_1^2.$$

It is shown that these sequential procedures all exist in the sense that the risks of accepting H_0 when H_1 is true and of accepting H_1 when H_0 is true are approximately α and β respectively and that these sequential procedures terminate with probability unity. Some of these situations have been generalized to give simultaneous tests on the means and covariance matrix of a sample.

No expressions yet exist for the OC or ASN functions although some conjectured values have been determined for the latter and suggest, in comparison with their corresponding fixed-sample tests, substantial reductions in the sample sizes required when either H_0 or H_1 is true.

The general problem of tolerances is discussed and then some of these procedures are demonstrated with a numerical example drawn from the field of ballistic missiles.

The determination of p_{1n}/p_{0n} is quite laborious for both the sequential χ^2- and T^2-tests since it requires the evaluation of a hypergeometric function each time an observation is made. It would be better for each value of n, given p, α, β and λ^2 under H_1, to compute the values of χ_n^2 or T_n^2 which would correspond to the boundaries of the tests indicated by $\beta/(1-\alpha)$ and $(1-\beta)/\alpha$. Tables to facilitate both the sequential χ^2- and T^2-tests are given for $p = 2,3,\ldots,9$; $\lambda^2 = 0.5$, 1.0, 2.0; $\alpha = \beta = .05$ for n ranging from the minimum value necessary to reach a decision to 30, 45 and 60 for $\lambda^2 = 0.5$, 1.0 and 2.0 respectively. These tables were prepared on the IBM 650 computer using the Newton-Raphson iterative procedure.

Finally, a discussion is given for the hypergeometric function ${}_0F_1(c;x)$ and a table given of this function for $c = .5(.5)5.0$ and $x = .1(.1)1(1)\ 10(10)100(50)1000$.