TRANSFORMATIONS PRESERVING TAME SETS

by

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I. INTRODUCTION

In this paper we investigate functions from a complex into a complex which preserve some property of tame subsets. If f is a function from E^3 into E^3 which takes tame sets onto tame sets in E^3, then f may be a homeomorphism or have as image a tame graph. The present research extends this observation. Theorem 3.3 characterizes local homeomorphisms on triangulated n-manifolds which preserve tame sets. Corollary 4.2 characterizes linear maps on Euclidean spaces and Theorem 4.6 characterizes simplicial maps on triangulated manifolds which preserve tame sets. For functions defined on topological spaces onto polyhedra Theorem 5.3 gives a new definition of continuity. A function from a triangulated n-manifold to a triangulated n-manifold whose image is of dimension greater than one is a homeomorphism if and only if f takes tame arcs onto arcs or points, Theorem 6.17. There are maps which take tame sets onto tame sets which raise the dimension of tame subsets, Theorem 8.1. Finally there are 1:1 maps from a complex into a complex which fail to take some tame set onto a tame set, Example 9.4.

An n-manifold M^n is a separable metric space such that each point $p \epsilon M^n$ lies in a neighborhood homeomorphic to

Euclidean n-space, E^n. An n-manifold M^n with boundary is a separable metric space such that each point $p \varepsilon M^n$ lies in a neighborhood whose closure is homeomorphic to the closed n-cube. The set of points of an n-manifold M^n with boundary, which have neighborhoods homeomorphic to E^n is called the interior of the manifold and denoted $\mathrm{int}(M^n)$. The boundary of the manifold is defined to be $\mathrm{Bd}(M^n) = M^n \setminus \mathrm{int}(M^n)$. Note that an n-manifold is an n-manifold with boundary but not conversely. All manifolds will be assumed to be connected.

A topological space X will be said to be triangulatable with triangulation $\Delta(K,h)$ if there is a locally finite geometric complex K and a homeomorphism h of K onto X. A simplex of X is the image of a simplex of K. A triangulated space will be called a complex or topological complex. A polyhedron in X is the image under h of a finite subcomplex of the complex K. If X is a complex with a triangulation Δ and P is a homeomorph of a polyhedron in X with respect to Δ, then P is tame in X if and only if there exists a homeomorphism $h' \circlearrowleft X$ and a triangulation Δ' of X in which $h'(P)$ is a polyhedron. A set K in a complex C is locally tamely imbedded if for each point $p \varepsilon K$ there is a neighborhood N of p and homeomorphism h_p of $\overline{N}$ onto a polyhedron in C such that $h_p(\overline{N} \cap K)$ is a polyhedron.

Let f be a function from a complex X into a complex Y. If for each tame set $P \subset X$, f(P) is tame in Y, then f will be called tame with respect to Y. If Y is a topological space and f(X) a complex then f will be called tame with respect to f(X) if for each tame set $P \subset X$, f(P) is tame in f(X). When it is possible to give two theorems, which have analogous proofs, one for functions tame with respect to f(X) and another for functions tame with respect to Y, we will combine the theorems by calling f tame. In both cases if f is tame we will say that f preserves tame sets.

II. REVIEW OF LITERATURE

The study of transformations, particularly continuous transformations, has been a central problem of mathematics. Recently the study of one topological space, especially polyhedra, imbedded in another has proved fruitful in topology. In this paper will be considered a natural connection between these two studies which is outlined below.

A topological space X is said to be imbedded in a topological space Y by the imbedding h if h is a homeomorphism of X into Y. If there are two imbeddings h_1 and h_2 of X into Y such that $h_1(X)$ and $h_2(X)$ can be distinguished with respect to a property P, then P is called an imbedding property. If P is an imbedding property and if $h_1(X)$ and $h_2(X)$ can not be distinguished with respect to P then $h_1(X)$ and $h_2(X)$ are said to be equivalent imbeddings of X in Y with respect to P. Let now h be an imbedding of X in Y which has property P. Let T be a function from Y into Z such that T(h(X)) has an imbedding property F in Z. Then what is the nature of T or what is the largest class of functions which transforms a particular imbedding property P to F? In order to get meaningful results in terms of important imbedding properties the problem must be rephrased somewhat as follows: Let $\{(X_i, h_i, P_i, Y, F_i)\}$ be a collection consisting of topological spaces X_i with corresponding imbeddings h_i in a topological space Y and with corresponding imbedding properties P_i and F_i. Let T be a function from Y into Z such that $T(h_i(X_i))$ has a property F_i in Z when X_i has property P_i in Y. Then what is the nature of T or what is the largest class of functions which transforms the imbedding properties P_i to F_i?

Given two imbeddings h_1 and h_2 of X in Y, then there may or may not exist a homeomorphism h of Y onto itself such that $h(h_1(X)) = h_2(X)$. If h exists then we say that $h_1(X)$ and $h_2(X)$ are relatively tamely or equivalently imbedded in Y and if h does not exist we say that $h_1(X)$ and $h_2(X)$ are relatively wild. In this paper we develop what is meant by the class of tame subsets of certain topological spaces and ask the question which can be paraphrased, "What functions preserve the tameness of sets?" This question which has escaped notice in the literature was suggested by Dr. P. H. Doyle. When the work commenced he contributed Theorems 3.1 and 3.2.

III. TAMENESS PRESERVING MAPS

As noted in Section I, the collection of all functions $f:E^3 \to E^3$ which are tame with respect to E^3 contain all homeomorphisms of E^3 into E^3 and all maps that have a tame graph as image. If h is a homeomorphism of E^3 into E^3 and if P is tame in E^3, then f(P) is locally tame in E^3 and hence tame as locally tame sets are tame, Bing [1] or Moise [7]. A similiar result holds for every 3-manifold as they are triangulatable, Bing [1], and as the image of each tame set is locally tame. The result for tame graphs generalize.

Theorem 3.1. (Doyle). Let f map K^n into K^m where K^n and K^m are a n-complex and a m-complex respectively. If $f(K^n)$ is a tame graph then f preserves tame sets.

Theorem 3.2. (Doyle). If a map $f:E^n \to E^{n \geq 2}$ is tame with respect to $f(E^n)$, then f may have as image any 1:1 continuous image of E^1 in any Euclidean space $E^{p \geq 2}$.

Proof: E^n can be mapped onto a line which can then be mapped in this way.

Let f be a function from X into Y, X and Y topological spaces. Then f is said to be locally a homeomorphism if for each $x \epsilon X$ there exists a neighborhood $U(x)$ of x such that $f|U$ is a homeomorphism. For example the open cylinder $(0,1)x(0,1)x(0,1)$ can be mapped onto the open solid torus $(0,1)x(0,1)xS'$, S' the 1-sphere, in this way. But in such examples tameness may not be preserved.

Theorem 3.3. Let M_1 and M_2 be triangulated n-manifolds, $n>1$. If a local homeomorphism $f:M_1 \to M_2$ is tameness preserving, then it is a homeomorphism.

Proof: As f is locally a homeomorphism, it is open for if U is open in M_1, then for each $p \epsilon U$ there is an open homeomorph N_p of E^n containing p on which f is a homeomorphism. Then $f(U) = f(\bigcup_{p \epsilon U} N_p) = \bigcup_{p \epsilon U} f(N_p)$ is open by Brouwer's Invariance of Domain. Thus suppose f is not 1:1 so that for some point $p \epsilon f(M_1)$, $\{f^{-1}(p)\}$ is not a point. For any two points of $\{f^{-1}(p)\}$, say x and y, there exists disjoint neighborhoods N_x and N_y respectively on which f is a homeomorphism. Now there exists an arc A in N_x whose image is of the form $\xi \sin \frac{1}{\xi}$ with endpoint x and an arc B in N_y such that $A \cup B$ is tame by a result of Cantrell but $f(A) \cup f(B)$ is not tame or polyhedral, Figure 1. Thus f is 1:1 and

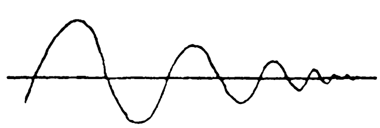

$f(A) \cup f(B)$

Figure 1

therefore a homeomorphism.

As can be seen from the proof, the above theorem is true if f preserves arcs instead of tame sets. Moreover:

Corollary 3.4. If f preserves tameness or arcs and fails to be a homeomorphism, then there exists an $x \varepsilon M_1$ such that f is not locally a homeomorphism at x.

There are open maps which do not map tame sets onto tame sets (projections of Euclidean spaces onto subspaces). And there are 1:1 maps which do not take tame sets to tame sets (see Example 9.4). Furthermore, as would be expected, there are maps which map tame sets onto tame sets which are not open (easy construction using E^1). And there are maps which take tame sets to tame sets which are not 1:1 (project E^n onto E^1). However, light open maps of a 2-manifold into itself which preserve tameness can be characterized.

Theorem 3.5. Let f be a light open map of a 2-manifold M into itself. Then f preserves tameness if and only if f is a homeomorphism.

Proof: Each point of M has a neighborhood W on which $f|W$ is topologically equivalent to the map $z \to z^n$ of E^2 to E^2, Church and Hemmingsen [2] or Stailow [11]. Let p be a point for which $n \neq 1$. Now consider the situation given in Figure 2

$$\begin{tikzcd} W \arrow[r, "f|W"] \arrow[d, "h"] & W \\ E^2 \arrow[r, "g", "z \to z^n"'] & E^2 \arrow[u, "h^{-1}"'] \end{tikzcd}$$

Figure 2

where h is a homeomorphism. Let $q \varepsilon E^2$ be a point such that $\{g^{-1}(q)\}$ is not a point. Let $x, y \varepsilon \{g^{-1}(q)\}$ be two distinct points and let N_x and N_y be disjoint ε-neighborhoods of x and y respectively such that $g|N_x$ and $g|N_y$ are homeomorphisms. Consider $N = g(N_x) \cap g(N_y) \neq \emptyset$ which is open. Construct in $h^{-1}(N)$ a set like that of Figure 1 consisting of arcs A_1 and A_2. Consider both the arcs $A_1' = h^{-1}[g^{-1}h(A_1) \cap N_x] \subset h^{-1}[g^{-1}(N) \cap N_x]$ and $A_2' = h^{-1}[g^{-1}h(A_2) \cap N_y] \subset h^{-1}[g^{-1}(N) \cap N_y]$. They are disjoint and locally tame. However by construction $f(A_1' \cup A_2') = h^{-1}gh(A_1' \cup A_2') = (A_1 \cup A_2)$ which is not tame. Therefore n=1 for all $p \varepsilon M$. Therefore f is a local homeomorphism and thus by Theorem 3.3 f is a homeomorphism.

Let $g: X \to Y$ be a map of X into Y. The branch set of g, denoted by B_g, is the set of points of X at which g fails to be a local homeomorphism.

Theorem 3.6. Let f be a light map of a triangulated n-manifold M^n into itself, $n \geq 2$. If f preserves tameness, then either f is a homeomorphism or for some $p \varepsilon B_f$ every open neighborhood of f(p) fails to meet $f(B_f)$ in a tame (n-2)-cell.

Proof: This follows from Theorem 4.1 [2] or Theorem 1.2 [3], Church and Hemmingsen using a similar argument to that of Theorem 3.5. If n=3 use Theorem 9 of Bing [1] and if $n \geq 4$ use Homma's Theorem [6] Mutatus Mutandis.

Considering now the domain of maps which preserve tame sets the following is a characterization of compact complexes among connected n-complexes.

Theorem 3.7. Let K^n be a connected n-complex. Then K^n is compact if and only if every tameness preserving map of K^n into K^n has a polyhedral image.

Proof: If K^n is compact then K^n is a polyhedron and so $f(K^n)$ is a polyhedron. If K^n is not compact retract it onto I^1, a closed unit interval, and then map this line onto a homeomorph of the set in Figure 3 in the appropriate way.

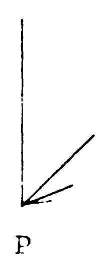


Figure 3

The set in Figure 3 is a countably infinite collection of lines in the plane meeting in a point P as indicated. The first line is of unit length. The second line is at an angle of $\frac{\pi}{4}$ radians to the first and of $\frac{1}{2}$ unit length. The n-th line is at an angle of $\frac{\pi}{2^n}$ radians to the (n-1)-st line and is of $\frac{1}{2^{n-1}}$ units length.

The star of a geometric simplex σ, $St(\sigma)$, is the union of all simplices of which σ is a face. The open star of σ is the interior of $St(\sigma)$.

Let $f:S^n \to S^n$ be a tameness preserving map while $1 < \dim f(S^n) \leq n-1$ at some point p. Since $f(S^n)$ is a polyhedron the dimension of it at p is equal to the highest dimensional simplex in the open star of p. Let this be $1<k\leq n-1$. If σ^k is such a simplex, we have $f^{-1}(\text{int } \sigma^k) = U'$ is an open set in S^n. U' has a countable number of components and so U' has a component U'', f(U'') contains an open k-cell in int σ^k. Thus there is an open n-cell $U \subset U''$ and $f(U) \subset \text{int } \sigma^k$ contains an open k-cell; for U'' is a union of basic elements of S^n and if each in a covering of U'' were (k-1)-dimensional in its image, f(U'') would be (k-1)-dimensional. So we have a theorem.

Theorem 3.8. (Doyle). Let $f_1:S^n \to S^n$ be an into tameness preserving map that has a local dimension of k at some point p, where $1<k\leq n-1$. Then there is a tameness preserving

map $g:E^n \to E^n$ such that $g(E^n) \subset E^n$, and $\dim g(E^n) = k$.

Proof: Let $m:E^n \to U$ be a homeomorphism of E^n onto U above for f_1 replacing f. Let $f_1 m = g$.

IV. LINEAR AND SIMPLICIAL MAPS

In this section we characterize linear and simplicial maps which preserve tame sets. A linear map for Euclidean n-space E^n is a map f of E^n into E^n such that if x_1 and x_2 are in E^n and λ_1 and λ_2 are real numbers then $f(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 f(x_1) + \lambda_2 f(x_2)$.

Theorem 4.1. Let E^n be Euclidean n-space and L a linear map of E^n into itself. Then L preserves tameness if and only if either the range of L is topologically E^1, a point, or L is a homeomorphism.

Proof: First suppose n=2 for if n=1 tameness is clearly preserved. As L is linear, if it is onto it is also 1:1. In this case it is locally a homeomorphism and is therefore a homeomorphism. Thus tameness is preserved. If L is not onto then the range is topologically E^1 and tameness is consequently preserved. So for a linear map of E^2 into E^2 tameness is preserved.

Now let L be a linear map on E^n, n>2. E^n is the direct sum of the null space of L, $\eta(L)$, and the space span by any set of progenitors of the range of L, P(L), ie. $E^n = \eta(L) \oplus P(L)$. A set of progenitors for the range of L, Q(L), is a set $x_{\nu+1}, x_{\nu+2}, \text{---}, x_n$ such that

$Lx_{\nu+1}$, $Lx_{\nu+2}$, --- , Lx_n is a basis for the range of L, R(L). The above decomposition for E^n is a direct corollary to the proof of the theorem which states dim E^n = dim $\eta(L)$ + dim R(L). Assume that dim R(L)$\geq$2, otherwise tameness is trivially preserved. If dim R(L) = n, then L is 1:1, onto, and locally a homeomorphism and therefore a homeomorphism. Thus assume dim $\eta(L) \neq 0$. Let x_{γ_1} and x_{γ_2} belong to the progenitors of R(L). Now x_{γ_1} and x_{γ_2} are independent, for suppose not, ie. $x_{\gamma_1} = rx_{\gamma_2}$. Then $Lx_{\gamma_1} = rLx_{\gamma_2}$, a contradiction. Therefore x_{γ_1} and x_{γ_2} span a subspace M of dim 2 which is mapped by L onto a subspace L(M) of dim 2. Therefore L is 1:1, locally a homeomorphism from M onto L(M) and hence is a homeomorphism. Let x belong to the basis of $\eta(L)$. Now x+M is a linear variety such that L(x+M) = L(M) so x+M is mapped homeomorphically onto L(M). Now construct in M a set of the form given in Figure 4 consisting of

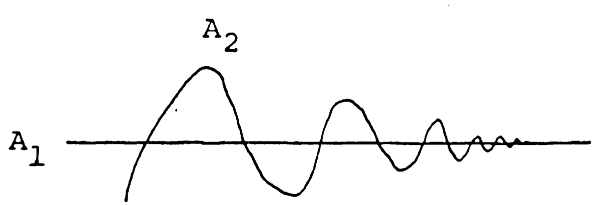


Figure 4

an arc A_1 and an arc A_2 of the form $\xi \sin \frac{1}{\xi}$. Let $\bar{A}_1$ be a homeomorphic copy of A_1 in x+M. Now by a result of Cantrell

$\bar{A}_1 \cup A_2$ is tame. But by construction $L(\bar{A}_1) \cup L(A_2)$ is homeomorphic to $A_1 \cup A_2$ which is not a polyhedron.

Corollary 4.2. Let E^n be Euclidean n-space and L a linear map of E^n into itself. Then L preserves tameness if and only if either the range of L is topologically E^1, a point or one of the following equivalent conditions hold:

(i) L is a homeomorphism,

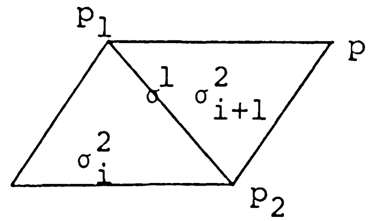
(ii) L is onto,

(iii) L is 1:1.

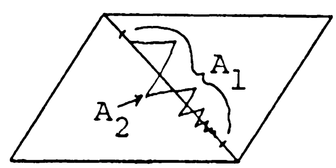
A map from a complex K_1 into a complex K_2 is simplicial if the image of each simplex of K_1 is a simplex of K_2. A complex K is said to be n-chain connected if every two points lie on a geometric n-chain. A geometric n-chain is a finite, simply ordered collection of incident alternately closed n- and (n-1)- simplexes terminating in an n-simplex.

Theorem 4.3. (Doyle). Let K^2 be a 2-chain connected 2-complex and let K be a complex. If $f:K^2 \to K$ is a simplicial map that preserves dimension ($\dim f(K^2) = 2$) and if some 2-simplex collapses, f does not preserve tame sets.

Proof: Let $\sigma^2 \varepsilon K^2$ be such that $f|\sigma^2$ is a homeomorphism (as dimension is preserved). Now suppose $\sigma_1^2 \varepsilon K^2$ and $\dim (f(\sigma_1^2)) \neq 2$. Let $\sigma^2, \sigma_2^2, \text{---}, \sigma_k^2, \sigma_1^2$ be a regular chain from σ^2 to σ_1^2. Now there exists a σ_i^2 such that $f|\sigma_i^2$ is a homeomorphism while $f|\sigma_{i+1}^2$ is not. Let $\sigma_i^2 \cap \sigma_{i+1}^2 = \sigma^1$. Let $\sigma^1 = (p_1, p_2)$ and $\sigma_{i+1}^2 = (p_1, p_2, p)$.



As f is simplicial f(p) is identified with $f(p_1)$ or $f(p_2)$ and so (p_1, p) or (p_2, p) is mapped onto $f(\sigma^1)$. In either case $f|\sigma_i^2 \cup \sigma_{i+1}^2$ does not preserve tame sets for if A_1 is a subarc of σ^1 we can construct a set like that in Figure 4 by choosing appropriate arc A_2. Let A_3 be an arc



in (p_1, p) or (p_2, p) which is mapped onto $f(A_1)$ by f. Now $A_2 \cup A_3$ is tame but $f(A_2) \cup f(A_3)$ is nota polyhedron.

Theorem 4.4. Let K^n be a 2-chain connected n-complex and let K^m be an m-complex. If $f: K^n \to K^m$ is a simplicial map such that some q-simplex has a q-simplex image $q \geq 2$ and if some p-simplex collapses $p \geq 2$, f does not preserve tame sets.

Proof: As $p \geq 2$, if some p-simplex collapses, then some 2-simplex face σ_o^2 of this simplex collapses. As some q-simplex has a q-simplex image a 2-simplex face σ^2 of the q-simplex is mapped homeomorphically onto a 2-simplex. Let σ^2, σ_1^2, σ_2^2, --- , σ_k^2, σ_o^2 be a regular chain from σ^2 to σ_o^2 in the two skeleton of K^n. This chain is a triangulated 2-manifold and Theorem 4.3 above applies.

Theorem 4.5. If a n-complex $n \geq 2$ is not collapsed into a 1-complex by a simplicial map, then some 2-simplex is mapped onto a 2-simplex.

Proof: There is a simplex of dimension 2 or greater that is not collapsed to a one simplex. Therefore it is mapped onto a simplex with three or more vertices. Consider the inverse image of three of these vertices intersecting with the vertices of the original simplex. Choose a vertex from each set. These three vertices determine a 2-simplex which is mapped onto a 2-simplex.

Theorem 4.6. A simplicial map f of a triangulated n-manifold with boundary M^n into itself preserves tameness if and only if either $f(M^n)$ lies in a 1-complex or f is a homeomorphism into.

Proof: By the previous two theorems if $f(M^n)$ does not lie in a 1-complex in M^n and f preserves tame sets, then f does not collapse simplices of dimension two or greater. This implies that f is locally a homeomorphism. Therefore f is a homeomorphism.

V. TRANSFORMATIONS AND TAME SETS

To prove Theorems 3.3., 3.5., 3.6., 4.1., and 4.6. use was made of some tame arc whose image failed to be polyhedral. In this section will be considered transformations which preserve or whose inverses preserve some property of polyhedra for tame arcs. First we investigate the connectedness of the image of such transformations and then the continuity of the transformations.

A topological space X is said to be strongly arcwise connected provided every infinite subset of X intersects some arc of X in infinitely many points. Analogously, a complex K will be called strongly tame-arcwise connected provided every infinite subset of K intersects some tame arc of K in infinitely many points.

Theorem 5.1. Let K be a strongly tame-arcwise connected complex and let f be a function defined on K to a topological space. If f takes tame arcs onto sets consisting of a finite number of components, then f(K) consists of at most a finite number of components.

Proof: Suppose f(K) consists of more than a finite number of components. Let $\{C_i\}_{i=1}^{\infty}$ be a countably infinite collection of these components. In each $\{f^{-1}(C_i)\}$ choose a point p_i. Pass a tame arc A through a convergent subsequence of $\{p_i\}$. Now f(A) consists of an infinite number of components. But this is contrary to our assumption.

A strongly tame-arcwise connected complex is compact. Suppose not. Let $\{x_i\}$ be a divergent sequence of points in the complex such that no subsequence converges. As the complex is strongly tame-arcwise connected there is an arc in the complex which meets infinitely many of the points and hence the sequence converges.

Theorem 5.2. Let K be a compact connected complex and let f be a function defined on K to a complex K'. If f and f^{-1} takes tame sets onto tame sets then f(K) is connected.

Proof: Suppose f(K) consists of a finite number of disjoint topological polyhedral components, each closed. Denote them by $C_1, C_2, \text{---}, C_n$. Now $f^{-1}(C_i)$ is closed and $f^{-1}(C_i) \cap f^{-1}(C_j) = \emptyset$ if $i \neq j$. Thus $K = \bigcup_{i=1}^{m} f^{-1}(C_i)$ is a decomposition of K into disjoint sets. But this is not possible.

Let p and q be two points in S^n, the n-sphere. Consider the function which is the identity on q and retracts $S^n \setminus q$ onto p. Then the image of each tame set is tame but the function is not continuous. However, if the inverse of a function is considered it is possible to characterize continuous functions on complexes as follows.

Theorem 5.3. Let f be a function defined on a topological space X and let $f(X)$ be a strongly tame-arcwise connected complex. Then a necessary and sufficient condition that f be continuous is that f^{-1} take tame arcs onto closed sets.

Proof: Let $\{x_i\}$ be a sequence in x converging to x such that $\{f(x_i)\}$ does not converge to $f(x)$. Let Λ be an infinite set of positive integers such that $\{f(x_j):j\varepsilon\Lambda\}$ does not have $f(x)$ as a limit point. In $f(X)\setminus f(x)$ pass a tame arc A' through infinitely many $\{f(x_j):j\varepsilon\Lambda\}$. Now $f^{-1}(A')$ is closed. So $f^{-1}(A')$ contains infinitely many x_i and therefore x, contrary to assumption.

A complex K is said to be strongly tame-arcwise connected with respect to compact sets provided every infinite compact subset of K intersects some tame arc of K in infinitely many points.

Theorem 5.4 Let K be a complex which is strongly tame-arcwise connected with respect to compact sets and let f be a function defined on K with range in a complex. If f and f^{-1} take tame arcs onto tame sets, then f is continuous.

Proof: Suppose f is not continuous. Then there is a convergent sequence of points $\{x_i\}$ with limit x in K such that $\{f(x_i)\}$ does not converge to f(x). Let $\{f(x_n)\}$ be an infinite subset of $\{f(x_i)\}$ which does not have f(x) as a limit point. As $\{x_n\} \cup x$ is a compact set, pass a tame arc A through infinitely many $\{x_n, x\}$. Now f(A) consists of a finite union of tame components. As f is not continuous there is an ε-open star neighborhood N_ε of f(x) in f(A) such that almost all $\{f(x_n)\}$, lie in $f(A) \setminus N_\varepsilon = P$. Each component of P is strongly tame-arcwise connected. Thus let A^1 be a tame arc in P meeting infinitely many of the $\{f(x_n)\}$. Now $A^1 \cap f(x) = \emptyset$ but as $f^{-1}(A^1)$ contains infinitely many x_i, $f^{-1}(A^1)$ contains x, contrary to assumption.

In complexes compactness is equivalent to strongly tame-arcwise connectedness. Let K be a compact complex and let B be any infinite subset of K. As K is compact let σ be the smallest dimensional closed simplex containing a countably infinite subset of B. Let p be a limit point of this subsequence in σ. Let $\{p_i\}$ be a monotone converging subsequence in σ converging to p. Let A be a tame arc from p to p_1. Now it is easy to construct an isotopy of σ onto itself taking $\{p_i\}$ onto A without moving p and p_1 which is the identity on the boundary of σ. Hence the complex is strongly tame-arcwise connected.

Moreover, every complex is strongly tame-arcwise connected with respect to compact sets. Let B be an infinite compact set of a complex K. As B is compact some closed simplex of K contains an infinite subset of B and hence the results of the preceeding paragraph yields the desired conclusion.

The previous theorems now become:

Theorem 5.1. Let K be a compact connected complex and f a function defined on K to a topological space. If f takes tame arcs onto sets consisting of a finite number of components, then f(K) consists of at most a finite number of components.

Theorem 5.2. Let K be a compact connected complex and let f be a function defined on K to a complex K'. If f and f^{-1} takes tame sets onto tame sets then f(K) is connected.

Theorem 5.3. Let f be a function defined on a topological space X and let f(X) be a compact connected complex. Then a necessary and sufficient condition that f be continuous is that f^{-1} take tame arcs onto closed sets.

Theorem 5.4. Let K be complex and let f be a function defined on K with range in a complex. If f and f^{-1} take tame arcs onto tame sets, then f is continuous.

Let $p \varepsilon S^n$ be a point. The map of S^n onto p and its inverse takes tame sets onto tame sets but is not a homeomorphism.

Let π be the parallel projection of S^n onto its lower hemisphere D. Let r be a retraction of D onto a semi-great circle. Then the map $f = r\pi$ and its inverse takes tame sets onto tame sets but f is not a homeomorphism as $f(S^n)$ is an arc.

The following theorem will be used in the next section to characterize maps of S^n into S^n preserving tame sets which do not raise the dimension of an arc.

Theorem 5.5. Let M^n be a triangulated n-manifold with boundary and let f be a function defined on M^n to a complex K. If f takes tame arcs onto connected tame sets and if for every tame arc $A \varepsilon M^n$ f(A) contains no triod, then either the image of each tame arc is an arc or point or $f(M^n)$ is a simple closed curve.

Proof: Suppose $f(M^n)$ is not a simple closed curve and that there is a tame arc A such that f(A) is not an arc or point. If the dimension of f(A) is larger than one, as f(A) is tame, f(A) contains a simplex of dimension greater than one. Thus f(A) contains a triod. As this is not possible assume that f(A) is a simple closed curve. As $f(M^n)$ is itself not a simple closed curve, let p' be a point not contained in f(A). Choose a point p in $f^{-1}(p')$. Now $A \cup p$ is tame and hence lies on a tame arc A' in M^n. So f(A') contains $f(A) \cup p'$ and hence a triod.

VI. ARC AND TAME ARC PRESERVING TRANSFORMATIONS

In this section it is shown that the set of transformations from a triangulated n-manifold into a n-manifold of ranges of dimension greater than one, which take tame arcs onto arcs or points, is the class of homeomorphisms. As a consequence of this result, the class of maps of S^n into itself which map tame arcs onto tame sets and which do not take a tame arc onto a set containing a triod is also the set of homeomorphisms. Thus appealing to Theorem 5.3. or 5.4., if f is a function of S^n into S^n such that it and its inverse take tame sets onto tame sets and if the image of no tame arc contains a triod, then f is a homeomorphism provided $f(S^n)$ is not an arc or point. That this latter result can not be improved upon in terms of the no-triod condition is a consequence of the example in Section VIII.

A function defined on a topological space to a topological space is said to be arc preserving if the image of every arc is an arc or a point. An (n-1)-sphere S^{n-1} in E^n is bicollared if there exists a homeomorphism h from $S^{n-1} x I$ into E^n such that $h(S^{n-1} \times \frac{1}{2}) = S^{n-1}$.

Theorem 6.1. Let f be a function whose domain is a n-manifold M_1 and whose range is in a n-manifold M_2. If f is arc preserving and dim $f(M_1) > 1$, then f is a homeomorphism.

Proof: As M_1 is a manifold cover it with a countable collection, $\{C_i\}$, of closed n-cells with bicollared boundaries. Now there exists a n-cell C' in $\{C_i\}$ such that dim $(f(C')) > 1$. For suppose not, as f preserves arcs $f(C_i)$ must be connected and is hence either a point or an arcwise connected one dimensional set. We may suppose that $f(C_i)$ is a subset of an arc, for if not, $f|\overline{C}_i$ is a homeomorphism by 4.1 of Hall and Puckett [4]. Thus $f(M_1) = f(\bigcup_{i=1}^{\infty} C_i) = \bigcup_{i=1}^{\infty} f(C_i)$ can be expressed as a countable union of closed arcs or points as each $f(C_i)$ is either a point or expressible as a countable union of arcs. However, this is not possible by the Sum Theorem for Dimension. Therefore at least one C_i, say C', is such that dim $(f(C')) > 1$. Then $f|\overline{C}'$ is a homeomorphism by 4.1 Hall and Puckett [4]. Suppose now that f is not locally a homeomorphism at $p \varepsilon M_1$. Then $p \cup C'$ lies in a n-cell C" with bicollared boundary by a result of Doyle and Hocking. Thus $f|C''$ is a homeomorphism by 4.1 of Hall and Puckett. Consequently f is locally a homeomorphism. Suppose f is not 1:1. Let p and q be contained in M_1 and such that $f(p) = f(q)$. Then $p \cup q$ lies interior to a closed n-cell, $\overline{C}''' \supset C''$, in M_1. Therefore $f|\overline{C}'''$ is a homeomorphism.

Thus $f(p) \neq f(q)$ and so f is 1:1. To see that f is open and continuous observe that, as f is locally a homeomorphism, it preserves local dimension and $f(M_1)$ is locally Euclidean by Browers Invariance of Domain. Thus by Invariance of Domain f and f^{-1} are open. Therefore f is a homeomorphism.

An analogous theorem for manifolds with boundaries does not hold. But it does follow that:

Theorem 6.2. Let f be a function whose domain is an n-manifold M^n with boundary and whose range is an n-manifold. If f preserves arcs and if dim $(f(M^n))>1$, then f is 1:1 and $f|\mathrm{int}(M^n)$ is a homeomorphism.

Proof: Let p be such that dim $(f(p))>1$. Suppose $p \varepsilon \mathrm{int}(M^n)$ then by Theorem 6.1., $f|\mathrm{int}(M^n)$ is a homeomorphism. Suppose $p \varepsilon \mathrm{Bd}(M^n)$. Then there exists a closed and compact n-cell C such that $p \varepsilon \mathrm{Bd}(C)$. Then by 4.1 Hall and Puckett [4] $f|C$ is a homeomorphism. Therefore there exists a point $q \varepsilon \mathrm{int}(M^n)$ such that dim $(f(q))>1$ and therefore $f|\mathrm{int}(M^n)$ is a homeomorphism.

Suppose f is not 1:1. Let x, $y \varepsilon M^n$ be such that $f(x) = f(y)$. Not both x and y belong to $\mathrm{int}(M^n)$. Let A be an arc connecting x and y such that if $A \cap \mathrm{Bd}(M^n) \neq \emptyset$ then either this intersection is x, y or $x \cup y$. From above we see if $x \varepsilon \mathrm{Bd}(M^n)$ then there exists a closed n-cell C such that $x \in Bd(C)$. Thus f is locally a homeomorphism on M^n. Therefore f(A) is a simple closed curve. But this is a contradiction to the assumption that arcs are taken to arcs.

A function defined on a complex to a topological space is said to preserve tame arcs if the image of every tame arc is an arc or point. What follows in this section is a proof that the hypothesis of Theorem 6.1. can be weakened to functions which preserve tame arcs in case of triangulated manifolds. Many of the theorems on pages 33 through page 39 are analogous to those of Hall and Puckett [4] for arc preserving functions. When in the following theorems the range of a function is not mentioned it is to be assumed to be a topological space. Some of the theorems which follow in this section are true for complexes as well as for manifolds. However, they will not be given as their proofs usually require more work.

Theorem 6.3. Let $\{\alpha_i\}_1^q$ be a finite collection of arcs and simple closed curves with a tame union A in a compact triangulated manifold M^n, $n > 1$. If $\{M_j\}$ is a countably infinite collection of disjoint subsets of M^n, then there exists a tame arc α_{q+1} meeting infinitely many of the M_j such that $\alpha_{q+1} \cup A$ is tame.

Proof: If an α_i meets infinitely many of the M_j, choose one such α_i. Suppose no α_i meets infinitely many M_j. From each $M_{j'}$ such that $M_{j'} \cap A = \emptyset$, choose a point $x_{j'}$ to form a sequence $\{x_{j'}\}$. As M^n is a compact manifold, $\{x_{j'}\}$ contains a monotone convergent subsequence, say $\{y_k\}$, with $y_k \to p$. Now $\{y_k\} \cap A = \emptyset$ but $\{p\} \cap A$ may or may not be empty. If $\{p\} \cap A = \emptyset$, then there exists an α_{q+1} with the desired properties as can be seen below from the proof of the other case. Without loss of generality, we may regard A as a standard simplicial polyhedron. First suppose A separates M^n, then at least one component of $M^n \setminus A$ contains an infinite subsequence $\{y'_{k'}\}$ converging to p. From this it can be seen that the argument for the case where A does not separate M^n suffices. Let B_ε be an $\frac{\varepsilon}{2\pi}$ ball about p which meets a minimum number of α_i, i = 1, 2, ---, q and such that every ε'-ball, $\varepsilon' < \varepsilon$, meets precisely the same number of α_i. Assume $\{y'_{k'}\} \subseteq B_\varepsilon$. As $\{y'_{k'}\}$ forms a monotone converging sequence, there exists a monotone sequence $\{\varepsilon_{k'}\}$ of real numbers and a monotone sequence of $\varepsilon_{k'}$-balls $\{B_{k'}\}$ such that $y'_{k'}$ belongs to the interior of $B_{k'} \setminus (B_{k'+1} \cup A)$. Let a be a polygonal arc in B_ε which meets A at p such that $A \cup a$ is tame. Choose a point in each $\text{int}(B_{k'} \setminus (B_{k'+1} \cup A)) \cap a$, say $p_{k'}$. Now, as $p_{k'}$ can be joined to $y'_{k'}$ by a polygonal arc in $\text{int}(B_{k'} \setminus (B_{k'+1} \cup A))$, $p_{k'}$ can be moved to $y'_{k'}$ by a $\varepsilon_{k'}$-isotopy in $\text{int}(B_{k'} \setminus B_{k'+1} \cup A)$ that does not disturb A.

As the resulting sequence of isotopies satisfy the desired uniformity condition, α_{q+1} can be constructed by the resulting isotopy.

Theorem 6.4. If M^n is a compact triangulated n-manifold and if $f|M^n$ is a 1:1 function which preserves tame arcs, then $f|M^n$ is a homeomorphism.

Proof: As M^n is compact we need only show $f|M^n$ is continuous. Let $\{x_i\}$ be a sequence of points converging to x in M^n. By Lemma 3, Rosen [10], $\{x_i\}$ lies on a tame arc α in M^n. For each m let α_m be the irreducible subarc of α containing $x \cup (\bigcup_{i=m}^{\infty} x_i)$; then $\bigcap \alpha_m = x$. As f is 1:1, $f(\bigcap \alpha_m) = \bigcap f(\alpha_m)$. Thus $f(x) = \bigcap f(\alpha_m)$ is the intersection of a monotone decreasing sequence of arcs. Thus $f(x_m)$ converges to f(x), since $f(x_m)$ is contained in $f(\alpha_m)$ for every m.

A complex is tame-arcwise connected with respect to compact countable sets if every compact countable set lies on a tame arc in the complex.

Corollary 6.5. If K is a compact complex which is tame arcwise connected with respect to compact countable sets and if $f|K$ is a 1:1 function which preserves tame arcs, then $f|K$ is homeomorphism.

Theorem 6.6. Let f be a function which preserves tame arcs whose domain is a triangulated n-manifold M^n. If ω is a fixed tame arc in M^n with endpoints $p^* \epsilon f^{-1}(p')$ and $q^* \epsilon f^{-1}(q')$, let $G = \{\alpha\}$ be the set of all tame arcs in M^n with endpoints p^* and q^* such that if $\alpha \epsilon G$ then $\omega \cup \alpha$ is tame. If the image of every tame simple closed curve in M^n is an arc, then $\bigcap f(\alpha)$ contains an arc joining p' and q'.

Proof: Let α^* be any arc in G. Let β_1' and β_2' be subarcs of $f(\omega)$ and $f(\alpha^*)$ respectively which have p' and q' as endpoints. Suppose there exists a point $x' \epsilon \beta_1'$ not contained in β_2', then $\beta_1' \cup \beta_2'$ contains a simple closed curve J'. Therefore as $f^{-1}(x')$ is disjoint with α^* and as $\omega \cup \alpha^*$ is tame, $\omega \cup \alpha^*$ contains a tame simple closed curve J such that $f(J) \supset J'$, contrary to hypothesis.

Theorem 6.7. Let f be a function which preserves tame arcs whose domain is a triangulated n-manifold M^n. If ω is a fixed tame arc in M^n with end points $p^* \epsilon f^{-1}(p')$ and $q^* \epsilon f^{-1}(q')$, let $G = \{\alpha\}$ be the set of all tame arcs with endpoints in $f^{-1}(p')$ and $f^{-1}(q')$ such that $\alpha \cup \omega$ is tame. If the image of every simple closed curve in M^n is an arc, then $\bigcap f(\alpha)$ contains an arc joining p' and q'.

Proof: From Theorem 6.6. if $G^* = \{\alpha^*\}$ is the collection of all arcs of G which have endpoints of p^* and q^*, then $\bigcap f(\alpha^*)$ contains an arc β between p' and q'. Let $p^*q^* \cup q^*p$ be an arc of G with $p \varepsilon f^{-1}(p')$. Now $f(p^*q^*) \supset \beta$ as p^*q^* is an arc of G^*. Further $f(q^*p)$ must contain β for otherwise $f(p^*q^* \cup q^*p)$ would contain a simple closed curve. Because of symmetry of the argument, $\bigcap f(\alpha)$ contains β.

Theorem 6.8. If $f|M^n$, M^n a triangulated n-manifold, preserves tame arcs and if J is a tame simple closed curve, then $f(J) = J'$ is a homeomorphism or J' is an arc or point.

Proof: Suppose J' is not an arc or point. Then, as J is a compact 1-dimensional manifold, we need only show $f(J) = J'$ is 1:1 by virture of Theorem 6.4. If $f(J) = J'$ is not 1:1 there exists x and y in J such that $f(x) = f(y)$. Express $J = \alpha \cup \beta$ where α and β are tame arcs such that $\alpha \cap \beta = x \cup y$. Now there exists in α points p and q, $p \neq q$, whose images are the endpoints of $f(\alpha)$ (If $f(\alpha)$ is a point choose p and q to be x and y respectively). Let the points be so named that $\alpha = xp \cup pq \cup qy$ where any two of the arcs on the right have at most a common end point. Now the endpoints of $f(\alpha)$ lie in its subarcs $f(xp)$ and $f(qy)$, which have a common point. Therefore $f(xp \cup qy) = f(xp) \cup f(qy) = f(\alpha)$.

Therefore the arcs $\gamma = px\cup\beta\cup qy$ is such that $f(\gamma) = J'$ contrary to the hypothesis that J' is not an arc.

An arc axb with end points a and b is said to span a point set M provided $M\cap axb = a\cup b$.

Theorem 6.9. If $f|M^n$ preserves tame arcs, M^n a compact triangulated n-manifold, and if there exists a tame simple closed curve $J\subset M^n$ such that $f(J) = J'$ is not an arc or point, then f is 1:1.

Proof: As J' is not an arc or point it follows by Theorem 6.8. that $f|J$ is a homeomorphism. Let $z\varepsilon J$ and suppose $z \neq f^{-1}f(z)$. Then there exists a point $z_1 \varepsilon M^n \setminus J$ such that $f(z_1) = f(z)$. Now there exists a homeomorphism h of M^n onto itself such that h(J) is rectilinear. As M^n has no cutpoints and as Ayre's Three Point Theorem holds in M^n, h(J) can be spanned in M^n by an arc A containing $h(z_1)$. As A is compact it can be approximated within ε by a polygonal arc, say $c'h(z_1)d'$. Thus there is an arc cz_1d in M^n spanning J such that $d \neq z$ and such that the subarcs of $cz_1d\cup J$ are tame. Write J as the sum of two simple arcs α and β having precisely the points z and d in common. Now $f(z) = f(z_1)$ and f is a homeomorphism on α. Therefore, since $f(z_1d\cup\alpha)$ is a tame arc, we have $f(z_1d)$ containing $f(\alpha)$. Therefore $f(z_1d\cup\beta)$ contains J', contrary to the fact that J' is not an arc. Therefore for every point $z \varepsilon J$ we have $z = f^{-1}f(z)$.

Now let z be a point of $M^n \setminus J$ and let α be a tame arc through z spanning J such that $\alpha \cup J$ is tame and such that α divides J into two arcs β and γ. From above and Theorem 6.8. it follows that as $\alpha \cup \beta$ is tame $f(\alpha \cup \beta)$ is a simple closed curve. Therefore $z = f^{-1}f(z)$ and therefore f is 1:1 on M^n.

Theorem 6.10. If $f|M^n$ preserves tame arcs, where M^n is a compact triangulated n-manifold, and if there exists a tame simple closed curve J is M^n such that $f(J) = J'$ is not an arc or point then f is a homeomorphism on M^n.

Proof: This is an immediate corollary of Theorem 6.9. and Theorem 6.4.

An arc α is a free arc of M provided α spans $\overline{M \setminus \alpha}$.

Theorem 6.11. Let J be a tame simple closed curve in a compact triangulated n-manifold M^n and let $f|M^n$ preserve tame arcs. Suppose $f|M^n$ is not a homeomorphism, then $f(J) = J'$ is a free arc or point of $f(M^n)$.

Proof: From Theorem 6.8. we see that $f(J) = J'$ is a homeomorphism or J' is an arc or point. If $f(J) = J'$ is a homeomorphism, then by Theorem 6.10. we see that $f|M^n$ is a homeomorphism. Hence assume J' is an arc a'x'b' of $f(M^n)$ which is not a free arc of $f(M^n)$. To obtain the desired contradiction we prove:

(1) There exists a tame arc uv in M^n such that f(uv) contains a nondegenerate subarc u'v', having exactly the point v' in common with J', where v' is an interior point of J'. We may further suppose $uv \cup J$ is tame.

Proof of (1): As J' is not a free arc of $f(M^n)$ there exists a sequence of points $\{x_n'\}$ of $f(M)^n \setminus J'$ converging to an interior point x' of J'. As M^n is a manifold, via Theorem 6.3. there exists a tame arc N in M^n meeting infinitely many of the $f^{-1}(x_n')$ such that $N \cup J$ is tame. Then f(N) contains a nondegenerate subarc u'v' satisfying the conditions of (1) and thus we obtain the arc uv as a subarc of N joining a point of $f^{-1}(u')$ to a point of $f^{-1}(v')$. This establishes (1).

Now as M^n is a manifold there exists a tame arc H in M^n intersecting $f^{-1}(u')$ and having endpoints in $f^{-1}(a')$ and $f^{-1}(b')$ such that $H \cup J$ is tame via Ayre's Three Point Theorem and by replacing an arc by a polygonal arc. Now f(H) contains J'. For suppose not; then by Theorem 6.7. there exists a tame simple closed curve J'' in M^n such that f(J'') is not an arc or point. Then by Theorem 6.10. $f|M^n$ is a homeomorphism contrary to hypothesis. Now f(H) contains both u' and v' and hence a subarc joining these points. Since f(H) is an arc, this subarc contains either a' or b', hence assume it contains a' and denote it by u'a'v'. From Theorem 6.7. it follows that if M is any tame arc in M^n having its endpoints in $f^{-1}(u')$ and $f^{-1}(v')$ such that $M \cup J$ is tame, then f(M) contains u'a'v'. But this says that the image of the tame arc uv given by (1) must contain a simple closed curve. This contradiction completes the proof.

Theorem 6.12. Let $\{\alpha_i\}_{i=1}^{q}$ be a finite collection of tame arcs in a compact triangulated manifold M^n such that their union A is tame. If a, b and c are three distinct points of M^n then there exists a tame arc α_{q+1} meeting a, b and c such that $\alpha_{q+1} \cup A$ is tame.

Proof: As A is tame there exists a homeomorphism $h: M^n \to M^n$ taking A to a standard polyhedron. Now there exists a tame arc uv which meets h(a), h(b) and h(c). As M^n is a manifold we may replace uv by a polyhedral arc u'v' such that $u'v' \cup h(A)$ is a polyhedron. Thus $h^{-1}(u'v') = \alpha_{q+1}$ is the desired arc.

Theorem 6.13. Let $f|M^n$ preserve tame arcs, where M^n is a compact triangulated n-manifold. If $f(M^n)$ is not an arc or point, then f is a homeomorphism.

Proof: Suppose f is not a homeomorphism. Then by Theorem 6.11. the image of every tame simple closed curve J of M^n is a free arc or point of $f(M^n)$ and consequently every two points of $f(M^n)$ lies on a free arc of $f(M^n)$. As M^n is a compact manifold it is strongly tame arcwise connected and therefore $f(M^n)$ is strongly arcwise connected and is hence compact and locally connected. Thus $f(M^n)$ is a simple closed curve.

Let ω be a tame arc in M^n. Then we show that if z' is any point in $f(M^n)$ then there exists a tame arc β of M^n such that $\omega \cup \beta$ is tame and z' is an interior point of $f(\beta)$. If z' is an interior point of $f(\omega)$ we are through. First suppose z' is an endpoint of $f(\omega) = y'z'$. Since $f(M^n)$ is a simple closed curve, it contains an arc p'z'q' having z' as an interior point. Suppose $f(\omega) \cap p'z' = z'$. Let $\{p_n'\}$ be a sequence of points of the arc p'z' converging monotonically to z'. Now $\omega \cap f^{-1}(p_n') = \emptyset$. Choose a sequence of points $\{x_n\}$, $x_n \varepsilon f^{-1}(p_n')$. As M^n is compact by Theorem 6.3. there exists a tame arc α passing through infinitely many of $\{x_n\}$ such that $\alpha \cup \omega$ is tame. Thus we obtain an arc $\gamma_p = z'x'$ which is a subarc of both z'p' and the image of a tame arc of M^n where $\alpha \cup \omega$ is also tame. As M^n is a manifold by Theorem 6.12. let β be a tame arc in M^n intersecting $f^{-1}(x')$, $f^{-1}(y')$ and $f^{-1}(z')$ such that $\beta \cup \omega \cup \alpha$ is tame. By virtue of Theorem 6.7. $f(\beta)$ contains $\gamma_p \cup f(\omega)$ an arc which has z' as an interior point.

Now suppose $z' \cap f(\omega) = \emptyset$. As $f(M^n)$ is a simple closed curve there exists an arc $p'z'q'$ having z' as an interior point. Let $\{p_n'\}$ be a sequence of points in the arc $p'z'$ converging monotonically to z'. As M^n is a manifold let α be a tame arc in M^n intersecting infinitely many of the sets $f^{-1}(p_n')$ such that $\alpha \cup \omega$ is tame by Theorem 6.3. Thus we obtain an arc $\gamma_p = z'x'$ which is a subarc of both $z'p'$ and the image of a tame arc of M^n. Likewise we obtain an arc $\gamma_q = z'y'$ which is a subarc of $z'q'$ and the image of a tame arc α' in M^n such that $\alpha' \cup \alpha \cup \omega$ is tame. By Theorem 6.12. let β be a tame arc in M^n intersecting $f^{-1}(x')$, $f^{-1}(y')$ and $f^{-1}(z')$ such that $\beta \cup \alpha' \cup \alpha \cup \omega$ is tame. By virture of Theorem 6.7. $f(\beta)$ contains $\gamma_p \cup \gamma_q$ an arc which has z' as interior point. Now if $z' = f(\omega)$, the same construction will yield the desired β.

We now wish to show if $\alpha_1, \alpha_2, ---, \alpha_n$ is a collection of tame arcs in M^n whose union is tame such that no $f(\alpha_i)$ is a point and if $z' \varepsilon f(M^n)$ does not lie in the interior of the image of one of these arcs, then there exists a tame arc α_{n+1} such that $\alpha_1, \alpha_2, ---, \alpha_n, \alpha_{n+1}$ is a collection of tame arcs whose union is tame and such that z' lies in the interior of $f(\alpha_{n+1})$. First if $z' \cap \bigcup_{i=1}^{n} f(\alpha_i) = \emptyset$ then we can construct α_{n+1} as β was constructed in the preceeding paragraph. Second if z' is the endpoint of one of the $f(\alpha_i)$ but for no j is it contained in the interior of $f(\alpha_i) \cup f(\alpha_j)$, then the construction used in the second paragraph above will give the desired α_{n+1}. Thus suppose z' is the endpoint of $f(\alpha_i)$ and $f(\alpha_j)$, $i \neq j$, and that z' belongs to the $\text{int}(f(\alpha_i) \cup f(\alpha_j))$. As $\alpha_1 \cup \alpha_2$ is tame and as $f(\alpha_1 \cup \alpha_2) = p'z'q'$, an arc containing z', there exists a tame arc α_{n+1} with the desired properties by Theorem 6.12. and Theorem 6.7. passing through $f^{-1}(p')$, $f^{-1}(z')$ and $f^{-1}(q')$.

Thus it follows from the Heine-Borel Theorem that there exists a finite number of tame arcs α_1, α_2, ---, $\alpha_n (n \geq 2)$ in M^n such that their union is tame and such that $\bigcup_{i=1}^{n} f(\alpha_i) = \bigcup_{i=1}^{n} \alpha_i' = f(M^n)$. Moreover, the α_i may be so selected and named that $\alpha_i' \cap \alpha_k' = \emptyset$ except for k=i-1, i, i+1(n+1 = 1). Let p_i' and q_i' be the end points of α_i' and assume them so named that p_{i+1}' is a point of α_i' and while q_{i+1}' is not. Select in M^n a tame arc $\beta_1 = p_1q_1 \cup q_1q_2$, where the points p_1, q_1, and q_2 are in $f^{-1}(p_1')$, $f^{-1}(q_1')$ and $f^{-1}(q_2')$ respectively such that $\beta_1 \cup \alpha_1 \cup \alpha_2 \cup \text{---} \cup \alpha_n$ is tame. By Theorem 6.7. $f(p_1q_2)$ contains α_1' and consequently p_1q_2 contains a point p_2 of $f^{-1}(p_2')$. The subarc p_2q_2 of β_1 has an image which contains α_2' by Theorem 6.7. Therefore $f(\beta_1)$ contains $\alpha_1' \cup \alpha_2'$. To complete the induction assume that a tame arc β_r has been obtained in M^n such that $\beta_r \cup \beta_{r-1} \cup \alpha_r \cup \alpha_{r+1} \cup \text{---} \cup \alpha_n$

is tame and such that $f(\beta_r)$ contains $\alpha_1'\cup\alpha_2'\cup\text{---}\cup\alpha_{r+1}'$. Select in M^n a tame arc $\beta_{r+1} = p_1q_{r+1}\cup q_{r+1}q_{r+2}$, where the points p_1, q_{r+1} and q_{r+2} are in $f^{-1}(p_1')$, $f^{-1}(q_{r+1}')$ and $f^{-1}(q_{r+2}')$ respectively such that $\beta_{r+1}\cup\beta_r\cup\alpha_r\cup\text{---}\cup\alpha_n$ is tame. By applying Theorem 6.7. as above it follows that $f(p_1q_{r+1})$ contains $\alpha_1'\cup\text{---}\cup\alpha_{r+1}'$ and finally $f(\beta_{r+1})$ contains $\alpha_1'\cup\alpha_2'\cup\text{---}\cup\alpha_{r+2}'$. Therefore, by induction, there exists a tame arc β_{n-1} in M^n such that $f(\beta_{n-1}) = f(M^n)$ which as a consequence of our supposition is a simple closed curve.

Theorem 6.14. If C is a compact n-cell and $f|C$ is a 1:1 function which preserves tame arcs, then $f|C$ is a homeomorphism.

Proof: As C is compact we need only establish continuity. Suppose $f|C$ is not continuous. Assume $\{x_i\}$ is a sequence of points converging to x such that $\{f(x_i)\}$ does not converge to f(x). Let $\{x_j\}$ be an infinite subsequence of $\{x_i\}$ such that f(x) is not a limit point of $\{f(x_j)\}$. We can pass a tame arc through infinitely many points of $\{x_j\}$. But a repetition of the argument used in Theorem 6.7 yields the contradiction that a subsequence of $\{f(x_j)\}$ does converge to f(x).

Theorem 6.15. If C is a closed n-cell, $f|C$ is a function which preserves tame arcs and if dim $(f(C))>1$, then $f|C$ is a homeomorphism.

Proof: Without loss of generality suppose C is a closed n-cube. Let $p\varepsilon C$ be a point such that dim $(f(p))>1$. Suppose $f|C$ is not 1:1. Let x,y be such that $f(x) = f(y)$. But $x\cup y\cup p$ can be made to lie on the boundary of a n-cube. Therefore, by Theorem 6.13. $f(x) \neq f(y)$ and so f is 1:1. Thus, by Theorem 6.14., f is a homeomorphism.

Theorem 6.16. Let f be a function whose domain is a triangulated n-manifold M_1 and whose range is in a n-manifold M_2. If f preserves tame arcs and dim $(f(M^n))>1$, then f is a homeomorphism.

Proof: Analogous proof to Theorem 6.1. can be given Mutis Mutandis with Theorem 6.15. replacing 4.1 of Hall and Puckett [4].

Conclusion:

Theorem 6.17. Let f be a function whose domain is a triangulated n-manifold M_1 with range in a n-manifold M_2 and suppose dim $f(M^n)>1$. Then the following are equivalent:

(a) f preserves tame arcs;

(b) f is a homeomorphism;

(c) f preserves arcs.

Corollary 6.18. A manifold M^n, $n>1$, fails to be triangulatalbe if and only if every function f from a triangulatable manifold onto M fails in one of the following equivalent ways:

(a) f fails to take some tame arc onto an arc or point;

(b) f fails to take some arc onto an arc or point;

(c) f fails to be a homeomorphism.

Theorem 6.19. Let M^n be a triangulated n-manifold or a triangulated compact n-manifold. If f is a function defined on M^n such that f takes tame arcs onto connected tame sets and such that for each tame arc $A \varepsilon M^n$ $f(A)$ contains no triod, then f is a homeomorphism if $f(M^n)$ is not an arc or point and in addition for the non-compact case is not a simple closed curve.

Proof: If M^n is compact, this theorem is a corollary to Theorems 5.5. and 6.13. If M^n is not compact this theorem is a corollary to Theorems 5.5. and 6.16.

Corollary 6.20. Let M^n be a triangulated n-manifold. If f is a map defined on M^n such that f takes tame arcs onto tame sets and such that for each arc $A \varepsilon M^n$ $f(A)$ does not contain a triod, then f is a homeomorphism if $f(M^n)$ is not an arc, point or simple closed curve.

Corollary 6.21. If f is a map defined on S^n such that f takes tame arcs onto tame sets and such that for each tame arc $A \varepsilon S^n$, $f(A)$ does not contain a triod, then f is a homeomorphism if $f(S^n)$ is not an arc or point.

VII. DEGREE OF SKEWNESS

If C is a set in E^n or S^n while C lies on a strong homeomorph of an $E^p \subset E^n$ or flat $S^p \subset S^n$ and if this is not true for any dimension less than p, then the degree of skewness of C is p.

Theorem 7.1. Consider E^n (or S^n) and let f be a map defined on E^n (or S^n) such that f takes tame arcs onto tame sets. Then the image of no tame arc contains a triod if and only if the degree of skewness of every tame arc is preserved, provided $f(E^n)$ (or $f(S^n)$) is not a point, arc or simple closed curve.

Proof: Theorem 6.16. contains one part of the implication. If the degree of skewness of an arc is preserved then the image of the arc can not contain a triod and the theorem follows.

This theorem, in light of Theorem 6.19., indicates that the no-triod condition is richer than the degree of skewness condition. As both the no-triod and the degree of skewness condition imply that the dimension of subsets is preserved, perhaps preservation conditions on the dimension of subsets could be used to study larger classes of tame functions.

VIII. TAME DIMENSION RAISING MAPS

There are tame maps which raise the dimension of sets.

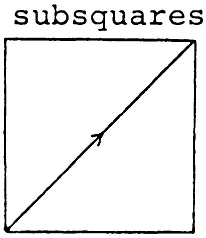
Theorem 8.1. Let K be a n-dimensional complex and let S be the unit square $[0,1]x[0,1]$. There exists a map F of K onto S such that the image of every tame set is tame.

Proof: As we can retract K onto a closed line, the following two theorems suffice.

Theorem 8.2. If I is the unit interval and S the unit square there exists a map f from I onto S such that the image of every subinterval is a 2-cell.

Proof: Consider E. H. Moore's Crinkly Curve C^3, [8]. It is a map $f:I \to S$ given as the uniform limit of a sequence of continuous maps $f_n(t) = (\phi_n(t), \psi_n(t))$ whose images are arcs C_n^3. Below we indicate f_0, f_1 and f_2 by giving C_0^3, C_1^3, and C_2^3.

3^0 subsquares

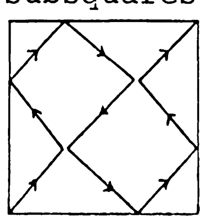


C_0^3

0 1

3^0 intervals

3^2 subsquares

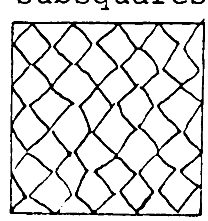


C_1^3

0 1

3^2 intervals

3^4 subsquares



C_2^3

0 1

3^4 intervals

The construction of C_n^3 is such that if $f_n(t_\alpha)$ is an end-point of a line segment in a square corresponding to the subdivision of S for C_n^3 then $f_m(t_\alpha) = f_n(t_\alpha)$ for all $m \geq n$. Further, if I' is an interval of the partition of I corresponding to S' of the partition of S for C_n^3, then $f(I') = S'$ (see Hilbert [5]).

Let [a,b] be any subinterval of [0,1] which is not a subinterval in any of the partitions of [0,1] or a finite union of subintervals from a number of partitions. Let N be the smallest integer such that [a,b] contains an interval $I_N^1 = [a_1, b_1]$ of the N-th partition of [0,1]. First consider $[a, a_1]$ which may contain other intervals of this partition. If so set $I_N^i = [a_i^N, a_{i-1}]$, where i = 2, ---, r(N); $r(N) \leq 8$. The interval $[a, a_{r(N)}^N]$ may not be empty. In this case let N+p(1) be the smallest integer such that $[a_1, a_{r(N)}^N]$ contains an interval of the N+p(1)-st partition of [0,1]. As above set

$$I_{N+p(1)}^1 = [a_1^{N+p(1)}, a_{r(N)}^N] \text{ and } I_{N+p(1)}^i = [a_i^{N+p(1)}, a_{i-1}^{N+p(1)}]$$

where i = 2, ---, r(N+p(1)); $r(N+p(1)) \leq 8$. So define

$$I_{N+p(j)}^1 = [a_1^{N+p(j)}, a_{r(N+p(j-1))}^{N+p(j-1)}] \text{ and } I_{N+p(j)}^i = [a_i^{N+p(j)},$$

$a_{i-1}^{N+p(j)}]$ where i = 2, ---, r(N+p(j)); $r(N+p(j)) \leq 8$; if

$[a, a_{r(N+p(j-1))}^{N+p(j-1)}]$ is not empty. Set $\{I^\alpha\} = \{I_N^1, I_N^2, ---, I_N^{r(N)},$

$I_{N+p(1)}^1, ---, I_{N+p(j)}^{r(N+p(j))}, ---\}$.

Now similarly define $\{J^\beta\} = \{I_N^1 = J_N^1, ---, J_N^{r(N)}, J_{N+q(1)}^1,$ $---, J_{N+q(1)}^{r(N+q(1))}, ---, J_{N+q(j)}^{r(N+q(j))}, ---\}$ where $J_{N+q(j)}^{r(N+q(j))} \subset$ $[b_1,b]$. Note the two sets $\{I^\alpha\}$ and $\{J^\beta\}$ may be empty, finite or countably infinite. From above we assume at least one is countably infinite.

Let $p = \lim_{\alpha\to\infty} S^\alpha$ and $q = \lim_{\beta\to\infty} S^\beta$ where $S^\alpha = f(I^\alpha)$ and $S^\beta = f(J^\beta)$ in case $\{I^\alpha\}$ and $\{J^\beta\}$ are infinite. To justify this we observe that by construction these limits exist and are single points as S^α and $S^{\alpha+1}$ meet in the face of $S^{\alpha+1}$ on the face of S^α and $\lim_{\alpha\to\infty} (\operatorname{diam}(S^\alpha)) = 0$. We will show that both p and q are limit points of $S \setminus f([a,b])$ and hence belong to $Bd(f([a,b]))$. Consider q and $f([0,b])$. Let M be the smallest integer such that $\overline{S \setminus f([0,b])}$ contains a square of M-th partition of S. Order the squares in the M-th partition in the natural way and let $S_M^{s(M)}$ be the first square of this partition that does not contain a point of $f([0,b])$ in its interior and let $S_M^{s(M)-1}, ---, S_M^2, S_M^1;$ $s(M) \leq 8$, be the other squares of this partition in $\overline{S \setminus f([0,b])}$ where their numbering is the reverse of their natural order. Let $M + u(1)$ be the smallest integer such that

$$\overline{S \setminus [f([0,b]) \cup (\bigcup_{i=1}^{s(M)} S_M^i)]}$$ contains a square of the $M + u(1)$-st partition of S. As before, construct a sequence of squares of this partition $S_{M+u(1)}^{s(M+u(1))}, ---, S_{M+u(1)}^2, S_{M+u(1)}^1;$

$s(M+u(1)) \leq 8$. Note this new sequence may be null. In general if $S \setminus [f([0,b]) \cup (\bigcup_{i=1}^{s(M)} S_M^i) \cup (\bigcup_{i=1}^{s(M+u(1))} S_{M+u(1)}^i) \cup --- \cup (\bigcup_{i=1}^{s(M+u(v))} S_{M+u(v)}^i)] \neq \emptyset$ then construct as before the sequence of squares $S_{M+u(v+1)}^{s(M+u(v+1))}, ---, S_{M+u(v+1)}^2, S_{M+u(v+1)}^1$; $s(M+u(v+1)) \leq 8$. Combining these sequences of squares form the sequence $\{S^\gamma\} = \{S_M^1, ---, S_M^{s(M)}, ---, S_{M+u(1)}^{s(M+u(1)}, ---\}$. Observe $\{S^\gamma\}$ and $\{S^\beta\}$ are finite or infinite as the other is finite or infinite. Again assume the sequence is infinite and define $o = \lim_{\gamma\to\infty} S^\gamma$. This follows from the construction used to obtain $\{S^\gamma\}$. Suppose for the points q and o, $q \neq o$. Consider the set $\{f(J_N^1), ---, f(J_N^{r(N)}), ---, f(J_{N+q(j)}^{i(N+q(j))}), ---\} \cup \{S_M^1, ---, S_M^{s(M)}, ---, S_{M+u(v)}^{s(M+u(v))}, ---\}$ from which we construct the sequence $\{S^\theta\}$ using the natural lexographic order obtained by first ordering the partition numbers, then giving $f(J_R)$'s precedence over S_R's in case there are J's and S's in the same partition, and by finally, in the given partition, giving the constructed order to the squares. Notice that diam $(S \setminus [f([0,a_1]) \cup (\bigcup_{\theta=1}^{v} S^\theta)]) \to 0$ as $v \to \infty$ by virtue of the nested partitions used to construct C^3. This implies $q = o$. Thus q is a limit point of $f([a,b])$ and its exterior.

To show that int(f([a,b])) is simply connected let C be an arbitrary cycle in int(f([a,b])). Thus C lies in the interior of the connected union of a finite number of squares. For if not p or q lie on C and hence C is not in the interior of f([a,b]). Thus int(f([a,b])) is simply connected if and only if it contains no "holes". Suppose H is an hole in int(f([a,b])). Let M_1 be the smallest integer such that H contains a square, say H_1, of the M_1-st partition of S. Let M_2 be the smallest integer such that f([a,b]) contains a square S_{M_2} of the M_2-st subdivision of S. Set $M = \max(M_1, M_2)$. Consider C^3_M with the natural order given to the open segments in the squares of the M-th partition of S. Suppose if $a \varepsilon \operatorname{int}(H_1 \cap C^3_M)$ and $b \varepsilon \operatorname{int}(S_{M_2} \cap C^3_M)$ then $a<b$ in the order given to the segments of C^3_M above. The segments of C^3_M induce a natural order in the set of squares of the M-th partition of S. Consider the subsequence of all subsquares $\{S^i_M\}$ in the M-th partition which precede H_1 or the subsquares of H_1 as the case may be. Now the interior of the union of this set of subsquares intersected with int(f([a,b])) (ie. $\operatorname{int}(\bigcup\limits_{i=1} S^i_M) \cap \operatorname{int}(f([a,b])) \neq \emptyset$) can not be empty, for otherwise the hole H would meet Bd(S) at the point (0,0) which is not possible. Thus let M_3 be the smallest integer such that $\overline{\operatorname{int}(f([a,b]) \cap (\bigcup\limits_{i=1} S^i_M)}$ contains a square S_{M_3} of the M_3-th subdivision of S. Set

$M_a = \max(M_1, M_2, M_3)$. Consider the set of subsquares of the M_a-th partition of S with their natural order. Clearly, a subsquare (or square) of H_1 separates a subsquare (or square) of S_{M_3} from a subsquare (or square) of S_{M_2} in this natural order which is not possible as int(f([a,b])) is connected in the sense of such a sequence of squares.

Now by the Hahn-Mazurkiewicz Theorem f([a,b]) is a Peano Space and hence compact and locally connected. Thus by Theorem 13.1 page 160 Newman [9] int(f([a,b])) is uniformly locally connected. Hence, by Theorem 16.2 page 167 Newman [9], Bd(f([a,b])) is a simple closed curve as int(f([a,b])) is simply connected and uniformly locally connected. Thus by the Schoenflies Theorem f([a,b]) is a 2-cell.

Theorem 8.3. Let $f: I \to S$ be the map for Moore's Crinkly Curve C^3 given above and let $\{R_i\}_1^p$ be a finite disjoint sequence of closed intervals in I. Then $\bigcup_{i=1}^{p} f(R_i) = f(\bigcup_{i=1}^{p} (R_i))$ is polyhedral.

Proof: By the previous theorem each $f(R_i)$ is a 2-cell. By construction $f(R_i)$ and $f(R_j)$ meet in their boundaries if their intersection is not null. If $f(R_1) \cap f(R_2)$ consists of a finite number of components then $f(R_1) \cup f(R_2)$ is a polyhedral set. If $[f(R_1) \cup f(R_2)] \cap f(R_3)$ is a finite number of components then $f(R_1) \cup f(R_2) \cup f(R_3)$ is polyhedral and $f(R_1) \cap f(R_3)$ and $f(R_2) \cap f(R_3)$ both consist of a finite number of components and conversely. By induction, this holds for all $i \leq p$. Hence it is sufficient to show that $f(R_i) \cap f(R_j)$ consists of a finite number of components.

By construction $f(R_i)$ and $f(R_j)$ are 2-cells which meet in their boundaries. Suppose $f(R_i) \cup f(R_j)$ contains two or more holes, say H_1 and H_2 in the evident sense. Let $S_{N(1)}$ and $S_{N(2)}$ be the largest squares in H_1 and H_2 respectively belonging to partitions $N(1)$ and $N(2)$ of S. Let $N = \max (N(1), N(2))$. Form the sequence $\{S_N^\omega\}$ of squares by taking the squares of the N-th partition in their natural order. Suppose in this sequence the finite subsequence $\{S_{N(1)} \cap S_N^\omega\}$ precedes the finite subsequence $\{S_{N(2)} \cap S_N^\omega\}$. Let $S_N^{\omega(1)}$ be the last square in $\{S_{N(1)} \cap S_N^\omega\}$ and let $S_N^{\omega(2)}$ be the first square in $\{S_{N(2)} \cap S_N^\omega\}$. Consider the subsequence of $\{S_N^\omega\}$, consisting of the squares between $S_N^{\omega(1)}$ and $S_N^{\omega(2)}$ ie, $\{S_N^{\omega(1)}, S_N^{\omega(1)+1}, ---, S_N^{\omega(2)}\}$. Now $\bigcup_{i=\omega(1)}^{\omega(2)} (\mathrm{int}(S_N^i))$ must meet $f(R_i) \cup f(R_j)$ in a set with non-void interior; otherwise $H_1 = H_2$. Let N_3 be the smallest integer such that there exists a square $S_{N(3)}$ of $N(3)$-rd partition in $[f(R_i) \cup f(R_j)] \cap \bigcup_{i=\omega(1)}^{\omega(2)} (S_N^i)$. Similarly, consider the sequences of squares $\{S_N^1, S_N^2, ---, S_N^{\omega(1)}\}$ and $\{S_N^{\omega(2)}, S_N^{\omega(2)+1}, ---, S_N^{3^{N+1}}\}$ and hence let $N(4)$ and $N(5)$ be the smallest integers corresponding to the largest squares $S_{N(4)}$ and $S_{N(5)}$ in $[f(R_i)\cup f(R_j)]\cap(\bigcup_{i=1}^{\omega(1)} S_N^i)$ and $[f(R_i)\cup f(R_j)]\cap(\bigcup_{i=\omega(2)}^{3^{2N}} S_N^i)$ respectively. Let $M = \max(N, N(3), N(4), N(5))$. From this M-th partition, form a sequences of squares in their natural order. Now $f(R_i)\cup f(R_j)$ meets this sequence of squares in at least three non-adjacent (in the sequence) subsequences of squares as the sequence meets $S_{N(4)}$, $S_{N(3)}$ and $S_{N(5)}$. However, the images of the intervals in R_i and R_j corresponding to the M-th partition of [0,1] form two subsequences of adjacent squares. This contradiction establishes that there is at most one hole.

Suppose now $f(R_i)\cap f(R_j)$ is not just two components. Coherently orient the boundaries of $f(R_j)$ and $f(R_i)$. Let $(ab)_i = [a,b]$ be an irreducible subarc of $Bd(f(R_i))$ with order agreeing with that of $Bd(f(R_i))$ which contains $f(R_i)\cap f(R_j)$. Similarly define $(ab)_j = [a,b]$ to be the irreducible subarc for $Bd(f(R_j))$. Now $f(R_i)\cap f(R_j)$ is a closed subset of both $(ab)_i$ and $(ab)_j$. Hence the complement of $f(R_i)\cap f(R_j)$ in each arc is a countable union of open arcs. Let $(a_1b_1)_i = (a_1,b_1)$ and $(a_2b_2)_i = (a_2,b_2)$ be two of these open intervals in $(ab)_i$. Now they must correspond to arcs $(a_1b_1)_j = (a_1,b_1)$ and $(a_2b_2)_j = (a_2,b_2)$ in $(ab)_j$. So $C_1 = \overline{(a_1b_1)_i}\cup(a_1b_1)_j$ and $C_2 = \overline{(a_2b_2)_i}\cup(a_2b_2)_j$

are simple closed curves bounding holes in $f(R_i) \cup f(R_j)$. This is not possible. Thus $f(R_i) \cap f(R_j)$ consists of at most two closed components.

To see that the inverse of the map f above does not take tame sets onto tame sets, consider the inverse image of the diagonal $(f(0), f(1))$ of S. This set is not a polyhedron.

IX. K-R MANIFOLDS AND ONE-TO-ONE MAPS

Most of the results in Section VI are for manifolds. Similiar results for manifolds with boundaries are not true, even if the tame arc preserving functions are replaced by 1:1 maps which certainly preserve all arcs. In this section are some elementary results for 1:1 maps on spaces more general than manifolds. We give an example of a 1:1 non-topological map which does not preserve tame sets. Also are characterized the 1:1 maps from a 2-dimensional manifold with a single boundary component onto a 2-dimensional manifold with a single boundary component.

Theorem 9.1. Let M_1 and M_2 be two n-dimensional manifolds with boundaries and let $f:M_1 \to M_2$ be a 1:1 map, then $f(\text{int}(M_1)) \subset \text{int}(M_2)$.

Proof: Let $p \varepsilon \text{int}(M_1)$ and let $B_p(\varepsilon)$ be a closed ε-ball centered at p. Now $f|B_p(\varepsilon)$ is a homeomorphism and hence $f(\text{int}(B_p(\varepsilon)))$ is topologically E^n.

Corollary 9.2. In the above theorem, $f|\text{int}(M_1)$ is a homeomorphism.

Proof: This follows as f is a 1:1 map of a topological E^n into a topological E^n.

Example 9.3. There is a 1:1 non-topological map from the 3-manifold with boundary $X = Dx[0,1]$, D the open unit 2-disk, to the 3-manifold $Y = DxS^1$, S^1 the one sphere, which preserves tame sets.

Proof: To see this, separate Y at Dx0 so that the resulting cylinder is X. The inverse of this separating operation is easily seen to yield a 1:1 non-topological map.

Let f be this 1:1 pasting map, let P be a tame set in X, and let C^3 be a polyhedral 3-cell in X in which P is tamely imbedded. Now $f(C^3)$ is a 3-cell in Y. Note that $C^3 \cap Bd(X)$ can be taken to be a tame disk. By Bing's or Moise's paper on locally tameness in 3-manifolds, since $f(C^3) \subset Y$ is locally tame, $f(C^3)$ is tame in Y.

A manifold with boundary M^n whose interior $int(M^n)$ is E^n and whose single boundary component $Bd(M^n)$ is E^{n-1} is an n-dimensional K-R manifold.

Example 9.4. (Doyle). There is a 1:1 non-topological map of a 3-dimensional K-R manifold onto E^2xS^1 that fails to preserve tame sets.

Proof: Let T^3 be the standard open solid torus in E^3 and D an almost polyhedral wild disk in T^3 that spans $Bd(T^3)$.

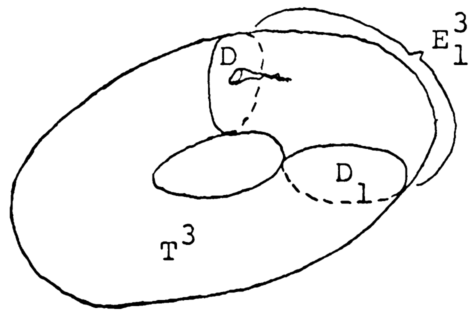


Figure 5

Let E_1^3 be a copy of E^3 as shown in Figure 5 and let D_1 be a nice disk as shown. As D is wild in T^3, $M_1^3 = E_1^3 \cup D_1$ is a pathological K-R manifold. If now we cut T^3 by separating D and E_1^3 we obtain a K-R manifold M_2^3 which is homeomorphic to M_1^3 as D is tame in M_2^3 by Theorem 5, Bing [1]. This homeomorphism induces a 1:1 map f of M_1^3 onto T^3. As D is wild in T^3 and D_1 is tame in M_1^3, f does not preserve tame sets.

We are able to characterize the 1:1 maps of a 2-dimensional K-R manifold onto a 2-dimensional K-R manifold.

Theorem 9.5. Let M_1 and M_2 be two 2-dimensional K-R manifolds. If f is a 1:1 map of M_1 onto M_2 then f is a homeomorphism.

Proof: Suppose $int(M_2) \setminus f(int(M_1)) \neq \emptyset$. As f is onto, there exists $p' \varepsilon [int(M_2) \setminus f(int(M_1))] \cap f(Bd(M_1))$. Let $q' \varepsilon Bd(M_2)$. Then there exists an arc $p'q' \varepsilon f(Bd(M_1))$. Order $p'q'$ from p' to q' and let r' be the first point of $p'q' \cap Bd(M_2))$. Now $p'r'$ meets $Bd(M_2)$ in the interior of a one disk D. Then $p'r' \cup \overline{D}$ is a triod. Let p', r_1' and r_2' be the ends of the triod. Then if $p = f^{-1}(p')$, $r_1 = f^{-1}(r_1')$ and $r_2 = f^{-1}(r_2')$, $p \cup r_1 \cup r_2$ lies in a closed 1-disk D_1 in $Bd(M_1)$. Thus as $f|D_1$ is a homeomorphism $p' \cup r_1' \cup r_2'$ lies on an arc in $f(Bd(M_1))$. But this arc does not lie in $p'r' \cup D$ as it is a triod. Therefore, $f(Bd(M_1))$ contains a simple closed curve. This cycle must separate M_2 and hence $f(int(M_1))$ as $f(int(M_1))$ is dense. This is not possible as $f(int(M_1))$ is connected.

It is conjectured that this theorem can not be improved upon by raising the dimension of the K-R manifolds.

Theorem 9.6. Let M_1 and M_2 be two n-dimensional spaces with boundaries obtained from two K-R manifolds by deleting from each K-R manifold all of its boundary except a point. If f is a 1:1 map of M_1 onto M_2 then $int(M_1)$ maps onto $int(M_2)$ and is a homeomorphism on the interior of M_1.

Proof: As in Theorem 9.1. $f(\text{int}(M_1)) \subset \text{int}(M_2)$. This implies as the boundaries of M_1 and M_2 are single points that $f(\text{Bd}(M_1)) = \text{Bd}(M_2)$. Thus $f(\text{int}(M_1)) = \text{int}(M_2)$.

Example 9.7. Let X be a 3-cell with all of its boundary removed except a point. Then there is a map f of X onto X which is 1:1 but is not a homeomorphism.

Proof: Consider the complex carried by the triangle $v_1v_2v_4$ illustrated in Figure 6.

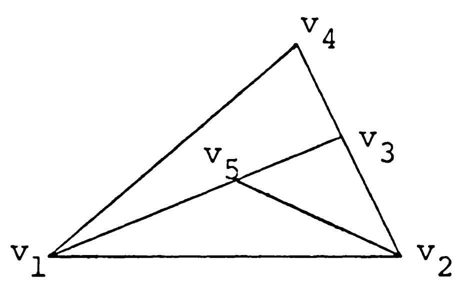


Figure 6

Let ϕ be the map of $v_1v_2v_4$ onto itself, defined by $\phi(v_1) = v_1$, $\phi(v_2) = v_2$, $\phi(v_3) = \phi(v_4) = v_4$ and $\phi(v_5) = v_5$ in the natural way. ϕ induces a natural map ϕ' on the cone K of $v_1v_2v_4$ obtained by suspending $v_1v_2v_4$ at v_6 not in the plane of the triangle. Set $f = \phi' | \{(K \setminus \text{Bd}(K)) \cup v_4\}$. It can be seen that by construction f is 1:1, and continuous.

A topological space X has a n-cell union if there is a sequence of closed n-cells $\{C_i\}$ such that $X = \bigcup_i C_i$ and $C_i \subset C_{i+1}$. If for each $x \varepsilon C_i$ there is an open set in C_{i+1} containing x, then X is said to have a strong n-cell union. Two topological spaces X and Y have similiar n-cell unions if there is a strong n-cell union for $X = \bigcup_i C_i$, an n-cell union for $Y = \bigcup_i C_i'$ and a sequence of homeomorphisms $\{h_i\}$ such that $h_i(C_i) = C_i'$ and $h_{i+1}|C_i = h_i$.

Theorem 9.8. (Doyle). If X has a strong n-cell union, then two topological spaces X and Y have similiar n-cell unions if and only if there is a 1:1 map of X onto Y.

Proof: Let h be a 1:1 map of X onto Y. Now $h|C_i$ is a homeomorphism as C_i is compact. Thus $h(C_i)$ is an n-cell. Therefore as $h(C_i) \subset h(C_{i+1})$, Y has a n-cell union similiar to X.

Let X and Y have similiar unions. Set $h = \lim_i h_i$. Suppose h is not 1:1. Let x and y be points such that $h(x) = h(y)$. We get a contradiction, for there exists an i such that $x, y \varepsilon C_i$. To show continuity, let $\{x_i\}$ be a convergent sequence in X, $x_i \to x$. Then there exists a j such that $x \varepsilon C_j$. By hypothesis there is an open set about x in C_{j+1}. Therefore, there is a subsequence of all but a finite number of $\{hx_i\}$ converging to $h(x)$ as there is an open set about x on which h is a homeomorphism.

As a consequence of this theorem, if X is a manifold with boundary and Y is a manifold and they have similiar n-cell unions, then there is a 1:1 non-topological map from X onto Y.

Theorem 9.9. Let X and Y have similiar n-cell unions. Then $h = \lim_i h_i$ is a homeomorphism on a neighborhood of each compact subset of X.

Proof: Suppose there is a compact set $K \subset X$ such that for no i is $K \subset C_i$. Thus for all i, $K \setminus C_i \neq \emptyset$. Choose $x_i \in K \setminus C_i$. Let $x_i \to x \in K$ as $x_i \in K$. But then there exists a j such that $x \in C_j$ and therefore there exists an open set $U \subset C_{j+1}$ containing x. So U contains all but a finite number of the x_i. This is a contradiction, for there exists a $N>0$ such that if $k \geq N$ then $x_k \in C_{j+1}$. Therefore h is a homeomorphism on a neighborhood of K.

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TRANSFORMATIONS PRESERVING TAME SETS

Harvey Johnson Charlton

An Abstract

If X is a complex with a triangulation and if P is a homeomorph of a polyhedron in X with respect to this triangulation, then P is tame in X if there is a homeomorphism h of X onto itself and another triangulation of X in which h(P) is a polyhedron. A function from one complex X into a complex is called tame and is said to preserve tame sets if for each tame set $P \subset X$, f(P) is tame.

Tame local homeomorphisms from triangulated n-manifolds into triangulated n-manifolds and tame light open maps of 2-manifolds into themselves are homeomorphisms. Connected complexes are compact if and only if every tame map of the complex into itself has a polyhedral image. Tame linear maps of Euclidean spaces and tame simplicial maps on triangulated n-manifolds with boundaries are homeomorphisms if their images are of dimension greater than one.

Functions from polyhedra into topological spaces which take tame arcs onto sets consisting of finite number of components have images of, at most, a finite number of components. If the function and its inverse takes tame sets onto tame sets then the image is connected, provided its image is in a complex. If the function is from a

topological space into a polyhedron, then it is continuous if and only if its inverse takes tame arcs onto closed sets. Finally a function from a complex to a complex is continuous if its inverse takes tame sets onto tame sets.

A function from an n-manifold into an n-manifold which has an image of dimension greater than one and which takes arcs onto arcs or points is a homeomorphism. A function from a compact triangulated n-manifold into a topological space which takes tame arcs onto arcs or points and whose image is not an arc or point is a homeomorphism. A function from a triangulated n-manifold into an n-manifold which takes tame arcs onto arcs or points and whose image is of dimension greater than one is a homeomorphism. A function from a triangulated n-manifold into a triangulated n-manifold which takes tame arcs onto connected tame sets such that the image of no tame arc contains a triod is a homeomorphism if its image set is not a point, arc or simple closed curve.

Finally there are tame maps which raise the dimension of sets. And there are 1:1 maps which do not preserve tame sets. A K-R manifold is a n-manifold with boundary whose interior is E^n and whose boundary is E^{n-1}. A 1:1 map of a 2-dimensional K-R manifold onto a 2-dimensional K-R manifold is a homeomorphism.