AN INVESTIGATION OF THE USE OF INERTIA AS A PERTURBATION PARAMETER AND CONTINUED FRACTIONS IN LINEAR VIBRATION PROBLEMS

by

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TABLE OF CONTENTS

I. INTRODUCTION

The response of linear elastic systems to time-dependent excitation is a much-studied and very important engineering problem. Many methods exist for casting the equations governing the motion of such systems into forms suitable for solution by digital computation. These equations may be based on the theory of linear elasticity, the methods of strength of materials, or idealized models of the system.

This dissertation contributes to the engineering literature a new formulation of these problems, which adds insight to certain existing methods, and has some computational advantage for certain types of problems.

The principal concepts which are employed in this development are:

1. The use of mass as a perturbation parameter in linear vibration problems, wherein the first term of a series solution is the static, or massless, response of the system, and each successive term is the solution of the equation governing the static deformation of the system due to a forcing function which is some function of the previous term. This concept is due to Dr. R. U. Chicurel.

2. The expansion of the Laplace transform of a problem in powers of the transform parameter.

3. The use of continued fractions to obtain approximate, and in some cases, exact solutions from the series generated by the above methods.

In what follows a study will be made of the generation and interpretation of power series solutions of linear vibration problems for which static solutions are either known or easily obtained. The interpretation of the series will be concerned with obtaining natural frequencies, mode shapes, and impulse responses, and in this context, several methods will be examined. These are, the systematic removal of the singularities of the series, the location of a point which will not be a node at any frequency, and the use of continued fractions. These concepts are put in proper perspective, with respect to the existing literature, in Chapter II.

II. REVIEW OF THE LITERATURE

Perturbation techniques have been used since the time of Poincaré (1)[1] to obtain solutions of nonlinear differential equations. Extensive work in recent years (2) has led to a marked increase in the power of such methods for solving very difficult nonlinear problems. During this same course of time little attention has been given to the use of perturbation methods for solving vibration problems which are governed by linear differential equations.

What may be called the classical perturbation technique is the procedure of assuming that the unknown function in a problem can be expressed as a power series in some parameter which appears in the problem. The assumed series is substituted in the governing equation and the boundary and initial conditions. A set of governing equations and boundary and initial conditions is obtained by gathering the coefficients of like powers of the perturbation parameter from all the expansions. Each problem in the set is (hopefully) solved recursively, and substituted in the original power series, which now represents the solution of the original problem. Proof has been given (1, 3) that the solution must depend analytically on the perturbation parameter if the method is to be successful.

Mass, or inertia, has appeared in the literature as a perturbation parameter, or a factor in a perturbation parameter. (For example,

[1]Numbers in parentheses refer to the references on page 60.

see (9).) However, the use of mass, or inertia, in the context of this dissertation is apparently new. The motivation for using the perturbation technique is the hope of obtaining a series of problems, each of which is more simple to solve than the original equation. A linear vibration problem is indeed simplified if the inertia is deleted. The result is a "static" problem in which every point in the system responds instantaneously to and in phase with whatever forcing function might be applied. With this observation as motivation, the solution to a linear vibration problem is assumed to be expressible as a power series in the inertia, which may be discrete or continuous. Each problem in the set will be a static problem in which some function of the solution for the problems of lower order in the perturbation parameter will appear as a loading.

The equations resulting from the perturbation expansion may involve lower order derivatives than the original equation, and hence it might not be possible to satisfy all the boundary and initial conditions. In this sense, the perturbation problem may be singular. However, initial conditions are not needed for determining the eigenvalues and eigenfunctions of a linear vibration problem. However, methods (7, 11) have been derived for treating such singular perturbation problems.

Also, truncation of the series solution for a given problem will give a solution which will be valid only for a limited range of some combination of the perturbation parameter and an independent variable. This is another way in which the perturbation solution is singular, and the methods discussed by Nayfeh (7, 11) can be considered. However, the method of continued fractions (8, 10) is used in this dissertation to extend the validity of the series, and it appears that continued fractions will have application to other singular perturbation problems.

Iterative methods of solution of linear vibration problems in which the mass is eliminated in the first approximation and then introduced by correcting the displacements according to the effect of the inertia forces, are known (4), but are not directly applicable to the determination of natural frequencies and normal modes. Soong (5) discusses the use of the perturbation method based on mass as the parameter; however, the unperturbed system is one in which the mass is uniformly distributed rather than altogether absent, and only certain types of lumped systems are treated. The perturbation theory associated with the determination of eigenvalues of matrices usually centers on the question of error analysis (6), where the behavior of the system is studied only very near a particular eigensolution.

Energy techniques (12) have been applied extensively in the solution of linear vibration problems, and in many cases are the most practical means of solution. The concept of transfer matrices has been exploited in a recent text by Pestel and Leckie (13).

Indeed, transfer matrix methods are presently one of the most popular means of solving linear vibration problems.

The concept of mobility has been widely used since being introduced by Firestone (14), and is extensively treated in a book edited by R. Plunkett (15). One application of the method of continued fractions, as developed in this dissertation, will be to facilitate an approximate structural mobility method considered in a recent paper by Neubert (16). Continued fractions have been used by Wall (10) as an alternative to the Routh-Hurwitz stability criterion. Dr. R. U. Chicurel and the writer have submitted a paper for publication which contains part of this dissertation.

III. LUMPED SYSTEMS

1. Two-Degree-of-Freedom System

The problem of determining the natural frequencies and modal configurations of a two-degree-of-freedom-system will serve to explain the salient features of the method. Consider the system shown in Figure 1, where, for simplicity, the masses and springs are identical, each having mass, m, and modulus, k. The displacements, x_A and x_B, are measured from the equilibrium positions of the masses. The equations of motion for free, undamped vibrations are

$$\frac{1}{\omega_o^2}\ddot{x}_A + \frac{1}{\omega_o^2}\ddot{x}_B = -x_A \qquad (1)$$

$$\frac{1}{\omega_o^2}\ddot{x}_A + \frac{2}{\omega_o^2}\ddot{x}_B = -x_B \qquad (2)$$

where $\omega_o^2 = \frac{k}{m}$ and dots indicate differentiation with respect to time. If harmonic response is assumed, then $x_A = X_A \cos \omega t$, and $x_B = X_B \cos \omega t$. Calling $\mu = \omega^2/\omega_o^2$, the equations become

$$\mu X_A + \mu X_B = X_A \qquad (3)$$

$$\mu X_A + 2\mu X_B = X_B \qquad (4)$$

The parameter μ, which is proportional to the mass, will be used as a perturbation parameter. If series expansions for X_A and X_B are assumed as

$$X_A = \sum_{n=o}^{\infty} X_{An}\,\mu^n \quad \text{and} \quad X_B = \sum_{n=o}^{\infty} X_{Bn}\,\mu^n \quad ,$$

and are substituted into Eqs. (3) and (4), it is seen that $X_{An} = X_{Bn} = 0$, and the method fails. The usual techniques for obtaining natural frequencies and mode shapes would begin with the homogeneous system of equations

$$X_A(-1+\mu) + \mu\, X_B = 0 \tag{5}$$

$$\mu\, X_A + X_B\,(2\mu - 1) = 0 \tag{6}$$

A system of non-homogeneous equations will be obtained, if a harmonic excitation, F cos ωt, is applied to mass B. With this forcing function, Eqs. (3) and (4) become

$$\mu\, X_A + \mu\, X_B = X_A - \frac{F}{k} \tag{7}$$

$$\mu\, X_A + 2\mu\, X_B = X_B - 2\,\frac{F}{k} \tag{8}$$

Substituting the assumed series for X_A and X_B, and equating coefficients of like powers of μ, a set of recursion relations for X_{An} and X_{Bn} results. These are

$$X_{Ao} = \frac{F}{k}\,,\; X_{Bo} = 2\,\frac{F}{k}$$

$$X_{Ai} = X_{A(i-1)} + X_{B(i-1)},\; X_{Bi} = X_{A(i-1)} + 2X_{B(i-1)},\; i = 1,\, 2,\, ..$$

With these relations, the following series are obtained:

$$X_A = \frac{F}{k}\,(1 + 3\mu + 8\mu^2 + 21\mu^3 + \ldots) \tag{9}$$

$$X_B = \frac{F}{k}\,(2 + 5\mu + 13\mu^2 + 34\,\mu^3 + \ldots) \tag{10}$$

The situation wherein an ordinary perturbation expansion will not yield a solution will occur in every linear vibration problem whose formulation is homogeneous. This difficulty can, in every case, be circumvented by applying external loads which will produce a non-homogeneous system of governing equations.

Now, consider the series for the displacement of mass B. Solution of Eqs. (7) and (8) by Cramer's rule would yield a quotient of polynomials as the solution for X_B, namely,

$$X_B = \frac{p_B(\mu)}{q(\mu)}$$

where $p_B(\mu)$ is at least one degree lower in μ than $q(\mu)$. Two natural frequencies exist for this simple system, and can be calculated from $q(\mu) = 0$. Also, the possibility exists that mass B might be a node, which implies $p_B(\mu) = 0$ has one real root. Hence, X_B can be written as

$$X_B = C\,\frac{(\mu - \bar{\mu})}{(\mu - \mu_1)(\mu - \mu_2)} \tag{11}$$

where μ_1 and μ_2 are the natural frequencies, $\bar{\mu}$ is the frequency at which X_B becomes a node[2] and C is a constant. Equation (11) indicates that the series for X_B should converge up to μ_1 (assuming $\mu_1 < \mu_2$), which is its first pole. On the other hand, the reciprocal of the series should converge up to $\bar{\mu}$, which can be seen by writing

$$\frac{1}{X_B} = \frac{1}{C} \; \frac{(\mu - \mu_1)(\mu - \mu_2)}{(\mu - \bar{\mu})} \tag{12}$$

The values of μ_1 and $\bar{\mu}$ can be obtained by truncating the corresponding series at some power of μ, and finding the lowest real root of the resulting polynomial.

Equating the right hand side of Eq. (10) to zero, and truncating the series at the powers of μ indicated below, gives these results for $\bar{\mu}$:

μ^1: $\bar{\mu} = - 0.4$ (ω is imaginary)

μ^2: a pair of complex conjugate roots.

The series for $\frac{1}{X_B}$, obtained from Eq. (10), is

$$\frac{1}{X_B} = \frac{k}{F} \left(\frac{1}{2} - \frac{5}{4} \mu - \frac{1}{8} \mu^2 - \frac{1}{16} \mu^3 \; . \; . \; . \right) \tag{13}$$

Equating Eq. (13) to zero, and truncating the series at the powers of μ indicated below gives these results for μ_1:

μ^1: $\mu_1 = 0.4$

μ^2: $\mu_1 = 0.39$.

[2] X_B is a node implies $X_B = 0$.

Comparing the results for μ_1 and $\bar{\mu}$, it appears that μ_1 is less than $\bar{\mu}$. Indeed, the exact solution for μ_1 is 0.382, to three significant figures, and the correct value of $\bar{\mu}$ is 2.0.

Referring to Eq. (11), the pole at μ_1 can be removed by multiplying the series for X_B by $(\mu - \mu_1)$. Performing this multiplication yields

$$(0.39 - \mu)\, X_B = 0.78 - 0.05\,\mu + 0.07\,\mu^2 + 0.26\,\mu^3 \ldots \qquad (14)$$

Following the procedure used to find μ_1, yields

μ^1: $\bar{\mu} = 15.6$

μ^2: complex conjugate roots .

It is immediately seen that this series will converge (if at all) much slower than that of Eq. (13), the reason being that μ_1 has not been removed exactly. If convergence can be obtained, more terms of the series will have to be included as the desired value of μ increases. This inherent truncation error cannot be avoided, and hence detracts, somewhat, from the procedure of removing the singularities, one at the time, from the displacement and its reciprocal series. Leaving the investigation of the series for X_B at this point, the series for X_A will be examined.

Division of unity by Eq. (9) yields

$$\frac{1}{X_A} = \frac{k}{F}\,(1 - 3\mu + \mu^2) \qquad (15)$$

Not only is Eq. (15) a finite polynomial, but the term in parentheses has roots corresponding to resonant conditions. Examination of Figure 1 reveals that mass A cannot become a node since neither spring could deflect under such a condition. This implies that if X_A were to be expressed in a form similar to Eq. (11), the numerator would be a polynomial of zero order. Hence, division by zero would not arise when the reciprocal series, $\frac{1}{X_A}$, is computed, and the resulting series will converge for all values of μ. Also, division of Eq. (10) by Eq. (9) gives

$$\frac{X_B}{X_A} = 2 - \mu \quad ,$$

and, therefore

$$X_B = X_A \left(\frac{X_B}{X_A} \right) = \frac{F}{k} \; \frac{(2 - \mu)}{(1 - 3\mu + \mu^2)}$$

That the division gives a finite polynomial is a remarkable result, which will be extremely useful when a point that cannot become a node can be identified in a vibrating system. This identification will not always be possible, but fortunately, an alternative procedure has been found for investigating the displacement series.

If the series for X_B is expanded into a continued fraction by the method of Viskovatoff (8), the procedure for computing the coefficients yields (see Appendix I):

α_{00}:	1	0	0	0	0	0	0	---
α_{10}:	2	5	13	34	89	233	610	---
α_{20}:	- 5	- 13	- 34	- 89	- 233	- 610	---	---
α_{30}:	1	3	8	1	---	---	---	---
α_{40}:	2	6	- 84	---	---	---	---	---
α_{50}:	0	---	---	---	---	---	---	---

where dashes indicate uncomputed terms. Since $\alpha_{50} = 0$, the continued fraction terminates at that point, and

$$\frac{k}{F} \cdot X_B = \cfrac{2}{1 - \cfrac{5\mu}{2 + \cfrac{\mu}{-5 + 2\mu}}}$$

When the terms are combined, the result is

$$X_B = \frac{F}{k} \; \frac{(2 - \mu)}{(1 - 3\mu + \mu^2)} \qquad ,$$

which is identical with the previous expression for X_B. The same procedure will give the correct expression for X_A. The concept of continued fraction representation of infinite series is seen to be a very desirable, and almost necessary, complement to the success of the perturbation procedure.

2. The N-Degree-of-Freedom System

The equations of motion of a linear system of N-degree-of-freedom can be formulated as

$$x_i = \sum_{j=1}^{N} a_{ij} F_j, \; i = 1, 2, \ldots, N \tag{16}$$

where x_i is the displacement of mass m_i from its equilibrium position, F_i is the force acting on mass m_i, and a_{ij} is a flexibility influence coefficient which is defined as the displacement of mass m_i due to a unit force applied to mass m_j in the direction of x_j. The force, F_i, can be expressed as

$$F_i = f_i'(t) - m_i \ddot{x}_i(t) \tag{17}$$

where $f_i'(t)$ is a known, externally applied, time-dependent force, and $-m_i\ddot{x}_i(t)$ is the inertia force acting on mass m_i. If $f_i'(t)$ and $x_i(t)$ are assumed to vary harmonically with time, then $x_i = X_i \cos \omega t$ and $f_i' = f_i \cos \omega t$, where ω is the frequency. Substitution of these relations and Eq. (17) in Eq. (16) yields

$$X_i = \sum_{j=1}^{N} a_{ij} f_j + \sum_{j=1}^{N} a_{ij} m_j X_j \omega^2, \; i = 1, 2, \ldots, N \tag{18}$$

If m and k designate a characteristic mass and stiffness, respectively, then

$$a_{ij}\, m_j\, \omega^2 = \frac{a'_{ij}}{k}\, m\, M_j\, \omega^2 = \frac{1}{k/m}\, a'_{ij}\, M_j\, \omega^2 = \frac{\omega^2}{\omega_o^{\ 2}} = \mu\, a'_{ij}\, M_j$$

where $\omega_o^{\ 2} = \frac{k}{m}$, and $\mu = \frac{\omega^2}{\omega_o^{\ 2}}$. If $a'_{ij}\, M_j$ is called A_{ij}, then Eq. (18) becomes

$$X_i = \sum_{j=1}^{N} a_{ij}\, f_j + \mu \sum_{j=1}^{N} A_{ij}\, X_j,\ i = 1, 2, \ldots, N \tag{19}$$

Assuming a solution for X_i of the form

$$X_i = \sum_{Q=0}^{\infty} \mu^Q\, X_i^{\ Q},\ i = 1, 2, \ldots, N \quad , \tag{20}$$

and substituting this equation in Eq. (19), and collecting coefficients of like powers of μ will give the following recursion formulas for the $X_i^{\ Q}$:

$$X_i^o = \sum_{j=1}^{N} a_{ij}\, f_j,\ X_i^{\ Q} = \sum_{j=1}^{N} A_{ij}\, X_j^{\ Q-1},\ Q \geq 1,\ i = 1, 2, \ldots, N \tag{21}$$

Using Eq. (21), the expanded form of Eq. (20) becomes

$$X_i = \sum_{j=1}^{N} a_{ij}\, f_j + \mu \sum_{j=1}^{N} \sum_{m=1}^{N} A_{ij}\, a_{jm}\, f_m +$$

$$\mu^2 \sum_{j=1}^{N} \sum_{m=1}^{N} \sum_{n=1}^{N} A_{ij} a_{jm} a_{mn} f_n + \ldots,\ i = 1, 2, \ldots, N \tag{22}$$

Based on Cramer's rule, the form of X_i must be

$$X_i = \frac{p_i(\mu)}{q(\mu)} \tag{23}$$

where $p_i(\mu)$ is a polynomial of at least one degree less in μ than $q(\mu)$, and $q(\mu)$ must be of degree N. An exception to the latter statement occurs when $p_i(\mu)$ and $q(\mu)$ have a common factor. This situation would mean that a nodal frequency is coincident with a resonant frequency. Even then $p_i(\mu)$ will be of at least one degree less in μ than $q(\mu)$. The continued fraction representation of Eq. (22) will be used to obtain the numerator and denominator polynomials in Eq. (23).

Referring to Appendix I, it will be convenient to write Eq. (22) in the form

$$X_i = \sum_{r=o}^{\infty} \alpha_{1r}^{i} \mu^r$$

where

$$\alpha_{10}^{i} = \sum_{j=1}^{N} a_{ij} f_j, \; \alpha_{11}^{i} = \sum_{j=1}^{N} \sum_{m=1}^{N} A_{ij} a_{jm} f_m, \; \ldots \; , \text{etc.}$$

It is now agreed to drop the index, i, with the understanding that the X in question is X_i. Hence, the expression for X becomes

$$X = \sum_{r=o}^{\infty} \alpha_{1r} \mu^r$$

As shown in Appendix I,

$\alpha_{0i} = 0,\ i \geq 1,\ \alpha_{00} = 1,\ \alpha_{20} = -\alpha_{00}\,\alpha_{11}$, and

$$\alpha_{mn} = \alpha_{m-1,0}\ \alpha_{m-2,n+1} - \alpha_{m-2,0}\ \alpha_{m-1,n+1},\quad m \geq 2,\ n \geq 0\ .$$

Using these relations, X can be written as

$$X = \frac{\alpha_{10}}{\alpha_{00}} + \frac{\alpha_{20}}{\alpha_{10}}\,\mu + \frac{\alpha_{30}\,\mu}{\alpha_{20}} + \frac{\alpha_{40}\,\mu}{\alpha_{30}} + \cdot\cdot\cdot\cdot \tag{24}$$

For a system having N degrees of freedom, the α's need be calculated only through $\alpha_{2N,0}$. Equation (24) can be rewritten with

$$p_i(\mu) = P_{2N-1} = \alpha_{2N-1,0}(P_{2N-2}) + \mu\,\alpha_{2N,0}\,(P_{2N-3}) \tag{25}$$

$$q(\mu) = Q_{2N-1} = \alpha_{2N-1,0}\,(Q_{2N-2}) + \mu\,\alpha_{2N,0}\,(Q_{2N-3}) \tag{26}$$

If an explicit expression for $p_i(\mu)$ or $q(\mu)$ is desired, this can be obtained by carrying out the algebra indicated by Eqs. (25) and (26) without giving μ a numerical value. This is not necessary if the values of the roots of $p_i(\mu) = 0$ and $q(\mu) = 0$ are to be found on a digital computer. The calculational procedure would be similar to any routine procedure for finding the real roots of an N^{th} degree polynomial, with Eqs. (25) or (26) taking the place of the polynomial expression.

The normal modes of the N-degree-of-freedom system can be calculated by determining the ratio of all the X_i to, say, X_m. For example, the previously considered two-degree-of-freedom system gives

$$\frac{X_B}{X_A} = 2 - \mu$$

which is known to be the correct amplitude ratio for the normal modes. That the above procedure will always give the correct result can be justified by considering how the particular solution to the forced vibration problem is usually developed. Following Chen (17), if only one external force is applied, then

$$X_i = \sum_{r=1}^{N} \frac{\phi_{ir}\,\phi_{1r}}{\omega_r^2 - \omega^2} \cos \omega t$$

where ω_r is the r^{th} natural frequency, and the ϕ's are coefficients of the normal modes as determined from the homogeneous problem. The ratio of X_i to X_m can be written as

$$\frac{X_i}{X_m} = \frac{(\omega_R^2 - \omega^2) \sum_{r=1}^{N} \frac{\phi_{ir}\,\phi_{1r}}{\omega_r^2 - \omega^2}}{(\omega_R^2 - \omega^2) \sum_{r=1}^{N} \frac{\phi_{mr}\,\phi_{1r}}{\omega_r^2 - \omega^2}}$$

where ω_R is a particular natural frequency.

As ω approaches ω_R, this ratio becomes

$$\frac{X_i}{X_m} = \frac{\phi_{iR}}{\phi_{mR}}$$

which is the ratio of the normal mode coefficients at $\omega = \omega_R$.

3. An Example

The three-degree-of-freedom system shown in Figure 2 will be used as an example of the method. The desired quantities are the natural frequencies and the amplitude ratios. For this system,

$$M_i \Rightarrow \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} , \quad a_{ij} \Rightarrow \frac{1}{k} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 7 \end{pmatrix} , \quad A_{ij} \Rightarrow \begin{pmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{pmatrix}$$

If a force, $f \cos \omega t$, is applied to mass m_1, then

$$f_i \Rightarrow \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}$$

Hence, from Eq. (21),

$$X_i^o = \sum_{j=1}^{3} a_{ij} f_j = a_{i1} f, \quad X_i^m = \sum_{j=1}^{3} A_{ij} X_j^{(m-1)}, \quad m \geq 1, \quad \text{or}$$

$$X_i^m = A_{i1} X_1^{(m-1)} + A_{i2} X_2^{(m-1)} + A_{i3} X_3^{(m-1)} .$$

Substituting numerical values in these relations yields

$$X_1^o = X_2^o = X_3^o = \frac{f}{3k} \qquad \text{and}$$

$$X_1^m = 4\,X_1^{(m-1)} + 2\,X_2^{(m-1)} + X_3^{(m-1)}$$

$$X_2^m = 4\,X_1^{(m-1)} + 8\,X_2^{(m-1)} + 4\,X_3^{(m-1)}$$

$$X_3^m = 4\,X_1^{(m-1)} + 8\,X_2^{(m-1)} + 7\,X_3^{(m-1)}$$

From the latter equations,

$$X_1^1 = \frac{f}{3k}\,(4 + 2 + 1) = 7\,\frac{f}{3k}$$

$$X_2^1 = \frac{f}{3k}\,(4 + 8 + 4) = 16\,\frac{f}{3k}$$

$$X_3^1 = \frac{f}{3k}\,(4 + 8 + 7) = 19\,\frac{f}{3k}\,,$$

$$X_1^2 = 79\,\frac{f}{3k}\,,\ X_2^2 = 232\,\frac{f}{3k}\,,\ X_3^2 = 289\,\frac{f}{3k}\,,\ \text{etc.}$$

The series for X_1 is

$$X_1 = \frac{f}{3k}\,(1 + 7\mu + 79\,\mu^2 + 1{,}069\,\mu^3 + 15{,}127\,\mu^4 + 216{,}133\,\mu^5 + + 3{,}094{,}351\,\mu^6 + \ldots) \qquad (27)$$

The series in parentheses on the right hand side of Eq. (27) is now expressed as a continued fraction by the scheme of Viskovatoff outlined in Appendix I. First, the α_{ij} are computed as outlined below.

1	0	0	0	0	0	0
1	7	79	1,069	15,127	216,133	3,094,351
-7	-79	-1,069	-15,127	-216,133	-3,044,351	---
30	516	7,644	110,244	1,581,420	---	---
+1,242	21,438	317,848	4,585,950	---	---	---
-2,268	-43,092	655,452	---	---	---	---
4,898,880	93,078,720	---	---	---	---	---
0	---	---	---	---	---	---

The dashes above indicate uncomputed quantities which are unnecessary for the computation of the quantities in the first column, and the expansion terminates with α_{60}. From the first column in the table,

$\alpha_{00} = 1$, $\alpha_{10} = 1$, $\alpha_{20} = -7$, $\alpha_{30} = 30$, $\alpha_{40} = +1{,}242$, $\alpha_{50} = -2{,}268$, and $\alpha_{60} = 4{,}898{,}880$.

Hence,

$$\frac{3k}{f} X_1 = \frac{1}{1 +} \frac{-7\mu}{1 +} \frac{30\mu}{-7 +} \frac{+1{,}242\,\mu}{30 +} \frac{-2{,}268\,\mu}{+1242\,\mu +} \frac{4{,}898{,}880\,\mu}{-2{,}268}$$

or in unabridged notation

$$3\,\frac{k}{f}\,X_1 = \frac{1}{1} + \cfrac{-\,7\mu}{1 + \cfrac{30\mu}{-\,7 + \cfrac{1242\,\mu}{30 + \cfrac{-\,2268\,\mu}{1242 + \cfrac{4,898,880\,\mu}{-\,2268}}}}}$$

Now, referring to the development of Eqs. (25) and (26) in Appendix I,

$$P_o = \alpha_{10} = 1$$

$$P_1 = \alpha_{10}\,\alpha_{10} = 1$$

$$P_2 = \alpha_{20}\,P_1 + \mu\,\alpha_{30}\,P_0 = -\,7(1) + \mu(30)(1) = -\,7 + 30\,\mu$$

$$P_3 = 30(-7 + 30\mu) + 1242(1) = -\,210 + 2142\,\mu$$

$$P_4 = -\,240,820 + 2,676,240\,\mu - 68,040\,\mu^2$$

$$\begin{aligned} P_5 &= (1-\,2,268)(-260,820 + 2,676,240\,\mu - 68,040\,\mu^2) \\ &\quad + \mu(4,898,880)(-\,210 + 2142\,\mu) \\ &= 591,539,760 - 7,098,477,120\,\mu + 10,647,715,680\,\mu^2 \\ &= 591,539,760\,(1 - 12\mu + 18\,\mu^2) \end{aligned}$$

$$Q_o = \alpha_{00} = 1$$

$$Q_1 = \alpha_{00}\,\alpha_{10} + \alpha_{20}\,\mu = (1)(1) - 7\mu = 1 - 7\,\mu$$

$$Q_2 = \alpha_{20}\,Q_1 + \mu\,\alpha_{30}\,Q_0 = -\,7(1-7\mu) + \mu(30)(1) = -\,7 + 19\mu$$

$$Q_3 = 30\ (-\ 7 + 79\mu) + \mu(1{,}242)(1 - 7\mu) = -\ 210 + 3{,}612\mu - 8{,}694\ \mu^2$$

$$Q_4 = 1{,}242\ (-\ 210 + 3{,}612\ \mu - 8{,}694\ \mu^2) + (-\ 2{,}268)\ \mu\ (-7 + 79\ \mu)$$

$$= -\ 260{,}820 + 4{,}501{,}980\ \mu - 10{,}977{,}120\ \mu^2$$

$$Q_5 = -\ 2{,}268\ (-\ 260{,}820 + 4{,}501{,}980\ \mu - 10{,}977{,}120\ \mu^2)$$

$$+\ 4{,}898{,}880\ \mu\ (-\ 210 + 3{,}612\ \mu - 8{,}694\ \mu^2)$$

$$Q_5 = 591{,}539{,}760 - 11{,}239{,}255{,}400\ \mu + 42{,}590{,}866{,}720\ \mu^2$$

$$-\ 42{,}590{,}862{,}\ 720\ \mu^3$$

$$Q_5 = 591{,}539{,}760\ (1 - 19\mu + 72\ \mu^2 - 72\ \mu^3)$$

Hence,

$$X_1 = \frac{f}{3k}\ \frac{P_5}{Q_5} = \frac{f}{3k} \cdot \frac{(1 - 12\mu + 18\ \mu^2)}{(1 - 19\mu + 72\ \mu^2 - 72\ \mu^3)}$$

An advantage of the method of continued fractions is that the partial convergents (Q_i) give ordered approximations to the roots (μ) which correspond to the natural frequencies. For example, when Q_3 is equated to zero, the roots are 0.0699 and 0.3455 as compared to the correct values (to the same number of digits) of 0.0698 and 0.3333, respectively.

4. Response to an Arbitrary Forcing Function

If there is only one external force, f', in Eq. (19), then the quantity, f cos ωt will appear in the series for X_i. If ω is replaced by (i s), ($i = \sqrt{-1}$), then the series for X_i, aside from the factor f cos ωt, will be the Laplace transform of the response due to the application of a unit impulse at the point of application of f cos ωt, for the case of quiescent initial conditions. Hence, the response to this impulse can be found by resolving $\frac{p_i(s)}{q\ (s)}$ into partial fractions, and inverting the resulting terms. Use of the convolution integral will enable the calculation of the response to any forcing function. For a system having no more than, say, three-degrees-of-freedom, it may be desirable to carry this calculation out on a desk calculator. The following procedure is well-suited for the solution of a system having $N > 3$ on a digital computer.

Since

$$X_i = C\,\frac{p_i(\mu)}{q(\mu)},$$

where C is a constant ($\frac{1}{3k}$ in the example of Article 3), X_i can be written as

$$X_i = C \cdot \frac{p_i(\mu)}{(\mu - \mu_1)(\mu - \mu_2)\ .\ .\ .\ (\mu - \mu_N)}$$

First, the right hand side of the latter equation is expanded by partial fractions as

$$C \cdot \frac{p_i(\mu)}{(\mu - \mu_1)(\mu - \mu_2)\ .\ .\ (\mu - \mu_i)\ .\ .\ (\mu - \mu_N)} =$$

$$\frac{C_1}{\mu - \mu_1} + \frac{C_2}{\mu - \mu_2} + .\ .\ + \frac{C_i}{\mu - \mu_i} + .\ .\ + \frac{C_N}{\mu - \mu_N},\ 1 < i < N$$

and hence

$$C_i = \frac{C\, p_i(\mu_i)}{(\mu_i - \mu_1)(\mu_i - \mu_2) \ . \ . \ . \ (\mu_i - \mu_N)}$$

where there are N - 1 factors in the denominator. The expression for X_i becomes

$$X_i = \sum_{n=1}^{N} \frac{C_n}{\mu - \mu_n}$$

If (i s) is substituted for ω, the Laplace transform of the response of the i^{th} mass to an impulse becomes

$$X_i(s) = \sum_{n=1}^{N} \frac{-\,C_n}{\frac{s^2}{\omega_o^2} + \mu_n} = \sum_{n=1}^{N} \frac{-\,\omega_o^2\, C_n}{s^2 + \mu_n\, \omega_o^2}$$

Each term of this expression can be inverted to give

$$X_i = \sum_{n=1}^{N} \frac{1}{\sqrt{\mu_n}\, \omega_o} \cdot (-\,\omega_o^2\, C_n) \sin \sqrt{\mu_n}\, \omega_o\, t$$

or

$$X_i = -\sum_{n=1}^{N} C_n \frac{\omega_o}{\sqrt{\mu_n}} \sin \omega_o \sqrt{\mu_n} \cdot t$$

5. Damping

That the method is applicable when odd order derivatives occur will be demonstrated by including viscous damping in the two-degree-of-freedom example. If a dashpot of damping coefficient, C, is placed in parallel with each of the springs in Figure 1, and a harmonic forcing function is applied to mass A, the equations of motion become

$$m\,\ddot{x}_A = k\,(x_B - x_A) - k\,x_A + C(\dot{x}_B - \dot{x}_A) + P$$

$$m\,\ddot{x}_B = -k\,(x_B - x_A) - C(\dot{x}_B - \dot{x}_A)$$

If $P = p\,e^{i\omega t}$, $x_A = X_A\,e^{i\omega t}$, $x_B = X_B\,e^{i\omega t}$, and $\mu = \frac{m\omega^2}{k}$, where X_A and X_B are complex-valued functions, the equations of motion become

$$\mu\,X_A = X_A + \alpha\,X_A - \alpha\,X_B - \frac{p}{k}$$

$$\mu\,X_B = \alpha\,X_B - \alpha\,X_A \quad ,$$

where $\alpha = 1 + i \cdot \frac{\omega C}{k}$.

Assuming that

$$X_A = X_A^o + \mu\,X_A^1 + \mu^2\,X_A^2 + \ldots$$

$$X_B = X_B^o + \mu\,X_B^1 + \mu^2\,X_B^2 + \ldots$$

the usual perturbation procedure gives

$$X_A^o = X_B^o = \frac{p}{k}$$

$$X_A{}^n = X_A{}^{n-1} + X_B{}^{n-1}, \; X_B{}^n = X_A{}^{n-1} + (1 + \frac{1}{\alpha})\,X_B{}^{n-1}, \; n \geq 1 \quad .$$

Hence,

$$X_A^o = \frac{p}{k} \qquad X_B^o = \frac{p}{k}$$

$$X_A^1 = 2\,\frac{p}{k} \qquad X_B^1 = \frac{p}{k}\,(1 + \beta)$$

$$X_A^2 = \frac{p}{k}\,(3 + \beta) \qquad X_B^{\;2} = \frac{p}{k}\,(2 + \beta + \beta^2)$$

$$X_A^{\;3} = \frac{p}{k}\,(5 + 2\beta + \beta^2) \qquad X_B^{\;3} = \frac{p}{k}\,(3 + 3\beta + \beta^2 + \beta^3)$$

$$X_A^{\;4} = \frac{p}{k}\,(8 + 5\beta + 2\beta^2 + \beta^3)$$

$$\vdots \qquad\qquad \vdots$$

where $\beta = 1 + \frac{1}{\alpha}$.

Considering, for example, the series for X_A, the scheme for writing X_A as a continued fraction is:

α_{00}:	1	0	0	0	0
α_{10}:	1	2	$(3+\beta)$	$(5+2\beta+\beta^2)$	$(8+5\beta+2\beta^2+\beta^3)$
α_{20}:	-2	$-(3+\beta)$	$-(5+2\beta+\beta^2)$	$-(8+5\beta+2\beta^2+\beta^3)$	---
α_{30}:	$-1+\beta$	$(-1+\beta^2)$	$(-2+\beta+\beta^3)$	---	
α_{40}:	$(1-2\beta+\beta^2)$	$(1-\beta-\beta^2+\beta^3)$	---		
α_{50}:	0				

Again, as in the case of no damping, the continued fraction representation of the series terminates. The expression for X_A is

$$X_A = \frac{p}{k}\left(\frac{1}{1} + \frac{-2\mu}{1} + \frac{(-1+\beta)\mu^2}{(-2\mu)} + \frac{(1-2\beta+\beta^2)\mu^3}{(-1+\beta)}\right),$$

and contraction yields

$$X_A = \frac{p}{k} \frac{1 + \mu(1-\beta)}{1 - (1+\beta)\mu + (1-\beta)\mu^2}$$

Replacing β by $1 + \frac{1}{\alpha}$, and α by $1 + i \cdot \frac{\omega C}{k}$ gives

$$X_A = \frac{p}{k} \frac{A(1-\mu) + \beta\theta \ + i \ \ A\theta - (1-\mu) B}{A^2 + B^2}$$

where $A = 1 - 3\mu + \mu^2$, $B = (1 - 2\mu)\ \theta$, and $\theta = \frac{\omega C}{k}$.

The magnitude of X_A squared is

$$X_A^{\ 2} = \frac{(1-\mu)^2 + \left(\frac{\omega C}{k}\right)^2}{(1 - 3\mu + \mu^2)^2 + (1 - 2\mu)^2 \left(\frac{\omega C}{k}\right)^2} \cdot \left(\frac{p}{k}\right)^2$$

Note that if C = 0, then

$$X_A = \frac{p}{k} \frac{1-\mu}{1 - 3\mu + \mu^2},$$

and the denominator is identical to that obtained before.

This example shows that viscous damping can be handled by this method of analysis, and it is the writer's future plan to include damping in the formulation of the N-degree-of-freedom problem.

IV. CONTINUOUS SYSTEMS

1. The Axially Loaded Rod

The solution of vibration problems involving continuous systems will be illustrated by applying the previously developed concepts to the simple case of the axially loaded rod shown in Figure 3. If E is the modulus of elasticity and ρ the mass-density, the equation of motion for the axial displacement, u, of any section of the rod, based on the one dimensional formulation is

$$E\,u'' = \rho\,\ddot{u} \tag{28}$$

where primes denote partial differentiation with respect to x, and dots with respect to time. If a force, $F \cos \omega t$, is applied to the end $x = L$ of the rod, and a harmonic response, $u = U \cos \omega t$, is assumed, where $U = U(x)$, Eq. (28) becomes

$$E\,\frac{d^2U}{dx^2} = -\,\rho\,\omega^2\,U \tag{29}$$

If $\tilde{u} = U/L$, $\tilde{x} = x/L$, and $\mu = \rho\,\omega^2\,L^2/E$, then the equation of motion is

$$\frac{d^2\tilde{u}}{d\tilde{x}^2} = -\,\mu\,\tilde{u}\;, \tag{30}$$

and the boundary conditions are

$$\tilde{u}(o) = 0, \; \tilde{u}'(1) = \frac{F}{AE} = \alpha \tag{31}$$

assuming

$$\tilde{u} = \tilde{u}_o + \mu\, \tilde{u}_1 + \mu^2\, \tilde{u}_2 + \ldots$$

and substituting this series into Eqs. (30) and (31) yields

$$\left(\frac{d^2\tilde{u}_o}{d\tilde{x}^2} + \mu \frac{d^2\tilde{u}_1}{d\tilde{x}^2} + \mu^2 \frac{d^2\tilde{u}_2}{d\tilde{x}^2} + \ldots \right) = - \mu\, \tilde{u}_o - \mu^2 \tilde{u}_1 - \mu^3\, \tilde{u}_2 - \ldots \tag{32}$$

$$0 = \tilde{u}_o(0) + \mu\, \tilde{u}_1(0) + \mu^2 \tilde{u}_2(0) + \ldots, \quad \alpha = \tilde{u}_o'(1) + \mu \tilde{u}_1'(1) + \mu^2 \tilde{u}_2'(1) + \ldots \tag{33}$$

Equating coefficients of like powers of μ in Eqs. (32) and (33) yields a set of differential equations and boundary conditions, the solutions to which are found recursively as follows:

$$\mu^o: \quad \frac{d^2\tilde{u}_o}{dx^2} = 0, \; \frac{d\tilde{u}_o}{dx} = \alpha, \; \tilde{u}_o = \alpha\, \tilde{x}$$

$$\mu^1: \quad \frac{d^2\tilde{u}_1}{d\tilde{x}^2} = - \tilde{\mu}_o = - \alpha\tilde{x}, \; \frac{d\tilde{u}_1}{d\tilde{x}} = - \alpha \frac{\tilde{x}^2}{2} + \frac{\alpha}{2}, \; \tilde{u}_1 = - \alpha\left(\frac{\tilde{x}^3}{6} + \frac{\tilde{x}}{2} \right),$$

and so on.

Hence,

$$\tilde{u} = \alpha\tilde{x} + \mu\,\alpha\left(\frac{\tilde{x}}{2} - \frac{\tilde{x}^3}{6}\right) + \mu^2\left(\frac{5}{24}\tilde{x} - \frac{\tilde{x}^3}{12} + \frac{\tilde{x}^5}{120}\right)$$

$$+ \mu^3\,\alpha\left(\frac{61}{720}\tilde{x} - \frac{5}{144}\tilde{x}^3 + \frac{1}{240}\tilde{x}^5 - \frac{1}{5{,}040}\tilde{x}^7\right)$$

$$+ \mu^4\,\alpha\left(\frac{1{,}385}{40{,}320}\tilde{x} - \frac{61}{4{,}320}\tilde{x}^3 + \frac{1}{576}\tilde{x}^5 - \frac{1}{10{,}080}\tilde{x}^7 + \frac{1}{362{,}880}\tilde{x}^9\right)$$

$$+ \;.\;.\;.\;.\;.\;.\;. \qquad (34)$$

Suppose the motion of the end $\tilde{x} = 1$ is to be obtained. Then Eq. (34) becomes

$$\tilde{u}(1) = \alpha\left(1 + \frac{1}{3}\mu + \frac{2}{15}\mu^2 + \frac{17}{315}\mu^3 + \frac{62}{2{,}835}\mu^4 + \ldots\right) \qquad (35)$$

The desirable form of Eq. (35) would be the ratio of two series, where the zeros of the denominator series would correspond to the natural frequencies of the rod, and the zeros of the numerator series would correspond to the frequencies at which the end, $\tilde{x} = 1$, becomes a node. Such a quotient can be obtained by observing that a point infinitesimally close to the fixed end of the rod cannot become a node, because the fixed end, $\tilde{x} = 0$, is always a node and, in fact, an isolated node, since $\tilde{u}$, which is continuous and has continuous derivatives for a harmonic excitation, is not identically zero in the neighborhood of $\tilde{x} = 0$. Retaining only terms containing the first power of $\tilde{x}$ in Eq. (34), and denoting the resulting series by $\tilde{g}$, gives

$$\tilde{g} = \alpha \tilde{x} \; (1 + \frac{1}{2}\mu + \frac{5}{24}\mu^2 + \frac{61}{720}\mu^3 + \frac{1,385}{40,320}\mu^4 + \ldots)$$

$$= \alpha \tilde{x} \quad 1/(1 - \frac{1}{2}\mu + \frac{1}{24}\mu^2 - \frac{1}{720}\mu^3 + \frac{1}{40,320}\mu^4 + \ldots)$$

The operations suggested by the following equations are performed to obtain the desired form of Eq. (35):

$$\tilde{u}(1) = \frac{\tilde{u}(1)}{\tilde{g}} \cdot \tilde{g}$$

First,

$$\frac{\tilde{u}(1)}{\tilde{g}} = \frac{1}{\tilde{x}} (1 - \frac{1}{6}\mu + \frac{1}{120}\mu^2 - \frac{1}{5,040}\mu^3 \ldots)$$

and finally,

$$\tilde{u}(1) = \alpha \cdot \frac{1 - \frac{1}{6}\mu + \frac{1}{120}\mu^2 - \frac{1}{5,040}\mu^3 \ldots .}{1 - \frac{1}{2}\mu + \frac{1}{24}\mu^2 - \frac{1}{720}\mu^3 + \frac{1}{40,320}\mu^4 + \ldots}, \tag{36}$$

which is the desired form. The numerator and denominator series in Eq. (36) are recognizable as $\frac{1}{\sqrt{\mu}} \sin\sqrt{\mu}$ and $\cos\sqrt{\mu}$, respectively. Hence, Eq. (36) can be written as

$$\tilde{u}(1) = \alpha \frac{1}{\sqrt{\mu}} \tan\sqrt{\mu} \tag{37}$$

Equation (36) can be obtained in yet another way. Equation (34) can be written as

$$\frac{\tilde{u}}{\alpha} = \tilde{x} \quad (1 + \frac{\mu}{2} + \frac{5}{24}\mu^2 + \frac{61}{720}\mu^3 + \ldots)$$

$$+ \tilde{x}^3 \; (- \frac{\mu}{6} - \frac{\mu^2}{12} - \frac{5}{144}\mu^3 - \ldots\ldots)$$

$$+ \tilde{x}^5 \; (\frac{1}{120}\mu^2 + \frac{1}{240}\mu^3 + \frac{1}{576}\mu^4 + \ldots)$$

$$+ \begin{matrix} . \\ . \\ . \\ . \end{matrix}$$

or

$$\frac{\tilde{u}}{\alpha} = \tilde{x}_1 \; g_1 + \tilde{x}^3 \; g_2 + \tilde{x}^5 \; g_3 + \ldots\ldots$$

where the g's are the corresponding power series in μ. Now, at a natural frequency, $\tilde{u} \to \infty$, and $\frac{1}{\tilde{u}} \to 0$, and the previous equation gives

$$\frac{\alpha}{\tilde{u}} = \frac{1}{g_1\tilde{x}} - \frac{g_2}{g_1^2} x + (\frac{g_2^2}{g_1^3} - \frac{g_3}{g_1^2}) x^3 + \ldots$$

where

$$g_1 = 1 + \frac{\mu}{2} + \frac{5}{24}\mu^2 + \frac{61}{720}\mu^3 + \ldots$$

Since g_1 is common to every denominator, $\frac{1}{\tilde{u}} \to 0$ as $\frac{1}{g_1} \to 0$, and by long division,

$$\frac{1}{g_1} = 1 - \frac{\mu}{2} + \frac{1}{24}\mu^2 - \frac{1}{720}\mu^3 \ldots$$

which is the same as the denominator of Eq. (36). Equation (36) is obtained by solving for G, where

$$1 + \frac{1}{3}\mu + \frac{2}{15}\mu^2 + \frac{17}{315}\mu^3 + \ldots = G/(1 - \frac{\mu}{2} + \frac{1}{24}\mu^2 - \frac{1}{720}\mu^3 \ldots).$$

Hence,

$$G = 1 - \frac{1}{6}\mu + \frac{1}{120}\mu^2 - \frac{1}{5{,}040}\mu^3 - \ldots$$

which is the numerator of Eq. (36).

The immediate question which arises from the above discussion is whether or not, in any problem, it will be possible to find a point which will not become a node at any frequency, or if a common factor, such as g_1, will occur. In general, it will not always be possible to obtain these conditions. This is to say that not only a point whose deflection will not vanish cannot, in general, be found, but no derivative (such as a slope) can be found that does not vanish at any frequency. In a paper submitted for publication by Dr. R. U. Chicurel and the writer, certain beam vibration problems are discussed where, for example, it is possible to identify a position where the bending moment will not vanish at any frequency, and hence, the reciprocal of the series for the moment will have no singularities.

This lack of singularities permits the calculation and removal of the zeros of the series, one at the time.

Use will now be made of continued fractions to extract certain information from Eq. (35). Following the method outlined in Appendix I, the following convergents are obtained for the series in parentheses on the right hand side of Eq. (35):

$$P_o = 1 \qquad Q_o = 1$$

$$P_1 = 1 \qquad Q_1 = 1 - \frac{1}{3}\mu$$

$$P_2 = -\frac{1}{3} + \frac{1}{45}\mu \qquad Q_2 = -\frac{1}{3} + \frac{2}{15}\mu$$

$$P_3 = \frac{(10\mu - 105)}{15(21)(45)} \qquad Q_3 = \frac{-(\mu^2 - 45\mu + 105)}{105(135)}$$

$$P_4 = \frac{-(\mu^2 - 105\mu + 945)}{21(45)(45)(105)(135)} \qquad Q_4 = \frac{-(\mu^2 - 28\mu + 63)}{45(105)(135)(315)}$$

The values of μ for which the P_i and Q_i vanish correspond to the squares of the frequencies at which a node appears at $\tilde{x} = 1$, and the natural frequencies of the rod, respectively. The square roots of μ from the above approximations are:

P_o: ---	Q_o: ---
P_1: ---	Q_1: 1.732
P_2: 3.873	Q_2: 1.581
P_3: 3.240	Q_3: 1.572, 6.522
P_4: 3.153, 9.750	Q_4: 1.57, 5.053
(3.142), (6.284)	(1.571), (4.712)

The results of an "exact" analysis are shown, to four significant figures, in parentheses below the approximate results for P_4 and Q_4.

It must be kept in mind that if Eq. (35) is viewed as an infinite series, it will converge up to $\mu = \frac{\pi^2}{4}$, which is the first value of μ for which the value of the series approaches infinity (the first natural frequency). However, the use of continued fractions has produced information about values of μ which far exceed the range of convergence of the infinite series. Indeed, only the first five terms of the series have been used, and the second natural frequency, $\sqrt{\mu} = 5.053$, is within 7.25 per cent of the exact value, $\sqrt{\mu} = 1.5\ \pi$. The motion of the point, $\tilde{x} = 1$, has been used only for convenience, and any other position on the axis of the rod would serve as well.

Ordinarily, the mode shapes corresponding to the natural frequencies would be found, in order to find the response of the rod to an arbitrary forcing function by using the orthogonality of the mode shapes. Instead of this approach, an approximation to an integral transform technique will be used. The first step, in general, is to apply a harmonic forcing function having the same spatial variation as the forcing function for which a solution is desired. For example, if in the present problem, the response of the rod to a force applied at the free end is desired, then Eq. (34) would be developed. Next, the fact is used that if (i s), where $i = \sqrt{-1}$ and s is the Laplace transform parameter, is substituted for ω, Eq. (34) becomes the Laplace transform of the response of the rod to a unit impulse applied at $\tilde{x} = 1$. If the resulting expression were inverted term-by-term, the result would be a power series in time that, for any value of $\tilde{x}$, would be only a short-time approximation. However, continued fractions can be used to obtain an approximate expression for the transform as the ratio of two polynomials, and this expression can then be inverted by standard procedures. Once the response to a unit impulse is obtained, the response to any forcing function can be calculated by using the convolution integral.

Suppose the response of the end, $\tilde{x} = 1$, to a unit impulse applied at that point is to be found. The procedure discussed in the above paragraph can be immediately implemented by writing the ratio of, say P_3 to Q_3, and substituting (i s) for ω. This gives

$$\bar{\tilde{u}}\,(1,s) = \frac{\frac{1}{\beta}\,(105 + 10\,\beta\,5^2)}{\left(s^2 + (\frac{1.572}{\sqrt{\beta}})^2\right)\left(s^2 + (\frac{6.522}{\sqrt{\beta}})^2\right)}$$

where $\bar{\tilde{u}}\,(1, s)$ is the Laplace transform of $\tilde{x} = 1$ to the unit impulse, and $\beta = \frac{\rho}{E} L^2$. The inverse transform is

$$\tilde{u}(1,t) = \sqrt{\beta}\left(1.275 \sin \frac{1.572t}{\sqrt{\beta}} + 1.226 \sin \frac{6.522t}{\sqrt{\beta}}\right)$$

The complete solution for the response, $\tilde{u}(1,t)$, is the square-wave function, which has the following series representation:

$$\tilde{u}(1,\ t) = \sqrt{\beta}\ \ \frac{4}{\pi} \sum_{n\ \mathrm{odd}}^{\infty} \frac{1}{n} \sin \frac{n\ \pi/2\ t}{\sqrt{\beta}}$$

The first two terms of the series gives

$$\tilde{u}(1,t) \doteq \sqrt{\beta}\left(1.273 \sin \frac{1.571t}{\sqrt{\beta}} + 0.424 \sin \frac{4.712t}{\sqrt{\beta}}\right)$$

To improve the accuracy of the solution, the next odd convergents would have to be developed, because the ratio of two even convergents, such as P_4/Q_4, will have polynomials of the same degree in the numerator and denominator. The above inversion procedure requires a numerator of lower degree than the denominator.

The example of the axially loaded rod has demonstrated these features of the method of analysis:

(1) Any type of spatial variation for the harmonic forcing function will suffice for the determination of the natural frequencies.

(2) The quantity chosen (in the present case, $\widetilde{u}(1)$) to be developed into a series is a matter of convenience.

(3) The mode shapes are not necessary for the determination of the response to a given forcing function.

(4) Continued fractions can be used to invert Laplace transforms which appear as power series in the transform parameter.

(5) The method is well adapted, although certainly not limited, to linear vibration problems involving time-dependent boundary conditions.

(6) Continued fractions can be used to extend the accuracy of any solution which appears as a finite number of terms of an infinite series. In particular, the usefulness of many singular perturbation problems can be increased by making use of continued fractions.

These remarks are, of course, made in the sense of approximate solutions.

2. The Timoshenko Beam

When the effects of shear and rotatory inertia are considered, the usual Euler-Bernoulli formulation for a beam is modified and becomes (17)

$$\frac{\partial}{\partial x} \left(EI \frac{\partial \psi}{\partial x} \right) + k \left(\frac{\partial y}{\partial x} - \psi \right) = \rho I \frac{\partial^2 \psi}{\partial t^2}$$

$$\rho A \frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x} \left(k \left(\frac{\partial y}{\partial x} - \psi \right) \right) = w(x,t)$$

where

x = spatial coordinate along the neutral axis of the undeformed beam

t = time

E = Young's modulus of elasticity

G = shear modulus of elasticity

y = total deflection of the neutral axis

I = moment of inertia of the cross-section about the neutral axis

ρ = mass-density of the material

A = cross-section area

ψ = slope of the beam due to bending

k'= a numerical factor which relates the beam shear force to the shear strain at the neutral axis (21)

$k = k' AG$

w = external transverse load per unit length

V = beam shear force

M = internal bending moment

β = slope at the neutral axis due to shear .

Also,

$$M = EI \frac{\partial \psi}{\partial x}$$

$$V = k \left(\frac{\partial y}{\partial x} - \psi \right)$$

$$\frac{\partial y}{\partial x} = \psi + \beta$$

These equations have been studied rather extensively, and the literature on them is easy to find. Recently, Ames and Sontowski (18) have considered a multiparameter perturbation solution of the frequency equation for a pin-ended Timoshenko beam. It should be noted that the frequency equation they treat is an algebraic polynomial, which is obtained because the shape functions of the pin-ended beam are sine functions. Other end conditions will usually produce more complicated, transcendental shape functions, and the frequency equation will not be a polynomial.

The application of the method of analysis to Timoshenko beams will be illustrated by considering the problem shown in Figure 4. First, the governing equations will be nondimensionalized by substituting the relations

$$x = \xi L,\ y = uL,\ \mu = \frac{\rho}{E}\omega^2 L^2,\ \alpha = \frac{kL^2}{EI},\ s = \frac{L^2}{I/A},\ \gamma = \frac{M_o L}{EI}$$

After harmonic response to the applied end-moment, $M_o \cos \omega t$, is assumed, the equations become, for $w = 0$,

$$\frac{d}{d\xi}\left(\frac{d\psi}{d\xi}\right) + \alpha\left(\frac{du}{d\xi} - \psi\right) = -\mu$$

$$\frac{d}{d\xi}\left(\alpha\left(\frac{du}{d\xi} - \psi\right)\right) = -\mu\, s\, u\ ,$$

and the boundary conditions are

$$u(0) = 0,\quad \psi(0) = 0,\ u(1) = 0,\ \frac{\partial\psi(1)}{\partial\xi} = \frac{M_o L}{EI} = \gamma\ .$$

A beam of square cross-section ($A = 1\ \text{in}^2$, $L = 5$ in) and made of a material for which $E = \frac{8}{3} G$ will be chosen as an example. If $k' = 0.833$, the parameters in the problem become:

$$A = 1,\ \frac{I}{A} = \frac{1}{12}\ ,\quad s = 12(5)^2 = 300\ ,$$

$$\alpha = \frac{kL^2}{EI} = \frac{k'AGL^2}{EI} = 93.75,\ \ \frac{1}{\alpha} = 0.010666$$

Since γ is a common factor, it will be chosen as unity. The solution is assumed to be

$$u = u_o + \mu\, u_1 + \mu^2\, u_2 + \ldots$$
$$\psi = \psi_o + \mu\, \psi_1 + \mu^2\, \psi_2 + \ldots\ ,$$

and all the boundary conditions are homogeneous, except $\frac{\partial \psi_o(1)}{\partial \xi}$, which is unity. Following the usual procedure, the coefficients of μ^o yield

$$\frac{d}{d\xi}\left(\frac{d\psi_o}{d\xi}\right) + \alpha\left(\frac{du_o}{d\xi} - \psi_o\right) = 0$$

$$\frac{d}{d\xi}\quad \alpha\left(\frac{du_o}{d\xi} - \psi_o\right) = 0$$

Hence,

$$\alpha\left(\frac{du_o}{d\xi} - \psi_o\right) = \text{constant} = C_1$$

and

$$\frac{d}{d\xi}\left(\frac{d\psi_o}{d\xi}\right) = - C_1 \quad .$$

Therefore,

$$\frac{d\psi_o}{d\xi} = - C_1\ \xi + C_2$$

and

$$\frac{d\psi_o(1)}{d\xi} = 1 = - C_1 + C_2 \quad .$$

Thus,

$$\psi_o = - C_1 \frac{\xi^2}{2} + (1 + C_1)\ \xi + C_3 \quad ,$$

and C_3 vanishes since $\psi_o(0) = 0$.

The equation for determining u_o becomes

$$\frac{du_o}{d\xi} = - C_1 \frac{\xi^2}{2} + (1 + C_1)\ \xi + 0.010666\ C_1$$

Integrating, and applying the remaining boundary conditions results in

$$u_o = 0.242249\ \xi^3 - 0.226746\ \xi^2 - 0.015503\ \xi$$

$$\psi_o = 0.726746\ \xi^2 - 0.453491\ \xi$$

which is merely the static solution of the problem.

The equations which govern u_1 and ψ_1 are

$$\frac{d}{d\xi}\ \alpha\left(\frac{du_1}{d\xi} - \psi_1\right) = -\ 300\ u_o = -\ 72.6747\ \xi^3 + 68.0238\ \xi^2 + 4.6509\ \xi$$

$$\frac{d}{d\xi}\left(\frac{d\psi_1}{d\xi}\right) + \alpha\left(\frac{du_1}{d\xi} - \psi_1\right) = -\ \psi_o = -\ 0.726746\ \xi^2 - 0.453491\ \xi$$

along with homogeneous boundary conditions. The solution for u_1 and ψ_1 is straightforward, and was obtained with the aid of a desk calculator. Only the results will be shown here.

$$u_1 = 0.86518\ \xi^7 - 0.188955\ \xi^6 - 0.089627\ \xi^5 + 0.079357\ \xi^4 + \\ + 0.642156\ \xi^3 - 0.488888\ \xi^2 - 0.040566\ \xi$$

$$\psi_1 = 0.605623\ \xi^6 - 1.13373\ \xi^5 - 0.25435\ \xi^4 + 0.075582\ \xi^3 + \\ + 1.901664\ \xi^2 - 0.977760\ \xi\ .$$

Similarly,

$$u_2 = 0.003277\ \xi^{11} - 0.011248\ \xi^{10} - 0.013938\ \xi^9 + 0.028342\ \xi^8 + \\ + 0.237381\ \xi^7 - 0.416494\ \xi^6 - 0.235848\ \xi^5 + 0.171093\ \xi^4 + \\ + 1.159100\ \xi^3 - 0.848866\ \xi^2 - 0.0727933\ \xi$$

$$\psi_2 = 0.036050\ \xi^{10} - 0.112475\ \xi^9 - 0.090840\ \xi^8 + 0.140358\ \xi^7 + \\ + 1.613868\ \xi^6 - 2.448179\ \xi^5 - 0.665547\ \xi^4 + 0.162933\ \xi^3 + \\ + 3.412397\ \xi^2 - 1.697732\ \xi \quad .$$

The moment at $\xi = 0$ will be chosen as the quantity whose series development will be used to determine the natural frequencies. This moment is proportional to

$$f = -0.453491 - 0.977760\ \mu - 1.697732\ \mu^2 \ldots$$

Dividing through by the leading term yields

$$g = 1 + 2.156074\ \mu + 3.743695\ \mu^2 + \ldots$$

The scheme for developing g into a continued fraction becomes

α_{00}:	1	0	0	0
α_{10}:	1	2.156074	3.743695	---
α_{20}:	-2.156074	-3.743695	---	---
α_{30}:	-0.904960	---	---	---

$Q_o = 1$ $\qquad$ $P_o = 1$

$Q_1 = 1 - 2.156074\ \mu$ $\qquad$ $P_1 = 1$

$Q_2 = -2.156074 + 3.743695\ \mu$ $\qquad$ $P_2 = -2.156074 - 0.904960\ \mu$

The zeros of these expressions are:

Q_o:	---	P_o:	---
Q_1:	$\mu = -\ 0.463677$	P_1:	---
Q_2:	$\mu = 0.575921$	P_2:	$\mu = -2.382508$.

According to Q_2,

$$\omega^2 = \frac{E}{\rho L^2}\,(0.575921), \quad \omega = 0.758895\,\frac{1}{L}\sqrt{\frac{E}{\rho}}$$

The lowest value of $\bar{\omega}$, based on Euler-Bernoulli theory is (10),

$$\omega = 0.89015\,\frac{1}{L}\sqrt{\frac{E}{\rho}}$$

from which, shear and rotatory inertia are seen to reduce the lowest natural frequency by approximately 15 per cent.

Observing that P_2 gives a negative frequency for the moment in question to become zero indicates that the second natural frequency will occur before a frequency at which the moment vanishes. Hence, the singularity at $\mu = 0.575921$ can be removed by multiplying the series for g by $\frac{(0.575921 - \mu)}{0.575921}$. This yields

$$1 + 0.419724\,\mu + \ldots$$

Considering only the first two terms of the reciprocal of this series, there results

$$1 - 0.419724\,\mu \ldots$$

and, to this approximation, the value of μ corresponding to the second natural frequency is

$$\mu = 2.38251 \quad ,$$

and

$$\omega = 1.54354\,\frac{1}{L}\sqrt{\frac{E}{\rho}} \quad .$$

Reference (19) gives

$$\omega = 2.88472\,\frac{1}{L}\sqrt{\frac{E}{\rho}}$$

for the second natural frequency when shear and rotatory inertia are neglected.

3. Euler Beam with Variable Cross-Section

The tapered wedge shown in Figure 5 will be used as an example to demonstrate the use of the method of analysis for calculating the natural frequencies of members whose motion is governed by differential equations having variable coefficients. Following the nomenclature in section 2 of this chapter, the governing equation and boundary conditions are

$$\frac{\partial^2}{\partial x^2}\left(EI\,\frac{\partial^2 y}{\partial x^2} \right) + \rho A\,\frac{\partial^2 y}{\partial t^2} = w$$

$$(EI \frac{\partial^2 y}{\partial x^2})_{x=o} = 0, \frac{\partial}{\partial x}(EI \frac{\partial^2 y}{\partial x^2})_{x=o} = 0, (\frac{\partial y}{\partial x})_{x=L} = 0, (y)_{x=L} = 0 .$$

where

$$I = \frac{1}{12}(2b)^3 \frac{x^3}{L^3} = I_o x^3, \; A = 2b \frac{x}{L} = A_o x$$

Substituting the expressions for I and A in the above equations, assuming harmonic response to the loading function, $w = w_o x \cos \omega t$, and nondimensionalizing the equations as in section 2, yields

$$\frac{d^2}{d\xi^2}(\xi^3 u'') - \mu \xi u = \xi$$

$$\frac{\partial}{\partial \xi}(\xi^3 \frac{\partial^2 u}{\partial \xi^2})_{\xi=o} = 0, (\xi^3 \frac{\partial^2 u}{\partial \xi^2})_{x=o} = 0, (\frac{\partial u}{\partial \xi})_{\xi=1} = 0, (u)_{\xi=1} = 0$$

where $\mu = \frac{\rho A_o}{EI_o} \omega^2 L^2$, and $\frac{w_o L}{EI_o}$ has been chosen as unity. The reason for choosing $w = w_o x \cos \omega t$ will be explained later in the problem.

Assuming

$$u = u_o + \mu u_1 + \mu^2 u_2 + \mu^3 u_1 + \dots$$

and following the usual perturbation procedure, the equation for determining u_o, which is the static solution, is

$$(\xi^3 u_o'')'' = \xi$$

along with homogeneous boundary conditions. Successive integration and application of the boundary conditions yields

$$(\xi^3 u_o'') = \frac{\xi^3}{6}, \text{ or } u_o'' = \frac{1}{6}$$

Having chosen the loading proportional to ξ, the above equation can be divided through by ξ^3, which results in the absence of negative powers of ξ. If such terms were present, a singularity would occur at $\xi = 0$. This circumstance illustrates that the spatial distribution of the loading function is a matter of convenience when natural frequencies are to be determined.

Integrating $u_o'' = \frac{1}{6}$ twice, and applying the remaining boundary conditions gives

$$u_o = \frac{1}{12}\xi^2 - \frac{1}{6}\xi + \frac{1}{12}$$

The equation for determining u_1 is

$$(\xi^3 u_1'')'' = \xi u_o = \frac{1}{12}\xi^3 - \frac{1}{6}\xi^2 + \frac{1}{12}\xi$$

along with homogeneous boundary conditions. Applying successive integrations yields

$$u_1 = \frac{1}{4(720)}\xi^4 - \frac{1}{3(144)}\xi^3 + \frac{1}{144}\xi^2 - \frac{1}{120}\xi + \frac{1}{12(720)}$$

Similarly,

$$u_2 = \frac{1}{72(720)}\left(\frac{1}{70}\xi^6 - \frac{1}{5}\xi^5 + \frac{3}{2}\xi^4 - 6\xi^3 + \frac{29}{2}\xi^2 - \frac{563}{35}\xi + \frac{439}{70}\right)$$

and

$$u_3 \doteq \frac{1}{70(72)^2(720)} \left(\frac{\xi^8}{56} - \frac{3}{7}\xi^7 + 6\,\xi^6 - \frac{252}{5}\xi^5 + \frac{609}{2}\xi^4 - 1126\,\xi^3 + + 2634\,\xi^2 - \frac{20,227}{7}\xi + \frac{78,473}{70} \right) .$$

Choosing the deflection of the free-end, u(o), as the quantity whose series development will be used to determine the natural frequencies, the above equations yield

$$u(o) = \frac{1}{12} + \frac{29}{12(720)}\mu + \frac{439}{70(72)(720)}\mu^2 + \frac{78,473}{(70)^2(72)^2(720)}\mu^3 + \ldots,$$

and hence

$$u(o) \sim 1 + 0.04027777\,\mu + 0.00145179\,\mu^2 + 0.00005149\,\mu^3 + \ldots$$

The scheme for developing this series as a continued fraction becomes

α_{00}:	1	0	0	0
α_{10}:	1	0.04027777	0.00145179	0.00005149
α_{20}:	- 0.04027777	- 0.00145179	- 0.00005149	---
α_{30}:	- 0.00017050	- 0.00000699	---	---
α_{40}:	- 0.000000034	---	---	---

Hence,

$Q_o = 1$ $\qquad$ $P_o = 1$

$Q_1 = 1 - 0.04027777\ \mu$ $\qquad$ $P_1 = 1$

$Q_2 = -\ 0.04027777 + 0.00145179\ \mu$ $\qquad$ $P_2 = -\ 0.04027777 - 0.0001705\ \mu$

$Q_3 = 10^{-10}\ (68,673.6 - 2,815.3\mu + 13.69\ \mu^2)$ $\qquad$ $P_3 = -\ 6.86736 \times 10^{-6} - 0.42975\ \mu \times 10^{-8}$

The values of μ for which the Q_i and P_i vanish are:

Q_o: --- $\qquad$ P_o: ---

Q_1: 24.8276 $\qquad$ P_1: ---

Q_2: 27.7435 $\qquad$ P_2: - 236.23

Q_3: 28.283, 177.363 $\qquad$ P_3: - 1597.99

(28.228, 231.107)

The correct values of μ (19) are written in parentheses below the approximate values of μ given by $Q_3 = 0$. The lowest approximate value of μ agrees with the correct value to three figures, and the second value of μ is within 24 per cent of the correct value. As in the case of the Timoshenko beam, the values of μ for which the free-end becomes a node are negative, indicating that a nodal frequency for $\xi = 0$ will not occur between the first two natural frequencies. The singularity, $\mu = 28.238$, can be removed from the series to which u(o) is proportional by multiplying that series by $\dfrac{28.238 - \mu}{28.238}$. The first three terms of the resulting series are

$$1 + 0.004864\ \mu + 0.00002542\ \mu^2 + \ldots$$

The denominator of the continued fraction representation of this series gives

$Q_0 = 1$, ———

$Q_1 = 1 - 0.004864\ \mu$, (205.59)

$Q_2 = -\ 0.004864 + 25.42 \times 10^{-6}\ \mu$, (191.345) ,

where the zeros of the Q_i are indicated in parentheses. The correct value of μ corresponding to the second natural frequency is (19) 231.107. Now, the frequencies, ω, are proportional to the square root of μ. Therefore, the correct value of the second natural frequency is proportional to 13.318, whereas the approximate value is proportional to $\omega = (191.345)^{1/2} = 13.833$, the error being less than 4 per cent. It should also be noticed that the $\mu = 177.363$ gives ω proportional to 13.318, the error being approximately 12 per cent.

V. DISCUSSION

The objective of this dissertation is to show that the use of inertia as a perturbation parameter leads to a simple method for obtaining approximate solutions to linear vibration problems. The writer feels that the example problems support an affirmative conclusion for both lumped and distributed systems.

Lumped systems, without damping, have been shown to be completely amenable to this method of solution. Once the equations of motion have been formulated, only $2N$ terms of the displacement series for a N-degree-of-freedom system need be developed in order to obtain the frequency and, what may be called the nodal polynomials. The use of continued fractions has enabled the extraction of this information from only $2N$ terms of an easily developed series. As has been previously pointed out, the application of the mobility method to lumped systems results in a continuous repeating fraction. The present method has an advantage over the mobility method in that it can be terminated at any point, and useful results for natural and nodal frequencies can be obtained. More specifically, the latter statement refers to the calculation of the i^{th} partial convergent, $\frac{P_i}{Q_i}$, where $i < N$. The reason this situation exists is that the present method of analysis takes into account all the properties of the system in the static solution, whereas the mobility method, if terminated at any point in its procedure, will not take account of certain inertia and stiffness parameters.

The example of the damped two-degree-of-freedom system is included to show that the method of analysis can accommodate damping in a lumped system, with little more effort than is involved in the undamped case. It is the intention of the writer to extend the method to include damping in the N-degree-of-freedom problem.

The examples of linear vibration problems involving continuous systems show that the method of analysis is very useful for obtaining approximate natural frequencies, even when the system has variable properties. The series solution for the Euler beam with varying cross-section was developed with the aid of a desk-calculator. The calculations were routine integrations, but more difficult problems could be integrated on a digital computer.

In contrast to developing the series solution for some quantity in a problem, the frequency equation can be obtained in closed form for many practically important vibration problems. In a very recent paper, Kosko (20) has used the idea of reversing (or reversion of) the series expansion of the transcendental frequency equation for a beam carrying a lumped mass, but only the first natural frequency is obtained with much accuracy. In these situations, the power of the continued fraction representation of such series can be used to obtain the higher frequencies.

Once the approximate response of a particular point in either a lumped or distributed system has been obtained, the response of that point to an arbitrary forcing function, with the same spatial location as the impulse, can be found by using the superposition integral for linear systems. The accuracy of the impulse response depends on the accuracy present in the approximate values of the natural frequencies. These frequencies can be obtained to any desired degree of accuracy by taking more terms of the series being considered and determining the corresponding continued fraction representation. Indeed, the examination of the continued fraction representation of any series solution for a mechanics problem appears to be a worthwhile area of research.

Although it is not presented here, the method of analysis has been successfully used to obtain the buckling load for several types of Euler Columns, wherein the axial load is used as a perturbation parameter. Also, the writer has used the method of analysis to determine the natural frequencies of vibrations of a simply supported rectangular plate.

In conclusion, the concept of inertia as a perturbation parameter and the use of the continued fraction representation of the resulting series solution are original and useful contributions to the literature on linear vibration problems.

VI. ACKNOWLEDGMENTS

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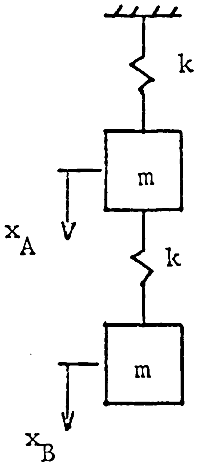

Figure 1
Two-Degree-of-Freedom System

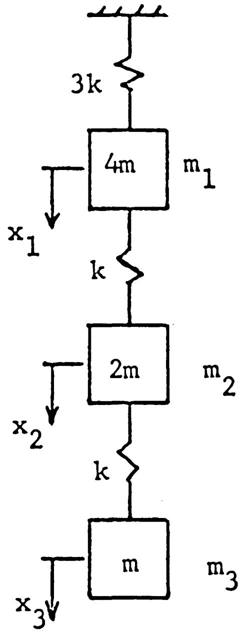

Figure 2
Three-Degree-of-Freedom System

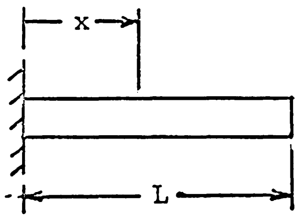

Figure 3
Axially Loaded Rod

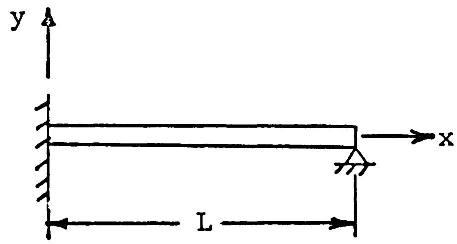

Figure 4
Timoshenko Beam

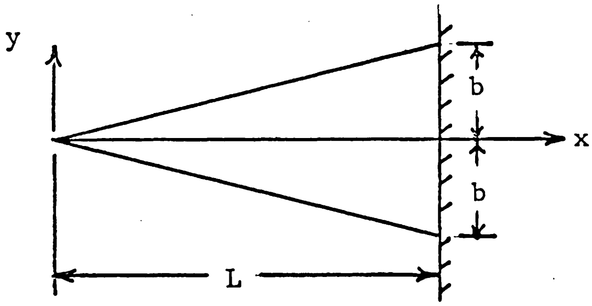

Figure 5
Euler Beam with Variable Cross-section

APPENDIX I

Transformation of Series Into Continued Fractions (8)

The development

$$b_o + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n + \ddots}}}$$

is called a continued fraction. In this dissertation, the development will be written in abridged notation as

$$b_o + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots \frac{a_n}{+ b_n +} \cdots\cdots$$

Suppose a function, f(x), is given as the ratio of two infinite power series in x. Then f(x) can be written as

$$f(x) = \frac{\alpha_{10} + \alpha_{11} x + \alpha_{12} x^2 + \alpha_{13} x^3 + \ldots}{\alpha_{00} + \alpha_{01} x + \alpha_{02} x^2 + \alpha_{03} x^3 + \ldots} \qquad \text{(A1)}$$

A direct method for transforming Eq. (A1) into a continued fraction is that of V. Viskovatoff. Equation (A1) is transformed in the following manner:

$$f(x) = \cfrac{1}{\cfrac{\alpha_{00} + \alpha_{01}\, x + \alpha_{02}\, x^2 + \alpha_{03}\, x^3 + \ldots}{\alpha_{10} + \alpha_{11}\, x + \alpha_{12}\, x^2 + \alpha_{13}\, x^3 + \ldots}}$$

$$= \cfrac{1}{\cfrac{\alpha_{00}}{\alpha_{10}} + \cfrac{\alpha_{00} + \alpha_{01}\, x + \alpha_{02}\, x^2 + \alpha_{03}\, x^3 + \ldots}{\alpha_{10} + \alpha_{11}\, x + \alpha_{12}\, x^2 + \alpha_{13}\, x^3 + \ldots} - \cfrac{\alpha_{00}}{\alpha_{10}}}$$

$$= \cfrac{\alpha_{10}}{\alpha_{00} + x \cdot \cfrac{(\alpha_{10}\, \alpha_{01} - \alpha_{00}\, \alpha_{11}) + (\alpha_{10}\, \alpha_{02} - \alpha_{00}\, \alpha_{12})x + \ldots}{\alpha_{10} + \alpha_{11}\, x + \alpha_{12}\, x^2 + \ldots}}$$

$$= \cfrac{\alpha_{10}}{\alpha_{00} + x \cdot \cfrac{\alpha_{20} + \alpha_{21}\, x + \alpha_{22}\, x^2 + \ldots}{\alpha_{10} + \alpha_{11}\, x + \alpha_{12}\, x^2 + \ldots}}$$

$$= \cfrac{\alpha_{10}}{\alpha_{00} + x \cdot \cfrac{\alpha_{20}}{\alpha_{10} + x \cdot \cfrac{(\alpha_{20}\alpha_{11} - \alpha_{10}\alpha_{21}) + (\alpha_{20}\alpha_{12} - \alpha_{10}\alpha_{22})x + \ldots}{\alpha_{20} + \alpha_{21}x + \alpha_{22}\, x^2 + \ldots}}}$$

$$= \frac{\alpha_{10}}{\alpha_{00} +}\,\frac{\alpha_{20}\ x}{\alpha_{10}\ \ +}\,\frac{\alpha_{30}\ x}{\alpha_{20}\ \ +} \ldots \tag{A2}$$

The α_{i0} can be computed by using the following scheme:

$$\begin{array}{lllll}
\alpha_{00} & \alpha_{01} & \alpha_{02} & \alpha_{03} & --------- \\
\alpha_{10} & \alpha_{11} & \alpha_{12} & \alpha_{13} & --------- \\
\alpha_{20} & \alpha_{21} & \alpha_{22} & \alpha_{23} & --------- \\
\alpha_{30} & \alpha_{31} & \alpha_{32} & \alpha_{33} & --------- \\
\cdot & & & & \\
\cdot & & & & \\
\cdot & & & & \\
\cdot & & & & \\
\cdot & & & &
\end{array}$$

The first two rows of the above array are the coefficients of the denominator and numerator, respectively, of Eq. (A1). Any other coefficient in the array is computed by

$$\alpha_{mn} = \alpha_{m-1,0}\ \alpha_{m-2,n+1} - \alpha_{m-2,0}\ \alpha_{m-1,n+1}, \quad m \geq 2,\ n \geq 0$$

If Eq. (A2) is truncated at, say, the second term, the right hand side becomes

$$\frac{\alpha_{10}}{\alpha_{00} +} \frac{\alpha_{20}\ x}{\alpha_{10}} = \frac{\alpha_{10}}{\alpha_{00} + \dfrac{\alpha_{20}\ x}{\alpha_{10}}} = \frac{\alpha_{10}\ \alpha_{10}}{\alpha_{00}\ \alpha_{10} + \alpha_{20}\ x}$$

The latter expression is called the first convergent of the continued fraction and is written as $\frac{P_1}{Q_1}$, where

$$P_1 = \alpha_{10}\ \alpha_{10}, \quad Q_1 = \alpha_{00}\ \alpha_{10} + \alpha_{20}\ x$$

Following this notational scheme,

$$P_o = \alpha_{10} \qquad\qquad Q_o = \alpha_{00}$$

$$P_1 = \alpha_{10}\,\alpha_{10} \qquad\qquad Q_1 = \alpha_{00}\,\alpha_{10} + \alpha_{20}\,x$$

$$P_2 = \alpha_{20}(P_1) + x\,\alpha_{30}(P_o) \qquad\qquad Q_2 = \alpha_{20}(Q_1) + x\,\alpha_{30}(Q_o)$$

$$\vdots \qquad\qquad \vdots$$

$$P_n = \alpha_{no}(P_{n-1}) + x\,\alpha_{n+1,o}(P_{n-2}) \qquad\qquad Q_n = \alpha_{no}\,(Q_{n-1}) + x\,\alpha_{n+1,o}(Q_{n-2})$$

$$\vdots \qquad\qquad \vdots$$

The above expressions for the numerator, P_n, and denominator, Q_n, of the n^{th} convergent of Eq. (A2) can be proved by induction.

In Chapter III of the dissertation, the solutions for the lumped systems are shown to be of the form

$$X_i = \frac{p_i(\mu)}{q\,(\mu)}$$

Consider the series expansion of such a function, say

$$f(x) = \frac{2 - x}{x^2 - 3x + 1}\ ,$$

which is found by long division to be

$$2 + 5x + 13x^2 + 34x^3 + 89x^4 + 233x^5 + \ldots$$

As it stands, this series will represent f(x) up to the lowest root of $x^2 - 3x + 1 = 0$. Following the scheme for computing the α_{ij},

1	0	0	0	0	0	--	--
2	5	13	34	89	233	--	--
-5	-13	-34	-89	-233	--	--	--
1	3	8	21	--	--	--	--
2	6	16	--	--	--	--	--
0	0	--	--	--	--	--	--

Hence, the α_{ij} terminate, and the non-zero results from the first column are:

$$\alpha_{00} = 1,\ \alpha_{10} = 2,\ \alpha_{20} = -5,\ \alpha_{30} = 1,\ \alpha_{40} = 2$$

Referring to the expressions for P_n and Q_n ,

$P_o = 2$

$P_1 = 4$

$P_2 = -5(4) + x(1)(2)$
$\quad = -20 + 2x$

$P_3 = 1(-20 + 2x) + x(2)(4)$
$\quad = -20 + 10x$

$P_4 = 2P_3$

.
.
.
.

$Q_o = 1$

$Q_1 = 2 - 5x$

$Q_2 = -5(2 - 5x) + x(1)(1)$
$\quad = -10 + 26x$

$Q_3 = 1(-20 + 26x) + x(2)(2-5x)$
$\quad = -20 + 30x - 10x^2$

$Q_4 = 2Q_3$

.
.
.
.

Thus,

$$\frac{P_3}{Q_3} = \frac{-10\ (2 - x)}{-10(1 - 3\ x + x^2)} = \frac{2 - x}{1 - 3x + x^2} = f(x)$$

Since the solutions for the X_I, as mentioned above, will be the ratio of two polynomials, it will be possible to represent the series expansion of X_i by a terminating continued fraction. Generalizing the above example, a system of N-degrees-of-freedom will require the computation of $2N - 1$ α_{io}'s beyond α_{00} and α_{10}, and 2N convergents. Hence,

$$p_i(\mu) = P_{2N-1} = \alpha_{2N-1,0}(P_{2N-2}) + x\ \alpha_{2N,0}\ (P_{2N-3})$$

$$q(\mu) = Q_{2N-1} = \alpha_{2N-1,0}(Q_{2N-2}) + x\ \alpha_{2N,0}\ (P_{2N-3})$$

APPENDIX II

Expansion of the Laplace Transform

An alternative to substituting (i s) for ω in the expression for the response of a system to harmonic excitation in order to find the response to a unit impulse is explained in this appendix. The single-degree-of-freedom system, which is governed by

$$m \ddot{x} + C \dot{x} + k x = f(t)$$

will be used to illustrate the procedure. First, the Laplace transform of the latter equation, with quiescent initial conditions, is

$$m s^2 \bar{x} + C s \bar{x} + k \bar{x} = \bar{f}(s)$$

where the symbols have been previously defined.

Calling

$$\frac{m}{k} = \mu, \ \frac{C}{k} = \alpha, \ 1 + \alpha s = \beta$$

the transform becomes

$$\mu s^2 \bar{x} + \beta \bar{x} = \frac{\bar{f}(s)}{k}$$

The expression for $\bar{x}$ could be written down immediately, but to illustrate the perturbation procedure, let it be assumed that

$$\bar{x} = \bar{x}_o + \mu \bar{x}_1 + \mu^2 \bar{x}^2 + . . .$$

and hence

$$\mu(s^2 \bar{x}_o + \mu s^2 \bar{x}_1 + \ldots) + \beta(\bar{x}_o + \mu \bar{x}_1 + \mu^2 \bar{x}_2 + \ldots) = \frac{\bar{f}(s)}{k}$$

The usual perturbation procedure yields

$$\bar{x}(s) = \frac{\bar{f}(s)}{k} \frac{1}{\beta} (1 - \frac{s^2}{\beta} + \frac{s^4}{\beta^2} - \frac{s^6}{\beta^3} + \ldots)$$

The series in parentheses could be treated by continued fractions, but is easily verified to be $\dfrac{1}{1 + \dfrac{s^2}{\beta}}$.

Finally,

$$\bar{x}(s) = \frac{1}{k} \frac{1}{\beta + s^2} \bar{f}(s)$$

which can, in principal, be inverted for a given f(t). In particular, if f(t) is the unit impulse, then $\bar{f}(s) = 1$.

The above procedure can be applied to any linear vibrating system.

AN INVESTIGATION OF THE USE OF INERTIA AS A PERTURBATION PARAMETER AND CONTINUED FRACTIONS IN LINEAR VIBRATION PROBLEMS

by

Jerry Counts

ABSTRACT

The eigenvalue problem associated with the determination of the natural frequencies, mode shapes and impulse response of a linear vibrating system is a classical and very important engineering problem. This dissertation presents a new technique for obtaining useful approximate, and in some cases, exact, solutions for such problems. The fundamental concepts on which the technique is based are:

1) The use of inertia (mass) as a perturbation parameter for developing series solutions for the response of a system to harmonic excitation.

2) The expansion of these series solutions as continued fractions.

The series solutions are obtained by applying the classical perturbation technique, which assumes the solution for the governing differential equation (say, for the case of one independent and one dependent variable) can be expressed as

$$w = w_0 + \mu w_1 + \mu^2 w_2 + \mu^3 w_3 + \ldots\ldots$$

where w is the dependent variable, and the w_i are unknown functions. μ is an inertia parameter that appears in the coefficients of the acceleration terms of the governing differential equation. The series for w is substituted in the governing differential equation, and the associated initial and boundary conditions. Since μ is arbitrary, the coefficients of like powers of μ are equated to zero. The result is an infinite set of differential equations, and boundary and initial conditions, each of which is (hopefully) easier to solve than the original problem. w_0 becomes the massless, or static, solution, in which the system responds instantaneously to, and in phase with, the applied excitation. The equation governing w_1 is the same as that for w_0, except that some function of w_0 appears as a loading function, and, in general, the equation governing w_i will involve some function of w_{i-1} as a loading function.

There are two ways in which the problem can become a so-called singular perturbation problem. First, if the order of the equations governing w_i is not as high as that of the original equation, it may not be possible to accommodate all the initial and boundary conditions. However, initial conditions are not necessary for determining the eigenvalues of a linear vibrating system. The second way in which a singular perturbation may arise is the limiting of the range of validity of the series solution to small values of some combination an independent variable and the perturbation parameter. The range of validity of the series solution can be extended by truncating the series after some term and converting the truncated series to the quotient of two polynomials by means of continued fractions. The zeros of the denominator polynomial will correspond to resonant conditions.

Lumped systems without damping are completely amenable to this method of solution, and a two-degree-of-freedom system with damping is solved. Approximate solutions for an axially loaded rod, a Timoshenko beam, and an Euler beam of variable cross-section illustrate the application of the method of analysis to continuous systems.