REES MATRIX SEMIGROUPS
OVER SPECIAL STRUCTURE GROUPS WITH ZERO

by

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1. Rees matrix semigroups over the trivial group with zero

We shall consider Rees matrix semigroups $M^o((e);I^+,I^+;P)$ over the trivial group with zero (e,0) and sandwich matrices P [1, p. 88].

1.1 Idempotent semigroups

Let I^+ be the set of all positive integers.
Consider Rees matrix semigroups $M^o((e);I^+,I^+;P)$ over the trivial group with zero (e, 0) and sandwich matrices P.

Lemma 1.1. The Rees matrix semigroup $M^o((e);I^+,I^+;P)$ is an idempotent semigroup if and only if P contains no zero entry.

Proof. Assume that $M^o((e);I^+,I^+;P)$ is an idempotent semigroup. Then every element $(e;i,u)$ of $M^o((e);I^+,I^+;P)$ is an idempotent by definition of an idempotent semigroup. Thus we must have $(e;i,u)o(e;i,u)=(ep_{ui}e;i,u)=(e;i,u)$, from which we get $p_{ui}=e\neq 0$ for every $i,u=1,2,\ldots$ Therefore the sandwich matrix P contains no zero entry. Conversely, if P contains no zero entry, then every element $(e;i,u)\in M^o((e);I^+,I^+;P)$ is clearly an idempotent. Thus $M^o((e);I^+,I^+;P)$ is an idempotent semigroup if P contains no zero entry.

Lemma 1.2. The number of nonzero idempotent elements of a Rees matrix semigroup $M^o((e);I^+,I^+;P)$ is equal to the number of nonzero entries of the sandwich matrix P.

P Proof. $(e;i,u)$ is idempotent if and only if $p_{ui}\neq 0$, where p_{ui} is an entry of P.

Lemma 1.3. There are no inverse idempotent semigroups of the type $M^o((e);I^+,I^+;P)$.

Hence there are no Brandt idempotent semigroups of the form $M^o((e);I^+,I^+;P)$.

Proof. If there is an inverse idempotent semigroup $S = M^o((e);I^+,I^+;P)$ then, by Lemma 1.1, P contains no zero entry. Consider $(e;i,i+1)o(e;i,i)=(ep_{i+1i}e;i,i)=(e;i,i)$ and $(e;i,i)o(e;i,i+1)=(ep_{ii}e;i,i+1)=(e;i,i+1)$. This shows that idempotents of $M^o((e);I^+,I^+;P)$ are not commutative. Therefore, it is not an inverse semigroup.

Definition 1.1. (i) Let $A=(a_{ij})$ be a nxm matrix over a group G with zero. A matrix $B=(b_{uv})$ is a submatrix of A if and only if B is made from A by removing a number of rows or a number of columns of A.

(ii) A matrix C over a group G with zero is regular if and only if each row and each column of C contains a nonzero entry.

(iii) A permutation matrix Q is a square matrix which in each row and in each column has some one entry e and all other entries zero.

Lemma 1.4. Let $S=M^o((e);I^+,I^+;P)$ be a Rees matrix semigroup over the one element group (e) with zero with a sandwich matrix P.

(i) Every element a of the semigroup is either an idempotent element or a nilpotent element.

(ii) Two nonzero idempotents $(e;i,u)$ and $(e;j,v)$ commute if and only if $p_{uj}=p_{vi}=0$.

(iii) S contains a proper inverse idempotent subsemigroup E if and only if P has a proper permutation submatrix Q of P.

Proof. (i) Let $a=(e;i,j)$. Then $aa=(e;i,u)o(e;i,u)$

$$= \begin{cases} (e;i,u)=a \text{ if } p_{ui}\neq 0, \\ 0 \quad \text{otherwise.} \end{cases}$$

(ii) Let us assume that $p_{uj}=p_{vi}=0$. Then we have the following:

$(e;i,u)o(e;j,v)=(ep_{uj}e;i,v)=(e0e;i,v)=(0)_{iv}=0$,

$(e;j,v)o(e;i,u)=(ep_{vi}e;j,u)=(e0e;j,u)=(0)_{ju}=0$.

Thus two idempotents $(e;i,u)$ and $(e;j,v)$ commute.

Conversely, if one of two entries p_{uj}, p_{vi} of P is not zero, say $p_{uj}\neq 0$, then $(e;i,u)o(e;j,v)=(ep_{uj}e;i,v)=$ $(e;i,v)$ and $(e;j,v)o(e;i,u)=(ep_{vi}e;j,u)$. The above The above shows the necessity of (ii).

(iii). Let $Q=(q_{ji})$ be a permutation submatrix of P. Define a set $E=((p_{ji}^{-1})_{ij}$:for all $q_{vu}=p_{ji}\neq 0)$, where q_{vu} are entries of Q. Then $E^{o}=E\cup 0$ is an inverse idempotent subsemigroup of S with zero.

Conversely, suppose that there is no a proper permutation submatrix Q of P, and assume that E^{o} is an inverse idempotent subsemigroup of S with zero. By Lemma 1.3, $E^{o}\neq S$. By (ii), the product of any two idempotents is the zero. If $(e)_{ij}$ belongs to E^{o}, then clearly we have that $p_{ji}=e$. From this define a set $Q=(p_{ji}:p_{ji}$ is an entry of P, $(e)_{ij}\in E)$. Let P_{o} be a submatrix of P formed by the set Q of entries of P. Then we can show that Q is a permutation matrix, which contradicts assumption.

This completes the proof.

Remark 1. The following properties are easily checked [8]. (i) S is rectangular.

(ii) S is totally noncommutative.

1.2 The fundamental formulae

We shall give a solution of the unsolved problem in Exercise 2 on page 61 of [1] in a special case.

We shall list the fundamental formulae for Rees matrix semigroups over a single element group (e) with zero. Let us write the problem here.

Problem 1. What are necessary and sufficient conditions for $LR=D$ in the following exercise.

"Let R be an r-class and L an l-class of a semigroup S such that $R\cap L$ contains an idempotent. Let D be the d-class containing R and L. Then $LR=D$. The condition that $R\cap L$ contains an idempotent is not necessary for $LR=D$."

For convenience we recall the following notations [1, p. 89], for a Rees matrix semigroup $S=M^{0}(G;I,\wedge;P)$ over a group with zero G^{0} and with a sandwich matrix $P=(p_{ui})$ for two given sets I and $\wedge$. Denote the elements of S by $(a)_{iv}$ with $a\in G^{0}$, for i in I, v in $\wedge$.

Let $R_i=((a)_{iv}: a\in G;\ v\in\wedge)$ and $R_i^0=R_i\cup 0$;

$L_v=((a)_{iv}: a\in G;\ i\in I)$ and $L_v^0=L_v\cup 0$;

$H_{iv}=R_i\cap L_v=((a)_{iv}: a\in G)$.

One of the terms in the next definition 1.2, "a semiregular matrix" is the generalization of " a regular matrix" in Definition 1.1.

Definition 1.2. (i) A matrix P over a group G with zero is called a right(left) semiregular matrix if and only if row(column) of P contains a nonzero entry.

(ii) If A is a set of elements of a groupoid S and a is an element of S such that the product ax of a by every element x of A is zero, we say that a is a left annihilator of the set A.

The definition of a right annihilator of a set A is analogous [6, p. 159].

(iii) A semigroup S with zero is called a zero semigroup if $SS=(0)$.

The following results are modifications of Lemma 3.2 of [1].

Lemma 1.5. In a Rees matrix semigroup $S=M^0((e);I^+,I^+,P)$ we have

(i) $(e)_{iu}$ is a left annihilator of S if and only if the uth row of the sandwich matrix P contains no nonzero entry.

(ii) $(e)_{iu}$ is a right annihilator of S if and only if the ith column of P contains no nonzero entry.

(iii) $(e)_{iu}$ is a two sided annihilator of S if and only if the uth row and the ith column of P contains no nonzero entry.

(iv) Every left(right) ideal $L_u(R_i)$ of S is a $l(r)$-class if and only if the sandwich matrix P is left(right) semiregular. If P is left semiregular then L_u^0 is a 0-minimal left ideal of S.

(v) Let $E(R_i^0)$ be the set of all idempotents of R_i^0, then $R_i^0\ E(R_i^0)$ is a zero subsemigroup of S.

Proof. (i) Let $(e)_{iu}$ be a left annihilator of S, then for every element $(e)_{hk}$ of S we must have $(e)_{iu}\circ(e)_{hk}=0$, which shows that $p_{uh}=0$ for every h in I^+. Thus every entry of the uth row of the sandwich matrix P is zero. The converse is also easy to prove. The proofs for (ii) and (iii) are analogous.

(iv) Suppose that P is left semiregular. Then by the Definition 1.2, every column of P contains a nonzero entry. Let L_j be a left ideal of S for $j\in I^+$. Then there exist k and h in I^+ such that $p_{kj}\neq 0\neq p_{hu}$ in P and $(e)_{uk}\circ(e)_{ij}=(e)_{uj}$, $(e)_{ih}\circ(e)_{uj}=(e)_{ij}$.

Therefore L_j is an l-class of S.

Conversely, assume that P is not left semiregular. Then there exists at least one column, say the uth column, of P which contains no nonzero entry. By (ii) there exists $(e)_{uk}$ in S such that $So(e)_{uk}=0$, which shows that a single element $(e)_{uk}$ forms an l-class for every k in I^+. Thus L_k is not an l-class for every k. The 0-minimality of L_k^0 when P is left semiregular is evident.

(v) Let $a=(e)_{ij}\in R_i^0\setminus E(R_i^0)$. Since a is not idempotent, we must have $p_{ji}=0$. Then $aa=0$. If $b=(e)_{ih}$ lies in $R_i\setminus E(R_i)$, then $p_{hi}=0$ and $bb=0$. From $ba=(e)_{ih}$ $o(e)_{ij}=(ep_{hi}e)_{ij}=0=ab$, it follows that $R_i^0\setminus E(R_i^0)$ is a zero semigroup.

The next lemma is the partial solution of the problem 1.

Lemma 1.6. In every regular Rees matrix semigroup $S=M^0((e);I,V;P)$ we have that $LR=D$ if and only if $R\cap L$ contains an idempotent, where L, R and D are l, r and d-classes of S, respectively.

Proof. We know that a Rees matrix semigroup S is regular if and only if P is regular by Lemma 3.1 in [1]. By Lemma 1.5-(iv), for an l-class L and a r-class R, there exist i in I and j in V such that $L=L_j$ and $R=R_i$. Suppose that $LR=D$. From $LR=L_jR_i=D$, we have that $p_{ji}\neq 0$. Clearly R_i L_j contains an idempotent $(e)_{ij}\neq 0$.

Conversely, assume that $R\cap L$ contains an idempotent. It follows from $R=R_i$ and $L=L_j$ that $R_i\cap L_j$ contains an idempotent $(e)_{ij}$, and hence $p_{ji}\neq 0$. Then $L_jR_i=D=LR$.

In order to state the next lemma we introduct new terms and notation.

Definition 1.3. Let P be a sandwich matrix of a Rees matrix semigroup over a group G with zero. Denote the ith column submatrix of P by Pxi and denote the jth row submatrix of P by Pjx.
$Pxi=(0)$ means that Pxi contains no nonzero entry of P.
$Pjx=(0)$ means that Pjx contains no nonzero entry of P.
$Pxi\neq(0)$ means that Pxi contains a nonzero entry.
$Pjx\neq(0)$ means that Pjx contains a nonzero entry.
$P\neq(0)$ means that P contains a nonzero entry.

Lemma 1.7. In a Rees matrix semigroup $S=M^{o}((e);I^{+},I^{+},P)$ we have

(i) $R_i^{o}S=R_i^{o}$ and $SL_v^{o}=L_v^{o}$ if $P\neq(0)$;

(ii) $R_i\cap L_v=(e)_{iv}$;

(iii) $R_i^{o}R_j^{o}=\begin{cases}(0) & \text{if } Pxj=(0),\\ R_i^{o} & \text{otherwise;}\end{cases}$

(iv) $R_i^{o}L_j^{o}=\begin{cases}(e)_{ij}\cup 0 & \text{if } P\neq(0),\\ (0) & \text{otherwise;}\end{cases}$

(v) $L_i^{o}L_j^{o}=\begin{cases}(0) & \text{if } Pix=(0),\\ L_j^{o} & \text{otherwise;}\end{cases}$

(vi) $L_i^{o}R_j^{o}=\begin{cases}S & \text{if } p_{ji}\neq 0,\\ (0) & \text{otherwise;}\end{cases}$

(vii) $(e)_{ij}R_k^{o}=\begin{cases}R_i^{o} & \text{if } p_{jk}\neq 0,\\ (0) & \text{otherwise;}\end{cases}$

(viii) $R_k^{o}(e)_{ij}=\begin{cases}(0) & \text{if } Pxi=(0),\\ (e)_{kj}\cup 0 & \text{otherwise;}\end{cases}$

(ix) $(e)_{ij}L_k^{o}=\begin{cases}(0) & \text{if } Pjx=(0),\\ (e)_{ik}\cup 0 & \text{otherwise;}\end{cases}$

(x) $L_k^{o}(e)_{ij}=\begin{cases}L_j^{o} & \text{if } p_{ki}\neq 0,\\ (0) & \text{otherwise.}\end{cases}$

We omit the proof.

1.3 Completely maximal elements in $M^o((e);I^+,I^+;P)$.

Two of main features of semigroups in comparison with groups (which are special semigroups) are that (1) the definition of an ideal in a group is fruitless (2) there are considerably more properties of elements in semigroups (for example, regular elements, idempotent elements, completely maximal elements and primitive idempotents.)

The next theorem is one regarding (2) above and it characterizes the completely 0-simple semigroups in Rees matrix semigroups which were introduced by D. Rees [12] to determine all completely (0-) simple semigroups.

Definition 1.4. (i) An element s in a semigroup S is said to be a completely maximal element if $SsS=S$ [3].

(ii) Let T be a subset of a semigroup S. Denote the set of all idempotents of T by E(T). Define $e\leq f$ (e,f in E(S)) to mean $ef=fe=e$. If $e\leq f$ we say that e is under f. An idempotent e of a semigroup is said to be primitive if the only idempotents under e are e itself and 0 (if S has a zero), and $e\neq 0$.

Lemma 1.8. If a semigroup M contains a completely maximal element and M is not simple, then $M\backslash S$ is a proper two sided ideal of M, where S is the set of all completely maximal elements of M [3].

Proof. Let $S^*=M\backslash S$. Let s^* be an arbitrary element of S^*. Suppose that $MS^*\not\subset S^*$. Then there exists an element ts^* in S for some element t in M. Since ts^* is a completely maximal element of M we have $Mts^*M=M$.

But then $M=M(ts^*)M\subset Ms^*M$, which implies that $M=Ms^*M$ because of the fact that $M\subset Ms^*M\subset M$. Then s^* is a completely maximal element of M, which is a contradiction of s^* in S^*.

Therefore S^* is a left ideal of M. We can show analogously that S^* is a right ideal of M. Consequently S^* is a two sided ideal of M.

Theorem 1.9. Let $S=M^0((e);I^+,I^+;P)$ be a Rees matrix semigroup.

(i) $(e)_{ij}$ is a completely maximal element of S if and only if $Pxi\neq(0)$ and $Pjx\neq(0)$.

(ii) $(e)_{ij}$ is a completely maximal element if and only if it is regular.

(iii) The set K of all completely maximal elements in S forms a completely 0-simple subsemigroup of S.

Proof. (i) Suppose that $Pxi=(0)$. Then by Definition 1.3 every entry of the ith column of P is zero and we have $S(e)_{ij}=0$ and $S(e)_{ij}S=0$. If $Pjx=(0)$, then we have $(e)_{ij}S=(0)$. Consequently, assume that $Pxi\neq(0)$ and $Pjx\neq(0)$. Then there exists $p_{jk}\neq 0$ for some k in I^+. By Lemma 1.7-(vii), $(e)_{ij}R_k^0=R_i^0$, since $p_{jk}\neq 0$. Pick $(e)_{im}$ from R_i. Since $Pxi\neq(0)$, there exists an entry $p_{hi}\neq 0$ for some $i\in I^+$. By Lemma 1.7-(vi), we have $S(e)_{ij}S\supset L_h^0(e)_{ij}R_k^0=L_h^0((e)_{ij}R_k^0)=L_h^0R_i^0=S$ and $S(e)_{ij}S=S$. Hence $(e)_{ij}$ is a completely maximal element.

(ii) Let us assume that the jth row and the ith column of P contains nonzero entries. Then $p_{jk}\neq 0\neq p_{hi}$ for some k and h in I^+.

Clearly $(e)_{ij}\circ(e)_{kh}\circ(e)_{ij}=(ep_{jk}ep_{hi}e)_{ij}=(e)_{ij}$, thus $(e)_{ij}$ is regular.

Conversely, suppose that $(e)_{ij}$ is regular. Then there is an element $(e)_{mn}$ in S such that $(e)_{ij}\circ(e)_{mn}\circ(e)_{ij}=(e)_{ij}$, which grarantees the existence of $p_{jm}\neq 0\neq p_{ni}$ for some m and n. Thus $Pjx\neq(0)\neq Pxi$. By (i), $(e)_{ij}$ is a completely maximal element in S.

(iii) Let T be the set of all completely maximal elements in S. Let $T^o=T\cup 0$. Let a, $b\in T$. By (ii), a and b are regular elements in S. Therefore we must have axa=a and byb=b for some nonzero elements x and y in S. If $ab.ab\neq 0$ then ab.ab=ab, because $bSb=\{b,0\}$, which is immediate from Lemma 1.7 -(vii) and (viii). If ab.ab=0, then ab is in T^o. Thus $TT\subset T^o$, and then $T^oT^o\subset T^o$. Thus T^o is a subsemigroup of S.

We shall show that T contains at least one nonzero primitive idempotent. Clearly T contains an idempotent. Let f be a nonzero idempotent and let g be any nonzero idempotent such that gf=fg=g and f, $g\in T$. Let $g=(e)_{ij}$ and $f=(e)_{mn}$. From $gf=(e)_{ij}\circ(e)_{mn}=fg=(e)_{mn}\circ(e)_{ij}=(e)_{ij}=g$, we infer that i=m, n=j, and hence f=g. Thus f is a nonzero primitive idempotent.

Now we only have to show the 0-simplicity of T^o. By Lemma 2.28 of [1], it is sufficient to show that $T^oaT^o=T^o$ for every nonzero element a in T^o.
Let a and b be arbitrary two elements of T. Since a is a completely maximal element of S we have SaS=S, from which it follows there exist t and s in S such that tas=b. By Lemma 1.8, $S\backslash T=K^o$ is an ideal of S. Therefore t and s could not be elements of K^o, if t and s weee in K^o, then tas=b lies in K^o, which

contradicts the fact that b in T. Thus $T^oaT^o \supset T^o$. Clearly $T^oaT^o \subset T^oT^o \subset T^o$, because, T^o is a subsemigroup of S and a is in T^o. Hence $T^oaT^o = T^o$, for every element a in T. This completes the proof.

Remark 2. If every regular element of every semigroup is a completely maximal element, then Theorem 1.9-(ii) may not be interesting. Therefore we shall give the following example to show that a regular element is not completely maximal.

Example 1. Let S be a set of elements e, f , g and a with the following multiplication table.

	e	f	g	a
e	e	e	e	e
f	f	f	f	f
g	g	g	g	g
a	e	e	f	e

We can check that S is a semigroup and f is an idempotent element of S. Hence f is regular, but it is not completely maximal, for $SfS = (e,f,g) \neq S$.

1.4 Inverse elements

We know that the inverse element of a nonzero element in a regular semigroup is not unique in general. Lemma 1.1 gives the actual number of inverses of each regular element in $M^0((e);I^+,I^+;P)$.

Lemma 1.10. Let $S=M^0((e);I^+,I^+;P)$ be a Rees matrix semigroup.

(i) Every idempotent of S is primitive.

(ii) For all a, b in S $aba=a\neq 0$ implies $bab=b$.

Proof. (i) For any two distinct nonzero idempotents f and g in S we have no relation $f\leq g$ or $f\geq g$. Thus every nonzero idempotent is primitive. Since 0 is primitive, every idempotent in S is primitive.

(ii) Let $a=(e)_{ij}$ and let $b=(e)_{mn}$, where e is the identity of the group. Assume that $bab=a\neq 0$. Then there exist nonzero entries p_{jm} and p_{ni} of the sandwich matrix P such that $aba=(e)_{ij}\circ(e)_{mn}\circ(e)_{ij}=(ep_{jm}ep_{ni}e)_{ij}=a$. Thus $bab=(e)_{mn}\circ(e)_{ij}\circ(e)_{mn}=(ep_{ni}ep_{jm}e)_{mn}=(e)_{mn}=b$.

The next example is for Lemma 1.10-(ii).

Example 2. Let R be the set of all positive rational numbers. In the Cartesian product RxR, define a binary operation "*" by the rule:

$$(a,b)*(c,d)=(ac,\max(bc,d)),\quad a,b,\ c \text{ and } d \text{ are in } R.$$

Then $S=(R\times R,*)$ is a semigroup.

Observe $(2,3)*(\frac{1}{2},1)*(2,3)=(2,3)$,

$(\frac{1}{2},1)*(2,3)*(\frac{1}{2},1)=(\frac{1}{2},3/2)$.

Hence $aba=a\neq 0$ does not implies $bab=b$ in this semigroup if $a=(2,3)$ and $b=(\frac{1}{2},1)$.

We shall use the following notations. We denote by $V(a)$ the set of all inverses of $a\neq 0$ in a semigroup S. Let T be a set. Denote by $|T|$ the cardinal number of the set T.

Lemma 1.11. Let m and n be positive integers, and let $a=(e)_{ij}$ be a regular element of a semigroup $S=M^{o}((e);m,n;P)$. Then
$|V(a)|$ =(number of nonzero entries of the jth row of P) .(number of nonzero entries of the ith column of P).

Proof. Assume that the jth row of P contains nonzero entries $(p_{jt_1},p_{jt_2},..,p_{jt_k})$ and the ith column of P contains nonzero entries $(p_{r_1 i},p_{r_2 i},..,p_{r_h i})$. Then $((e)_{t_u r_v}:u=1,2,..,k;\ v=1,2,..,h)$ is the set of all inverse elements of a, where $(t_u)_{u=1}^{k}$ and $(r_v)_{v=1}^{h}$ are subsets of $(1,2,..,m)$ and $(1,2,..,n)$, respectively. To show this take $(e)_{t_u r_v}$ and let us observe the calculation $(e)_{ij}o(e)_{t_u r_v}o(e)_{ij}=(ep_{jt_u}ep_{r_v i}e)_{ij}=(e)_{ij}$, which shows $(e)_{t_u r_v}$ is an inverse of a by Lemma 1.10. We can show that there are no inverses of a other than $((e)_{t_u r_v}:u=1,2,..,k;\ v=1,2,..,h)$. This completes the proof.

Definition 1.5. (i) The degree of a regular element x in a semigroup S is the cardinal number of the set V(x) of inverses of x in S.

(ii) A non-regular element x in S is called a singular element of S.

(iii) A semigroup S is called homogeneous n-regular if $|V(a)|$ =n for every nonzero element a of S, where n is a fixed positive integer.

By the above definition, an inverse semigroup is a homogeneous 1-regular semigroup.

Now we naturally raise a question:

Problem 2. Can we define a degree for a singular element in a semigroup S ?

Lemma 1.12. Let n be a fixed positive integer. A Rees matrix semigroup $M^0((e);I^+,I^+,P)$ is homogeneous n regular if and only if every row and every column of P contains precisely h nonzero entries and k nonzero entries of P such that $hk=n$, where h and k are positive integers.

The proof follows from Lemma 1.11.

Lemma 1.13. In a homogeneous $n(\geq 2)$-regular Rees matrix semigroup $M^0((e);I^+,I^+;P)$ for every nonzero element a in S there exists an element b in S such that $a\neq b$ and $V(a)\cap V(b)\neq \Box$.

Proof. Let a be a nonzero element in S and let $V(a)=(a_i:i=1,2,..,n)$. Consider $V(a_1)$. Let $V(a_1)=(y_i:i=1,2,..,n)$. We may assume that $y_2\neq a$. Then $y_2ay_2=y_2$ and $a_1y_2a_1=a_1$, $aa_1a=a$ and $a_1aa_1=a_1$. Thus $V(y_2)\cap V(a)$ contains a_1. Since a is an arbitrary nonzero element, this completes the proof.

1.5 Congruences on $M^o((e);I^+,I^+;P)$

If S is an inverse semigroup, then by [9], we have a congruence π on S defined by the rule that x y if and only if there exists a nonzero idempotent e in S such that $xe=ye$, and S/π is a group. But in general, a completely 0-simple semigroup is not an inverse semigroup. Therefore the above result is not applicable. But we do have congruences ρ_i on a completely 0-simple semigroup $S=M^o((e);I^+,I^+;P)$ such that S/ρ_i are right zero semigroups or left zero semigroups for i in I^+.

Theorem 1.14. Let $S=M^o((e);I^+,I^+;P)$ be a regular Rees matrix semigroup. For x in S and $R_i=((e)_{iv}:v\in I^+)$, define
$$f_i(x)=\begin{cases}(R_ix)\backslash 0 & \text{if } x\neq 0,\\ 0 & \text{if } x=0.\end{cases}$$
Define a relation ρ_i on S by the rule that $x\rho_i y$ if and only if $f_i(x)=f_i(y)$, for x, y in S. Then

(i) The relations ρ_i are congruences on S.

(ii) S/ρ_i is a right zero semigroup.

For x in S and $L_k=((e)_{ik}: i\in I^+)$, define
$$g_k(x)=\begin{cases}(xL_k)\backslash 0 & \text{if } x\neq 0,\\ 0 & \text{if } x=0.\end{cases}$$
Define a relation σ_k on S by the rule that $x\sigma_k y$ if and only if $g_k(x)=g_k(y)$. Then σ_k is a congruence on S and S/σ_k is a left zero semigroup.

Proof. (i) Since S is regular, by Lemma 1.7-(viii), $f_i(x)$ is a single valued function of x in S. To show that ρ_i is a congruence on S, since the reflexivity and symmetricity of ρ_i are clear, consider first the transitivity of ρ_i. Suppose that $x\rho_i y$ and $y\rho_i z$.

Let $x \in S\backslash 0$. From $f_i(x)=(R_i x)\backslash 0=(R_i y)\backslash 0=f_i(y)$ and $f_i(y)=(R_i y)\backslash 0=(R_i z)\backslash 0=f_i(z)$, it follows that $f_i(x)=(R_i x)\backslash 0=(R_i z)\backslash 0=f_i(z)$, and hence $x\rho_i z$. Thus ρ_i is an equivalence relation on $S\backslash 0$. Let $x\in S\backslash 0$. Assume that $x\rho_i y$. Then $f_i(x)=f_i(y)$, and $(R_i x)\backslash 0=(R_i y)\backslash 0$, from which we get $(R_i xz)\backslash 0=(R_i yz)\backslash 0$ for every z in $S\backslash 0$. Therefore ρ_i is a right congruence on $S\backslash 0$. Consider the left compatibility of ρ_i. By Lemma 1.7-(viii), for $z\in S\backslash 0$, $(R_i z)\backslash 0$ lies in R_i. But $f_i(x)$ is a single valued point function of x in S and it follows from $(R_i z)\backslash 0 \in R_i$, $(R_i zx)\backslash 0 \in (R_i x)\backslash 0$ and $(R_i zx)\backslash 0=(R_i x)\backslash 0$ that $(R_i zx)\backslash 0=(R_i x)\backslash 0=(R_i y)\backslash 0=(R_i zy)\backslash 0$. Thus $zx\rho_i zy$, and hence ρ_i is a proper congruence on S.

(ii) Since P is regular we have $R_i(e)_{11}=R_i(e)_{21}=\dots=R_i(e)_{n1}=$ .. Let $x\rho_i$ denote the ρ_i-class of S containing x in S. Then, by Lemma 1.7-(viii), $(e)_{11}\rho_i=((e)_{i1}\colon i \text{ in } I^+)=L_1$,

$(e)_{12}\rho_i=((e)_{i2}\colon i \text{ in } I^+)=L_2$, and so on.

By Lemma 1.7-(v), the family $(L_i\colon i \text{ in } I^+)\cup 0$ forms a right zero semigroup.

This completes the proof of (ii).

The proof of (iii) is analogous and we omit it.

1.6 A characterization of $M^0((e);m,n;P)$

Theorem 1.15. Let S^0 be a semigroup with zero and $|S|=mn$, where m and n are positive integers. S^0 is isomorphic to a Rees matrix semigroup over a single element group with zero if

(i) there exist disjoint right ideals $(R_i:i=1,2,..,m)$ of S^0 with $|R_i|=n$, for every $i=1,2,.,m$,

(ii) there exist disjoint left ideals $(L_j:j=1,2,.,n)$ of S^0 with $|L_j|=m$, for every j,

(iii) there exists at least one pair (R_i,L_j) such that $L_jR_i=S$, and $L_hR_k=S$ or (0), for every $h=1,2,..,n$, $k=1,2,..,m$,

(iv) $(R_iL_j)\setminus 0$ is a single element set of S and $(R_iL_j \cap R_uL_v)\setminus 0=\square$ if and only if $(i,j)\neq(u,v)$, for every $i,u=1,2,..,m$; $j,v=1,2,..,n$,

(v) $R_i^0S^0=R_i^0$ and $S^0L_j^0=L_j^0$, for every $i=1,2,..,m$; and $j=1,2,..,n$.

Proof. Let us assume that S^0 has right ideals $(R_i:i=1,2,..,m)$ and left ideals $(L_j:j=1,2,..,n)$ satisfying the conditions above. From $|S|=nm$ and $(s_{ij}:s_{ij}=R_iL_j\setminus 0,\ i=1,2,..,m;\ j=1,2,..,n)\subset S$, we have $S^0=(s_{ij}:s_{ij}=R_iL_j\setminus 0,\ i=1,2,..,m;\ j=1,2,..,n)\cup(0)$. Now what we want to show is the following:

$$s_{ij}s_{hk}=\begin{cases} s_{ik} & \text{if } L_jR_h=S,\\ 0 & \text{otherwise.}\end{cases}$$

To do this let us suppose that $s_{ij}s_{hk}=0$ and $L_jR_h=S$. Then $0=s_{ij}s_{hk}=(R_iL_j\setminus 0)(R_hL_k\setminus 0)$ implies $R_iL_jR_hL_k=(0)$; whence $R_i(L_jR_h)L_k=R_iSL_k=(0)$. Since $R_i^0S^0=R_i^0$, we have $(R_i^0S^0)\setminus 0=R_i$ and $R_iL_k=((R_i^0S^0)\setminus 0)L_k$. Similarly $R_iL_k=R_i((S^0L_k^0)\setminus 0)$.

Then $((R_i^o S^o)\backslash 0)L_k = [(R_i \cup 0)(S \cup 0)\backslash 0]L_k = ((R_i S \cup 0)\backslash 0)L_k \subset R_i S L_k = (0)$. Similarly $R_i((S^o L_k^o)\backslash 0) = R_i((SL_k \cup 0)\backslash 0) \subset R_i S L_k = (0)$. Hence $R_i L_k = 0$. But $R_i L_k = s_{ik} \neq 0$ in S. This contradiction shows that if $L_j R_h = S$, then $s_{ij} s_{hk} = s_{ik}$ for every i and k.

For the case $L_j R_h = 0$, it follows from $s_{ij} s_{hk} \in (R_i L_j)(R_h L_k) = R_i 0 L_k = (0)$, that $s_{ij} s_{hk} = 0$. Hence we have proved that
$$s_{ij} s_{hk} = \begin{cases} s_{ik} & \text{if } L_j R_h = S, \\ 0 & \text{otherwise.} \end{cases}$$

Let (e) be a group of a single element e. By (iii), we define an $n \times m$ matrix $P = (p_{ij})$ by the rule
$$p_{ij} = \begin{cases} e & \text{if } L_i R_j = S \\ 0 & \text{otherwise.} \end{cases}$$
Let $M^o = M^o((e); m, n; P) = ((e)_{ij} : i = 1, 2, .., m;\ j = 1, 2, ., n) \cup 0$.
Define a mapping $f: M^o \dashrightarrow S^o$ by
$$f(x) = \begin{cases} s_{ij} & \text{if } x = (e)_{ij} \\ 0 & \text{if } x = 0. \end{cases}$$
Clearly f is a single valued mapping M^o upon S^o.
To show $f(xy) = f(x)f(y)$ for every x, y in M^o, it is sufficient to consider the case for which $x \neq 0$ and $y \neq 0$.
Let $x = (e)_{ij}$ and let $y = (e)_{hk}$.
Then
$$f(xy) = f((e)_{ij} \circ (e)_{hk}) = \begin{cases} f(0) = 0 & \text{if } p_{jh} = 0 \\ f((e)_{ik}) = s_{ik} & \text{if } p_{jh} = e, \end{cases}$$
$$f(x)f(y) = s_{ij} s_{hk} = \begin{cases} s_{ik} & \text{if } p_{jh} = e \\ 0 & \text{if } p_{jh} = 0. \end{cases}$$
If one of x and y is zero, say $x = 0$, then $f(x) = 0$. For this case we also have $f(xy) = f(x)f(y)$. Hence $f(xy) = f(x)f(y)$ for every x, y in M^o, and the mapping f is a homomorphism of M^o upon S^o. Finally, to show that f is one to one, let us assume that $f(x) = f(y) = s_{ij} \neq 0$ in S^o, then we have $x = y = (e)_{ij}$ by the definition of f. If $f(x) = f(y) = 0$, for x, y in M^o, then we have $x = y = 0$.

2 Rees matrix semigroups over a cyclic groups

We shall use the following notation.
Let Im be a finite set of m elements, where m is a positive integer. Let $(a)^p=G$ denote the cyclic group of order p generated by a.

In this chapter we consider Rees matrix semigroups $M^o((a)^p;Im,In;P)$ over cyclic group $(a)^p$ with the generator a for a positive number p, and with sandwich matrices P.

2.1 The basic properties of Rees matrix semigroups.

For a Rees matrix semigroup $M^o((a)^p;Im,In;P)$ we can deduce the next lemma from the results in Chapter 1.

Lemma 2.1. Let $S=M^o((a)^p;Im,In;P)$.

(i) $(a)_{ij}$ is a nonzero idempotent if and only if $a=p_{ij}^{-1}\neq 0$. The number of nonzero idempotents of S is equal to the number of nonzero entries of P.

(ii) Two idempotents $e=(p_{ui}^{-1})_{iu}$ and $f=(p_{vj}^{-1})_{jv}$ $(e\neq f)$ commute if and only if $p_{uj}=p_{vi}=0$.

(iii) Every idempotent of S is primitive.

(iv) The degree of a regular element $(x)_{ij}$ is equal to the product of the number of nonzero entries of the jth row of P and the number of nonzero entries of the ith column of P.

(v) $(x)_{iu}$ is a left annihilator of S if and only if every entry of the uth row of P is zero.

(vi) Every left ideal L_v of S is an l-class if and only if P is left semiregular.

(vii) Define $E(R_1^o)G=((xa)_{ij}:(x)_{ij}\subset E(R_1^o),\ a\subset G)$. Then $R_1^o\backslash E(R_1^o)G$ is a zero semigroup.

Proof. (i) Let p_{ij} be a nonzero entry of P. Then p_{ij} takes the form $p_{ij}=g$ in G. Since g has the inverse g^{-1} in G, $(p_{ij}^{-1})_{ji}=(g^{-1})_{ji}$ in S is a nonzero idempotent of S. This shows the truth of (i).

The proof of the rest follows by using arguments similar to those in previous lemmas and theorems.

The next lemma is a corresponding form of Lemma 1.7.

We shall use the following notation in a Rees matrix semigroup $S=M^{o}(G;I,V;P)$ over a group G with zero and with sandwich matrix $P=(p_{vi})$.

Denote the elements of S by $(a)_{iv}$ with a in G, i in I, and v in V.

Let $G_{ij}=((x)_{ij}$: x in G), where $i\in I$, j in V, and i and j are fixed.

$G^{o}_{ij}=((x)_{ij}$: x in $G^{o})=G_{ij}\cup 0$.

Let T be a subset of S. We denote by E(T) the set of all idempotents of S in T.

Lemma 2.2. Let $S=M^{o}(G;I,V;P)$ be a Rees matrix semigroup.

(i) $R^{o}_{i}R^{o}_{j}=\begin{cases} R^{o}_{i} & \text{if } Pxj\neq(0), \\ (0) & \text{otherwise.} \end{cases}$

(ii) $R^{o}_{i}L^{o}_{j}=\begin{cases} G^{o}_{ij} & \text{if } P\neq(0), \\ (0) & \text{otherwise.} \end{cases}$

(iii) $L^{o}_{i}L^{o}_{j}=\begin{cases} (0) & \text{if } Pix=(0), \\ L^{o}_{j} & \text{otherwise.} \end{cases}$

(iv) $L_{i}R_{j}=\begin{cases} S\ 0 & \text{if } p_{ij}\neq 0, \\ (0) & \text{otherwise.} \end{cases}$

(v) $R_k^o(x)_{ij} = \begin{cases} G_{kj}^o & \text{if } Pxi \neq (0), \\ (0) & \text{otherwise.} \end{cases}$

(vi) $(x)_{ij}R_k^o = \begin{cases} R_i^o & \text{if } p_{jk} \neq 0, \\ (0) & \text{otherwise.} \end{cases}$

(vii) $(x)_{ij}L_k^o = \begin{cases} G_{ik}^o & \text{if } Pjx \neq (0), \\ (0) & \text{otherwise.} \end{cases}$

(viii) $L_k^o(x)_{ij} = \begin{cases} L_j^o & \text{if } p_{kj} \neq 0, \\ (0) & \text{otherwise.} \end{cases}$

(ix) $G_{ij}G_{mn} = \begin{cases} G_{in} & \text{if } p_{jm} \neq 0, \\ (0) & \text{otherwise.} \end{cases}$

(x) $G_{ij}R_k = \begin{cases} R_i & \text{if } p_{jk} \neq 0, \\ (0) & \text{otherwise.} \end{cases}$

(xi) $R_k^oG_{ij} = \begin{cases} G_{kj}^o & \text{if } Pxi \neq (0), \\ (0) & \text{otherwise.} \end{cases}$

(xii) $G_{ij}L_k^o = \begin{cases} G_{ik}^o & \text{if } Pjx \neq (0), \\ (0) & \text{otherwise.} \end{cases}$

(xiii) $L_kG_{ij} = \begin{cases} L_j & \text{if } p_{ki} \neq 0, \\ (0) & \text{otherwise.} \end{cases}$

(xiv) $G_{ij}(x)_{mn} = \begin{cases} G_{in} & \text{if } p_{jm} \neq 0, \\ (0) & \text{otherwise.} \end{cases}$

(xv) $(x)_{mn}G_{ij} = \begin{cases} G_{mj} & \text{if } p_{ni} \neq 0, \\ (0) & \text{otherwise.} \end{cases}$

(xvi) $G_{ij} \cong G$ if $E(G_{ij}) \neq 0$.

(xvii) $(e)_{mi}P(e)_{jn} = (p_{ij})_{mn}$.

Proof. (i) Since R_i^o is a right ideal of S, we have $R_i^oR_j^o \subset R_i^o$. Suppose that $Pxj \neq (0)$ and $p_{kj} \neq 0$ for some k. Let $(c)_{ih}$ be an arbitrary element of R_i^o and $c \neq 0$ in G.

We pick $(p_{kj}^{-1})_{ik}$ from R_i^0 and $(c)_{jh}$ from R_j^0.
Then we have $(c)_{ih}=(p_{kj}^{-1})_{ik}\circ(c)_{jh}$. Thus $R_i^0=R_i^0R_j^0$.
This completes the proof of (i).

We omit the proof of the rest.

We conclude this section with Theorem 2.3 for Problem 1.

Theorem 2.3. In a Rees matrix semigroup $S=M^0(G;I,V;P)$, we have that $L_iR_j=S\backslash 0$ if and only if $R_j\cap L_i$ contains an idempotent of $S\backslash 0$.

Proof. The necessity. By Lemma 2.2-(iv), $p_{ij}\neq 0$ of P. Then it is clear that the nonzero idempotent $(p_{ij}^{-1})_{ji}$ belongs to R_j and L_i.

The proof of the sufficiency follows Lemma 2.2-(iv).

This theorem is the most general and final form of Lemma 1.6 in the Rees matrix semigroup. Notice that it does not require the regularity of a semigroup.

2.2 The n regularity of semigroups

In this section we generalize the theorem by R. McFadden and Hans Schneider [7].

So far we have almost avoided touching the completely 0-simplicity of semigroups, since we already know the structure theorems: Theorem 3.5, Theorem 2.48 and Theorem 2.55 in [1].
But by the homogeneous n regularity of a semigroup in Definition 1.5-(iii) we can classify completely 0-simple semigroups in two ways, (1) Homogeneous n regular semigroups, where n=1,2,.., and (2) Non-homogeneous regular semigroups.

Furthermore, we can establish Theorem 2.9, the most general form of the theorem of McFadden and Schneider [7].

We shall use the following notation in a Rees matrix semigroup $M^0(G;I,V;P)$ over a group G with zero and with a sandwich matrix P.
Denote by Pjx the cardinal number of nonzero entries of the jth row of P. Pxi is the cardinal number of nonzero entries of the ith column of a sandwich matrix P.

The next lemma gives an inverse element of a regular element in a Rees matrix semigroup.

Lemma 2.4. Let $M^0(G;Im,In;P)$ be regular.

(i) If $(x)_{ij}$ is nonzero and $p_{hi}\neq 0\neq p_{jk}$, then $(p_{jk}^{-1}x^{-1}p_{hi}^{-1})_{kh}$ lies in $V((x)_{ij})$.

(ii) $|V((x)_{ij})| = |Pjx|\,|Pxi|$.

Proof. (i) $(x)_{ij}\circ(p_{jk}^{-1}x^{-1}p_{hi}^{-1})_{kh}\circ(x)_{ij}=(x)_{ij}$ and Lemma 1.10 prove the assertion (i).
(ii) follows from (i).

Definition 2.1. Let n be a positive integer.

(i) A sandwich matrix P is n regular if $|Pix|\,|Pxj|=n$ for every i and j.

(ii) An element $a\neq 0$ is said to be completely regular in a semigroup S if we can find in S an element x such that $axa=a$ and $ax=xa$.

(iii) A semigroup S consisting entirely of completely regular elements is siad to be completely regular.

Lemma 2.5. Let $S=M^{0}(G;I,V;P)$ be a homogeneous n regular semigroup.

(i) P is n regular.

(ii) Every R_i contains $|Pxi|$ nonzero idempotents.

(iii) Every L_j contains $|Pjx|$ nonzero idempotents.

(iv) For $a\neq 0$ in S, there exist two sets $(e_i:i=1,2,.,h_n)$ and $(f_j:j=1,2,.,k_n)$ of nonzero idempotents of S such that $e_i a=a=af_j$ $(i=1,2,.,h_n;j=1,2,.,k_n)$ and $h_n k_n=n$.

(v) There exists a completely regular element.

(vi) There are no two distinct inverses x, y of a such that $xa=ax$ and $ya=ay$, for a in $S\backslash 0$.

(vii) $a=(x)_{ij}$ is completely regular if and only if $R_i\cap L_j$ contains an idempotent.

(viii) $a=(x)_{ij}$ is completely regular if and only if $p_{ji}\neq 0$.

(ix) If n is a prime number, then P is a kn x k or k x kn matrix.

Proof. (i) is clear by Lemma 2.4-(ii). By Lemma 2.1-(i), (ii) and (iii) are evident.

(iv) Let $V(a)=(x_i : i=1,2,..,n)$ for $a \neq 0$ in S. Define $e_i=ax_i$ and $f_j=x_ja$ $(i,j=1,2,..,n)$. Then $e_ia=a=af_j$. It follows from $Sa=Saf_j \subset Sf_j$ and Lemma 2.47 [1] that $Sa=Sf_j$ $(j=1,2,..,n)$. By (ii), (iii) and Definition 2.1, (iv) follows.

(v) follows from the fact that every nonzero idempotent is completely regular.

(vi) Assume that there are two inverses x and y of a such that $x=y$, $ax=xa$ and $ay=ya$. Then $e_1=ax=xa=f_1$ and $e_2ay=ya=f_2$ are units of a, that is, $e_2a=af_2=a$ and $e_1a=af_1=a$. Then $x=xax=x(ax)=x(ay)=(xa)y=axy=yay=y$, that is, $x=y$, but we assumed that $x \neq y$. This proves (vi).

(vii) Let $a=(x)_{ij}$ be completely regular in S. Then there exists an element b such that $aba=a$ and $ba=ab$. Let $b=(y)_{hk}$. From $ab=ba$, we have $(x)_{ij} \circ (y)_{hk}=(y)_{hk} \circ (x)_{ij}$, which implies that $i=h$ and $j=k$. Thus $b=(y)_{ij}$. From $aba=a$, it follows that $p_{ji} \neq 0$. Then $(p_{ji}^{-1})_{ij}$ belongs to $(R_i \cap L_j)$.

Conversely, if $R_i \cap L_j$ contains an idempotent, say $(g)_{ij}$, then $g=p_{ji}^{-1} \neq 0$. Define $b=(p_{ji}^{-1}x^{-1}p_{ji}^{-1})_{ij}$. Then we have $aba=a$ and $ba=ab$. Hence a is completely regular.
Incidentally, (viii) is shown in the proof of (vii).

(ix) follows from (i), (ii), (iii) and Lemma 2.4-(ii).

We shall need the following lemmas.

Lemma 2.6. For all a, b in a Rees matrix semigroup $S=M^0(G;I,V;P)$ $aba=a\neq 0$ implies $bab=b$.

Every completely 0-simple semigroup has this property by Theorem 3.5 [1].

Proof. Let $a=(g)_{ij}$ and let $b=(g')_{mn}$, where g and g' are nonzero elements of the group with zero G^0, and for $i,m\in V$, $j,n\in I$. Assume that $aba=a\neq 0$. Then there exist nonzero entries p_{jm} and p_{ni} of the sandmatrix P such that $aba=(g)_{ij}\circ(g')_{mn}\circ(g)_{ij}=(gp_{jm}g'p_{ni}g)_{ij}=(g)_{ij}=a$. Hence we get $p_{jm}g'p_{ni}g=e$, and $g'p_{ni}=p_{jm}^{-1}g^{-1}$, where e is the identity of the group G. Therefore we have $bab=(g')_{mn}\circ(g)_{ij}\circ(g')_{mn}=(g'p_{ni}gp_{jm}g')_{mn}=(p_{jm}^{-1}g^{-1}gp_{jm}g')_{mn}=(g')_{mn}=b$, or $bab=b$. This completes the proof.

Lemma 2.7. In a completely 0-simple semigroup S, for a nonzero idempotent e and a nonzero element a in S such that $ea=a$, the equation $ax=e$ has a solution x in Se.

Proof. From $ea=a$, we have $eaS=aS\subset eS$. Since S is completely 0-simple, by Lemma 2.47 [1] we have $aS=eS$. Let $Se=L$ and $aS=R$. Since $ea=a$ in LR, by Lemma 2.46 [1] $RL=R\cap L$ is a group with zero, and clearly e lies in $(R\cap L)$. By $aL=aSe=aSSe=RL=R\cap L$, we can solve the equation $ax=e$ for x in $L=Se$. This completes the proof of the lemma.

Lemma 2.8. Let a and b be nonzero elements in a completely 0-simple semigroup D. Then $aS\cap Sb$ contains at most one nonzero idempotent of S.

Proof. By Corollary 2.49 and Lemma 2.45 in [1], $aS=R$ and $Sb=L$ are 0-minimal right and left ideals of S, respectively. By Lemmas 2.43 and 2.15 in [1], $R \cap L$ contains at most one nonzero idempotent of S.

Theorem 2.9. Let S be a 0-simple semigroup and let n be a fixed positive integer. Then the following are equivalent.

(i) S is a homogeneous n regular and completely 0-simple semigroup.

(ii) For every a in $S\backslash 0$, there exist precisely n distinct nonzero elements $(x_i:i=1,2,..,n)$ such that $ax_ia=a$ for $i=1,2,..,n$ and for all c, d in S $cdc=c\neq 0$ implies $dcd=d$.

(iii) For every a in $S\backslash 0$, there exist precisely h distinct nonzero idempotents $E_a=(e_i:i=1,2,..,h)$ and k distinct nonzero idempotents $F_a=(f_j:j=1,2,..,k)$ such that $e_ia=a=af_j$ for every i and j, $hk=n$, E_a contains every nonzero idempotent which is a left unit of a, F_a contains every nonzero idempotent which is a right unit of a and $E_a \cap F_a$ contains at most one element.

(iv) For every a in $S\backslash 0$, there exist precisely k nonzero principal right ideals $(R_i:i=1,2,..,k)$ and h nonzero principal left ideals $(L_j:j=1,2,..,h)$ such that R_i and L_j contain h and k inverses of a, respectively, every inverse of a is contained in a suitable set $R_i \cap L_j$ for $i=1,2,.,k$; $j=1,2,.,h$, and $R_i \cap L_j$ contains at most one nonzero idempotent, where $hk=n$.

(v) Every nonzero principal right ideal R contains precisely h nonzero idempotents and every nonzero principal left ideal L contains precisely k nonzero idempotents such that $hk=n$, and $R\cap L$ contains at most one nonzero idempotent.

(vi) S is completely 0-simple. For every 0-minimal right ideal R there exist precisely h 0-minimal left ideals $(L_i;\ i=1,2,..,h)$ and for every 0-minimal left ideal L there exist precisely k 0-minimal right ideals $(R_j:j=1,2,..,k)$ such that $L\ R_j=L_iR=S$, for every $i=1,2,.,h$, $j=1,2,.,k$, where $hk=n$.

(vii) S is completely 0-simple. Every 0-minimal right ideal R of S is the union of a right group with zero G^0, a union of h disjoint groups except zero, and a zero subsemigroup Z which annihilates the right ideal R on the left, and every 0-minimal left ideal L of S is the union of a left group with zero G'^0, a union of k disjoint groups except zero, and a zero subsemigroup Z' which annihilates the left ideal L on the right and $hk=n$.

(viii) S contains at least n nonzero distinct idempotents, and for every nonzero idempotent e there exists a set $E=(e_i:i=1,2,.,n)$ of nonzero idempotents of S such that eE is a right zero semigroup containing precisely h nonzero idempotents, Ee is a left zero semigroup containing precisely k nonzero idempotents of S, $e(E(S)\setminus E)=(0)=(E(S)\setminus E)e$, and $eE\cap Ee=(e)$, where $hk=n$.

Proof. (i) implies (ii). This is clear by the definition of homogeneous n regular semigroup and Lemma 2.6.

(ii) implies (iii).

We shall prove the existence of a nonzero primitive idempotent of S. Let a be a nonzero element of S. By (ii) there exist $(x_i : i=1,2,..,n)$ in S such that $ax_ia=a$ and $x_iax_i=x$ for $i=1,2,..,n$. Choose $ax_1=e_1$. Clearly $0\neq e_1 \in E(S)$. Let f be any nonzero idempotent such that $fe_1=e_1f=f$. Then $fe_1f=(fe_1)f=ff=f$. By the assumption of (ii), we also have $e_1fe_1=e_1$. But we have $e_1fe_1=e_1(fe_1)=e_1f=f$. Hence we conclude $e_1=f$, and e_1 is a nonzero primitive idempotent of S. Thus S is completely 0-simple.

Let $V(a)=(x_i : i=1,2,..,n)$ be the set of all inverses of a in $S\backslash 0$. Then clearly every e_i in $E_a=(ax_i=e_i : x_i \text{ in } V(a))$ is a left unit of a, that is, $e_ia=a$ $(i=1,2,..,n)$. Also every f_j in $F_a=(x_ia=f_i :$ x_i in $V(a))$ is a right unit of a, that is, $af_j=a$, for every $j=1,2,.,n$. It is evident that $E_a \cup F_a \subset E(S)$. If we denote by $E_a^*=(e_{i_1}, e_{i_2}, .., e_{i_k})$ the set of all distinct elements in E_a then $E_a^*=E_a$ and $k\leq n$. Analogously, we can assume that $F_a^*=(f_{j_1}, f_{j_2}, .., f_{j_h})$ is the set of all distinct elements in F_a and $F_a^*=F_a$, where $h\leq n$.

Define a set $X_m=(x_{j_1}ax_{i_m}, x_{j_2}ax_{i_m}, .., x_{j_h}ax_{i_m})$ for $m=1,2,..,k$. It is easy to see that $X_m \subset V(a)$. Let $X=\bigcup_{m=1}^{k} X_m$. We shall now show that $X=V(a)$ and $kh=n$. We claim that $X_u \cap X_v=\square$ for $u\neq v$, $u, v \in (1,2,.,k)$. To show this let us assume that $X_u \cap X_v \neq \square$ and $u\neq v$. Choose an arbitrary element z in $X_u \cap X_v$, say; $z=x_{j_s}ax_{i_u}$ $=x_{j_t}ax_{i_v}$, for some s and t in $(1,2,..,h)$.

Then $az=a(x_{j_s}ax_{i_u})=ax_{i_u}=e_{i_u}$, $az=a(x_{j_t}ax_{i_v})=ax_{i_v}=e_{i_v}$, and $e_{i_u}=e_{i_v}$ which contradicts the fact that $e_{i_u}\neq e_{i_v}$ in E_a^* for $u\neq v$. Thus we have proved the claim.

Define $Y_m=(x_{j_m}ax_{i_1}, x_{j_m}ax_{i_2}, .., x_{j_m}ax_{i_k})$, for $m=1,2,.,h$. Clearly we see that $X=\bigcup_{m=1}^{k} X_m = \bigcup_{m=1}^{h} Y_m \subset V(a)$.

Analogously, we can easily show that $Y_u \cap Y_v=\square$ if $u\neq v$, u, $v\in(1,2,..,h)$. Thus the cardinality of X is hk. Suppose that $V(a)\setminus X$ is not empty. Then we choose an element x_o in $V(a)\setminus X$. We would have $ax_o=e_o\in E_a^*$ and $x_oa=f_o\in F_a^*$. Let $e_o=e_{i_t}\in E_a^*$ and $f_o=f_{j_s}\in F_a^*$ for some t and s. By direct calculation $x_{j_s}ax_{i_t}=x_{j_s}(e_{i_t})=x_{j_s}e_o=(x_{j_s}a)x_o=f_{j_s}x_o=f_ox_o=x_oax_o=x_o$, we find that $x_o=x_{j_s}ax_{i_t}$ belongs to X. This is a contradiction, because of the fact that x_o is in $V(a)\setminus X$. Therefore we must have $V(a)=X$ and $hk=n$.

Now we want to show that E_a contains every nonzero idempotent e which is a left unit of a. To do this let $e\neq 0$ be an idempotent of S such that $ea=a$. By Lemma 2.7, there exists an inverse y of a, and hence $e=ay\in E_a$. Analogously, we can show that F_a contains every nonzero idempotent f in S which is also a right unit of a.

Finally, we have to show that for every a in $S\setminus 0$, $E_a\cap F_a$ contains at most one element. Suppose that $E_a\cap F_a$ contains g_1 and g_2. Let $g_1=e_{i_u}=f_{j_v}=ax_{i_u}=x_{j_v}a$, $g_2=e_{i_s}=f_{j_t}=ax_{i_s}=x_{j_t}a$ for suitable positive integers u,v, s and t. From the two equalities above we have g_1 and g_2 lie in $aS\cap Sa$. By Lemma 2.8, we conclude that $g_1=g_2$. This completes the proof.

(iii) implies (iv). In order to show the existence of a nonzero primitive idempotent of S, let e be a nonzero idempotent and let f be any nonzero idempotent of S with $fe=ef=f$. By (iii), for $f\neq 0$ there exist two sets E_f and F_f of left and right units of f, respectively. Since $ff=f$, we have f is in $E_f \cap F_f \neq \square$. From $fe=ef=f$, e is also a unit of f, and hence e is in $E_f \cap F_f$, since $E_f(F_f)$ contains every nonzero idempotent which is at the same time a left(right) unit of f. By (iii), we have $e=f$. Thus e is a nonzero primitive idempotent of S, and hence S is completely 0-simple.

Let $a \in S\backslash 0$. By (iii), let $E_a=(e_i:i=1,2,..,h)$ and $F_a=(f_j:j=1,2,..,k)$ be sets of left and right units of a and $E_a \cup F_a \subset E(S)$.

Now we are ready to show the existence of k nonzero principal right ideals each of which contains h inverses of a and the existence of h nonzero principal left ideals each of which contains k inverses of a.

Let $Se_i=L_i$ for e_i in E_a. Then by Lemma 2.7, we can solve the equation $ax=e_i$ for $i=1,2,..,h$, for x in L_i. We denote by x_i in L_i one of the solutions of the equation $ax=e_i$. Since $ax_ia=e_ia=a$ for every i, we have $x_i\neq x_j$ for $i\neq j$ and $X_0=(x_i:i=1,2,..,h) \subset V(a)$ by Lemma 2.6. Analogously, we get a set $Y_0=(y_j:j=1,2,..h)$ of k distinct solutions of the equations $ya=f_j$, for $j=1,2,..,k$. Consider $X=\bigcup_{m=1}^{h} X_m=\bigcup_{i=1}^{k} Y_i$, where

$X_m=(y_1ax_m,y_2ax_m,..,y_kax_m)$, $Y_i=(y_iax_1,y_iax_2,..,y_iax_h)$,

for $m=1,2,..,h$, $i=1,2,..,k$.

It is not hard to show that X is a set of inverses of a with the cardinality $hk=n$.

From Y_i, it is evident that $y_iS=R_i$ contains the set Y_i of h inverses of a. Similarly, $Sx_j=L_j$ contains the set X_j of k inverses of a. To show that $L_u \neq L_v$ for $u \neq v$, assume that $L_u=L_v$ for $u \neq v$, L_u, L_v in the set $(L_i:i=1,2,..,h)$.
From $L_u=L_v$, $Sx_i=Se_i=L_i$, we have $Se_u=Se_v$. Since $e_ua=e_va=a$, it follows that $e_uaS=e_vaS=aS \subset e_uS$, whence $e_uS=aS=e_vS$ by the 0-minimality of e_uS. Then $e_uS \cap Se_v$ contains e_u and e_v, which is impossible since $e_u \neq e_v$ in E_a for $u \neq v$. Hence the cardinality of the set $(L_j:j=1,2,..,h)$ is h.

Analogously, the cardinality of the set $(R_i:i=1, 2,..,k)$ is k.

It is easy to show that $X=V(a)$.

Let R be an arbitrary nonzero principal right ideal of S which contains an inverse z_0 of a. Let $z_0=y_iax_j$ in $X=V(a)$. We may assume that $R=cS$ for c in $S\backslash 0$. For c, there exists an element x in S such that $cxc=c$ by the argument at the beginning of this proof. Setting $e=cx$, $ec=cxc=c$, from which we have $ecS=cS \subset eS$. Since e is primitive and by Lemma 2.47 [1], we have $cS=eS=R$. Since z_0 is in R, let $z_0=eb$ for some b in S. Then $ez_0=eeb=eb=z_0$, which implies $ez_0S=z_0S \subseteq eS$. Since eS is a 0-minimal right ideal and $z_0S \neq 0$, $z_0S=eS$. Letting $y_ia=f_i$, we can show that $z_0S=f_iS$. Consequently $R=cS=eS=z_0S=f_iS=R_i$.
Therefore we have proven that if a nonzero principal right ideal R of S contains an inverse of a, then R is one of $(R_i;\ i=1,2,..,k;\ R_i=y_iS)$.

Analogously we can show that if a nonzero principal left ideal L of S contains an inverse of a, then it is a member of $(L_j : j=1,2,..,h;\ L_j=Sx_j)$. By Lemma 2.8, $R_i \cap L_j$ contains at most one nonzero idempotent.

Since $a\neq 0$ is arbitrary in S, this completes the proof.

(iv) implies (v). To show that S is completely 0-simple, let us first assume that e is a nonzero idempotent and let f be any nonzero idempotent of S with $fe=ef=f$. Since e is a nonzero idempotent, e is an inverse of itself. By (iv), for e there exist a nonzero principal right ideal R and a nonzero principal left ideal L such that $R\cap L$ contains e which is an inverse of itself. From $f=ef\in RS\subset R$ and $f=fe\in SL\subset L$, we have that f lies in $R\cap L$. Since $e\cup f\subset (R\cap L)$, we must have $e=f$ by (iv). Hence e is a nonzero primitive idempotent and S is completely 0-simple.

Next we shall show that every nonzero principal right ideal aS contains h nonzero idempotents and every nonzero principal left ideal Sa contains k nonzero idempotents of S such that $hk=n$, where a is an arbitrary nonzero element in S. To do this we want verify that every nonzero principal right ideal contains at least one nonzero idempotent of S. Let $aS=R$ be a nonzero principal right ideal of S. By (iv), for $a\neq 0$ in S, there exist inverses of a, whence $V(a)\neq \square$. Let $x_0\in V(a)$. Then $ax_0=e_0$ is a nonzero idempotent in $R=aS$. Similarly, we can show that every nonzero principal left ideal of S contains at least one nonzero idempotent.

In order to show that every nonzero principal right ideal R contains precisely h nonzero idempotents, let us assume that R is a nonzero principal right ideal of S containing h' distinct nonzero idempotents $(e_i:i=1,2,..,h')$, where $h'\neq h$ is a positive integer. Let $R=bS$ for b in $S\backslash 0$. Then by Corollary 2.49, Lemmas 2.43 and 2.14 in [1], we have $e_ib=b$ for every $i=1,2,..,h'$. By Lemma 2.7, the equation $by=e_i$ has at least one solution y in $Se_i=L_i$. We denote by y_i one of the solutions of the equation $by=e_i$ above. Clearly, $(y_i:i=1,2,..,h')$ is a set of h' distinct inverses of b by Lemma 2.6. Also it is clear that each $Se_i=L_i$ contains an inverse of b, hence by (iv), L_i contains precisely k inverses of b. We shall show that $(L_i:i=1,2,..,h')$ is a set of h' distinct nonzero principal left ideals of S. To do this, suppose that $L_i=L_j$ for $i\neq j$ in $(1,2,..,h')$. Then $Se_i=L_i=L_j=Se_j$ and $R\cap L_i=R\cap L_j$ contains e_i and e_j, but $e_i\neq e_j$ for $i\neq j$. This is impossible according to Lemma 2.8; hence the cardinality of the set $(L_i:i=1, 2,..,h')$ is h'. If $h'>h$, then $h'k>hk$, which does not satisfy hypothesis (iv). If $h'<h$, then by (iv) there must exists a set $(L_{h'+1},..,L_h)$ of nonzero principal left ideals of S each of which contains k inverses of b such that $(L_i:i=1,2,..,h')\cap(L_i:i=h'+1, h'+2,..,h)=(0)$ and $L_i\neq L_j$ for $i\neq j$ in $(h'+1,..,h)$. Let y_0 be an inverse of b contained in L_h. Then $by_0=e_0$ lies in $R=bS$ and e_0 in $(e_i:i=1,2,..,h')$. But if e_0 is in $(e_i:i=1,2,..,h')$, then we would have L_h lies in $(L_i:i=1,2,..,h')$. Therefore no such set $(L_{h'+1},..,L_h)$ exists, and then $h'<h$ implies $h'k<hk$.

This fails to satisfy hypothesis (iv).

Hence we have shown that every nonzero principal right ideal of S contains precisely h nonzero idempotents. To show that every nonzero principal left ideal of S contains precisely k nonzero idempotents is similar.

The last assertion of (v) follows immediately from Lemma 2.8.

(v) implies (vi). We shall show first that S is completely 0-simple. Clearly $E(S) \neq (0)$. Let e be a fixed nonzero idempotent and let f be any idempotent in $S \setminus 0$ with $ef = fe = f$. Then $efS = fS \subset eS$ and $Sf = Sfe \subset Se$, which implies f lies in $(eS \cap Se)$. But e is also in $(eS \cap Se)$. By (v), we conclude $e = f$, and e is a primitive idempotent. Thus S is completely 0-simple. Since S is completely 0-simple, every nonzero principal right ideal $R = aS$ for a in $S \setminus 0$ is a 0-minimal right ideal of S. Also every nonzero principal left ideal $Sa = L$ $(a \in S \setminus 0)$ is a 0-minimal left ideal of S. By (v), every 0-minimal right ideal R of S contains precisely h nonzero idempotents $(e_i : i = 1, 2, .., h)$, for each e_i there exists a unique 0-minimal left ideal $Se_i = L_i$ of S containing e_i. By Lemma 2.46[1], we have $L_i R = S$ for $i = 1, 2, .., h$.

The proof for 0-minimal left ideals is analogous.

(vi) implies (vii). We have $hk = n$ from (vi). Let R be a 0-minimal right ideal of S, then by (vi) there exists a set of 0-minimal left ideals $(L_i : i = 1, 2, .. h)$ such that $L_i R = S$. By Lemma 2.46[1], $R \cap L_i = RL_i$ is a group with zero for $i = 1, 2, .., h$.

Let $G^{0} = \cup G_{i}^{0} = \bigcup_{i=1}^{h}(R \cap L_{i}) = \bigcup_{i=1}^{h}(RL_{i})$ be the union of h groups with zero. Let Z be the complement of the nonzero part of G^{0} in R. Then $R = G^{0} \cup Z$, and $ZR = (0)$ since each element of Z belongs to a left ideal L' for which $L'R = (0)$ by Lemma 2.46[1] and $L_{i}R = S$. Therefore Z is a zero subsemigroup of S. In general G^{0} is not a group with zero since $h \geq 1$, and G^{0} contains h nonzero idempotents of S, each of which is the identity of a group $G_{i}^{0} = RL_{i}$ with zero for some $i = 1, 2, .., h$. By Exercise 2[1, p. 39], it suffices to show that E(G), the set of nonzero idempotents in G, is a right zero subsemigroup of G^{0}. Let e_{i} and e_{j} be two elements in E(G). Since R is a 0-minimal right ideal of S containing e_{i} and e_{j}, by Lemmas 2.43 and 2.14[1], we have $e_{i}e_{j} = e_{j}$ and $e_{j}e_{i} = e_{i}$. Thus E(G) is a right zero subsemigroup of G^{0}.

The proof of the rest is similar to the preceeding argument.

(vii) implies (viii). Let $e \neq 0$ in E(S). Let R be a 0-minimal right ideal of S containing e. By (vii), there exist precisely h distinct nonzero idempotents $E(R \backslash 0) = (e_{i} : i = 1, 2, .., h) = \bar{E}$ in R such that $e_{i}e_{j} = e_{j}$ for $i, j = 1, 2, .., h$, and e in $\bar{E}$. For each e_{i} in $\bar{E}$, there exists a 0-minimal left ideal L_{i} such that $L_{i}R = S$. By (vii), every L_{i} contains precisely k nonzero idempotents, say $E(L_{i} \backslash 0) = (f_{i1}, f_{i2}, .., f_{ik})$ for $i = 1, 2, ., h$. Since $L_{i}R = S$, it follows that L_{i} contains e_{i}. Hence it is immediate that $L_{i} \cap L_{j} = (0)$ if $i \neq j$. Thus $F = \bigcup_{i=1}^{h} E(L_{i} \backslash 0)$ contains precisely $hk = n$ nonzero idempotents of S, hence incidentally, it is shown that S contains at least n nonzero idempotents. Since e is in F, there exists L_e in $(L_i : i=1,2,..,h)$ such that e is in L_e. By (vii), $E(L_e\setminus 0)e=E(L_e\setminus 0)$. From $Fe \subset SL_e \subset L_e$, it follows that $Fe=E(L_e\setminus 0)$, and Fe contains precisely k nonzero idempotents.

Setting $F^*=E(S)\setminus F$, we claim that $F^*.e=(0)$. To prove this, assume that $ge=h\neq 0$ for some g in F^*. Let L be a 0-minimal left ideal of S containing g. Clearly L is not a member of $(L_i : i=1,2,..,h)$, since g is in F^*. From $ge\neq 0$ in LR, it follows that LR=S and $RL=R\cap L$ is a group with zero, by Lemma 2.46[1]. Hence there exists the identity f in RL, which implies that L is a member of $(L_i : i=1,2,..,h)$, because of the fact that f is in $E(R\setminus 0)\subset F$. This is a contradiction, and hence we conclude $F^*.e=(0)$.

If one begins with a 0-minimal left ideal L of S containing e, then there exist precisely k 0-minimal right ideals $(R_j : j=1,2,..,k)$ of S such that $LR_j=S$ for every $j=1,2,..,k$.

Let $E=\bigcup_{j=1}^{k} E(R_j\setminus 0)$, then the cardinality of E is hk=n and eE contains precisely h nonzero idempotents such that $e(E(S)\setminus E)=(0)$ are clear.

Finally, in order to prove that $eE\cap Fe=(e)$, let us assume that $(eE\cap Fe)$ contains g, a nonzero idempotent of S. Let R_e be the member of $(R_j : j=1,2,..,k)$ containing e. It is easy to see that

$$eE=E(R_e\setminus 0) \text{ and } eE\cap Fe=E(R_e\ 0)\cap E(L_e\setminus 0)\subset (R_e\cap L_e)\setminus 0.$$

By Lemma 2.8, e=g.

(viii) implies (i). To show the existence of a primitive idempotent of $S\backslash 0$, let e be a nonzero idempotent and let f be any nonzero idempotent of S with $fe=ef=f$. By (viii), for e there exist two sets E and F of nonzero idempotents such that $e(E(S)\backslash E)=(0)=(E(S)\backslash F)e$, and $eE\cap Fe=(e)$. Clearly, f is in E, for if not then we would have $ef=0$. Thus f is in E. Similarly, f is in F. Hence $fe=ef=f$ lies in $(eE\cap Fe)$. But, by (viii), $eE\cap Fe=(e)$; hence $f=e$. We conclude e is a nonzero primitive idempotent and S is completely 0-simple.

Secondly, to show the n regularity of the semigroup S, let $a\neq 0$ be in S and let R be a 0-minimal right ideal of S containing a. Since S is completely 0-simple, for R there exists at least one 0-minimal left ideal L of S such that RL is a group with zero. Let e be the identity of RL. By (viii), for e there exist E and F such that eE and Fe contain precisely h and k nonzero idempotents, respectively.
Let $eE=(e_i:i=1,2,..,h)$ and let $Fe=(f_j:j=1,2,..,k)$. From $eE\subset R=eS$, it follows that $eS=e_iS=aS=R$ for every $i=1,2,..,h$. By Lemmas 2.43 and 2.14[1], $e_ia=a$ holds for every $i=1,2,..,h$. By Lemma 2.7, there exists a solution x of the equation $ax=e_i$ for x in $Se_i=L_i$, and denote by x_i a solution x of $ax=e_i$. Then by Lemma 2.6, x_i is in $V(a)$, and $X=(x_i:i=1,2,..,h)$ is a set of h distinct inverses of a. For if $x_i=x_j$ when $i\neq j$, then $ax_i=ax_j=e_i=e_j$ for $i\neq j$. This is a contradiction in eE. Thus $x_i\neq x_j$ for $i\neq j$. Analogously, we will get a set $Y=(y_j:j=1,2,..,k)$ of distinct solutions of the equations $ya = f_j$ from Fe. By Lemma 2.6, we have $Y\subset V(a)$.

It is not hard to show that YaX is a set of hk=n distinct inverses of a and YaX=V(a).

Since a is arbitrary in $S\backslash 0$, this completes the proof.

If n=1, then the theorem above takes the same form of R. McFadden and Hans Schneider's theorem [7], except (iv).

2.3 Brandt congruences on n regular semigroups

"A problem of general interest in the algebraic theory of semigroups is that of determining those congruences ρ on a semigroup S which have the property that the quotient semigroup S/ρ is some preassigned type." is a starting sentence of the paper Brandt Congruences on Inverse semigroups, by W. D. Munn [10].

Here we give a congruence ρ on a regular semigroup S such that S/ρ is a Brandt semigroup.

Notice that we do not require that the semigroup be an inverse semigroup; Munn [10] does in his paper. First of all we shall give the definition of Brandt congruence on a semigroup.

Definition 2.2. A congruence ρ on a semigroup S is a Brandt congruence if S/ρ is a Brandt semigroup. (See: page 158 [10].)

Before establishing a theorem, we shall give an example of a congruence ρ on a regular semigroup S which is not an inverse semigroup such that S/ρ is a Brandt semigroup.

Example 3. Let $G=(a)$ be a cyclic group of order 3 with a generator a. Consider the Rees matrix semigroup $S=M^o(G;I_3,I_9,P)$, where

$$P = \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ a^2 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^2 \\ 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & e \\ 0 & 0 & a^2 \end{pmatrix},$$

and e is the identity of the group G.
Define ρ on S such that $x\rho y$ if and only if $V(x)=V(y)$, where x and y are in S. Then the relation ρ on S is a congruence relation on S such that S/ρ is a Brandt semigroup. Notice S is not an inverse semigroup, but a regular semigroup. Let a_ρ be the congruence class containing a nonzero element a in S.

If we put

$b_1=(e)_{11\rho}$ $c_1=(e)_{21\rho}$ $d_1=(e)_{31\rho}$ $=((e)_{31},(e)_{37},(a^2)_{33})$
$b_2=(e)_{12\rho}$ $c_2=(e)_{22\rho}$ $d_2=(e)_{32\rho}$ $=((e)_{32},(a^2)_{34},(a^2)_{36})$
$b_3=(e)_{13\rho}$ $c_3=(e)_{23\rho}$ $d_3=(e)_{33\rho}$ $=((e)_{33},(a)_{31},(a)_{37})$
$b_4=(e)_{14\rho}$ $c_4=(e)_{24\rho}$ $d_4=(e)_{34\rho}$ $=((e)_{34},(e)_{36},(a)_{32})$
$b_5=(e)_{15\rho}$ $c_5=(e)_{25\rho}$ $d_5=(e)_{35\rho}$ $=((e)_{35},(e)_{39},(a^2)_{38})$
$b_6=(e)_{18\rho}$ $c_6=(e)_{28\rho}$ $d_6=(e)_{38\rho}$ $=((e)_{38},(a)_{35},(a)_{39})$
$b_7=(a)_{13\rho}$ $c_7=(a)_{23\rho}$ $d_7=(a)_{33\rho}$ $=((a)_{33},(a^2)_{31},(a^2)_{37})$
$b_8=(a)_{14\rho}$ $c_8=(a)_{24\rho}$ $d_8=(a)_{34\rho}$ $=((a)_{34},(a)_{36},(a^2)_{32})$
$b_9=(a)_{18\rho}$ $c_9=(a)_{28\rho}$ $d_9=(a)_{38\rho}$ $=((a)_{38},(a^2)_{35},(a^2)_{39})$,

then we have the following multiplication table of elements in S/ρ .

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9
b_1	b_3	b_5	b_7	b_8	b_6	b_9	b_1	b_2	b_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
b_2	0	0	0	0	0	0	0	0	0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	0	0	0	0	0	0	0	0	0
b_3	b_7	b_8	b_1	b_2	b_9	b_5	b_3	b_4	b_6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
b_4	0	0	0	0	0	0	0	0	0	b_3	b_4	b_7	b_8	b_6	b_9	b_1	b_2	b_4	0	0	0	0	0	0	0	0	0
b_5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	b_7	b_8	b_1	b_2	b_9	b_5	b_3	b_4	b_6
b_6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
b_7	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
b_8	0	0	0	0	0	0	0	0	0	b_7	b_8	b_1	b_2	b_9	b_5	b_3	b_4	b_6	0	0	0	0	0	0	0	0	0
b_9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	b_3	b_4	b_7	b_8	b_6	b_9	b_1	b_2	b_5
c_1	c_3	c_5	c_7	c_8	c_6	c_9	c_1	c_2	c_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
c_2	0	0	0	0	0	0	0	0	0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	0	0	0	0	0	0	0	0	0
c_3	c_7	c_8	c_1	c_2	c_9	c_5	c_3	c_4	c_6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
c_4	0	0	0	0	0	0	0	0	0	c_3	c_4	c_7	c_8	c_6	c_9	c_1	c_2	c_4	0	0	0	0	0	0	0	0	0
c_5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	c_7	c_8	c_1	c_2	c_9	c_5	c_3	c_4	c_6
c_6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
c_7	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
c_8	0	0	0	0	0	0	0	0	0	c_7	c_8	c_1	c_2	c_9	c_5	c_3	c_4	c_6	0	0	0	0	0	0	0	0	0
c_9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	c_3	c_4	c_7	c_8	c_6	c_9	c_1	c_2	c_5
d_1	d_3	d_5	d_7	d_8	d_6	d_9	d_1	d_2	d_4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
d_2	0	0	0	0	0	0	0	0	0	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	0	0	0	0	0	0	0	0	0
d_3	d_7	d_8	d_1	d_2	d_9	d_5	d_3	d_4	d_6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
d_4	0	0	0	0	0	0	0	0	0	d_3	d_4	d_7	d_8	d_6	d_9	d_1	d_2	d_4	0	0	0	0	0	0	0	0	0
d_5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	d_7	d_8	d_1	d_2	d_9	d_5	d_3	d_4	d_6
d_6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9
d_7	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
d_8	0	0	0	0	0	0	0	0	0	d_7	d_8	d_1	d_2	d_9	d_5	d_3	d_4	d_6	0	0	0	0	0	0	0	0	0
d_9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	d_3	d_4	d_7	d_8	d_6	d_9	d_1	d_2	d_5

From the table we can easily see that every nonzero element is completely maximal in S/ρ and every element except zero has precisely one inverse element.. $E(S/\rho)=(b_7, c_2, d_6, 0)$ and idempotents commute. Since idempotents commute, it is clear that every nonzero idempotent of S/ρ is primitive. By Theorem 3.9 of [1], S/ρ is a Brandt semigroup.

Definition 2.3. (i) A congruence ρ on a semigroup S with zero is proper if (0) is a ρ-class of S [10] .

(ii) Let P be an m x n matrix with entries in a group G with zero. Let $Pxi=(p_{1i}, p_{2i}, .., p_{mi})$ be the ith column submatrix of P.
Define the inner product set of two column submatrices Pxi and Pxj by

$$(Pxi, Pxj)=\bigcup_{k=1}^{m}(p_{ki}p_{kj}).$$

(iii) An m x n matrix P over a group G with zero is called a column orthogonal matrix if $(Pxi, Pxj)=(0)$ for every $i\neq j$. The definition of the row orthogonal matrix P is analogous. P is orthogonal if P is row and column orthogonal.

(iv) Let S be a homogeneous n regular semigroup with zero. S is called h_n-k_n type if and only if for every nonzero element a in S there exist precisely k_n distinct nonzero principal right ideals $(R_i : i=1,2,.,k_n)$ and precisely h_n distinct nonzero principal left ideals $(L_j : j=1,2,..,h_n)$ such that R_i and L_j contain h_n and k_n inverses of a, respectively, where $h_n k_n=n$.

Lemma 2.10. Let $S=M^{0}(G;Ik,Ikn;P)$ be a homogeneous n regular semigroup. The following conditions are equivalent.

(i) S is n-1 type.

(ii) For every nonzero element a there exist two sets $E_a=(e_i:i=1,2,..,n)$ and $F_a=(f)$ of idempotents in S such that $e_i a=a=af$ for $i=1,2,..,n$.

(iii) Every 0-minimal right ideal of S contains precisely n nonzero idempotents and every 0-minimal left ideal of S contains just one nonzero idempotent such that $E(R_i)E(R_j)=E(R_j)E(R_i)=\begin{cases} 0 \text{ if } R_i\cap R_j=(0) \\ E(R_i) \text{ otherwise,}\end{cases}$

where R_i and R_j are 0-minimal right ideals of S.

(iv) P is column orthogonal, $|Pxi|=n$ and $|Pjx|=1$ for every i and j.

Proof. (i) implies (ii). Let a be a nonzero element in S, and let $V(a)=(x_i:i=1,2,..,n)$. Define $f_i=x_i a$ and $e_i=ax_i$ $(i=1,2,..,n)$. We shall show that $f_i=f_j$ for all $i,j=1,2,..,n$. To show this, let us assume that $f_i\neq f_j$ for $i\neq j$. Then $x_i ax_i=x_j ax_i$. Clearly the nonzero principal left ideal Sx_i contains two distinct inverses $x_i ax_i$ and $x_j ax_i$ of a. This contradicts the n.1 regularity of S, and hence we conclude $f_i=f_j$ for every i and j.
Now we claim that $e_i\neq e_j$ for every $i\neq j$. To prove this, let us assume that $e_i=e_j$ for $i\neq j$. Then

$$x_i=x_i ax_i=x_i(ax_i)=x_i e_i=x_i e_j=x_i(ax_j)=(x_i a)x_j=(f_i)x_j$$

$=(f_j)x_j=(x_j a)x_j=x_j$, which contradicts the fact that $x_i\neq x_j$. Thus $(e_i:i=1,2,..,n)$ is the set of n distinct left units of a.

(ii) implies (iii). Suppose that R is a 0-minimal right ideal of S containing $m(\neq n)$ nonzero idempotents $(e_i : i=1,2,..,m)$. Let a be a nonzero element in R. Then we have $R=e_iS=aS$ and $e_ia=a$ $(i=1,2,..,m)$. By (ii), for $a\neq 0$, there exists a unique set $E_a=(f_j : j=1,2,..,n)$ of idempotents with $f_ja=a$. It follows from $f_ja=a$ that $aS=f_jS$. Hence we have $R=e_iS=f_jS$ and R contains n nonzero idempotents E_a. This contradiction implies that R contains precisely n nonzero idempotents.

Analogously we can show that every 0-minimal left ideal L of S contains just one nonzero idempotent.

Finally, in order to show that $E(R_i)E(R_j)=(0)$ for two distinct 0-minimal right ideals R_i and R_j, let us assume that $E(R_i)E(R_j)\neq(0)$ and $R_i\cap R_j=(0)$. Let $ef\neq 0$ for $e\in E(R_i)$ and $f\in E(R_j)$. Since $SefS\neq(0)$, $fSSe=fSe$ is a group with zero. Let g be the identity of the group fSe. Then g takes the form $g=fse$ for some s in S. From this we have $Sg=Se$ and $gS=fS$. By (ii), $g=e$ and $g\in fS=R$ by the 0-minimality of fS. Thus we have $e\in(R_i\cap R_j)$ and $(R_i\cap R_j)\neq(0)$, which is a contradiction. Thus we conclude $E(R_i)E(R_j)=(0)$ if R_i and R_j are distinct 0-minimal right ideals of S. By Theorem 2.9, $E(R_i)$ is a right zero semigroup.

(iii) implies (iv). By Lemma 2.5-(ii) and (iii), we have $|Pxi|=n$, $|Pjx|=1$ for every i and j. If P is not column orthogonal, then there exist at least two Pxi and Pxj with $(Pxi,Pxj)\neq(0)$. Let $p_{ui}p_{uj}\neq 0$. Then the cardinality of all nonzero entries in Pux is greater than 2.

Hence P must be column orthogonal.

(iv) implies (i). By Lemma 2.5, every 0-minimal right ideal contains precisely n nonzero idempotents, and every 0-minimal left ideal contains just one nonzero idempotent.
The proof follows from Theorem 2.9-(iv).

The next condition has appeared on page 156 of [10].

Condition 1. If a,b and c are elements of S such that $abc=0$ then either $ab=0$ or $bc=0$.

We shall use Lemma 2.11 below to prove Theorem 3.12.

Lemma 2.11. Let S be a n-1 type regular Rees matrix semigroup with zero.

(i) If S satisfies Condition 1 and $aeb=0$ then either $ab=0$ or $ae=0$, where a,e and b in S and $ee=e\neq 0$.

(ii) If S is a Brandt semigroup and x, y and e are elements of S such that $ee=e$, $xe=ye\neq 0$, then $x=y$ [10].

(iii) If ρ is a proper congruence on S defined by $a\rho b$ if and only if there exists a set $(e_i : i=1,2,..,n)$ of nonzero idempotents in S such that $ae_i=be_i$ for every $i=1,2,..,n$, then, if e is a nonzero idempotent in a 0-minimal right ideal R of S, we have that $E(R\backslash 0)$ is the ρ-class containing e.

Proof. (i) Let a,b and e be elements in S with $ee=e$ and $aeb=0$. If $a=0$ there is nothing to prove.

Let $a\neq 0$ and let $V(a)=(x_i:i=1,2,..,n)$. Let $f_i=x_ia$. If f_i and e lie in the same 0-minimal right ideal, then by Lemma 2.10-(iii) we have $ef_i=f_i$ and $f_ie=e$. For this case, it follows from $aeb=(ax_ia)eb=af_ieb$ $=aef_ieb=0$ that either $ae=0$ or $ef\ eb=0$ by Condition 1. In the latter case we have $0=ef_ieb=(ef_i)eb=f_ieb=eb$. Consider ab. If $b=0$, then $ab=0$. If $b\neq 0$, let $V(b)=(y_j:j=1,2,..,n)$ and let $by_j=g_j$. Then $ab=ax_iab$ $=af_ib=aef_i(by_jb)=aef_ig_jb$. If $f_ig_j=0$, then $ab=0$. If $f_ig_j\neq 0$, then $f_ig_j=g_j$. From this $ab=aef_ig_jb=$ $aeg_jb=aeby_jb=0$ since $eb=0$. Hence $ab=0$.

If f_i and e do not lie in a 0-minimal right ideal, then $f_ie=ef_i=0$. Since $x_ia\neq 0$, $f_ie=x_iae=0$ implies that $ae=0$ by Condition 1.
This completes the proof.

(ii) Since S is a Brandt semigroup we may assume that $S=M^o(G;I,I,V)$, where V is a diagonal matrix with all entries 1, the identity of the group G. Let $x=(a)_{ij}$, $y=(b)_{hk}$ and $e=(1)_{mm}$. From $xe=ye\neq 0$ we have $j=m$, $k=m$, $i=h$ and $a=b$. This gives the required result.

(iii) Suppose that ρ is a proper congruence on S defined by the rule in (iii) above. Let e be a nonzero idempotent in a 0-minimal right ideal R and let $E(R\backslash 0)=(e_i:i=1,2,..,n)$. If f in $E(R\backslash 0)$, then $ee_i=e_i=fe_i$. Hence $E(R\backslash 0)\subset\rho_e$, the ρ-class containing e.

On the other hand, if we assume that $g\rho e$, then g belongs to $E(eS\backslash 0)$. To do this let us assume that $ge_i=ee_i=e_i$ for every $i=1,2,..,n$.

Clearly $e_ige_i=e_i$ and ge[illegible], whence e_ig belongs to $E(S\backslash 0)$. Let $f=e_ig$. It follows from $fS=e_igS\subset e_iS$ that the idempotent f belongs to $\cap e_iS$, whence $e_iS=e_jS$ for all $i,j=1,2,..,n$. Hence f lies in $E(eS\backslash 0)$, which implies $f=e_j$ for some j, because of the fact that $E(eS\backslash 0)=(e_i:i=1,2,..,n)$. Thus $f=e_ig=e_j$. Then $g=ge_ig=g(e_ig)=ge_j=e_j$, which shows that g belongs to $E(eS\backslash 0)$. This completes the proof of the lemma.

The next theorem is important.

Theorem 2.12. Let $S=M^o(G;I,V;P)$ be a n-1 type homogeneous n regular semigroup.

Define a relation ρ on $S\backslash 0$ in such a way that a b if and only if there exists a set $(e_i:i=1,2,.,n)$ of n distinct nonzero idempotents with $ae_i=be_i\neq 0$ for every $i=1,2,..,n$. Then ρ is an equivalence on $S\backslash 0$. If we extend ρ to S by defining (0) to be a ρ-class on S, then ρ is a proper Brandt congruence on S.

Further, if σ is any congruence on S with the property that S/σ is a Brandt semigroup, then $\rho\subset\sigma$.

Proof. Let a be a fixed nonzero element in S and let b be any nonzero element in S such that $ab\neq 0$. The existence of b follows from $SaS=S$. Analogous to the result of (ii) in Lemma 2.10, there exists a set $(e_i:i=1,2,..,n)$ of n distinct nonzero idempotents such that $e_ib=b$ for every $i=1,2,..,n$. It follows from $ab=a(e_ib)=(ae_i)b$ that $ae_i\neq 0$ $(i=1,2,..,n)$.

The symmetric property is immediate from the definition. To show that ρ is transitive, let $a\rho b$ and $b\rho c$, for a,b and c in $S\backslash 0$. Then there exist two sets $E=(e_i:i=1,2,..,n)$ and $F=(f_j:j=1,2,.,n)$ of idempotents in S such that $ae_i=be_i\neq 0$, $bf_j=cf_j=0$, for every $i,j=1,2,..,n$. Now consider $ae_if_j=(be_i)f_j$. Suppose that $be_if_j=0$. Then, by Lemma 2.11, either $be_i=0$ or $bf_j=0$, each of which provides a contradiction. Hence $be_if_j\neq 0$ and $e_if_j\neq 0$. This shows that e_i and f_j are in the same 0-minimal right ideal, say R, whence $e_if_j=f_j$. Then $0\neq be_if_j=b(e_if_j)=bf_j=cf_j$ and $0\neq be_if_j=(be_i)f_j=(ae_i)f_j=a(e_if_j)=af_j$ which implies $af_j=cf_j\neq 0$ and $a\rho c$. Hence ρ is an equivalence on S.

Now let ρ be extended to be an equivalence on S by defining (0) to be a ρ-class.

Let $a\rho b$ (a,b in S) and let $x\in S$. To show that ρ is a congruence on S we have to prove that $xa\rho xb$ and that $ax\rho bx$. If $a=b=0$ these results clearly hold. Hence we suppose that there is a set $(e_i:i=1,2,..,n)$ of nonzero idempotents in S such that $ae_i=be_i\neq 0$ for every $i=1,2,..,n$. From $ae_i=be_i$ we have $ae_ix=be_ix$. Consider first the case $ae_ix\neq 0$. Then $e_ix\neq 0$, which indicates that e_i and x lie in the same 0-minimal right ideal of S. For if e_i and x are not contained in the same 0-minimal right ideal, assume that $e_i\in R_i$, $x\in R_j$ and $R_i\cap R_j=(0)$, where R_i, R_j are 0-minimal right ideals of S. Let f be a nonzero idempotent of R_j, then $fx=x$. From $e_ix=e_i(fx)=(e_if)x$ we have $e_ix=0$ by Lemma 2.10-(iii). Hence we have shown that e_i and x belong to a 0-minimal right ideal.

It follows from $e_ix=x$ that $ax=ae_ix=be_ix=bx$ and $ax=bx\neq 0$. Choose a set $(g_k:k=1,2,..,n)$ of n nonzero idempotents of S such that $xg_k\neq 0$ for every k.

By Condition 1, $axg_k=bxg_k\neq 0$ for every $k=1,2,..,n$.

On the other hand, if $ae_ix=0$, then by Lemma 2.11-(i), either $ax=0$ or $ae_i=0$. But the latter gives a contradiction and so $ax=0$.

Similarly, $bx=0$ and therefore ax bx. Again, since $ae_i=be_i$ we have $xae_i=xbe_i$ for every $i=1,2,..,n$. If $xae_i=0$, then by Condition 1, either $xa=0$ or $ae_i=0$. But $ae_i\neq 0$ and so $xa=0$. Analogously, $xb=0$.

If $xae_i\neq 0$ then $xae_i=xbe_i\neq 0$. Consequently we have shown that $xa\rho xb$.

Thus ρ is a congruence on S, and it is evidently proper, for if $0\rho x$, then $x=0$.

It is clear that S/ρ is a regular semigroup since S is regular.

We claim that S/ρ is an inverse semigroup. It is sufficient to show that idempotents commute in S/ρ. To prove this let us assume A is a nonzero idempotent in S/ρ. What we want to show is that there exists a nonzero idempotent a in S such that $\bar{\rho}(a)=A$, where $\bar{\rho}$ is the natural homomorphism of S onto S/ρ. For A there exists an element a in S with $\rho(a)=A$. We proceed to verify that a is a nonzero idempotent in S.

Since $AA=A$ we have $\bar{\rho}(a)\bar{\rho}(a)=\bar{\rho}(a)$ and $\bar{\rho}(aa)=\bar{\rho}(a)$, whence $a^2\rho a$. By the definition of ρ there exists a set $(h_i:i=1,2,..,n)$ of n nonzero idempotents of S with $a^2h_i=ah_i\neq 0$ for every $i=1,2,..,n$. Thus $a^2\neq 0$. Since S is n regular there exists the set $V(a)=(y_i:i=1,2,..,n)$ of n inverses of $a\neq 0$ with $ag_i=a$, where $g_i=y_ia$. We observe that $0\neq ah_i=ay_jah_i=a(g_jh_i)$ implies that $g_jh_i\neq 0$. By Lemma 2.10, g_j and h_i belong to the same 0-minimal right ideal of S, for if g_j and h_i do not belong to a 0-minimal right ideal, then g_jh_i would be zero. Since the set of all idempotents of a 0-minimal right ideal forms a right zero subsemigroup of S we have $h_ig_j=g_j$ for every i and j. It then follows from $aa=a(ay_ja)=a^2g_j=a^2(h_ig_j)=(a^2h_i)g_j=(ah_i)g_j=a(h_ig_j)$

$=ag_j=ay_ja=a$ that a is a nonzero idempotent of S.

Thus we have shown that if A is a nonzero idempotent in S/ρ then there exists a nonzero idempotent a in S such that $\bar{\rho}(a)=A$.

By Lemma 2.11-(iii) for A there exists a 0-minimal right ideal R_i of S containing the nonzero idempotent a such that $\bar{\rho}(a)=E(R_i\setminus 0)$, and hence $E(R_i\setminus 0)=A$.

Similarly, if B is a nonzero idempotent in S/ρ different from A, then there is a 0-minimal right ideal R_j different from R_i such that $B=E(R_j\setminus 0)$. By Lemma 2.10-(iii), we have $E(R_i\setminus 0)E(R_j\setminus 0)=(0)=E(R_j\setminus 0)E(R_i\setminus 0)$. Since ρ is a proper congruence on S we conclude $AB=0=BA$. Hence any two idempotents in S/ρ commute with each other and S/ρ is an inverse semigroup.

Now we show that S/ρ is 0-simple. Let A and X be any two nonzdro elements in S/ρ. Let $\bar{\rho}(a)=A$ and let $\bar{\rho}(x)=X$ for some elements a and x in S.

According to Lemma 2.28[1] we have $SaS=S$ from which we may assume that $x=uav$ for u, v in $S\setminus 0$. From $x=uav$ and $X=\bar{\rho}(u)A\bar{\rho}(v)$ we conclude that S/ρ is 0-simple.

Since the product of any two distinct nonzero idempotents in S/ρ is zero, every nonzero idempotent is a nonzero primitive idempotent in S/ρ.

Thus S/ρ is a Brandt semigroup with zero.

Finally, let σ be any proper Brandt congruence on S. Let a and b be elements in S with $a\rho b$. By ρ there exists a set $(e_i:i=1,2,..,n)$ of n nonzero idempotents in S such that $ae_i=be_i\neq 0$ for $i=1,2,..,n$. Let $\bar{\sigma}$ be the natural homomorphism of S onto S/σ. It follows from $ae_i=be_i\neq 0$ that $\bar{\sigma}(a)\bar{\sigma}(e_i)=\bar{\sigma}(b)\bar{\sigma}(e_i)\neq 0$. Since S/σ is a Brandt semigroup, by Lemma 2.11-(ii), we have $\bar{\sigma}(a)=\bar{\sigma}(b)$, that is, $a\sigma b$.

What we have shown is that $a\rho b$ implies $a\sigma b$, or equivalently, $\rho\subset\sigma$. This proves the theorem.

Remark 3. If S is a 1-n type regular Rees matrix semigroup with zero, then define a relation ρ on S such a way that $a\rho b$ (a, b in S) if and only if there exists a set $(f_i:i=1,2,..,n)$ of n distinct nonzero idempotents in S with $f_ia=f_ib\neq 0$ $(i=1,2,..,n)$. Then ρ is an equivalence on S. If we extend ρ to S by defining (0) to be a ρ-class on S, then ρ is a proper congruence on S such that S/ρ is a Brandt semigroup. Further, ρ is the finest such congruence, in the sense that if σ is any proper Brandt congruence on S then $\rho\subset\sigma$.

If p if a prime number, then every homogeneous p regular Rees matrix semigroup S with zero has a Brandt congruence ρ on S.

2.4 Partitions of regular semigroups

In this section we shall consider partitions of n regular Rees matrix semigroups by their homogeneous regular components.

The first main result (Theorem 2.14) is that a homogeneous regular component S(m) of a n regular Rees matrix semigroup S with zero is a subsemigroup of S with zero if and only if it is a direct union of all nonnull $h_m k_m = m$ type regular components $S(h_m, k_m)$ of S.

The second significant theorem (Theorem 2.16) gives a necessary and sufficient condition that S have a semigroup decomposition by its homogeneous regular components S(m).

Definition 2.4. Let S be a regular semigroup with zero and let n be a positive integer.

(i) S is called n regular if every nonzero element a in S has $m(\geq n)$ inverses of a.

(ii) Let $S[m]$ be the set of all elements a in S such that the cardinality of V(a) is m. $S[m]$ is called the (homogeneous) m regular component of S. $S(m) = S[m] \cup 0$ is called the (homogeneous) m regular component of S with zero.

(iii) A nonzero element a in S is said to be a $h_m k_m (=m)$ type regular element if the principal right ideal of S generated by a contains precisely h_m nonzero idempotents, and the principal left ideal of S generated by a contains precisely k_m nonzero idempotents with $h_m k_m = m$.

Lemma 2.13. Let S be an n regular Rees matrix semigroup with zero. Let a be $h_m k_m (=m)$ type regular and let b be $h_u k_u (=u)$ type regular in S. If $ab=c\neq 0$, then c is a $h_m k_u$ type regular element in S.

Proof. Let a and b be respectively $h_m k_m = m$ and $h_u k_u = u$ type regular elements in S. By the definition of the $h_m k_m$ type regular element a, we have that aS Sa contain h_m and k_m nonzero idempotents of S, respectively. Similarly, bS and Sb contain h_u and k_u nonzero idempotents of S, respectively. If $ab=c\neq 0$, then ab lies in $(aS \cap Sb)$, and $ab=c$ is a $h_m k_u$ type regular element in S since $aS=abS$ and $Sb=Sab$ by Lemma 2.45 1 .

Definition 2.5. Let $m \geq n$ and let S be an n regular semigroup.

(i) We denote the set of all $h_m k_m = m$ type regular elements in S by $S[h_m, k_m]$.
Let $S(h_m, k_m) = S[h_m, k_m] \cup 0$.
$S(h_m, k_m)$ is called the $h_m k_m$=type regular component of S, or a simple type component of S.
Clearly $S(h_m, k_m) \subset S(m)$.

(ii) S(m) is called a main component of S if S(m) contains a nonzero idempotent of S.
$S(h_m, k_m)$ is called a main component if it contains a nonzero idempotent.

(iii) Let S be a semigroup with zero and let $F(S)=(S_m$: m in M) be a family of subsets of S.
S is a direct union of its 0-disjoint subsets S_m in F(S) if $S=\bigcup_{m\in M} S_m$, $S_u S_v=(0)$ and $S_u \cap S_v=(0)$ for every $u\neq v$.

Theorem 2.14. Let S be an n regular Rees matrix semigroup with zero.

(i) There exists an index set of positive integers $N=(n_1, n_2, ..)$ such that $S=\bigcup_{n_i \in N} S(n_i)$

(ii) If a is a $h_m k_m = m$ type regular element in S, then the cardinality of V(a) is m.

(iii) $S(h_m, k_m)$ is a homogeneous m regular subsemigroup of S.

(iv) S(m) is a subsemigroup of S if and only if S(m) is a direct union of 0-disjoint simple type m components of S.

Proof. (i) For every nonzero element a in S there exists a component S(m) to which a belongs.

(ii) By Theorem 2.9, if a is an $h_m k_m = m$ type regular element, then a has precisely m inverses of a.

(iii) Let $S(h_m, k_m)$ be a simple type m regular component. Let a and b be two arbitrary elements in $S(h_m, k_m)$. If ab=0, there is nothing to prove. Hence let us assume that $ab=c\neq 0$. It follows from Lemma 2.13 that ab=c is $h_m k_m = m$ type regular, and hence c belongs to $S(h_m, k_m)$. Thus $S(h_m, k_m)$ is a subsemigroup of S. By (ii) every nonzero element of $S(h_m, k_m)$ has precisely m inverse elements. Therefore $S(h_m, k_m)$ is a homogeneous m regular sub-semigroup of S with zero.

(iv) If S(m) is simple type, say $S(m)=S(h_m, k_m)$, then there is nothing to prove. Let $S(m)=\bigcup_{i=1}^{N} S(h_m^i, k_m^i)$ where $N\geq 2$ and $(h_m^i, k_m^i)\neq(h_m^j, k_m^j)$ for $i\neq j$.

Suppose now that $S(m)$ is a subsemigroup with zero. Then we shall show that $S(h_m^i,k_m^i)S(h_m^j,k_m^j)=(0)$ and $S(h_m^i,k_m^i)\cap S(h_m^j,k_m^j)=(0)$ for every $i\neq j$. To avoid the notational confucion we clarify that the set

$((h_m^i,k_m^i)$: $i=1,2,..,N)$ means that $h_m k_m=m$ for every i and $(h_m^i,k_m^i)\neq(h_m^j,k_m^j)$ if $i\neq j$.

To prove the foregoing without ambiquity in the notation let us assume that $S(h_m,k_m)$ and $S(r_m,s_m)$ are two distinct simple components of S. From $(h_m,k_m)\neq(r_m,s_m)$, either $h_m > r_m$ or $h_m< r_m$. We may assume that $h_m>r_m$ which causes no loss of generality. Then from $h_m k_m=m=r_m s_m$ and $h_m>r_m$ we have $k_m<s_m$. Suppose that $S(h_m,k_m)S(r_m,s_m)=(0)$, and pick two non-elements a in $S(h_m,k_m)$ and b in $S(r_m,s_m)$ with $ab=c\neq 0$. By Lemma 2.13, c is $h_m s_m$ type regular. Since $h_m s_m\neq m$, denoting $h_m s_m=t$, c is $t(\neq m)$ regular and c does not lie in $S(m)$.

What we have shown is this: a and b are in $S(m)$ and $ab=c\neq 0$, but $ab=c$ does not lie in $S(m)$. Since $S(m)$ is a subsemigroup, this contradiction implies that $S(h_m,k_m)S(r_m,s_m)=(0)$ as desired. Thus we conclude that $S(h_m^i,k_m^i)S(h_m^j,k_m^j)=(0)$ for every $i\neq j$.

Conversely, suppose that $S(m)$ is the direct union of simple type m regular components of S. To show that $S(m)$ is a subsemigroup, let a and b be elements of $S(m)$. There are two cases.

Case 1. a and b belong to the same simple type m component. Since every simple type component is a subsemigroup, there is nothing to prove in this case.

Case 2. a and b lie in two distinct simple type m regular components $S(h_m,k_m)$ and $S(r_m,s_m)$, respectively. By hypothesis, $S(h_m,k_m)S(r_m,s_m)=(0)$ implies $ab=0$. Similarly $ba=0$. Hence $S(m)$ is a subsemigroup of S with zero.

Definition 2.6. Let $S=M^o(G\ ;I,V;P)$ be n regular.

(i) The Cartesian product IxV will be called the coordinate plane of a semigroup, and an element (i,j) in IxV is called a coordinate of the plane IxV. (i,j) in IxV is called an m-coordinate if Pxi and Pjx contain respectively h_m and k_m nonzero entries of P with $h_mk_m=m$. For this case, the coordinate (i,j) is called an $h_mk_m(=m)$ type coordinate of IxV.
We denote by (IxV)(m) the set of all m coordinates (i,j) in IxV. Similarly we define
$(IxV)(h_m,k_m)=((i,j)\in IxV\colon (i,j)$ is $h_mk_m=m$ type).

(ii) p_{ji} is an m entry of P if the coordinate (i,j) is an m coordinate of IxV. p_{ji} is an $h_mk_m=m$ type m entry of P if (i,j) is of type h_mk_m in IxV. $P(m)=(p_{ji}\in P\colon p_{ji}$ is an m entry) will be called an m component of P if P(m) is a subarray of P. $P(h_m,k_m)=(p_{ji}\colon p_{ji}$ is h_mk_m type) will be called a simple type component of P if it is a subarray of P. We shall use $P(h_m,k_m)$ and P(m) as subarrays of P.

(iii) $S(h_m,k_m)$ and $S(h_u,k_u)$ are called a non-reduced pair if $h_mk_u=h_uk_m$, $h_mk_m=m$, $h_uk_u=u$ and $u\neq m$. A nonreduced pair is called non-trivial if one of the pair is a main component of S.

Remark 4. It is evident that a component $P(h_m,k_m)$ is a submatrix of given sandwich matrix P of an n regular Rees matrix semigroup with zero. But a component P(m) is not a submatrix of P, in general. The following example exhibits the idea of the definition of an m component P(m) of a sandwich matrix P.

Example 4. Let $P=\begin{pmatrix} e & e & 0 & 0 \\ e & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & e \end{pmatrix}$ be a sandwich matrix over a group G=(e, a, 0) with 0, where e is the identity of G.

Then $P(1) = \begin{pmatrix} a & 0 \\ 0 & e \end{pmatrix}$ $\quad P(4)= \begin{pmatrix} e & e \\ e & a \end{pmatrix}$

$$P(2) = \begin{pmatrix} & & 0 & 0 \\ & & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & & \end{pmatrix}.$$

Clearly P(2) is not a submatrix of P.

Lemma 2.15. Let S be an n regular Rees matrix semigroup $S=M^o(G;I,V;P)$.

(i) $(g)_{ij}$ is an $|E(R_i\backslash 0)|\,|E(L_j\backslash 0)|$ type regular element in S.

(ii) If $(g)_{ij}$ belongs to $S(h_m,k_m)$, then G_{ij} is contained in $S(h_m,k_m)$. Hence $S(h_m,k_m)=\bigcup_{JxU} G_{ij}$ for a suitable subset UxJ of VxI.

(iii) If $S(h_m,k_m)$ includes G_{ij} and G_{rs}, then $S(h_m,k_m)$ includes G_{is} and G_{rj}.

(iv) If $S(h_m,k_m)$ and $S(h_u,k_u)$ are a nontrivial nonreduced pair, then, denoting $t=h_mk_u$, S(t) is not a subsemigroup of S.

(v) $S(h_m,k_m)S(h_u,k_u)=\begin{cases} S(h_m,k_u) \text{ if } S(h_u,k_m) \text{ is} \\ \quad \text{a main component,} \\ (0) \quad \text{otherwise.} \end{cases}$

(vi) $(g)_{ij}$ is $h_m k_m$ type regular if and only if (i,j) is an $h_m k_m$ type coordinate of $I \times V$.

Proof. (i) Since $(g)_{ij}$ belongs to $R_i \cap L_j$ and S is regular, R_i and L_j are respectively the principal right and left ideals generated by $(g)_{ij}$. Hence it is an $|E(R_i \backslash 0)|\,|E(L_j \backslash 0)|$ type regular element.

(ii) Let $(g)_{ij}$ be an element in $S(h_m,k_m)$. Then every nonzero element of G_{ij} is of the same type as $(g)_{ij}$ by (i). Hence G_{ij} is a subset of $S(h_m,k_m)$ because $S(h_m,k_m)$ contains the zero element.

(iii) Assume that $S(h_m,k_m)$ contains two sets G_{ij} and G_{rs}. Consider the set $G_{is}=R_i \cap L_s$. By (i), R_i and L_s contain h_m and k_m nonzero idempotents of S. Hence a nonzero element $(g)_{is}$ is an $h_m k_m$ type regular element. By (ii), G_{is} is contained in $S(h_m,k_m)$. The proof for G_{rj} is analogous.

(iv) Let $S(h_m,k_m)$ and $S(h_u,k_u)$ be a nontrivial nonreduced pair. Then by the definition of the nonreduced and nontrivial pair, we can assume that $m \neq u$, $h_m k_u = h_u k_m$ and one of these components is a main component of S. Let $S(h_m,k_m)$ be a main component of S. Then $S(h_m,k_m)$ contains a nonzero idempotent. Let $(p_{ji}^{-1})_{ij}$ be a nonzero idempotent lying in $S(h_m,k_m)$. Let $G_{ij} \subset S(h_m,k_m)$ and let $G_{rs} \subset S(h_u,k_u)$. Then we have $i \neq r$ and $j \neq s$. For if $i=r$, then by (i), $(g)_{ij}$ and $(g')_{rs}$ are respectively $h_m k_m$ and $h_m k_u$ type regular, whence $h_m=h_u$ and $k_m=k_u$. Then $m=h_mk_m=h_uk_u=u$, which contradicts the fact that $m\neq u$. Likewise, $j=s$ leads to a contradiction. Let $t=h_mk_u=h_uk_m$. Then $t\neq m$ and $t\neq u$. For if $t=m$, then $m=h_mk_m=h_mk_u=h_uk_m$, and hence $k_m=k_u$, $h_m=h_u$ and $m=u$. The proof for $t\neq u$ is similar.

Now consider $S(t)$.

We shall show that $S(t)$ contains G_{is} and G_{rj}. Since every nonzero element of G_{rj} is an h_uk_m type regular element and $h_uk_m=t$, then G_{rj} is a subset of $S(t)$. Similarly, we have that $G_{is}\subset S(t)$.

The proof will be concluded when we show that there exist two nonzero elements in $S(t)$ such that the product of these two elements does not belong to $S(t)$. To verify this let g and g' be nonzero group elements in G. It follows from $(g)_{rj}\circ(g')_{is}=(gp_{ji}^{-1}g')_{rs}\neq 0$ that $(gp_{ji}^{-1}g')_{rs}$ lies in G_{rs}. But, since G_{rs} is a subset of $S(h_u,k_u)$ and $S(u)\cap S(t)=(0)$, $0\neq(g)_{yj}\circ(g')_{is}$ does not lie in $S(t)$. Thus $S(t)$ is not a subsemigroup of S.

(v) By Lemma 2.13, we have $S(h_m,k_m)S(h_u,k_u)$ $S(h_m,k_u)$. Let us assume that $S(h_u,k_m)$ is a main component of S, that is, $S(h_u,k_m)$ contains a nonzero idempotent. Let $(g)_{ij}$ be a nonzero idempotent in $S(h_u,k_m)$, then $g=p_{ji}^{-1}$ is a nonzero group element of G, whence $p_{ji}\neq 0$. Let $(g_1)_{st}$ be arbitrary in $S(h_m,k_u)$.

We shall show that there exist two nonzero elements $(g_2)_{sj}$ and $(g_3)_{it}$ lying respectively in $S(h_m,k_m)$ and $S(h_u,k_u)$ such that $(g_2)_{sj}\circ(g_3)_{it}=(g_1)_{st}$.

Since $(g)_{ij}$ lies in $S(h_u,k_m)$, the minimal right

and left ideals R_i and L_j of S contain precisely h_u and k_m distinct nonzero idempotents of S, respectively. Similarly, it follows from $(g_1)_{st} \in S(h_m,k_u)$ that the minimal right and left ideals R_s and L_t, respectively contain h_m and k_u nonzero idempotents of S. Then $(g_2)_{sj}$ is an $h_mk_m=m$ type regular element, whence $(g_2)_{sj}$ belongs to $S(h_m,k_m)$. Analogously, $(g_3)_{it}$ is in $S(h_u,k_u)$.
By explict calculation, if we choose $g_2=p_{ji}^{-1}$ and $g_3=g_1$, then $(g_2)_{sj} \circ (g_3)_{it}=(g_2p_{ji}g_3)_{st}=(p_{ji}^{-1}p_{ji}g_1)_{st}=(g_1)_{st}$.

Thus we have established the first half of the equality, that is, $S(h_m,k_m)S(h_u,k_u)=S(h_m,k_u)$ if $S(h_u,k_m)$ is a main component of S.

Finally, we shall show that if $S(h_u,k_m)$ contains no nonzero idempotent then $S(h_m,k_m)S(h_u,k_u)=(0)$. Suppose that $S(h_m,k_m)S(h_u,k_u)\neq(0)$ and $S(h_u,k_m)$ contains no nonzero idempotent. We choose nonzero elements $(g)_{sj}$ in $S(h_m,k_m)$ and $(g')_{it}$ in $S(h_u,k_u)$ such that $(g)_{sj}\circ(g')_{it}\neq 0$. This implies that $p_{ji}\neq 0$ and that p_{ji}^{-1} is in G. Then $(p_{ji}^{-1})_{ij}$ is a nonzero idempotent, and it is an h_uk_m type regular element. Hence $S(h_u,k_m)$ does contain a nonzero idempotent of S. This contradiction proves the desired equlity above.

(vi) follows from Definitions 2.4, 2.6 and Lemma 2.5-(ii).

Remark 5. Notice that there is a one to one correspondence between the set $F(S)=(S(h_m,k_m):m\in M)$ of all simple type components of S and the set

$F(P)=(P(h_m,k_m):m\in M)$ of all simple type components of P. We use the symbol "$\sim$" to indicate this correspondence. For example, if P is a sandwich matrix of an n regular Rees matrix semigroup S with zero, then we have $S(h_m,k_m)\sim P(h_m,k_m)$.

The following figure helps us to prove Lemma 2.15-(v).

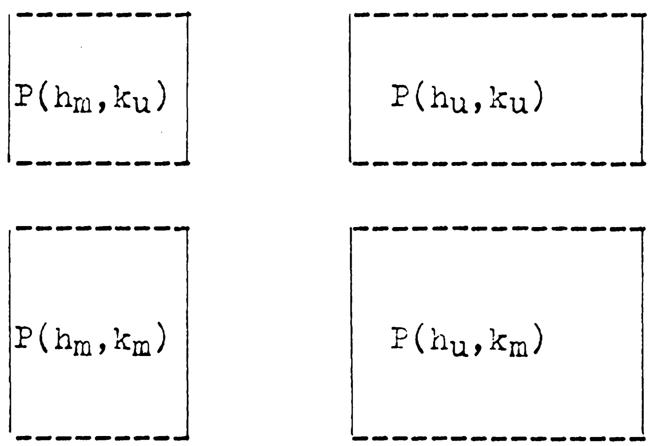


Figure 1.

Definition 2.7. If S is a semigroup with zero and if S has a decomposition by its subsemigroups $(S_m;\ m\in M)$ with zero, then we say that this decomposition $S=\bigcup_{m\in M} S_m$ is a semigroup decomposition if $S_m\cap S_u=(0)$ for $m\neq u$ in M.

We know that every simple type regular component of an n regular Rees matrix semigroup with zero is a

subsemigroup with zero and that two distinct simple type components are 0-disjoint.
Thus S has a semigroup decomposition by its simple type regular components. Hence our only interest in a semigroup decomposition by its regular components is the case for which there is a regular component S(m) of S which is not a simple type regular component.

The next theorem describes about this decomposition.

Theorem 2.16. Let S be an n regular and completely 0-simple semigroup.

(i) S has a semigroup decomposition by its homogeneous regular components S(m) if and only if every nonnull component S(m) is a direct union of all its simple type subcomponents $S(h_m^i, k_m^i)$, $i \in N$, an index set for S(m).

(ii) S has a semigroup decomposition by its homogeneous m regular components S(m) of S if and only if there is no nontrivial nonreduced pair in the family of all simple type regular components of S.

Proof. The proof of (i) follows from Theorem 2.14-(iv).

(ii). First assume that $S = \bigcup_{m \in N} S(m)$ is a semigroup decomposition by its homogeneous regular components of S, where N is a suitable index set for S.
If there is a nontrivial nonreduced pair in the family F*(S) of all simple type regular components of S with zero, then by Lemma 2.15-(iv) there exists a homogeneous

regular component S(t) which is not a subsemigroup with zero. Then the decomposition $S = \bigcup_{m \in N} S(m)$ would not be a semigroup decomposition. This contradicts the assumption. Hence there is no such nontrivial nonreduced pair of simple type components.

Conversely, assume that there is no nontrivial nonreduced pair in the family $F^*(S)$ of all simple type regular components. Suppose that $S = \bigcup_{m \in N} S(m)$ is not a semigroup decomposition of S by its homogeneous regular components of S. But it is clear that $S(m) \cap S(u) = (0)$ for $m \neq u$. For if $S(m) \cap S(u) \neq (0)$ for $m \neq u$, then choose a nonzero element a in $(S(m) \cap S(u))$. From $a \in S(m)$, the cardinality of the set V(a) is m, but from $a \in S(u)$, the cardinality of the set V(a) is u. This is a contradiction because the cardinality of the set V(a) is unique for a fixed element $a \neq 0$.

The foregoing indicates that there must exists at least one component S(m) which is not a subsemigroup of S, otherwise, the decomposition $S = \bigcup S(m)$ would be a semigroup decomposition.
Suppose that S(m) is not a subsemigroup of S. Then there must exist two nonzero elements a and b in S(m) such that the product $ab = c$ of the two elements does not lie in S(m). By Theorem 2.14-(iii) every simple type homogeneous regular component is a subsemigroup, and hence S(m) is not a simple type component. Thus we may assume that there exist two distinct simple type components $S(h_m^i, k_m^i)$ and $S(h_m^j, k_m^j)$ such that a lies in $S(h_m^i, k_m^i)$, b lies in $S(h_m^j, k_m^j)$,

$h^i_m k^i_m = m = h^j_m k^j_m$ and $(h^i_m, k^i_m) = (h^j_m, k^j_m)$.

Since $ab = c \neq 0$, by applying Lemma 2.15-(v), we have

$S(h^i_m, k^i_m) S(h^j_m, k^j_m) = S(h^i_m, k^j_m)$. Let $h^i_m k^j_m = t$.
Clearly $c \in S(h^i_m, k^j_m) \subset S(t)$. Notice that $S(t) \neq S(h^i_m, k^j_m)$, in general.

We claim that $S(h^i_m, k^j_m)$ and $S(h^j_m, k^i_m)$ are a nontrivial and nonreduced pair of simple type components. To do this, first we shall show that $S(h^i_m, k^j_m) \neq S(h^j_m, k^i_m)$. Assume, by way of contradiction, that $S(h^i_m, k^j_m) = S(h^j_m, k^i_m)$. Then we have $h^i_m k^j_m = h^j_m k^i_m$. From this we proceed to a contradiction. Since $h^i_m k^j_m = h^j_m k^i_m$, we have

$$\frac{h^i_m}{k^i_m} = \frac{h^j_m}{k^j_m} = r \ .$$

Then $h^i_m = k^i_m \cdot r$ and $h^j_m = k^j_m \cdot r$.

By substituting $k^i_m r$ for h^i_m in m, we have

$m = h^i_m k^i_m = (k^i_m \cdot r) k_m = (k^i_m)^2 r$ and $m = h^j_m k^j_m = (k^j_m \cdot r) k^j_m = (k^j_m)^2 r$.

From this we deduce that $k^i_m = k^j_m$ and $h^i_m = h^j_m$, whence

$S(h^i_m, k^i_m) = S(h^j_m, k^j_m)$. This contradicts the fact that

$S(h^i_m, k^i_m) \neq S(h^j_m, k^j_m)$. Hence we must infer that

$h^i_m k^j_m \neq h^j_m k^i_m$, and hence $S(h^i_m, k^j_m) \neq S(h^j_m, k^i_m)$.

Therefore the two simple type components $S(h^i_m, k^j_m)$ and $S(h^j_m, k^i_m)$ are a nonreduced pair.
Since $S(h^i_m, k^i_m) S(h^j_m, k^j_m) = S(h^i_m, k^j_m)$, by Lemma 2.15-(v), $S(h^j_m, k^i_m)$ contains a nonzero idempotent of S, and hence it is a nontrivial main component of S.

Finally we conclude that $S(h_m^i, k_m^j)$ and $S(h_m^j, k_m^i)$ are a pair of nontrivial nonreduced components of S.
This is a contradiction because we have assumed that there is no nontrivial nonreduced pair in the family $F^*(S)$ of all simple type regular components.
Thus the decomposition $S = \cup S(m)$ is a semigroup decomposition by its homogeneous m regular components of S.
This completes the proof.

2.5 Semigroups with graphs

Since for every n x n square matrix P over a groupoid G, we can define a graph as shown below, it is natural to consider relationships between Rees matrix semigroups with zero and their associated graphs.

Let $P=(p_{ij})$ be any n x n matrix over a group G, and consider any n distinct points $A_1, A_2, .., A_n$ in the plane (or in relative general position in space), which we call vertices. For every nonzero entry $p_{ij} \neq 0$ of the matrix P, we connect the vertex A_i to the vertex A_j by means of a path $\overline{A_iA_j}$, which we shall call an edge (a loop if $i=j$) directed from A_i to A_j.
In this way, with every n x n matrix P can be associated a finite directed graph G(P), see Example 4.

Definition 2.8. (i) A directed graph is strongly connected if, for any ordered pair of vertices A_i and A_j, there exists a directed path $\overline{A_iA_{h_1}}$, $\overline{A_{h_1}A_{h_2}}$, ..,

$\overline{A_{h_{k-1}}A_{h_k}}$, $h_k=j$, connecting A_i to A_j. We shall say that such a path has length k.

(ii) Let $S=M^o(G;I_n,I_n;P)$ be a Rees matrix semigroup. Then the graph G(P) is called the associated graph of S, or simply it is the graph G(P) of S.

(iii) Let A be a vertex of a graph G(P). The local degree at a vertex A is the number of edges having A as one end point. In a directed graph there are two types of edges at each vertex A. The outgoing edges from A and the incoming edges to A. Correspondingly, we have two local degrees: the number $\phi(A)$ of outgoing edges and the number of $\phi^*(A)$ of incoming edges.

Example 5. Let $P = \begin{pmatrix} a & e & a^2 & a & e \\ a & 0 & 0 & 0 & 0 \\ a^3 & 0 & 0 & 0 & 0 \\ e & 0 & 0 & 0 & 0 \\ a^2 & 0 & 0 & 0 & 0 \end{pmatrix}$

be a sandwich matrix of a Rees matrix semigroup $S=M^o((a)^4;5,5;P)$. Then S has its associated graph G(P).

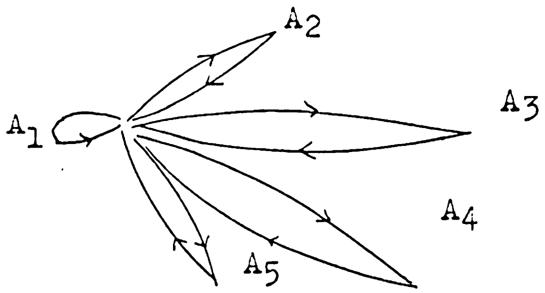


Figure 2.

Lemma 2.17. Let $G(P)$ be a graph of a semigroup $S=M^{o}(G;In,In;P)$.

(i) Then every vertex A of the graph $G(P)$ has at most one loop of $G(P)$. Let B be a vertex of the graph $G(P)$ and $A\neq B$, then there is at most one edge $E(A,B)$ from A to B.

(ii) S is regular if and only if for every vertex A in the graph $G(P)$ of S, $\phi(A).\phi^*(A)\geq 1$.

(iii) If the graph $G(P)$ is strongly connected, then S is completely O-simple.

(iv) S is n regular if and only if for every vertex A of the graph $G(P)$, $\phi(A).\phi^*(A)\geq n$, where n is a positive integer.

(v) Let A_i be a vertex of the graph $G(P)$. $\phi(A_i)=m$ if and only if the ith row of P contains exactly m distinct nonzero entries $(p_{ij_1},..,p_{ij_m})$.

(vi) Let A_i be a vertex of the graph $G(P)$. $\phi^*(A_i)=k$ if and only if the ith column of P contains exactly k nonzero entries.

(vii) S is a Brandt semigroup if and only if $\phi(A)=\phi^*(A)=1$ for every vertex A of the graph $G(P)$.

(viii) S is a homogeneous m regular semigroup if and only if $\phi(A).\phi^*(A)=m$ for every vertex A of the graph $G(P)$.

Proof. The proof of (i) is clear.

(ii) Let P be an n x n sandwich matrix of S. Let $(A_1,A_2,..,A_n)$ be the set of all vertices of the graph $G(P)$. Suppose that $\phi(A_i)\gtrsim 1$. By Definition of $\phi(A_i)$, Pix contains at least one nonzero entry and vice versa. Analogously, if $\phi^*(A_i)\geq 1$, then Pxi contains at least one nonzero entry of P and vice versa.

Thus P is regular and S is completely 0-simple if and only if $\phi(A_i).\phi^*(A_i)\geq 1$ for every $i=1,2,..,n$.

(iii) Let A_i and A_j be two arbitrary distinct vertices of the graph G(P) of S. Since G(P) is strongly connected there exists at least one directed path $\overline{A_iA_{h_1}}, \overline{A_{h_1}A_{h_2}},..,\overline{A_{h_{k-1}}A_{h_k}}$, $h_k=j$, connecting A_i to A_j. Thus $p_{h_{k-1}h_k}\neq 0$, which shows that the jth column of P contains a nonzero entry. Since j is arbitrary in In, every column of P contains a nonzero entry. From the directed path above, we have $p_{ih_1}\neq 0$, and hence every ith row Pix contains at least one nonzero entry. Thus P is regular.
By Lemma 3.1 and Theorem 3.3 in [1], S is completely 0-simple.
We omit the proof of (iv).

(v) Let $\phi(A_i)=m$. Then by the definition of a local degree $\phi(A_i)$ of the vertex A_i there exist vertices $(A_{j_1},A_{j_2},..,A_{j_m})$ such that every directed edge $E(A_i,A_{j_k})$ for $k=1,2,..,m$, is non-empty. From the non-empty edge $E(A_i,A_{j_k})$, we have $p_{ij_k}\neq 0$ for every $k=1,2,..,m$.

Conversely, if $p_{ij_k}\neq 0$ then there exists a non-empty edge $E(A_i,A_{j_k})$, where A_{j_k} is a vertex of the graph G(P). Hence if the ith row of P contains exactly m nonzero elements then $\phi(A_i)=m$.
The proof of the rest is easy.

Corollary 2.18. A Rees matrix semigroup $S=M^0(G;In,In;P)$ is homogeneous m^2 regular if the directed graph G(P) of the semigroup S is regular of degree m.

The proof of the corollary follows from the definition of <u>regular</u> of degree m of a directed <u>graph</u> [11, p.11] and Lemma 2.17-(viii).

Corollary 2.18 above gives a good relationship between homogeneous n regular and Rees matrix semigroups and regular directed graphs.

Problem 3. For a rectangular matrix which is not a square matrix, how one can define a graph G of a given matrix ?

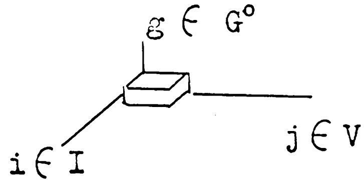
In general, sandwich matrices P of Rees matrix semigroups are not square, therefore if we can define it in Problem 3; then we will be able to establish a considerable number of results for Rees matrix semigroups.

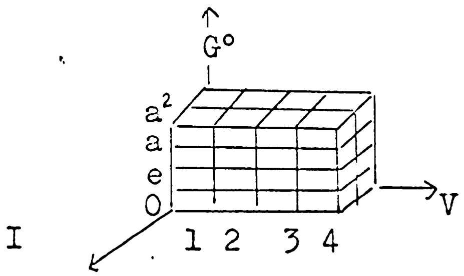
3 Rees matrix semigroups over finitely generated abelian groups with zero

From the study of Rees matrix semigroups it seems that there would be a considerable number of interesting properties of these semigroups depending only upon properties of their structure groups G with zero. Explicitly, if G has just one proper subgroup then does S have only one proper subsemigroup ? The answer to this question is negative.

The first motivation for this chapter is the study of horizontal sections of a rectangular brick prism of a Rees matrix semigroup. Before giving the formal definition of a prism of a Rees matrix semigroup we shall describe the ideas of a rectangular (brick) prism of a Rees matrix semigroup and its horizontal sections by taking the following simple semigroup with zero.

Let us take a brick as an element of a Rees matrix semigroup $S=M^{o}(G;I_2,I_4;P)$ over a cyclic group $G^{o}=(e,a,a^2)\cup 0$ of order 3 with zero. Since $S=\cup G^{o}_{ij}$ $(i=1,2, j=1,2,3,4)$ we can build a rectangular brick prism with a 2 by 4 rectangular base of eight bricks, each of which is counted as the zero of the semigroup S. Then there are eight vertical sections $G^{o}_{ij}=(0,(e)_{ij},(a)_{ij},(a^2)_{ij})$ for $i=1,2,$ $j=1,2,3,4$. It is well known that if G_{ij} contains a nonzero idempotent then it is a group with zero.





Now define a horizontal section of the prism. $S(i,I_4,g)=((g)_{i1},(g)_{i2},(g)_{i3},(g)_{i4})$ is a horizontal section of the prism (S) of i-g coordinate, where g is a nonzero group element in G. For example, $S(1,I_4,e)=((e)_{11},(e)_{12},(e)_{13},(e)_{14})$ is the horizontal of the prism (S), which is in the first row of the upstairs of the prism. See the picture on page 70.

$S(I_2,j,g)=((g)_{1j},(g)_{2j})$ is a g-j horizontal section of the prism, where g is a nonzero element in G.

Does a horizontal section of the prism of a Rees matrix semigroup with zero form a group with zero ?

Does a horizontal section of a prism of a Rees matrix semigroup with zero form a subsemigroup ?

For the former question we shall give the complete solution in Theorem 3.2. For the latter problem we shall have a very interesting result in Theorem 3.6.

Gerard Lallement and Mario Petrich [4] found the equivalent conditions for a 0-rectangular Rees matrix semigroup and this comes into our general problem in the study of horizontal sections of the prisms of Rees matrix semigroups.

3.1 Horizontal sections of the prisms

We give the formal definition of a prism of a Rees matrix semigroup with zero.

Definition 3.1. Let $S=M^{0}(G;I,V;P)$ be a Rees matrix semigroup.

(i) The cartesian product IxVxG is called the coordinate space of the semigroup S.

(ii) We identify an element (i,j,g) in IxVxG with $(g)_{ij}$, in other word, $(g)_{ij}=(i,j,g)$.
Denote (S) the set of all elements (i,j,g) in the coordinate space IxVxG piled as a rectangular brick prism according to their coordinates in the space IxVxG. (S) is called a rectangular prism of the semigroup S, or simply a prism of S.

(iii) We shall use the following notations.
$S(I,j,g)=((g)_{ij}:i \text{ in } I)$, for a fixed j in V, a fixed g in G. $S(i,V,g)=((g)_{ij}: j \text{ in } V)$, $S(i,j,G^{0})=G^{0}_{ij}$,
$S(I,j,G)=((g)_{ij}; g \text{ in } G, i \text{ in } I)$,
$S(I,V,g)=((g)_{ij}: i \text{ in } I, j \text{ in } V)$,
$S(i,j,G^{0})$ is called a simple vertical section of (S).
$S(I,j,g)$ is called a left horizontal section of the prism, or a horizontal g-j section of (S).
$S(i,V,g)$ is called a right horizontal section of (S).
$S(I,V,g)$ is called a horizontal g section of (S).

The next lemma includes known properties of Rees matrix semigroup looked at from the point of view of the prism.

Lemma 3.1. Let $S=M^0(G;I,V;P)$ and let $|I|\geq 2, |V|\geq 2$.

(i) Every simple vertical section $S(i,j,G^0)$ is either a group with zero or a nilpotent semigroup.

(ii) $S(I,j,G^0)$ and $S(i,V,G^0)$ are left and right ideals of S, respectively.

(iii) Every left(right) horizontal section of (S) contains no the identity.

(iv) A left horizontal section $S(I,j,g)\cup 0$ with zero is a subsemigroup with zero if and only if every nonzero entry of the jth row of P is g^{-1}.

(v) A right horizontal section $S(i,V,g)\cup 0$ with zero is a subsemigroup with zero if and only if every nonzero entry of the ith column of P is g^{-1}. If every entry of Pxi is g^{-1}, then $S(i,V,g)$ is a right zero semigroup.

(vi) Every horizontal section $S(I,V,g)\cup 0$ with zero contains no the identity.

(vii) A horizontal g-section $S(I,V,g)\cup 0$ with zero is a subsemigroup of S if and only if every nonzero entry of P is g^{-1}. If every entry of P is g^{-1}, then $S(I,V,g)$ is an idempotent subsemigroup of S.

Proof. (i) It suffices to show that if G^0_{ij} contains no nonzero idempotent, then it is a nilpotent subsemigroup. Let us assume that G_{ij} contains no nonzero idempotent. Then $p_{ji}=0$. It follows from $p_{ji}=0$ that $G^0_{ij}G^0_{ij}=(0)$. Hence G^0_{ij} is a zero semigroup.

(ii) follows from $L^0_j=S(I,j,G^0)$ and $R^0_i=S(i,V,G^0)$.

The proof of (iii) is evident.

(iv) Assume that a left horizontal section $S(I,j,g)\cup 0$ with zero is a subsemigroup with zero.

Suppose that there is at least one nonzero entry in the jth row of the sandwich matrix P, and let it be $p_{ji}\neq 0$. Choose $(g)_{hj}$ and $(g)_{ij}$ in $S(I,j,g)$ and consider the product $(g)_{hj}\circ(g)_{ij}=(gp_{ji}g)_{hj}$. By the assumption, $(gp_{ji}g)_{hj}$ is equal to $(g)_{hj}$. This implies that $p_{ji}^{-1}=g$ and $g^{-1}=p_{ji}$.

Conversely, if every nonzero entry of the jth row of P is g^{-1}. Then

$$(g)_{hj}\circ(g)_{kj}=\begin{cases}(g)_{hj} & \text{if } p_{jk}\neq 0,\\ 0 & \text{otherwise.}\end{cases}$$

Hence $S(I,j,g)$ is a subsemigroup of S with zero. The foregoing shows that if every entry of the jth row of P is g^{-1}, then $S(I,j,g)$ is a left zero subsemigroup.

The proof of (v) follows by the dual argument in the proof of (iv).

(vii) First we assume that $S(I,V,g)\cup 0$ is a subsemigroup of S. Let p_{ij} be any nonzero entry of a nonzero sandwich matrix P. Choose $(g)_{hi}$ and $(g)_{jk}$. Consider the product $(g)_{hi}\circ(g)_{jk}=(gp_{ij}g)_{hk}$. This element must be equal to $(g)_{hk}$ since $S(I,V,g)\cup 0$ is a subsemigroup of S with zero. Thus $p_{ij}=g^{-1}$. This proves the necessity of (vii).

If every nonzero entry of P is g^{-1}, then clearly $S(I,V,g)\cup 0$ is a subsemigroup of S.

If every entry of P is g^{-1}, then $S(I,V,g)$ is an idempotent semigroup without zero.

3.2 Subgroups of a Rees matrix semigroup

The next theorem is fundamental in a study of subgroups of Rees matrix semigroups.

Theorem 3.2. In a Rees matrix semigroup $S= M^{o}(G;I,V;P)$ there are two types of subgroups.

(i) $S(i,j,G)=G_{ij}$ with $E(G_{ij})\neq 0$.

(ii) A subgroup of the group G_{ij}.

Proof. Let H be a subgroup of S. Then H contains the identity e. Let $e=(p_{ji}^{-1})_{ij}$. Assume that $H\backslash e$ is not empty. Choose a nonzero element a in $H\setminus e$. Let $a=(g)_{hk}$. From $ae=a=ea$, we have $(g)_{hk}o(p_{ji}^{-1})_{ij}=(g)_{hk}=(p_{ji}^{-1})_{ij}o(g)_{hk}$, which implies that $j=k$ and $i=h$. Hence $a=(g)_{hk}=(g)_{ij}$, and a belongs to G_{ij}. Since a is an arbitrary element of H, it follows that H is a subset of G_{ij}, and hence H is a subgroup of G_{ij}. If $H=(e)$, it is a single element group. This completes the proof.

By Theorem 3.2 the hereditary properties of a Rees matrix semigroup $M^{o}(G;I,V;P)$ from a structure group G are clear in some point of view. In other words, we must not try to seek subgroups or subgroups with zero of a Rees matrix semigroup since there are two types of subgroups of a Rees matrix semigroup, one is a form G_{ij}, and the other is a subgroup of a group G_{ij}.

3.3 Subsemigroups of Rees matrix semigroups

Since all possible types of subgroups of a Rees matrix semigroup are known by Theorem 3.2, therefore we shall study mainly subsemigroups of Rees matrix semigroups with zero.

Among sections $S(I,j,g)$, $S(i,V,g)$, $S(i,j,G^0)$, $S(I,j,G^0)$, $S(i,V,G^0)$, and $S(I,V,g)$, we know that $S(i,j,G^0)$, $S(I,j,G^0)$ and $S(i,V,G^0)$ are respectively a subgroup with zero or a nilpotent subsemigroup, left and right ideals of S. By Lemma 3.1, $S(I,j,g)\cup 0$, $S(i,V,g)\cup 0$ and $S(I,V,g)\cup 0$ are subsemigroups of S if and only if every nonzero entry of the jth column, every nonzero entry of the ith row and every nonzero entry of P are g^{-1}, respectively.

We need more definitions to extend our study of various subsemigroups of a semigroup.

Definition 3.2. Let $S=M^0(G;I,V;P)$.

(i) Define $S(i,V\times G^0)=((g)_{ij}:j\in V, g\in G^0, (g)_{ij}=(g')_{ij}$ implies $g=g')$ for a fixed i, and it is called a mixed x=i horizontal section of the prism (S).

(ii) Let $S(I\times G^0,j)=((g)_{ij}: i\in I, g\in G^0, (g)_{ij}=(g')_{ij}$ implies $g=g')$, $S(I\times V\times G^0)=((g)_{ij}: i\in I, j\in V, g\in G^0, (g)_{ij}=(g')_{ij}$ implies $g=g')$. $S(I\times V\times G^0)$ is called a horizontal section of the prism (S).

The mixed horizontal section $S(i,V\times G^0)$ corresponds to $S(i,V,g)$ in Definition 3.1. $S(I\times G^0,j)$ and $S(I\times V\times G^0)$ are corresponding sets $S(I,j,g)$ and $S(I,V,g)$, respectively.

Lemma 3.3. Let $S=M^{0}(G;I,V;P)$.

(i) For every i in I, there exists a mixed horizontal section $S(i,V\times G^{0})$ of the prism such that $S(i,V\times G^{0})\cup 0$ is a subsemigroup of S. If the ith column of the sandwich matrix P contains no zero entry, then there exists a unique horizontal section, which is a subsemigroup of S. This is the case, it is a right zero semigroup.

(ii) For every j in V, there exists a mixed horizontal section $S(I\times G^{0},j)$ of the prism such that $S(I\times G^{0},j)\cup 0$ is a subsemigroup of S. If the jth row of P contains no zero entry, then $S(I\times G^{0},j)$ is a unique left zero semigroup.

Proof. (i) Let $P\times i=(p_{ai},p_{bi},\ldots)$ be the ith column of P. Define a set $S(i,V\times G^{0})$ of all elements of the forms $(g)_{ij}$ defined by the following:

$$(g)_{ij} = \begin{cases} (p_{ji}^{-1})_{ij} & \text{if } p_{ji}\neq 0 \\ (g)_{ij} & \text{if } p_{ji}=0 \text{ for every } j \text{ in } V, \end{cases}$$

where g is a nonzero element of G. To show that $0\cup S(i,V\times G^{0})$ is a subsemigroup with zero, consider the product of any two nonzero elements $(g')_{ic}$ and $(g'')_{id}$ in $S(i,V\times G^{0})$,

$$(g')_{ic}\circ(g'')_{id}= \begin{cases} (g'p_{ci}g'')_{id}=(g'')_{id} & \text{if } p_{ci}\neq 0, \\ 0 \quad \text{otherwise.} \end{cases}$$

Hence $S(i,V\times G^{0})\cup 0$ is a subsemigroup with zero and the cardinal number of the set $S(i,V\times G^{0})$ is equal to the cardinal number of the set V.

Consider the case for which the ith column of P contains no zero entry.

If we define a set $S(i,VxG)=((p_{ji}^{-1})_{ij}:\ p_{ji}\in Pxi)$, then $S(i,VxG)\cup 0$ forms a subsemigroup of S with zero.

To show the uniqueness of such a mixed x=i horizontal section, let us suppose that K is a mixed x=i horizontal section of the prism (S) such that K is a subsemigroup without zero and $K\neq S(i,VxG)$. Let $a=(g)_{ic}$ be in $K\setminus S(i,VxG)$. Then $(g)_{ic}\circ(g)_{ic}=(gp_{ci}g)_{ic}$ belongs to K. Since K is a semigroup without zero, we have $(gp_{ci}g)_{ic}\neq 0$ and hence $p_{ci}\neq 0$. Since $((g)_{ic},(gp_{ci}g)_{ic})\subset K$, $(g)_{ic}=(gp_{ci}g)_{ic}$ and $g=p_{ci}^{-1}$. Then $a=(g)_{ic}=(p_{ci}^{-1})_{ic}$, which shows that a belongs to $S(i,VxG)$, contrary to $a\in K\setminus S(i,VxG)$. Hence $S(i,VxG)$ is a unique subsemigroup of S without zero.

The dual argument to the foregoing can be proved (ii).

Now consider subsemigroups of a structure group G of a Rees matrix semigroup S. If the structure group G is finite order, then every subsemigroup of G is a subgroup of G. If G contains an element a of order zero, then there is a subsemigroup of G which is not a subgroup of G. $T_n(a)=(a^{n+i}:i=1,2,..)$ for $n\geq 1$, is such a one.

A semigroup is called monogenic if it has a generating set consisting of a single element[5, p.106].

The next lemma tells us that if K is a subsemigroup of S, then K is a union of monogenic subsemigroups and zero subsemigroups of S.

A semigroup K is said to be a zero semigroup if $KK=(0)$ [1, p.4].

Lemma 3.4. Let $S=M^{o}(G;I,V;P)$.

(i) If a in S is a nilpotent element of order finite, then $aa=0$.

(ii) If K is a subsemigroup of S, then K is a union of monogenic subsemigroups of groups and zero subsemigroups of S.

Proof. (i) Assume that $a\neq 0$ and $a^m=0$ for $m>1$. Putting $a=(g)_{ij}$, we have $(g)_{ij}\circ(g)_{ij}\circ\cdots\circ(g)_{ij}=0$ which implies that $p_{ji}=0$. Hence $aa=0$.

(ii) Let K be a subsemigroup of S and let $a=(g)_{ij}$ be a nonzero element of K. Consider $a^m=$

$$((g)_{ij})^{m-2}\circ((g)_{ij}\circ(g)_{ij})=((g)_{ij})^{m-2}\circ(gp_{ji}g)_{ij}$$
$$=\begin{cases}((gp_{ji})^{m-1}g\ \ )_{ij} & \text{if } p_{ji}\neq 0,\\ 0 & \text{otherwise.}\end{cases}$$

If $a^m=0$, then $a^2=0$ by (i), and hence $G^o_{ij}G^o_{ij}=(0)$. For this case, define $K_{ij}=(K\cap G^o_{ij})$, then K_{ij} is a subsemigroup of the zero semigroup G^o_{ij}.
If $a^m\neq 0$, $T_1(a)$ is a monogenic subsemigroup of the group G^o_{ij} without zero. Define

$$K_1(a)=\begin{cases}K_{ij}=(K\cap G^o_{ij}) & \text{if } a^2=0, \text{ for a in K}, \quad a\in G^o_{ij},\\ T_1(a) & \text{otherwise.}\end{cases}$$

Then $K=\bigcup_{a\in K}K_1(a)$, which proves (ii).

We conclude this section by exhibiting three examples.

Example 6. Let $S=M^{o}(G;I,V;P)$.

If four nonzero entries p_{ij}, p_{im}, p_{km} and p_{kj} of P are subjected to a condition $p_{ij}^{-1}p_{im}p_{km}^{-1}p_{kj}=e$, where e is the identity of the structure group G. Then $K_1=((p_{ij}^{-1})_{ji},(p_{im}^{-1})_{mi},(p_{km}^{-1})_{mk},(p_{kj}^{-1})_{jk})$ is a subsemigroup of S without zero.

A multiplication table of the semigroup K_1 is the following. Let $a=(p_{ij}^{-1})_{ji}$, $b=(p_{kj}^{-1})_{jk}$, $c=(p_{km}^{-1})_{mk}$ and $d=(p_{im}^{-1})_{mi}$.

	a	b	c	d
a	a	b	b	a
b	a	b	b	a
c	d	c	c	d
d	d	c	c	d

If $p_{ij}=0$, $p_{im}\neq 0$, $p_{km}\neq 0\neq p_{kj}$ are entries of P, then $K_2=((p_{im})_{mi},(p_{km})_{mk},(p_{kj})_{jk},(p_{kj}p_{km}p_{im})_{ji},0)$ is a subsemigroup of S with zero.

A multiplication table of K_2, if we write $a=(p_{im}^{-1})_{mi}$, $b=(p_{km}^{-1})_{mk}$, $c=(p_{kj}^{-1})_{jk}$, and $d=(p_{kj}^{-1}p_{km}p_{im}^{-1})$, is the following.

	a	b	c	d	0
a	a	b	0	0	0
b	a	b	b	a	0
c	d	c	c	d	0
d	d	c	0	0	0
0	0	0	0	0	0

If $p_{ij}=0$, $p_{im}\neq 0$, $p_{km}=0$ and $p_{kj}\neq 0$ are entries of P, then $K_3=((p_{kj}^{-1})_{jk},(p_{im}^{-1})_{mi},0)$ is a subsemigroup with zero.

3.4 The g-cellular sandwich matrices

Problem (A). Under what condition on S, does a horizontal section $S(I\times V\times G^{0})$ of a prism of a Rees matrix semigroup $S=M^{o}(G;I,V;P)$ form a subsemigroup ?

The answer to the problem (A) is made in Lemma 40-(vii); that is, if every nonzero entry of a sandwich matrix P is equal to a nonzero group element g of a structure group G of a Rees matrix semigroup S, then $S(I\times V,g^{-1})\cup 0$ is a subsemigroup of S with zero.

We shall give the second answer to the problem (A) imposed above in Theorem 3.5.

Problem (B). Under what condition on S, does a union of 0-disjoint horizontal section $S(I\times V\times G^{0})$ of the prism of a Rees matrix semigroup $S=M^{o}(G;I,V;P)$ form a subsemigroup ?

We shall have a very interesting solution (Theorem 3.6) to the problem (B).

Definition 3.3. (i) A semigroup S with zero is said to be 0-rectangular if it has the property that all the products at the vertices of a closed polygonal line (with a finite number of vertices) of the multiplication table are all but one equal to a nonzero element m and the remaining product is not zero, then it is also equal to m [4].

(ii) If four coordinates of four entries of P are four vertices of a rectangle in the coordinate plane IxV, then these four entries are called rectangular four entries in P, where P is a sandwich matrix of a Rees matrix semigroup $M^{o}(G;I,V;P)$.

(iii) Rectangular four entries in P are called cellular four entries in P if four coordinates of these are vertices of a minimal rectangle in the coordinate plane.

(iv) A sandwich matrix P is called an e-rectangular matrix if (1) P contains no zero entry and (2) every rectangular four entries p_{ij}, p_{im}, p_{um} and p_{uj} satisfy the condition $p_{ij}^{-1}p_{im}p_{um}^{-1}p_{uj}=e$, where e is the identity of the group G.

(v) A sandwich matrix P is called a g-cellular matrix if (1) P contains no zero entry and (2) every cellular four entries p_{ij}, p_{im}, p_{um} and p_{uj} satisfy the condition $p_{ij}^{-1}p_{im}p_{um}^{-1}p_{uj}=g$, where g is a nonzero group element in G.

The next theorem includes known properties [4, Theorem 1] of Rees matrix semigroups looked at from the point of view of the prism.

Theorem 3.5. If P is an e-rectangular sandwich matrix of a Rees matrix semigroup $S=M^o(G;I,V;P)$, and e is the identity of the structure group G. Then

(i) $M^o(G;I,V;P')$ is isomorphic with S, where P' is a VxI sandwich matrix with all entries e.

(ii) S is rectangular.

(iii) S has a subsemigroup S(IxVxG) intersecting each H-class of S in exactly one element.

Proof. (i) Let $P=(p_{vu})$. Pick a coordinate (i,j) in IxV. Choose rectangular four entries p_{ji}, p_{jk}, p_{uk} and p_{ui}, where i and j are fixed.

By the definition of a e-rectangular sandwich matrix , we have $p_{ji}^{-1}p_{jk}p_{uk}^{-1}p_{ui}=e$, and hence $p_{uk}=p_{ui}p_{ji}^{-1}p_{jk}$, for every u in V, and every k in I.
Then we have

$$P=\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & p_{j-1i}p_{ji}^{-1}p_{ji-2} & p_{j-1i}p_{ji}^{-1}p_{ji-1} & p_{j-1i} & p_{j-1i}p_{ji}^{-1}p_{ji-1} & \cdot \\ \cdot & p_{ji-2} & p_{ji-1} & p_{ji} & p_{ji-1} & \cdot \\ \cdot & p_{j-1i}p_{ji}^{-1}p_{ji-2} & p_{j-1i}p_{ji}^{-1}p_{ji-1} & p_{j-1i} & p_{j-1i}p_{ji}p_{ji-1}^{-1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Take two diagonal VxV and IxI matrices A and B

$$A=\begin{pmatrix} \cdot & & & & & \\ & p_{ji}p_{j-1i}^{-1} & & & & \\ & & e & & & \\ & & & p_{ji}p_{j-1i}^{-1} & & \\ & & & & \cdot & \\ & & & & & \cdot \end{pmatrix}$$

$$B=\begin{pmatrix} \cdot & & & & & \\ & \cdot & & & & \\ & & p_{ji-1}^{-1} & & & \\ & & & p_{ji}^{-1} & & \\ & & & & p_{ji-1}^{-1} & \\ & & & & & \cdot \\ & & & & & & \cdot \end{pmatrix}$$

and then APB=

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & e & e & e & e & e & e & \cdot \\ \cdot & e & e & e & e & e & e & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

By Theorem 3.6 in [1], the proof follows.

(ii) By (i), we have that $M^o(G;I,V;P)$ is isomorphic with $M^o(G;I,V;APB)=S'$, where APB is given on page 83. Hence we shall prove (ii) in S' in the case of four vertices of a closed polygonal line of the multiplication table.

Let a,b,x and y be elements of the semigroup S' and let four elements ax, ay, bx and by are at the vertices of a rectangle in the multiplication table of S'.

	x	y
a	ax	ay
b	bx	by .

Assume that any three are equal to m, say ax=ay=by=m. Then we shall show that bx=m. Let $m=(g)_{ij}$.
We may assume that $a=(g_1)_{ih}$, $b=(g_2)_{ik}$, $x=(g_3)_{sj}$ and $y=(g_4)_{tj}$. From ax=by=ay=m,

$$ax=(g_1)_{ih}\circ(g_3)_{sj}=(g_1p_{hs}g_3)_{ij}=(g)_{ij},$$
$$by=(g_2)_{ik}\circ(g_4)_{tj}=(g_2p_{kt}g_4)_{ij}=(g)_{ij} \quad \text{and}$$
$$ay=(g_1)_{ih}\circ(g_4)_{tj}=(g_1p_{ht}g_4)_{ij}=(g)_{ij}$$

we have $g=g_1g_3=g_1g_4=g_2g_4$, and hence $g_1=g_2$ and $g_3=g_4$.
Thus $bx=(g_2)_{ik}\circ(g_3)_{sj}=(g_2p_{ks}g_3)_{ij}=(g_2g_3)_{ij}=(g_2g_4)_{ij}=(g)_{ij}=m$.
The proof of (ii) for the general case is analogous.

(iii) Define a horizontal section of the prism $S(IxVxG)=((p_{ij}^{-1})_{ji}:p_{ij}$ is an entry of P). It is easy to show that S(IxVxG) is a subsemigroup of S. By the definition of S(IxVxG), S(IxVxG) intersects H-class of S in exactly one element.
This completes the proof of Theorem 3.5.

Definition 3.4. Let $S=M^{o}(G;I,V;P)$.

(i) Let $V=(v_i:i=1,2,..,n)$ be a set of vertices of a closed polygonal line Q of the multiplication table of the semigroup S. We denote by $R=(i_2,i_3,...i_n)$ a permutation on the set of integers $(2,3,.,n)$, and $(i_2,i_3,..,i_n)=R$ shall be called a path of Q.

(ii) Define a product $T(R)=v_1v_{i_2}...v_{i_n}v_1$. Define a path number $Z(R)$ of a closed polygonal line Q in a multiplication table with a vertex set $V=(v_i : i=1,2,..,n)$ of a polygone of the Q by the following way that, for a product $T((i_1,i_2,..,i_n))=(a_1)_{j_1k_1}o$
$(a_2)_{j_2k_2}o...o(a_n)_{j_nk_n}o(a_1)_{j_1k_1}$,
$Z((i_2,i_3,.,i_n))=Z(R)=(k_1-1)(j_2-1)+(k_2-1)(j_3-1)+...$
$..+(k_n-1)(j_1-1)$,
where $v_i=(a_i)_{j_ik_i}$.

(iii) A Rees matrix semigroup S with zero is said to be g-polygonal if every product $T((i_2,i_3,.,i_n))=v_1v_{i_2}..v_{i_n}v_1$ of n+1 vertices $(v_i:i=1,2,.,n)$ with a path $R=(i_2,i_3,..,i_n)$, and with the first and the last factors v_1, of a closed polygonal line Q(with a finite number of vertices) of the multiplication table, is a function of the path number $Z(R)$ defined above.

Theorem 3.6. Let P be a g-cellular matrix of a Rees matrix semigroup $S=M^{o}(G;In,Im;P)$ over a finitely generated abelian group with zero. Let g be a nonzero element of the group G of order m, where m is a positive integer greater than 1. Let $g^{-1}=q$.

(i) $S \approx M^{o}(G;In,Im;P')$, where P' is given by

$$P' = \begin{pmatrix} e & e & e & \cdots & e \\ e & q & q^2 & \cdots & q^{n-1} \\ e & q^2 & q^4 & \cdots & q^{2(n-1)} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ e & q^{m-1} & \cdot & \cdots & q^{(m-1)(n-1)} \end{pmatrix}.$$

(ii) $K=((g^u p_{j1}^{-1})_{ij} : i \in In,\ j \in Im,\ u=1,2,\ldots,m-1)$ is a subsemigroup of S and it is a union of m horizontal section $S(In \times Im \times G)$.

(iii) S is g-polygonal.

Proof. (i) Let $P=(p_{ij})$. We shall show that

$$P=\begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots p_{1n} \\ p_{21} & p_{21}p_{11}^{-1}p_{12}q & p_{21}p_{11}^{-1}p_{13}q^2 & \cdots p_{21}p_{11}^{-1}p_{1n}q^{n-1} \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ p_{m1} & p_{m1}p_{11}p_{12}q^{m-1} & p_{m1}p_{11}p_{13}q^{2(m-1)} & \cdots p_{m1}p_{11}p_{1n}q^{(n-1)(m-1)} \end{pmatrix}$$

By the definition of a cellular matrix P, it follows from $p_{11}^{-1}p_{12}p_{22}^{-1}p_{21}=g$ that $p_{22}=p_{21}p_{11}^{-1}p_{12}q$. Notice that G is abelian. Let v be a fixed positive integer such that $4 \leq v \leq m+n$. Assume that $p_{hk}=p_{h1}p_{11}^{-1}p_{1k}q^{(h-1)(k-1)}$, for every $h+k \leq v$. Since P is cellular, we have

$p_{h-1k}^{-1}p_{h-1k+1}p_{hk+1}^{-1}p_{hk}=g$, from this

$p_{hk+1}=p_{hk}p_{h-1k}^{-1}p_{h-1k+1}q=p_{h1}p_{11}^{-1}p_{1k}q^{(h-1)(k-1)}(p_{h-11}\cdot$
$p_{11}^{-1}p_{1k}q^{(h-2)(k-1)})^{-1}(p_{h-11}p_{11}^{-1}p_{1k+1}q^{(h-2)(k)})q=$
$=p_{h1}p_{11}^{-1}p_{1k+1}q^{(h-1)(k-1)-(h-2)(k-1)+(h-2)k+1}$
$=p_{h1}p_{11}^{-1}p_{1k+1}q^{(h-1)k}$.

Thus we have proved that P takes the form indicated above by induction.
Now choose two diagonal Im x Im and In x In matrices

$$V = \begin{pmatrix} e & & & & & \\ & p_{11}p_{21}^{-1} & & & \bigcirc & \\ & & p_{11}p_{31}^{-1} & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & \bigcirc & & & & p_{11}p_{m1}^{-1} \end{pmatrix}$$

and

$$U = \begin{pmatrix} p_{11}^{-1} & & & & \\ & p_{12}^{-1} & & \bigcirc & \\ & & p_{13}^{-1} & & \\ & \bigcirc & & \cdot & \\ & & & & p_{1n}^{-1} \end{pmatrix}.$$

Then

$$VPU = \begin{pmatrix} e & e & e & \cdot & \cdot & \cdot & e \\ e & q & q & \cdot & \cdot & \cdot & q^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ e & q^{m-1} & q^{2(m-1)} & & \cdot & \cdot & q^{(m-1)(n-1)} \end{pmatrix}$$

By Theorem 3.6 [1], we have that S is isomorphic with $M^{o}(G;In,Im;P')$, where $P'=VPU$.

(ii) Let $K=((g^{u}p_{ji}^{-1})_{ij}$: i in In, j in Im, u=1,2, ,..,m). Then it is easy to check that K is a subsemigroup of S. Since g is an element of order m in the group G, for each coordinate (i,j) in In x Im, the set K has precisely m elements. Thus K is a union of m distinct horizontal sections S(In x Im x G).

(iii) Let $V=((a_i)_{h_ik_i}:i=1,2,..,n)$ be a set of n vertices of a polygonal line in a multiplication table of the semigroup S.
Let $R_1=(h_2k_2,..,h_nk_n)$ be a path of the polygonal line Q. Then $T(R_1)=T((h_2k_2,..,h_nk_n))=$

$=(a_1)_{h_1k_1}o(a_2)_{h_2k_2}o\cdots o(a_n)_{h_nk_n}o(a_1)_{h_1k_1}$

$=(a_1a_2\cdots a_na_1(p_{11}^{-1})^n p_{k_11}p_{k_21}\cdot\cdot p_{k_n1}p_{1h_1}p_{1h_2}\cdot\cdot p_{1h_n}\cdot q^{(k_1-1)(h_2-1)+\cdots\cdots+(k_n-1)(h_1-1)})_{h_1k_1}$

$=(mq^{Z(R_1)})_{h_1k_1}$,

where $m=a_1a_2\cdot\cdot a_na_1(p_{11}^{-1})^n p_{k_11}p_{k_21}\cdot\cdot p_{k_n1}p_{1h_1}p_{1h_2}\cdot\cdot p_{1h_n}$.

Let $R_i=(s_2t_2,..,s_nt_n)$ be an arbitrary path on $(h_2k_2,..,h_nk_n)$.

Consider $T(R_i)=(a_1)_{h_1k_1}o(b_2)_{s_2t_2}\cdot\cdot o(b_n)_{s_nt_n}o(a_1)_{h_1k_1}$

$=(a_1a_1b_2b_3\cdot\cdot b_np_{k_1s_2}p_{t_2s_3}\cdot\cdot p_{t_nh_1})_{h_1k_1}$

$=(a_1a_1a_2\cdots a_n(p_{11}^{-1})^n p_{k_11}p_{t_21}\cdot\cdot p_{t_n1}p_{1h_1}\cdot p_{1s_n}q^{\cdots\cdots})_{h_1k_1}$

$=(a_1a_1a_2\cdot\cdot a_n(p_{11}^{-1})^n p_{k_11}p_{k_21}\cdot\cdot p_{k_n1}p_{1h_1}\cdot\cdot p_{1h_n}q^{(k_1-1)\cdot(s_2-1)+(t_2-1)(s_3-1)+\cdot\cdot+(t_n-1)(h_1-1)})_{h_1k_1}$.

Since $(s_i,t_i)=(h_j,k_j)$ for some $j=2,3,..,n$, for each i.
Thus we have $T(R_i)=(mq^{Z(R_i)})_{h_1k_1}$.
This completes the proof of Theorem 3.6.

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Abstract

Rees matrix semigroups over special structure groups with zero

by Jin Bai Kim

Let S be a semigroup with zero and let a S\0. Denote by V(a) the set of all inverses of a, that is, $V(a)=(x\in S$: $axa=a$, $xax=x)$. Let n be a fixed positive integer. A semigroup S with zero is said to be homogeneous n regular if the cardinal number of the set V(a) of all inverses of a is n for every nonzero element a in S. Let T be a subset of S. We denote by E(T) the set of all idempotents of S in T.
The next theorem is a generalization of R. McFadden and Hans Schneider's theorem [1] .

Theorem 1. Let S be a 0-simple semigroup and let n be a fixed positive integer. Then the following are equivalent.
(i) S is a homogeneous n regular and completely 0-simple semigroup.
(ii) For every $a\neq 0$ in S there exist precisely n distinct nonzero elements $(x_i)_{i=1}^{n}$ such that $ax_ia=a$ for $i=1,2,..,n$ and for all c, d in S $cdc=c\neq 0$ implies $dcd=d$.
(iii) For every $a\neq 0$ in S there exist precisely h distinct

nonzero idempotents $(e_i)_{i=1}^{h}=E_a$ and k distinct nonzero idempotents $(f_j)_{j=1}^{k}=F_a$ such that $e_ia=a=af_j$ for $i=1,2,...,h$, $j=1,2,..,k$, $hk=n$, E_a contains every nonzero idempotent which is a left unit of a, F_a contains every nonzero idempotent which is a right unit of a and $E_a \cap F_a$ contains at most one element.

(iv) For every $a\neq 0$ in S there exist precisely k nonzero principal right ideals $(R_i)_{i=1}^{k}$ and h nonzero principal left ideals $(L_j)_{j=1}^{h}$ such that R_i and L_j contain h and k inverses of a, respectively, every inverse of a is contained in a suitable set $R_i \cap L_j$ for $i=1,2,..,k$, $j=1,2,..,h$, and $R_i \cap L_j$ contains at most one nonzero idempotent, where $hk=n$.

(v) Every nonzero principal right ideal R contains precisely h nonzero idempotents and every nonzero principal left ideal L contains precisely k nonzero idempotents such that $hk=n$, and $R \cap L$ contains at most one nonzero idempotent.

(vi) S is completely 0-simple. For every 0-minimal right ideal R there exist precisely h 0-minimal left ideals $(L_i)_{i=1}^{h}$ and for every 0-minimal left ideal L there exist precisely k 0-minimal right ideals $(R_j)_{j=1}^{k}$ such that $LR_j=L_iR=S$, for every $i=1,2,.,h$, $j=1,2,..,k$, where $hk=n$.

(vii) S is completely 0-simple. Every 0-minimal right ideal R of S is the union of a right group with zero G^0, a union of h disjoint groups except zero, and a zero subsemigroup Z which annihilates the right ideal R on the left and every 0-minimal left ideal L of S is the union of a left group with zero G'^0, a union of k disjoint groups except zero, and a zero subsemigroup Z' which annihilates the left ideal L on the right and $hk=n$.

(viii) S contains at least n nonzero distinct idempotents, and for every nonzero idempotent e there exists a set E of n distinct nonzero idempotents of S such that eE is a right zero subsemigroup of S containing precisely h nonzero idempotents, Ee is a left zero subsemigroup of S containing precisely k nonzero idempotents of S, $e(E(S)\setminus E)=(0)=(E(S)\setminus E)e$, and $eE\cap Ee=(e)$, where $hk=n$.

S is said to be h-k type if every nonzero principal left ideal of S contains precisely k nonzero idempotents and every nonzero principal right ideal of S contains precisely h nonzero idempotents of S.

W. D. Munn defined the Brandt congruence [2]. A congruence ρ on a semigroup S with zero is called a Brandt congruence if S/ρ is a Brandt semigroup.

Theorem 2. Let S be a 1-n type homogeneous n regular and completely 0-simple semigroup. Define a relation ρ on S in such a way that $a\rho b$ if and only if there exists a set $(e_i)_{i=1}^{n}$ of n distinct nonzero idempotents such that $e_ia=e_ib\neq 0$, for every $i=1,2,.,n$. Then ρ is an equivalence on $S\setminus 0$. If we extend ρ on S by defining (0) to be a ρ-class on S, then ρ is a proper Brandt congruence on S. Furthermore, if σ is any proper Brandt congruence on S, then $\rho\subset\sigma$.

Let $P=(p_{ij})$ be any n x n matrix over a group with zero G^0, and consider any n distinct points $A_1,A_2,.,A_n$ in the plane, which we shall call vertices. For every nonzero entry $p_{ij}\neq 0$ of the matrix P, we connect the vertex A_i to the vertex A_j by means of a path $\overline{A_iA_j}$, which we shall call an edge (a loop if $i=j$) directed from A_i to A_j. In this way, with every n x n matrix P can be associated a finite directed graph G(P).

Let $S=M^0(G;In,In;P)$ be a Rees matrix semigroup. Then the graph G(P) is called the associated graph of the semigroup S, or simply it is the graph G(P) of S.

Theorem 3. A Rees matrix semigroup $S=M^0(G;In,In;P)$ is homogeneous m^2 regular if the directed graph G(P) of the semigroup S is regular of degree m [3, p. 11] .

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