COMPLEXES WITH INVERT POINTS

by

Vyron Martin Klassen, B. S., M. S.

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I. INTRODUCTION

Let X be a topological space, and let $x \in X$. Then X is said to be invertible at x if for every neighborhood U of x in X there is a homeomorphism h on X onto X which is such that $h(X - U) \subseteq U$. If X is invertible at x, x is said to be an invert point of X. The set of invert points of X will be denoted by I(X). The space X is said to be invertible if $X = I(X)$.

This paper considers certain spaces with non-null invert sets. Most of the spaces will be finite geometric simplicial complexes; however, some of the results will be for locally compact Hausdorff spaces which have open cone neighborhoods of the invert points. An open cone neighborhood of $x \in X$ is defined to be the homeomorphic image in X of a set $A \times [0,1)$, with $A \times 0$ identified with a point p which is equivalent to x under the homeomorphism.

Examples of such spaces with non-null invert sets include the n-sphere $S^n = \{(x_1, x_2 \ldots, x_{n+1}) \mid x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1\}$ in E^{n+1}, and the n-cell $B^n = \{(x_1, x_2, \ldots, x_n) \mid x_1^2 + x_2^2 + \ldots + x_n^2 \leqq 1\}$ in E^n. E^n denotes Euclidean n-space. S^n is invertible, while $I(B^n) = \{(x_1, x_2, \ldots, x_n) \mid x_1^2 + x_2^2 + \ldots + x_n^2 = 1\} = S^{n-1}$. Another example, used repeatedly, is what will be called the "pinched annulus", A, defined as the product $S^1 \times [0,1]$, with $p \times [0,1]$ identified with a point q, for $p \in S^1$. Then $I(A) = \{q\}$. Other examples will be presented in the body of the paper to illustrate the various discussions.

The first spaces considered are the connected 1-complexes with non-null invert set, in Section 3.

Section 4 contains theorems on locally compact Hausdorff spaces which have open cone neighborhoods of the invert points. Theorem 4.3 states that if such a space has two or more invert points, it is a suspension, while Theorem 4.7 says that if the space has one invert point p, and U, a neighborhood of p, is such that U - p is not connected, then the space is a suspension, with suspension points identified. A space X is continuously invertible at a point $p \in X$ if, for every neighborhood U of p, there is an isotopy $\{h_i\}$, $0 \leq i \leq 1$ on X onto X such that $h_1(X - U) \subseteq U$. The space X is continuously invertible if it is continuously invertible for all $x \in X$. Also in Section 4, it is proved that for a locally compact space with invert point p which is mapped onto itself by the inverting homeomorphisms, if p has an open cone neighborhood, the space is continuously invertible at p.

Two-complexes with invert points are discussed in Section 5. Here, 2-complexes with exactly one invert point p are characterized, and it is proved that every such 2-complex C, such that C - p is connected, may be expressed as a monotone union of closed 2-cells.

Section 6 includes several results on the invert points of subcomplexes with invert points.

Finally, Section 7 studies the homology and homotopy groups of complexes with invert points.

Let $U = A \times [0,1)$, with $A \times 0$ identified with a point p, be an open cone neighborhood of p in a space X. Then the link of U is defined to be $L = Cl(U) \cap (X - U)$. If X is a complex, U may be considered to be the open star of p, $\overset{\circ}{St}(p)$, while $L = Cl(U) - U$. If U is an open cone neighborhood of p, it contains a closed cone neighborhood, $A \times [0,t]$, $t < 1$, with $A \times 0$ identified with p. It will be assumed throughout this paper that all open cone neighborhoods used in proofs will have closures which are closed cone neighborhoods.

II. REVIEW OF LITERATURE

The concept of an invertible space was introduced by P. H. Doyle and J. G. Hocking [3]. Prior to this, they proved the theorem which, stated in terms of invertibility, says that the only n-manifold without boundary with non-null invert set is the n-sphere [2]. Then, in [4], Doyle and Hocking proved that if $M - E^{n+1}$ is continuously invertible and contains an n-sphere S^n, then $M = S^n$.

In [5], Doyle, Hocking, and R. P. Osborne showed that if an n-manifold with non-empty boundary has an invert point, then it is topologically an n-cell.

Concerning complexes with invert points, Doyle has proved that, for a fixed triangulation of a complex K, I(K) is a subcomplex [1]. Also in this paper, I(K) is proved to be null, a point, or a subcomplex which is a sphere.

III. 1-COMPLEXES

Let C_1 be a 1-complex with $I(C_1) \neq \emptyset$. As noted in Section II, $I(C_1)$ must be a 1-sphere, a 0-sphere, or a point.

Theorem 3.1. If C_1 is a 1-complex and $I(C_1) = S^1$, a 1-sphere, then $C_1 = S^1$.

Proof: Since S^1 is a subcomplex of C_1, there is a $p \epsilon S^1$ such that p lies interior to a 1-simplex A_1 in S^1. Let U be a neighborhood of p in A_1, and let h be the inverting homeomorphism on C_1 such that $h(C_1 - U) \subseteq U$. Since U is Euclidean at each point, $C_1 - U$ must be also. Thus, C_1 is a 1-manifold, and by the characterization of an n-sphere stated in Section II, C_1 is a 1-sphere.

Theorem 3.2. If C_1 is a connected 1-complex and $I(C_1) = S^0$, a 0-sphere, then C_1 is the suspension over a finite number of points $a_1, \dots, a_n$.

Proof: Let $p \epsilon S^0$ be an invert point, and let U be a neighborhood of p in C_1. Then U must be the union of, say n, semi-open line intervals meeting in p. But then, since $C_1 - U$ is carried into U by an inverting homeomorphism, $C_1 - U$ must be the union of, say $m \leqq n$ closed intervals, meeting in q, where $q \epsilon S^0 - p$. (Note that, since $p \cup q = S^0$, p and q must be interchanged by each inverting homeomorphism.) The same argument could have been used for q as in the invert point, so $n \leqq m$. But then $m = n$, and C_1 must be a suspension over n points.

Theorem 3.3. (Doyle) Let C_1 be a 1-complex with $I(C_1) = p$. Then C_1 is the union of n 1-spheres which intersect in p.

Proof: Let U be a neighborhood of p. Then U is the union of, say m, semi-open intervals meeting in p. Let h be the inverting homeomorphism such that $h(C_1 - U) \subseteq U$. Since $I(C_1) = p$, $h(p) = p$. Thus, $h(C_1 - U) \subseteq U - p$, and $C_1 - U$ must be a union of n disjoint closed arcs. But $C_1 - p$ is topologically $C_1 - Cl(U)$. This may be seen by mapping $Cl(U)$ onto p by a pseudo-isotopy on C_1. Since the one-point compactification of a locally compact Hausdorff space is unique, C_1 is seen to be the union of n 1-spheres which intersect in p.

The complex C_1 in this proof will be called an n-leafed rose.

A characterization of the 1-sphere is of interest. First, however, a definition is necessary. A space X is said to be continuously homogeneous if, for any $x, y \in X$, $x \neq y$, there is an isotopy $\{h_t\}$, $0 \leq t \leq 1$, such that $h_1(x) = y$.

Theorem 3.4. (Doyle). A space X is the 1-sphere if and only if it is continuously invertible, continuously homogeneous, locally compact and locally metrizable at some point, and if it contains two points a and b which separate X.

Proof: If X is the 1-sphere, the necessity is obvious.

To prove the sufficiency, first note that X must be compact and metrizable, since it is invertible, as proved by Doyle and Hocking [3] .

Suppose that $\{a,b\}$ separates X into A and B. A and B are open, since $A \cup B$ is open, and A and B are separated. Also, Cl(A) is connected. For, suppose $Cl(A) = C \cup D$, where C and D are separated. If $a, b \in C$, then $Cl(D) \cap C \neq \emptyset$, or $Cl(D) \cap (B \cup C) = \emptyset$, and X is not connected. If $a \in C$, $b \in D$, then $b \in Cl(C)$, or a is a cutpoint for X. But this is impossible, for Doyle and Hocking have proved that no continuously invertible Hausdorff continuum has a cutpoint [4]. Thus, Cl(A) is connected.

By invertibility, a monotone decreasing sequence of closed connected neighborhoods of a point $x \in X$, whose intersection is x, may be obtained. Let $y \in X$ be any other point. Then by homogeneity, there is also such a monotone decreasing sequence of neighborhoods of y. Then for any $\epsilon > 0$, there is a covering of X by connected sets of diameter less than ϵ, and by the fact that X is compact, a finite number of these sets cover X. But this is exactly the definition of property S, and a theorem in Wilder [10] says that a metric space is locally connected if it has property S. Thus, X is locally connected, and is therefore a Peano continuum.

A theorem proved by Doyle and Hocking [4] says that a continuously invertible Peano continuum contains a 1-sphere. Let $S \subseteq X$ be such a set.

Since $Cl(A) - A = a \cup b$, a 0-sphere, X is 1-dimensional at every point.

If $S = X$ the theorem is proved. So suppose that there is an $x \in X - S$. Let U_i be a neighborhood of x, with diameter less than $\frac{1}{i}$, for $i = 1, 2, \ldots$, which is disjoint from S, and such that $U_i \subsetneqq U_{i-1}$. Then there is an isotopy $\{h_t\}$, $0 \leqq t \leqq 1$, on X such that $S^1 = h_1(S) \subseteq U_1 - U_2$. Inductively, an isotopy $\{h_t\}$, $(i - 1) \leqq t \leqq i$, may be defined on X, such that $h_i\left[h_{i-1}(S^{i-1})\right] \subseteq U_i - U_{i+1}$, where $S^i = h_i(S^{i-1})$. Thus, $\{h_t\}$, $0 \leqq t < \infty$, is an isotopy, which demonstrates that S is contractible in X.

But a theorem in Hurewicz and Wallman [8], states that a space X has dimension $\leqq n$ if and only if for each closed set C and mapping f of C into S^n, there is an extension of f over X. Such a function f may be taken to be the identity, for $C = S$, with $n = 1$. This would imply that the 1-sphere is contractible on itself. This is a contradiction.

Therefore, $S = X$, and the theorem is proved.

IV. THEOREMS CONCERNING SPACES WITH A LOCAL TRIANGULABILITY CONDITION

Consider the two examples which have been presented above of spaces which are invertible at exactly one point, namely, the n-leafed rose and the "pinched" annulus. It may be seen that these spaces are each continuously invertible at their invert point. The following theorem shows that this occurs for all such complexes, and for many other spaces.

Theorem 4.1. Let S be a locally compact space which is invertible at single point p. Then if p has an open cone neighborhood C in S, S is continuously invertible at p.

Proof: Let $C = A \times [0,1)$, with $A \times 0$ identified with p. Let U be a neighborhood of p in S. Let $C' = A \times [0,\epsilon]$, with $A \times 0$ identified with p, where $\epsilon > 0$ is chosen such that $C' \subseteq U$.

Now, let h be the inverting homeomorphism for p and C'. $h(p) = p$. There exists a pseudo-isotopy $\{r_t\}$, $0 \leqq t \leqq 1$, on S onto itself, such that r_t is the identity on $S - C$, for all $t \in [0,1]$, while $r_1(C') = p$. By continuity, there exists a $T \in (0,1)$ such that $h\left[r_T(C')\right] \subseteq U$.

But $S - C' \subseteq h(C')$. So $h^{-1}(S - C') \subseteq C'$. Thus $\{q_t\} = \{hr_th^{-1}\}$, $t \in [0,T]$ is an isotopy on S onto itself such that q_0 is the identity on S, while q_T carries $S - U$ into U, and the theorem is proved.

Corollary 4.2. Let S be a space which is invertible at a pointp, and suppose that p has an open cone neighborhood in S. Then if the inverting homeomorphisms on S fix p, S is continuously invertible at p.

If a complex has two or more invert points, it is not necessarily continuously invertible. A simple example of this is a suspension over three points. However, the following two theorems characterize certain spaces which have two or more invert points as suspensions.

Theorem 4.3. Let S be a locally compact Hausdorff space which is invertible at points p and q. Then if p has an open cone neighborhood U in S, S is a suspension.

Proof: Since S has two invert points, at least one of the invert points is moved by every inverting homeomorphism. Suppose $h(p) = r$, where h is an inverting homeomorphism for U that moves p to $r \neq p$.

Now $U = U' \times [0,1)$, where $U' \times 0$ is identified with p. Let $U_i = U' \times [0,\frac{1}{i})$, with $U' \times 0$ identified with p, where $i \in I$, I a monotonic sequence of integers, determined below.

With each U_i there is associated an inverting homeomorphism h_i such that $S - U_i \subseteq h_i(U_i)$. The h_i may be chosen so that $h_i(p) = r$, for all $i \in I$.

Now, if $T_i = h_i(U_i)$, $\bigcup_{i \in I} T_i$ can be made a monotone union, for proper choice of I. The T_i are open cone neighborhoods of r, and are all homeomorphic, since they are each homeomorphic to some U_i, which are each, in turn, homeomorphic to U. In fact, Kwun [9] has proved the following two theorems:

1. Let U and V be any two open cone neighborhoods of a point x in a locally compact Hausdorff space X. Then there is a homeomorphism of V onto U which leaves a neighborhood of x pointwise fixed.

2. With x and X as in Theorem 1, if $U^1 \subseteq U^2 \subseteq \ldots$ is a sequence of open cone neighborhoods of x then $U = \bigcup_{i=1}^{\infty} U^i$ is also an open cone neighborhood of x homeomorphic to each U^i.

Thus, not only is T_i homeomorphic to T_{i+1}, but the homeomorphism may be taken to fix T_{i-1}, for all $i \in I$, and $X = \bigcup_{i \in I} T_i$ an open cone, say $V \times [0,1)$, with $V \times 0$ identified with r.

But X = S - p, and since the one point compactification of a locally compact Hausdorff space is unique, S is topologically $V \times [0,1]$, with $V \times 0$ identified with r, $V \times 1$ identified with p. Thus, S is a suspension.

Theorem 4.4. Let S be a suspension. Then S is invertible at two or more points.

Proof: Let S be $S' \times [0,1]$, with $S' \times 0$ identified with p, $S' \times 1$ identified with q. Let $U = S' \times (0,\epsilon) \cup p$ be a neighborhood of p in S. Let $U' = S' \times [0,\epsilon)$.

Now, let h_1 be the homeomorphism on $S' \times [0,1]$ onto $S' \times [0,1]$ defined by $h_1(y \times t) = y \times (1 - t)$. Then $h_1(U') = S' \times [1 - \epsilon, 1]$. Let h_2 be the homeomorphism on $S' \times [0,1]$ such that $h_2[S' \times (1 - \epsilon)]$ $= S' \times \epsilon$, and $h_2(S' \times 0) = S' \times 0$, $h_2(S' \times 1) = S' \times 1$. Then $h = h_2 h_1$ is a homeomorphism on $S' \times [0,1]$ onto itself which carries $S' \times [0,1] - U'$ onto U', and $S' \times 0$ onto $S' \times 1$ and $S' \times 1$ onto $S' \times 0$. The corresponding homeomorphism h' on S onto itself, defined by $h'(p) = q$, and $h'(y \times t) =$ $h(y \times t)$ for $y \times t \in S' \times (0,1)$, is the desired inverting homeomorphism.

These two theorems may be combined to give Theorem 4.5.

Theorem 4.5. Let S be a locally compact Hausdorff space which has an open cone neighborhood at points p and q. Then S is invertible at p and q if and only if S is a suspension, with suspension points p and q.

Note that it is necessary that the invert points p and q of a locally compact Hausdorff space have open cone neighborhoods. For consider the following example [5]:

Let

$$K_i = \begin{cases} \sum_{j=1}^{i} \frac{1}{2^j} + \frac{1}{2^{j+1}}, & \text{for } i \geqq 1, \\ 0, & \text{for } i = 0 \\ -\sum_{j=1}^{|i|} \frac{1}{2^j} + \frac{1}{2^{j+1}}, & \text{for } i \leqq -1. \end{cases}$$

Then, let $S_i = \left\{(x,y) \mid (x - K_i)^2 + y^2 = \frac{1}{2^{2\ i+1}}\right\}$,

for i any integer. It may be seen that the S_i are 1-spheres in E^2, that S_i is tangent to S_{i-1} and S_{i+1}, and $S = \bigcup_{i=-\infty}^{\infty} S_i$ has (2,0) and (-2,0) as limit points. Let S' = Cl(S). S' is invertible at (2,0) and (-2,0), but neither of these points have open cone neighborhoods in S'.

Now, consider again a locally compact Hausdorff space X with exactly one invert point p. If X - p is not connected, then each of the components, together with p, must be invertible in its own right.

This fact is an easy consequence of Theorem 4.1. Thus, X is the union of spaces which intersect in exactly one point, which is a point of continuous invertibility of each of the spaces. As a result, it suffices to study the closure of one component of X - p.

It may be that the closure K of a component of X - p has two or more invert points. In this case, K is a suspension, by Theorem 4.3. Examples of X, for such K, would be the n-leafed rose, and a "bow tie" complex, which is the union of two 2-simplices which meet in a vertex of each. James Chew pointed out that the union is invertible at the common vertex.

Thus, it is now necessary to consider a space S, such that I(S) = p, and such that S - p is connected. Theorem 4.7. deals with certain of these spaces, but first it is necessary to prove a lemma.

Lemma 4.6. Let S be a locally compact Hausdorff space with exactly one invert point p. Let U' be an open cone neighborhood of p in S. Suppose that S - p is connected. Then there exists an open cone neighborhood U of p in S such that U - p has at most two components.

Proof: Let $U' - p = A \times (0,1)$, and let $U - p = A \times (0, \frac{1}{2})$. There is a pseudo-isotopy on S onto itself which is the identity on S - U' and carries U onto p. Thus, since S - p is connected, S - U is connected also.

Now, suppose that U - p has n-components. Let $U_1, \ldots, U_n$ be the n open cones which, with p deleted, constitute these components.

Let h be an inverting homeomorphism on S which is such that $h(S - U) - U$. Then $S - U$ is carried into one component of $U - p$, say U_1, by h, since $S - U$ is connected. Also, for the same reason, either U_1 or some other, say U_2, must be carried by h to cover $S - U$.

Let L_i be the link of U_i, for $i = 1, \ldots, n$. Then $L = \bigcup_{i=1}^{n} L_i$ is the link of U, and is such that $L \subseteq S - U$. Thus $h(L) \subseteq U_1$.

Now, suppose that $n \geqq 3$. Also, suppose that $h(U_i) \; U_j \neq \emptyset$, for some $i \geqq 3$, $j \geqq 2$. Trivially, $h(U_i) \cap U_1$ and $h(U_i) \cap U_j$ are both in $h(U_i)$. But $U_1 - p$ and $U_j - p$ are separated by $S - U$ in $S - p$, as are all the $U_k - p$, $k = 1, \ldots, n$. Thus, there exists a $q \in h(U_i)$ such that $q \in S - U$. But then $h^{-1}(q) \in U_2$ or U_1. But this is impossible, since the $U_k - p$ are all disjoint. Therefore, $h(U_i) \subseteq U_1$, for all $i = 3, \ldots, n$.

Now there exists an open cone neighborhood U_1^* of p in U_1 which is such that $h(S - U) \cap U_1^* = \emptyset$. If $U_1 - p = A_1 \times (0,1)$, let $U_1^* = A_1 \times (0,\epsilon) \cup p$, for $\epsilon > 0$ sufficiently small. But then $U_1^* \subseteq \bigcup_{i=1}^{n} h(U_i)$, and since U_1^* is connected, there exists an i such that $U_1^* \subseteq h(U_i)$. However, since p is a limit point of the $h(U_i)$, $i = 1, \ldots, n$, $U_1^* \cap h(U_i - p) \neq \emptyset$, for all $i = 1, \ldots, n$. But the $h(U_i)$ are disjoint, so this is a contradiction. Thus, $n \leqq 2$, and the Lemma is proved.

Theorem 4.7. Let S be a locally compact Hausdorff space which is invertible at exactly one point p, with $S - p$ connected. Let U be an open cone neighborhood of p in S, and suppose that $U - p$ has two components. Then S is the one point compactification of $L \times (0,1)$, with L the link of one component of $U - p$ in S.

Proof: Let h be an inverting homeomorphism for S and U. As proved in the above Lemma, S - U is connected, so h(S - U) is a subset of U_1 or U_2. Say $h(S-U) \subseteq U_2$.

Then $S-U \subseteq h(U_1)$. For suppose $S-U \subseteq h(U_2)$. As in the proof of Lemma 4.6, $h(U_1)$ would have to have points in U_1 and U_2, and thus be separated by $h(U_2)$. But this is impossible.

Let L be the link of U_1. Let $U_1^i = L \times (0,\frac{1}{i}) \cup p$. As in the proof of Theorem 4.3, a monotone sequence of open cone neighborhoods $h(U_1^i)$ is obtained, all homeomorphic to U_1. Thus, by the above stated theorem of Kwun[8], $\bigcup_{i=1}^{n} h(U_1^i)$ is an open cone $L \times (0,1) \cup p$. But this set is topologically S. Thus, S is the one point compactification of $L \times (0,1)$.

The converse of this theorem is also true.

Theorem 4.8. Let $S = A \times [0,1]$, with $A \times 0$ and $A \times 1$ identified with a point p. Then S is invertible at p.

Proof: By theorem 4.4, $S' = S \times [0,1]$, with $A \times 0$ identified with p', $A \times 1$ with q', is invertible at p' and q'.

Let U be a neighborhood of p in S, and let U' be the corresponding neighborhood of $p' \cup q'$ in S'. Then there is an inverting homeomorphism h' on S' such that $h'(p') = q'$ and $h'(S' - U') \subseteq U'$. But then if h is the homeomorphism defined by $h(x) = h'(x)$ for $x \in S - p$, and $h(p) = p$, it is evident that h is an inverting homeomorphism on S.

Theorems 4.7 and 4.8 give the following theorem.

Theorem 4.9. Let S be a locally compact Hausdorff space, with open cone neighborhood U of p such that U - p is not connected, while S - p is connected. Then S is invertible at p if and only if S is a suspension with suspension points identified.

Suspensions with suspension points identified do not exhaust the possibilities for complexes with exactly one invert point, however, as shown by the following example:

Let A be a "pinched" annulus and let S be a 2-sphere. Let S_1 be a 1-sphere in A which passes through the invert point p and does not separate A. Let S_2 be a 1-sphere in S. Then, let K be the space formed by sewing A and S together by identifying S_1 and S_2. K is invertible at p, but it is not a suspension with suspension points identified. However, this complex may be obtained in another manner which will have added meaning in conjunction with Theorem 5.3. The construction is as follows:

Let T be a triod, and let a, b, and c be the vertices of T which are faces of exactly one 1-simplex. Consider $T \times [0,1]$, with $T \times 0 \cup T \times 1 \cup a \times [0,1] \cup b \times [0,1]$ identified with a point p. This space is topologically equivalent to K above.

This procedure may be generalized to give many spaces with an invert point.

Theorem 4.10. Let A be a space, and let A' be a subspace of A. Then the space $A \times [0,1]$, with $A \times 0 \cup A \times 1 \cup A' \times [0,1]$ identified with a point p is invertible at p.

Proof: Let S be the space formed from $A \times [0,1]$ by the identification, and let U be a neighborhood of p in S. Corresponding to U in S there is a set U' in $A \times [0,1]$ which is a neighborhood of $A \times 0 \cup A \times 1 \cup A' \times [0,1]$. Since U' is a neighborhood of $A \times 0 \cup A \times 1$, there is a homeomorphism h' on $A \times [0,1]$ which carries $A \times [0,1] - U'$ into U', leaves $A \times 0$ and $A \times 1$ pointwise fixed, and $A' \times [0,1]$ setwise fixed. The corresponding homeomorphism h on S is an inverting homeomorphism for S and U.

The decomposition carried out on the space X, above, to obtain subspaces S which were such that $S - p$ was connected, may be carried further in the case where the space is a complex.

Let C_n be an n-complex with exactly one invert point p. Let C_{n-1} be the subcomplex consisting of all $(n - 1)$-simplices of C_n which are faces of no, one, three, or more n-simplices.

Theorem 4.11. Let C_n, C_{n-1}, and p be as above. Then C_{n-1} is invertible at p.

Proof: By Theorem 4.1, C_n is continuously invertible at p. Let $x \in C_{n-1}$, and let V be a neighborhood of x in C_n. V cannot be a Euclidean neighborhood, by the choice of C_{n-1}. Thus, every inverting isotopy on C_n must carry x onto another point in C_{n-1}, for points in $C_n - C_{n-1}$, in a neighborhood of C_{n-1}, must all have Euclidean neighborhoods. But if C_{n-1} must be carried onto itself by the inversion isotopies on C_n, C_{n-1} must be invertible at p in its own right.

This theorem gives the means of further decomposing C_n. C_{n-1} separates C_n into possibly several components. Let C_n^1 be the closure of one of these components. Since C_n is invertible by an isotopy, C_n^1 must be invertible under the same isotopy, restricted to C_n^1.

Thus, the study of an n-complex may be reduced to the study of the various C_n^1.

Another decomposition of an n-complex with an invert point may be informative. First, a definition is necessary.

Let X be a space, and $x \in X$. Let H(X) be the set of all homeomorphisms on X onto X. Then the orbit of x, O(x) is the set of points $y \in X$ such that there exists an $h \in H(X)$ for which $h(x) = y$.

Let X be a space. Then X may be decomposed into disjoint orbits. If X is invertible at a point p, then the closure of each orbit in X must also be invertible at p.

The usefulness of this approach is shown by the following two theorems.

Theorem 4.12. Let C^n be an n-complex, and let O(x) be the orbit of a point in C^n. Then O(x) is a k-manifold, for $k \leqq n$.

Proof: It will suffice to prove the theorem for x in S^n, a closed n-simplex in C^n minus its (n - 1)-skeleton. For if this is done, the only orbits for which the theorem will not have been proved will be in the (n - 1)-skeleton. Then the following procedure may be repeated.

Let $x \in S^n$ be as above. A theorem which Doyle has proved [1] states, in the above notation, that $S^n \subseteq O(x)$. Thus, there is a neighborhood U of x in O(x) which is topologically E^n. But then $y \in O(x)$ has a neighborhood equivalent to E^n also. Thus, O(x) is an n-manifold. This procedure may be repeated for other n-simplices which are disjoint from O(x), until all points in the various S^n in C^n have been included.

A space X is said to be 1-moveable if, for every compact proper subset $C \subsetneq X$, there is a homeomorphism on X onto X which carries C into X - C.

Theorem 4.13. Let C^n be an n-complex which is invertible at a point p. Then $O(x)$ is 1-moveable for all $x \in C^n - p$.

Proof: Let $x \in C^n - p$, and let K be a proper compact subset of $O(x)$. Let U be a neighborhood of p in $C^n - K$. Then $V = U \cap O(x)$ is an open set in $O(x)$, disjoint from K, and from the fact that C^n is invertible at p, there is a homeomorphism h on $O(x)$ onto itself such that $h(K) \subseteq V$. Thus, the theorem is proved.

The above theorems show that, not only are the orbits manifolds, but they have a property which holds for all product spaces, that of 1-moveability. This indicates that the orbits should be similar to products.

Another concept will aid in understanding complexes with invert points. This is the inversion set. Let X be a space, and let $Y - X$ be such that, for every neighborhood U of Y, there exists a homeomorphism h on X onto X such that $h(X - U) \subseteq U$ and, for all $p \in Y$, $h(p) = p$. It is evident that if Y is an inversion set of X, then the decomposition space obtained by identifying Y with a point is invertible at that point.

Let C^n be an n-complex. The (n - 1)-skeleton C^{n-1} of C^n is an inversion set, as shown by Doyle [1]. It is desired to define a procedure for simplifying this to a less complicated set of invertibility. The definition is given inductively, by removing a single (n - 1)-simplex from the inversion set, in such a way that the resulting subcomplex is also an inversion set for C^n, until no more such (n - 1)-simplices may be found.

There is no loss of generality in assuming that C^n consists only of n-simplices and their faces. Principal (n - k)-simplices, for $k > 0$, may be considered separately in the same manner as follows for C^n.

Let $p_1, \ldots, p_m$ be the barycenters of the m n-simplices $S_1^n, \ldots, S_m^n$ in C^n. Let C_*^n be the n-complex obtained from C^n by adding, as vertices, $p_1, \ldots, p_m$ to the other vertices of C^n. If S_k^{n-1} is an (n - 1)-simplex in C^{n-1}, and a face of S_j^n in C^n, let $p_j S_k^{n-1}$ be the n-simplex in C_*^{n-1} determined by the vertices of S_k^{n-1} and p_j.

Let C_s^{n-1} be an (n - 1)-subcomplex of C^n, which is an inversion set of C^n. Suppose that there are two components A^s and B^s of $C^n - C_s^{n-1}$ which are such that $Cl(A^s) \cap Cl(B^s)$ contains an (n - 1)-simplex S_i^{n-1} which is a face of exactly two n-simplices, one in A^s and the other in B^s. Let these two n-simplices be S_q^n and S_r^n, respectively. Let U' be a neighborhood of $C_s^{n-1} - S_i^{n-1}$ in C^n whose closure is disjoint from the p_i, and let U be a neighborhood of C_s^{n-1} such that $U - (p_q S_i^{n-1} \cup p_r S_i^{n-1} \cup S_i^{n-1}) = U'$, and such that $p_q, p_r \notin Cl(U)$.

Suppose that there is a homeomorphism h_q^s on C^n onto C^n, pointwise fixed on $C^n - A^s$, such that $h_q^s(A^s - U) \subseteq U - p_q S_i^{n-1}$. Also, suppose that there is a homeomorphism h_r^s on C^n onto C^n, pointwise fixed on

$C^n - B^s$, such that $W = h_r^s\left[(B^s - U) \cup p_r S_i^n\right] \subseteq p_r S_i^{n-1}$. A homeomorphism h^s may be defined on C^n onto C^n, pointwise fixed on $C^n - (p_r S_i^{n-1} \cup p_q S_i^{n-1})$, such that $h^s(W) - p_q S_i^{n-1} \subseteq U$ and $h^s(p_q S_i^{n-1} - U) \subseteq p_q S_i^{n-1} - U$. Then $C_s^{n-1} - S_i^{n-1}$ is a set of invertibility, with homeomorphism $\pi^s = h_q^s h^s h_r^s$.

This procedure may be repeated until it is impossible to find such sets A^s and B^s, and homeomorphisms h_q^s and h_r^s. It may be that another sequence of deletions would give a set of invertibility which contains fewer simplices. Since there are a finite number of n-simplices in C^n, a best sequence of deletions may be obtained.

V. TWO-COMPLEXES WITH INVERT POINTS

After the more general spaces of Chapter IV, attention will now focus on complexes. This chapter will deal with 2-complexes.

The 2-sphere and 2-cell are two simple examples of 2-complexes with invert points, but, aside from the n-leafed rose, all the examples which have been presented previously are such complexes also.

Another example is the "pinched" torus. This is the space formed from S' x S', where S' is a 1-sphere, with S' x or x S' identified with a point p, for $x \in S'$.

The following theorems characterize 2-complexes with invert points. First, consider 2-complexes with exactly one invert point.

Theorem 5.1. Let C^2 be a 2-complex which is invertible at exactly one point p. Then $C^2 = (\bigcup_{i=1}^{n} C_i^2) \cup (\bigcup_{j=1}^{r} S_j^1)$, where C_i^2 is a 2-cell, a 2-sphere, a "pinched" annulus, or a "pinched" torus, such that $C_i^2 \cap C_j^2 = p$ or B_k^1, a union of 1-spheres containing p, for all i, $j = 1, \ldots, n$, $i \neq j$, and the S_j^1 are 1-spheres which are disjoint from $\left[(\bigcup_{i=1}^{n} C_i^2) \cup (\bigcup_{k \neq j} S_k^1)\right] - p$, for all $j = 1, \ldots, r$.

Proof: Let C* be the subcomplex of C^2 determined by the 1-simplices which are faces of no, one, three, or more 2-simplices in C^2. By Theorem 4.11, C* is invertible at p, and thus is an m-leafed rose. Certain of the 1-spheres, $S_1^1, \ldots, S_r^1$ in C* are determined by those 1-simplices of C^2 which are the faces of no 2-simplices. These 1-spheres are the S_j^1 in the statement of the theorem. Let $S = \bigcup_{j=1}^{r} S_j^1$, and let $C^1 = (C^* - S) \cup p$.

As noted in Chapter IV, C^1 separates $(C - S) \cup p$ into, say, n components, whose closures form subcomplexes, $C_1^2, \ldots, C_n^2$, each of which is invertible at p. Note that $C_i^2 \cap C_j^2$ is either p or a union of 1-spheres in C^1. Thus, all that is necessary to complete the proof is to show that C_i^2 is a 2-sphere, a 2-cell, a "pinched" torus, or a "pinched" annulus, for $i = 1, \ldots, n$.

Let U_i be an open cone neighborhood of p in C_i^2, and let L_i be the link of U in C_i^2, if $U_i - p$ is connected. Otherwise, let L_i be one component of the link of U_i. Now L_i is a graph in C_i^2, and by the choice of C_i^2, has no vertices which are faces of three or more 1-simplices in L_i. Thus, L_i is either a 1-sphere or a 1-cell.

Suppose that L_i is a 1-sphere. Also, suppose that L_i equals the link of U_i. Then U_i is a 2-cell, and by Doyle and Hocking's characterization of the n-sphere which was stated in Chapter II, C_i^2 is a 2-sphere. On the other hand, suppose that L_i is a component of the link of U_i. Then, by Theorem 4.9, C_i^2 is a suspension over L_i, with suspension points identified. Thus, C_i^2 must be a "pinched" torus.

Now, suppose that L_i is a 1-cell, and first suppose that L_i is the link of U_i. Then U_i is a 2-cell, and by a theorem of Doyle, Hocking, and Osborne, also stated in Chapter II, C_i^2 is a 2-cell. Finally, suppose that L_i is one component of the link of U_i. Then C_i^2 is a suspension over L_i, with suspension points identified. This must be a "pinched" annulus. Thus, the theorem is proved.

Theorem 5.2. Let C^2 be a 2-complex which is invertible at a single point p. Then C^2 is topologically equivalent to $C^1 \times [0,1]$, with $C^1 \times 0$, $C^1 \times 1$, and $C^0 \times [0,1]$ identified with the point p, where C^0 is a finite set of points in C^1, a 1-complex.

Proof: Let U be an open cone neighborhood of p in C^2, and let L be the link of U. Then $C^2 - (U \cup L)$ is topologically equivalent to $C^2 - p$. Thus, if it can be shown that $C^2 - U$ is topologically $C^1 \times [0,1]$, with L topologically $C^1 \times 0 \cup C^1 \times 1 \cup C^0 \times [0,1]$, for some C^1 and C^0, the theorem will be proved.

Let C_i^2 be as in the proof of Theorem 5.1. Let $L_i = L \cap C_i^2$. If L_i has two components, C_i^2 is a suspension, and so $C_i^2 - U$ is topologically $K_i \times [0,1]$, for K_i one component of L_i.

So suppose that L_i has one component. As in the proof of Theorem 5.1, L_i is a 1-cell or a 1-sphere, and C_i^2 is a 2-cell or a 2-sphere. In either case, $C_i^2 - U$ is a 2-cell.

Suppose that C_i^2 is a 2-cell. Let a and b be two points in L_i. They divide L_i into three arcs A_1, A_2, and A_3, such that A_2, say, has a and b as endpoints. Then C_i^2 may be expressed as $A_1 \times [0,1]$, with A_2 equivalent to $a \times [0,1]$, and such that $A_1 \times 0 \cup A_1 \times 1 \cup a \times [0,1]$ is equivalent to L_i.

On the other hand, suppose that C_i^2 is a 2-sphere. Except for p, C_i^2 must then be disjoint from the other C_j^2. For otherwise, C_i^2 would contain one of the B_k^1, which was the union of 1-spheres. Then B_k^1 would separate C_i^2. But this cannot happen, by the choice of C_i^2.

So let A be an arc in L_i. Then $C_i^2 - U$ may be expressed as $A \times [0,1]$, with $A \times 0 \cup A \times 1 \cup a \times [0,1] \cup b \times [0,1] = L_i$, where a and b are the endpoints of A.

Since $C_i^2 \cap C_j^2 = B_k^1$ is a union of 1-spheres, $B_k^1 - U$ are paremeterized also, and thus $C^2 - U$ is the desired product. Thus, the theorem is proved.

Theorem 5.2 and Theorem 4.10 give the following theorem:

Theorem 5.3. Let C^2 be a 2-complex. Then C^2 is invertible at a point p, which is fixed under the inversion, if and only if C^2 is topologically $C^1 \times [0,1]$, with $C^1 \times 0 \cup C^1 \times 1 \cup C^0 \times [0,1]$ identified with p, for C^0 a finite set of points in C^1, a 1-complex.

Now, consider 2-complexes with more than one invert point. Theorem 4.5 characterized such complexes as suspensions. Let C^2 be such a complex. Then L^1, the complex which, suspended, gives C^2, must be a 1-complex. Let L^0 be the subcomplex in L^1 consisting of points which are vertices of no, one, three, or more 1-simplices in L^1. Then L^0 separates L^1 into components whose closures are 1-cells and 1-spheres. The suspension over L^0 likewise separates C^2 into components whose closures are 2-cells and 2-spheres. Thus, the following theorem is obtained, analogous to Theorem 5.1.

Theorem 5.4. Let C^2 be a 2-complex which is invertible at two or more points. Then $C^2 = (\bigcup_{i=1}^{n} C_i^2) \cup (\bigcup_{j=1}^{r} S_j^1)$, where C_i^2 is a 2-sphere or a 2-cell, such that $C_i^2 \cap C_j^2 = p$ or D_k^1, a union of 1-cells which intersect in $I(C^2)$, and the S_j^1 are 1-cells which are disjoint from $\left[(\bigcup_{i=1}^{n} C_i^2) \cup (\bigcup_{k \neq j} S_k^1)\right] - I(C^2)$, for all $j = 1, \ldots, r$.

The union $\bigcup_{i=1}^{n} C_i$ is a monotone union if $C_1 \subseteq C_2 \subseteq C_3 \ldots$. The next theorem characterizes certain 2-complexes as the monotone union of closed 2-cells.

Theorem 5.5. Let C^2 be a 2-complex with non-null invert set $I(C^2)$, such that $C^2 - I(C^2)$ is connected (or null). Then C^2 may be expressed as a monotone union of closed 2-cells.

Proof: If $C^2 - I(C^2) = \emptyset$, C^2 is a 2-sphere, and the result is obvious.

Otherwise, let p be an invert point. Theorem 5.1 or Theorem 5.4 expresses C^2 as a union of certain C_i^2 which intersect in unions of 1-spheres or 1-cells, respectively.

In either case, the C_j^2 intersect in a 1-complex, say C^1. An arc A may be found which has p as one endpoint, intersects C^1 a finite number of times, and intersects each $C_j^2 - C^1$. Choose A such that, if $q \in A \cap C^1$, A intersects two of the $C_j^2 - C^2$ in every neighborhood of q. Let $A \cap C^1 = \{p, q_1, \ldots, q_m\}$.

Construct a 2-cell D in C^2 which contains A and is such that $D \cap C_1 = Q_1 \cup \cdots \cup Q_m \cup p$, where the Q_j are disjoint 1-cells, such that $q_j \in Q_j$, for all $j = 1, \ldots, m$.

Now D separates C_i^2 into several, say n_i, components. Let the closures of these components be $C_{i_1}^2, \cdots C_{i_n}^2$. Then each of the $C_{i_j}^2$ is a 2-cell.

Let the $C_{i_j}^2$ be ordered. Let them be represented as $E_1, \ldots, E_k$.

Consider E_i. Let F_i be a 1-cell in $D \cap E_i$, such that $F_i \cap C^1 = \emptyset$. Let $G_i = E_i \cap C^1$. G_i is the union of disjoint 1-cells.

Now, consider E_1. Let $\{D_{1_i}\}$ be a monotone sequence of 2-cells in E_1, such that $D \cap D_{1_i} = F_1$. The D_{1_i} may be chosen so that $E_1 - (D \cup G_1) \subseteq \bigcup_{i=1}^{\infty} D_{1_i}$. In fact, the D_{1_i} may be chosen so that $G_1 \subseteq \bigcup_{i=1}^{\infty} D_{1_i}$ also.

Next, construct a monotone sequence $\{D_{2_i}\}$ in the same way, except that the D_{2_i} must be disjoint from G_1. Thus, $E_2 - (D \cup G_2) \subseteq \bigcup_{i=1}^{\infty} D_{2_i}$, and $G_2 - G_1 \subseteq \bigcup_{i=1}^{\infty} D_{2_i}$. Repeat this construction until each of the E_i are covered.

Now $D \cup (\bigcup_{j=1}^{k} D_{j_i})$ is a 2-cell, so $\bigcup_{i=1}^{\infty} \left[D \cup (\bigcup_{j=1}^{k} D_{j_i})\right]$ is a monotone union of 2-cells, and it equals C^2.

Next, attention turns to the problem of which 2-complexes may be embedded in E^3.

Theorem 5.6. Let C^2 be a 2-complex which is invertible at two or more points. Then let p be an invert point, and U an open cone neighborhood of p. If the link of U is planar, C^2 may be embedded in E^3.

Proof: Let L be the link of U. Then C^2 is the suspension over L, and since L may be embedded in $E^2 \subseteq E^3$, C^2 may be embedded in E^3.

Lemma 5.7. Let C^2 be a 2-complex which is invertible at exactly one point p, with $C^2 - p$ connected. Let U be an open cone neighborhood of p, and let L be the link of U. Then if U - p has two components and if L is planar, C^2 may be embedded in E^3.

Proof: By Theorem 4.7, C^2 is a suspension over L, with suspension points identified. Let L be embedded in $E^2 \subseteq E^3$, and let C be the suspension over L, with suspension points p_1 and p_2. Then C is in E^3. But p_1 and p_2 may be identified in $E^3 - \left[C - (p_1 \cup p_2)\right]$. Thus, C^2 is in E^3.

Theorem 5.8. Let C^2 be a 2-complex which is invertible at exactly one point with $C^2 - p$ connected. Let U be an open cone neighborhood of p in C^2, and let L be the link of U. Then if L is planar, C^2 may be embedded in E^3.

Proof: Once again, the decomposition of Theorem 5.1 is used.

If U - p has two components, Lemma 5.7 applies. So suppose that U - p is connected. Then $C^2 = \bigcup_{i=1}^{n} C_i^2$, and at least one of the C_i^2 must be a 2-sphere or a 2-cell. But if C_i^2 is a 2-sphere, it must equal C^2, since $C^2 - p$ is connected, and if another C_j^2 intersected C_i^2, it would separate C_i^2 into two components.

So suppose r of the C_i^2 are 2-cells, while the other C_j^2 are "pinched" tori and "pinched" annuli. Let the C_i^2 be so ordered that the first r are 2-cells. Let $L_i = L \cap C_i^2$, and let a_i and b_i be points in $L_i - C^1$. Here, C^1 is as it was in the proof of Theorem 5.1, the set of 1-simplices in C^2 which are faces of one, three, or more 2-simplices. Let A_i be a 1-cell in L_i, which has a_i and b_i as endpoints, and let H_i be the open cone over $a_i \cup b_i$. Then $S_i = A_i \cup H_i$ is a 1-sphere in $C_i^2 - C^1$. S_i separates C_i^2 into two components one of whose closures is a 2-cell which is disjoint from $C^1 - p$. Let M_i be this component.

Now the inversion isotopies on C^2 may be taken so that S_i is carried onto itself under every inversion. Thus, $C = C^2 - \bigcup_{i=1}^{r} M_i$ is a 2-complex which is invertible at p. But, by construction, $(U - p) \cap C$ has two components. Thus, by Lemma 5.7, C may be embedded in E^3 with $L \cap C$, the link of $U \cap C$, in $E^2 \subseteq E^3$. Note that L is separated by $\bigcup_{i=1}^{r} A_i$ into two components. Let $L - \bigcup_{i=1}^{r} A_i = L^1 \cup L^2$, where each of the L^j is connected. Note that the L^j are topologically equivalent.

By the construction of C, $U \cap C$ is on one side of E^2 in E^3, and $C - U$ is on the other side.

Now since $a_i \in L^1$, $b_i \in L^2$, say, there is an arc A_i' in E^2 with endpoints a_i and b_i, disjoint from C except for a_i and b_i, for all i = 1, ..., r. Such arcs may be found, for suppose that one of the L^j, say L^1 separated a_i from b_i. Then $L^1 - L_i$ would also separate a_i from b_i in E^2. Thus, since $C - U$ is topologically $L^1 \times [0,1]$, $(L^1 - L_i) \times [0,1] = K_i$ would have to separate a_i and b_i in that component of $E^3 - E^2$ determined by $C - U$.

But A_i is an arc joining a_i and b_i in $E^3 - K_i$. Let the A_i' be so chosen that they are disjoint, for all i.

Then $A_i \cup A_i' = S_i'$, a 1-sphere. This 1-sphere spans a 2-cell R_i, such that $R_i \cap C = S_i'$, and such that $R_i - A_i'$ is in that component of $E^3 - E^2$ determined by $C - U$. For if not, then there would be a 1-sphere in K_i which would link S_i'. But this is impossible, by the same reasoning as above.

Now, let T_i be the 2-cell formed by taking the cone over A_i' from the vertex p. Then $R_i \cup T_i = M_i'$ is a 2-cell, and $C \cup (\bigcup_{i=1}^{r} M_i')$ is topologically equivalent to C^2. But this complex has been embedded in E^3.

Note that if L is not planar, the above theorems will not hold. For the suspension over any skew graph, one which cannot be placed in a plane, cannot be embedded in E^3.

Next, the union of two 2-complexes, each of which has a non-null invert set, is considered.

One definition is necessary. A book is a union of 2-cells $\bigcup_{i=1}^{n} D_i$, which are such that $D_i \cap D_j = A$, a 1-cell, for all $i \neq j$.

Theorem 5.9. Let C_1 and C_2 be 2-complexes such that $C_1 \cap C_2 = S_1$, a 1-sphere which is a subcomplex of each, and such that each is continuously invertible at $p \in S_1$. Suppose that for each open set U, $p \in U$, there exist inversion isotopies $\{h_{1t}\}$ and $\{h_{2t}\}$ for C_1 and C_2, respectively, such that $h_{it}(S_1) = S_1$, $i = 1, 2$, and for all t. Then $C_1 \cup C_2 = C$ is continuously invertible at p.

Proof: Let U be a neighborhood of p in C. There is no loss of generality in assuming that $\{h_{1t}\}$ and $\{h_{2t}\}$ carry $S_1 - (U \cap S_1)$ into the same component of $(U \cap S_1) - p$. For if not, $\{h_{1(1-t)}h_{11}^{-1}\}$ and $\{h_{2t}\}$ do.

Either some 1-simplex which is not in S_1 has a face in $S_1 - p$, or else no 2-simplex or 1-simplex does. If the latter occurs, the result is clear, for the isotopy on C_1 may be taken to be $\{h_{1t}\}$ while that on $C_2 - S_1$ is $\{h_{2t}\}$.

If there is a 1-simplex not in S_1, which has a face in S_1, then there is a book of 2-simplices in C, whose back is in S_1. The isotopies and their inverses slide this book in C, along all of $S_1 - (S_1 \cap U)$. Thus, in some neighborhood of $S_1 - p$, say V, there is a book B whose back contains $S_1 - (U \cap S_1)$. In fact, B may be taken to include $S_1 - (W \cap S_1)$, where W is any neighborhood of p in C. Thus, there is a singular book B' in C, which is the union of "pinched" annulii which intersect in S_1. Let these "pinched" annulii be $A_1, \ldots, A_n$.

For each A_i, let k_i be a homeomorphism on A_i into A_i, fixed on $A_i \cap \mathrm{Cl}(C - B')$, which carries $S_1 - p$ into $A_i - S_1$, in such a way that $A_i - k_i(A_i)$ is a collar E_i for S_1 in A_i. Then $\bigcup_{i=1}^{n} \mathrm{Cl}(E_i)$ is a singular book, with back S_1.

Let

$$g_t(x) = \begin{cases} k_i\, h_{1t}\, k_i^{-1}(x), & x \in A_i - E_i, \text{ if } A_i \subseteq C_1 \\ k_i\, h_{2t}\, k_i^{-1}(x), & x \in A_i - E_i, \text{ if } A_i \subseteq C_2 \\ h_{1t}(x), & x \in C_1 - B' \\ h_{2t}(x), & x \in C_2 - B' \end{cases}$$

Now g_t is a homeomorphism on $C - \bigcup_{i=1}^{n} E_i$ onto itself, and g_1 carries $(C - U) \cap (C - \bigcup_{i=1}^{n} E_i)$ into U. Let $g_t(x) = h_{1t}(x)$, for $x \in S_1$. Then g_t is a homeomorphism on $\mathrm{Cl}(E_i) - E_i$, for $i = 1, \ldots, n$. Thus, g_t may be extended to a homeomorphism on E_i, for all i. This extension may be accomplished in such a way that $E_1 - U$ is carried inside U by g_1. Thus, $\{g_t\}$ is an inversion isotopy on C, and the theorem is proved.

This theorem offers a method of constructing 2-complexes with invert points from simpler 2-complexes.

VI. SUBCOMPLEXES OF n-COMPLEXES WITH INVERT POINTS.

It has been noted in several proofs that certain subcomplexes in n-complexes which have invert points inherit the inversion properties. These subcomplexes are those determined by the (n - 1)-simplices which are faces of no, one, three, or more n-simplices. In this section, however, the i-skeleton of such an n-complex is considered.

An example of an n-complex with non-null invert set whose (n - 1)-skeleton has a non-null invert set is a closed n-simplex. Its (n - 1)-skeleton is an (n - 1)-sphere.

It is not hard to construct other simple examples. The "bow tie" complex described in Chapter 4 is one. However, it turns out that the n-complex must be relatively uncomplicated. First, a lemma is needed, which also gives insight into the sort of n-complexes which have non-null invert sets.

Lemma 6.1. Let C^n be an n-complex which contains one or more n-spheres. Then C^n is not invertible at any point which is disjoint from any of the n-spheres.

Proof: Suppose that p is an invert point. Let S^n be a n-sphere which is disjoint from p, and let U be an open cone neighborhood of p which is disjoint from S^n. By the invertibility of C^n at p, there must be an n-sphere in U. But then this n-sphere in U must define a non-bounding cycle in $H_n(U)$, since U contains no (n + 1)-simplices. However, a theorem from algebraic topology says that, if U is a cone, then $H_i(U) = 0$, for all i [6]. Thus, there cannot be such an n-sphere in U, and therefore p cannot be an invert point.

Theorem 6.2. Let C^n be an n-complex, and let C^{n-1} be its (n - 1)-skeleton. Suppose that C^n contains m closed n-simplices $A_1, \ldots, A_m$, which are such that $\bigcap_{i=1}^{m} A_i = \emptyset$. Then C^{n-1} is not invertible at any point.

Proof: Let $S_1, \ldots, S_m$ be the (n - 1)-spheres which are, respectively, determined by the faces of the $A_1, \ldots, A_m$. Suppose that $p \in C^{n-1}$ is an invert point. Then by Lemma 6.1, $p \in \bigcap_{i=1} A_i$. This is a contradiction.

Theorem 6.3. Let C^n be an n-complex, and let C^{n-2} be its (n - 2)-skeleton. Then C^{n-2} is not invertible at any point.

Proof: For $n = 1$, the result is true vacuously. For $n = 2$, the (n - 2)-skeleton is a union of three or more points. This cannot be invertible at any of them.

So suppose that $n \quad 2$. Let A^n be a principal closed n-simplex in C^n, and let A^{n-2} be its (n - 2)-skeleton. A^{n-2} contains n + 1 (n - 2)-spheres, whose intersection is null. But then, by Theorem 6.2, C^{n-2} cannot be invertible at any point.

VII. HOMOLOGY AND HOMOTOPY PROPERTIES

This chapter considers the homology and homotopy properties of n-complexes with invert points. First, consider an n-complex C^n with at least two invert points. By Theorem 4.5, C^n is a suspension. Let C^{n-1} be the (n - 1)-complex which, suspended, gives C^n. The following theorem is a restatement, in terms of invertibility, of Theorem 8.5.10 in Hilton and Wylie [6].

Theorem 7.1. Let C^n and C^{n-1} be as above. Then there is an isomorphism $E_* : H_{k-1}(C^{n-1}) \cong H_k(C^n)$.

The next theorem concerns the homotopy groups of an n-complex with a k-sphere invert set.

Theorem 7.2. Let C^n be an n-complex whose set of invert points is a k-sphere S^k. Then $\pi_i(C^n)$ is trivial, for $1 \leqq i < k$, and $\pi_k(C^n)$ and $H_k(C^n)$ are isomorphic.

Proof: Let S^i be the continuous image of an i-sphere in C^n, for $1 \leqq i < k$. Let $p \in S^k - S^i$, and let U be a neighborhood of p which is such that $S^i \subseteq C^n - U$. Then if h is an inverting homeomorphism on C^n which carries $C^n - U$ into U, $h(S^i) \subseteq U$. But $h(S^i)$ is homotopically trivial in U, so S^i is homotopically trivial in C^n. Thus, $\pi_i(C^n)$ is trivial.

The Hurewicz Theorem [6] states that if C^n is such that $\pi_i(C^n)$ is trivial, for $1 \leqq i < k$, then $\pi_k(C^n)$ is isomorphic to $H_k(C^n)$. Thus, the theorem is proved.

Now, let C^n be any n-complex which has a non-null invert set.

A result similar to Theorem 7.1 is obtained. However, it is not as strong.

Theorem 7.3. Let C^n be an n-complex which is invertible at a point p. Let U be an open cone neighborhood of p in C^n, and let C^{n-1} be the link of U in C^n. Then $H_{k+1}(C^n)$ is isomorphic to a subgroup of $H_k(C^{n-1})$.

Proof: The Meyer-Vietoris sequence

$$\cdots \xrightarrow{\theta_p} H_p(C^{n-1}) \xrightarrow{\phi_p} H_p\left[Cl(U)\right] \oplus H_p(C^n - U) \xrightarrow{\psi_p} H_p(C^n) \xrightarrow{\theta_{p-1}} H_{p-1}(C^{n-1}) \xrightarrow{\phi_{p-1}} \cdots$$

is exact [6]. But H_p Cl(U) $= 0$, since Cl(U) is a closed cone, so the sequence becomes

$$\cdots \xrightarrow{\theta_p} H_p(C^{n-1}) \xrightarrow{\phi_p} H_p(C^n - U) \xrightarrow{\psi_p} H_p(C^n) \xrightarrow{\theta_{p-1}} H_{p-1}(C^{n-1}) \xrightarrow{\phi_{p-1}} \cdots$$

Now $\psi_p\left[H_p(C^n - U)\right] = 0$, since $C^n - U$ is homologically trivial in C^n. But since the sequence is exact, $\psi_p\left[H_p(C^n - U)\right]$ is the kernel of θ_{p-1}. So θ_{p-1} is an isomorphism into.

The following example shows that $H_p(C^n)$ may be isomorphic to a proper subgroup of $H_{p-1}(C^{n-1})$. Let S^n be the n-sphere, and let A' and A" be parameter n-cells in S^n such that $A' \cap A'' = p$. Let h be a map on S^n such that $h(A'') = A'$, and h is the identity on $S^n - A''$. Let $h(S^n) = C^n$. Let U be an open cone neighborhood of p" in C^n. Then the link of U, C^{n-1}, consists of two disjoint (n - 1)-spheres, joined by an (n - 1)-cell. Thus, $H_{n-1}(C^{n-1})$ is the free abelian group on two generators.

Now, let f be the map on S^n onto a space X, such that $f(A) = p''$, for $A = A' \cup A''$, while $f(x) = x$, for $x \in S^n - A$. Then X is an n-sphere. Let g be a map on C^n onto a space Y such that $g(A') = p''$, and $g(x) = x$, for $x \in C^n - A'$. Then $f = gh$, so $X = Y$.

But the Vietoris Mapping Theorem [7] says that, for M and N compact metric spaces, and f an n-monotone mapping of M onto N, $H_p(M)$ is isomorphic to $H_p(N)$, for all $p \leqq n$. Here, an n-monotone map f is one such that, for $x \in N$, $f^{-1}(x)$ is homologically trivial for all $p \leqq n$, and is connected.

Then g is an n-monotone map on C^n onto S^n, so $H_n(C^n)$ is isomorphic to $H_n(S^n)$, the free abelian group on one generator. Thus, $H_n(C^n)$ is isomorphic to a proper subgroup of $H_{n-1}(C^{n-1})$.

The Vietoris Mapping Theorem may be applied to other examples which may be mapped onto a space such as C^n by a monotone function.

Corollary 7.4. Let C^n, p, U, and C^{n-1} be as above. If $H_p(C^{n-1}) = 0$, $H_{p+1}(C^n) = 0$.

Now, these theorems will be applied to the special case that $n = 2$. Let C^2 be a 2-complex which is invertible at exactly one point p. Suppose that $C^2 - p$ is connected. Let U be an open cone neighborhood of p in C^2. If $U - p$ has two components, C^2 is a suspension over a connected 1-complex, with vertices identified, by Theorem 4.7. Thus, $H_1(C^2)$ is the free group with one generator. If $U - p$ is connected, then the link of U, C^1, is connected. By the Theorem 7.3, $H_1(C^2)$ is a subgroup of $H_0(C^1)$ and since $H_0(C^1) = 0$, $H_1(C^2) = 0$.

Thus, if $C^2 - p$ is not connected, $H_1(C^2)$ is the free abelian group with k generators, where k is the number of components of $C^2 - p$ which are such that $U \cap (C^2 - p)$ has two components.

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COMPLEXES WITH INVERT POINTS

Vyron Martin Klassen

An Abstract

A topological space X is invertible at $p \in X$ if for every neighborhood U of p in X, there is a homeomorphism h on X onto X such that $h(X - U) \subseteq U$. X is continuously invertible at $p \in X$ if for every neighborhood U of p in X there is an isotopy $\{h_t\}$, $0 \leqq t \leqq 1$, on X onto X such that $h_1(X - U) \subseteq U$.

It is proved that, if X is a locally compact space which is invertible at a point p which has an open cone neighborhood, and if the inverting homeomorphisms may be taken to be the identity at p, then X is continuously invertible at p.

A locally compact Hausdorff space X, invertible at two or more points which have open cone neighborhoods in X, is characterized as a suspension. A locally compact Hausdorff space X which is invertible at exactly one point p, which has an open cone neighborhood U such that U - p has two components, while X - p is connected, is characterized as a suspension with suspension points identified.

Let C^n be an n-complex with invert point p. Let U be an open cone neighborhood of p in C^n, and let L be the link of U in C^n. Then it is shown that $H_p(C^n)$ is isomorphic to a subgroup of $H_{p-1}(L)$.

Invertibility properties of the i-skeleton of an n-complex are discussed, for $i < n$. Also, a method is described by which an n-complex which is invertible at certain points may be expressed as the union of subcomplexes, each of which is invertible at the same points.

One-complexes with invert points are characterized as either a suspension over a finite set of points or a union of simple closed curves $\bigcup_{i=1}^{n} S_i$, such that $S_i \cap S_j = p$, for $i \neq j$.

It is proved that, if C^2 is a 2-complex with one invert point p, C^2 may be expressed as the monotone union of closed 2-cells. Also, if the link of an open cone neighborhood of an invert point in a 2-complex C^2 is planar, C^2 may be embedded in E^3.