ON AXIALLY SYMMETRIC ELASTIC WAVE PROPAGATION IN A FLUID-FILLED CYLINDRICAL SHELL

by

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Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in candidacy for the degree of

DOCTOR OF PHILOSOPHY

in

Engineering Mechanics

June, 1965

Blacksburg, Virginia

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LIST OF SYMBOLS

Dimensional Quantities

Velocity of sound in the fluid	C_0
Skalak water-hammer velocities	C_1, C_2
Joukowsky water-hammer velocity	C_J
Young's modulus of the shell material	E
Shearing modulus of the shell material	G
Thickness of the shell	H
Bulk modulus of the fluid	K
Moment stress resultants in the shell	M_X, M_θ
Membrane stress resultants in the shell	N_X, N_θ
Pressure in the fluid	P
Average pressure in the fluid	$\bar{P}$
Fluid pressure at the inner surface of the shell	P^*
Transverse shear stress resultant in the shell	Q_X
Radial coordinate	R
Radius of the middle surface of the cylindrical shell	R_0
Time	T
Radial and axial components of displacement of a point in the shell	U_R, U_X

Axial displacement of a point in the middle surface of the shell. . . . . U

Radial and axial components of velocity of the fluid . . . . V_R, V_X

Average axial velocity component of the fluid. . . $\bar{V}_X$

Radial displacement of a point in the middle surface of the shell. . . . . W

Axial coordinate X

Radial distance of a point from the middle surface of the shell ($Z=R-R_0$) . . . . Z

Density of the fluid and of the shell material respectively. . . . . ρ_f, ρ_s

Components of stress in the shell. . . . . σ_{XX}, σ_{RR}, $\sigma_{\theta\theta}$, σ_{RX}

Velocity potential for the fluid . . . . Φ

Dimensionless Quantities

$c_1 = C_1/C_0$, $c_2 = C_2/C_0$ . . . . c_1, c_2

$c_p = C_p/C_0$ . . . . c_p

Velocity of propagation of shear disturbances according to the Herrmann-Mirsky shell theory ($c_s = \sqrt{\kappa^2 g/\rho}$). . . . . c_s

$e = E/K$ . . . . e

$g = G/K$ . . . . g

$h = H/R_0$ . . . . h

Bessel functions . . . . J_0, J_1

$m_x = \dfrac{M_X}{KR_0^2}$ . . . . m_x

$n_x = \dfrac{N_X}{KR_0}$, $n_\theta = \dfrac{N_\theta}{KR_0}$ . . . . n_x, n_θ

$p = \dfrac{P}{K}$ . . . . p

$\bar{p} = \dfrac{\bar{P}}{K}$ . . . . $\bar{p}$

$p^* = \dfrac{P^*}{K}$ . . . . p^*

$p_j = -\dfrac{\partial \psi_j}{\partial t}$ . . . . p_j

$q_x = \dfrac{Q_X}{KR_0}$ . . . . q_x

$r = R/R_0$ . . . . r

$t = C_0T/R_0$ . . . . t

$u = U/R_0$ . . . . u

$v_r = V_R/C_0$, $v_x = V_X/C_0$ . . . . v_r, v_x

$\bar{v}_x = \bar{V}_X/C_0$ . . . . $\bar{v}_x$

$v_j = \dfrac{\partial \psi_j}{\partial x}$ . . . . v_j

$w = W/R_0$ . . . . w

$x = X/R_0$ x

Non-negative roots of $\gamma J_0' [\gamma(1-h/2)] = 0$. . . .	γ_j
See Figure 15	(δt), $(\delta t)_f$, $(\delta t)_s$
Components of strain in the shell	ϵ_{XX}, ϵ_{RR}, $\epsilon_{\theta\theta}$, ϵ_{RX}
Dimensionless phase velocity.	η
Cylindrical coordinate.	θ
Constant depending upon Poisson's ratio	κ^2
Dimensionless wavelength.	λ
Poisson's ratio for the shell material.	ν
Dimensionless wave number.	ξ
$\rho = \rho_s/\rho_f$	ρ
$\varphi = \Phi/R_0C_0$	φ
Finite Hankel Transform of φ.	χ_j
See page 56	ψ_j
Dimensionless frequency	ω

INTRODUCTION

General Statement of the Problem

The dynamic behavior of a structure is often influenced by the interaction of the structure with a fluid which constitutes part of its environment. Conversely, motion of the fluid medium is influenced by motion of the structure. The manner in which this coupling is included in the analysis of an engineering problem depends upon the goal of the analysis; the structural analyst is not generally interested in the details of the fluid motion, while the acoustician often is not concerned with the details of the motion of a structure which serves as a source of acoustic waves. Moreover, certain problem parameters, mass and elastic properties of the fluid and of the structure, govern the engineering significance of the coupling. Hence many studies of the vibrations of structures surrounded by air are conducted by ignoring the fluid medium, and many problems in fluid mechanics may be adequately treated by ignoring the flexibility of an adjacent structure. On the other hand the field of aeroelasticity is devoted to the study of the interaction between structures and air, and if the fluid involved is a liquid, coupling between the fluid and a structure is even more likely to be significant. Therefore a substantial body of literature is devoted to problems involving the interaction of water and naval structures [1][1].

This thesis is concerned with a particular structure-fluid interaction problem, transient axially symmetric wave propagation in an elastic cylindrical shell filled with an inviscid compressible fluid. Water hammer phenomena have provided the principal stimulus for studies of this nature, although certain problems of naval structural mechanics fall into this category. For example, blast waves from underwater explosions can be propagated into elements of a ship's piping system which are open to the sea. Hence, although particular attention will be given here to water hammer problems, the analysis presented in this thesis may be adapted easily to a larger class of wave propagation problems.

Review of the Literature

The elementary theory of water hammer, developed by

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[1]Numbers in brackets refer to the Bibliography.

Joukowsky [2], predicts the propagation of longitudinal pressure waves, without change of shape (no dispersion), at a velocity

$$C_J = \frac{C_0}{\sqrt{1 + \frac{K}{Eh}}}$$

where C_0 is the velocity of propagation of acoustic disturbances in an infinite body of fluid, K is the bulk modulus of the fluid, E is Young's modulus for the shell material, and h is the thickness to radius ratio of the cylindrical shell. This theory was developed by neglecting longitudinal stresses (bending and membrane) and shearing stresses in the shell and by neglecting the inertial forces associated with radial motion of the shell and radial motion of the fluid. Linearized (acoustic) governing equations were used to describe motion of the fluid. The model predicts that the fluid pressure is uniform over any section taken normal to the axis of the tube and that the shell experiences radial deflections corresponding to the instantaneous pressure in the fluid and predicted by the statical membrane theory of shells. Hence this elementary formulation predicts that a sharp disturbance generated in the fluid is accompanied by discontinuous radial displacements in the shell.

The elementary theory, which predicts that disturbances are governed by the one-dimensional wave equation, has been applied to problems of water hammer in complex systems with a great deal of success over the last fifty years. Parmakian [3] has presented many important results in a monograph, and recently there have been extensions of the theory to include consideration of fluid friction [4] and the effects of elastic material surrounding the tube [5]. In addition, Wood and Stelson [6] have presented an analysis based on energy, rather than momentum, considerations to account for nonuniform fluid particle velocities over a cross section normal to the axis of the tube. These extensions of the Joukowsky theory, however, are all based essentially upon the elementary formulation and do not take radial inertia of the fluid into account explicitly; nor do they provide a more palatable formulation of the behavior of the shell.

Application of the elementary water hammer theory has been paralleled in the literature by the development of governing equations which are more meaningful physically, but less tractable mathematically, than those given by the elementary formulation. In fact Joukowsky's work was preceded by a study by Korteweg [7] in 1878 in which the radial inertias associated with the fluid and shell motions were included. The solution of transient problems according to this theory is difficult, since it predicts that waves continually change in shape (dispersion) as they are propagated. However a particular solution, corresponding to an infinite train of sinusoidal waves, is obtained easily. The phenomenon of dispersion is here manifested in that the phase velocity for the propagation of a steady train of sinusoidal longitudinal waves is predicted to be dependent on the wavelength. Korteweg's formulation yields the Joukowsky velocity, C_J, as the only finite long-wavelength asymptote of the phase velocity.

In 1898 Lamb [8] contributed to the theory by extending Korteweg's formulation to account for longitudinal stresses (membrane)and longitudinal inertia in the tube. Hence motion of the shell was taken to be governed by the dynamical equations of the classical membrane theory for a cylindrical shell. For the propagation of a steady train of sinusoidal waves, Lamb's theory predicts two long-wavelength asymptotes of phase velocities. The smaller, C_l, of the two, associated with the first branch of the frequency equation, is very close to the Joukowsky velocity, and the larger, C_2, associated with the second branch of the frequency equation, is very close to the so-called "plate velocity".

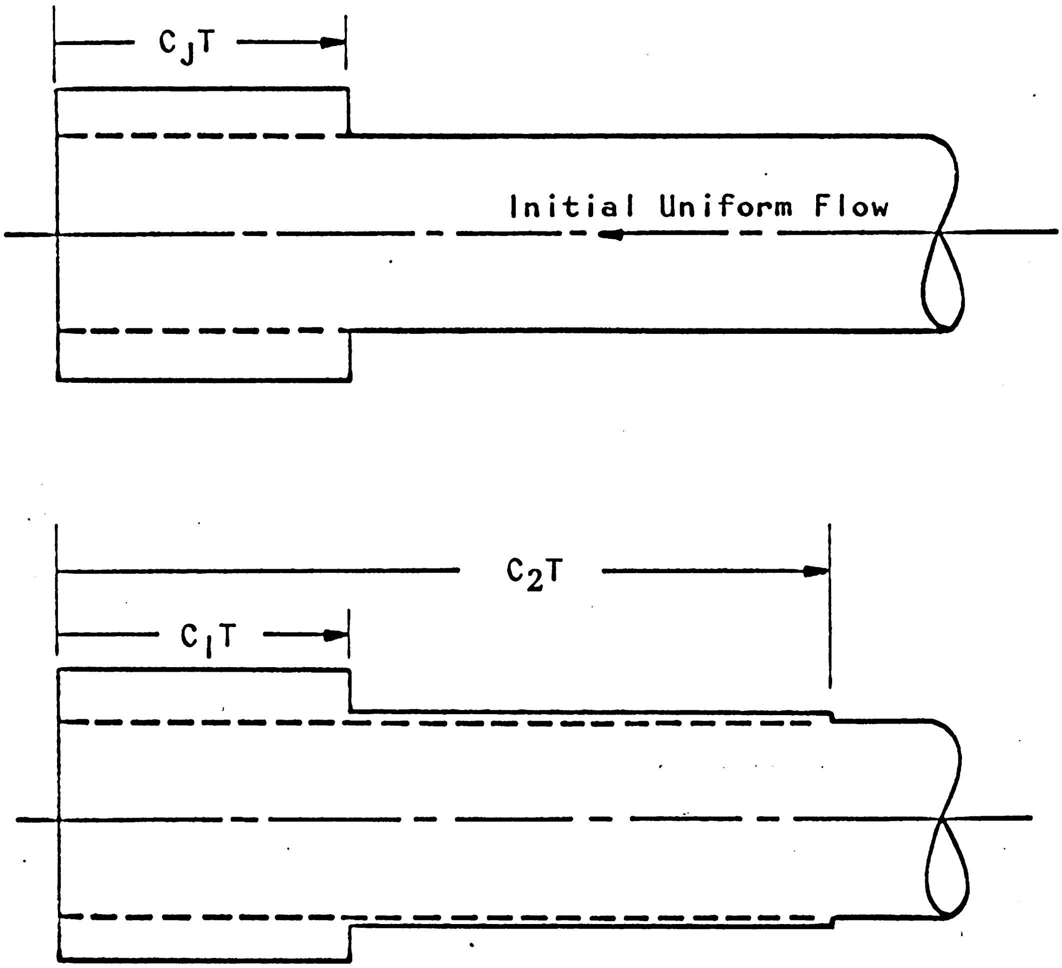
$$C_P = \sqrt{\frac{E}{\rho_s(1-\nu^2)}} \quad ,$$

where ρ_s is the density of the shell material and ν is Poisson's ratio for the shell material. Among modern contributions to the study of the propagation of sinusoidal waves in a fluid-filled cylinder are the work of Thomson [9], which included the effect of bending stresses in the shell, and the work of Lin and Morgan [10], which included the effects of rotary inertia and shear deformation in the shell. These effects were found to be of importance only for the short wavelengths. A recent monograph by Redwood [11] contains a rather complete bibliography of similar studies.

Of particular significance to this thesis is a paper by Skalak [12]. He found that, if the elementary water hammer theory is modified by including longitudinal membrane stresses and longitudinal inertia in the shell, the modified theory predicts the propagation of two waves without change of shape. The propagation velocities are the long-wavelength velocities, C_1 and C_2, predicted by Lamb, Skalak's extension of the Joukowsky theory being equivalent to neglecting radial inertia in Lamb's formulation. For the sudden extinction of a uniform flow at the end of a semi-infinite tube, two sharp waves of radial deflection, pressure, etc. are propagated along the axis of the shell. Figure 1 shows qualitatively the radial deflection predicted by the Joukowsky theory and by Skalak's extension. The wave propagated at velocity C_2 is essentially a longitudinal tension wave, and the stresses, radial displacements and pressures associated with it are so small that they have negligible engineering significance.

Skalak also analyzed a transient problem according to a more complete theory in which motion of the fluid was governed by the axisymmetric wave equation. Motion of the shell was described by equations which accounted for radial, longitudinal, and rotary inertia as well as bending and membrane stresses. The solution to a water hammer problem was obtained formally by multiple integral transforms. The transformed solution was inverted approximately to yield an asymptotic solution valid for points sufficiently distant from the end of the shell and after a sufficient amount of time has elapsed following initiation of the disturbance. From this long-time solution Skalak concluded the following:

(a) Joukowsky Theory



(b) Skalak Theory

——— Deformed Tube

----- Undeformed Tube

Figure 1

Predictions of Water Hammer

by Elementary Theories

"1. If the initial conditions specify a sharp wave front of pressure, this front gradually will disperse over a finite length as it proceeds. This dispersion continues indefinitely so that no steady wave shape is ever reached.

2. Two waves will be generated in general; one which is similar to the Joukowsky result and one which is essentially a tension wave in the pipe wall. The pressure change due to this precursor is very small.

3. After a sufficiently long time has elapsed since the generation of a wave there will be one point in each wave front which has a constant deflection and fluid pressure and moves at a constant velocity. This velocity is very close to the Joukowsky velocity for the main pressure wave. For the precursor this velocity is slightly less than the velocity of sound in the pipe wall.

4. The bending stresses in the pipe wall have very little effect on the characteristic velocities and rates of dispersion after a sufficient time has elapsed. Such stresses can only be significant in the early stages of the motion in regions where the wave front is sharp so that variation of pressure is very rapid along the length of the pipe.

5. The relations between fluid velocity, pressure, and hoop stress given by the Joukowsky theory are substantially correct, but there are also longitudinal stresses in the pipe not predicted by the usual theory."

Skalak's long-time solution indicates that dispersive effects are essentially local (near the wave fronts), and at great distances from the fronts the asympototic solution agrees with Skalak's extension of the Joukowsky theory. In addition radial inertia of the fluid and of the shell were determined to have the most significant influences on the dispersion. In fact Skalak's long-time solution shows explicitly no other dispersive effects.

The weaknesses of the analysis described above may be summarized as:

1. The integral transforms used by Skalak do not allow the imposition of constraints on the radial motion of the end of the shell. It was assumed by Skalak that such constraints would have a negligible influence upon the water hammer disturbance at large distances from the end of the shell.
2. The solution is not valid during the early stages of development of the disturbance.
3. The solution is not applicable near the end of the tube where the disturbance is generated.

The last two comments are of particular significance to the structural analysis of the shell, since it is reasonable to expect maximum stresses to occur during the early stages of the motion, before dispersion has materially affected the motion. Moreover, if the radial motion of the end of the shell is inhibited by external constraints, stresses near the end of the shell are likely to be of interest to the structural analyst. This thesis will be devoted, in part, to the inadequacies of Skalak's solution of the water hammer problem.

It should be noted that many of the difficulties encountered in the analysis of the water hammer problem are also encountered in analyses of the transient motions of cylindrical shells in vacuo. Berkowitz [13] has recently considered the problem of longitudinal impact of a semi-infinite cylindrical shell within the framework of the classical membrane theory of shells. A formal solution to the problem was obtained by the use of the Laplace transform, but inversion could only be accomplished asymptotically for large values of time. Spillers [14] showed that a short-time solution can be obtained easily if the governing partial differential equations are reduced, by the method of characteristics, to a system of ordinary differential equations and integrated numerically.

The response of a thin cylindrical shell to a moving pressure wave has been investigated recently by Bhuta [15] and by Tang [16]. Bhuta considered a simply-supported shell of finite length according to the classical bending theory of shells, and, by neglecting longitudinal membrane stresses, he was able to obtain a solution, in the form of an infinite trigonometric series, valid for all values of time. Bhuta's analysis can be adapted easily to other end conditions for the shell; however if longitudinal membrane stresses are included, the formulation does not lend itself to a series solution except in the special cases for which the modes of free vibration can be determined by inspection to be described by trigonometric functions. On the other hand, Tang found that if rotary inertia and shear deformation are included in the formulation of the equations of motion of the shell, then the governing equations can be integrated, by utilizing the method of characteristics, for any of the commonly used engineering constraints on the motion of the end, or ends, of the shell. Tang considered in detail the response of a semi-infinite shell to a moving pressure wave.

In summary, the literature of water hammer and related problems is rather exhaustive with regard to steady state problems and with regard to asymptotic solutions valid long after a disturbance is generated. However, the literature is notably deficient with regard to the early stages of development of a disturbance, when stresses in the shell are likely to be the largest.

Scope of the Investigation

A water hammer problem, typical of a larger class of problems, is to be considered here in detail. The sudden termination of a uniform flow at the end of a semi-infinite tube is to be investigated. A thin-shell theory will be utilized, and linearized (acoustic) equations governing the fluid motion will be employed. The point of view taken in this study will be that of the structural analyst; that is, the details of the fluid motion will be of secondary importance.

A short-time solution to the water hammer problem will be obtained, and the analysis will be applicable for all of the usual engineering boundary conditions which describe external constraints at the end of the shell.

GOVERNING EQUATIONS AND STEADY STATE SOLUTIONS

Governing Equations for the Shell Motion

The axially symmetric motion of a thin elastic cylindrical shell will be described by equations which include the effects of rotary inertia and shear deformation. Such equations have been developed by Herrmann and Mirsky [17], by Lin and Morgan [18], and by Naghdi and Cooper [19]. We shall utilize here the development of Herrmann and Mirsky.

It is assumed that motion of the shell can be analyzed within the framework of classical linear elasticity for a homogeneous isotropic material. For axially symmetric motion the equations of motion in cylindrical coordinates (R, θ, X) are [20] :

$$\frac{\partial \sigma_{RR}}{\partial R} + \frac{\partial \sigma_{RX}}{\partial X} + \frac{1}{R} \left(\sigma_{RR} - \sigma_{\theta\theta} \right) = \frac{\partial^2 U_R}{\partial T^2} \qquad (1a)$$

$$\frac{\partial \sigma_{RX}}{\partial R} + \frac{\partial \sigma_{XX}}{\partial X} + \frac{1}{R} \sigma_{RX} = \frac{\partial^2 U_X}{\partial T^2} \qquad (1b)$$

where U_R and U_X are components of displacements, σ_{RR}, σ_{RX}, σ_{XX}, and $\sigma_{\theta\theta}$ are stress components, ρ_s is the density of the shell material and T is time. The strain-displacement relations are:

$$\epsilon_{RR} = \frac{\partial U_R}{\partial R} \qquad (2a)$$

$$\epsilon_{XX} = \frac{\partial U_X}{\partial X} \qquad (2b)$$

$$\epsilon_{RX} = \frac{1}{2}\left(\frac{\partial U_X}{\partial R} + \frac{\partial U_R}{\partial X}\right) \qquad (2c)$$

and the stress-strain relations are given by:

$$E \quad \epsilon_{XX} = \sigma_{XX} - \nu\,(\sigma_{RR} + \sigma_{\theta\theta}) \qquad (3a)$$

$$E \quad \epsilon_{\theta\theta} = \sigma_{\theta\theta} - \nu\,(\sigma_{XX} + \sigma_{RR}) \qquad (3b)$$

$$E \quad \epsilon_{RR} = \sigma_{RR} - \nu\,(\sigma_{XX} + \sigma_{\theta\theta}) \qquad (3c)$$

$$2G\,\epsilon_{RX} = \sigma_{RX} \qquad (3d)$$

where E is Young's modulus, G is the shearing modulus, and ν is Poisson's ratio.

Attention is now focused upon a circular cylindrical shell of mean radius R_0 and constant thickness H. With $Z = R - R_0$, the displacement components U_R (R, X, T) and U_X (R, X, T) are approximated by

$$U_R = W(X, T)$$

and $$U_X = U(X, T) + Z\beta(X, T),$$

where β is the angular displacement of a line fixed in the shell and normal to the middle surface of the undeformed shell. If Equation (1a) is multiplied by $(R_0 + Z)\,dZ$ and integrated from $Z = -H/2$ to $Z = +H/2$, we obtain

$$\frac{\partial Q_X}{\partial X} - \frac{1}{R_0} N_\theta - \rho_s H \frac{\partial^2 W}{\partial T^2} + (1 - H/2R_0) \quad P^* = 0$$

where P* is the pressure acting on the inner surface of the shell (the outer surface is assumed to be stress free and the inner surface is assumed free of shearing stresses). The stress resultants, Q_X and N_θ , are defined by

$$Q_X = \int_{-H/2}^{H/2} \sigma_{RX} (1 + Z/R_0) \, dZ$$

and
$$N_\theta = \int_{-H/2}^{H/2} \sigma_{\theta\theta} \, dZ \; .$$

A similar integration of Equation (1b) yields

$$\frac{\partial N_X}{\partial X} = \int_{-H/2}^{H/2} \rho_s (1 + Z/R_0) \left(\frac{\partial^2 U}{\partial T^2} + Z \frac{\partial^2 \beta}{\partial T^2} \right) dZ,$$

where the stress resultant N_X is given by

$$N_X = \int_{-H/2}^{H/2} (1 + Z/R_0) \, \sigma_{XX} \, dZ \; .$$

For a thin shell $Z/R_0 << 1$; hence the term Z/R_0 will be neglected compared to unity in the integration of the approximated longitudinal acceleration. This approximation will be employed several times in the development of the governing equations, and it marks the only deviation from the presentation of Herrmann and Mirsky. Thus the integrated form of Equation (1b) is

$$\frac{\partial N_X}{\partial X} = \rho_s \, H \, \frac{\partial^2 U}{\partial T^2}$$

Finally if Equation (1b) is multiplied by $Z\,(R_0 + Z)\,dZ$ and integrated from $Z = -H/2$ to $Z = +H/2$, we obtain

$$\frac{\partial M_X}{\partial X} - Q_X = \frac{\rho_s \, H^3}{12} \, \frac{\partial^2 \beta}{\partial T^2} \quad ,$$

where again Z/R_0 has been neglected compared to unity in the integrals involving the longitudinal acceleration, and the stress resultant M_X is given by

$$M_X = \int_{-H/2}^{H/2} Z(1 + Z/R_0)\, \sigma_{XX} \, dZ \quad .$$

The geometry of the shell and the stress resultants which act on an element of the shell are shown in Figure 2. The stress resultant M_θ which does not appear explicitly in the equations of motion is defined by

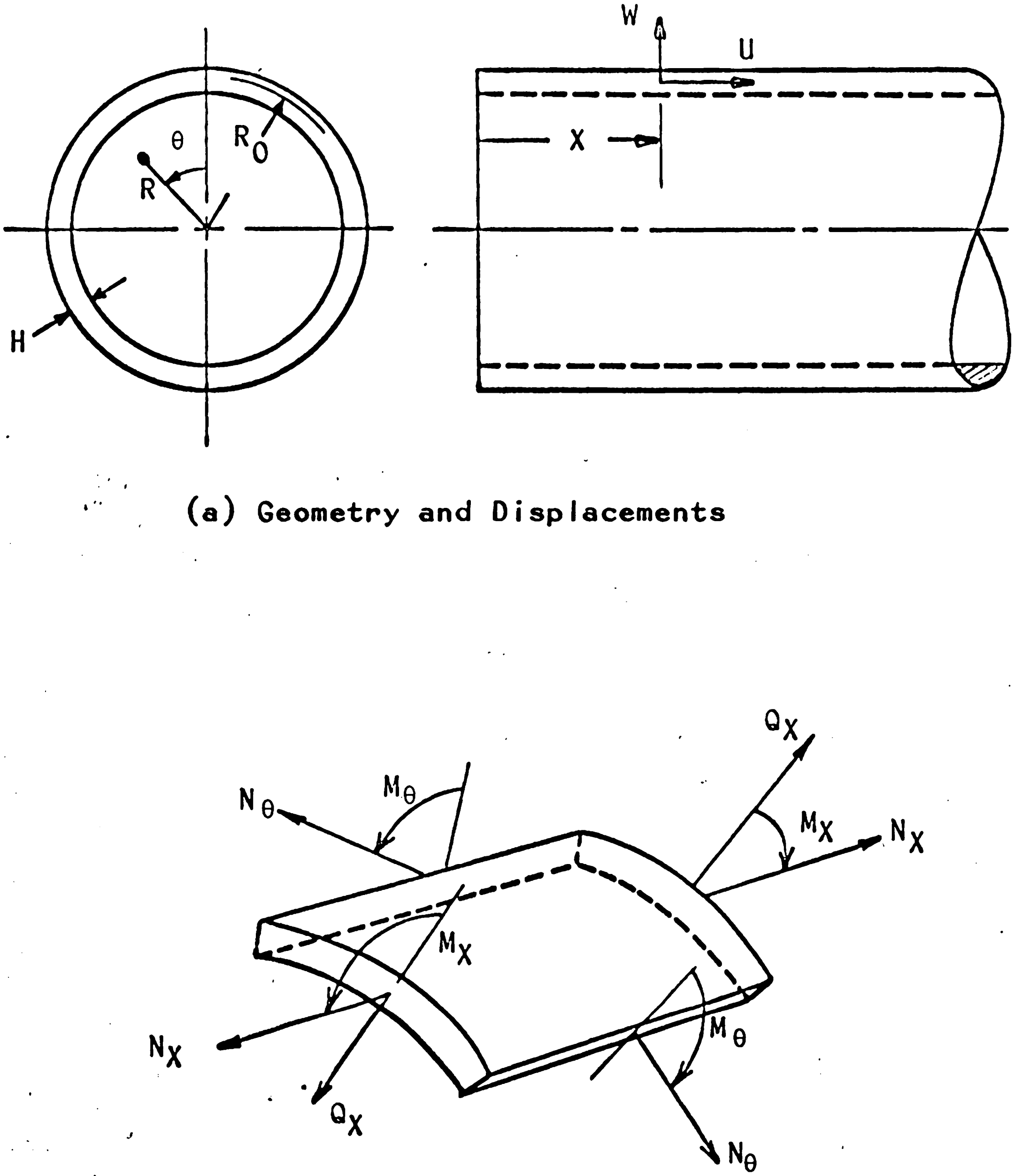
$$M_\theta = \int_{-H/2}^{H/2} Z \, \sigma_{\theta\theta} \, dZ \quad .$$

If Equations (3a) and (3b) of the stress-strain relations are solved for σ_{XX} and $\sigma_{\theta\theta}$, there result

$$\sigma_{XX} = \frac{E}{1-\nu^2} [\epsilon_{XX} + \nu \, \epsilon_{\theta\theta}] + \frac{\nu}{1-\nu} \, \sigma_{RR}$$

and $$\sigma_{\theta\theta} = \frac{E}{1-\nu^2} [\epsilon_{\theta\theta} + \nu \epsilon_{XX}] + \frac{\nu}{1-\nu} \, \sigma_{RR} \quad .$$

In terms of the approximate displacements the stress-strain



(a) Geometry and Displacements

(b) Stress Resultants

Figure 2

Variables Associated with the Shell

relations are

$$\sigma_{XX} = \frac{E}{1-\nu^2}\left[\frac{\partial U}{\partial X} + Z\frac{\partial \beta}{\partial X} + \frac{\nu W}{(R_0 + Z)}\right] + \frac{\nu}{1-\nu}\sigma_{RR}$$

and $$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left[\frac{W}{(R_0 + Z)} + \nu\left(\frac{\partial U}{\partial X} + Z\frac{\partial \beta}{\partial X}\right)\right] + \frac{\nu}{1-\nu}\sigma_{RR}.$$

Integration of these relations over the thickness of the shell yields the stress resultant-displacement relations,

$$N_\theta = \frac{EH}{1-\nu^2}\left[\frac{W}{R_0} + \nu\frac{\partial U}{\partial X}\right],$$

$$N_X = \frac{EH}{1-\nu^2}\left[\frac{\partial U}{\partial X} + \nu\frac{W}{R_0}\right],$$

and $$M_X = \frac{EH^3}{12(1-\nu^2)}\frac{\partial \beta}{\partial X}.$$

where the integrals involving σ_{RR} have been neglected, and Z/R_0 has been neglected again in comparison to unity in the integrals involving the approximate displacements.

The shearing strain, ϵ_{RX}, in terms of the approximate displacements, is

$$2\,\epsilon_{RX} = \frac{\partial W}{\partial X} + \beta .$$

A correction factor, κ^2, is introduced by modifying the above expression to read

$$2\,\epsilon_{RX} = \kappa^2\left(\frac{\partial W}{\partial X} + \beta\right).$$

Integration of the shearing stress-shearing strain relation yields the stress resultant-displacement expression

$$Q_X = \kappa^2 \; GH \left(\frac{\partial W}{\partial X} + \beta \right) .$$

The correction factor, κ^2, is a function of Poisson's ratio, and it is determined by matching the short-wave-length asymptote of the first branch of the phase velocity-wavelength spectrum for the shell theory with that obtained from the three-dimensional theory of elasticity [17]. Similar correction factors have been employed by Lin and Morgan [18] and Naghdi and Cooper [19], but the values used were established by criteria different from that described above. For $\nu = 0.3$, κ^2 was determined to be 0.86 by Herrmann and Mirsky, while the corrections determined by Lin and Morgan and by Naghdi and Cooper were 8/9 and 5/6 respectively.

The governing equations may now be summarized:

$$\frac{\partial N_X}{\partial X} - \rho_s H \frac{\partial^2 U}{\partial T^2} = 0 \tag{4a}$$

$$\frac{\partial Q_X}{\partial X} - \frac{1}{R_0} N_\theta - \rho_s H \frac{\partial^2 W}{\partial T^2} + \left(1 - \frac{H}{2R_0}\right) P^* = 0 \tag{4b}$$

$$\frac{\partial M_X}{\partial X} - Q_X - \frac{\rho_s H^3}{12} \frac{\partial^2 \beta}{\partial T^2} = 0 \tag{4c}$$

$$N_X = \frac{EH}{1-\nu^2} \left(\frac{\partial U}{\partial X} + \frac{\nu W}{R_0} \right) \tag{4d}$$

$$N_\theta = \frac{EH}{1-\nu^2} \left(\frac{W}{R_0} + \nu \frac{\partial U}{\partial X} \right) \tag{4e}$$

$$Q_X = \kappa^2 GH \left(\frac{\partial W}{\partial X} + \beta \right) \tag{4f}$$

$$M_X = \frac{EH^3}{12(1-\nu^2)} \frac{\partial \beta}{\partial X} \quad . \tag{4g}$$

Throughout the remainder of the text these relations will be referred to as the simplified Herrmann-Mirsky equations.

It should be noted that the simplification which we have introduced to the Herrmann-Mirsky development eliminates one form of coupling between radial and longitudinal motions of the shell. This coupling is due to the shape of a constant thickness element of the shell which is bounded by a pair of planes defined by constant values of the axial coordinate, X, and by a pair of radial planes defined by constant values of the angular coordinate, θ . Hence the element is bounded by two cylindrical surfaces, two rectangular surfaces, and two surfaces which approach trapezoids as the angle between the radial planes approaches zero. By neglecting the term Z/R_0 in comparison to unity in integrals involving approximations to the shell displacements, we have ignored the trapezoidal shape of the element faces described above; that is we have treated the trapezoidal faces as plane rectangular surfaces. In other words no distinction has been made between the locus of mass centers of elements, such as the one described above, and the middle surface of the shell.

The classical equations of motion for a thin cylindrical shell (i.e. the equations obtained by adding longitudinal and transverse inertia to the classical bending theory of shells [21]) can be determined by suppressing the effects of shear deformation and rotary inertia in the simplified Herrmann-Mirsky equations. This is effected by neglecting $\frac{\partial^2 \beta}{\partial T^2}$ in Equation (4c), setting $\frac{\partial W}{\partial X} = -\beta$, and ignoring the stress resultant-displacement relation involving Q_X (Equation (4f)). The dynamical equations of the classical bending theory are

$$\frac{\partial N_X}{\partial X} - \rho_s H \frac{\partial^2 U}{\partial T^2} = 0 ,$$

$$\frac{\partial Q_X}{\partial X} - \frac{1}{R_0} N_\theta - \rho_s H \frac{\partial^2 W}{\partial T^2} + \left(1 - \frac{H}{2R_0}\right) P^* = 0 ,$$

$$\frac{\partial M_X}{\partial X} - Q_X = 0 ,$$

$$N_X = \frac{EH}{1-\nu^2} \left[\frac{\partial U}{\partial X} + \nu \frac{W}{R_0}\right],$$

$$N_\theta = \frac{EH}{1-\nu^2} \left[\frac{W}{R_0} + \nu \frac{\partial U}{\partial X}\right],$$

and $$M_X = \frac{-EH^3}{12(1-\nu^2)} \frac{\partial^2 W}{\partial X^2} .$$

Finally the dynamical equations of the classical membrane theory of shells are obtained by ignoring M_X and Q_X:

$$\frac{\partial N_X}{\partial X} - \rho_s H \frac{\partial^2 U}{\partial T^2} = 0$$

$$\frac{1}{R_0} N_\theta + \rho_s H \frac{\partial^2 W}{\partial T^2} - (1-H/2R_0)P^* = 0$$

$$N_X = \frac{EH}{1-\nu^2} \left[\frac{\partial U}{\partial X} + \nu \frac{W}{R_0} \right]$$

$$N_\theta = \frac{EH}{1-\nu^2} \left[\frac{W}{R_0} + \nu \frac{\partial U}{\partial X} \right] .$$

The Acoustic Equations for the Fluid Motion

For axially symmetric motion of a compressible inviscid fluid, the linearized (acoustic) equations of motion are [22]

$$\frac{\partial V_R}{\partial T} + \frac{1}{\rho_f} \frac{\partial P}{\partial R} = 0 \tag{5a}$$

and

$$\frac{\partial V_X}{\partial T} + \frac{1}{\rho_f} \frac{\partial P}{\partial X} = 0 , \tag{5b}$$

and the linearized continuity equation is

$$\frac{1}{K} \frac{\partial P}{\partial T} + \frac{1}{R} \frac{\partial}{\partial R} [R V_R] + \frac{\partial V_X}{\partial X} = 0 , \tag{5c}$$

where V_X and V_R are the axial and radial components of the fluid particle velocity, P is the pressure in the fluid, K is the bulk modulus of the fluid, and ρ_f is the fluid density in the undisturbed (reference) state. Acoustic waves are propagated in an infinite body of fluid at the velocity,

$C_0 = \sqrt{K/\rho_f}$. It is assumed that the fluid is homogeneous and that cavitation does not occur.

If a velocity potential, Φ, is introduced so that

$$V_R = \frac{\partial \Phi}{\partial R} ,$$

$$V_X = \frac{\partial \Phi}{\partial X} ,$$

and $$P = - \rho_f \frac{\partial \Phi}{\partial T} ,$$

then the equations of motion are satisfied identically and the continuity equation becomes the wave equation,

$$\frac{\partial^2 \Phi}{\partial X^2} + \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial \Phi}{\partial R} \right) - \frac{1}{C_0^2} \frac{\partial^2 \Phi}{\partial T^2} = 0 .$$

Compatibility of the fluid and shell motions at the inner surface of the shell requires that

$$[V_R]_{R=R_0-H/2} = \frac{\partial W}{\partial T} , \tag{6}$$

and the fluid pressure at the inner surface of the shell is

$$[P]_{R=R_0-H/2} = - \rho_f \left[\frac{\partial \Phi}{\partial T}\right]_{R=R_0-H/2} = P^* .$$

Note that P^* is the loading which appears in the equation governing the radial motion of the shell. The two relations above are the mathematical expressions for the coupling between the fluid and shell motions.

It is of interest to consider average values of the fluid pressure and axial velocity given by

$$\overline{V_X} = \frac{1}{\pi(R_0-H/2)^2} \int_0^{R_0-H/2} 2\pi R\, V_X(R,X,T)dR$$

$$\overline{P} = \frac{1}{\pi(R_0-H/2)^2} \int_0^{R_0-H/2} 2\pi R\, P(R,X,T)dR \ .$$

If each term of Equations (5b) and (5c) is multiplied by $2\pi R\, dR$ and integrated from $R = 0$ to $R = R_0-H/2$, the following relations result:

$$\frac{\partial \overline{V}_X}{\partial T} + \frac{1}{\rho_f} \frac{\partial \overline{P}}{\partial X} = 0$$

and
$$\frac{1}{K} \frac{\partial \overline{P}}{\partial T} + \frac{\partial \overline{V}_X}{\partial X} + \frac{2}{(R_0-H/2)} \frac{\partial W}{\partial T} = 0 \ .$$

These are the basic equations used to describe the fluid motion in the elementary water hammer theory, except that in the elementary formulation no distinction is made between the middle surface and the inner surface of the shell, i.e. $H/2$ is neglected in comparison to R_0. Hence the inertia associated with radial fluid motion is neglected. A first approximation to the effect of this radial inertia can be constructed by letting

$$V_R = \frac{R}{(R_0-H/2)} \frac{\partial W}{\partial T}$$

which satisfies Equation (6). Hence Equation (5a) yields

$$P = - \frac{\rho_f \quad R^2}{2(R_0-H/2)} \frac{\partial^2 W}{\partial T^2} + F(X,T) \ .$$

Integrating this expression to obtain the average pressure we find that $F(X,T)$ is given by

$$F(X,T) = \overline{P} + \frac{\rho_f \ (R_0-H/2)}{4} \frac{\partial^2 W}{\partial T^2} \quad .$$

Hence the pressure acting on the shell is given by

$$P^* = \overline{P} - \frac{\rho_f \, (R_0-H/2)}{4} \frac{\partial^2 W}{\partial T^2} \quad .$$

If this approximation is used, the radial inertia of the fluid appears explicitly as an additional inertia term in one of the equations governing motion of the shell. Therefore this approximation will be called the virtual mass formulation throughout the remainder of the thesis.

Dimensionless Forms of the Governing Equations

It is convenient to introduce the following dimensionless variables and parameters:

$$r = \frac{R}{R_0} \ , \quad x = \frac{X}{R_0} \ , \ t = \frac{C_0 T}{R_0} \ ,$$

$$p = \frac{P}{K} \ , \quad p^* = \frac{P^*}{K} \ , \ \overline{p} = \frac{\overline{P}}{K}$$

$$v_r = \frac{V_R}{C_0}, \quad v_x = \frac{V_X}{C_0}, \quad \bar{v}_x = \frac{\bar{V}_X}{C_0},$$

$$\varphi = \frac{\Phi}{R_0 C_0}, \quad w = \frac{W}{R_0}, \quad u = \frac{U}{R_0},$$

$$h = \frac{H}{R_0}, \quad \rho = \frac{\rho_s}{\rho_f}, \quad e = \frac{E}{K},$$

$$g = \frac{G}{K}, \quad n = \frac{N_\theta}{KR_0}, \quad n_x = \frac{N_X}{KR_0},$$

$$q_x = \frac{Q_X}{KR_0}, \quad m_x = \frac{M_X}{KR_0^2}.$$

Therefore the fluid motion is governed by

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial \varphi}{\partial r}\right] - \frac{\partial^2 \varphi}{\partial t^2} = 0, \tag{7}$$

and the fluid and shell motions are coupled through the boundary conditions

$$\left[\frac{\partial \varphi}{\partial r}\right]_{r = 1-h/2} = \frac{\partial w}{\partial t} \tag{8a}$$

and $$\left[\frac{\partial \varphi}{\partial t}\right]_{r = 1-h/2} = -p^* . \tag{8b}$$

The dimensionless average pressure and axial fluid velocity must satisfy

$$\frac{\partial \bar{v}_x}{\partial t} + \frac{\partial \bar{p}}{\partial x} = 0 \tag{9a}$$

and $$\frac{\partial \bar{p}}{\partial t} + \frac{\partial \bar{v}_x}{\partial x} + \frac{2}{(1-h/2)}\frac{\partial w}{\partial t} = 0, \tag{9b}$$

and the virtual mass approximation for the effects of radial inertia becomes

$$p^* = \bar{p} - \frac{(1-h/2)}{4} \frac{\partial^2 w}{\partial t^2} . \tag{10}$$

The various sets of equations governing motion of the shell are expressed in dimensionless form as follows:

Simplified Herrmann-Mirsky Theory:

$$\frac{\partial n_x}{\partial x} - \rho h \frac{\partial^2 u}{\partial t^2} = 0 \tag{11a}$$

$$\frac{\partial q_x}{\partial x} - n_\theta - \rho h \frac{\partial^2 w}{\partial t^2} + (1-h/2)p^* = 0 \tag{11b}$$

$$\frac{\partial m_x}{\partial x} - q_x - \frac{\rho h^3}{12} \frac{\partial^2 \beta}{\partial t^2} = 0 \tag{11c}$$

$$n_x = \frac{eh}{1-\nu^2} \left(\frac{\partial u}{\partial x} + \nu w \right) \tag{11d}$$

$$n_\theta = \frac{eh}{1-\nu^2} \left(w + \nu \frac{\partial u}{\partial x} \right) \tag{11e}$$

$$q_x = \kappa^2 \; gh \left(\frac{\partial w}{\partial x} + \beta \right) \tag{11f}$$

$$m_x = \frac{eh^3}{12(1-\nu^2)} \frac{\partial \beta}{\partial x} \tag{11g}$$

Classical Bending Theory:

$$\frac{\partial n_x}{\partial x} - \rho h \frac{\partial^2 u}{\partial t^2} = 0 \tag{12a}$$

$$\frac{\partial q_x}{\partial x} - n_\theta - \rho h \frac{\partial^2 w}{\partial t^2} + (1-h/2) p^* = 0 \tag{12b}$$

$$\frac{\partial m_x}{\partial x} - q_x = 0 \tag{12c}$$

$$n_x = \frac{eh}{1-\nu^2}\left(\frac{\partial u}{\partial x} + \nu\ w\right) \tag{12d}$$

$$n_\theta = \frac{eh}{1-\nu^2}\left(w + \nu\frac{\partial u}{\partial x}\right) \tag{12e}$$

$$m_x = -\frac{eh^3}{12(1-\nu^2)}\ \frac{\partial^2 w}{\partial t^2} \tag{12f}$$

Classical Membrane Theory:

$$\frac{\partial n_x}{\partial x} - \rho h \frac{\partial^2 u}{\partial t^2} = 0 \tag{13a}$$

$$n_\theta + \rho h \frac{\partial^2 w}{\partial t^2} - (1-h/2)p^* = 0 \tag{13b}$$

$$n_x = \frac{eh}{1-\nu^2}\left(\frac{\partial u}{\partial x} + \nu w\right) \tag{13c}$$

$$n_\theta = \frac{eh}{1-\nu^2}\left(w + \nu\frac{\partial u}{\partial x}\right) \tag{13d}$$

Propagation of a Steady Train of Waves

The transient motion of an infinite fluid-filled shell due to nonhomogeneous initial conditions (initial value problem) has been determined formally by Skalak [12]. The axisymmetric wave equation governing the fluid motion was reduced to an ordinary differential equation by the successive application of Laplace and Fourier transforms. This technique may be applied to certain problems involving a semi-infinite tube, that is, whenever a semi-infinite tube can be replaced by an equivalent symmetric or antisymmetric (with respect to the axial coordinate) infinite tube. This rather stringent restriction on the problems that may be analyzed by this technique is due to the manner in which the fluid and shell motions are coupled. If the fluid pressure is specified at the end of the shell, then, according to the classical bending theory of shells, the moment stress resultant and radial displacement of the shell must be specified at the end. Similarly if the fluid velocity is specified at the end of the shell, then the shear stress resultant and the slope of the shell middle surface must be specified at the end. Hence, the only transient solution, including dispersive effects, which appears in the literature is that corresponding to a pure initial value problem for an infinite tube.

Although, as was mentioned previously, Skalak was able to invert his formal solution only asymptotically, he was able to show that displacements, pressure, etc. could each be expressed as an infinite series of Fourier integrals. The integration was to be carried out with respect to the parameter of the original Fourier transform, and each Fourier integrand was a solution to the governing equations for a steady train of sinusoidal waves. Thus some insight into our problem can be obtained by considering here the propagation of steady trains of sinusoidal waves in an infinite fluid-filled tube.

Letting the fluid velocity potential take the form $\varphi = A(r) \exp[i(\xi x + \omega t)]$, where ξ is the wave number and ω is the frequency of the train of waves, the wave equation, (7), yields $\frac{1}{r}\frac{d}{dr}\left[r\frac{dA}{dr}\right] + (\omega^2 - \xi^2)A = 0$ which is Bessel's equation. The appropriate solution for A, and hence φ, to be finite at $r = 0$ is $A(r) = B_1\, J_0[r(\omega^2 - \xi^2)^{\frac{1}{2}}]$.

Equations (12), the classical bending theory of shells, may be combined to yield

$$\frac{eh^3}{12(1-\nu^2)}\frac{\partial^4 w}{\partial x^4} + \frac{eh}{(1-\nu^2)}\left(w + \nu\frac{\partial u}{\partial x}\right) + \rho h\frac{\partial^2 w}{\partial t^2} = (1-h/2)p^* \tag{14a}$$

and $$\frac{e}{1-\nu^2}\left(\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial w}{\partial x}\right) - \rho\frac{\partial^2 u}{\partial t^2} = 0 \tag{14b}$$

Letting $w = B_2 \exp[i(\xi x + \omega t)]$ and $u = B_3 \exp[i(\xi x + \omega t)]$ and substituting these along with $\varphi = B_1\, J_0[r(\omega^2 - \xi^2)^{\frac{1}{2}}]$ into Equations (8) and (14), we obtain the frequency equation

$$(\omega^2 - \xi^2)^{\frac{1}{2}} J_1[(1-h/2)(\omega^2 - \xi^2)^{\frac{1}{2}}]\left\{\left[\frac{eh^3\xi^4}{12(1-\nu^2)} + \frac{eh}{1-\nu^2} - \rho h\omega^2\right]\left[\rho\omega^2 - \frac{e\xi^4}{1-\nu^2}\right] + \frac{e\nu^2 h\xi^4}{1-\nu^2}\right\} + \left(\rho\omega^2 - \frac{e^2}{1-\nu^2}\right) J_0[(1-h/2)(\omega^2 - \xi^2)^{\frac{1}{2}}] = 0. \tag{15}$$

The transcendental equation, (15), was solved with the following parameters, which correspond to a steel shell filled with water:

$$e = 100, \qquad \rho = 7.84,$$
$$\nu = 0.3, \qquad h = 0.02.$$

For the first five branches of the frequency equation, (15), the frequency (ω) - wave number (ξ) spectrum is given in Figure 3, and the phase velocity (η) - wavelength (λ) spectrum is given in Figure 4. The phase velocity is given by $\eta = \omega/\xi$ and the wavelength is given by $\lambda = 2\pi/\xi$. Of particular interest in Figures 3 and 4 are the behaviors of the first two branches of the frequency equation for small wave numbers (large wavelengths). It will be recalled that it was upon this behavior that Skalak's long-time solution was based.

The minimum phase velocity (Figure 4) of the first branch is due to the inclusion of bending in the equations governing the shell motion. Omission of bending (membrane theory) produces a phase velocity-wavelength spectrum similar to Figure 4 except that no minimum of the first-branch phase velocity occurs. The phase velocity for the first branch of the membrane theory approaches zero as the wavelength approaches zero.

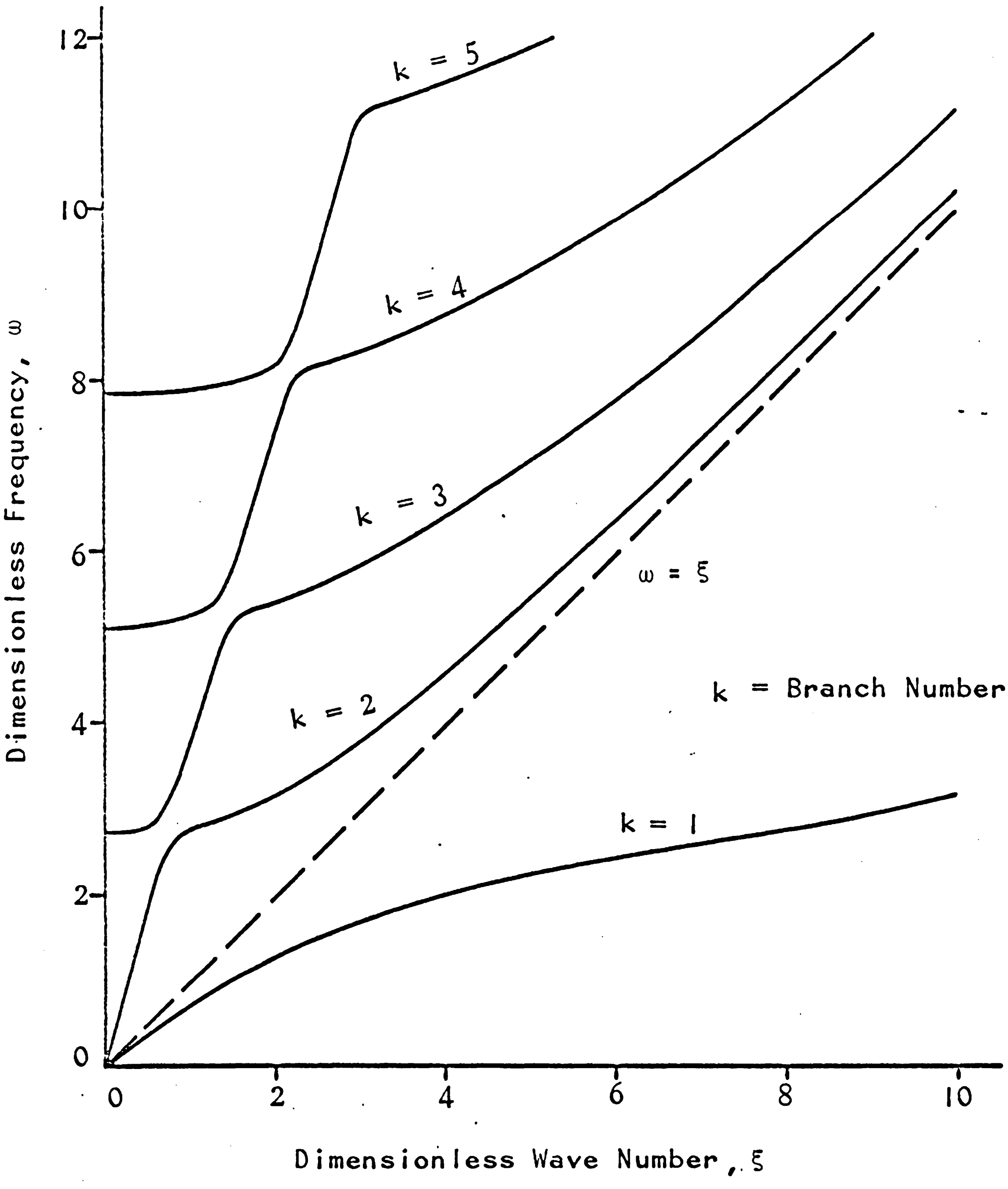


Figure 3

Frequency - Wave Number Relations:

Classical Bending Theory

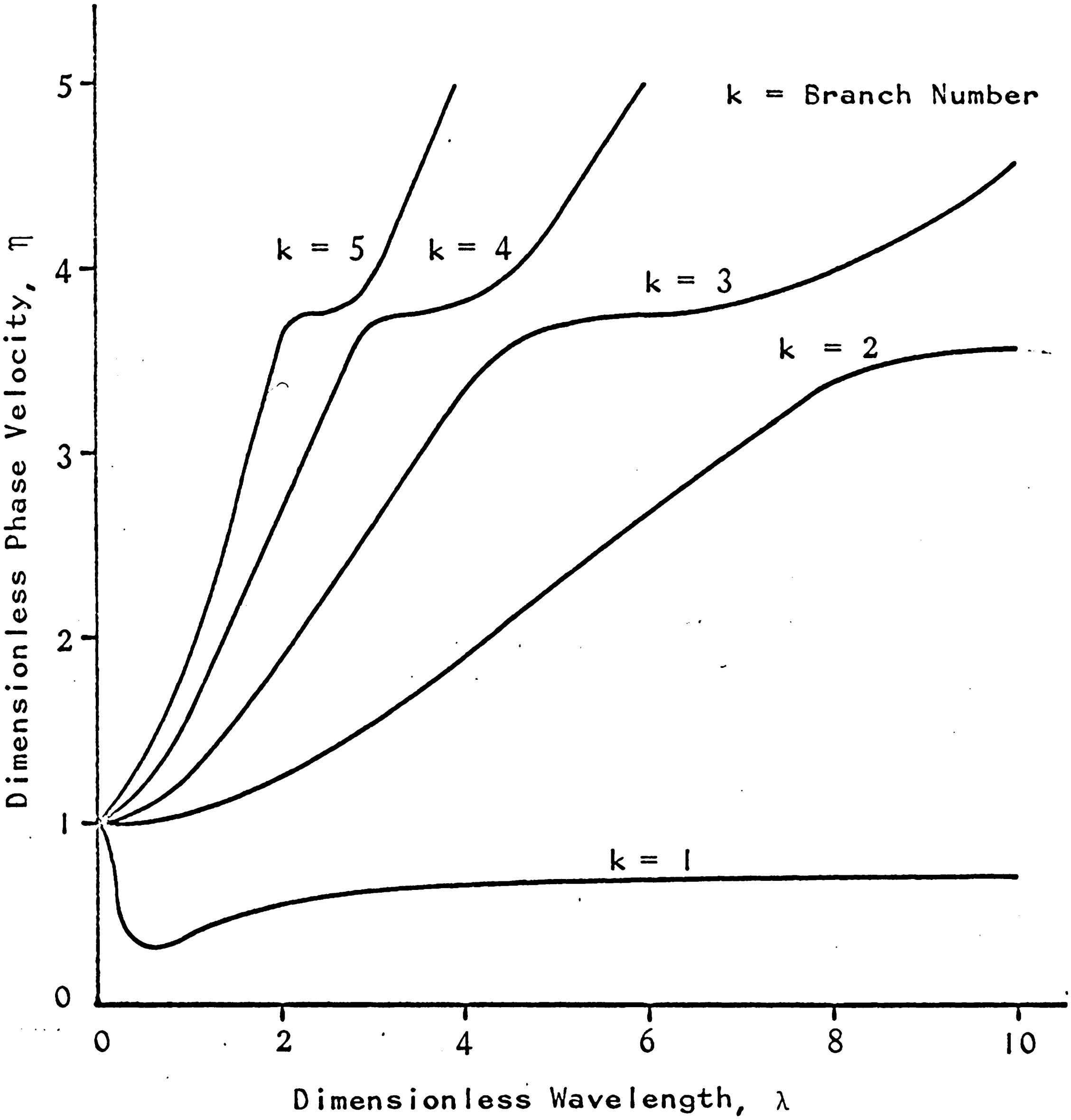


Figure 4

Phase Velocity - Wavelength Relations:

Classical Bending Theory

The influence of the longitudinal motion on the dispersion curves is shown in Figure 5. The frequency-wave number spectrum is given for the classical bending theory in which the longitudinal motion has been excluded by neglecting the membrane stress resultant n_x. We see that coupling between longitudinal and radial motion of the shell has a minor influence upon the frequency-wave number spectrum.

Lin and Morgan [10] showed that rotary inertia has a negligible influence on the roots of the frequency equation, and shear deformation causes first-branch frequencies to be slightly lowered at short wavelengths.

On the Applicability of the Virtual Mass Approximation to Long-Time Solutions

Skalak's asymptotic solution of a water hammer problem depended critically upon the behavior, for large wavelengths, of those branches of the frequency equation which exhibit finite phase velocities for an infinite wavelength [12]. As can be seen in Figures 3, 4 and 5, the first two branches of Equation (15) have this character. Skalak's long-time solution was developed by the use of the first two terms of a power series, in the wave number,

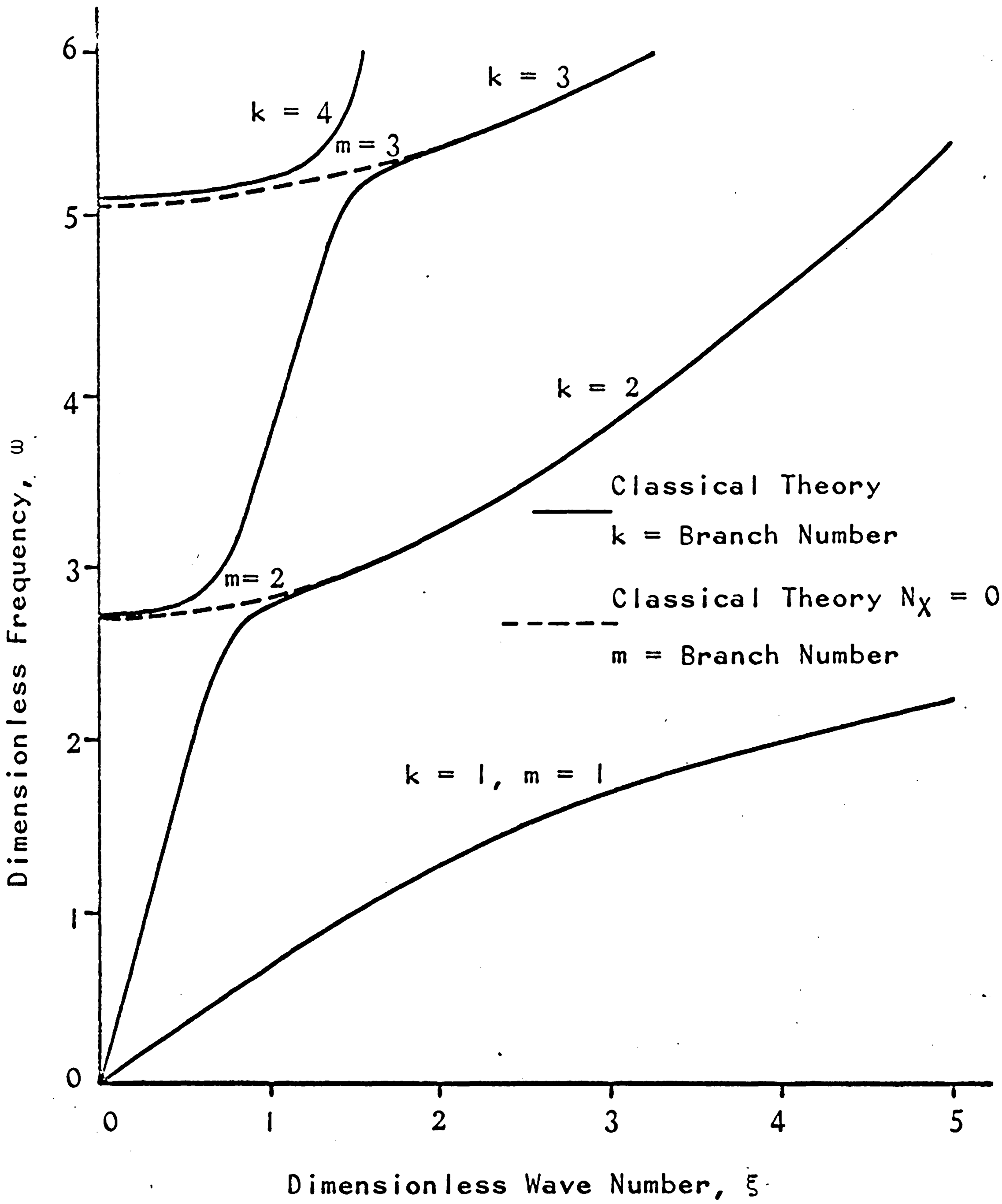


Figure 5

Influence of Longitudinal Inertia

of the Shell on Frequencies

to represent the long-wavelength behavior of each of the first two branches of an equation similar to (15). Hence, if, for these branches, we represent the frequency, ω, by

$$\omega = a\xi + b\xi^3 + \dots$$

and substitute this series into equation (15), using the series forms for the Bessel functions in (15), the coefficients of ξ^4, ξ^6, ξ^8, etc. must each vanish, since the transcendental frequency equation must be satisfied for all values of ξ.

Equating the coefficient of ξ^4 to zero yields

$$\left(\frac{2}{\rho h} + c_p^2\right) a^4 - \left[(1-\nu^2)c_p^4 + \left(1 + \frac{2}{\rho h}\right)c_p^2\right] a^2 + (1-\nu^2)c_p^4 = 0 \qquad (16)$$

where c_p is the dimensionless plate velocity, $c_p = \sqrt{\dfrac{e}{\rho(1-\nu^2)}}$.

The roots of (16) are $a = \pm c_1$, and $a = \pm c_2$, where c_1 and c_2 are the dimensionless propagation speeds which arise in Skalak's extension of the Joukowsky water hammer theory.

Equating the coefficient of ξ^6 in (15) to zero, we obtain

$$b = a\left[4 + \frac{(1-h/2)^2}{\rho h}\right] \left[a^4 - (1 + c_p^2)a^2 + c_p^2\right]. \qquad (17)$$

Let us now consider the dispersion curves which arise when the virtual mass approximation is utilized in the description of the fluid motion. The average pressure and axial velocity are governed by Equations (9), and according to the virtual mass approximation the pressure at the surface of the shell is given by Equation (10). Motion of the shell is to be described again by the classical bending theory, Equations (14). Assuming harmonic solutions to Equations (9), (10), and (14), we obtain the following frequency equation which has three branches:

$$(\omega^2 - \xi^2)\{(\frac{\omega^2}{c_p^2} - \xi^2)[(1 + \frac{(1-h/2)^2}{4\rho h})\omega^2 - (1 + \frac{h}{12}\xi^4)c_p^2]$$
$$- \nu^2 c_p^2 \xi^2\} - \frac{2}{\rho h}\omega^2(\frac{\omega^2}{c_p^2} - \xi^2) = 0 . \qquad (18)$$

The first two branches of the roots of (18) exhibit finite phase velocities for an infinite wavelength. If the frequency for each of these branches is represented, for large wavelengths, by a power series in the wave number, ξ, then Equation (18) yields the same leading terms which were generated above by the more exact formulation.

It remains to be shown that the pressure at the surface of the shell is correctly predicted by the virtual mass approximation. Recall that for steady propagation of a sinusoidal train of waves the axially symmetric velocity potential is given by

$$\varphi = B_1 J_0 [r(\omega^2 - \xi^2)^{\frac{1}{2}}] \exp [i(\xi x + \omega t)]$$

and the pressure is

$$p = i \omega B_1 J_0 [r(\omega^2 - \xi^2)^{\frac{1}{2}}] \exp [i(\xi x + \omega t)].$$

Compatibility of the fluid and shell motions at the inner surface of the shell requires that

$$\left[\frac{\partial \varphi}{\partial r}\right]_{r = 1-h/2} = \frac{\partial w}{\partial t} .$$

Therefore, since the radial displacement, w, is given by

$$w = B_2 \exp [i(\xi x + \omega t)] ,$$

then

$$B_1 = \frac{-i \omega B_2}{(\omega^2 - \xi^2)^{\frac{1}{2}} J_1 [(1-h/2)(\omega^2 - \xi^2)^{\frac{1}{2}}]} .$$

Hence the pressure in the fluid at the inner surface of the shell may be written

$$p^* = \frac{-\omega^2 B_2 J_0 [(1-h/2)(\omega^2 - \xi^2)^{\frac{1}{2}}]}{(\omega^2 - \xi^2)^{\frac{1}{2}} J_1 [(1-h/2)(\omega^2 - \xi^2)^{\frac{1}{2}}]} \exp [i(\xi x + \omega t)],$$

and the average pressure in the fluid is

$$\bar{p} = \frac{-2 \omega^2 B_2}{(1-h/2)(\omega^2 - \xi^2)} \exp [i(\xi x + \omega t)] .$$

Using the series forms of the Bessel functions we find that

$$p^* - \bar{p} = \omega^2 B_2 \left\{ \frac{(1-h/2)}{4} + \sum_{n=1}^{\infty} D_n (\omega^2 - \xi^2)^n \right\}$$

$$\exp [i(\xi x + \omega t)] . \qquad (19)$$

Hence if the motion corresponds to a branch of the frequency equation for which $\omega \to 0$ as $\xi \to 0$, the difference between the pressure at the shell surface and the average pressure at long wavelengths is

$$p^* - \bar{p} = -\frac{(1-h/2)}{4} \frac{\partial^2 w}{\partial t^2} ,$$

since

$$\frac{\partial^2 w}{\partial t^2} = - \omega^2 B_2 \exp [i(\xi x + \omega t)] .$$

The long-wavelength expression given above is, of course, the virtual mass approximation.

Since only the first term on the right side of Equation (19) was used in Skalak's asymptotic solution, the virtual mass approximation for the fluid motion will produce, for a transient problem, the same large-time solution.

The adequacy of the virtual mass approximation in describing motion associated with the early stages of a transient disturbance is determined in part by the dispersion curves over the entire wavelength spectrum. We now consider the virtual mass approximation to the fluid motion along with the classical bending theory of shells with longitudinal shell motion excluded (n_x is neglected). Assuming harmonic solutions to these equations we obtain a frequency equation with two branches. The roots of this equation are compared in Figures 6 and 7 with those of the first two branches obtained by using the axisymmetrical wave equation to describe motion of the fluid.

The approximate formulation predicts the first branch reasonably well over the entire wavelength spectrum, and agreement between the first branches is excellent for wavelengths greater than the diameter of the shell. Agreement between the second branches is not good except for moderately small wavelengths. Moreover, the second-branch phase velocity according to the virtual mass approximation does not have a finite asymptote as the wavelength approaches zero, while the more exact formulation predicts an asymptote of unity. It should be noted that this effect is due to the inclusion of bending in the analysis and due to the fact that there exist only two branches of the frequency equation for the virtual mass formulation. On the other hand, the more exact theory predicts an infinite set of branches, each having a short-wavelength asymptote for the phase velocity of unity (velocity of propagation of a plane acoustic wave in an infinite body of fluid). Hence the short-wavelength influence of bending is suppressed when the more exact formulation for the fluid motion is used.

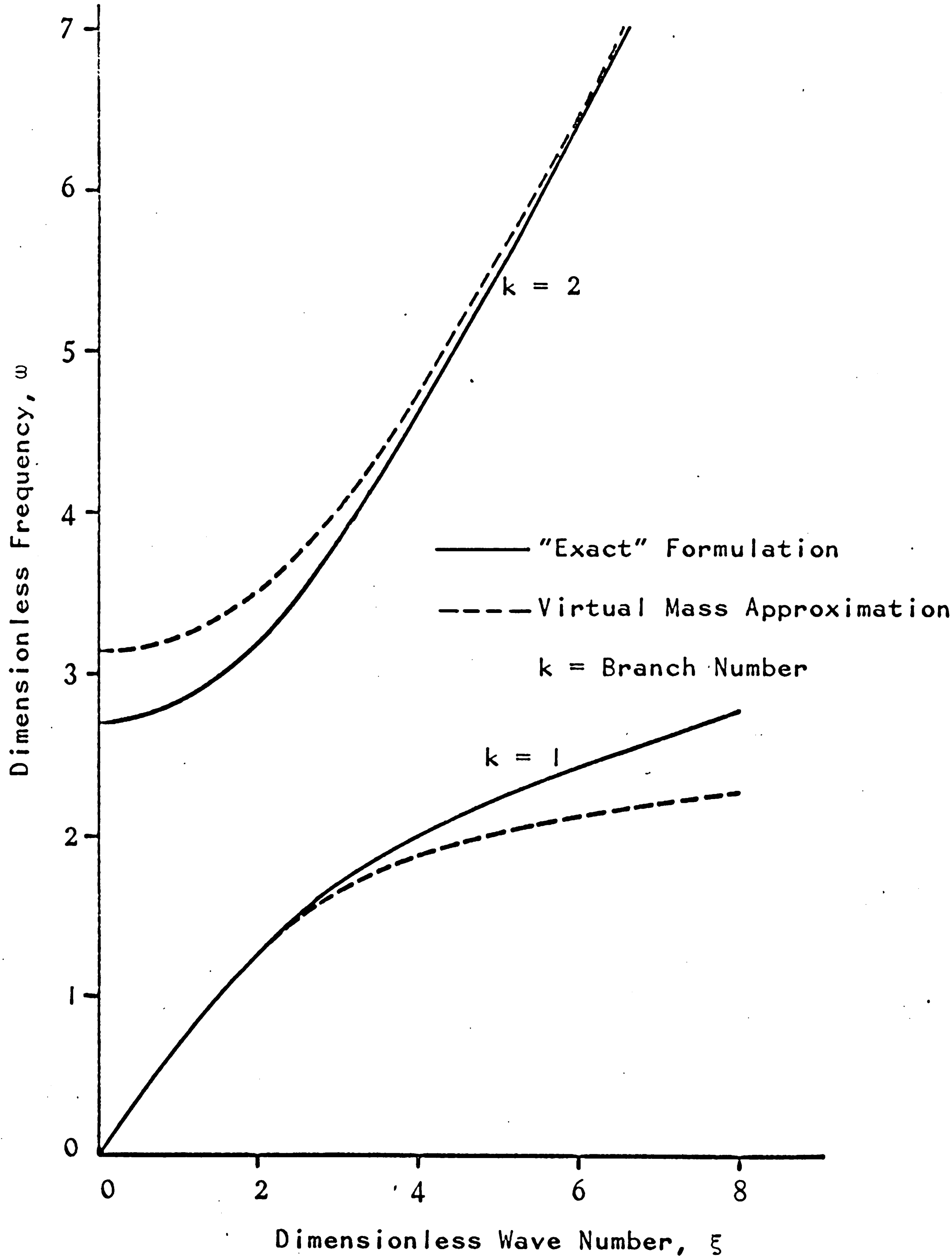


Figure 6

Effect of the Virtual Mass Approximation on Frequencies: Classical Bending Theory with $N_x = 0$

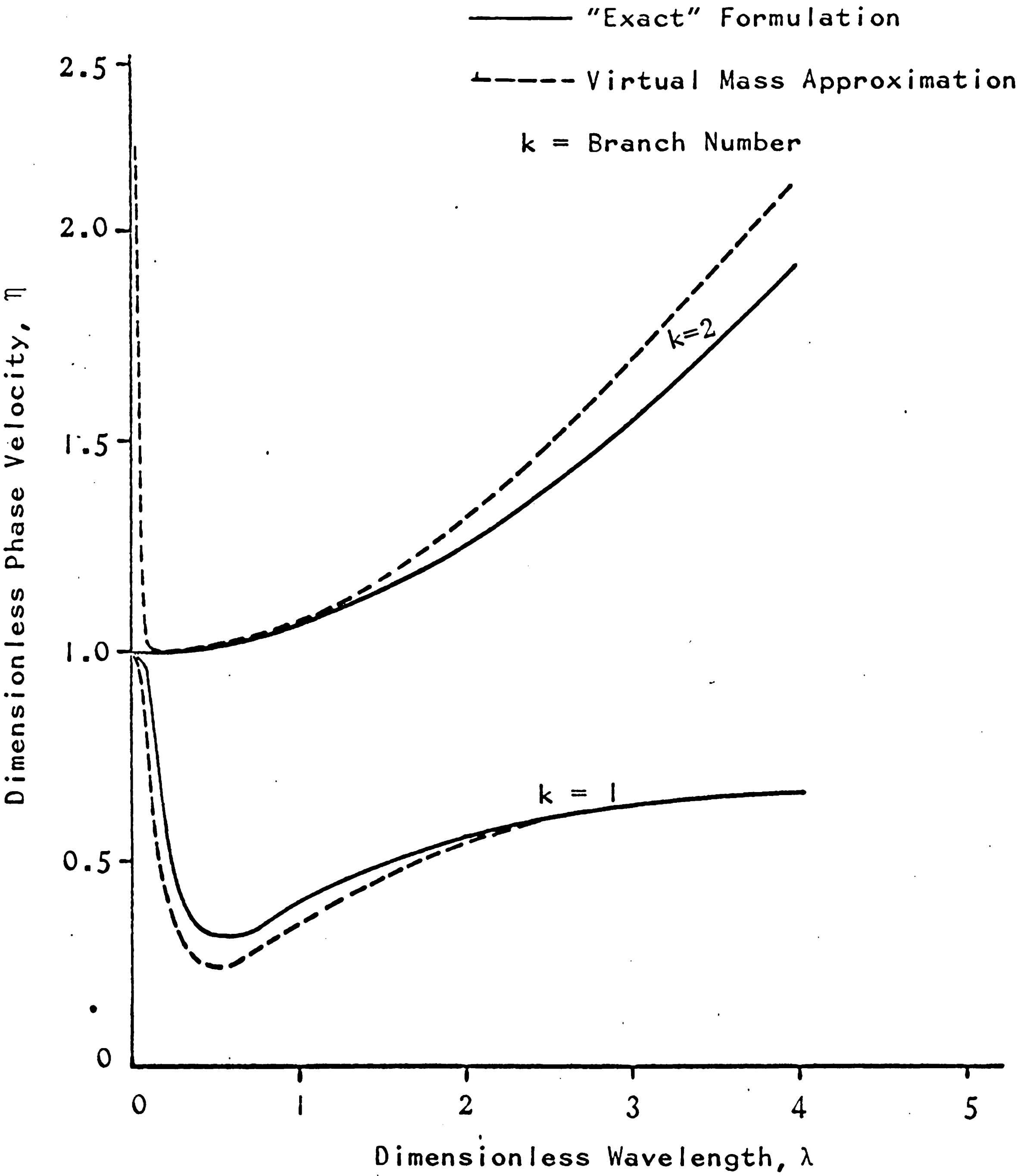


Figure 7

Effect of the Virtual Mass Approximation on Phase Velocities: Classical Bending Theory with $N_x = 0$

Summarizing, the virtual mass approximation yields the long-time solution obtained by Skalak for a water hammer problem. There is a substantial lack of agreement between the dispersion curves associated with the virtual mass approximation and the more exact formulation based upon the axisymmetrical wave equation. An assessment of the value of the virtual mass formulation, in describing the early stages of a disturbance, will be made in the next section.

SMALL-TIME SOLUTION I: MEMBRANE ANALYSIS

Representation of the Fluid Motion by an Infinite System of Wave Equations

In a previous section equations governing motion of the shell and motion of the fluid were developed. Attention was focused upon shell theories rather than upon the theory of elasticity, since solutions, according to the theory of elasticity, have been obtained for only the simplest problems of elastic wave propagation. For axially symmetric motion the shell equations constitute a system of partial differential equations in two independent variables (x, t), while the elasticity relations involve three independent variables (r, x, t). It is to be expected that an a priori elimination of the dependence of the fluid equations on the radial coordinate, r, will facilitate solution of the coupled fluid-shell governing equations. This, of course, was the motivation behind the introduction of the virtual mass approximation to account for radial inertia of the fluid. In this section the axially symmetric wave equation is to be replaced by an infinite system of one-dimensional wave equations, thus eliminating the explicit dependence on the radial variable, r, of the equations governing the fluid motion.

Motion of the fluid is governed by the wave equation,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial \varphi}{\partial r} \right] - \frac{\partial^2 \varphi}{\partial t^2} = 0 . \qquad (7)$$

The independent variable, r, is now to be suppressed by the application of a finite Hankel Transform. Multiplying each term in Equation (7) by $rJ_0(\gamma r)dr$, and integrating from $r = 0$ to $r = 1-h/2$, we obtain [23, p. 89]

$$\frac{\partial^2 \chi_j}{\partial x^2} - \frac{\partial^2 \chi_j}{\partial t^2} - \gamma_j^2 \chi_j + (1-h/2) \; J_0 \left[(1-h/2) \gamma_j \right] \left[\frac{\partial \varphi}{\partial r} \right]_{r=1-h/2} = 0$$

for $j = 0, 1, 2, 3, \ldots$

where $\chi_j(\gamma_j, x, t) = \int_0^{1-h/2} rJ_0(\gamma_j r) \varphi(r, x, t) dr,$

and the γ_j are the non-negative roots of

$$\gamma J_0' \left[(1-h/2) \gamma \right] = 0$$

where the prime denotes differentiation with respect to the argument of the Bessel function. Inversion of the Hankel transform yields [23, p. 84]

$$\varphi(r, x, t) = \frac{2}{(1-h/2)^2} \sum_{j=0}^{\infty} \frac{J_0 [r\gamma_j]}{\{ J_0 [(1-h/2)\gamma_j] \}^2} \chi_j(\gamma_j, x, t) .$$

It is convenient to introduce new variables, ψ_j, such that

$$\psi_j = \frac{2 \chi_j}{(1-h/2)^2 \; J_0 \left[(1-h/2) \gamma_j \right]}, \quad j = 0, 1, 2, \ldots .$$

in which case the pressure of the fluid at the inner surface of the shell is given by

$$[p]_{r = 1-h/2} = - \sum_{j=0}^{\infty} \frac{\partial \psi_j}{\partial t} ,$$

and the ψ_j are governed by

$$\frac{\partial^2 \psi_j}{\partial x^2} - \frac{\partial^2 \psi_j}{\partial t^2} - \gamma_j^2 \psi_j + \frac{2}{(1-h/2)} \left[\frac{\partial \varphi}{\partial r} \right]_{r = 1-h/2} = 0 .$$

Note that $\gamma_0 = 0$, and γ_j, $j = 1, 2, 3, \ldots$, are the positive roots of $J_1[\gamma (1-h/2)] = 0$. According to the shell theories considered in this investigation, compatibility of the fluid and shell motions at the inner surface of the shell requires that

$$\left[\frac{\partial \varphi}{\partial r} \right]_{r = 1-h/2} = \frac{\partial w}{\partial t} .$$

Therefore

$$\frac{\partial^2 \psi_j}{\partial x^2} - \frac{\partial^2 \psi_j}{\partial t^2} - \gamma_j^2 \psi_j + \frac{2}{(1-h/2)} \frac{\partial w}{\partial t} = 0 . \qquad (20)$$

Introducing new variables, p_j and v_j, by

$$p_j = - \frac{\partial \psi_j}{\partial t} \qquad (21a)$$

and $$v_j = \frac{\partial \psi_j}{\partial x} , \qquad (21b)$$

then $$\frac{\partial p_j}{\partial x} + \frac{\partial v_j}{\partial t} = 0 \qquad (22a)$$

and $$\frac{\partial v_j}{\partial x} + \frac{\partial p_j}{\partial t} - \gamma_j^2 \psi_j + \frac{2}{(1-h/2)} \frac{\partial w}{\partial t} = 0. \qquad (22b)$$

Thus the pressure at the inner surface of the shell is given by

$$[p]_{r=1-h/2} = p^* = \sum_{j=0}^{\infty} p_j \quad .$$

We note that p_0 and v_0 represent the average pressure and the average axial velocity of the fluid over a section normal to the axis of the tube. They are governed by

$$\frac{\partial p_0}{\partial x} + \frac{\partial v_0}{\partial t} = 0$$

and $$\frac{\partial v_0}{\partial x} + \frac{\partial p_0}{\partial t} + \frac{2}{(1-h/2)} \frac{\partial w}{\partial t} = 0 \; ,$$

which are the equations governing fluid motion in the elementary water hammer theory. Hence $\sum_{j=1}^{\infty} p_j$ represents the effect of radial inertia of the fluid upon the pressure acting on the shell.

The wave equation, (7), involving three independent variables has been replaced by Equations (20), an infinite set of wave equations in two independent variables, or, alternatively, by an infinite system of pairs of first order partial differential equations. Equations (20) or (21) and (22) must now be integrated along with appropriate equations governing motion of the shell. This integration is facilitated by the application of the method of characteristics to Equations (22), which yields (see Appendix) the following system of ordinary differential equations:

Along $\frac{dx}{dt} = 1$,

$$\frac{dp_j}{dt} + \frac{dv_j}{dt} = \gamma_j^2 \psi_j - \frac{2\dot{w}}{(1-h/2)}, \quad j = 0, 1, 2, 3, . . \quad (23a)$$

Along $\frac{dx}{dt} = -1$,

$$\frac{dp_j}{dt} - \frac{dv_j}{dt} = \gamma_j^2 \psi_j - \frac{2\dot{w}}{(1-h/2)}, \quad j = 0, 1, 2, 3, . . \quad (23b)$$

where $\dot{w} = \frac{\partial w}{\partial t}$. Integration of the above equations, coupled with equations describing the shell motion, will be practicable if the infinite system of equations is approximated by a finite subset of the system.

Membrane Analysis of a Water Hammer Problem

Some insight into the interaction phenomenon can be obtained by utilizing a very simple model to describe motion of the shell. We shall employ here the classical membrane theory of shells, Equations (13), with the longitudinal membrane stress resultant, n_x, set equal to zero. Hence radial motion of the shell is governed by

$$\rho h \frac{\partial \dot{w}}{\partial t} + ehw = (1-h/2)p^* ,$$

which may be interpreted as an ordinary differential equation in $\dot{w}$, w, and p^* to be satisfied along lines of x = constant.

Using the first five of Equations (23) as an approximation for the infinite system, we obtain the following sets of ordinary differential equations:

Along $\frac{dx}{dt} = 1$,

$$\frac{dp_j}{dt} + \frac{dv_j}{dt} = \gamma_j^2 \psi_j - \frac{2\dot{w}}{(1-h/2)}, \quad j = 0, 1, 2, 3, 4 \tag{24a}$$

Along $\frac{dx}{dt} = -1$,

$$\frac{dp_j}{dt} - \frac{dv_j}{dt} = \gamma_j^2 \psi_j - \frac{2\dot{w}}{(1-h/2)}, \quad j = 0, 1, 2, 3, 4 \tag{24b}$$

Along $\frac{dx}{dt} = 0$,

$$\rho h \frac{d\dot{w}}{dt} + ehw = (1-h/2) \sum_{j=0}^{4} p_j = (1-h/2)p^* \tag{25a}$$

$$\frac{dw}{dt} = \dot{w} \tag{25b}$$

and $$\frac{d\psi_j}{dt} = -p_j \qquad j = 1, 2, 3, 4 . \tag{25c}$$

The equations above are to be integrated numerically for the classical water hammer problem corresponding to sudden stoppage of a uniform flow at the end of a semi-infinite tube. The problem is defined in the first quadrant of the x-t plane, and the appropriate intial and boundary conditions are:

Initial Conditions, $x>0$, $t = 0$:

$$\dot{w} = w = 0$$

$$p_j = v_j = \psi_j = 0 \qquad j = 1, 2, 3, 4$$

$$p_0 = 0, \qquad v_0 = -v(0)$$

Boundary Conditions, $x = 0$, $t>0$:

$$v_j = 0 \qquad j = 0, 1, 2, 3, 4$$

According to the governing equations employed in this section the highest velocity at which longitudinal disturbances can be propagated is unity, which is the velocity of propagation of acoustic waves in an infinite body of fluid. Hence the solution to the water hammer problem in the region $x-t>0$ is dependent only upon the initial conditions of the problem. For the problem which we have posed here the initial conditions constitute the solution throughout the region $x-t>0$.

We note that the origin ($x = 0$, $t = 0$) is a point of discontinuity for the initial and boundary conditions on v_0. Hence it is to be expected that the characteristic line $x = t$ is a line across which discontinuities in p_0 and v_0 occur. Equations (24) predict, assuming $\dot{w}$ and the ψ_j to be continuous, that discontinuities in $(p_j + v_j)$ propagate undiminished along lines for which $\frac{dx}{dt} = 1$. The only such discontinuity possible in our problem is in $(p_0 + v_0)$ along $x = t$. The first of Equations (24b) predicts that with the initial conditions of our water hammer problem, $\lim_{t \to 0^+} p_0 = v(0)$ at $x = 0$. Hence $x = t$ is a line across which a jump in $(p_0 + v_0)$ occurs, and the magnitude of the jump is $2\,v(0)$. Utilizing the solution of the problem in the region $x-t > 0$, the first of Equations (24b) yields a jump in v_0, from $-v(0)$ to zero, and a jump in p_0, from zero to $v(0)$, as the line $x = t$ is crossed in going from the region $x-t > 0$ to the region $x-t < 0$. Thus, if we associate the line $x = t$ with the region $x-t < 0$, no discontinuities occur within $x-t \leq 0$. To obtain a solution in the "disturbed" region, $x-t \leq 0$, the initial conditions may be replaced by the following conditions imposed along $x = t$:

$$\dot{w} = w = 0$$

$$p_j = v_j = \psi_j = 0 \qquad j = 1,\ 2,\ 3,\ 4$$

$$v_0 = 0$$

$$p_0 = v(0)\ .$$

Numerical integration of the differential equations was effected in the triangular region shown in Figure 8. Values of the dependent variables were computed at the intersections of regularly spaced characteristic lines

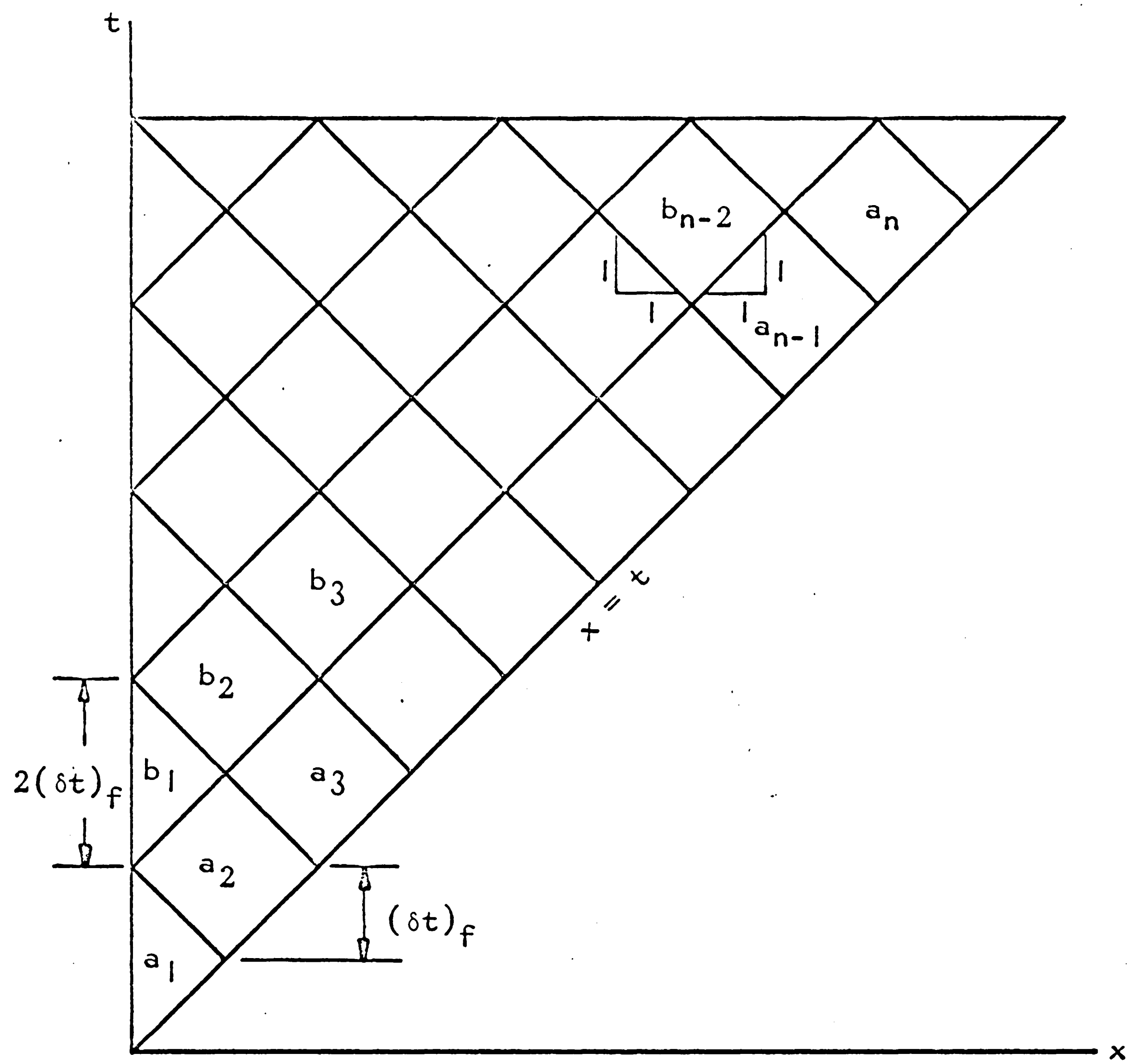


Figure 8

Net for Numerical Calculations:

Membrane Analysis

($\frac{dx}{dt} = 1$, $\frac{dx}{dt} = -1$). These characteristic lines are shown in Figure 8. Integration of Equations (24) and (25) was accomplished by linear approximation of the behavior of the dependent variables between adjacent grid points.

The dependent variables at points on the boundary, x = 0, are determined by use of the triangular element shown in Figure 9a. It is assumed that all of the dependent variables are known at the points on the element labelled "1" and "2". Letting $(w)_3$ stand for the radial displacement at point "3", the dependent variables at point "3" are obtained from the following relations:

$$(p_j)_3 - (v_j)_3 = (p_j)_2 - (v_j)_2 + \gamma_j^2 \left[\frac{(\psi_j)_2 + (\psi_j)_3}{2} \right] (\delta t)_f$$

$$- \frac{2}{(1-h/2)} \left[\frac{(\dot{w})_2 + (\dot{w})_3}{2} \right] (\delta t)_f$$

$$(\psi_j)_3 = (\psi_j)_1 - [(p_j)_3 + (p_j)_1] (\delta t)_f$$

$$\rho h (\dot{w})_3 = \rho h (\dot{w})_1 - eh [(w)_3 + (w)_1] (\delta t)_f$$

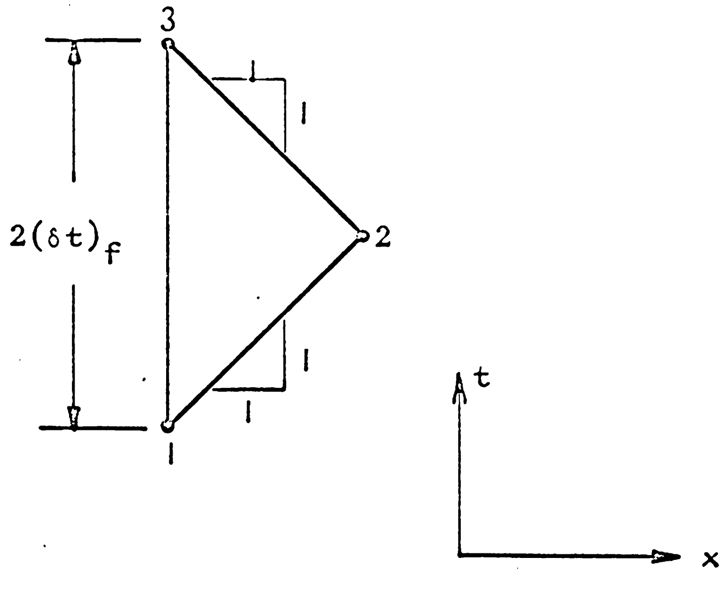
$$+ (1-h/2) [(p^*)_3 + (p^*)_1] (\delta t)_f$$

$$(w)_3 = (w)_1 + [(\dot{w})_3 + (\dot{w})_1] (\delta t)_f$$

$$(p^*)_3 = \sum_{j=0}^{4} (p_j)_3 \quad .$$

Employing the boundary condition, $(v_j)_3 = 0$, we have twelve

(a) Boundary Element



(b) Interior Element

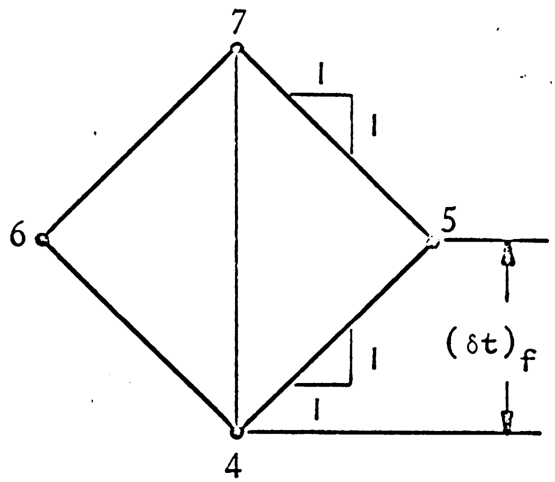


Figure 9

Elements of Computational Net:

Membrane Analysis

linear algebraic equations in twelve unknowns, $(w)_3$, $(\dot{w})_3$, $(p_0)_3$, $(p_j)_3$, $(\psi_j)_3$, and $(p^*)_3$ where $j = 1, 2, 3, 4$. Combining these equations to eliminate $(w)_3$ and the $(\psi_j)_3$, we obtain seven equations in seven unknowns. These are solved easily by iteration: $(\dot{w})_3$ is approximated initially by $(\dot{w})_2$, the $(p_j)_3$ and $(p^*)_3$ are computed, and finally $(\dot{w})_3$ is computed and the cycle repeated. Iteration is continued until two successively computed values of $(\dot{w})_3$ differ by less than some predetermined number (0.000001 throughout this work). Then the other eleven dependent variables are calculated to complete the computation at the boundary point.

At interior points of the region of computation (Figure 8) the dependent variables are determined by use of the square element of Figure 9b. It is assumed that these variables are known at the points on the element labelled "4", "5", and "6". The dependent variables at point "7" are obtained from the following expressions:

$$(p_j)_7 - (v_j)_7 = (p_j)_5 - (v_j)_5 + \gamma_j^2 \left[\frac{(\psi_j)_5 + (\psi_j)_7}{2}\right] (\delta t)_f$$

$$- \frac{2}{(1-h/2)} \left[\frac{(\dot{w})_5 + (\dot{w})_7}{2}\right] (\delta t)_f \, , \quad j = 0, 1, 2, 3,$$

$$(p_j)_7 + (v_j)_7 = (p_j)_6 + (v_j)_6 + \gamma_j^2 \left[\frac{(\psi_j)_6 + (\psi_j)_7}{2} \right] (\delta t)_f$$

$$- \frac{2}{(1-h/2)} \left[\frac{(\dot{w})_6 + (\dot{w})_7}{2} \right] (\delta t)_f, \quad j = 0, 1, 2, 3, 4$$

$$\rho h(\dot{w})_7 = \rho h(\dot{w})_4 - eh\left[(w)_7 + (w)_4\right] (\delta t)_f$$

$$+ (1-h/2) \left[(p^*)_7 + (p^*)_4\right] (\delta t)_f$$

$$(w)_7 = (w)_4 + \left[(\dot{w})_7 + (\dot{w})_4\right] (\delta t)_f$$

$$(p^*)_7 = \sum_{j=0}^{4} (p_j)_7 \quad .$$

Thus for an interior point we have seventeen linear algebraic equations to be solved simultaneously. Combining these equations to eliminate $(w)_7$, the $(\psi_j)_7$, and the $(v_j)_7$, we obtain seven equations in seven unknowns. These are solved by iteration in the manner described previously for computations at a boundary point.

The calculations described above were carried out in the water hammer problem to obtain the values of the dependent variables at all the grid intersections of Figure 8. This was accomplished by utilizing successively the elements $a_1, a_2, \ldots, a_n$, then successively the elements $b_1, b_2, \ldots, b_{n-2}$, etc. until the solution was obtained completely within the triangular "disturbed" region. Parameters, $\rho = 7.84$,

$e = 100$, corresponding to a water-filled steel shell were used, and calculations were performed for two thickness ratios, $h = 0.10$ and $h = 0.02$. The computations were performed on a Burroughs B-5500 computer for time intervals $(\delta t)_f = 0.01$ and $(\delta t)_f = 0.02$ with no significant differences in results.

The analysis based upon the membrane theory of shells retains the effects of radial inertia of the shell and radial inertia of the fluid, these representing the principal contributions to dispersion in water hammer problems. Results of the membrane analysis are to be used to determine whether or not $\sum_{j=0}^{4} p_j$ is an adequate representation of the pressure on the surface of the shell, given exactly by $\sum_{j=0}^{\infty} p_j$. In addition the analysis is a convenient vehicle for an evaluation of the virtual mass approximation to the radial inertia of the fluid. To invoke the virtual mass approximation here we need only set $p_j = 0$, $j \neq 0$, replace the coefficient, ρh, of the radial acceleration of the shell by $\rho h + \frac{(1-h/2)^2}{4}$, and proceed with the numerical calculations as outlined in this section. It should be noted that, according to the virtual mass approximation, for h = 0.10 radial inertia of the shell is more important than radial inertia of the fluid, while for h = 0.02 the reverse is true.

Results of the Membrane Analysis

For h = 0.02 the contributions, p_j, to the pressure acting on the inner surface of the shell are shown in Figure 10. As expected the significance of a contribution, p_j, diminishes as the index j increases, and the pressure acting on the surface of the shell appears to be approximated adequately by $\sum_{j=0}^{4} p_j$. Moreover, the significance of a p_j, $j \neq 0$, generally diminishes with increasing time; this is consistent with the results obtained by Skalak.

For h = 0.10 the p_j are given in Figure 11. The contributions p_3 and p_4 are not shown for t = 1.0 and t = 2.0 since they are insignificant. The effect of radial inertia of the fluid, as reflected by the p_j, $j \neq 0$, is not as significant as in the case of the thinner shell (h = 0.02). This is of course principally due to the relative stiffness of the thicker shell (h = 0.10).

An evaluation of the virtual mass approximation is based upon a comparison of radial shell displacements

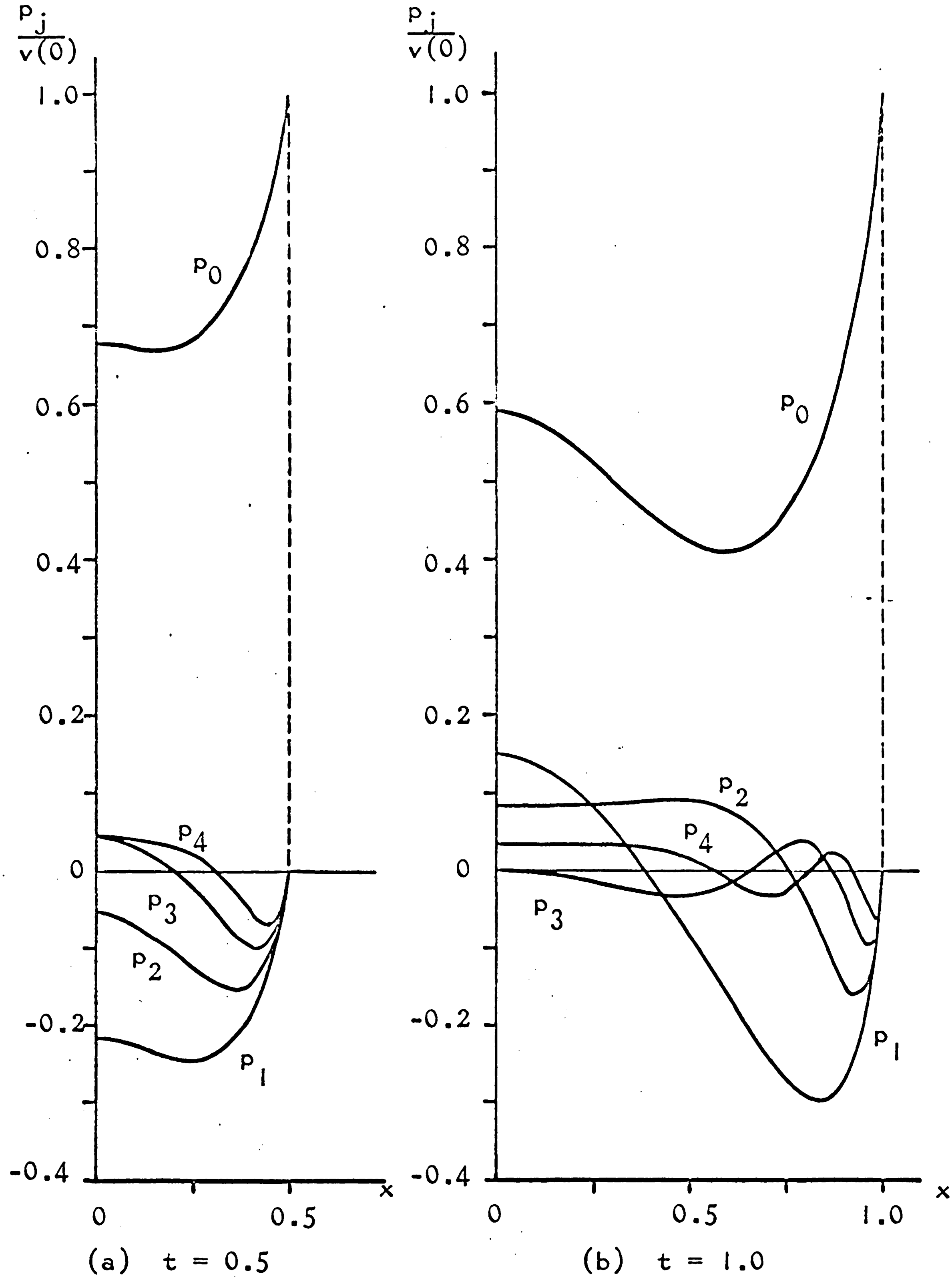
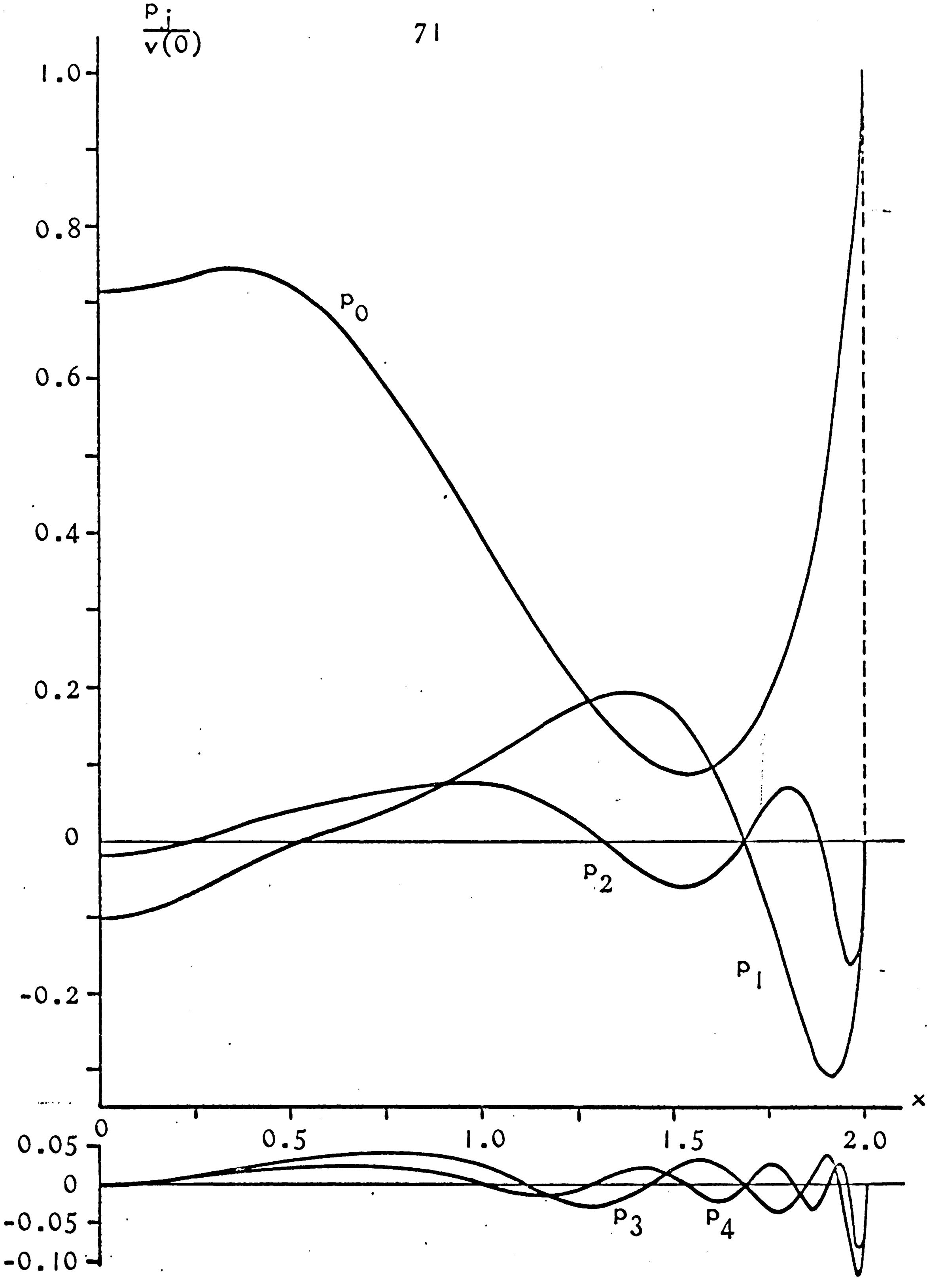


Figure 10

Contributions to Surface Pressure: Membrane Analysis of Water Hammer in a Steel Shell with h = 0.02



(c) t = 2.0

Figure 10 (Continued)

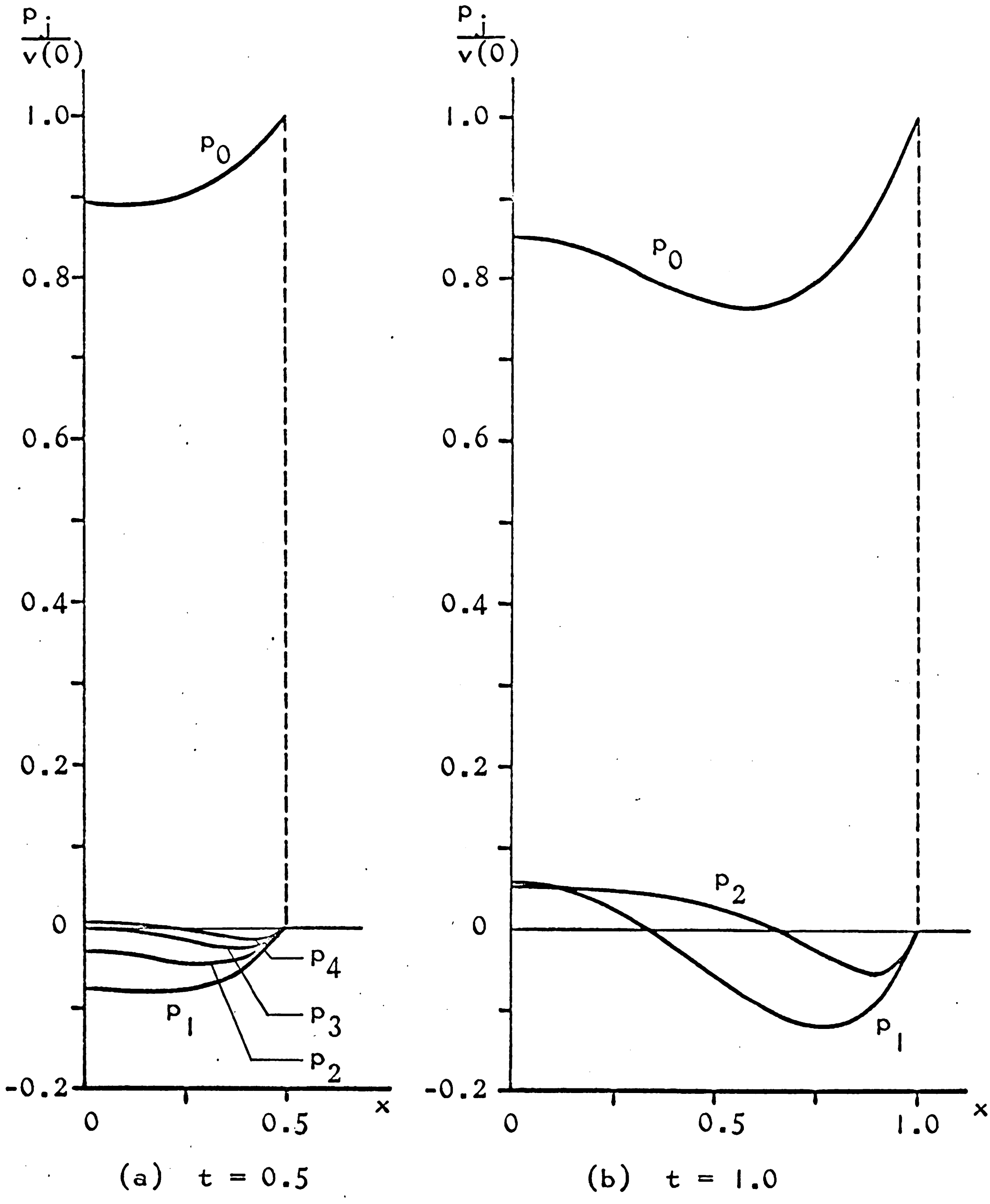
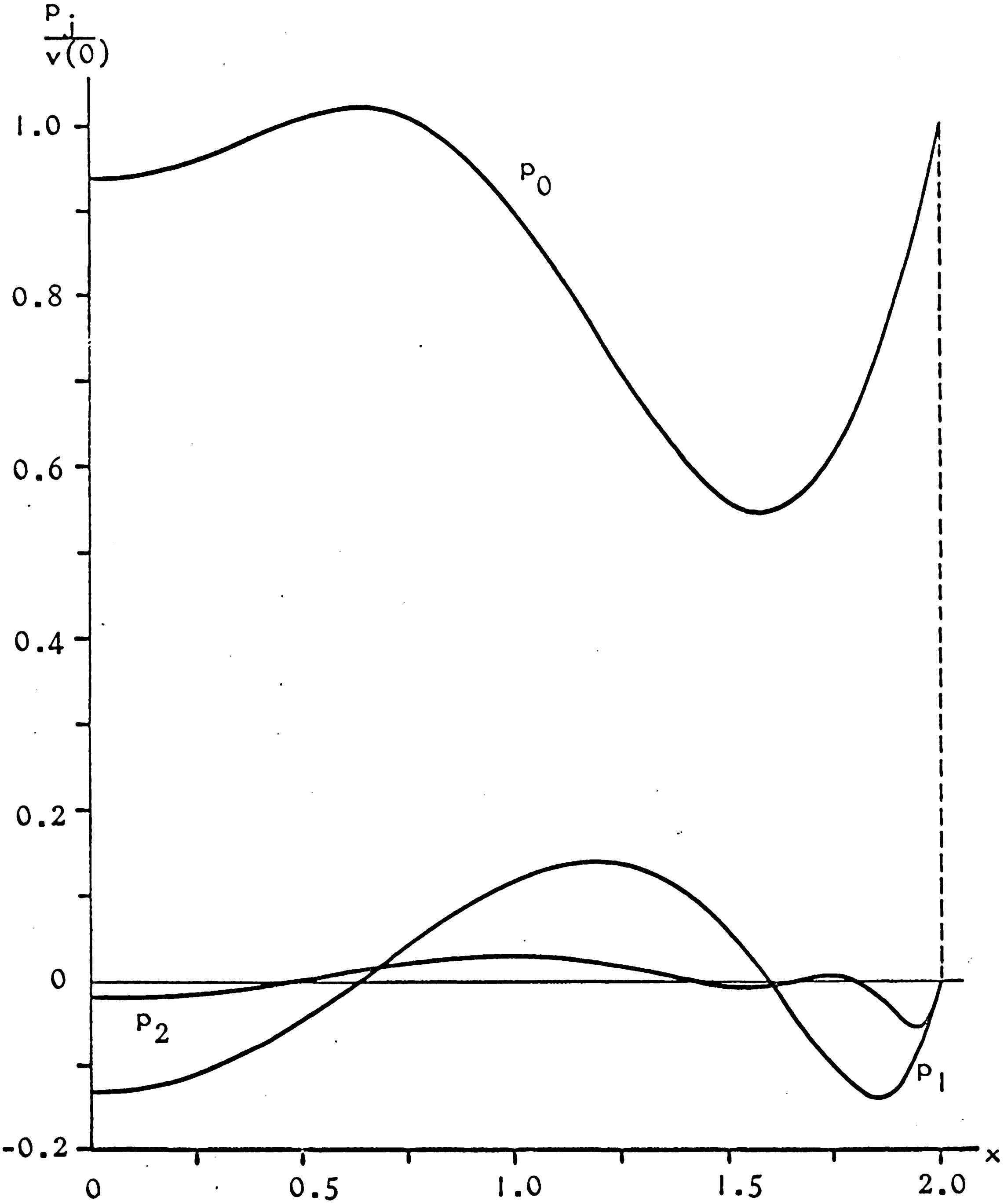


Figure 11

Contributions to Surface Pressure: Membrane Analysis of Water Hammer in a Steel Shell with h = 0.10



(c) t = 2.0

Figure 11 (Continued)

predicted by the "exact" formulation ($p^* = \sum_{j=0}^{4} p_j$) and the displacements predicted by the virtual mass formulation ($p^* = p_0 - \frac{(1-h/2)}{4} \frac{\partial^2 w}{\partial t^2}$). These displacements are shown in Figure 12 for h = 0.02 and in Figure 13 for h = 0.10. There are enough discrepancies between the displacements predicted by the two formulations to cast doubt upon the value of the virtual mass approximation in an analysis of the early stages of a water hammer disturbance.

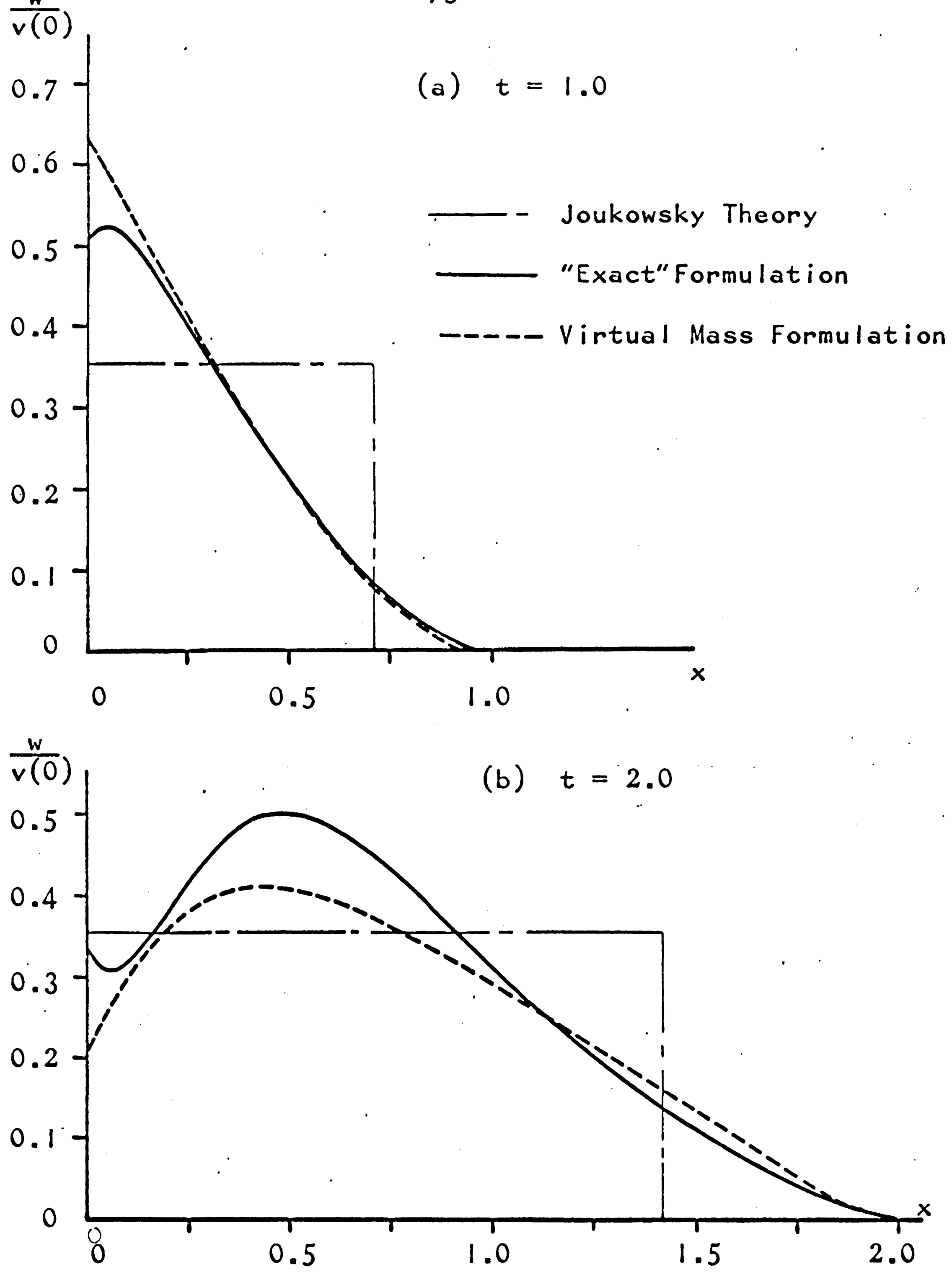


Figure 12

Radial Displacements of the Shell Middle Surface:

Membrane Analysis of Water Hammer

in a Steel Shell with h = 0.02

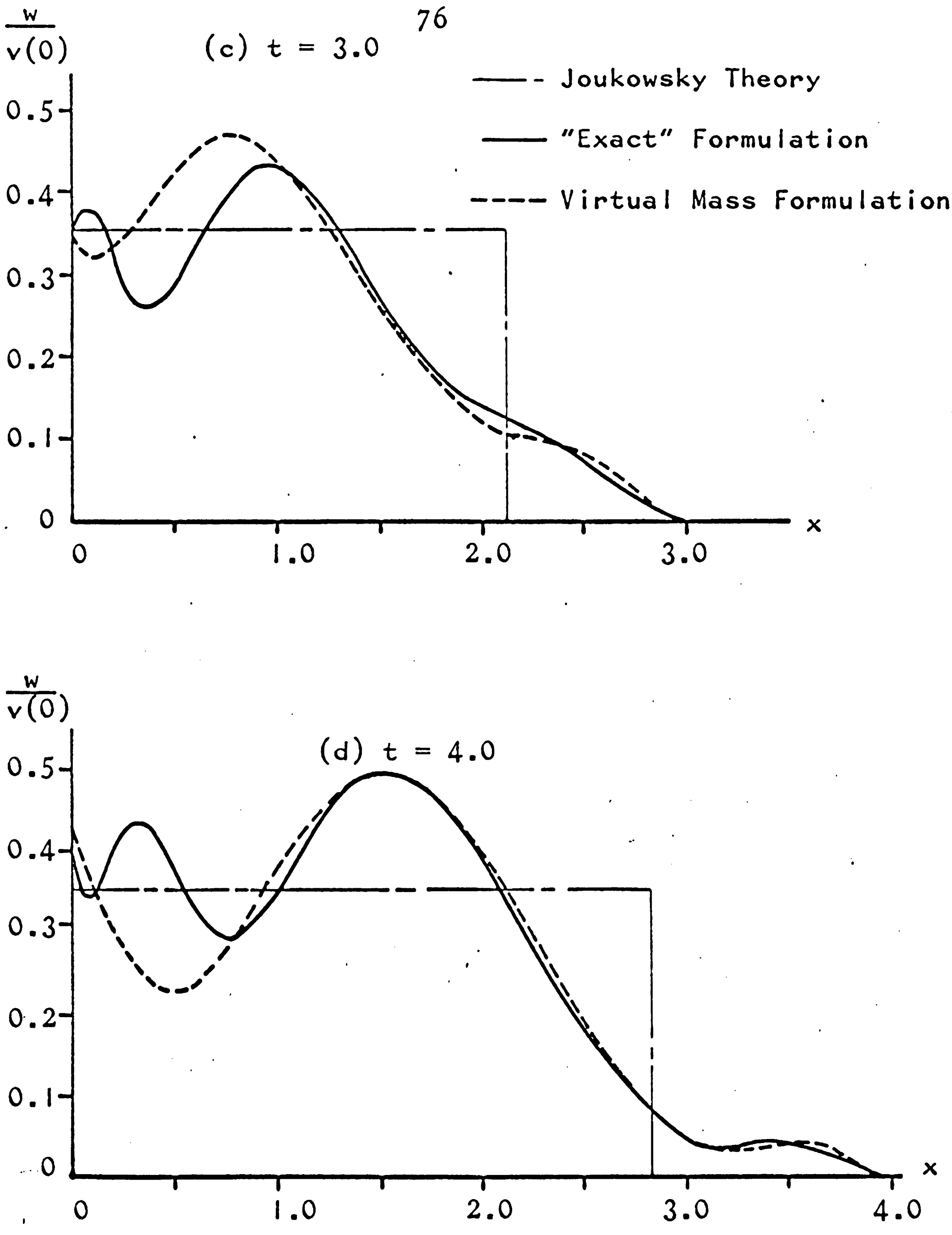
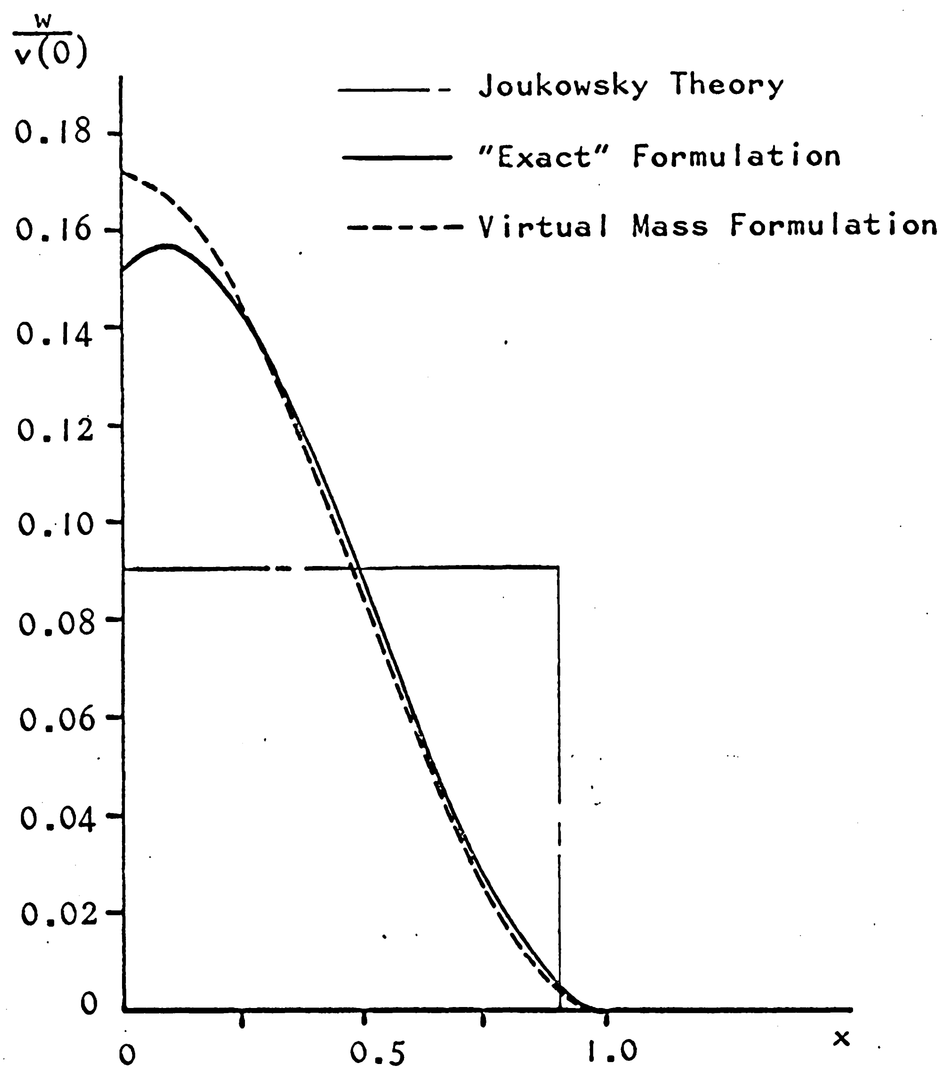


Figure 12 (Continued)

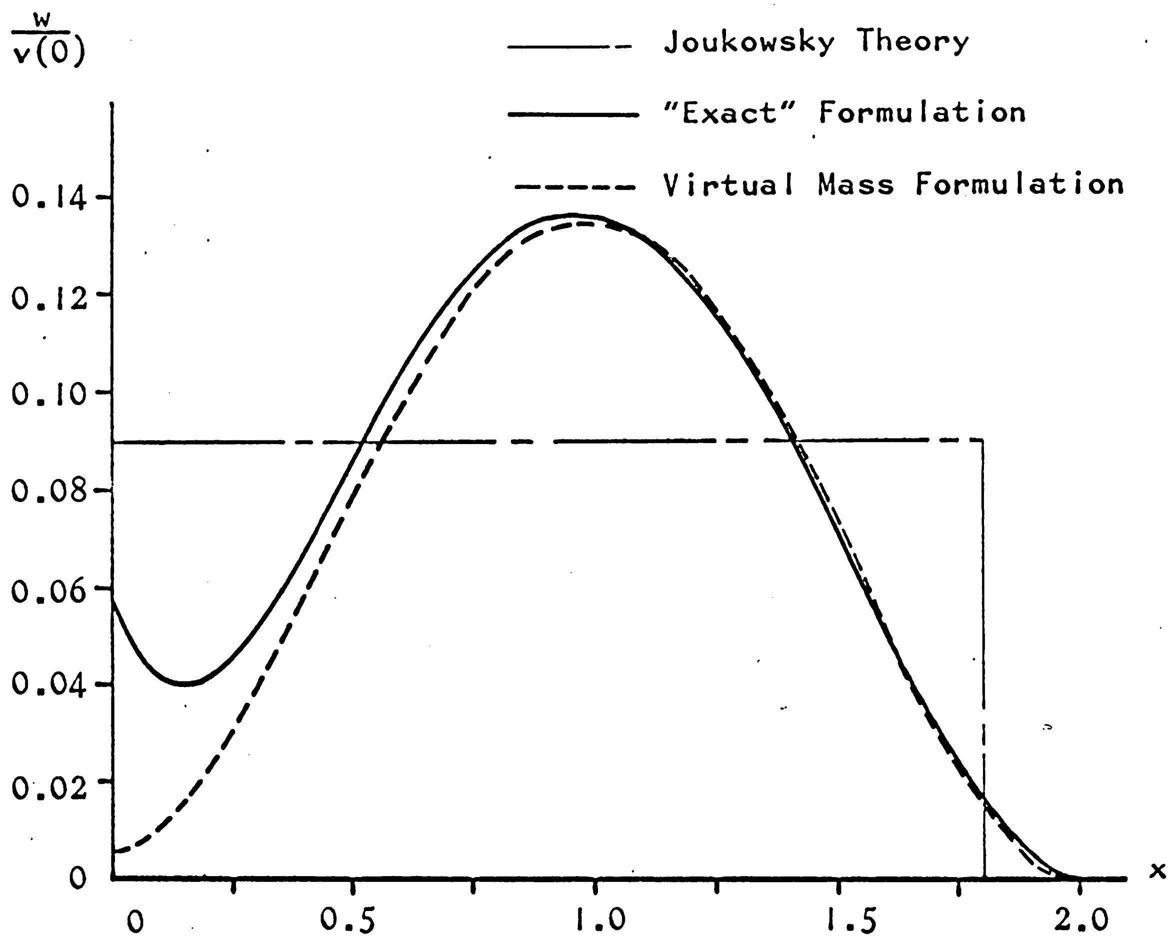


(a) t = 1.0

Figure 13

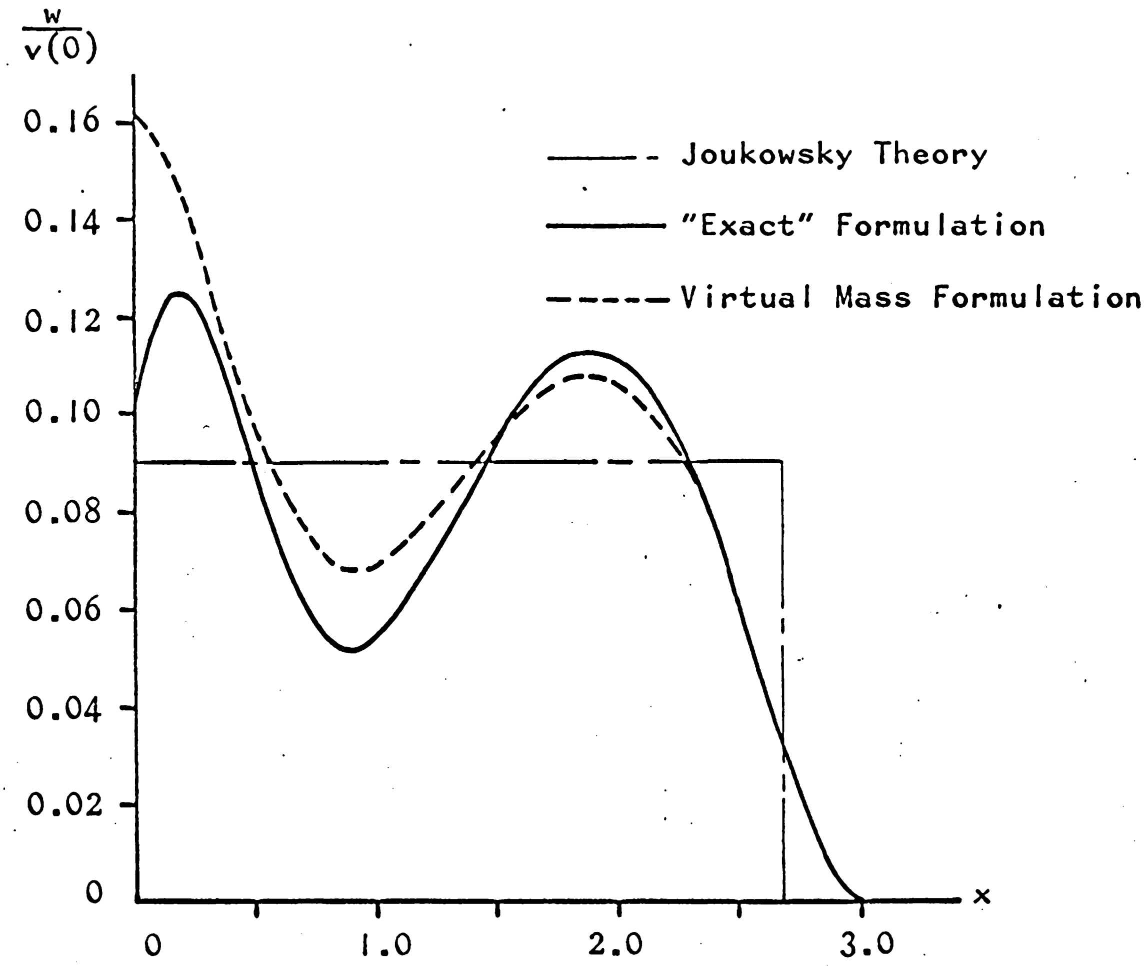
Radial Displacements of the Shell Middle Surface:

Membrane Analysis of Water Hammer in a Steel Shell with h=0.10



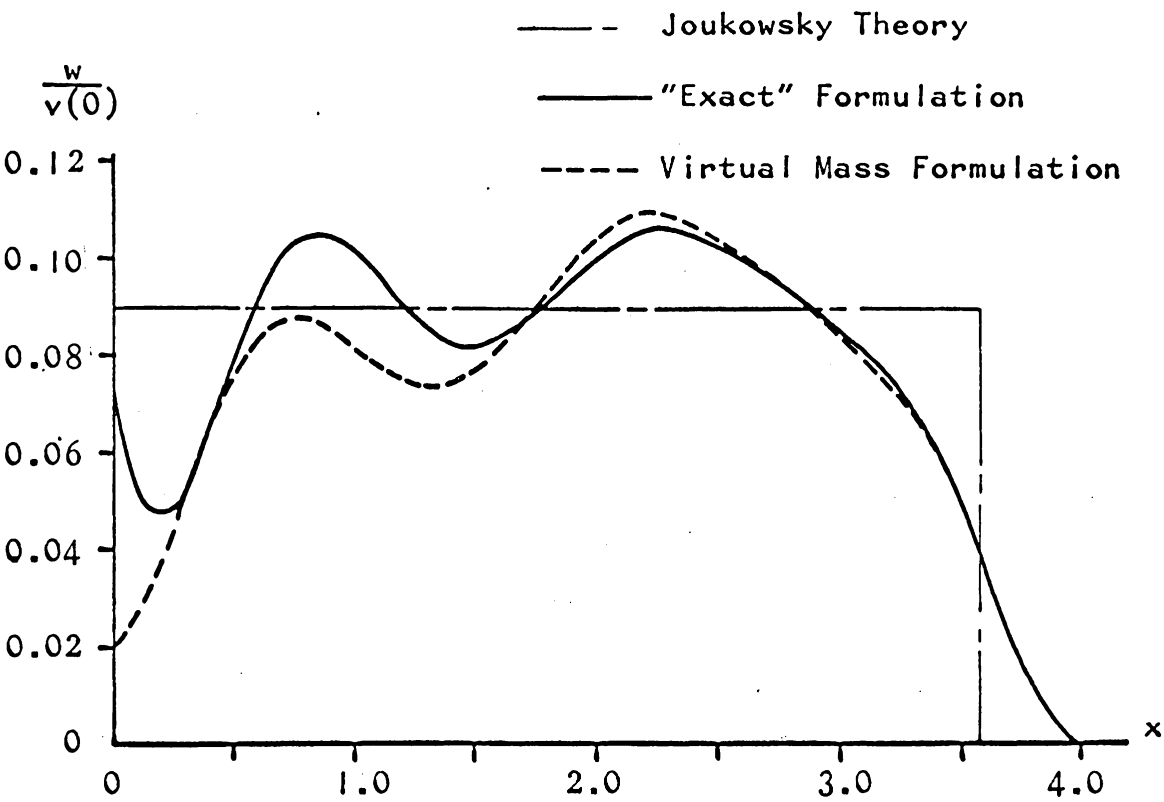
(b) $t = 2.0$

Figure 13 (Continued)



(c) t = 3.0

Figure 13 (Continued)



(d) t = 4.0

Figure 13 (Continued)

SMALL-TIME SOLUTION II:
BENDING ANALYSIS

Representation of the Shell Motion by a System of Ordinary Differential Equations

Although the analysis based upon the membrane theory of shells is of some intrinsic interest, it does not allow consideration of external constraints on motion of the end, or ends, of a cylindrical shell. In most problems of the type considered here the shell is loaded essentially by a pressure wave which moves at a velocity considerably less than either of the characteristic velocities (dilatation and shear) associated with three-dimensional elastic wave propagation in the shell material. Hence it would appear that an analysis based upon the classical bending theory of shells would be sufficient to overcome the objectionable features of a membrane analysis. However, when motion of the shell is described by equations which include bending, but exclude rotary inertia and shear deformation, the equations have a parabolic character in that the theory predicts the propagation of bending disturbances at infinite speed. This was displayed in the dispersion curves of Figure 7 by the absence of a finite asymptote of the short-wavelength phase velocity associated with the second branch of the frequency equation. On the other hand the simplified Herrmann-Mirsky equations, which include the effects of rotary inertia and shear deformation, are hyperbolic and predict the propagation of disturbances at finite speeds. Moreover, as was seen in the preceding section, a system of hyperbolic partial differential equations can be reduced to a system of ordinary differential equations by use of the method of characteristics.

The simplified Herrmann-Mirsky equations, Equations (11), may be written in the following form:

$$\frac{\partial n_x}{\partial x} - \rho h \frac{\partial \dot{u}}{\partial t} = 0 \tag{26a}$$

$$\frac{\partial n_x}{\partial t} - \frac{eh}{1-\nu^2}\frac{\partial \dot{u}}{\partial x} = \frac{\nu\, eh}{1-\nu^2}\dot{w} \tag{26b}$$

$$\frac{\partial m_x}{\partial x} - \frac{\rho h^3}{12}\frac{\partial \dot{\beta}}{\partial t} = q_x \tag{27a}$$

$$\frac{\partial m_x}{\partial t} - \frac{eh^3}{12(1-\nu^2)}\frac{\partial \dot{\beta}}{\partial x} = 0 \tag{27b}$$

$$\frac{\partial q_x}{\partial x} - \rho h \frac{\partial \dot{w}}{\partial t} = \nu n_x + ehw - (1-h/2)p^* \tag{28a}$$

$$\frac{\partial q_x}{\partial t} - \kappa^2 gh \frac{\partial \dot{w}}{\partial x} = \kappa^2 gh\, \dot{\beta} \ , \tag{28b}$$

where use has been made of

$$n_\theta = ehw + \nu\, n_x$$

$$\dot{u} = \frac{\partial u}{\partial t}$$

$$\dot{w} = \frac{\partial w}{\partial t}$$

and $$\dot{\beta} = \frac{\partial \beta}{\partial t} .$$

If Equations (26) are regarded as a system of hyperbolic partial differential equations in the dependent variables n_x and $\dot{u}$, the characteristic directions are given by $\frac{dx}{dt} = \pm c_p$ where c_p is the dimensionless plate velocity (see Appendix). The following differential equations must be satisfied along the characteristics:

Along $\frac{dx}{dt} = c_p$

$$\frac{dn_x}{dt} - c_p \rho h \frac{d\dot{u}}{dt} = \nu c_p^2 \rho h \dot{w} \tag{29a}$$

Along $\frac{dx}{dt} = -c_p$

$$\frac{dn_x}{dt} + c_p \rho h \frac{d\dot{u}}{dt} = \nu c_p^2 \rho h \dot{w} . \tag{29b}$$

Equations (27) yield the same characteristics, $\frac{dx}{dt} = \pm c_p$, and the following differential equations:

Along $\frac{dx}{dt} = c_p$

$$\frac{dm_x}{dt} - \frac{h^3}{12} \rho c_p \frac{d\dot{\beta}}{dt} = c_p q_x \tag{30a}$$

Along $\frac{dx}{dt} = -c_p$

$$\frac{dm_x}{dt} + \frac{h^3}{12} \rho c_p \frac{d\dot{\beta}}{dt} = -c_p q_x . \tag{30b}$$

The characteristic directions of Equations (28) are given by $\frac{dx}{dt} = \pm c_s$, where $c_s = \sqrt{\frac{\kappa^2 G}{\rho}}$ is the propagation velocity of transverse shear disturbances according to the Herrmann-Mirsky theory. The ordinary differential equations which must be satisfied along the characteristics are:

Along $\frac{dx}{dt} = c_s$

$$\frac{dq_x}{dt} - \rho h c_s \frac{d\dot{w}}{dt} = \rho^2 h c_s \dot{\beta} + c_s [\nu n_x + ehw - (1-h/2)p^*] \tag{31a}$$

Along $\frac{dx}{dt} = -c_s$

$$\frac{dq_x}{dt} + \rho h c_s \frac{d\dot{w}}{dt} = \rho h c_s^2 \dot{\beta} - c_s [\nu n_x + ehw - (1-h/2)p^*] \tag{31b}$$

In addition we have the following differential equations along $\frac{dx}{dt} = 0$:

$$\frac{dw}{dt} = \dot{w}$$

$$\frac{du}{dt} = \dot{u}$$

$$\frac{d\beta}{dt} = \dot{\beta} .$$

This description of the shell motion by a system of ordinary differential equations was employed by Tang [16] in his analysis of the response of a cylindrical shell to a moving pressure wave. He found it useful to neglect the longitudinal membrane stress resultant, n_x; it is included here since no advantage is gained by its neglect in our problem.

Analysis of the Water Hammer Problem

The equations developed above are now to be integrated numerically in conjunction with the ordinary differential equations describing the fluid motion. The boundary and initial conditions for the fluid motion are the same as those utilized in the membrane analysis, but we now have the capability to impose external constraints on motion of the end of the shell. In particular we select boundary conditions which correspond to a built-in end of the shell. That is we impose at $x = 0$ the conditions $u = w = \beta = 0$. Note that the condition $\beta = 0$ requires that there be no rotation of a normal to the middle surface of the shell. Other commonly used engineering boundary conditions can be applied with equal facility; those above were chosen because they correspond to a high degree of constraint on motion of the end of the shell.

As before we consider the water hammer problem associated with the sudden extinction of an initially uniform flow. If we prescribe homogeneous initial conditions on motion of the shell, there will be again "disturbed" and "undisturbed" regions in the x -t plane. For $c_p > 1$ and $c_p > c_s$, which is normally the case in problems of this type, the "disturbed" region is given by $x - c_p t < 0$. The initial conditions (all zero except $v_0 = v(0)$ constitute the solution for $x - c_p t > 0$. The solution is to be obtained in the region bounded by $x = 0$, $x = c_p t$, and t = constant as shown in Figure 14. We see that the grid to be used in the calculations is constructed from the characteristic lines given by $\frac{dx}{dt} = \pm c_p$.

The membrane analysis predicted discontinuities in the average pressure and average axial velocity of the fluid, the jumps occurring across the line $x = t$. The same discontinuities result from the present formulation, but now they cause some computational difficulties since $x = t$ is within, rather than a boundary of, the "disturbed" region. These difficulties are avoided by simply ignoring the discontinuities, which is equivalent to requiring that the

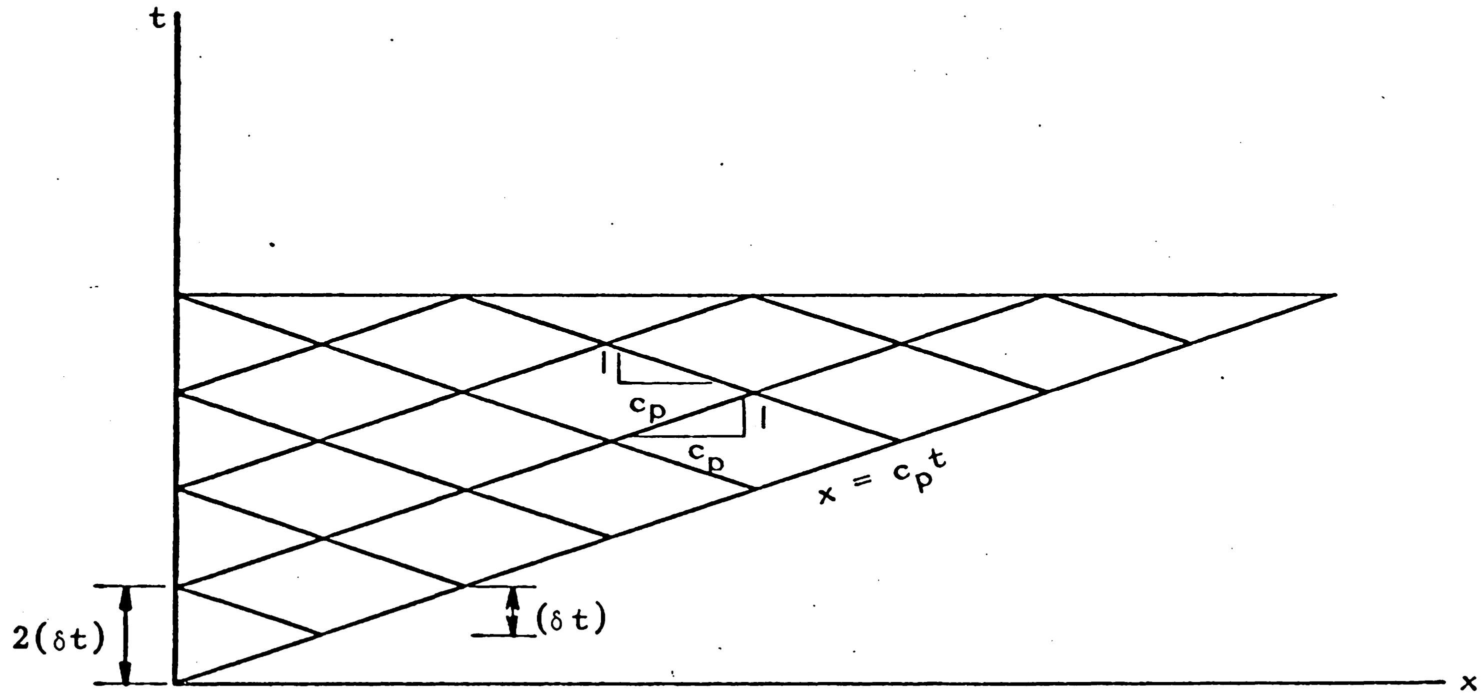


Figure 14

Net for Numerical Calculations: Bending Analysis fluid be brought to rest in a short, but finite, interval of time, thus forcing continuity of the initial-boundary conditions. It is anticipated, and verified by calculations based on the membrane analysis, that the structural response of the shell is negligibly affected.

The water hammer problem to be considered is stated in terms of the boundary conditions ($x = 0$, $t > 0$):

$$u = w = \beta = 0$$

$$v_j = 0 \qquad j = 0, 1, 2, 3, 4$$

and the initial conditions ($x > 0$, $t = 0$):

$$u = \dot{u} = w = \dot{w} = \beta = \dot{\beta} = 0$$

$$n_x = n_\theta = m_x = q_x = 0$$

$$v_j = \psi_j = 0 \qquad j = 1, 2, 3, 4$$

$$p_j = 0 \qquad j = 0, 1, 2, 3, 4$$

$$v_0 = -v(0).$$

Since the initial conditions constitute the solution for $x > c_p t$ and there are no discontinuities across $x = c_p t$, the initial conditions are imposed on the line $x = c_p t$.

The ordinary differential equations obtained by the method of characteristics are to be integrated numerically using the triangular boundary element in Figure 15a and the rhombic interior element in Figure 15b. It is assumed that the dependent variables are known at points "1" and "2" on

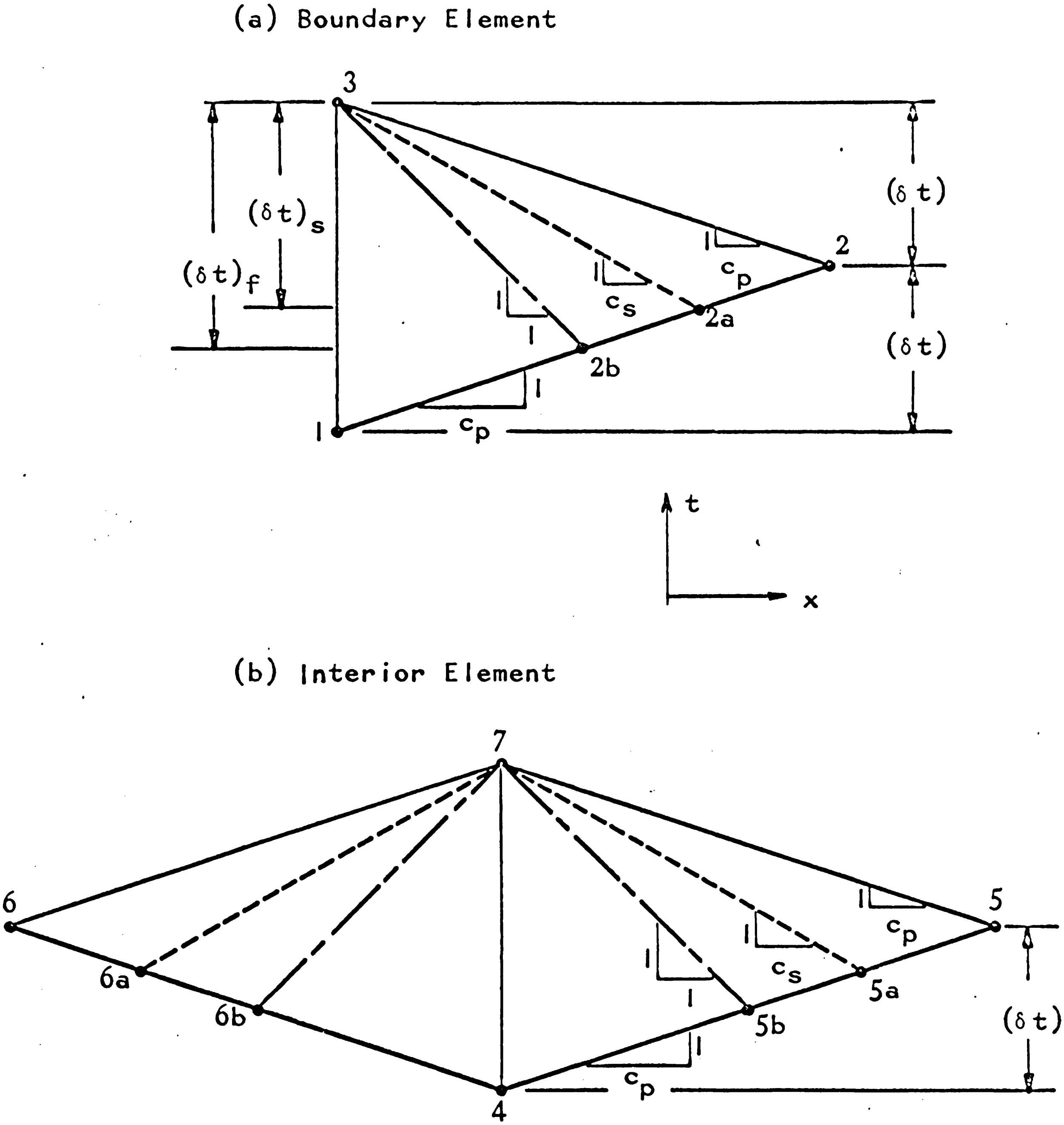


Figure 15

Elements of Computational Net:

Bending Analysis

a boundary element and at points "4", "5", and "6" on an interior element. To accomplish the integration of the differential equations which hold along $\frac{dx}{dt} = \pm 1$ or along $\frac{dx}{dt} = \pm c_s$ it is necessary that the dependent variables be known at "2a", "2b", "5a", "5b", "6a", and "6b". This is accomplished by linear interpolation of the variables between "1" and "2", between "4" and "5", and between "4" and "6". Numerical integration of the ordinary differential equations governing motion of the fluid and motion of the shell is effected in the same manner as before (membrane analysis).

The dependent variables at a boundary point, "3", are determined from the following linear algebraic equations (results of approximate integration):

$$(p_j)_3-(v_j)_3 = (p_j)_{2b}-(v_j)_{2b}+ \gamma_j^2 \left[\frac{(\psi_j)_{2b}+(\psi_j)_3}{2}\right](\delta t)_f$$

$$-\frac{2}{(1-h/2)}\left[\frac{(\dot{w})_{2b}+(\dot{w})_3}{2}\right](\delta t)_f \quad j = 0, 1, 2, 3, 4$$

$$(\psi_j)_3 = (\psi_j)_1 - [(p_j)_1+(p_j)_3](\delta t) \quad j = 0, 1, 2, 3, 4$$

$$(p^*)_3 = \sum_{j=0}^{4} (p_j)_3$$

$$(n_x)_3+ c_p\rho h(\dot{u})_3= (n_x)_2+ c_p\rho h(\dot{u})_2+ c_p^2 \rho h\left[\frac{(\dot{w})_3+(\dot{w})_2}{2}\right](\delta t)$$

$$(m_x)_3 + \frac{\rho h^3}{12} c_p (\dot\beta)_3 = (m_x)_2 + \frac{\rho h^3}{12} c_p (\dot\beta)_2$$

$$- c_p \left[\frac{(q_x)_3 + (q_x)_2}{2}\right] (\delta t)$$

$$(q_x)_3 + \rho h c_s (\dot w)_3 = (q_x)_{2a} + \rho h c_s (\dot w)_{2a}$$

$$+ c_s (\delta t)_s \left\{\frac{\rho h c_s}{2}\left[(\dot\beta)_3 + (\dot\beta)_{2a}\right] - \nu \left[\frac{(n_x)_3 + (n_x)_{2a}}{2}\right]\right.$$

$$\left. - eh\left[\frac{(w)_3 + (w)_{2a}}{2}\right] + (1-h/2)\left[\frac{(p^*)_3 + (p^*)_{2a}}{2}\right]\right\}.$$

These are now solved in conjunction with the boundary conditions $(\dot u)_3 = 0$, $(w)_3 = 0$, $(\dot w)_3 = 0$, $(\dot\beta)_3 = 0$, and $(v_j)_3 = 0$. For this set of boundary conditions, the algebraic equations are weakly coupled and explicit expressions are found easily for each of the variables at "3" in terms of known quantities.

The dependent variables at an interior point, "7", are determined from the following relations:

$$(p_j)_7 - (v_j)_7 = (p_j)_{5b} - (v_j)_{5b} + \gamma_j^2 \left[\frac{(\psi_j)_{5b} + (\psi_j)_7}{2}\right] (\delta t)_f$$

$$- \frac{2}{(1-h/2)} \left[\frac{(\dot w)_{5b} + (\dot w)_7}{2}\right] (\delta t)_f, \quad j = 0, 1, 2, 3, 4$$

$$(p_j)_7 + (v_j)_7 = (p_j)_{6b} + (v_j)_{6b} + \gamma_j^2 \left[\frac{(\psi_j)_{6b} + (\psi_j)_7}{2}\right] (\delta t)_f$$

$$- \frac{2}{(1-h/2)} \left[\frac{(\dot w)_{6b} + (\dot w)_7}{2}\right] (\delta t)_f, \quad j = 0, 1, 2, 3, 4$$

$$(\psi_j)_7 = (\psi_j)_4 - [(p_j)_7 + (p_j)_4](\delta t), \quad j = 1, 2, 3, 4$$

$$(p^*)_7 = \sum_{j=0}^{4} (p_j)_7$$

$$(n_x)_7 + \rho h c_p (\dot{u})_7 = (n_x)_5 + \rho h c_p (\dot{u})_5 + \nu \rho h c_p^2 \left[\frac{(\dot{w})_7 + (\dot{w})_5}{2}\right](\delta t)$$

$$(n_x)_7 - \rho h c_p (\dot{u})_7 = (n_x)_6 - \rho h c_p (\dot{u})_6 + \nu \rho h c_p^2 \left[\frac{(\dot{w})_7 + (\dot{w})_5}{2}\right](\delta t)$$

$$(m_x)_7 + \frac{\rho h^3}{12} c_p (\dot{\beta})_7 = (m_x)_5 + \frac{\rho h^3}{12} c_p (\dot{\beta})_5 - c_p\left[\frac{(q_x)_7 + (q_x)_5}{2}\right](\delta t)$$

$$(m_x)_7 - \frac{\rho h^3}{12} c_p (\dot{\beta})_7 = (m_x)_6 - \frac{\rho h^3}{12} c_p (\dot{\beta})_6 + c_p\left[\frac{(q_x)_7 + (q_x)_6}{2}\right](\delta t)$$

$$(q_x)_7 + \rho h c_s (\dot{w})_7 = (q_x)_{5a} + \rho h c_s (\dot{w})_{5a} + c_s(\delta t)_s \left\{ \frac{\rho h}{2} c_s \left[(\dot{\beta})_7 + (\dot{\beta})_{5a}\right] - \nu\left[\frac{(n_x)_7 + (n_x)_{5a}}{2}\right] - eh\left[\frac{(w)_7 + (w)_{5a}}{2}\right] + (1 - h/2)\left[\frac{(p^*)_7 + (p^*)_{5a}}{2}\right]\right\}$$

$$(q_x)_7 - \rho h c_s (\dot{w})_7 = (q_x)_{6a} - \rho h c_s (\dot{w})_{6a}$$
$$+ c_s (\delta t)_s \{ \frac{\rho h}{2} c_s x [(\dot{\beta})_7 + (\dot{\beta})_{6a}] + \nu [\frac{(n_x)_7 + (n_x)_{6a}}{2}]$$
$$+ eh [\frac{(w)_7 + (w)_{6a}}{2}] - (1 - h/2) [\frac{(p^*)_7 + (p^*)_{6a}}{2}]\}$$

$$(w)_7 = (w)_4 + [(\dot{w})_7 + (\dot{w})_4] (\delta t) .$$

These 21 linear simultaneous equations are solved by an iteration scheme similar to that employed in the membrane analysis (actually it was found desirable to iterate on two variables, q_x and $\dot{w}$, here). Note that expressions for n_θ, u, and β are not given explicitly here since the former is easily written in terms of w and n_x, and the latter two are of minor interest to this investigation.

Numerical computations were performed for a water-filled steel shell. The parameters used were $\rho = 7.84$, $e = 100$, $\nu = 0.30$, $\kappa^2 = 0.86$, and $h = 0.10$. The relatively thick ($h = 0.10$) shell was chosen for the following reasons:

1. The membrane solution presented in an earlier section indicates that for this case the pressure acting on the shell is represented adequately by $p^* = \sum_{j=0}^{4} p_j$.

2. Anticipating from solutions of statical problems that the moment and shear stress resultants vary rapidly in the neighborhood of the end of the shell, the distance (along t = constant) between adjacent grid points in the x-t plane probably should be less than the thickness of the shell. Hence for a relatively thin shell a large number of computations are necessary to provide information about the motion at points several radii away from the end of the shell. That a very thin shell is not treated here is a concession to the limitations of the computing facilities available to the author.

3. The influences of moment and shear stress resultants are likely to be significant in the case of a relatively thick shell.

Calculations were performed on a Burroughs B-5500 computer using a maximum of approximately 22,500 grid points and 15 minutes of computer time. Results were collected along the boundaries of the triangular region (Figure 14) of the x-t plane in which the calculations were performed.

It was feasible to determine the system response from $t = 0$ through $t = 3.00$. For $t = 0$ through $t = 1.50$ a grid given by $\delta t = 0.005$ was used, and for $t = 1.75$ through $t = 3.00$ the grid size ranged from $\delta t = 0.0075$ to $\delta t = 0.011$ in order to accommodate the collection of results along lines of t = constant at time intervals of 0.25. It should be recalled that $\delta t = 0.1$ is the time required for an acoustic wave in the fluid to traverse one shell thickness, and during that time a disturbance travelling at the plate speed, c_p, traverses slightly less than four thicknesses.

The effects of grid size on the computations are illustrated in Figures 16, 17, and 18. The radial displacement and bending stress resultant in the shell and the fluid pressure were computed at $t = 1.5$ using $\delta t = 0.005$ and using $\delta t = 0.010$. Calculations of the average pressure and the radial displacement (Figures 16 and 17) are seen to be rather insensitive to the grid size in regions where magnitudes of the water hammer disturbances are significant. On the other hand there are slightly more significant differences in the bending stress resultants at points relatively distant from the end of the shell; the larger grid leads to underestimates of the moment peaks in regions

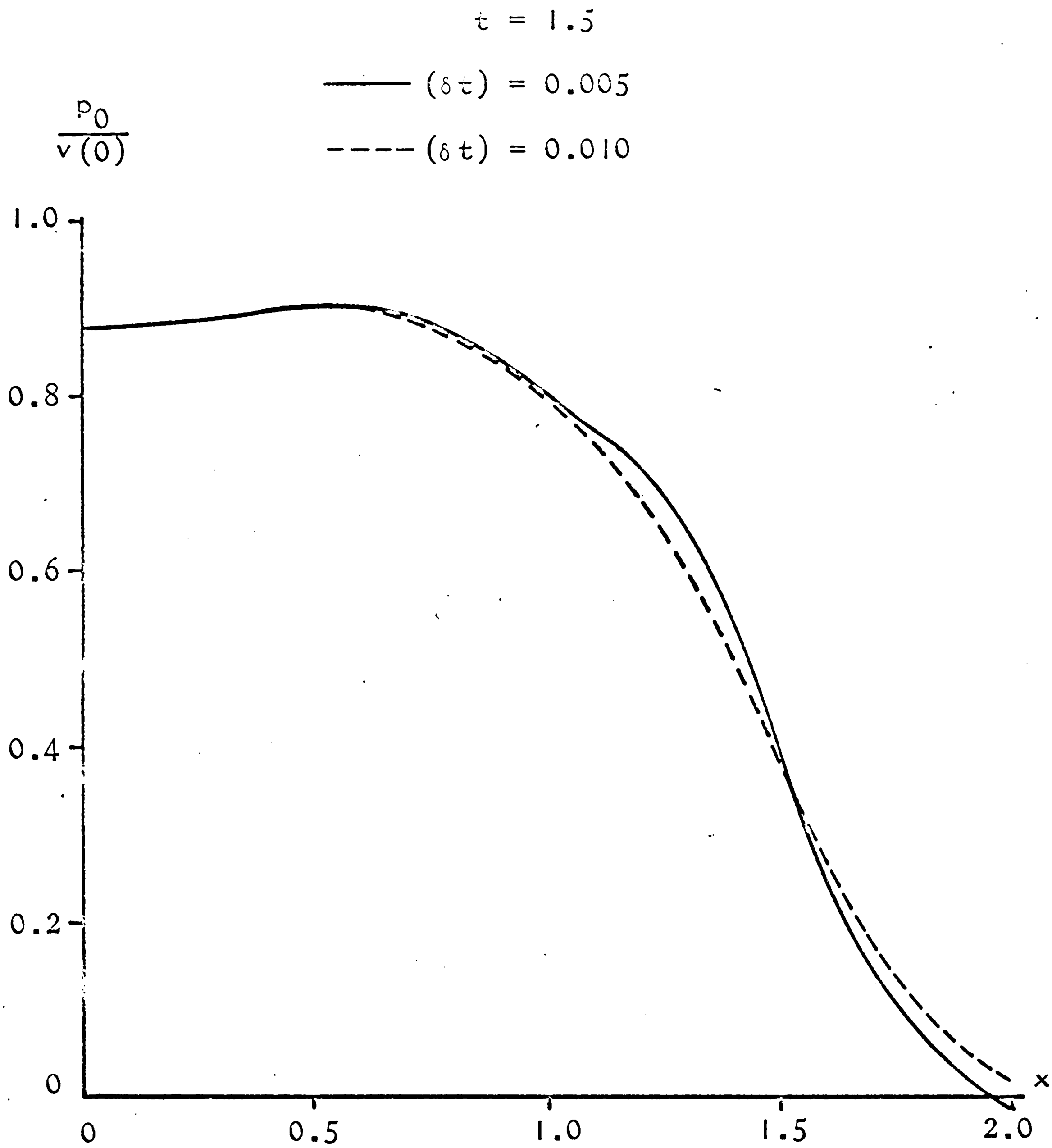


Figure 16

Effect of Grid Size on the Computation of Average Pressures: Bending Analysis of Water Hammer in a Steel Shell with Built-in End

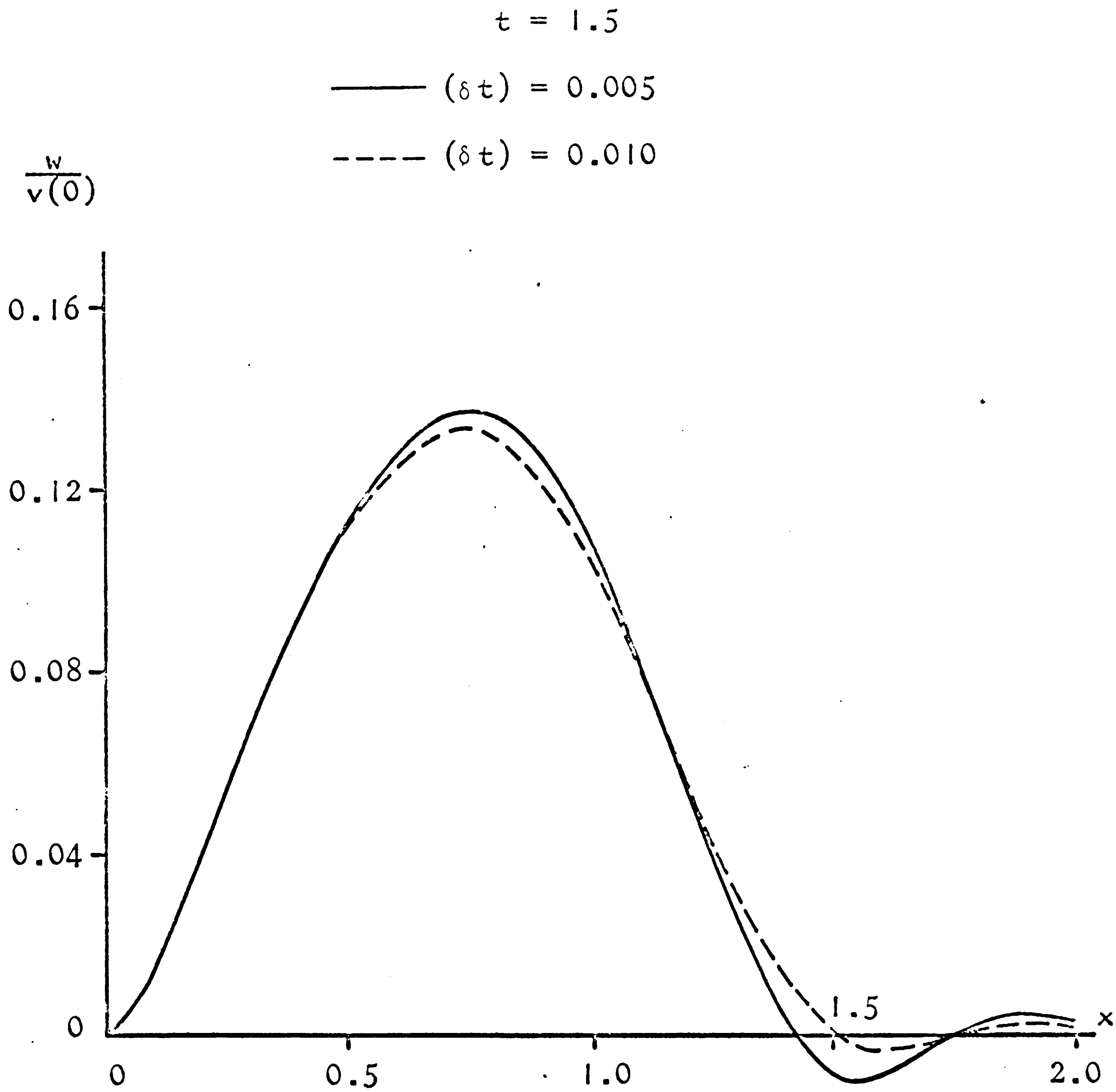


Figure 17

Effect of Grid Size on the Computation of Radial Displacements: Bending Analysis of Water Hammer in a Steel Shell with Built-in End

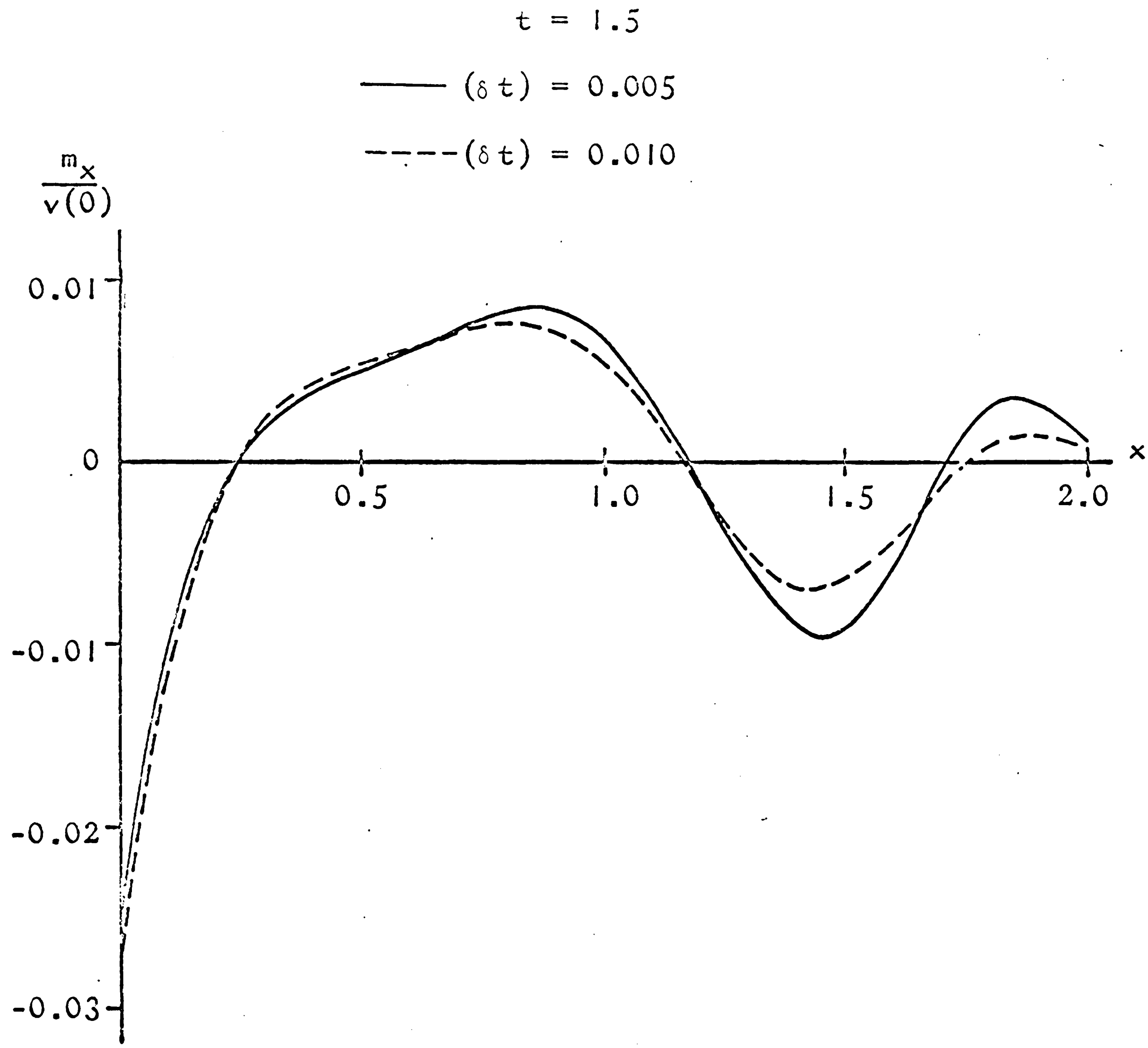


Figure 18

Effect of Grid Size on the Computation of Bending Moments: Bending Analysis of Water Hammer in a Steel Shell with Built-in End where the moment oscillates rapidly. Fortunately values of the moment computed near the end of the shell are in close agreement. Summarizing, it appears that the grid used is sufficiently small to predict behavior of the system in the region $x < t$ where the principal water hammer disturbance occurs, and the results of computations based on the smallest grid ($\delta t = 0.005$) are expected to be valid throughout the x-t plane.

Results of Numerical Calculations

The average pressure in the fluid and the pressure acting on the shell at various times are shown in Figure 19. At the end of the shell for $t \geq 1.0$ each of these pressures is very close to that predicted by the Joukowsky theory, and the initially steep fronts spread out continuously with increasing time as expected.

Of particular interest is the structural response of the shell (Figures 20 through 24), about which the following observations can be made:

1. Maximum moment and shear stress resultants (Figures 20 and 21) occur at the end of the shell. Each of these stress resultants decays

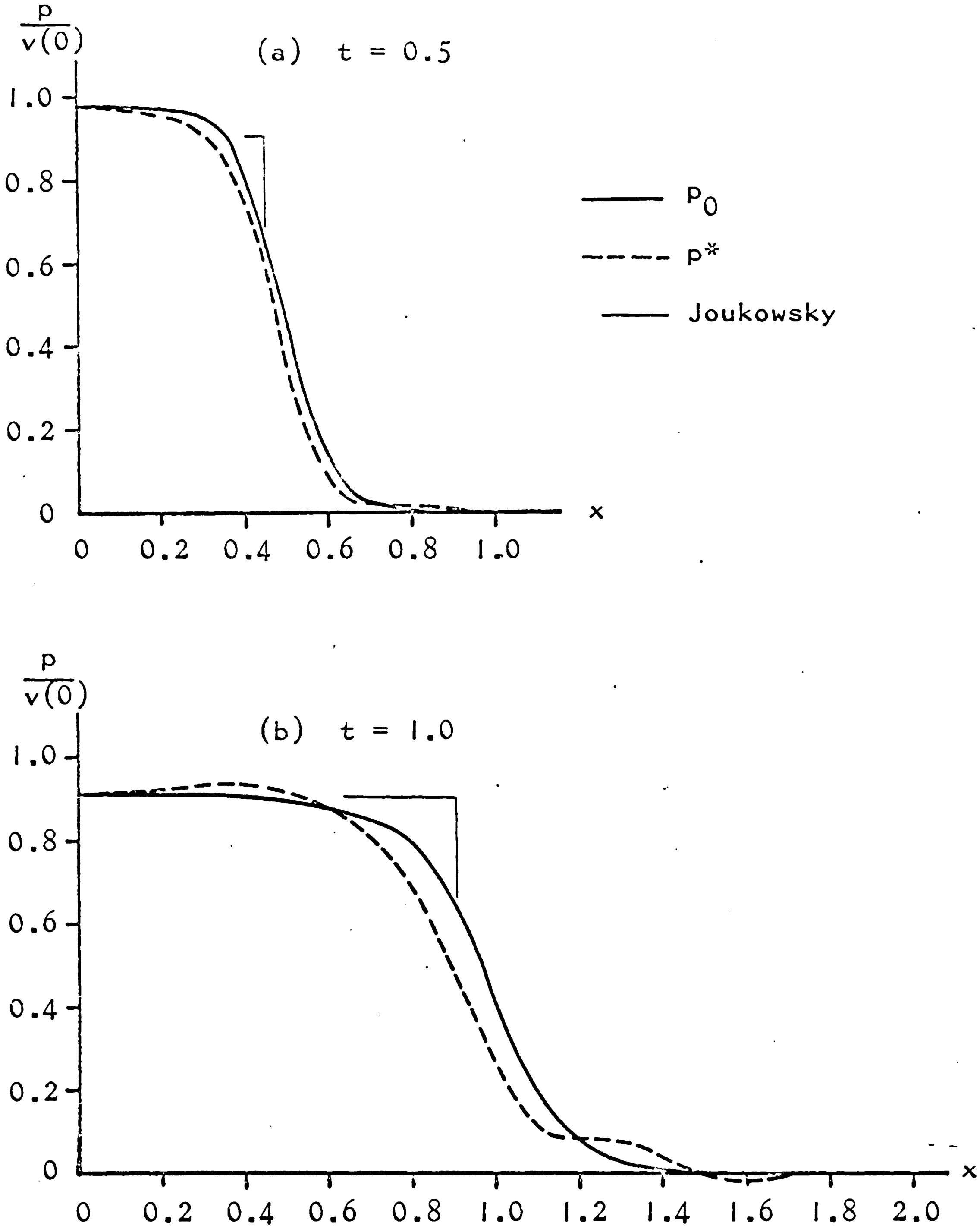


Figure 19

Pressures: Bending Analysis of Water Hammer in a Steel Shell with Built-in End

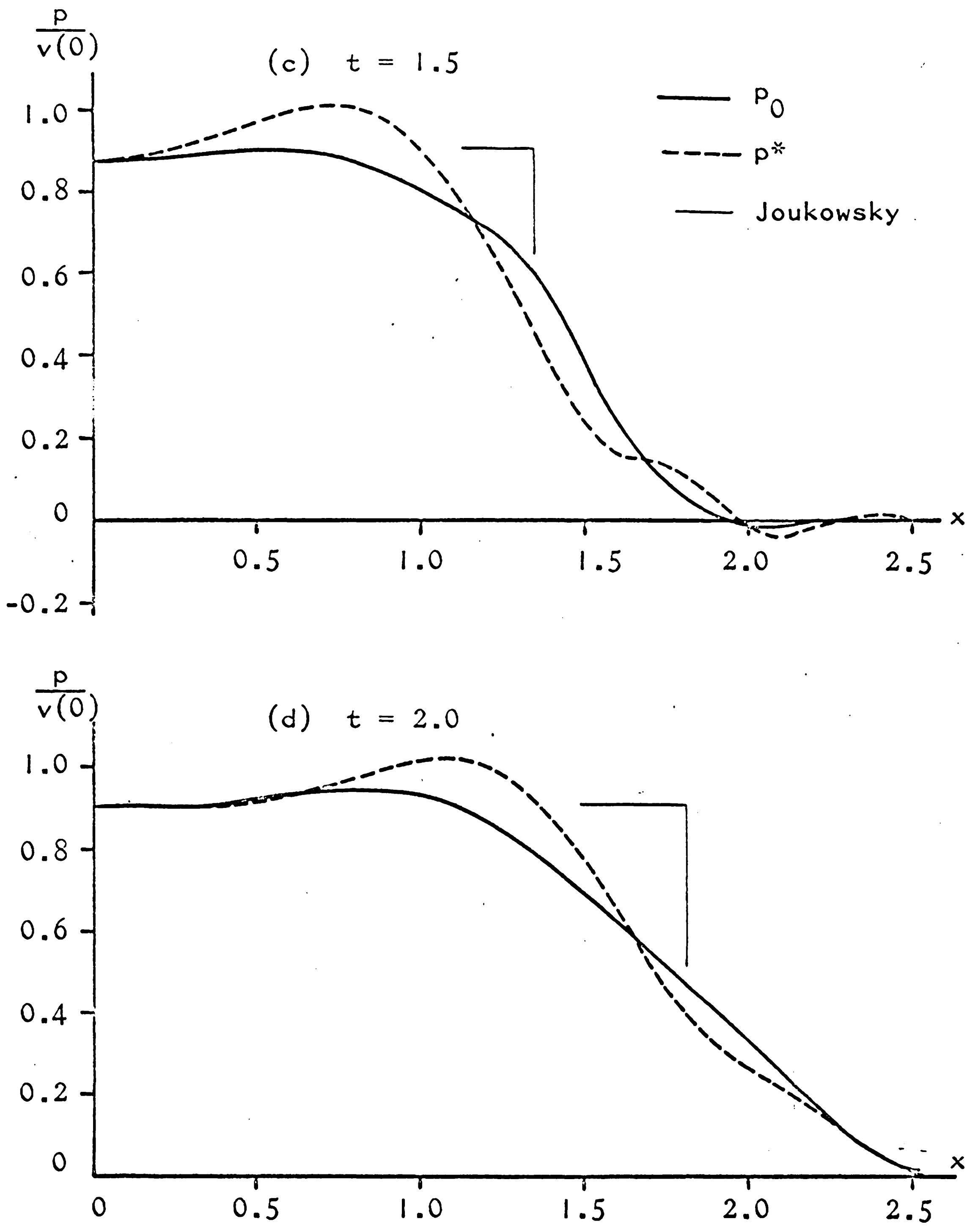


Figure 19 (Continued)

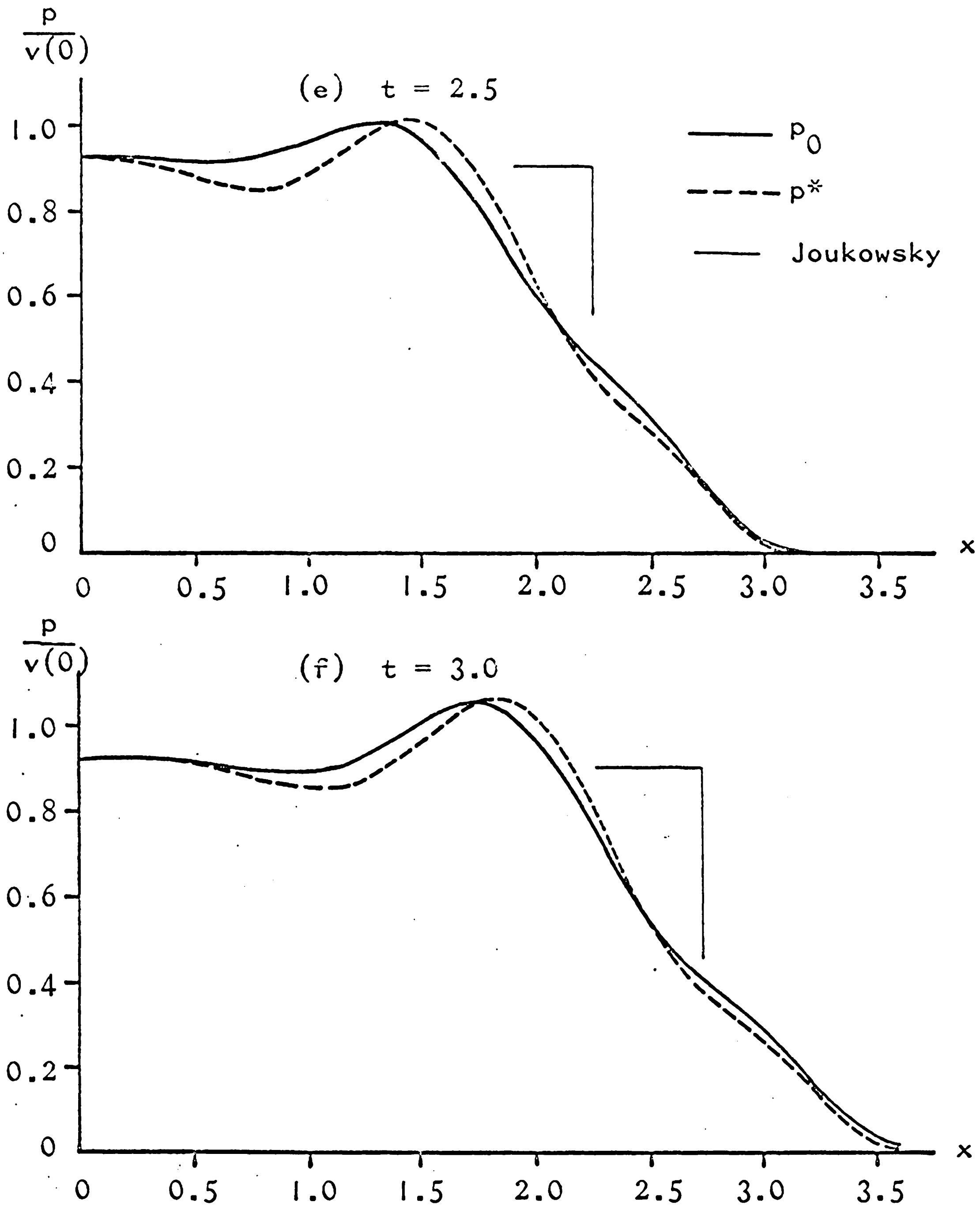


Figure 19 (Continued)

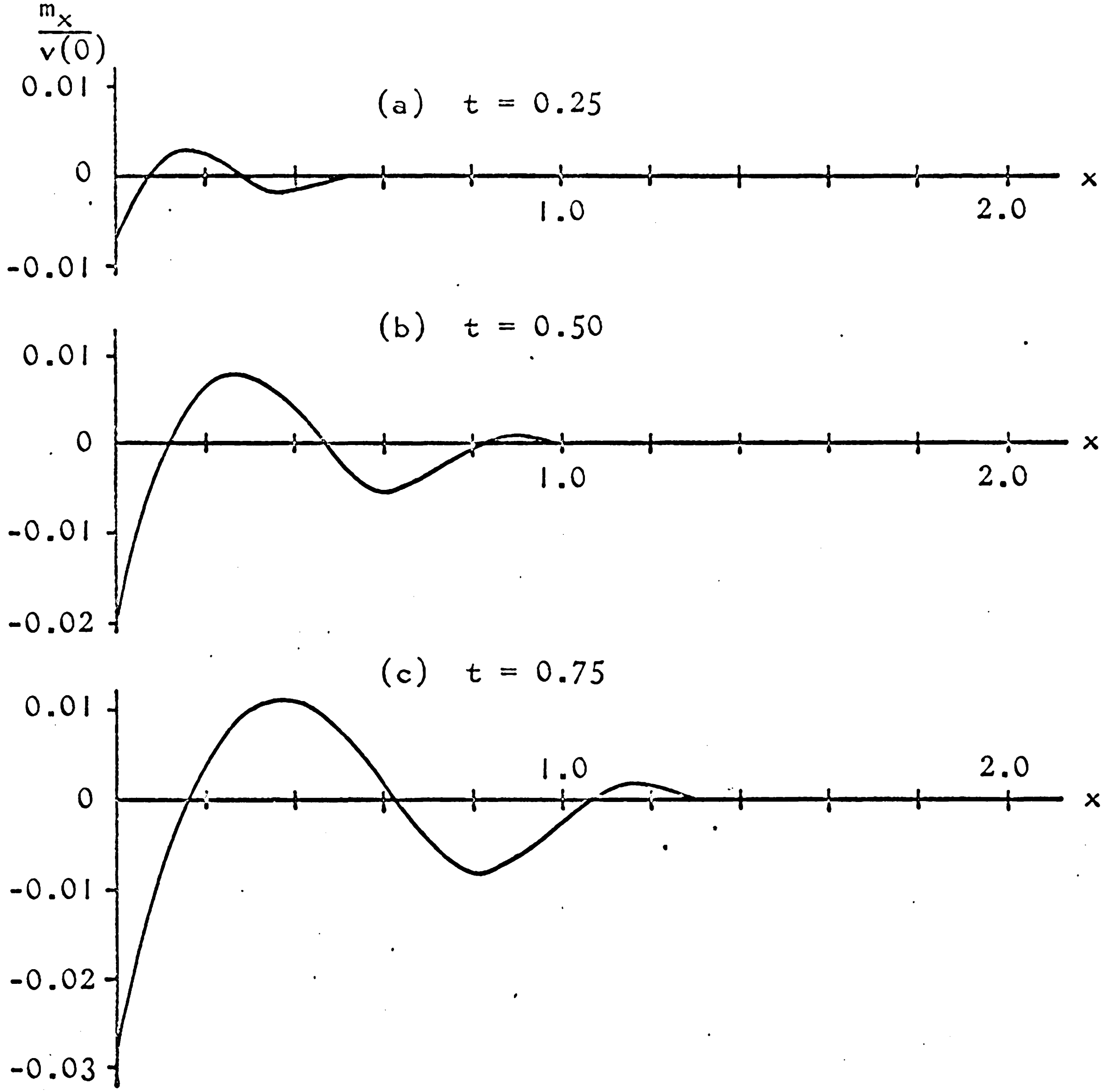


Figure 20

Bending Moment Stress Resultants: Bending Analysis of Water Hammer in a Steel Shell with Built-in End

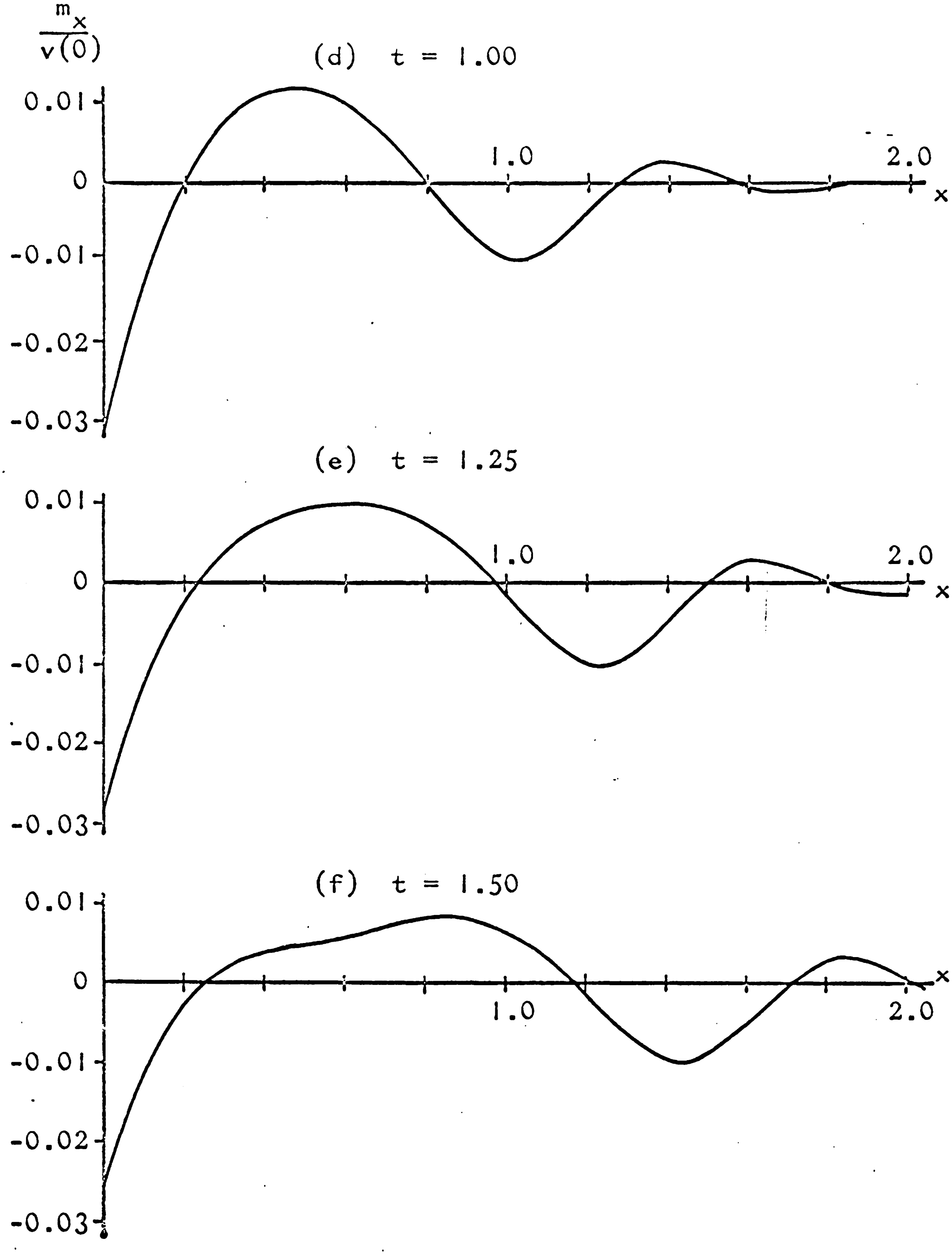


Figure 20 (Continued)

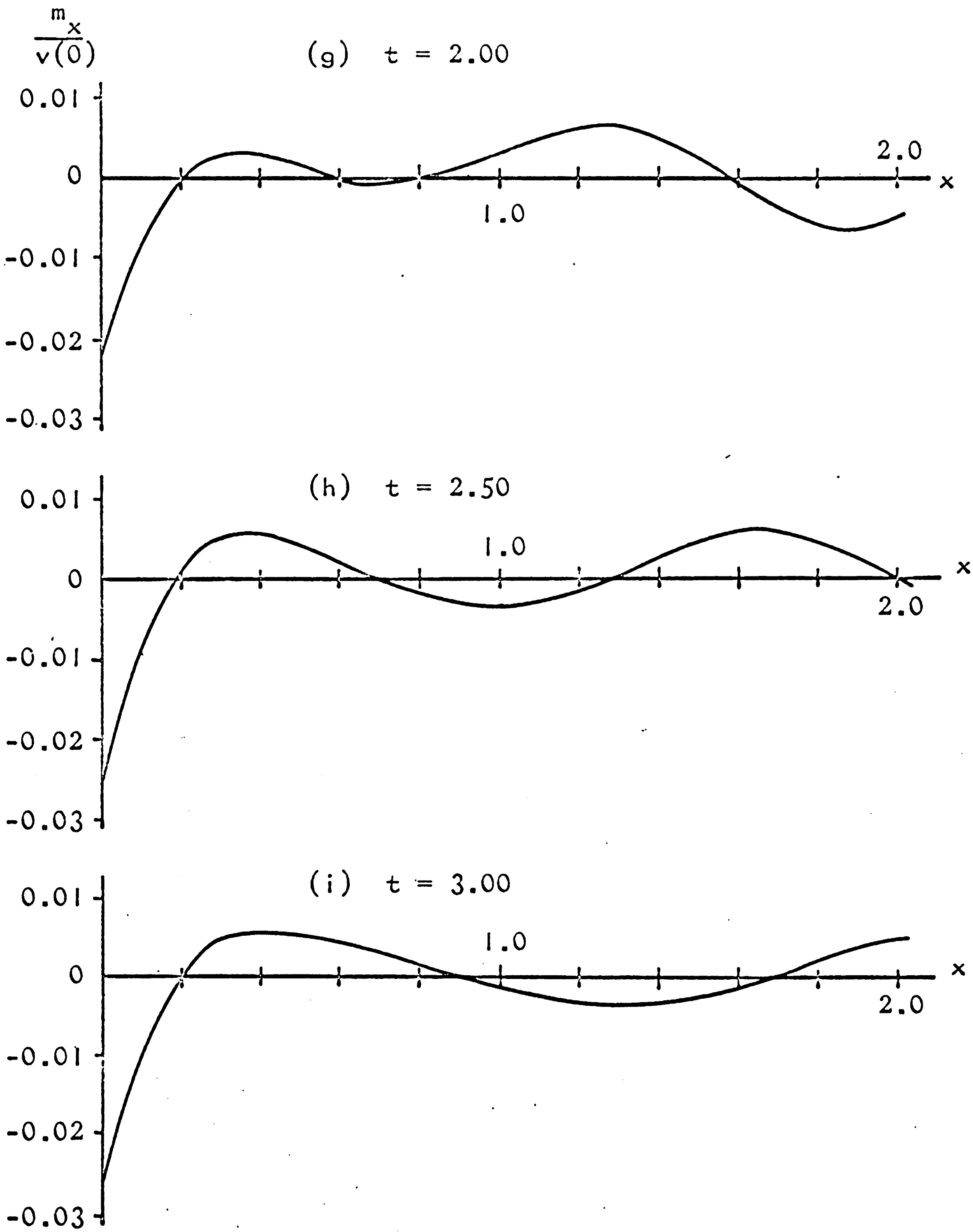


Figure 20 (Continued)

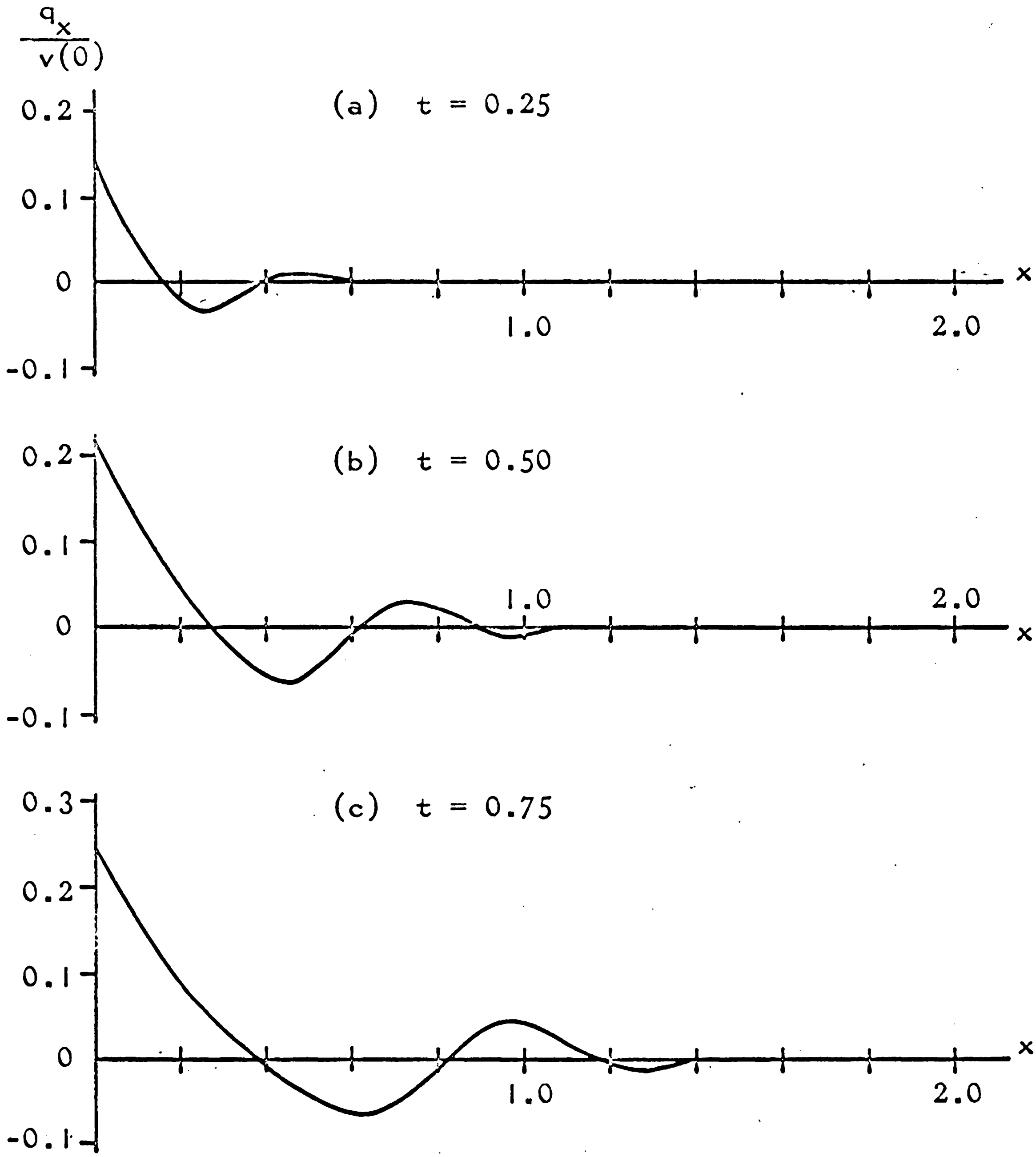


Figure 21

Transverse Shear Stress Resultants: Bending Analysis of Water Hammer in a Steel Shell with Built-in End

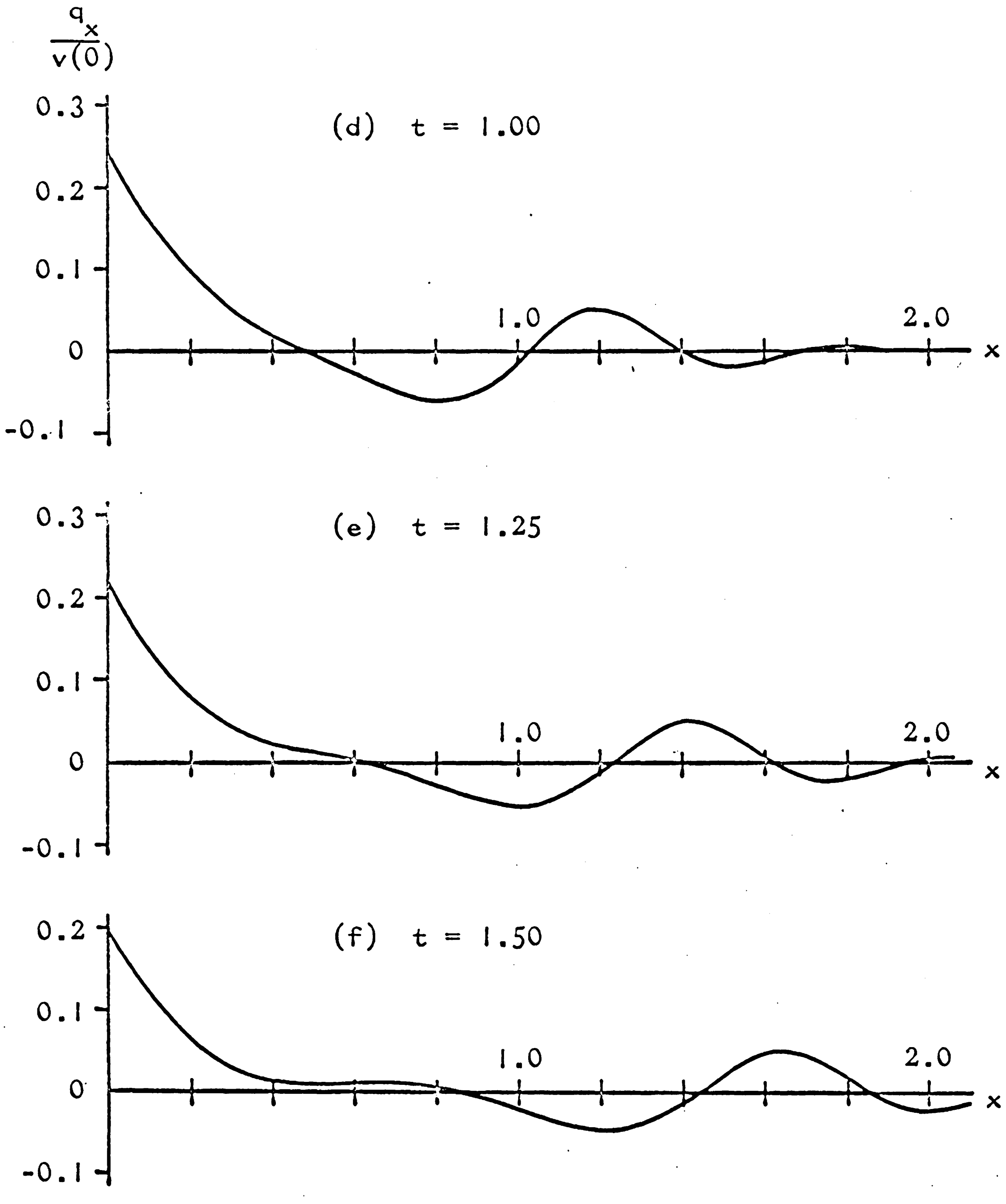


Figure 21 (Continued)

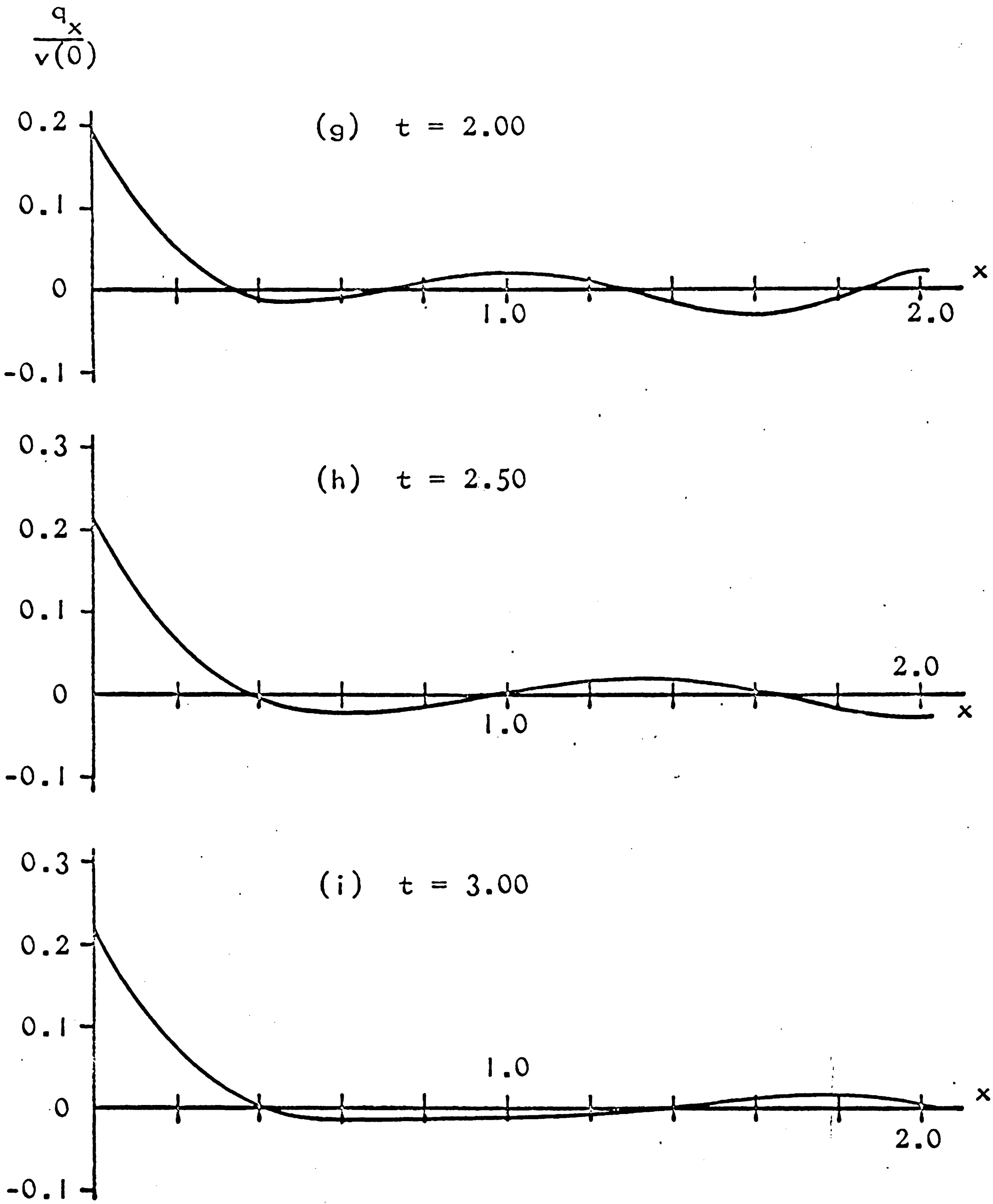


Figure 21 (Continued)

rapidly from the end of the shell and simultaneously undergoes spatial oscillations.

2. The longitudinal membrane stress resultant, n_x, (not plotted) behaves in a manner similar to that of m_x or q_x. However, the average longitudinal stresses, n_x/h, are of negligible magnitudes.

3. At the end of the shell m_x and q_x are given as functions of time in Figures 22 and 23. They appear to approach asymptotically, for increasing time, the values obtained by generating the solution according to the Joukowsky theory and then applying statically moment and shear stress resultants to the end of the shell to produce the desired boundary conditions. These asymptotes are labelled "static" in Figures 22 and 23.

4. The maximum radial displacement of the shell (Figure 24) exceeds the value predicted by the elementary water hammer theory by about fifty percent. This maximum occurs at approximately $t = 1.5$.

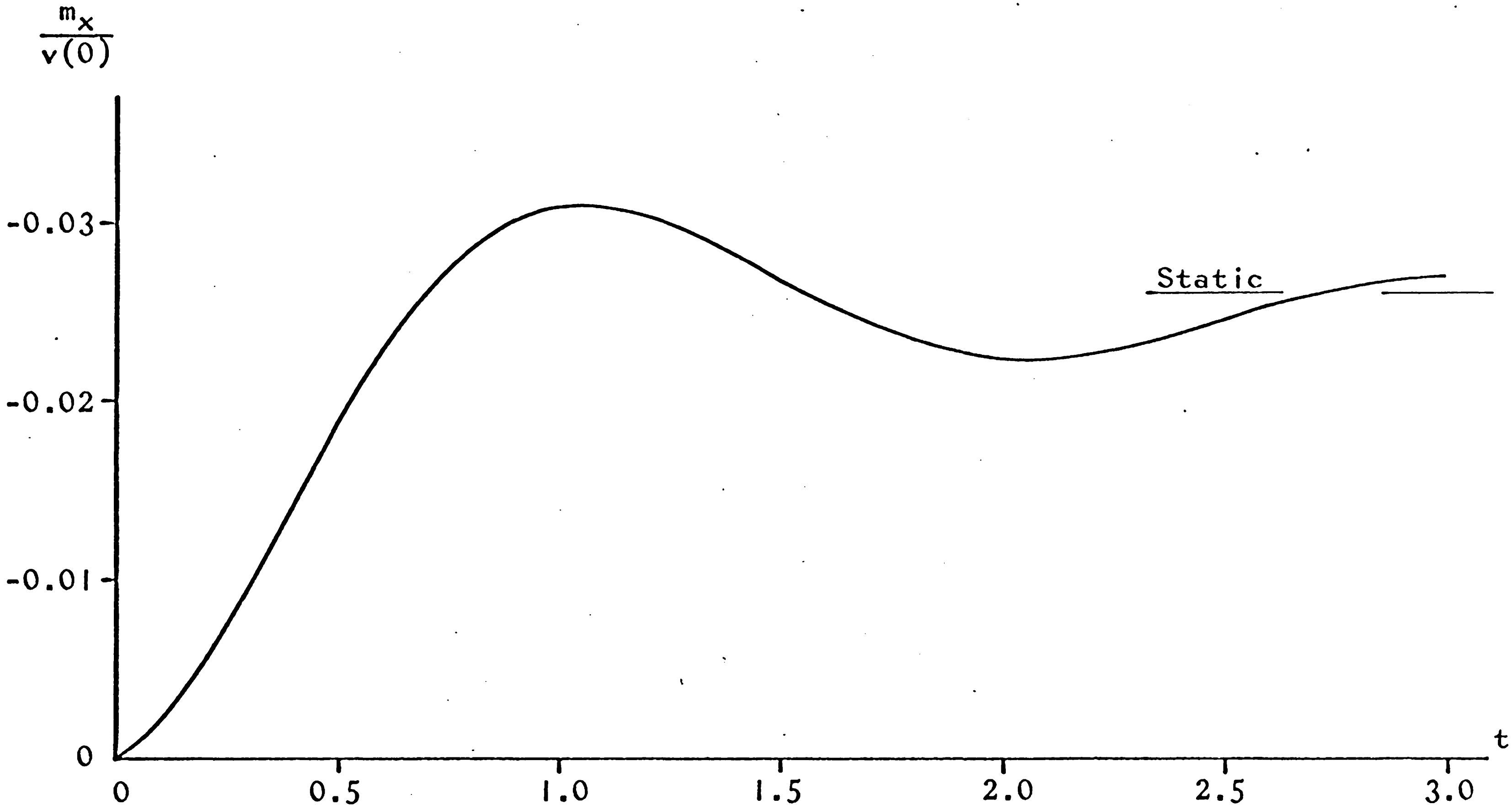


Figure 22

Moment Stress Resultant at Built-in End:

Bending Analysis of Water Hammer

in a Steel Shell

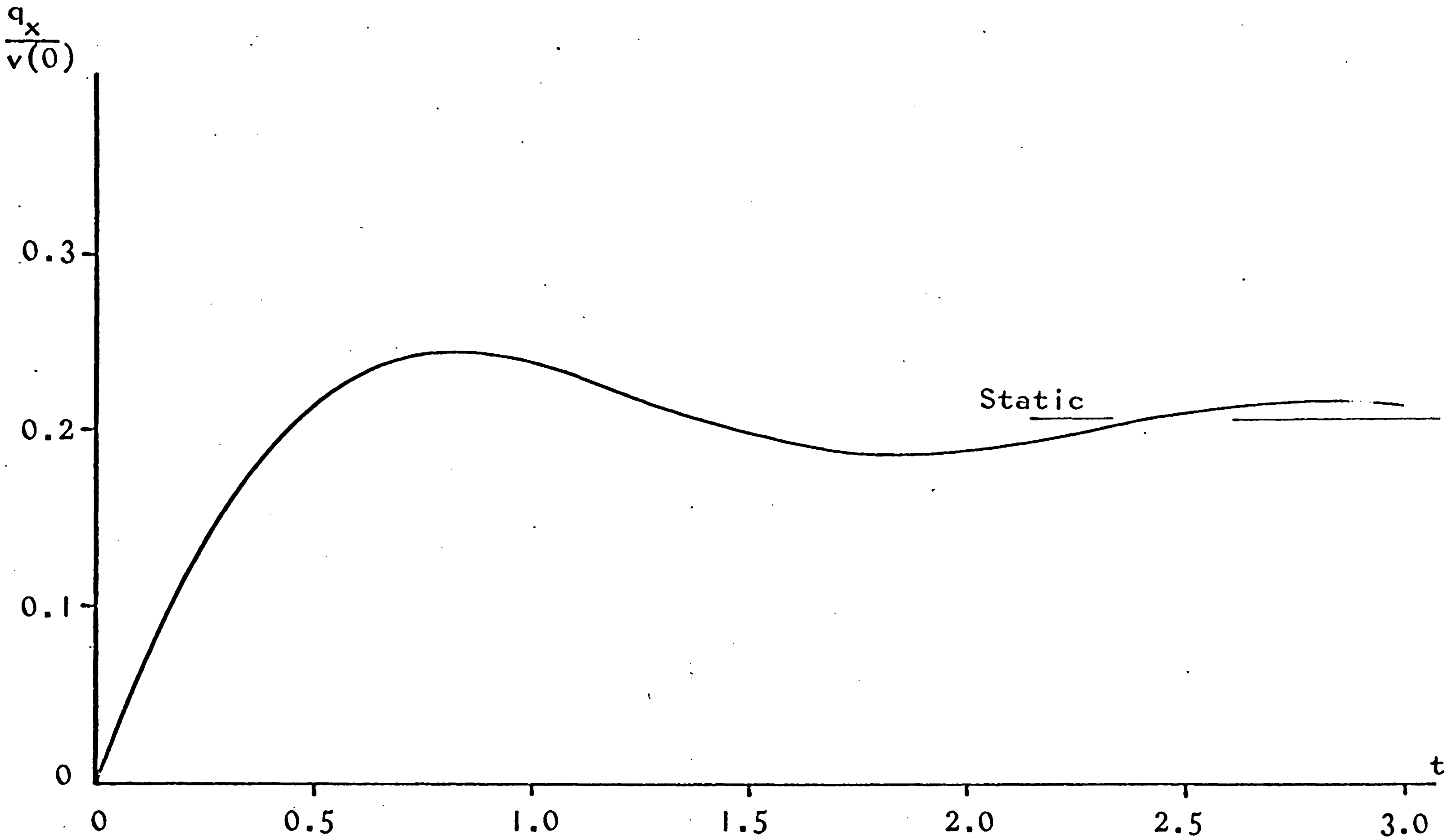
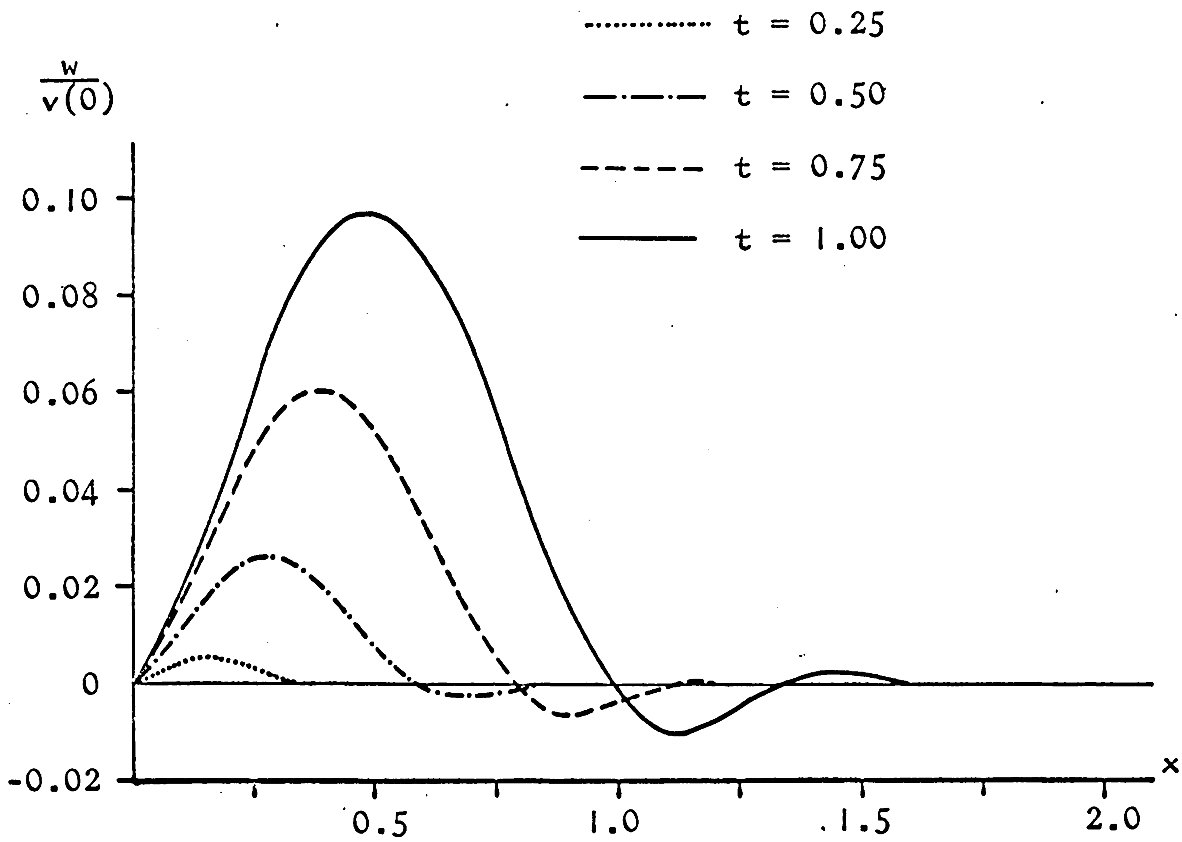


Figure 23

Transverse Shear Stress Resultant at Built-in End:

Bending Analysis of Water Hammer in a Steel Shell

III



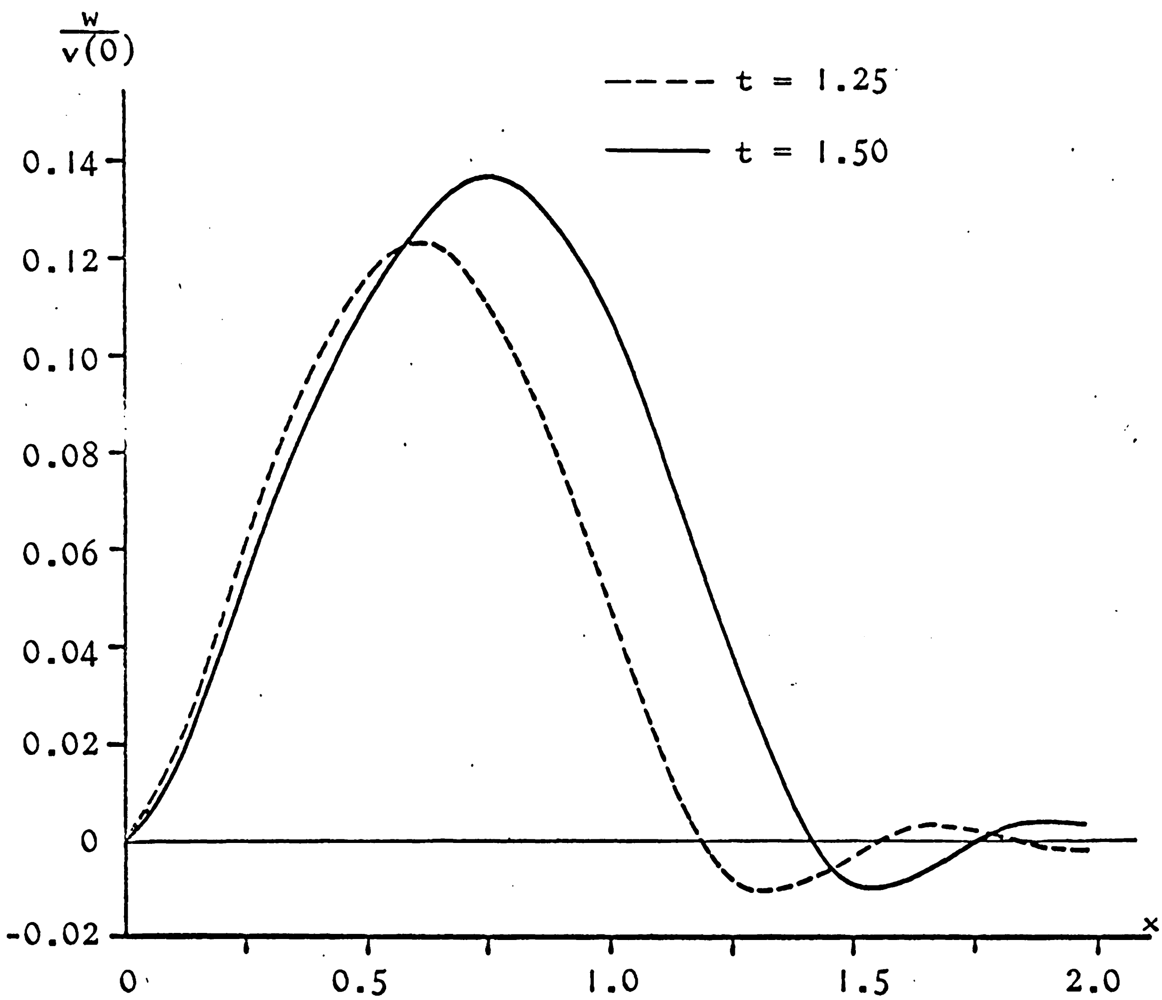
(a) t = 0.25 - t = 1.00

Figure 24

Radial Displacements of the Shell Middle Surface:

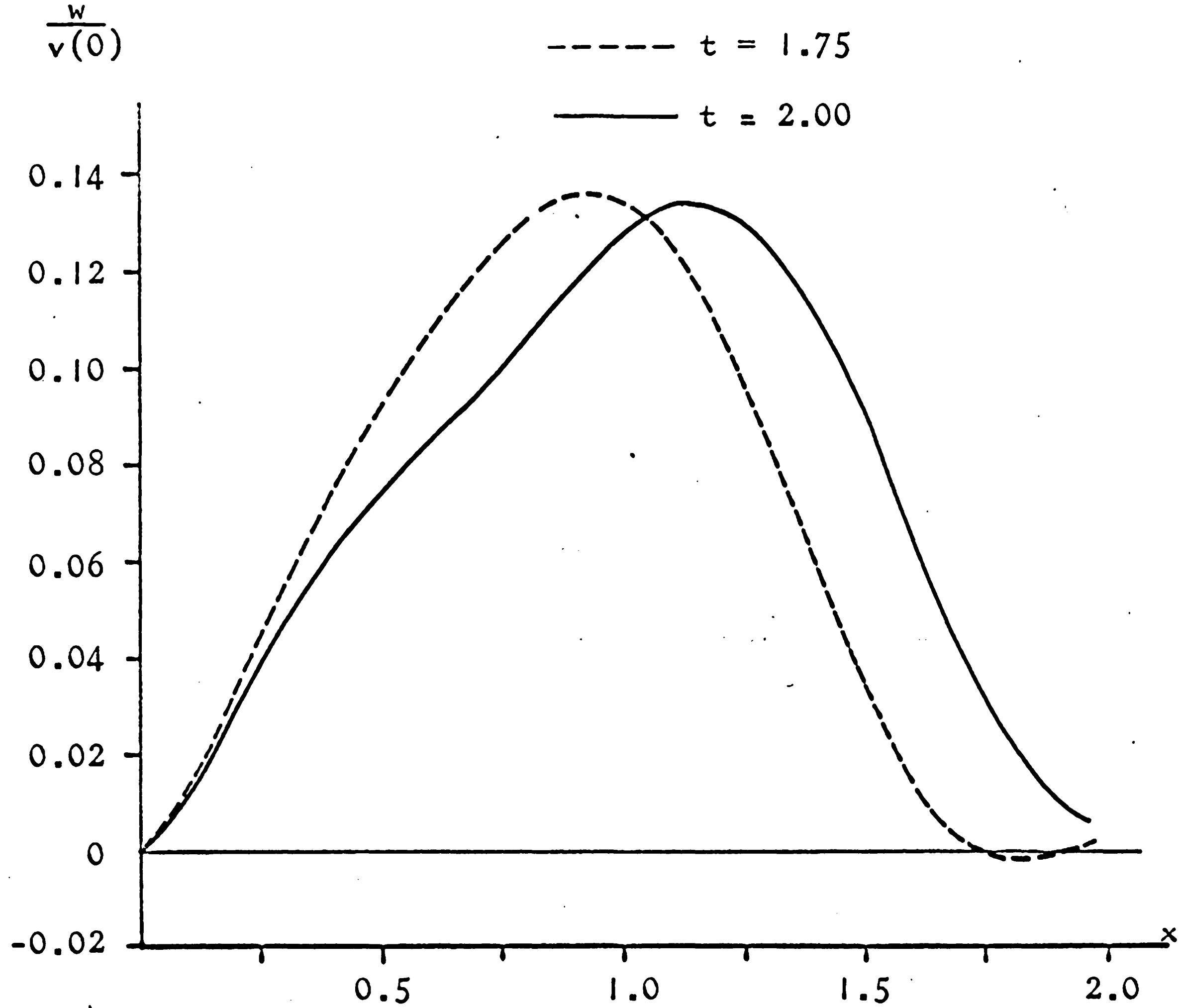
Bending Analysis of Water Hammer in a

Steel Shell with Built-in End



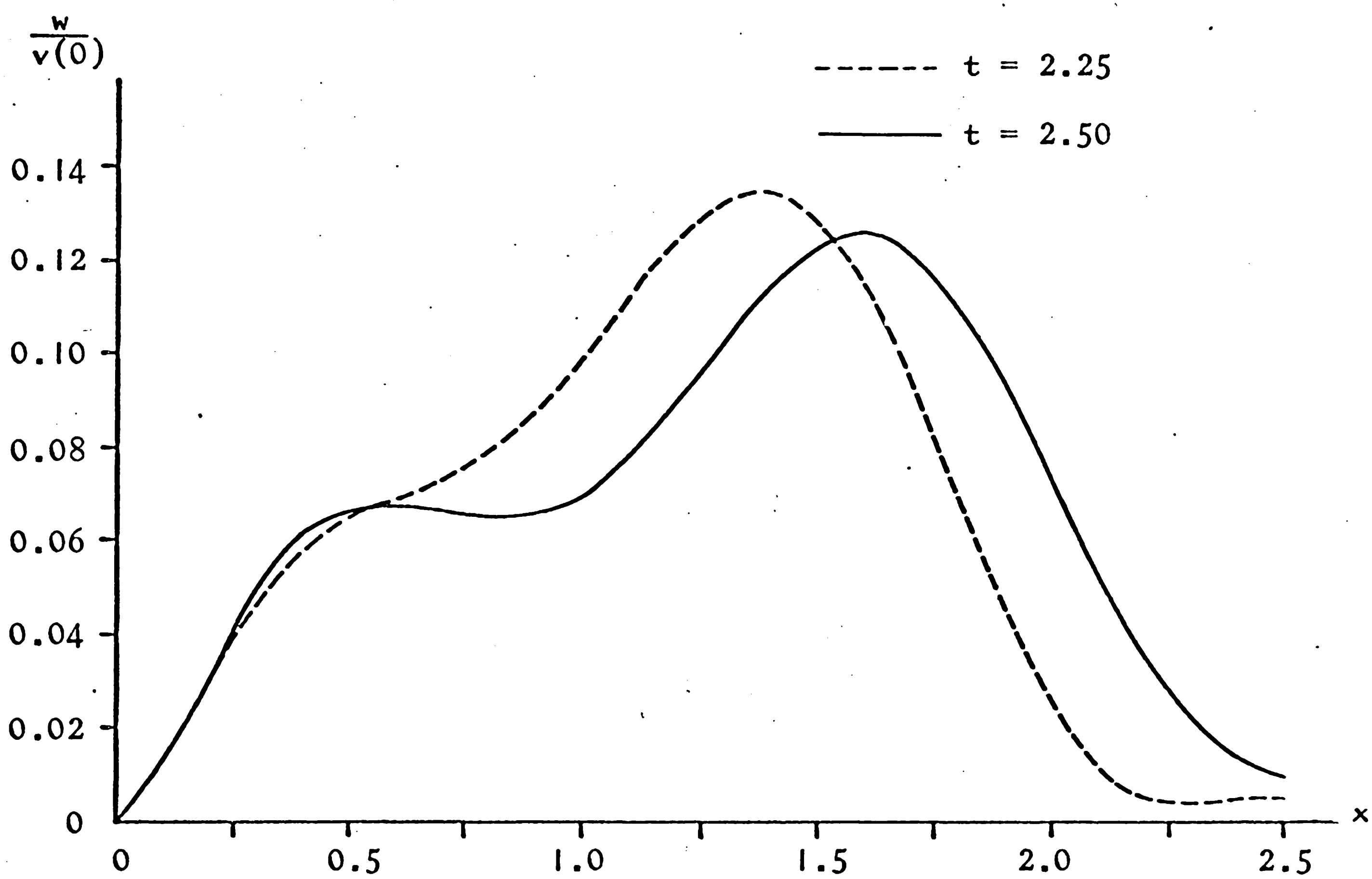
(b) t = 1.25 - t = 1.50

Figure 24 (Continued)



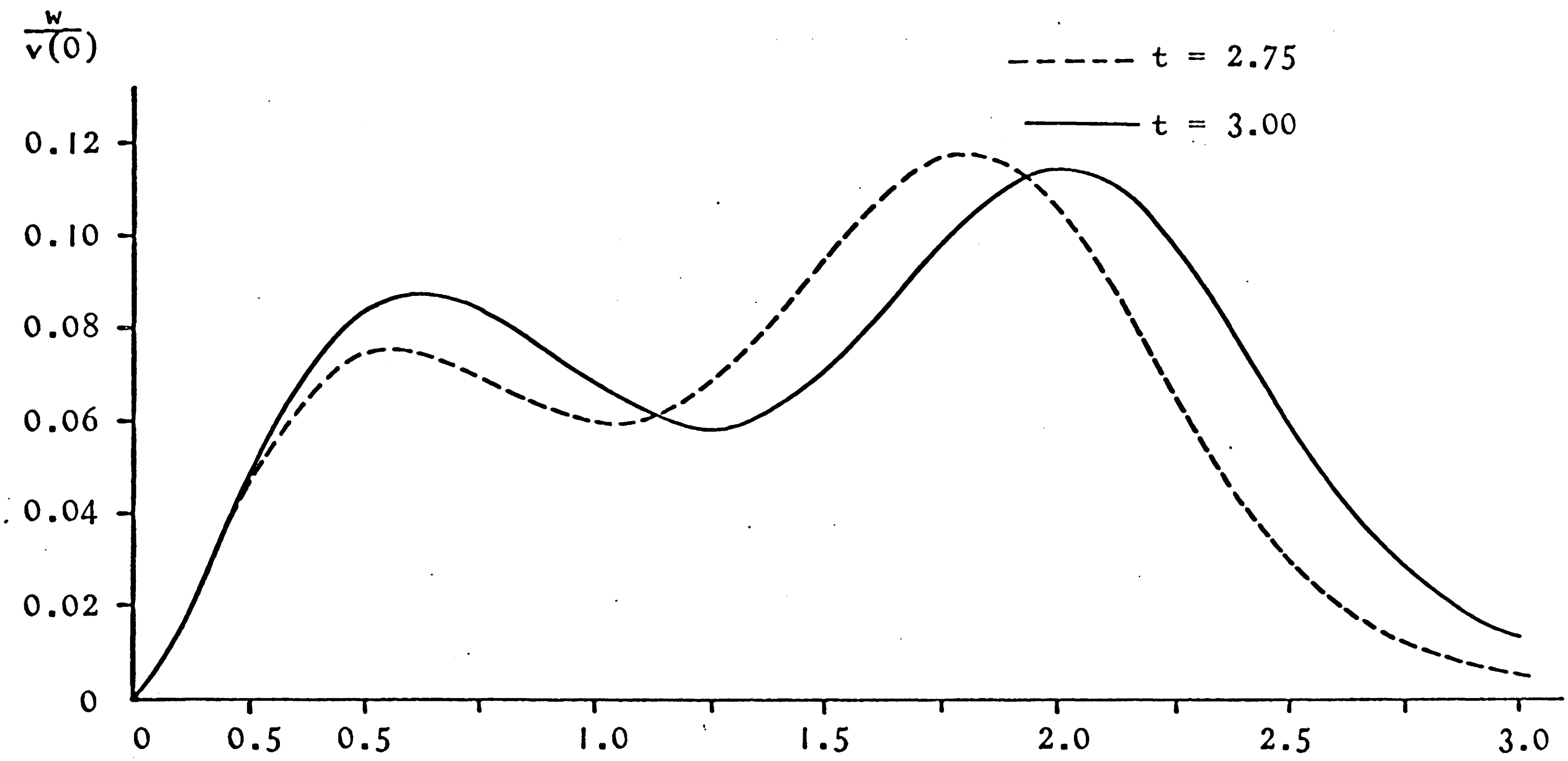
(c) t = 1.75 - t = 2.00

Figure 24 (Continued)



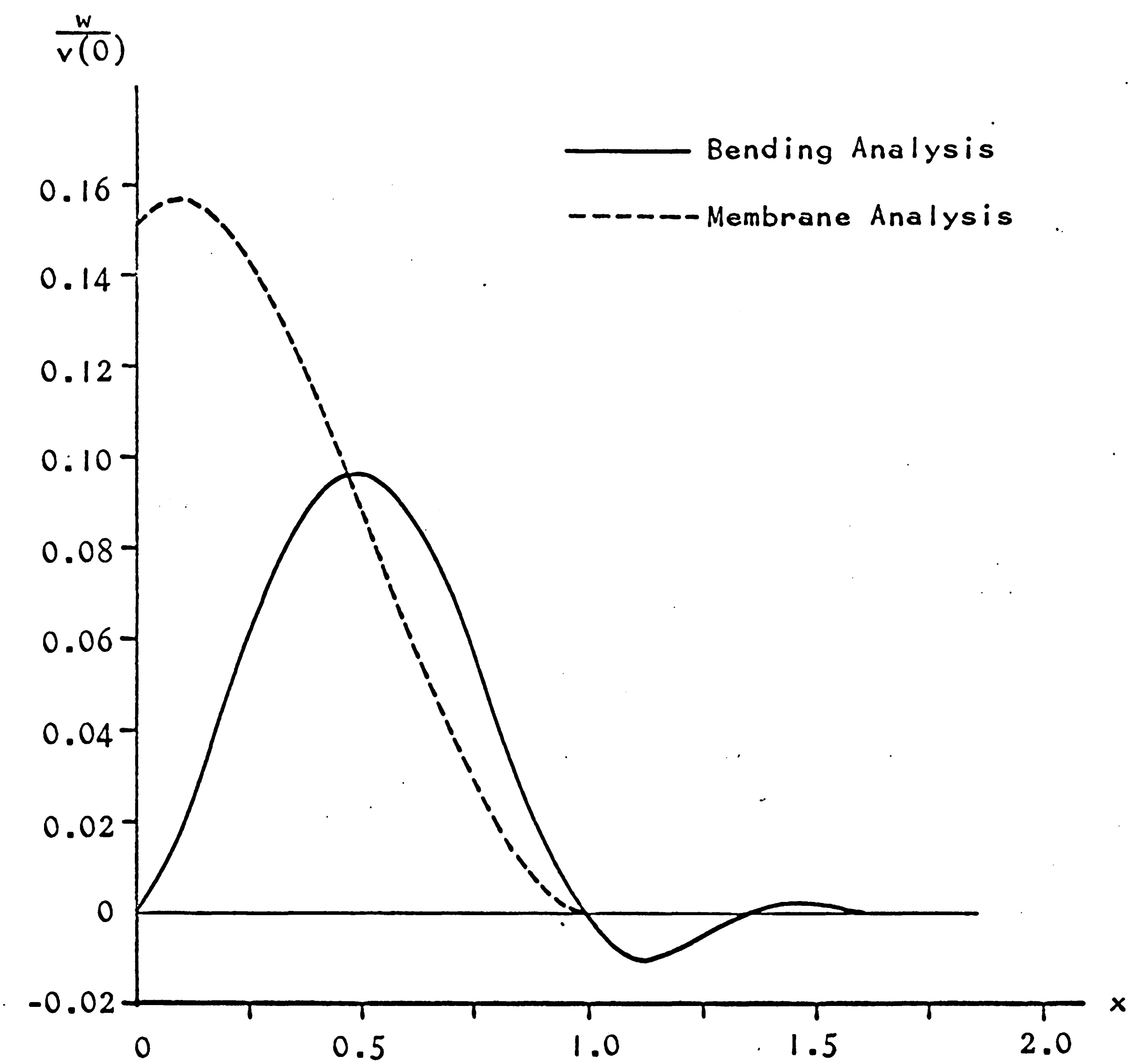
(d) t = 2.25 - t = 2.50

Figure 24 (Continued)



(e) t = 2.75 - t = 3.00

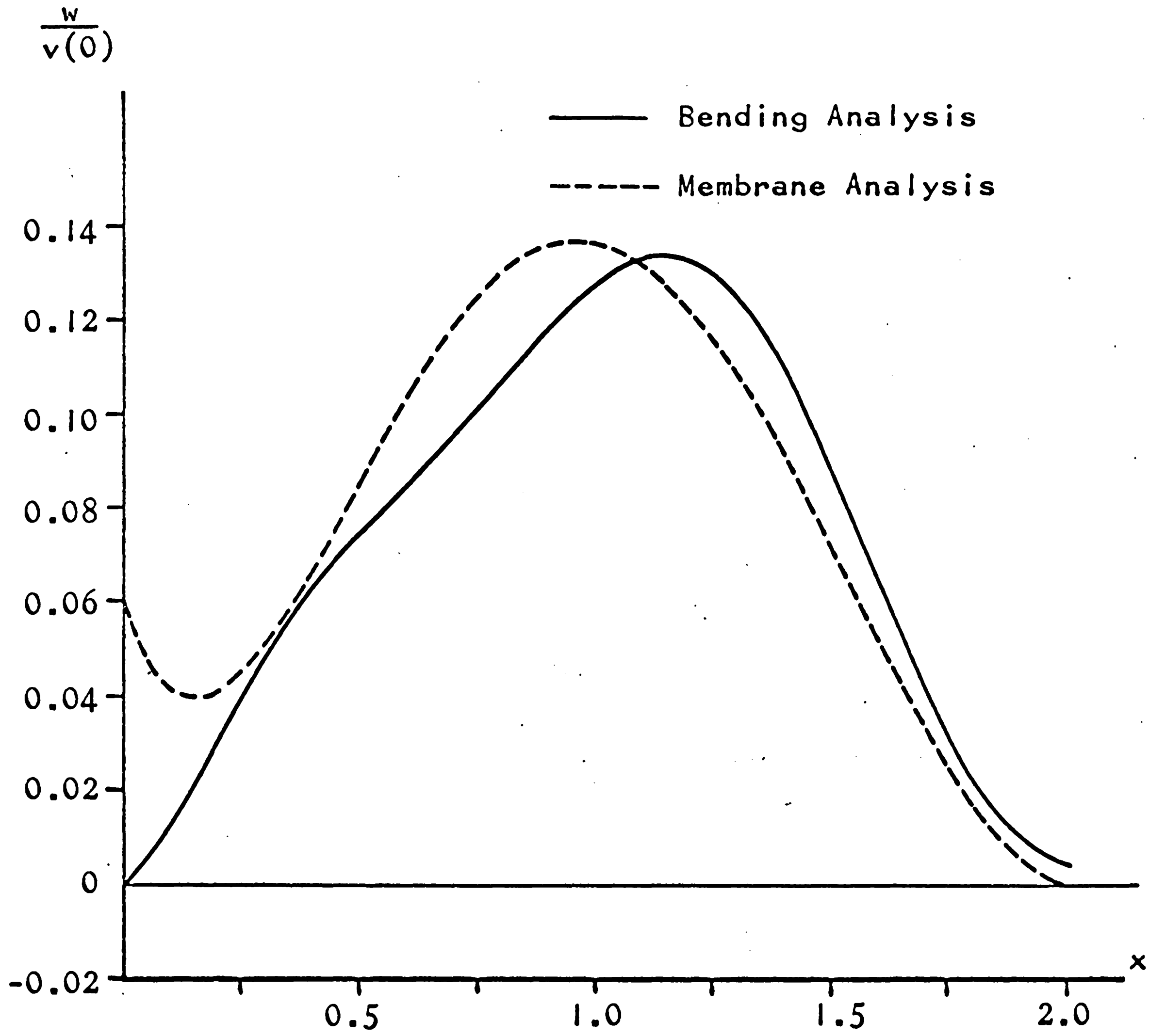
Figure 24 (Continued)



(a) t = 1.0

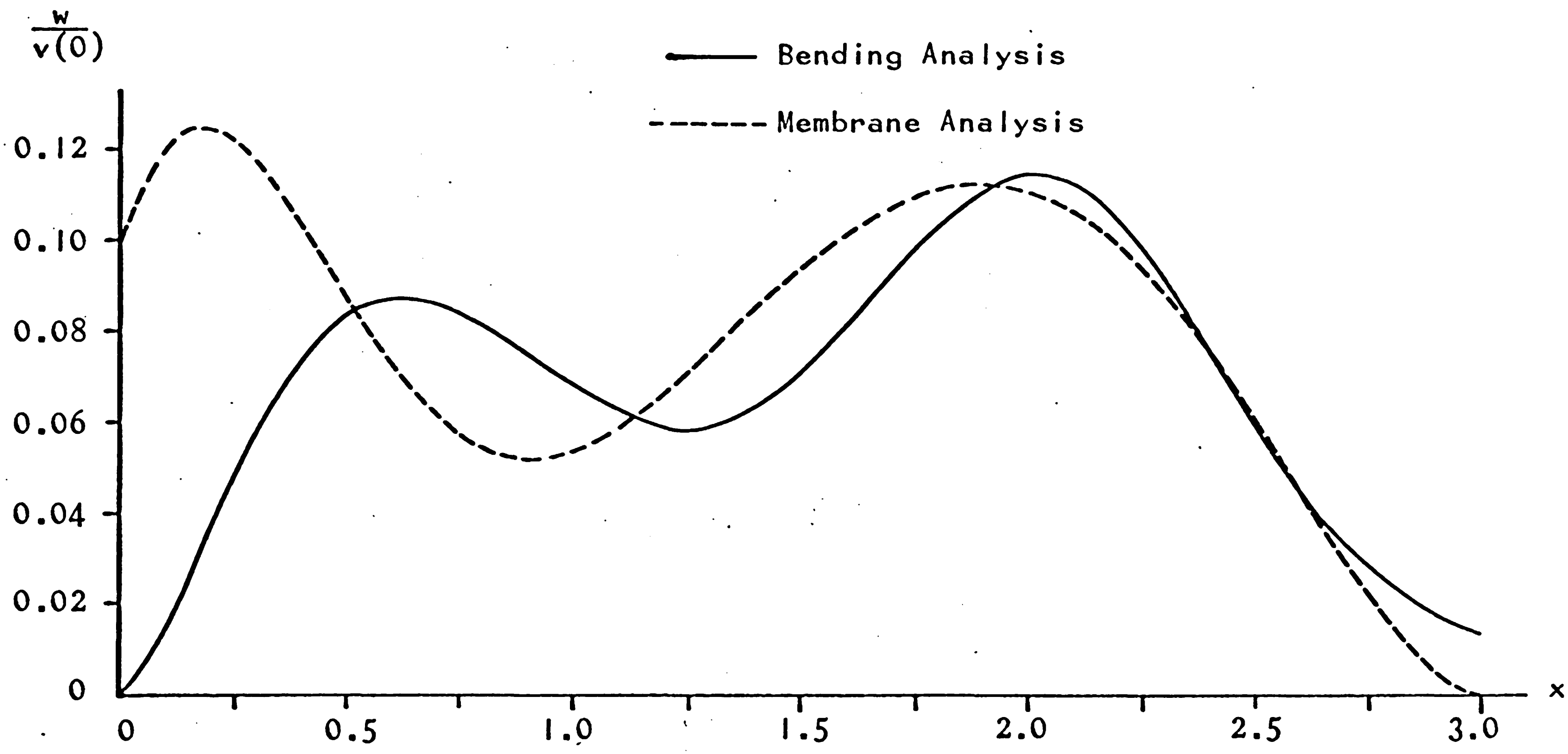
Figure 25

Radial Displacements of the Shell Middle Surface: Membrane and Bending Analyses of Water Hammer in a Steel Shell



(b) t = 2.0

Figure 25 (Continued)



(c) t = 3.0

Figure 25 (Continued)

5. Maximum stresses n_θ/h and $6m_x/h^2$ arising from the maximum radial displacement and maximum moment are of the same order of magnitude (note that $n_\theta = ehw$ when n_x is negligible). On the other hand, the maximum average transverse shearing stress, q_x/h, is about ten percent of n_θ/h or $6m_x/h^2$.

6. It appears that maximum values of the radial shell displacement and the important stress resultants occur within the interval, $t \leq 1.5$, for which the most accurate calculations were made.

Comparisons of radial displacements obtained from the bending analysis with the radial displacements obtained from the membrane analysis are presented in Figure 25. As expected, there are substantial differences near the end of the shell. It appears that, as the first peak displacement moves away from the end of the shell, differences between the results diminish in the vicinity of this leading displacement "wave". This is consistent with the predictions of Skalak's analysis of the water hammer problem.

SUMMARY AND CONCLUSIONS

The early stages of propagation of a water hammer disturbance have been investigated, water hammer constituting a special case of elastic wave propagation in a semi-infinite fluid-filled cylindrical shell. While the disturbance considered was considerably sharper than to be expected in a practical water hammer problem, e.g. sudden valve closure, there are, in naval structural mechanics, mathematically similar problems in which a very sharp disturbance is of interest. Water hammer was chosen as a vehicle for this study because of the substantial body of literature devoted to that problem. The analysis can be adapted easily to the propagation of any transient axially symmetric acoustic disturbance in the fluid.

Many of the objectionable features of the elementary water hammer theory have been removed, the critical point of the formulation here being the replacement of the axially symmetric wave equation governing fluid motion by an infinite system of one-dimensional wave equations. A finite subset of this system together with hyperbolic partial differential equations governing the motion of the shell provided the basis for numerical solution of the problem by the method of characteristics.

An analysis which includes bending of the shell was employed to consider the case of a shell with a "built-in" end. For a relatively thick steel shell filled with water it was found that bending stresses and transverse shearing stresses at the end of the shell are significant, but that nowhere are there significant longitudinal membrane stresses. Moreover, maximum stresses and displacements were found to occur within the time required for an acoustic wave in unbounded fluid to traverse one diameter of the shell. These stresses and displacements occur either at the end of the shell or near (within one diameter) the end. The maximum radial displacement of the middle surface of the shell was found to exceed the value predicted by the elementary (Joukowsky) theory by about fifty percent.

While dependence of computed quantities on the grid size employed in the numerical integration was not clearly established, it appears that the principal features of the motion were adequately described. For the results obtained, approximately fifteen consecutive minutes were required on a Burroughs B-5500 computer, and a further reduction of grid size or the use of a more sophisticated technique of numerical integration would materially increase the computer time required.

A solution based on the classical membrane theory of shells, neglecting longitudinal stresses, also was obtained by numerical integration of the ordinary differential equations arising from application of the method of characteristics to the governing partial differential equations. Since only one set of characteristic lines (as opposed to three sets in the bending analysis) arise, interpolation between successive points in the computational grid was not required, and numerical calculations were easily performed.

The membrane analysis was employed principally to show the adequacy of using the first five of the infinite set of one-dimensional wave equations governing motion of the fluid. It should be noted that this analysis takes account of two important omissions of the elementary theory, namely radial inertia of the fluid and radial inertia of the shell. The membrane formulation is to be recommended in cases where the end of the shell is free from external constraints.

Finally, a representation of the fluid motion by a single one-dimensional wave equation was investigated.

Radial inertia of the fluid is taken into consideration by attributing additional mass to the shell. This formulation which agrees with the best long-time asymptotic solution available in the literature is not adequate during early stages of the motion.

ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation for the encouragement and guidance of his thesis advisor, Dr. Ricardo Chicurel.

The thesis problem was suggested by Dr. Daniel Frederick, who directed the initial phase of the work under the auspices of a research program supported by the Underwater Explosions Division of the David Taylor Model Basin. This direction and support are gratefully acknowledged.

The author is indebted to Dr. George W. Swift for the use of his computer program for the determination of roots of transcendental equations.

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APPENDIX

Application of the Method of Characteristics to Equations Governing Motion of the Fluid

The method of characteristics is to be applied to a system of first order hyperbolic partial differential equations in order to obtain an equivalent system of ordinary differential equations. The presentation here is similar to that given by Hildebrand [24], while a more thorough exposition is given by Courant [25].

Motion of the fluid is described by

$$\frac{\partial p_j}{\partial x} + \frac{\partial v_j}{\partial t} = 0 \tag{22a}$$

and
$$\frac{\partial p_j}{\partial x} + \frac{\partial v_j}{\partial t} = \gamma_j^2 \psi_j - \frac{2\dot{w}}{(1-h/2)} \tag{22b}$$

where $j = 0, 1, 2, 3, \ldots$. Treating the ψ_j and $\dot{w}$ as known quantities, the equations above represent an infinite set of pairs of simultaneous linear partial differential equations describing the p_j and the v_j. We consider here a typical pair from this set.

Suppose that solutions, p_j and v_j, are known along a curve in the x-t plane, given in parametric form by $x = x(\zeta)$ and $t = t(\zeta)$. Then along this curve

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \frac{dt}{d\zeta} & \frac{dx}{d\zeta} & 0 & 0 \\ 0 & 0 & \frac{dt}{d\zeta} & \frac{dx}{d\zeta} \end{bmatrix} \begin{bmatrix} \frac{\partial p_j}{\partial t} \\ \frac{\partial p_j}{\partial x} \\ \frac{\partial v_j}{\partial t} \\ \frac{\partial v_j}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 \\ \gamma_j^2 \psi_j - \frac{2\dot{w}}{(1-h/2)} \\ \frac{dp_j}{d\zeta} \\ \frac{\partial v_j}{d\zeta} \end{bmatrix}$$

or, more compactly, $AY = B$, where Y and B are column matrices, and A is the four by four coefficient matrix. If A is nonsingular $Y = A^{-1}B$, and in particular $\frac{\partial p_j}{\partial t} = \frac{|A_1|}{|A|}$, where $|A|$ is the determinant of A, and $|A_1|$ is the determinant of the matrix obtained by substituting B for the first column of A.

Characteristic directions in the x-t plane are defined by $|A| = 0$, in which case the existence of a solution for $\frac{\partial p_j}{\partial t}$ requires that $|A_1| = 0$ (compatibility condition).

For the case under consideration the characteristic directions are given by

$$\begin{vmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \frac{dt}{d\zeta} & \frac{dx}{d\zeta} & 0 & 0 \\ 0 & 0 & \frac{dt}{d\zeta} & \frac{dx}{d\zeta} \end{vmatrix} = 0,$$

or $\left(\frac{dx}{d\zeta}\right)^2 - \left(\frac{dt}{d\zeta}\right)^2 = 0$,

or $\frac{dx}{dt} = \pm 1$.

The compatibility condition may be written

$$\begin{vmatrix} 0 & 1 & 1 & 0 \\ \{\gamma_j^2 \psi_j - \frac{2\dot{w}}{(1-h/2)}\} & 0 & 0 & 1 \\ \frac{dp_j}{d\zeta} & \frac{dx}{d\zeta} & 0 & 0 \\ \frac{dv_j}{d\zeta} & 0 & \frac{dt}{d\zeta} & \frac{dx}{d\zeta} \end{vmatrix} = 0,$$

or $$\frac{dp_j}{d\zeta}\frac{dt}{d\zeta} + \frac{dv_j}{d\zeta}\frac{dx}{d\zeta} = \{\gamma_j^2 \psi_j - \frac{2\dot{w}}{(1-h/2)}\} \left(\frac{dx}{d\zeta}\right)^2 ,$$

or, choosing the parameter $\zeta = t$,

$$\frac{dp_j}{dt} + \frac{dv_j}{dt}\frac{dx}{dt} = \{\gamma_j^2 \psi_j - \frac{2\dot{w}}{(1-h/2)}\} \left(\frac{dx}{dt}\right)^2 .$$

Hence along the characteristic lines, $\frac{dx}{dt} = \pm 1$, the following ordinary differential equations must be satisfied:

Along $\frac{dx}{dt} = 1$

$$\frac{dp_j}{dt} + \frac{dv_j}{dt} = \gamma_j^2 \psi_j - \frac{2\dot{w}}{(1-h/2)} \tag{24a}$$

Along $\frac{dx}{dt} = -1$

$$\frac{dp_j}{dt} - \frac{dv_j}{dt} = \gamma_j^2 \psi_j - \frac{2\dot{w}}{(1-h/2)} . \tag{24b}$$

Application of the Method of Characteristics to Equations Governing Motion of the Shell

According to the simplified Herrmann-Mirsky theory the membrane stress resultant, n_x, and the axial velocity, $\dot{u}$, of the shell middle surface are given by

$$\frac{\partial n_x}{\partial x} - \rho h \frac{\partial \dot{u}}{\partial t} = 0 \tag{26a}$$

and
$$\frac{\partial n_x}{\partial t} - \frac{eh}{(1-\nu^2)} \frac{\partial \dot{u}}{\partial x} = \frac{\nu eh}{(1-\nu^2)}\dot{w} , \tag{26b}$$

if $\dot{w}$ is regarded as known. As before we presume the existence of solutions n_x and $\dot{u}$ along a curve, $x = x(\zeta)$, $t = t(\zeta)$, and seek the first derivatives of n_x and $\dot{u}$ with respect to the independent variables, x and t. Hence we consider

$$\begin{bmatrix} 0 & 1 & -\rho h & 0 \\ 1 & 0 & 0 & -\dfrac{eh}{(1-\nu^2)} \\ \dfrac{dt}{d\zeta} & \dfrac{dx}{d\zeta} & 0 & 0 \\ 0 & 0 & \dfrac{dt}{d\zeta} & \dfrac{dx}{d\zeta} \end{bmatrix} \begin{bmatrix} \dfrac{\partial n_x}{\partial t} \\ \dfrac{\partial n_x}{\partial x} \\ \dfrac{\partial \dot{u}}{\partial t} \\ \dfrac{\partial \dot{u}}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{eh}{(1-\nu^2)}\dot{w} \\ \dfrac{dn_x}{d\zeta} \\ \dfrac{d\dot{u}}{d\zeta} \end{bmatrix}$$

Thus characteristic directions are given by

$$\begin{vmatrix} 0 & 1 & -\rho h & 0 \\ 1 & 0 & 0 & -\dfrac{eh}{(1-\nu^2)} \\ \dfrac{dt}{d\zeta} & \dfrac{dx}{d\zeta} & 0 & 0 \\ 0 & 0 & \dfrac{dt}{d\zeta} & \dfrac{dx}{d\zeta} \end{vmatrix} = 0 ,$$

which yields

$$\frac{dx}{dt} = \pm \sqrt{\frac{e}{\rho(1-\nu^2)}} = \pm c_p .$$

Compatibility requires that

$$\begin{vmatrix} 0 & 1 & -\rho h & 0 \\ \dfrac{\nu eh\dot{w}}{(1-\nu^2)} & 0 & 0 & -\dfrac{eh}{(1-\nu^2)} \\ \dfrac{dn_x}{d\zeta} & \dfrac{dx}{d\zeta} & 0 & 0 \\ \dfrac{d\dot{u}}{d\zeta} & 0 & \dfrac{dt}{d\zeta} & \dfrac{dx}{d\zeta} \end{vmatrix} = 0 ,$$

or $$\frac{dn_x}{d\zeta}\frac{dt}{d\zeta} - \rho h \frac{d\dot{u}}{d\zeta}\frac{dx}{d\zeta} = \nu\rho h\dot{w}\left(\frac{dx}{d\zeta}\right)^2 ,$$

or, choosing the parameter $\zeta = t$,

$$\frac{dn_x}{dt} - \rho h \frac{d\dot{u}}{dt}\frac{dx}{dt} = \nu\rho h\dot{w}\left(\frac{dx}{dt}\right)^2 .$$

Thus we obtain the following ordinary differential equations:

Along $\dfrac{dx}{dt} = c_p$

$$\frac{dn_x}{dt} - \rho h c_p \frac{d\dot{u}}{dt} = \nu\rho h c_p^2 \dot{w} \tag{29a}$$

Along $\frac{dx}{dt} = -c_p$

$$\frac{dn_x}{dt} + \rho h c_p \frac{d\dot{u}}{dt} = \nu \rho h c_p^{2} \dot{w} \ . \qquad (29b)$$

In like fashion Equations (30) are obtained from Equations (27), and Equations (31) are obtained from Equations (28).

ON AXIALLY SYMMETRIC ELASTIC WAVE PROPAGATION IN A FLUID-FILLED CYLINDRICAL SHELL

by

Wilton Wayt King

Abstract

The early stages of propagation of a water hammer disturbance are investigated, water hammer constituting a special case of axially symmetric elastic wave propagation in a fluid-filled cylindrical shell. Many of the objectionable features of the elementary (Joukowsky) water hammer theory are removed, and particular emphasis is placed upon consideration of the effects of radial inertia of the fluid and of the shell. The formulation is appropriate for consideration of any axially symmetric acoustic disturbance which originates in the fluid and any of the usual engineering boundary conditions which describe constraints on motion of the end, or ends, of the shell.

Motion of the shell is described by a thin-shell theory, and motion of the fluid is described by the axially symmetric wave equation, non-homogeneous boundary conditions providing coupling of the fluid and shell motions. Application of a finite Hankel transform to the axially symmetric wave equation yields an infinite system of one-dimensional wave equations representing motion of the fluid. Integration of a finite set of these wave equations in conjunction with equations governing motion of the shell is accomplished numerically after a straight-forward application of the method of characteristics.

An analysis which includes bending, rotary inertia, and shear deformation in the shell is conducted for the case of sudden termination of uniform flow in a semi-infinite shell with a "built-in" end. For a relatively thick steel shell filled with water it is found that bending stresses and transverse shearing stresses at the end of the shell are significant, but that nowhere are there significant longitudinal membrane stresses. Maximum stresses and displacements are found to occur within the time required for an acoustic disturbance in unbounded fluid to traverse one diameter of the shell. The maximum radial displacement of the middle surface of the shell is found to exceed the value predicted by the elementary theory by about fifty percent.

A solution based on the classical membrane theory of shells, neglecting longitudinal stresses, also is obtained by numerical integration of the ordinary differential equationsarising from application of the method of characteristics to the governing partial differential equations. Considerable simplification of the numerical calculations results from the fact that only one pair of families of characteristic lines are involved in the membrane analysis as compared to three pairs of such families in the bending analysis. The membrane analysis is employed principally to show the adequacy of using the first five of the infinite set of one-dimensional wave equations governing motion of the fluid. The membrane formulation takes account of two important omissions of the elementary theory, namely radial inertia of the fluid and radial inertia of the shell.

A representation of the fluid motion by a single one-dimensional wave equation is investigated. Radial inertia of the fluid is taken into account by attributing additional mass to the shell. This formulation is shown to produce the same results as the best available long-time asymptotic solution to a water hammer problem, but it is found, on the basis of an analysis employing the membrane theory of shells, to be inadequate for describing the early stages of a water hammer disturbance.