A FUNCTIONAL ANALYTIC APPROACH
TO MULTIGROUP TRANSPORT THEORY

by

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I. INTRODUCTION

Ever since Case[1] successfully applied the method of the singular eigenfunction expansion to the time-independent, one-speed, one-dimensional neutron transport equation with isotropic scattering, there have been efforts to generalize the treatment to the multigroup case. The basic equation considered for N groups is the following:

$$s\frac{\partial}{\partial x}f(x,s) + \Sigma f(x,s) - \int_{-1}^{1} C(s,t)f(x,t)dt = q(x,s), \quad -\infty<x<\infty; -1\leq s\leq 1, \qquad (1\text{-}1)$$

where f is a column vector whose elements f_i represent the neutron angular density in the i^{th} group, i=1,2..,N, and Σ is an invertible diagonal matrix whose i^{th} element σ_i is the total neutron cross section in the i^{th} group. Conventionally the σ_i's are ordered such that $\sigma_1>\sigma_2>..>\sigma_N=1$. The elements C_{ij} of the transfer matrix C represent the neutron scattering from group j to group i. The parameter s represents the cosine of the angle between the positive x axis and the velocity of neutrons. q is the neutron source.

The somewhat simplified isotropic case (in which C is independent of angle) is treated in detail by Siewert and Zweifel[2] for a rather special situation physically relevant to radiative transfer, namely the determinant of every minor matrix of C of rank greater than one was assumed to vanish. The more general case, in which the determinant of C was assumed not to vanish has turned out to be rather difficult. Part of the problem is notational (a difficulty also encountered in reference 2), because the continuum eigenvalues are highly degenerate and because adjoint solutions must be introduced to calculate expansion coefficients. This leads to a bookkeeping task of no small magnitude. However, more fundamental is the difficulty that a satisfactory proof of the half-range completeness of the singular eigenfunctions hinges on the signum of the so called partial indices of the matrix Riemann-Hilbert problem, and these indices turn out to be difficult to pin down. A review of the partial index problem, and references to some attempts to deal with the problem have been given by Burniston et.al.[3]

The notational difficulty alluded to above was solved in 1968 by Yoshimura and Katsuragi,[4] who also proved the relevant full-range completeness and orthogonality theorems. The more general case, in which C is a function of angles, was discussed by Silvenionnen and Zweifel,[5] where the further notational difficulties are handled, and sufficient conditions for full-range completeness presented. Half-range completeness can be deduced, in fact, if C is a symmetric matrix, as one obtains for example in thermal neutron problems with Maxwellian weights.

In 1973, a new development occurred in transport theory, namely the publication of a paper by Larsen and Habetler[6] in which the full and half-range formulas, originally obtained in reference 1 by heuristic arguments, were derived rigorously through functional analytic techniques. A later paper by Larsen[7] extended these results to the anisotropic case, still however, in one-group theory. These papers served not only to mollify the mathematicians who objected to Case's and his disciples' cavalier treatment of the continuous spectrum, but also gave the hope of simplifying and generalizing the original results. For example, a paper by Larsen, Sancaktar and Zweifel[8] based on the results of reference 6 has extended the original expansion theorems of Case to an enormously larger class of functions (the details are contained in appendix B). Basically, reference 6 made it possible to deal with linear operators in a Banach space, where previously one had to consider rather involved singular integral equations. Furthermore, a comparison with reference 4 shows that the Larsen-Habetler technique is simpler and clearer than the standard technique of obtaining adjoint solutions using Schmidt orthogonalization procedures and then calculating a large number of normalization integrals. But the major advantage of the Larsen-Habetler technique is that it can be applied to obtain the half-range eigenfunction expansion which, as it is noted earlier, has been hung up as the "partial indices" problem for a number of years.[3] (One should note, incidentally, a paper by Pahor and Suhadolc[9] whose ideas have some similarity to that of the present work, but whose arguments are still based on the original Case approach.)

Another advantage of the Larsen-Habetler technique is its suitability to generalization. For example, the anisotropic case could be treated just as easily as the isotropic case. In fact, this point appears in the course of the present work.

In Case's approach, one first applies separation of variables technique to the homogenous transport equation to reduce the problem to an s-dependent one only. Then one determines the eigensolutions (from now on the set of the proper and continuum eigenfunctions will be referred to as eigensolutions) of the reduced operator heuristically. Afterwards, one attempts to expand the solution in terms of these eigensolutions. The resulting equation is a singular integral equation for the expansion coefficients. In order to determine these coefficients, one is forced to prove completeness and orthogonality of the set of eigensolutions. For this purpose, one has to consider the adjoint solutions and also determine normalization constants.

As opposed to the above approach, the Larsen-Habetler technique naturally yields the above expansion coefficients and the completeness property of the eigensolutions. The problem is treated by separating the transport operator into the space derivative and an operator K^{-1} acting only on the variable s. The bounded inverse K of this operator is then considered. Using the resolvent operator $(zI-K)^{-1}$ of K, the expansion formula for the solutions of the homogenous transport operator is obtained, as a so-called eigenfunction expansion of the operator K.[9a]

The purpose of this work is to apply the Larsen-Habetler technique to multigroup transport theory to find explicit solutions for full and half-range problems with various transfer kernels. To achieve this purpose, the first step is to derive the eigenfunction expansion formula appropriate to the operator K. It turns out that this expansion formula has the form

$$f(x,s) = \sum_i a_i(x)\phi_i(s) + \int \Phi(t,s)A(x,t)dt \tag{1-2}$$

where f is an element in the domain of K (to be defined later); the summation over i extends over all the relevant eigenvalues of K for the problem in question, and the integration limits are -1,1 and 0,1 for the full and half-range cases. The ϕ_i's are the eigenfunctions of K, and each column of the matrix Φ is a so-called continuum eigenfunction of K. The a_i's and the column vector A are the expansion coefficients of the function f.

The second step is to use the above result and express the boundary conditions of the homogenous equation as an eigensolution expansion. Then the inhomogenous equation can be treated either by the Green's function approach or by expanding the source term in terms of the same eigenfunction expansion. Finally, the expansion coefficients of the solution can be determined in terms of the expansion coefficients of the source and the boundary conditions.

An explicit solution to the half-range problem for a subcritical medium will be given in terms of the matrices X and Y which factor the dispersion matrix Λ. However, canonical matrices, i.e., matrices whose inverses exist in the entire complex plane, are not required. For this reason, the question of partial indices never arises. The existence of the non-canonical matrices has been proved by Mullikin,[10] and although explicit solutions have not been found, they can be determined as the solutions of certain non-linear, non-singular integral equations. The present approach is reminiscent of that used by Siewert, Burniston and Kriese[11] for the two-group problem, their work being based on earlier work of Siewert and Ishiguro.[12] These authors introduce matrices H and $\tilde{H}$ which are, in fact, closely related to the inverses of the X and Y matrices used in this work. (Along these lines, one should also note a paper by Burniston, Mullikin and Siewert.)[13]

In the following chapters, the solutions of the transport equation with various transfer kernels are treated. The reduced transport operator K is introduced as follows:

$$\frac{\partial}{\partial x} f(x,s) + K^{-1} f(x,s) = q_0(x,s), \ s \neq 0, \qquad (1\text{-}3)$$

where K^{-1} is an unbounded operator acting only on the variable s and is defined by the following equation:

$$K^{-1} f(x,s) = \frac{1}{s}[\Sigma f(x,s) - \int_{-1}^{1} C(s,t) f(x,t) dt] , \ 0<|s|\leq 1, \qquad (1\text{-}4)$$

and

$$q_0(x,s) = q(x,s)/s, \ s \neq 0 \ .$$

In the next two chapters, a constant invertible transfer matrix C is considered. The full and half-range expansions of the solutions of the homogenous transport equation are found in terms of the eigensolutions of the bounded operator K, for subcritical media. Expressing Eq. (1-3) as

$$K\frac{\partial}{\partial x}f(x,s) + f(x,s) = Kq_0(x,s) \quad , \tag{1-5}$$

one may solve this equation rather than the transport equation (1-3). For this purpose, one multiplies Eq. (1-3) by K to get (1-5). One also has to check that the properties of Eq. (1-3) are not changed by this multiplication and that the domain of K is properly defined. This is done on page 13.

In the fourth chapter, the transport equation with a separable transfer matrix is studied. In this case, the operator K^{-1} is given by

$$K^{-1}f(x,s) = \frac{1}{s}[\Sigma f(x,s) + S(s)\int_{-1}^{1} L(t)f(x,t)dt] \ . \tag{1-6}$$

The conditions that S and L are invertible matrices are imposed. Again, the eigenfunction expansion formula of the operator K is obtained for full and half-range cases. The analysis of the two previous chapters proves to be invaluable for finding the desired eigensolutions for the degenerate transfer kernel case, in which rather involved notational problems are to be tackled.

In the fifth chapter, a finite sum of separable transfer kernels is considered. In this case, K^{-1} is given by

$$K^{-1}f(x,s) = \frac{1}{s}[\Sigma f(x,s) - \sum_{i=1}^{P} S_i(s)\int_{-1}^{1} L_i(t)f(x,t)dt] \ , \qquad (1\text{-}7)$$

where each S_i and L_i is assumed to be invertible. In this case, however, it was only possible to obtain the discrete eigenfunctions for the full-range case. The eigensolutions associated with the branch cut turned out to be impractical (perhaps impossible) to calculate because of technical difficulties arising from the cumbersome form of the dispersion matrix Λ.

In chapter six, non-singular, non-linear coupled integral equations are derived for the X and Y matrices which factor the dispersion matrix:

$$\Lambda(z) = Y(-z)X(z) \ . \qquad (1\text{-}8)$$

The analysis of this chapter depends heavily on Mullikin's work.

Chapter seven contains conclusions.

In appendix A, a reduction of the results of chapter two to the two-group, full-range results as obtained by Siewert and Shieh[14] is given.

Appendix B contains the extension of the one-speed Case formulas to L_p spaces and the formation of the resolution of the identity of the operator K for the same problem. The same approach can be used to extend the range of the multigroup formulas, although the work has not yet been carried out.

II. CONSTANT TRANSFER MATRIX: FULL-RANGE

Introduction

In this chapter, the full-range multigroup problem with a constant invertible transfer matrix C will be considered. The homogenous transport equation can be written as (for $0 \leq |s| \leq 1$)

$$\frac{\partial}{\partial x} f(x,s) + K^{-1} f(x,s) = 0 \tag{2-1}$$

where

$$K^{-1} f(x,s) = \frac{1}{s}[\Sigma f(x,s) - C\int_{-1}^{1} f(x,t)dt] \ , \ s \neq 0 \ . \tag{2-2}$$

For a fixed s, one would demand that the elements f in the domain of the transport operator have the following properties: $f_i(x,s)$ and $\partial f_i(x,s)/\partial x$ are continuous in x, $-\infty<x<\infty$ for i-1,2,..,N. The properties of f for a fixed x will be defined in the next section. The x-dependence of the operator involved in Eq. (2-1) suggests an exponential dependence of f on x. For simplicity of notation, the functions f will be considered to be evaluated at a constant x and the x dependence will be suppressed for the calculations of the eigensolutions of the reduced transport operator K.

The following convenient notation, which will save space especially in the anisotropic case, will be introduced:

$$\{f\}_n = \int_{-1}^{1} s^n f(s) ds \ ,$$

$$\{D\}_n(z) = \int_{-1}^{1} s^n D(z,s)ds \ , \tag{2-3}$$

where f is a column vector and D is an NxN matrix whose elements may be functions of one or two variables.

Since K^{-1} is an unbounded operator, the Larsen-Habetler technique of integrating the resolvent operator around a contour which contains the spectrum of the initial operator cannot be applied to determine its eigensolutions, since the spectral radius of K^{-1} is infinite. One may, however, determine the bounded inverse K of K^{-1} and then use the operator identity

$$I = \frac{1}{2\pi i}\oint_\tau (zI-K)^{-1}dz \ , \tag{2-4}$$

where τ is a simple, closed contour which surrounds the spectrum of K. For f in the domain of K, one may write

$$f(s) = \frac{1}{2\pi i}\oint_\tau (zI-K)^{-1}f(s)dz \ . \tag{2-5}$$

The contour τ can be suitably broken into smaller contours each of which surrounds an eigenvalue of K and the continuous part of the spectrum separately. Afterwards, each of these contour integrations can be carried out and their results can be associated with eigensolutions of the operator K. As a result, one obtains an eigenfunction expansion of f(s) where the expansion coefficients are explicitly given.

Once the eigensolutions of the operator K are obtained, one may attempt to solve the inhomogenous equation (1-5) with given boundary conditions by writing

$$K\frac{\partial}{\partial x}f(x,s) + f(x,s) = Kq_o(x,s) \ ; \ q_o(x,s) = \frac{1}{s}q(x,s) \ . \qquad (2\text{-}6)$$

Then the source and the solution can be expanded in terms of the eigensolutions of the operator K and can be substituted into the above equation. Finally the expansion coefficients of the solution can be determined from the boundary conditions and the expansion coefficients of the source.

The Resolvent Operator $(zI-K)^{-1}$

With the help of the notation given by Eq. (2-3), the operator K^{-1} may be written as

$$K^{-1}f(s) = \frac{1}{s}[\Sigma f(s) - C\{f\}_0] \ . \qquad (2\text{-}7)$$

The operator K can be calculated in a straightforward manner from the above equation as follows: let $g\varepsilon$ domain of K:

$$K^{-1}f(s) = g(s) \ . \qquad (2\text{-}8)$$

Then one may solve for f(s) to get

$$f(s) = \Sigma^{-1}[sg(s) + C\{f\}_0] \ . \qquad (2\text{-}9)$$

To eliminate $\{f\}_0$ on the right-hand side of the above equation, one integrates the equation over s and solves for $\{f\}_0$ in the resulting equation to get

$$\{f\}_0 = (I - 2C)^{-1}\Sigma^{-1}\{g\}_1 \ , \qquad (2\text{-}10)$$

Substituting Eq. (2-10) into Eq. (2-9) one obtains

$$f(s) = s\Sigma^{-1}g(s) + B^{-1}\{g\}_1 \ , \qquad (2\text{-}11)$$

where the NxN matrix B is defined by the equation

$$B = (\Sigma - 2C)C^{-1}\Sigma \ . \qquad (2\text{-}12)$$

From Eq. (2-11) one determines the operator K to be

$$Kg(s) = s\Sigma^{-1}g(s) + B^{-1}\{g\}_1 \; . \tag{2-13}$$

To make K a bounded operator, one first demands that the determinant of B is not zero; namely

$$\det(\Sigma - 2C) \neq 0 \; . \tag{2-14}$$

Secondly, one defines the domain D(K) of K to be the space of functions f in the following manner: define a space $X = \bigotimes_{i=1}^{N} L_2[-1,1]$ for every fixed x, and let $D(K)\varepsilon X$ such that $D(K) = \{f | f\varepsilon X, sf_i$ is Hölder continuous in $s\varepsilon[-1,1]\}$ with the norm

$$||f||^2 = \sum_{i=1}^{N} \int_{-1}^{1} |sf_i(s)|^2 ds < \infty \; . \tag{2-15}$$

Thus the operator K is bounded on the space defined above.

Now one may justify use of Eq. (1-5) rather than Eq. (1-3) as follows:

a) It will be assumed that 0 is not an eigenvalue of K so that $K\frac{\partial}{\partial x}f \neq 0$.

b) If $f(x,s)\varepsilon X$, then $\frac{\partial}{\partial x}f\varepsilon X$. This assertion will be shown to be true on page 28.

The determinant condition given above, incidentally, is related to the criticality of an infinite medium[15] and would reduce to $c \neq 1$ in the one-speed limit. In the analysis of reference 6, it was also required that $c \neq 1$.

Using Eq. (2-13) one may calculate the resolvent operator $(zI-K)^{-1}$ of K by writing:

$$(zI - K)h(s) = (zI - s\Sigma^{-1})h(s) - B^{-1}\{h\}_1 = f(s) \ , \qquad (2\text{-}16)$$

and then inverting the last equation (as was done before to get the operator K) to find the resolvent operator:

$$(zI - K)^{-1}f(s) = h(s) \ . \qquad (2\text{-}17)$$

First, one casts Eq. (2-17) into the form

$$h(s) = (zI - s\Sigma^{-1})^{-1}[f(s) + B^{-1}\{h\}_1] \ . \qquad (2\text{-}18)$$

To simplify the notation, the following definitions will be introduced here:

$$D(z,s) = (zI - s\Sigma^{-1})^{-1} \ ,$$

$$\Lambda(z) = B - \{D\}_1(z),$$

$$\det\Lambda(z) = \Omega(z) \ ,$$

$$M(z) = \Lambda^{-1}(z)\{Df\}_1(z) \ . \qquad (2\text{-}19)$$

Multiplying Eq. (2-18) by s, integrating over s, and solving for $\{h\}_1$, one gets

$$\{h\}_1 = [I - \{D\}_1(z)B^{-1}]^{-1}\{Df\}_1(z) \ . \qquad (2\text{-}20)$$

One may substitute Eq. (2-20) into Eq. (2-18) to obtain

$$h(s) = D(z,s)[f(s) + \Lambda^{-1}(z)\{Df\}_1(z)] ,$$

$$= D(z,s)[f(s) + M(z)] , \quad (2\text{-}21)$$

which defines the resolvent of K as

$$(zI - K)^{-1}f(s) = D(z,s)[f(s) + M(z)] . \quad (2\text{-}22)$$

The spectrum of K can be determined from the resolvent operator above. It consists of the set S given by

$$S = \{[-1,1]\cup\{v_i\}_{i=1}^{2n} , \Omega(v_i) = 0\} . \quad (2\text{-}23)$$

Λ is called the dispersion matrix whose determinant Ω determines the eigenvalues v_i of K through the equation

$$\Omega(v_i) = 0 , i = 1,2,..., 2n . \quad (2\text{-}24)$$

In the literature, it has been customary to define the dispersion matrix Λ such that it can be written as a sum of the identity matrix and another matrix.[4] One notes that the above definition differs from the customary definition by a multiplicative factor $\Sigma C^{-1}\Sigma$ which does not alter the determinant condition given by the above equation. The elements of Λ are even functions of z. Thus the eigenvalues of K occur in pairs $\pm v_i$, i = 1,2,..,n (The number of eigenvalues of K is written as 2n to emphasize this fact.) In the following analysis, it will be assumed that each v_i is a simple zero of Ω and none of the v_i's lies in the interval $[-1,1]$. These restrictions will simplify the analysis without seriously limiting its scope.

The resolvent operator given by Eq. (2-22) can be substituted into the identity (2-5) to get

$$f(s) = \frac{1}{2\pi i}\oint_{\tau} D(z,s)[f(s) + M(z)]dz \;. \tag{2-25}$$

Now the contour τ can be broken into 2n+1 disjoint contours such that each contour Γ_i, $i = 1,2,..,2n$, surrounds an eigenvalue v_i and the contour Γ surrounds the branch cut $[-1,1]$ (see fig. 1). Then Eq. (2-25) may be written as

$$f(s) = \sum_{i=1}^{2n} f^i(s) + f_v(s) \;, \tag{2-26}$$

where

$$f^i(s) = \frac{1}{2\pi i}\oint_{\Gamma_i} D(z,s)[f(s) + M(z)]dz \;,\; i = 1,2,..,2n \;, \tag{2-27}$$

$$f_v(s) = \frac{1}{2\pi i}\oint_{\Gamma} D(z,s)[f(s) + M(z)]dz \;. \tag{2-28}$$

In the following sections, $f^i(s)$ and $f_v(s)$ will be calculated.

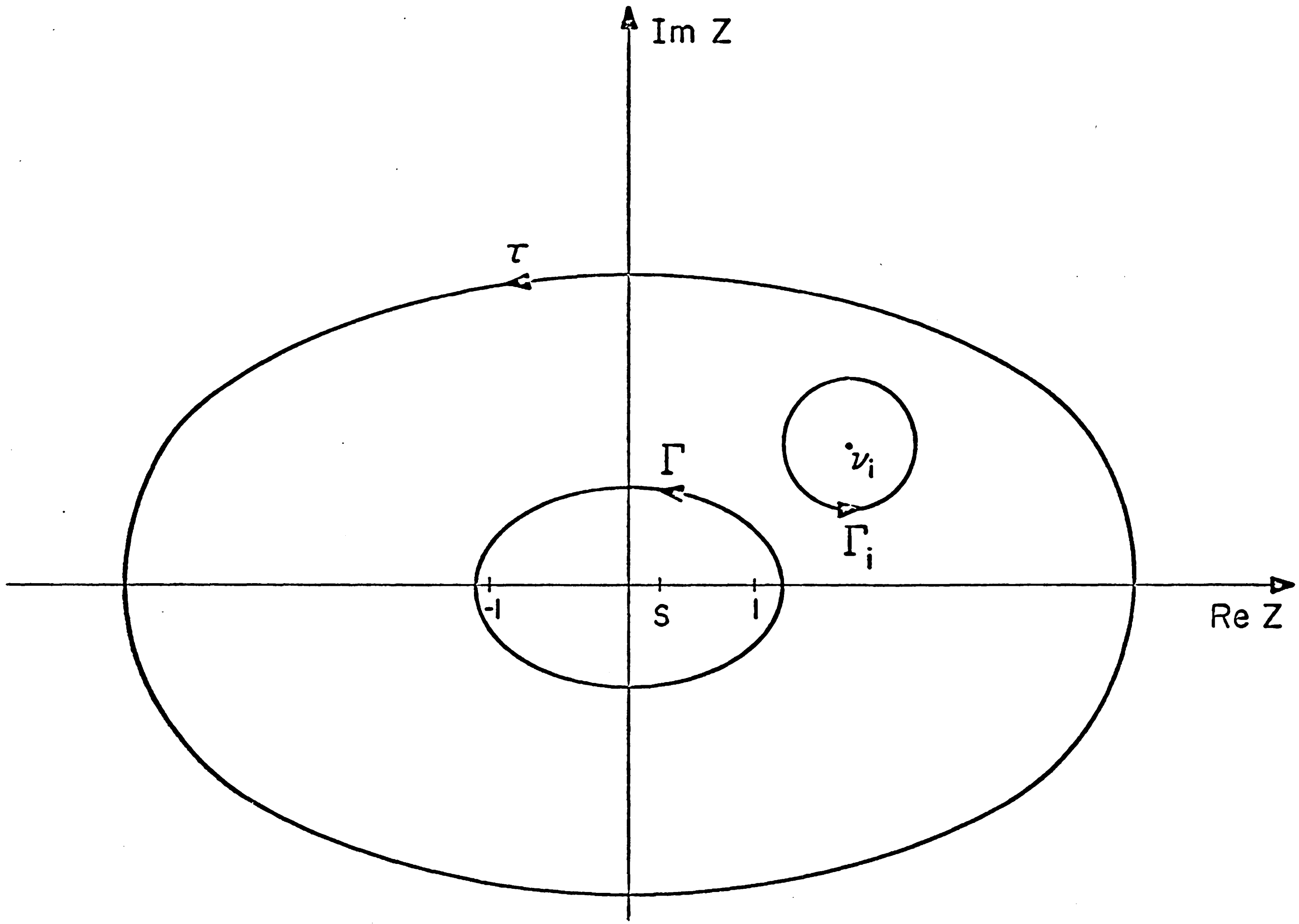

Fig. 1.

The contour τ , surrounding the spectrum of K is deformed into the contour Γ surrounding the interval [-1,1] plus contours Γ_i about each eigenvalue.

The Discrete Terms, f^i.

An application of the Cauchy theorem to the first term on the right hand side of Eq. (2-27) results in zero since D has no singularities inside Γ_i. Thus Eq. (2-27) reduces to

$$f^i(s) = \frac{1}{2\pi i}\oint_{\Gamma_i} D(z,s)M(z)dz \; . \tag{2-29}$$

The only singularity of the integrand in Eq. (2-29) is a simple pole at $z = v_i$, which arises in Λ^{-1} because of the Ω^{-1} term. With the help of Eq. (2-19), one may write Eq. (2-29) as

$$f^i(s) = \frac{1}{2\pi i}\oint_{\Gamma_i} D(z,s)\frac{[\Omega\Lambda^{-1}](z)}{\Omega(z)}\{Df\}_1(z)dz \; , \tag{2-30}$$

where the pole is explicitly shown by the existence of the Ω function in the denominator, the rest of the terms having no poles. Applying the Cauchy integral theorem once more, one may calculate the residue at v_i in Eq. (2-30) to be

$$f^i(s) = \frac{1}{\Omega'(v_i)}D(v_i;s)R(v_i)\{Df\}_1(v_i) \; , \; i = 1,2,..,2n \; , \tag{2-31}$$

where R is the cofactor matrix of Λ, namely

$$R_{ij} = (\Omega\Lambda^{-1})_{ij} \; , \tag{2-32}$$

R_{ij} being the cofactor of the ij^{th} element of Λ. Also

$$\Omega'(z) = \frac{d\Omega(z)}{dz} \; . \tag{2-33}$$

Now one can easily prove the following statement: f^i defined by Eq. (2-31) is proportional to the eigenvector of K associated with the eigenvalue v_i.

Proof: Using Eq. (2-13), one gets

$$(v_i I - K)f^i(s) = D^{-1}(v_i,s)f^i(s) - B^{-1}\{f^i\}_1 \ . \tag{2-34}$$

Calculating $\{f^i\}_1$ from Eq. (2-31) to be

$$\{f^i\}_1 = \{D\}_1(v_i)D^{-1}(v_i,s)f^i(s) \ , \tag{2-35}$$

and substituting into Eq. (2-34), one obtains

$$\begin{aligned}(v_i I - K)f^i(s) &= [D^{-1}(v_i,s) - B^{-1}\{D\}_1(v_i)D^{-1}(v_i,s)]f^i(s) \\ &= \Lambda(v_i)D^{-1}(v_i,s)f^i(s) = 0 \ .\end{aligned} \tag{2-36}$$

One notes that f^i given by Eq. (2-31) contains an arbitrary function $f\varepsilon D(K)$. One may factor out this f dependence and write a more familiar expression for the eigenfunction:

$$f^i(s) = a_i\phi_i(s) \ , \tag{2-37}$$

where the f dependence is isolated in the expansion coefficient a_i. Let β^i be the first nonzero column of $R(v_i)$, $i = 1,2,..,2n$. Then $\Lambda(v_i)\beta^i=0$ since $\Lambda(v_i)R(v_i)=0$. Also all the columns of R are proportional by the determinant condition (2-24). Let the j^{th} column of $R(v_i)$ be written as

$$R_j(v_i) = r_j^i\beta^i \ . \tag{2-38}$$

Then one may express Eq. (2-31) as

$$f^i(s) = \frac{1}{\Omega'(v_i)} \sum_{j=1}^{N} r_j [\{Df\}_1 (v_i)]_j D(v_i, s)\beta^i \ , \tag{2-39}$$

and define the eigenvector ϕ_i of K as

$$\phi_i(s) = D(v_i, s)\beta^i \ , \ i = 1,2,\ldots,2n \ . \tag{2-40}$$

in which case a_i becomes

$$a_i = \frac{1}{\Omega'(v_i)} \sum_{j=1}^{N} r_j [\{Df\}_1 (v_i)]j \ . \tag{2-41}$$

Thus

$$(v_i I - K)\phi_i(s) = 0 \ , \ i = 1,2,..,2n \ , \tag{2-42}$$

and all the eigenvectors of K and the expansion coefficients a_i of the function f are determined by Eqs. (2-40) and (2-41).

One may now attempt to evaluate the remaining contour integral around the interval [-1,1].

The Continuum Term, f_v

To evaluate the continuum contribution to f from the branch cut [-1,1] given by f_v of Eq. (2-27), namely

$$f_v(s) = \frac{1}{2\pi i}\oint_\Gamma D(z,s)[f(s) + M(z)]dz \;, \tag{2-43}$$

one can apply the Cauchy integral theorem to the first term on the right hand side of Eq. (2-43) to get f, reducing the equation to

$$f_v(s) = f(s) + \frac{1}{2\pi i}\oint_\Gamma D(z,s)M(z)dz \;. \tag{2-44}$$

The integral in the above equation can be evaluated after a further deformation of the contour Γ, as shown on figure II. Writing Γ as a union of contours $\Gamma_{-1} \cup \Gamma' \cup \Gamma_s \cup \Gamma_1$, one notes that near $z = \pm 1, M_i(z)$, the i^{th} element of M, behaves at worst as

$$M_i(z) = O|\log(z \pm 1)| \;, \tag{2-45}$$

since M_i behaves at worst as $\int_{-1}^{1} \frac{ds}{\sigma z - s}$. Thus the integrals over $\Gamma_{\pm 1}$ will be $O(\varepsilon|\log\varepsilon|)$ and in the limit $\varepsilon \to 0$ they will vanish, leaving Γ' and Γ_s to be evaluated. In order to evaluate these two contributions, the condition that $sf_i(s)$, $i = 1,2,..,2n$, are Hölder-continuous will be utilized. (The function $l(t)$ is said to satisfy a Hölder condition on an arc L if for any two points t_1 and t_2 of L,

$$|l(t_2) - l(t_1)| \leq A|t_2 - t_1|^m \;,$$

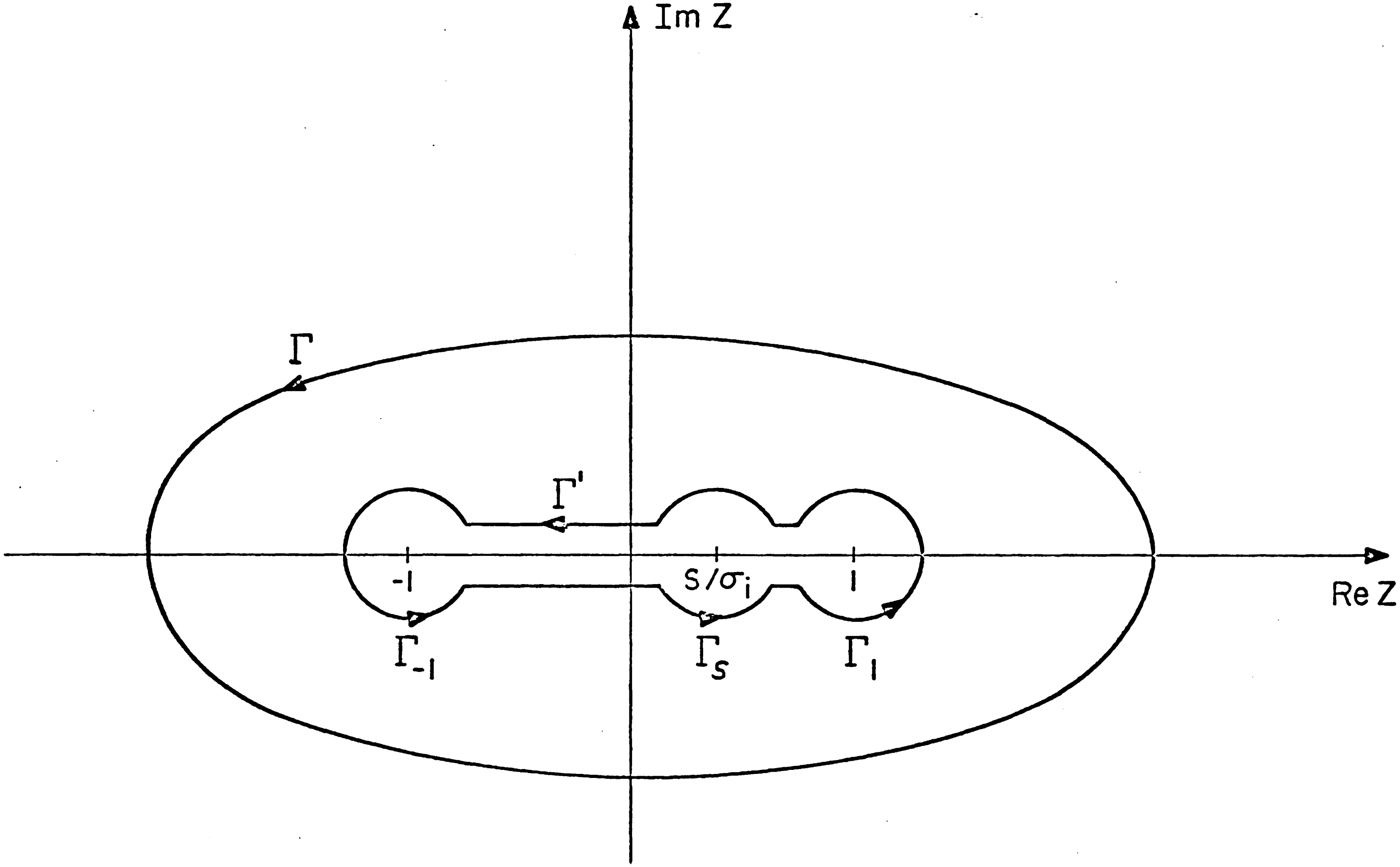

Fig. 11.

The contour Γ around $[-1,1]$ is squeezed down to the union of four contours $\Gamma'\cup\Gamma_{-1}\cup\Gamma_s\cup\Gamma_1$.

where A and m are positive constants. A is called the Hölder constant and m the Hölder index.)[16] Then one may use Muskhelishvili's results[16] to guarantee that M(z) would tend uniformly to $M^{\pm}(s)$ as $z \to s \pm i0$ for $s\varepsilon[-1,1]$.

One may evaluate these two contours in a straightforward manner to get

$$\oint_{\Gamma_s} D(z,s)M(z)dz = \frac{1}{2}(M^-(s) + M^+(s)) \ ,$$

$$\oint_{\Gamma'} D(z,s)M(z)dz = \frac{1}{2\pi i}\int_{-1}^{1} D(t,s)[M^-(t) - M^+(t)]dt \ . \tag{2-46}$$

By introducing the NxN diagonal matrix:

$$[\Delta(t,s)]_{ii} = \delta(\sigma_i t - s) \ , \tag{2-47}$$

where δ is the Dirac delta function, one may combine Eqs. (2-44), (2-46) to obtain

$$f_v(s) = \frac{1}{2\pi i}\int_{-1}^{1} [D(t,s)[M^-(t) - M^+(t)] +$$

$$\Delta(t,s)[2\pi i\Sigma f_\Sigma(t) + \pi i\Sigma[M^-(t) + M^+(t)]]]dt \ , \tag{2-48}$$

where the further notation (for the k^{th} component of the column $f_\Sigma(t)$),

$$[f_\Sigma(t)]_k = \begin{cases} f_k(\sigma_k t) & |t| \leq 1/\sigma_k \\ 0 & \text{otherwise} \end{cases} \ , \tag{2-49}$$

is introduced.

In an attempt to eliminate f_Σ from the right hand side of Eq.

(2-48), one refers to Eq. (2-19) and calculates the boundary values $(\Lambda M)^{\pm}$ of ΛM. For this purpose, one may use the Plemelj formulas:[17] if $F(z)$ is defined by

$$F(z) = \frac{1}{2\pi i}\int_a^b \frac{g(t)dt}{t - z} \quad , \tag{2-50}$$

where $g(t)$ is Hölder-continuous, then

$$F^{\pm}(s) = \frac{1}{2\pi i}P\int_a^b \frac{g(t)dt}{t - s} \pm \frac{1}{2}g(s) \tag{2-51}$$

holds, $F^{\pm}$ being the boundary values of F as z approaches the real axis (between a and b) from above and below, respectively. From this, one may conclude

$$F^{+}(s) + F^{-}(s) = \frac{1}{\pi i}P\int_a^b \frac{g(t)dt}{t - s} \tag{2-52}$$

and

$$F^{+}(s) - F^{-}(s) = g(s) \ . \tag{2-53}$$

Thus, an expression for f_{Σ} in terms of $(\Lambda M)^{\pm}$ can be calculated to be

$$2\pi i t\Sigma^2 f_{\Sigma}(t) = \Lambda^{-}(t)M^{-}(t) - \Lambda^{+}(t)M^{+}(t) \ . \tag{2-54}$$

Furthermore, $\Lambda^{\pm}$ can be calculated from Eq. (2-19) and the Plemelj formulas to be

$$\Lambda^{\pm}(t) = \Lambda_p(t) \pm \pi i t\Sigma^2 \ , \tag{2-55}$$

where

$$\Lambda_p(t) = B - P\int_{-1}^{1} sD(t,s)ds \ , \tag{2-56}$$

the integration being a principle value. Combining Eqs. (2-54) and (2-56), one gets

$$2\pi i\Sigma f_\Sigma(t) = \frac{1}{t}\Sigma^{-1}\Lambda_p(t)[M^-(t) - M^+(t)]$$

$$- \pi i\Sigma[M^+(t) + M^-(t)] \ . \tag{2-57}$$

Finally, substitution of Eq. (2-57) into Eq. (2-48) results in

$$f_v(s) = \int_{-1}^{1} \Phi(t,s)A(t)dt \ , \tag{2-58}$$

where

$$\Phi(t,s) = tD(t,s) + \Delta(t,s)\Sigma^{-1}\Lambda_p(t) \ , \tag{2-59}$$

and

$$A(t) = \frac{1}{2\pi i t}[M^-(t) - M^+(t)] \ . \tag{2-60a}$$

or

$$A(t) = \frac{[(\Lambda^{-1})^- - (\Lambda^{-1})^+]}{2\pi i t}P\int_{-1}^{1} sD(t,s)f(s)ds$$

$$+ [(\Lambda^{-1})^- + (\Lambda^{-1})^+]2\Sigma^2 f_\Sigma(t) \ . \tag{2-60b}$$

If each column of Φ is labeled by ψ_i, then Eq. (2-58) may also be written as

$$f_v(s) = \sum_{j=1}^{N} \int_{-1}^{1} A_j(t)\psi_j(t,s)dt , \tag{2-61}$$

where the only f dependence is contained in the coefficient A_j.

Also one may show that[8]

$$Kf_v(s) = \int_{-1}^{1} t\Phi(t,s)A(t)dt . \tag{2-62}$$

For this purpose, one may utilize the identity

$$(zI - K)^{-1}Kf(s) = z(zI - K)^{-1}f(s) - f(s) \tag{2-63}$$

Then, the previous analysis can be carried out identically for Kf(s) rather than f(s), using the right hand side of the above equation. The last term in that equation does not contribute to the result since it is analytic. The multiplicative factor z in the first term just causes a minor modification of the previous analysis to yield Eq. (2-62).

Uniqueness of the Expansion Coefficients:

From functional analysis,[17a] one knows that the operators

$$P_i = \frac{1}{2\pi i}\oint_{\Gamma_i}(zI - K)^{-1}dz , \quad i = 1,..,2n$$

and

$$P_0 = \frac{1}{2\pi i}\oint_{\Gamma}(zI - K)^{-1}dz$$

are projection operators. One may use this result to show that the expansion coefficients are unique. For this purpose, one may expand the 0 function in terms of an expansion:

$$0 = \sum_{i=1}^{2n} b_i\phi_i(s) + \int_{-1}^{1} \Phi(t,s)B(t)dt .$$

An application of P_i to the above equation results in

$$b_i = 0 , \quad i = 1,2,..,2n .$$

Thus the equation reduces to

$$0 = \int_{-1}^{1} \Phi(t,s)B(t)dt .$$

To show that B(t) is also zero, one notes that an operation of K^n on the above equation yields

$$0 = \int_{-1}^{1} t^n\Phi(t,s)B(t)dt ,$$

which implies that

$$0 = \int_{-1}^{1} p(t)\Phi(t,s)B(t)dt , \tag{2-64}$$

where p(t) is an arbitrary polynomial in t. Let $G(t) = \int_{-1}^{1} \Phi(t,s)ds$. Integrating Eq. (2-64) over s, one gets

$$0 = \int_{-1}^{1} p(t)G(t)B(t)dt .$$

Let H(t) = G(t)B(t). Then for each component $H_i(t)$ of H,

$$0 = \int_{-1}^{1} H_i(t)p(t)dt , \quad i = 1,..,N .$$

For a given i, pick a sequence of polynomials $p_n(t)$, such that $p_n(t) \to \overline{H}_i(t)$ in the norm. Then

$$0 = \lim_{n\to\infty}\int_{-1}^{1} H_i(t)p_n(t)dt = \int_{-1}^{1} |H_i(t)|^2 dt \ ,$$

which immediately yields

$$H(t) = G(t)B(t) = 0 \ .$$

One may simply calculate G to be $G(t) = \Sigma C^{-1}\Sigma$. Thus G is invertible and one finally gets

$$B(t) = 0 \ .$$

One concludes that the expansion coefficients are unique.

One may also verify assertion (b) on page 13 as follows: For the discrete terms, it is clear that $\frac{\partial}{\partial x} f \varepsilon X$. For the continuum term

$$\frac{\partial}{\partial x} f = - \int_{-1}^{1} \Phi(t,s)\frac{A(t)}{t} e^{-x/t} dt \ .$$

If one examines the term $\frac{A(t)}{t}$ in the limit $t \to 0$, one observes that, by using Eq. (2-60b), the worst behavior of any element of $\frac{A(t)}{t}$ will be of the form $\frac{1}{t} P \int_{-1}^{1} \frac{sf_i(s)}{\sigma_i t - s} ds$. But $sf(s)$ cannot have any singularity by the restriction in the domain. Thus $\frac{A_i(t)}{t}$ at worst goes as $\frac{1}{t}\tanh^{-1}\sigma_i t$ which behaves well in the limit $t \to 0$.

Finally one may state the following theorem:

THEOREM I: For $\det(\Sigma - 2C) \neq 0$, the operator K is bounded. If f(s) is defined on $0 \leq |s| \leq 1$ such that $sf_i(s)$ is Hölder continuous for $i = 1,2,..,N$, then there exists unique coefficients a_i and A_j such that

$$f(s) = \sum_{i=1}^{2n} a_i\phi_i(s) + \int_{-1}^{1} \Phi(t,s)A(t)dt , \qquad (2\text{-}65)$$

v_i being the simple zeros of Ω lying outside the interval [-1,1]. Also

$$Kf(s) = \sum_{i=1}^{2n} v_i a_i \phi_i(s) + \int_{-1}^{1} t\Phi(t,s)A(t)dt . \qquad (2\text{-}66)$$

Some Remarks on ψ_i's:

In the previous approaches to the multigroup transport theory, the so called degenerate continuum eigenfunctions related to the various parts of the interval [-1,1] were calculated.[4] The branch cut was treated as a union of N intervals each of which was defined by

$$S_i = [\frac{1}{\sigma_{i-1}} , \frac{1}{\sigma_i}]U[- \frac{1}{\sigma_i} - \frac{1}{\sigma_{i-1}}] , i = 1,2,..,N , \qquad (2\text{-}67)$$

with $1/\sigma_0 = 0$. Then $[-1,1] = Us_i$. In each interval s_i, the degeneracy of the continuum eigenfunctions was found to be (N+1-i). In the present work, the eigensolutions associated with the whole interval [-1,1] appear compactly to be N-fold degenerate. In fact, one may finally prove that

$$(tI - K)\psi_i(v,s) = 0 , i = 1,2,..,N , \qquad (2\text{-}68)$$

holds in a distributional sense.

Proof: Using Eq. (2-13), one may write

$$(tI - K)\Phi(t,s) = D^{-1}(t,s)\Phi(t,s) - B^{-1}\{\Phi\}_1(t) . \qquad (2\text{-}69)$$

$\{\Phi\}_1(t)$ may be evaluated from Eq. (2-59) to be

$$\{\Phi\}_1(t) = tD_1(t) + t\Lambda_p(t) . \qquad (2\text{-}70)$$

One must note that the integral involved in $\{D\}_1(t)$ is a principal value integral. Substituting Eq. (2-70) into Eq. (2-69) and noting that $D^{-1}\Delta = 0$, one gets

$$(tI - K)\Phi(t,s) = tB^{-1}[B - D_1(t) - \Delta_p(t)] = 0 . \qquad (2\text{-}71)$$

Each column ψ_i of Φ also satisfies the equation

$$(tI - K)\psi_i(v,s) = 0 . \qquad (2\text{-}72)$$

Thus, using the terminology of the Case approach, Eq. (2-61) gives the contribution of each degenerate continuum eigenfunction ψ_i to the expansion of the function f, with A_i's giving the expansion coefficients. (Also note that the column vector M is, within multiplicative constants, the Hilbert transform vector N as defined in reference 4.)

If one examines the behavior of these eigensolutions ψ_i for each of the intervals s_i, the coefficients A_i collapse together to give the degeneracies which agree with those of previous work . This fact is demonstrated explicitly in appendix A where it is shown that for N = 2 case, the continuum eigensolutions given by the columns ψ_j of Φ reduce to those calculated by Siewert and Shieh.[14]

III. CONSTANT TRANSFER MATRIX: HALF-RANGE

Introduction

For the half-range neutron transport problem, the boundary conditions are given at $x \geq 0$ in terms of incoming flux where $s > 0$. To find solutions to the half-range problem, it is desired to expand a set of functions defined on the interval [0,1] in terms of the eigensolutions of K. For this problem, the domain of K will be chosen to be the set of functions g(x,s) defined on $s \varepsilon [0,1]$ rather than $s \varepsilon [-1,1]$ and with identical properties to those of chapter II otherwise. The basic equation considered is the same. One would expect that the expansion will occur in terms of some of the eigensolutions found in the previous chapter, but with different expansion coefficients.

The technique of evaluating the half-range expansion of f depends heavily on the construction of an operator E, which maps the functions g defined on $s \varepsilon [0,1]$ to the functions f defined on [-1,1] in such a way that the properties

(a) $Eg(s) = g(s)$, $0 < s \leq 1$,

(b) $(zI - K)^{-1} Eg(s)$ is analytic in z for Re $z < 0$,

hold. The motivation is that the full-range expansion of Eg(s) will correspond to half-range expansion of g(s), since the contribution from the part of the spectrum of K in the left-half z-plane will not contribute to the contour integrals of the previous chapter because of the property (b) above. Then the full range expansion of Eg(s),

$$Eg(s) = \frac{1}{2\pi i}\oint_{\tau}(zI - K)^{-1}Eg(s)dz \ , \ -1 \leq s \leq 1 \tag{3-1}$$

will reduce to the half-range expansion of g(s)

$$g(s) = \frac{1}{2\pi i}\oint_{\tau'}(zI - K)^{-1}Eg(s)dz \ , \ 0 < s \leq 1 \ , \tag{3-2}$$

where the contour τ' is obtained from τ by squeezing it around the part of the spectrum of K on the right-half z-plane, after using the analyticity of the integrand on the left-half z-plane.

The main obstacle in solving the half-range problem was the difficulty in obtaining a factorization of the dispersion matrix in terms of two matrices, one of which has a branch cut [-1,0], and the other [0,1]. The matrices X and Y which factor the dispersion matrix Λ will be introduced here before the analysis is pursued any further. Later, more consideration will be given to calculation of integral equations for these matrices.

One demands that the matrices X and Y have the following properties:

i) $\Lambda(z) = Y(-z)\,X(z)$ (which holds for subcritical media).

ii) X and Y are analytic in the complex plane cut along [0,1].

iii) $\det X(v_i) = \det Y(v_i) = 0$ for $\mathrm{Re}\, v_i > 0$, $i = 1,2..,n$.

iv) $\lim_{|z| \to \infty} X(z) = \text{constant}$; $\lim_{|z| \to \infty} Y(z) = \text{constant}$

One should note that the factorization of Λ in terms of the matrices X and Y is not unique in the sense that if X and Y factor Λ, so would SX and YS^{-1} where S is a constant matrix. This fact, however, presents no problem since the half-space expansion formulas turn out to contain the product YX, so that the above mentioned non-uniqueness does not play any role.

It is important to distinguish the difference between the canonical factorization (sought before)[3] and the factorization used here. In the Case approach, one obtains the half-range eigenfunction expansion as the solution of a singular integral equation on the line [0,1], and requires a matrix R^{-1}, analytic in the complex plane cut along [0,1], such that

$$(R^-)^{-1}R^+ = (\Lambda^-)^{-1}\Lambda^+ .$$

One notes that R can be reconstructed from the X matrix as follows:

$$(\Lambda^-)^{-1}\Lambda^+ = (X^-)^{-1}X^+ \text{ for Re } z > 0$$

Thus

$$R(z) = \frac{X(z)}{\prod_{i=1}^{n} (v_i - z)}$$

so that det R will not vanish at $z = v_i$, $i = 1,2,..n$.

One should also notice that Case[1], in his original work, solved the Riemann-Hilbert problem which was

$$\frac{X^+}{X^-} = \frac{\Lambda^+}{\Lambda^-}$$

for the one-speed case. Then the factorization of the dispersion function was shown to hold[18] as follows:

$$\Lambda(z) = (1 - c)(v_0^2 - z^2)X(z)X(-z) \; .$$

On the other hand, Mullikin approached the problem by finding the factorization first and then the desired properties of R can be obtained by modifying the factorization matrix.

Integral equations for the X and Y matrices can be found in chapter VI, where it is also shown that these equations are valid for subcritical systems. Finally, in the same chapter, a uniqueness theorem for the solutions of these integral equations is presented.

After these remarks, the original analysis proceeds. First, the form of the operator E will be determined for $-1 \leq s \leq 0$. Then Eq. (3-2) will be used to find the expansion of g(s) in terms of the eigensolutions of K which are associated with the part of the spectrum of K lying in the right-half-plane. At the end of the chapter, the full and half-range formulas will be applied to solve the inhomogenous neutron transport equation.

Determination of the Operator E

Property (a) in the previous section defines $Eg(s)$ for $s > 0$ and property (b) is sufficient to determine $Eg(s)$ for $s \leq 0$. For this purpose, one writes (using Eqs. (2-22))

$$(zI - K)^{-1}Eg(s) = D(z,s)[Eg(s) + \Lambda^{-1}(z)T(z)] \tag{3-3}$$

where

$$T(z) = \{DEg\}_1(z) \ . \tag{3-4}$$

Also, one defines the following column vector

$$M'(z) = \Lambda^{-1}(z)T(z) \ . \tag{3-5}$$

Then Eq. (3-3) can be written as

$$(zI - K)^{-1}Eg(s) = D(z,s)[Eg(s) + M'(z)]. \tag{3-6}$$

In order for $(zI - K)^{-1}Eg(s)$ to be analytic in z for Re z < 0, (property (b)), one requires

A. $M'^{+}(s) = M'^{-}(s) = M'(s) \ , \ -1 \leq s < 0 \ ,$

B. $(M'(s))_i = -((Eg)_\Sigma(s))_i \ , \ -1 \leq \sigma_i s < 0 \ , \ i = 1,2,..,N.$

C. $M'(v_i) < \infty \ , \ \mathrm{Re}\,v_i < 0 \ , \ i = 1,2,..,n$

Now one attempts to find another representation of M' by defining the operator E and a column vector F below such that the column vector Q defined by

$$Q(z) = \int_{-1}^{1} sD(z,s)Eg(s)ds - Y(-z)\int_{0}^{1} \frac{1}{z-s}F(s)ds \ , \tag{3-7}$$

is identically zero. Since $Q(z)$ vanishes as $|z| \to \infty$ and is analytic, except perhaps for a cut along the interval $[-1,1]$, one needs only to require that $Q(z)$ be continuous across that interval. So demand that

$$Q^{+}(s) = Q^{-}(s) \text{ for } 0 < |s| \leq 1 \ . \tag{3-8}$$

Then one may use the property (a) of E and property ii of Y together with the Plemelj formulas to write $Q^{\pm}(s)$ along the branch cut. Using Eq. (3-8) and Q , one gets

$$F_i(s) = \begin{cases} [sY^{-1}(-s)\Sigma^2 g_\Sigma(s)]_i \ , & 0 < s \leq 1/\sigma_i \ , \\ 0 & 1/\sigma_i < s \leq 1 \ , \end{cases} \quad i = 1,\ldots,N. \tag{3-9}$$

and

$$[(Eg)_\Sigma(s)]_i = -[X^{-1}(s)\int_0^1 \frac{F(t)}{s-t}dt]_i \ , \ -1/\sigma_i \leq s < 0 \ , \ i = 1,\ldots,N. \tag{3-10}$$

With the operator E and the column F defined by the above equations, one may apply Liouville's theorem to the column vector function Q to get $Q = 0$. Then Eq. (3-7) yields

$$T(z) = Y(-z)\int_0^1 \frac{1}{z-s}F(s)ds \ . \tag{3-11}$$

Note that by virtue of Eq. (3-11) above, condition C for M' holds.

Eq. (3-10) together with the property a) of E is sufficient to determine the operator E for all $s\varepsilon[-1,1]$:

$$Eg(s) = \begin{cases} g(s) \ , \ 0 < s \leq 1 \\ [- X^{-1}(s')\int_0^1 \frac{1}{s' - t}F(t)dt]_{s' = \Sigma^{-1}s} \ , \ -1 \leq s < 0 \end{cases} \tag{3-12}$$

Using Eqs. (3-5) and (3-11), one may verify that the conditions A and B for M' hold. Namely,

$$\begin{aligned} M'(z) &= \Lambda^{-1}(z)Y(-z)\int_0^1 \frac{1}{z - s}F(s)ds \ , \\ &= X^{-1}(z)\int_0^1 \frac{1}{z - s}F(s)ds \ , \\ &= - (Eg)_\Sigma(s) \ , \end{aligned} \tag{3-13}$$

and

$$\begin{aligned} M'^{\pm}(s) &= X^{-1}(s)\int_0^1 \frac{1}{s - t}F(t)dt \\ &= M'(s) \ , \ -1 \leq s < 0 \ . \end{aligned} \tag{3-14}$$

Combining Eqs. (3-3), (3-4), (3-9) and (3-11), one gets

$$\begin{aligned} (zI - K)^{-1}Eg(s) &= D(z,s)[g(s) + X^{-1}(-z)\int_0^1 \frac{t}{z - t}Y^{-1}(-t)\Sigma^2 g_\Sigma(t)dt], \\ &= D(z,s)[g(s) + M'(z)] \ . \end{aligned} \tag{3-15}$$

Now one may use the identity given by Eq. (3-2) to get

$$g(s) = \frac{1}{2\pi i}\int_\tau D(z,s)[g(s) + X^{-1}(s)\int_0^1 \frac{1}{z - t}F(t)dt]dz \ . \tag{3-16}$$

As in chapter II, one may break the contour τ into smaller contours to express the contributions from the various parts of the spectrum. One uses property (b) to deduce that the only singularities which will contribute to the contour integral are in the set $(0,1) \cup \{v_i\}$, $i = 1, 2,..,n$, Re $v_i > 0$. The part of the spectrum in the left-half of the z plane would not contribute since $(zI - K)^{-1}Eg(s)$ is analytic for Re z ≤ 0. Thus the non-zero contributions to the expansion of g(s) from the various parts of the spectrum of K are given by

$$g(s) = \sum_{i=1}^{n} g^i(s) + g_v(s) , \tag{3-17}$$

where

$$g^i(s) = \frac{1}{2\pi i}\int_{\Gamma i} D(z,s)[g(s) + X^{-1}(z)\int_0^1 \frac{1}{z - t}F(t)dt]dz , \tag{3-18}$$

$$g_v(s) = \frac{1}{2\pi i}\int_{\Gamma} D(z,s)[g(s) + X^{-1}(z)\int_0^1 \frac{1}{z - t}F(t)dt]dz , \tag{3-19}$$

with Γ_i surrounding v_i, Re $v_i > 0$, $i = 1,2..,n$, and Γ surrounding the cut [0,1]. Now each contribution can be evaluated separately. The expansion obtained for g(s) will be only in terms of the eigensolutions associated with the elements of the spectrum in the right half complex plane. This is the desired half-range expansion formula.

Calculation of the Discrete and Continuum Terms

For a simple zero, v_i of Ω, the residue at v_i can be calculated in Eq. (3-18) to yield

$$g^i(s) = \frac{1}{\Omega'(v_i)} D(v_i,s) R(v_i) Y(-v_i) \int_0^1 \frac{1}{v_i - t} F(t)\,dt \;, \tag{3-20}$$

where property I is also utilized. One may write g^i above as

$$g^i(s) = a_i' \phi_i(s) \;, \tag{3-21}$$

with a_i' defined by

$$a_i' = \frac{1}{\Omega'(v_i)} \sum_{j=1}^{N} r_j(v_i) [Y(-v_i) \int_0^1 \frac{1}{v_i - t} F(t)\,dt]_j \;. \tag{3-22}$$

The last expression can be seen to be a close analog of Eq. (2-41) for the full-range expansion coefficient.

The continuum term g_v is given by

$$g_v(s) = \frac{1}{2\pi i} \int_\Gamma D(z,s) [g(s) + M'(z)]. \tag{3-23}$$

where M' is defined to be

$$M'(z) = X^{-1}(z) \int_0^1 \frac{1}{z - t} F(t)\,dt \;. \tag{3-24}$$

The details of the integration around the contour Γ are the same as in chapter II and the integration can be cast into the following form:

$$g_v(s) = \frac{1}{2\pi i} \int_0^1 [\Delta(t,s) [2\pi i \Sigma g_\Sigma(t) + \pi i \Sigma (M'^-(t) + M'^+(t))]$$

$$+ D(v,s)[M'^-(t) - M'^+(t)]]dt \ . \tag{3-25}$$

Using Eq. (3-24) one may solve for g in terms of $(XM')^{\pm}$ to get

$$2\pi i\Sigma g_\Sigma(t) = \frac{1}{t}\Sigma^{-1}[(\Lambda M')^-(t) - (\Lambda M')^+(t)] \ . \tag{3-26}$$

The last expression yields the following one upon substitution into Eq. (3-25):

$$g_v(s) = \int_0^1 \Phi(t,s)A'(t)dt \ , \tag{3-27}$$

where Φ is the same matrix as obtained in chapter II (Eq. (2-59)) and the expansion coefficient vector A' is defined by

$$A'(t) = \frac{1}{2\pi it}[M'^-(t) - M'^+(t)] \ , \tag{3-28}$$

Thus, the half-range expansion of the functions g(s) defined on the interval $0 < s \leq 1$, and with otherwise the same properties as the full-range functions, in terms of the eigensolutions of the operator K can be achieved by introduction of the matrices X and Y which factor the dispersion matrix Λ. One can now state the following theorem:

THEOREM II: The half-range expansion is given in terms of the eigensolutions of K which are associated with the part of the spectrum lying in the right half complex plane as follows:

$$g(s) = \sum_{i=1}^{n} a_i'\phi_i(s) + \sum_{j=1}^{N}\int_0^1 A_j(t)\psi_j(t,s)dt \ , \tag{3-29}$$

where the eigensolutions ϕ_i and ψ_j are given by Eqs. (2-40) and (2-59) and the half-range expansion coefficients a_i' and A_j' are given by Eqs. (3-22) and (3-28). The half-range expansion formula also satisfies the following identity:

$$Kg(s) = \sum_{i=1}^{n} v_i a_i' \phi_i(s) + \sum_{j=1}^{N} \int_0^1 A_j(t) \psi_j(t,s) dt \ .$$

Proof of the last equation is identical to that of Eq. (2-62).

Applications to Transport Problems

In this section, the results of the preceding analysis will be used to find solutions to the inhomogenous neutron transport equation with constant transfer matrix. Both the full and the half-range problems will be considered and the corresponding expansion formulas given by Theorems I and II will be used. The general procedure is to expand the solution and the source in terms of eigensolutions of the operator K. The expansion coefficients of the source are known. Then the unknown expansion coefficients of the solution can be determined in terms of those of the source and the boundary conditions by inserting the expanded functions into the inhomogenous equation.

A. Full-Range

In the context of a full-range expansion, only infinite medium problems are really relevant. The infinite medium Green's function will be determined here. One seeks the solutions of the homogenous equation

$$K\frac{\partial}{\partial x}f(x,s) + f(x,s) = 0 \ , \tag{3-31}$$

subject to the condition

$$\lim_{|x| \to \infty} f(x,s) = 0 \tag{3-32}$$

and

$$\lim_{\varepsilon \to 0 \ , \ \varepsilon > 0}[f(\varepsilon,s) - f(-\varepsilon,s)] = \frac{q(s)}{s} = q_0(s) \tag{3-33}$$

(the last equation is the jump condition.[12a]), where q is the source strength vector. q_0 is assumed to be Hölder continuous.

The x-dependence of Eq. (3-31) strongly suggests that separation of variables be sought in the form

$$f(x,s) = \int h(t,s)e^{-x/t}dt \ . \tag{3-34}$$

This leads to an equation

$$\int (tI - K)h(t,s)dt = 0 \ . \tag{3-35}$$

Thus, one is led to associate the vector h with the eigensolutions of K. Accordingly, one attempts to expand the solution of Eq. (3-31) as

$$f(x,s) = \sum_{i=1}^{n} a_i \phi_i(s) e^{-x/v_i} + \int_0^1 \Phi(t,s) e^{-x/t} A(t)dt \ , \ x > 0 \ ,$$

for Re $v_i \geq 0$, and (3-36)

$$f(x,s) = -\sum_{i=1}^{n} a_i \phi_i(s) e^{-x/v_i} - \int_{-1}^0 \Phi(t,s) e^{-x/t} A(t)dt \ , \ x < 0 \ ,$$

for Re $v_i < 0$. (3-37)

The form of these equations is chosen to satisfy Eq. (3-32). Here a_i's are constants and A is a column vector (recall that Φ is a matrix) to be determined.

Using Eqs. (3-36) and (3-37) together with Eq. (3-33), one finds that

$$q_0(s) = \sum_{i=1}^{2n} a_i \phi_i(s) + \int_{-1}^1 \Phi(t,s)A(t)dt \ . \tag{3-38}$$

Since q_0 is assumed to be Hölder-continuous, its expansion coefficients are determined by the results of Theorem I. (For example, in the case of an isotropic source $q = \frac{1}{4\pi}$ or $q_0 = \frac{1}{4\pi s}$, the expansion coefficients of the source are given by

$$a_i = \frac{1}{\Omega'(v_i)} \sum_{j=1}^{N} r_j [\frac{1}{4\pi}\int_{-1}^{1} D(v_i,s)ds]_j$$

and

$$A(t) = \frac{1}{(2\pi i t)4\pi}[(\Lambda^{-1})^-(t)[\int_{-1}^{1} D(t,s)ds]^- - (\Lambda^{-1})^+(t)[\int_{-1}^{1} D(t,s)ds]^+],$$

or

$$a_i = \frac{1}{\Omega'(v_i)} \sum_{j=1}^{N} r_j\sigma_j \ln(\frac{\sigma_j v_i + 1}{\sigma_j v_i - 1})$$

$$A(t) = \frac{1}{8\pi^2 it}[(\Lambda^{-1})^-(t)[\Lambda^+(t) - \Lambda^-(t)](\Lambda^{-1})^+(t)D_0(t)$$

$$+ \pi i(\Lambda^{-1})^-(t)[\Lambda^+(t) + \Lambda^-(t)](\Lambda^{-1})^+(t)\Sigma] \ .)$$

Thus, the function given by Eqs. (3-36) and (3-37) is indeed a solution since it obeys the boundary conditions and satisfies Eq. (1-5). To show this one notes for $x > 0$,

$$\int_0^1 \Phi(t,s)e^{-x/t}A(t)dt = \int_{-1}^1 \Phi(t,s)e^{-x/t}A'(t)dt$$

where

$$A'(t) = \begin{cases} A(t) \ , & t\varepsilon[0,1] \\ 0 \ , & \text{otherwise} \end{cases}$$

Then, one may use Theorem I to get

$$K\frac{\partial}{\partial x}\int_0^1 \Phi(t,s)e^{-x/t}A(t)dt = - \int_0^1 \Phi(t,s)e^{-x/t}A(t)dt \ .$$

The discrete terms also yield

$$K\frac{\partial}{\partial x}a_i\phi_i(s)e^{-x/v_i} = - a_i\phi_i(s)e^{-x/v_i} \ , \ i = 1,..2n.$$

Combining the above equations one sees that $f(x,s)$, $x > 0$, satisfies Eq. (1-5). The same argument follows for $x < 0$. One may now show that the solution is unique.

For the one-speed problem, uniqueness is well known.[18] For the energy dependent transport equation, a number of uniqueness theorems have been shown[18,19], and the multigroup case considered here can, with a little effort, be shown to be a special case of some of these treated there; the one-speed transport equation can be written as an integral equation[18] by introducing the Green's function. In exactly the same way, the multigroup transport equation with source q is equivalent to an equation for $p(x) = \int_{-1}^{1} f(x,s)ds$:

$$p(x) = \int_{-\infty}^{\infty} dyG(|x - y|)p(y) + \overline{Q}(x) \ , \tag{3-39}$$

where

$$\overline{Q}(x) = \int_{-\infty}^{\infty} dy\int_{-1}^{1} ds\frac{e^{-\Sigma|x-y|/|s|}}{|s|} q(y,s) \tag{3-40}$$

$$G(|x-y|) = E_1(\varepsilon|x-y|)C \tag{3-41}$$

E_1 being defined in reference 18 as

$$E_1(x) = \int_1^\infty e^{-xs} \frac{ds}{s} . \tag{3-42}$$

A Fredholm equation like (3-39) is known to possess a unique solution if the norm of the kernel G is less than unity. In chapter VI, this norm is computed and it is concluded that

$$||G|| \leq 2||\Sigma^{-1}C|| . \tag{3-43}$$

Then the solution given by Eqs. (3-36) and (3-37) is indeed unique if the infinite medium under consideration is subcritical.[15] This is precisely the condition at which one arrives in one speed-theory, i.e., $c < 1$.

B. Half-Range

One of the classic problems in neutron transport is the Albedo problem,[18] characterized by the following equation

$$\frac{\partial}{\partial x}f(x,s) + K^{-1}f(x,s) = 0 \text{ , } x > 0 \text{ , } |s| \leq 1 \text{ ,} \tag{3-44}$$

subject to the boundary conditions

$$f(0,s) = f_0(s) \text{ , } 0 < s \leq 1 \text{ ,}$$

$$\lim_{x \to \infty} f(x,s) = 0 .$$

Using the half-range expansion formula, one may write f as

$$f(x,s) = \sum_{i=1}^{n} e^{-x/v_i} a_i \phi_i(s) + \int_0^1 e^{-x/t} \Phi(t,s) A(t) dt \qquad (3\text{-}45)$$

One notes that Eq. (3-45) obeys the second boundary condition. If one demands that the first boundary condition is also to be satisfied, one gets

$$f_0(s) = \sum_{i=1}^{n} a_i \phi_i(s) + \int_0^1 \Phi(t,s) A(t) dt \,. \qquad (3\text{-}46)$$

One may immediately determine the expansion coefficients from the results of the half-range expansion formula. Namely,

$$a_i = \frac{1}{\Omega'(v_i)} \sum_{j=1}^{N} r_j(v_i) [Y(-v_i) \int_0^1 \frac{tdt}{v_i - t} Y^{-1}(-t) \Sigma^2 [f_0(s)]_{s = \Sigma t}]_j \qquad (3\text{-}47)$$

and

$$A_i(t) = \frac{1}{2\pi i t} [M^-(t) - M^+(t)]$$

with

$$M(v) = X^{-1}(v) \int_0^1 \frac{tdt}{v - t} Y^{-1}(-t) \Sigma^2 [f_0(s)]_{\Sigma t = s}$$

After some manipulation, one gets

$$A_i(v) = \frac{1}{2\pi i v} [(\Lambda^{-1})^+ (\Lambda^+ - \Lambda^-)(\Lambda^{-1})^-](v) \;] Y(-v) P \int_0^1 \frac{tdt}{v - t} Y^{-1}(-t) \Sigma^2$$

$$[f_0(s)]_{s = \Sigma t} + [(\Lambda^{-1})^+ (\Lambda^+ + \Lambda^-)(\Lambda^{-1})^-](v) \;] \Sigma^2 [f_0(s)]_{s = \Sigma v} . \qquad (3\text{-}48)$$

After a substitution of Eqs. (3-47) and (3-48) into Eq. (3-45), one obtains the solution of the problem.

Comments on f(x,s) given by Eq. (3.45): (a) f(x,s) is constructed to satisfy the boundary conditions; (b) it obeys the transport Eq. (3-44). One would observe that if one writes the transport equation after multiplication by K, as

$$K\frac{\partial}{\partial x}f + f = g \ ,$$

substitutes Eq. (3-45) into it, and uses Theorem II, one gets g = 0; (c) The solution given by Eq. (3-45) is unique. Assume not:

$$h(x,s) = \sum_{i=1}^{n} e^{-x/v_i} b_i \phi_i(s) + \int_0^1 e^{-x/t} \Phi(t,s) B(t) dt \ ,$$

is also a solution. Then the difference (f-h) evaluated at x = 0 must be identically zero to satisfy the boundary condition:

$$\sum_{i=1}^{n} (a_i - b_i)\phi_1(s) + \int_0^1 \Phi(t,s)[A(t) - B(t)]dt = 0$$

Now one uses the fact that the expansion coefficients are unique and the expansion coefficients of the zero function are zeros to get

$$a_i = b_i \ , \ i = 1,..n$$

and

$$A(t) = B(t) \ .$$

The above solution can be used to solve the Milne problem in the following manner: the Milne problem is characterized by zero incident distribution, with the boundary condition

$$\lim_{x \to \infty} e^{-x/v_m} f(x,s) = \phi_{-v_m}(s)$$

at $x = \infty$, v_m being the discrete eigenvalue with the largest real part.

One notes that the Milne problem can be reduced to the Albedo problem if one lets

$$f_0(s) = -\phi_{-v_m}(s) , \tag{3-49}$$

and

$$f_M(x,s) = f_A(x,s) + e^{x/v_m}\phi_{-v_m}(s) , \tag{3-50}$$

with $f_A \to 0$ as $x \to \infty$ (where M and A stand for Milne and Albedo).

Thus, one may use Eq. (3-49) in Eqs. (3-47) and (3-48) to determine the expansion coefficients of f_A and may substitute the result in Eq. (3-50) to get the solution to the Milne problem.

IV. DEGENERATE TRANSFER KERNEL

Introduction

A transfer kernel in the form SL, where S and L are invertible NxN matrices which depend on angles, is used in the following analysis. This kernel represents an anisotropic scattering among the groups. Then Eq. (1-1) can be written as

$$s\frac{\partial}{\partial x}f(x,s) + \Sigma f(x,s) - S(s)\int_{-1}^{1} L(t)f(x,t)dt = q(x,s) \ . \tag{4-1}$$

and the operator K^{-1} for this case is given by

$$K^{-1}f(s) = \frac{1}{s}[\Sigma f(s) - S(s)\int_{-1}^{1} L(t)f(t)dt]. \tag{4-2}$$

The domain of the transport operator will be taken to be the function space defined in chapter II with the minor modification that the functions $sL_{ij}(s)\ g_j(s)$ will be Hölder-continuous rather than $sg_j(s)$. Also $S_{ij}(s)$ will be assumed to be Hölder-continuous for the half-space analysis to be valid. In the following sections of this chapter, the details of the analysis will be kept at a minimum since one has already acquired an insight into the problem because of a similar analysis in the preceding chapters. The applications of the results are also similar to those of the constant transfer matrix problem.

Full-Range

The operator K^{-1} given by Eq. (4-2) can be inverted as in chapter II. First introduce the notation

$$\{h\}_n = \int_{-1}^{1} t^n L(t)\Sigma^{-1} h(t)dt \ , \tag{4-3}$$

where h may be an element of the domain of the transport operator or an NxN matrix. Then

$$K^{-1}f(s) = \frac{1}{s}[\Sigma f(s) - S(s)\{\Sigma f\}_0] = g(s) \ ,$$

$$\Sigma f(s) = sg(s) + S(s)\{\Sigma f\}_0 \ . \tag{4-4}$$

Integrating over s in Eq. (4-4), after multiplication by $L(s)\Sigma^{-1}$, one finds

$$\{\Sigma f(s)\}_0 = (I - \{S\}_0)^{-1}\{g\}_1 \ . \tag{4-5}$$

Substituting Eq. (4-5) into (4-4), one obtains the following expression for K:

$$Kg(s) = \Sigma^{-1}(sg(s) + S(s)(I - \{S\}_0)^{-1}\{g\}_1) \ . \tag{4-6}$$

By demanding that the determinant condition holds (which incidentally is the statement that the medium is non-critical), i.e., that

$$\det(I - \{S\}_0) \neq 0 \tag{4-7}$$

holds, one insures that the operator K is bounded. One may construct the resolvent operator of K in a straightforward manner by first forming $(zI - K)$, and then inverting this operator as was done before. If one introduces the dispersion matrix Λ:

$$\Lambda(z) = (I - \{S\}_0) - \{\Sigma^{-1}DS\}_1(z) , \tag{4-8}$$

one may write the resolvent operator of K as

$$(zI - K)^{-1}f(s) = D(z,s)[f(x) + \Sigma^{-1}S(s)\Lambda^{-1}(z)\{Df\}_1(z)]. \tag{4-9}$$

For compactness of notation, one may attach a separable s dependence to the dispersion matrix as follows:

$$\Lambda(z,s) = \Lambda(z)S^{-1}(s)\Sigma. \tag{4-10}$$

Then Eq. (4-9) can be written as

$$(zI - K)^{-1}f(s) = D(z,s)[f(s) + \Lambda^{-1}(z,s)\{Df\}_1(z)]. \tag{4-11}$$

By examining the form of the resolvent operator, one may determine the spectrum of K as the branch cut $[-1,1]$ and the zeros of the determinant Ω of the dispersion matrix Λ. Let the set $\{v_i\}_1^{2n}$ be defined by the following equation:

$$\Omega(v_i) = \det\Lambda(v_i) = 0 . \tag{4-12}$$

Since the elements of Λ are even functions, the zeros of Ω appear in pairs. For the simplification of the further analysis, one assumes that each v_i is a simple zero of Ω and does not lie in the interval [-1,1].

Using the identity in Eq. (2-5), one gets

$$f(s) = \frac{1}{2\pi i}\oint_{\tau} D(z,s)[f(s) + \Lambda^{-1}(z,s)\{Df\}_1(z)]\,dz \; . \tag{4-13}$$

The contour can be handled as in chapter II to give

$$f(s) = \sum_{i=1}^{2n} f^i(s) + f_v(s) \; . \tag{4-14}$$

where

$$f^i(s) = \frac{1}{\Omega'(v_i)} D(v_i,s)\Sigma^{-1}S(s)R(v_i)\{Df\}_1(v_i) \; ,$$

$$R(s) = (\Omega\Lambda^{-1})(s) \; . \tag{4-15}$$

To handle the f_v term, write

$$f_v(s) = \frac{1}{2\pi i}\oint_{\Gamma} D(z,s)[f(s) + \Sigma^{-1}S(s)M(z)]\,dz$$

$$= \frac{1}{2\pi i}\oint_{\Gamma} D(z,s)[f(s) + M(z,s)]\,dz \; . \tag{4-16}$$

where

$$M(z) = \Lambda^{-1}(z)\{Df\}_1(z) \; . \tag{4-17}$$

and $M(z,s)$ is defined in a similar way as Eq. (4-10), where a separable s dependence is attached for compactness of notation. With these definitions, one gets

$$f_v(s) = \frac{1}{2\pi i}\int_{-1}^{1} [\pi i \Sigma \Lambda(t,s)[2f_\Sigma(t) + (M^+(t,s) + M^-(t,s))]$$

$$+ D(t,s)[M^-(t,s) - M^+(t,s)]]dt \ . \tag{4-18}$$

Using the Plemelj formulas, the boundary values of Λ can be calculated along the branch cut:

$$\Lambda^\pm(t) = \frac{1}{2}[\Lambda^+(t) + \Lambda^-(t)] \pm \pi i t L^\Sigma(t) S_\Sigma(t) \ ,$$

$$= \Lambda p \pm \pi i t L^\Sigma(t) \ S_\Sigma(t) \ . \tag{4-19}$$

where

$$[L^\Sigma(t)]_{ij} = L_{ij}(\sigma_j t) \ , \ i,j = 1,2,..,N \tag{4-20}$$

and

$$[S_\Sigma(t)]_{ij} = S_{ij}(\sigma_i t) \ , \ i,j = 1,2,..,N \tag{4-21}$$

are defined.

From Eq. (4-17) and the Plemelj formulas, one may also calculate the following expression (to guarantee the existence of the boundary values along the cut, one demands that $sL_{ij}(s)f_j(s)$ is Hölder-continuous):

$$2f_\Sigma(t) = \frac{\Sigma^{-1}}{\pi i t}(L^\Sigma(t))^{-1}[\Lambda^-(t)M^-(t) - \Lambda^+(t)M^+(t)]. \tag{4-22}$$

With the help of Eqs. (4-19) and (4-22), one may eliminate the f_Σ term in Eq. (4-18) to obtain

$$f_v(s) = \int_{-1}^{1} \Phi(t,s)A(t)dt \,. \tag{4-23}$$

where

$$\Phi(t,s) = [tD(t,s)\Sigma^{-1}S(s) + \Delta(t,s)(L^{\Sigma}(t))^{-1}\Lambda_p(t)] \,,$$

and

$$A(t) = \frac{M^{-}(t) - M^{+}(t)}{2\pi i t} \,. \tag{4-25}$$

One notes that in the limit $L \to I$ and C constant, Eqs. (4-24) and (4-25) reduce to the results of chapter II.

One can show by straightforward substitution that the following identity holds:

$$(v_i I - K)f^i(s) = 0 \,. \tag{4-26}$$

Defining β^i to be the first non-zero column of $R(v_i)$, one may write each column R_j of the matrix R as $R_j = r_j\beta^i$ and introduce ϕ_i and a_i:

$$f^i(s) = a_i\phi_i(s) \,. \tag{4-27}$$

where

$$\phi_i(s) = D(v_i,s)\Sigma^{-1}S(s)\beta^i,$$

$$a_i = \frac{1}{\Omega'(v_i)} \sum_{j=1}^{N} r_j(v_i)[\{Df\}_1(v_i)]_j \,. \tag{4-28}$$

The ϕ_i's and each column of Φ are the Case-type eigensolution of the operator K. In fact by direct substitution, one may show that the identity

$$(tI - K)\Phi(t,s) = 0 \tag{4-29}$$

formally holds in a distributional sense.

The following theorem may now be stated:

THEOREM III: For $\det(1 - \{S\}_0) \neq 0$, the operator K is bounded. The functions $f(s)$ defined on $-1 \leq s \leq 1$, such that $sL_{ij}(s)f_j(s)$ is Hölder-continuous, can be expanded in terms of the eigensolutions of the operator K in the following way:

$$f(s) = \sum_{i=1}^{2n} a_i \phi_i(s) + \int_{-1}^{1} \Phi(t,s)A(t)dt \ . \tag{4-30}$$

where each ν_i is a simple zero of Ω and lies outside the interval [-1,1]. The following identity also holds:

$$Kf(s) = \sum_{i=1}^{2n} \nu_i a_i \phi_i(s) + \int_{-1}^{1} t\Phi(t,s)A(t)dt \ . \tag{4-31}$$

The eigensolutions ϕ_i and Φ are given by Eqs. (4-28) and (4-24) and the expansion coefficients are given by Eqs. (4-28) and (4-25).

Half-Range

The half-range problem can be tackled in an identical manner to chapter III after introduction of the projection operator E. Again the purpose is to expand a set of functions $g(s)$, defined along $0 \leq s \leq 1$, in terms of the eigenfunctions of K associated with the spectrum of K lying in the right half complex plane. With suitable modifications, the arguments are identical to those of chapter III and depend on the introduction of the X and Y matrices which factor the dispersion matrix Λ. The existence and the properties of these matrices are given in chapter VI.

One may write

$$(zI - K)^{-1}Eg(s) = D(z,s)[Eg(s) + \Lambda^{-1}(z,s)T(z)] \ , \qquad (4\text{-}32)$$

where

$$T(z) = \{DEg\}_1(z) \ . \qquad (4\text{-}33)$$

Also,

$$M'(z) = \Lambda^{-1}(z)T(z) \ . \qquad (4\text{-}34)$$

The properties of the X and Y matrices, the operator E and the column vector M' and the same as in chapter III except for the property B of M', which is replaced by the property

B. $(Eg)_\Sigma(s) = -\Sigma^{-1}S_\Sigma(s)M'(s)$.

As in the preceding chapter, one may introduce the column vector Q defined by

$$Q(z) = T(z) - Y(-z)\int_0^1 \frac{1}{z-t}F(t)dt \qquad (4\text{-}35)$$

The purpose is to find an expression for Eg(s), s < 0, by defining the column vector F suitably so that Q will identically be zero (after application of Liouville's theorem).

One may calculate the boundary values of Q(z) as z approaches the intervals $-1 \leq s < 0$ and $0 < s \leq 1$ by using Eq. (4-33) and the Plemelj formulas, together with the properties of the X and Y matrices. Then one demands that Q is continuous across the cut [-1,1] to get

$$(Ef)_\Sigma(s) = -\Sigma^{-1}C_\Sigma(s)Y(-s)\int_0^1 \frac{1}{s-t}F(t)dt \;,\; -1 < s < 0 \;, \qquad (4\text{-}36)$$

$$F_i(s) = \begin{cases} [sY^{-1}(-s)L^\Sigma(s)\Sigma g_\Sigma(s)]_i \;, & 0 < s \leq 1/\sigma_i \;, \\ 0 \;, & 1/\sigma_i < s \leq 1 \end{cases} \qquad (4\text{-}37)$$

where L^Σ is defined as in Eq. (4-20).

Eqs. (4-36) and (4-37) guarantee that Q is identically zero, as in chapter III, yielding

$$T(z) = Y(-z)\int_0^1 \frac{F(t)}{z-t}dt \;. \qquad (4\text{-}38)$$

where F is defined by Eq. (4-37). Using Eqs. (4-34) and (4-38), one observes that condition B holds.

Thus the operator E is completely determined as

$$Eg(s) = \begin{cases} g(s) \ , \ 0 \le s \le 1 \\ [-\Sigma^{-1}S_\Sigma(t)Y(-t)\int_0^1 \frac{1}{t - t'}F(t')dt']_{t = \Sigma^{-1}s} \ , \ -1 < s < 0 \ . \end{cases} \tag{4-39}$$

As a result of the preceding calculations, Eq. (4-32) can be written as

$$(zI - K)^{-1}Eg(s) = D(z,s)[g(s) + \Sigma^{-1}S(s)M'(z)] \ , \ s > 0 \ , \tag{4-40}$$

with

$$M'(z) = X^{-1}(z)\int_0^1 \frac{1}{z - t}F(t)dt \ . \tag{4-41}$$

One may find g_Σ from Eqs. (4-34) and (4-38) to be

$$2g_\Sigma(t) = \frac{1}{\pi i t}\Sigma^{-1}(L^\Sigma(t))^{-1}[(\Lambda M')^-(t) - (\Lambda M')^+(t)] \tag{4-43}$$

If one notes that the form of Eq. (4-40) is identical to Eq. (4-9) of the previous section, one may observe that g_v can be calculated in exactly the same way as in the previous section, with the exception that (4-17) is replaced by M'.

Thus the non-vanishing contour integrations around the part of the spectrum of K in the right half complex plane yields the usual discrete and continuum contributions to the expansion of g(s), given by

$$g^i(s) = \frac{1}{\Omega'(v_i)}D(v_i,s)\Sigma^{-1}S(s)R(v_i)Y(-v_i)\int_0^1 \frac{1}{v_i - t}F(t)dt \ . \tag{4-43}$$

and

$$g_v(s) = \int_0^1 \Phi(t,s)A'(t)dt \ , \tag{4-44}$$

where

$$A'(t) = \frac{1}{2\pi i t}[M'^-(t) - M'^+(t)] \ , \tag{4-45}$$

with

$$M'(t) = X^{-1}(t)\int_0^1 \frac{1}{t - s} SY^{-1}(-s)L^{\Sigma}(s)\Sigma g_{\Sigma}(s)\,ds \tag{4-46}$$

and Φ being the same matrix as calculated before (Eq. (4-24)).

As before, g^i can be written as

$$g^i(s) = a_i'\phi_i(s) \ , \tag{4-47}$$

with ϕ_i given by Eq. (4-28) and a_i' defined by

$$a_i' = \frac{1}{\Omega'(v_i)} \sum_{j=1}^{N} r_j(v_i)\left[Y(-v_i)\int_0^1 \frac{dt}{v_i - t} tY^{-1}(-t)L^{\Sigma}(t)\Sigma g_{\Sigma}(t)\right]_j . \tag{4-48}$$

One may state the following theorem for the half-range expansion formula:

<u>THEOREM IV</u>: The functions $g(s)$ defined on $0 \leq s \leq 1$, and satisfying a Hölder-continuity condition for $sL_{ij}(s)g_j(s)$, can be expanded in terms of the eigensolutions of K as follows:

$$g(s) = \sum_{i=1}^{n} a_i\phi_i(s) + \int_0^1 \Phi(t,s)A(t)dt \ , \ \mathrm{Re}\ v_i > 0 \ , \tag{4-49}$$

where the eigensolutions ϕ_i and Φ are given by Eqs. (4-28) and (4-24), and the expansion coefficients a_i' and A' are given by Eqs. (4-48) and (4-45). The following identity also holds:

$$Kg(s) = \sum_{i=1}^{n} v_i a_i \phi_i(s) + \int_0^1 t\Phi(t,s)A(t)dt \ . \qquad (4\text{-}50)$$

It is assumed that the eigenvalues of K are simple zeros of Ω and they lie outside the interval [-1,1].

V. TRANSFER KERNEL WITH A SUM OF DEGENERATE KERNELS

Introduction

In this chapter the following homogenous transport equation is considered:

$$S\frac{\partial}{\partial x}f(x,s) + \Sigma f(x,s) - \sum_{i=1}^{p} S_i(s)\int_{-1}^{1} L_i(t)f(x,t)dt = 0 , \qquad (5\text{-}1)$$

with each S_i and L_i being an invertible matrix. One notes that any compact transfer kernel of the type $C(s,t)$ can be approximated to a desired accuracy by a sum of degenerate kernels:

$$C(s,t) \cong \sum_{i=1}^{p} S_i(s)L_i(s) . \qquad (5\text{-}2)$$

Thus, solving Eq. (5-1) will exhaust the application of the Larsen-Habetler technique to the most general transfer kernel for an anisotropic problem in one dimension.

In this chapter, an expression for the resolvent operator $(zI-K)^{-1}$ is obtained for the problem in question. The integration around the branch cut [-1,1], however, is not calculated because of the difficulties in the handling of the dispersion matrix. The discrete terms can be obtained in a straightforward manner. In the opinion of this author, it is possible to reduce this problem to the case covered by the previous chapter by defining a suitable set of "super matrices", e.g., matrices whose elements themselves consist of matrices.

The Dispersion Matrix and the Resolvent Operator

The operator K^{-1} for this problem is given by

$$K^{-1}f(s) = \frac{1}{s}[\Sigma f(s) - \sum_{i=1}^{p} S_i(s)\int_{-1}^{1} L_i(t)f(t)dt] \ . \tag{5-3}$$

The following definition will be introduced to simplify the notation:

$$\{f\}_{in} = \int_{-1}^{1} t^n L_i(t)\Sigma^{-1}f(t)dt \ . \tag{5-4}$$

With this notation, one may write Eq. (5-3) as follows:

$$K^{-1}f(s) = \frac{1}{s}[\Sigma f(s) - \sum_{i=1}^{p} S_i(s)\{\Sigma f\}_{i0}] \ . \tag{5-5}$$

Now one may define the following super matrix $\underset{\sim}{C}_0$:

$$(\underset{\sim}{S}_0)_{ij} = \{S_i\}_{i0} \ . \tag{5-6}$$

Thus $\underset{\sim}{C}_0$ is actually an NPxNP matrix. Writing the NPxNP unit matrix as $\underset{\sim}{1}$, one can determine the operator K to be

$$Kf(s) = \Sigma^{-1}[sf(s) + \sum_{i,j} S_i(s)[\underset{\sim}{1} - \underset{\sim}{S}_0]^{-1}_{ij}\{f\}_{j1}] \tag{5-7}$$

If one imposes the condition that $\det(\underset{\sim}{1} - \underset{\sim}{C}_0)$ is not zero and defines the domain of K as in the previous chapter, except for the modification that $s[L_k(s)f(x,s)]_{ij}$ is Hölder continuous for all i,j,k. K will be a bounded operator defined on a suitable Banach space defined as in the previous chapter.

The resolvent operator of K can be found as

$$(zI - K)^{-1}f(s) = D(z,s)[f(s) + \sum_{i=1}^{p} M_i(z,s)] , \qquad (5\text{-}8)$$

where

$$M_i(z,s) = \sum_{j,k} S_i(s)[\underset{\sim}{I} - \underset{\sim}{S}_0]^{-1}_{ij}[\underset{\sim}{\Lambda}^{-1}(z)]_{jk}\{Df\}_{kl} , \qquad (5\text{-}9)$$

and the elements of the dispersion matrix are given by

$$[\underset{\sim}{\Lambda}(z)]_{ij} = \delta_{ij}I - \sum_{k}^{p}\{\Sigma^{-1}DS_k\}_{il}(z)[\underset{\sim}{I} - \underset{\sim}{S}_0]^{-1}_{kj} . \qquad (5\text{-}10)$$

One may define the determinant of the super matrix[5] $\underset{\sim}{\Lambda}$ as Ω and find the eigenvalues of the operator K as the zeros of Ω. As before, one assumes that these eigenvalues v_i are simple zeros of Ω, and they lie outside the interval [-1,1] which is in the spectrum of K.

The identity in Eq. (2-5) can be used and the functions f^i may be defined as before. To determine the f^i's, the residue at a simple zero v_i of Ω in the following equation for f^i can be calculated:

$$f^i(s) = \frac{1}{2\pi i}\oint_{\Gamma_i} D(z,s) \sum_{i=1}^{p} M_i(z,s)dz \qquad (5\text{-}11)$$

to yield

$$f^i(s) = \frac{D(v_i,s)}{\Omega'(v_i)} \sum_{i,j,k} S_i(s)[\underset{\sim}{I} - \underset{\sim}{S}_0]^{-1}[\Omega\underset{\sim}{\Lambda}^{-1}]_{jk}(v_i)\{Df\}_{kl}(v_i). \qquad (5\text{-}12)$$

By a straight forward substitution, it can be verified that

$$(v_iI - K)f^i(s) = 0 . \qquad (5\text{-}13)$$

Because of the cumbersome work involved, this problem will not be pursued any further. The author is hopeful, however, that a slightly different approach will enable the solution of this problem by reducing it to a single degenerate kernel.[20]

VI. SOME PROPERTIES OF X AND Y MATRICES

Introduction

In a recent paper[10] Mullikin has established the following factorization

$$[I - \hat{k}(z)]H_r(z)H_l(-z) = I, \quad \text{Im } z = 0 , \tag{6-1}$$

where

$$\hat{k}(z) = \int_{-\infty}^{\infty} k(x)\exp(ixz)\,dx ,$$

$$\int_{-\infty}^{\infty} |k_{ij}(x)|^n dx < \infty , \quad n = 1,2$$

$$\lim_{|x| \to \infty} k_{ij}(x) = 0 . \tag{6-2}$$

One may define an operator K_x, whose kernel is $k(x-y)$:

$$K_x f(x) = \int_0^{\infty} k(x-y)f(y)\,dy . \tag{6-3}$$

The matrices H_l and H_r are given in terms of the two nonlinear integral equations

$$H_r^{-1}(z) = I + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{H_l(t)\hat{k}(-t)}{t - z}dt , \quad \text{Im } z > 0 , \tag{6-4a}$$

$$H_l^{-1}(z) = I + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\hat{k}(t)H_r(t)}{t + z}dt , \quad \text{Im } z > 0 . \tag{6-4b}$$

These integral equations and the above factorization are valid if

$$\rho = \lim_{n \to \infty} ||M^n||^{1/n} < 1 , \tag{6-5}$$

where M is an NxN matrix whose entries are defined by

$$m_{ij} = \int_{-\infty}^{\infty} |k_{ij}(x)|\, dx \;, \tag{6-6}$$

and $||M|| = \sup_i \sum_{i=1}^{N} M_{ij}$.

To utilize the above results obtained by Mullikin, one may write the homogenous transport equation as an integral equation, as is done in the one-speed case[18]. For the constant transfer matrix, this equation is:

$$p(x) = \int_0^{\infty} G(x,y)\, p(y)\, dy + \bar{Q}(x) \tag{6-7}$$

where the various quantities are defined in chapter III, Eqs. (3-39) - (3-42). One notes that the Green's function for the multigroup equation can be directly obtained from the one-speed case, since the homogenous streaming equation consists of a diagonal operator which does not mix the components of the vector f. G is given by

$$G_{ij}(|x-y|) = E_1(\sigma_i |x-y|) C_{ij} \;, \quad i,j = 1,2,..,N$$

and E_1 is given by Eq. (3-42).

Eq. (6-7) is a Fredholm integral equation which has a unique solution if the norm of its kernel G is less than unity. In fact, effectively the same condition is required for the factorization of the dispersion matrix as will be obtained by use of Mullikin's work. This condition turns out to be related to the subcriticality of the medium under consideration, as shown by Bowden[15], and reduces to $c < 1$ for the one-speed problem.

In the next section, the integral form of the transport operator, K_x, given by Eq. (6-7) is used together with Mullikin's work to find coupled integral equations for the matrices X and Y which are shown to factor the dispersion matrix Λ. These equations are valid for $\rho < 1$. This bound can be related to the norm of K_x, as is done in the following section. The last section contains a uniqueness proof for the factorization matrices X and Y which satisfy the derived coupled integral equations.

Coupled Integral Equations for X and Y Matrices

First, one analytically extends the matrices $H_r(z)$ and $H_l(z)$ to the lower half of the z plane by defining

$$H^*(z) = \begin{cases} H_r(z) , & \mathrm{Im}\, z \geq 0 \\ [I - \hat{k}(z)]^{-1} H_l^{-1}(-z) , & \mathrm{Im}\, z < 0 \end{cases} , \tag{6-8}$$

and

$$H(z) = \begin{cases} H_l(z) , & \mathrm{Im}\, z \geq 0 \\ H_r^{-1}(-z)[I - \hat{k}(z)]^{-1} , & \mathrm{Im}\, z < 0 \end{cases} , \tag{6-9}$$

Now $H_r(z)$ is analytic for $\mathrm{Im}\, z > 0$ and $[I - \hat{k}(z)]^{-1}[H_l(-z)]^{-1}$ is analytic for $\mathrm{Im}\, z < 0$ except for a branch cut along $(-i, -i\infty)$, due to $[I - \hat{k}(z)]^{-1}$ and poles at the zeros of $\det[I - \hat{k}(z)]$. Since $H_r(z) = [I - \hat{k}(z)]^{-1}[H_l(-z)]^{-1}$, $\mathrm{Im}\, z = 0$, $H^*(z)$ is analytic everywhere in the complex plane except for the cut along $(-i, -i\infty)$ and poles at the zeros of $\det[I - \hat{k}(z)]$ in the lower half plane. Similar arguments follow for the matrix $H(z)$. Using Eqs. (6-8) and (6-9), one may easily show by direct substitution that

$$[I - \hat{k}(z)] H^*(z) H(-z) = I , \tag{6-10}$$

is valid for all z. Now one may define the X and Y matrices in terms of H and H* matrices and relate Λ matrix to $(I - \hat{k})$ (Eqs. (6-14) - (6-16)).

Case I. Constant Transfer Matrix.

To link Mullikin's results to the X and Y matrices used in this work, one writes the multigroup transport equation without source as an equation for $p(x) = \int_{-1}^{1} f(x,s)ds$:

$$p(x) = \int_0^\infty k(|x-y|)p(y)dy \ ,$$

$$k(|x-y|) = E_1(\Sigma|x-y|)C \ ,$$

$$[E_1(\Sigma|x-y|)]_{ij} = \delta_{ij}\int_0^1 \frac{1}{s} \exp[-\sigma_i \frac{|x-y|}{s}]ds \ . \tag{6-11}$$

Using this particular kernel, one calculates $\hat{k}$ and finds that it is related to the dispersion matrix Λ by the relationship

$$\Lambda(z) = \Sigma(I - \hat{k}(-\frac{i}{z}))C^{-1}\Sigma,$$

$$\hat{k}(y) = \int_{-1}^{1} (\Sigma - iysI)^{-1}Cds \ . \tag{6-12}$$

From Eqs. (6-10) and (6-12), one obtains

$$\Lambda(z) = \Sigma[H(\frac{i}{z})]^{-1}[H*(-\frac{i}{z})]^{-1}C^{-1}\Sigma. \tag{6-13}$$

If one now defines

$$X(z) = [H*(-\frac{i}{z})]^{-1}C^{-1}\Sigma \ , \tag{6-14}$$

and

$$Y(z) = \Sigma[H(-\frac{i}{z})]^{-1} \ , \tag{6-15}$$

one gets the factorization of the dispersion matrix Λ as

$$\Lambda(z) = Y(-z)X(z) . \tag{6-16}$$

To determine non-singular, coupled integral equations for the X and Y matrices, one considers the contour τ given in fig. III and notes that the integrands in Eqs. (6-4a) and (6-4b) are analytic inside τ and have a branch cut $(i, i\infty)$ due to $\hat{k}$. Since $\hat{k}$ vanishes as $z \to \infty$, one can write the integral in Eq. (6-4a) as

$$\lim_{R \to \infty} \frac{1}{2\pi i}\int_{-R}^{R} H_1(t)\hat{k}(-t)\frac{dt}{t+z} = \lim_{R \to \infty}\{\int_{iR}^{i} H_1(w)\frac{\hat{k}^{+}(-w)\,dw}{w+z}$$

$$+ \int_{i}^{iR} \frac{H_1(w)\hat{k}^{-}(-w)}{w+z}dw\} . \tag{6-17}$$

Using the last equation, Eq. (6-4a) can be expressed as

$$H^{-1}(-\frac{i}{z}) = I + \frac{z}{2\pi i}\int_0^1 \frac{ds}{s(z+s)}H_1(\frac{i}{s})[\hat{k}^{+}(\frac{i}{s}) - \hat{k}^{-}(\frac{i}{s})] . \tag{6-18}$$

where the limits of integration are now 0 and 1. One calculates $k^{+}(\frac{i}{s}) - k^{-}(\frac{i}{s})$ for $0 \le s \le 1$ from Eq. (6-12) and substitutes into Eq. (6-18) to get

$$H_r^{-1}(\frac{-i}{z}) = I + \frac{z}{2\pi i}\int_0^1 \frac{1}{s(z+s)}H_1(\frac{i}{s})\Sigma^{-1}[\Lambda^{+}(s)]\Sigma^{-1}C\,ds . \tag{6-19}$$

Identifying X and Y from their definitions in Eqs. (6-14) and (6-15), one gets the following integral equation for the X matrix:

$$X(z) = C^{-1}\Sigma + \frac{z}{2\pi i}\int_0^1 \frac{dt}{t(t-z)}Y^{-1}(-t)[\Lambda^{+}(t) - \Lambda^{-}(t)] , \quad \mathrm{Re}\, z < 0. \tag{6-20}$$

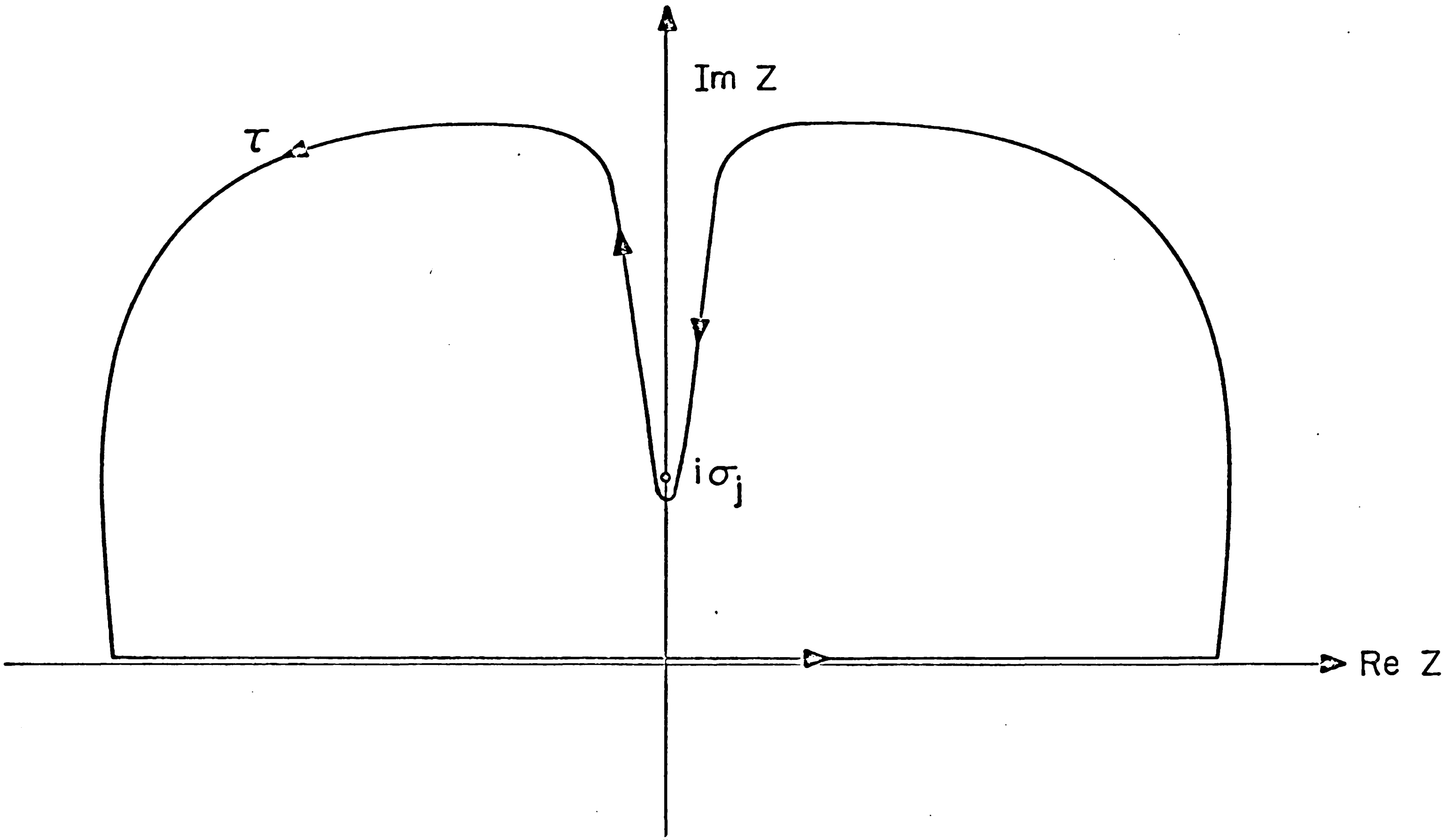

Fig. III.

The above contour is used to change the limits of integration from $\{-\infty,\infty\}$ to $\{0,1\}$ in Eq. (6-4).

Exactly the same analysis can be carried out for H_1^{-1}, Eq. (6-4b), to yield the integral equation for the Y matrix:

$$Y(z) = \Sigma + \frac{z}{2\pi i}\int_0^1 \frac{1}{t(t-z)}[\Lambda^+(t) - \Lambda^-(t)]X^{-1}(-t)dt \ , \ \mathrm{Re}\ z < 0 . \quad (6\text{-}21)$$

In Eqs. (6-20) and (6-21),

$$[\Lambda^+(t) - \Lambda^-(t)]_{ij} = \delta_{ij}2\pi i t\sigma_i^2 \ , \ |\sigma_i t| \leq 1 .$$

These coupled integral equations are non-singular, because each one holds for Re $z < 0$ thus making the denominator non-vanishing by virtue of the integration limits being 0 and 1. One should also note that at $t=0$, the difference $(\Lambda^+ - \Lambda^-)$ vanishes and the apparent singularity at that point does not exist.

Case II. <u>Degenerate Transfer Kernel</u>.

An identical analysis to the previous section will be carried out here to determine the X and Y matrices for the degenerate transfer kernel. The integral equation for the degenerate transfer kernel problem is

$$p(x) = \int_0^\infty dy \int_{-1}^1 L(t)G(x,y,t)S(t)p(y)dt \ , \quad (6\text{-}22)$$

where p is defined by

$$p(x) = \int_{-1}^1 L(t)f(x,t)dt \ , \quad (6\text{-}23)$$

and $G(x,y,s)$ is the Green's function for the streaming equation:

$$G_{ij}(x,y,s) = \begin{cases} \delta_{ij} \exp(-\sigma_i (x-y)/s)/|s| \ , \ \frac{x-y}{s} > 0 \ , \\ 0 \ , \ \text{otherwise} \ . \end{cases} \tag{6-24}$$

The integral form of the transport equation without a source can then be written as

$$K_x p(x) = \int_0^\infty k(\ x-y\ )p(y)dy \ , \tag{6-25}$$

with

$$k(|x-y|) = \int_{-1}^{1} L(t)G(x,y,t)S(t)dt \ . \tag{6-26}$$

As it is done in the previous case, one may calculate the Fourier transform of k given by

$$\hat{k}(z) = \int_{-\infty}^{\infty} k(x)\exp(izx)dx \ , \tag{6-27}$$

to get

$$\hat{k}(z) = \int_{-1}^{1} L(t)\int_{-\infty}^{\infty} G(x,t)e^{izx}dxS(t)dt \ ,$$

$$\hat{k}(-\ \tfrac{i}{z}) = \int_{-1}^{1} L(t)z\Sigma^{-1}D(z,t)S(t)dt \ . \tag{6-28}$$

Now one can identify the right hand side of the above equation with the dispersion matrix Λ to write

$$[I - \hat{k}(-\ \tfrac{i}{z})] = \Lambda(z) = Y(-z)X(z) \ . \tag{6-29}$$

Eq. (6-10) can be cast into the form

$$[I - \hat{k}(-\tfrac{i}{z})] = H^{-1}(\tfrac{i}{z})H^{*-1}(-\tfrac{i}{z}) \ . \tag{6-30}$$

Combining Eqs. (6-29) and (6-30), one gets

$$H^{-1}(\tfrac{i}{z})H^{*-1}(-\tfrac{i}{z}) = Y(-z)X(z) \ . \tag{6-31}$$

Then one may define

$$H^{-1}(\tfrac{i}{z}) = Y(-z) \text{ and } H^{*-1}(-\tfrac{i}{z}) = X(z) \ . \tag{6-32}$$

With these two definitions, the factorization of the dispersion matrix can be written as $\Lambda(z) = Y(-z)X(z)$.

The integral Eqs. (6-4a) and (6-4b) can be cast in a form where the integration limits are 0 and 1 (as it was done for the constant transfer matrix case) and the resulting coupled integral equations can be identified with X and Y matrices by using Eq. (6-32)

$$X(z) = I + \frac{z}{2\pi i}\int_0^1 \frac{1}{t(t - z)}Y^{-1}(-t)[\Lambda^+(t) - \Lambda^-(t)]dt \ , \tag{6-33a}$$

$$Y(z) = I + \frac{z}{2\pi i}\int_0^1 \frac{1}{t(t - z)}[\Lambda^+(t) - \Lambda^-(t)]X^{-1}(-t)dt \ . \tag{6-33b}$$

One notes that these integral equations are identical to those of the constant transfer matrix case except for different dispersion matrices (which causes the slight difference in the form of the two sets).

A Bound on the Norm of K_x

Case I. Constant Transfer Matrix.

In this section, a bound on the norm of K_x for the constant and degenerate transfer matrix cases will be obtained so that $||K_x|| < 1$. Thus Mullikin's condition that $\rho < 1$ (Eq. 6-5) will be satisfied. It is shown that $||K_x|| < 1$ results in $2||\Sigma^{-1}C||_M < 1$, where M stands for the matrix norm already defined (after Eq. (6-5)).

One may work with the Banach space

$$L = \bigotimes_{1}^{N} L_1 \text{ with the norm } ||f|| = \sum_{i=1}^{N} \int_0^\infty |f_i|dx \, .$$

Then, $||G|| = \sup\limits_{f\varepsilon L} \left|\left|\frac{Gf}{f}\right|\right| .$

In particular, if C is a matrix of constants, then

$$||C|| = \sup_k \sum_{j=1}^{N} |C_{jk}| = ||C||_M \, .$$

Now, if k is a matrix of operators, then each matrix element k_{ij} has an L_1 norm which is denoted by $||k_{ij}||_1$. By writing the operator $G = EC = (E\Sigma)\ (\Sigma^{-1}C)$, one concludes that

$$||G|| \leq ||E\Sigma||_1 \cdot ||\Sigma^{-1}C||_M \leq \sup ||E_{ii}\Sigma_{ii}||_1 \cdot ||\Sigma^{-1}C||_M \, ,$$

(since E and Σ are diagonal).

To compute $||E\Sigma||$ one may use Kato's criterion[21] which, for a difference kernel, reduces to

$$||E_{ii}\Sigma_{ii}|| \leq \sup_x \int_{-\infty}^{\infty} \sigma_i E_1(\sigma_i|x-y|)dy = 2 .$$

Thus, $||G|| \leq 2||\Sigma^{-1}C||_M$.

Eqs. (6-1) and (6-4) will be valid if

$$2||\Sigma^{-1}C||_M < 1 . \tag{6-34}$$

This result is exactly one of those derived by Bowden[15] as the subcriticality condition for an infinite medium. Thus one concludes that the factorization $\Lambda(z) = Y(-z)X(z)$ is valid for a subcritical medium.

Case II. Degenerate Transfer Kernel.

In this section an attempt will be made to find criteria for $||K_x|| < 1$ to hold for the degenerate transfer kernel problem. The kernel $k(|x-y|)$ of K_x is given by

$$k(|x-y|) = \int_{-1}^{1} L(t)G(x,y,t)S(t)dt . \tag{6-35}$$

The following result will be used:[22]

If $K_x(x,y)$ is an integral operator on L_2 with matrix kernel $k(x,y)$, then

$$||K_x|| \leq \frac{1}{2} \sup_x \{\int ||k(x,y)||_p dy + \int ||k(y,x)||_p dy\} \tag{6-36}$$

holds for K_x acting on functions of L_2.

Here the norms $||\quad||$ and $||\quad||_p$ are defined as

$$||K_x||^2 = \sup_{||f||=1} \sum_{i=1}^{N} |K_x f(x)|^2 dx ,$$

$$||k(x,y)||_p = \sup_{||f||=1\ ,\ x,y \text{ fixed}} k(x,y)f(x).$$

Thus

$$||K_x|| \le \int_{-1}^{1} ||L(t) \sup_x \int_0^\infty G(x,y,t)dyS(t)||_p dt \ ,$$
$$\le \int_{-1}^{1} ||L(t)\Sigma^{-1}S(t)||dt \ .$$

Demanding $K_x < 1$ so that Eqs. (6-33a) and (6-33b) will be valid (one notes that this restriction satisfies the $\rho < 1$ assumption of Mullikin and may hopefully be refined), one gets the condition

$$\int_{-1}^{1} ||L(t)\Sigma^{-1}S(t)||dt < 1 \ . \qquad (6\text{-}37)$$

Using a procedure similar to that of Bowden[15] who treated the case of isotropic scattering, one can verify that Eq. (6-37) is the condition that the infinite medium be subcritical. Thus, the half-range expansion exists under the same conditions as in the case of constant transfer matrix.

Further Properties of the X and Y Matrices

Lemma I: Any pair of matrices X and Y which satisfy Eqs. (6-20) and (6-21) provide a factorization of Λ according to Eq. (1-8).

Proof: Combining Eqs. (6-20) and (6-21) one gets

$$[Y(-z) - \Sigma][X(z) - C^{-1}\Sigma] = -\frac{z^2}{(2\pi i)^2}\int_0^1 ds\, \frac{[\Lambda^+(s) - \Lambda^-(s)]}{s(z+s)} \int_0^1 dt$$

$$X^{-1}(-s)Y^{-1}(t)\, \frac{[\Lambda^+(t) - \Lambda^-(t)]}{t(t-z)} \quad . \qquad (6\text{-}38)$$

Evaluating the right hand side of the last equation by partial fraction decomposition

$$\frac{1}{z+s}\,\frac{1}{t-z} = \left[\frac{1}{z+s} + \frac{1}{t-z}\right]\frac{1}{t+s}$$

and cancelling common terms on both sides, one obtains

$$Y(-z)X(z) = \Sigma C^{-1}\Sigma + \frac{z}{2\pi i}\int_0^1 \frac{[\Lambda^+(s) - \Lambda^-(s)]ds}{s(s-z)}$$

$$- \frac{z}{2\pi i}\int_0^1 \frac{[\Lambda^+(s) - \Lambda^-(s)]}{s(s+z)}ds \quad . \qquad (6\text{-}39)$$

After a change of variables in the last integral and use of the identity $\frac{z}{s-z} = \frac{s}{s-z} - 1$, one gets

$$Y(-z)X(z) = (\Sigma C^{-1}\Sigma - 2\Sigma) + \frac{1}{2\pi i}\int_{-1}^1 \frac{[\Lambda^+(s) - \Lambda^-(s)]ds}{(s-z)} \quad . \qquad (6\text{-}40)$$

Now one may substitute $\Lambda^+(s) - \Lambda^-(s) = 2\pi i s\Sigma^2$ and make a change of variables $s \to s/\sigma_i$ in each row of the matrices to finally get

$Y(-z)X(z) = \Lambda(z)$.

Lemma 2: Let $X_1(z)$ and $Y_1(z)$ satisfy Eqs. (6-20) and (6-21) and also satisfy the conditions (i) - (iv) on page 32. Let X_2 and Y_2 satisfy the same equations and conditions. Then $X_1 = X_2$ and $Y_1 = Y_2$.

Proof: The matrices

$$D_1(z) = [Y_2(z)]^{-1}Y_1(z) , \tag{6-41}$$

$$D_2(z) = X_1(z)[X_2(z)]^{-1} \tag{6-42}$$

are analytic everywhere in the complex plane except perhaps for a cut along [0,1] and poles at $\{+v_i\}$, $i = 1,..n$.

Also,

$$\lim_{|z| \to \infty} D_1(z) = \lim_{|z| \to \infty} D_2(z) = I , \tag{6-43}$$

because X_1, Y_1 and X_2, Y_2 satisfy Eqs. (6-20) and (6-21). Now one may calculate

$$D_1(-z)D_2(z) = [Y_2(-z)]^{-1}Y_1(-z)X_1(z)[X_2(z)]^{-1} = I \tag{6-44}$$

where lemma I is used. Similarly

$$D_1(z)D_2(-z) = I \tag{6-45}$$

The last two equations are valid for all z. Since $D_2(z)$ is analytic in the left half plane, it follows from Eq. (6-44) that $D_1(\ z)$ is also analytic in the right half plane. Similarly, from Eq. (6-45) one concludes that $D_2(z)$ is analytic in the right half plane. Thus, D_1 and D_2 are analytic everywhere and behave like I in the limit $(z) \to 0$. Using Liouville's theorem, one gets

$$D_1(z) = D_2(z) = I \tag{6-46}$$

or

$$[Y_2(z)]^{-1} Y_1(z) = I \tag{6-47}$$

and

$$X_1(z)[X_2(z)]^{-1} = I \, .$$

Now, one may redefine D_1 and D_2 to show

$$Y_1(z)[Y_2(z)]^{-1} = I \, , \tag{6-48}$$

$$[X_2(z)]^{-1} X_1(z) = I \, .$$

Combining Eqs. (6-47) and (6-48), one finally gets

$$X_1(z) = X_2(z) \text{ and } Y_1(z) = Y_2(z) \, . \tag{6-49}$$

VII. CONCLUSIONS

In this work, explicit full and half-range expansion formulas for the solutions of the multigroup neutron transport equation are found for various transfer kernels. These expansion formulas are given in terms of the eigensolutions of the operator K defined by

$$K^{-1}f(s) = \frac{1}{s}[\Sigma f(s) + \int_{-1}^{1} C(s,t)f(t)dt].$$

By writing the inhomogenous transport equation as

$$K\frac{\partial}{\partial x}f(x,s) + f(x,s) = Kq_0(x,s) \quad ,$$

one may apply these expansion formulas to solve transport problems with sufficiently smooth sources.

The following two transfer kernels are considered throughout the analysis:

i) $C(s,t) = C$,

C being a constant, invertible, NxN matrix.

ii) $C(s,t) = S(s)\, L(t)$,

where S and L are invertible NxN matrices.

Also a third type of transfer kernel is considered. Namely

iii) $C(s,t) = \sum_{i=1}^{p} S_i(s)\, L_i(t)$,

where each S_i and L_i is an invertible NxN matrices. In this case, however, only the discrete eigensolutions can be obtained because the expression for the resolvent operator proved to be too cumbersome. The author believes that with a suitable definition of some super matrices, it may be possible to reduce this case to the previous case, ii. Thus the results of chapter IV would then be used to find the eigensolutions for case iii.

In the first two cases, the functions f in the domain of K can be expanded in the general form

$$f(s) = \sum_i a_i \phi_i(s) + \int \Phi(t,s)A(t)dt \ ,$$

where the summation i extends over the eigenvalues of the operator K, and the limits of the integral are [-1,1] and [0,1] for the full and half-range cases, respectively. Thus, one may write the following expansion formula for the expansion of the functions f(x,s) in the domain of the transport operator:

$$f(x,s) = \sum_i a_i(s)\phi_i(s) + \sum_{j=1}^{N} \int A_j(x,t)\psi_j(t,s)dt$$

with the following useful identity:

$$Kf(x,s) = \sum_i v_i a_i(x)\phi_i(s) + \sum_{j=1}^{N} \int tA_j(x,t)\psi_j(t,s)dt$$

Here, the vectors ψ_i are the columns of the matrix Φ.

In solving the half range problem, the X and Y matrices which factor the dispersion matrix Λ;

$$\Lambda(z) = Y(-z)X(z) \ ,$$

are introduced. These two matrices are obtained as solutions of two coupled integral equations derived in chapter VI, and are given by

$$X(z) = I + \frac{z}{2\pi i}\int_0^1 \frac{Y^{-1}(-s)}{s(s - z)}[\Lambda^+(s) - \Lambda^-(s)]ds\,,$$

$$Y(z) = I + \frac{z}{2\pi i}\int_0^1 [\Lambda^+(s) - \Lambda^-(s)]\,\frac{X^{-1}(-s)}{s(s - z)}ds\ , \ \mathrm{Re}\ z < 0$$

(minor modifications might occur if the dispersion matrix is not defined with its leading term being the identity matrix).

Since the behavior of X and Y matrices at infinity is known, the problem of partial indices is avoided. One may apply numerical methods to solve these coupled equations, thus determining the half-range expansion coefficients. To guarantee the success of numerical methods in solving these equations, a uniqueness theorem for their solutions is presented in chapter VI.

Also the factorization of the dispersion matrix is shown to be valid for subcritical systems in the same chapter. This restriction arises to insure the uniqueness of the solutions of the integral form of the transport operator and also to achieve the factorization of the dispersion matrix.

As a conclusion the following remarks will be made about the present work:

a) In finding the continuum eigensolutions of K, no necessity to break the interval [-1,1] into smaller intervals arises. Thus the so called degenerate continuum eigenfunctions are given in a very compact manner by the NxN matrix Φ, as opposed to the rather cumbersome formation of these eigenfunctions in various parts of the interval [-1,1] as done in previous work.

b) The expansion coefficients are found naturally as a result of the analysis without recourse to adjoint solutions and normalization integrals and orthogonality arguments. Thus the singular integral equation method in determining the expansion coefficients is avoided.

c) The domain used for the s dependence of the functions could be extended to L_p spaces as done for the one group case and outlined in appendix B.

d) The half-range problem is presented in a form which is suitable for at least numerical evaluation.

e) A generalization to transfer kernels of the type $\sum_i S_i(s)L_i(t)$ is expected. Since any compact kernel can be approximated with the above mentioned type to any desired accuracy, this generalization should prove to be very useful.

f) The bound found in chapter VI for the validity of the results, hopefully can be generalized to include the case of a critical and supercritical medium, as has been achieved in the one-speed case.

APPENDIX A

Eigensolutions for the Two-Group, Isotropic Case

In this appendix, the results of chapter II will be applied to the two-group case to find the corresponding eigensolutions. These eigensolutions are shown to coincide with those obtained by Siewert and Shieh.[14]

For the two-group case with

$$\Sigma = \begin{bmatrix} \sigma & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} .$$

where det $C \neq 0$, one gets

$$\Lambda(z) = \frac{1}{c} \begin{bmatrix} \sigma^2(C_{22} - 2zcT(1/\sigma z)) & -\sigma C_{12} \\ -\sigma C_{21} & C_{11} - 2cT(1/z) \end{bmatrix}, \ z\varepsilon[-1,1],$$

after using Eq. (2-19). Here the function T stands for

$$T(x) = \tanh^{-1}(x) .$$

When $z\varepsilon[-1,1]$, the only modification in the dispersion matrix occurs in the functions T. Namely, they change to

$$T(1/\sigma s) \rightarrow T(1/\sigma s) \ ; \ T(1/s) \rightarrow T(s) \ \text{for} \ \frac{1}{\sigma} < |s| < 1 , \tag{A-1}$$

$$T(1/\sigma s) \to T(\sigma s) \;;\; T(1/s) \to T(s) \text{ for } |s| < \frac{1}{\sigma} \cdot$$

The determinant of Λ is calculated to yield

$$\Omega(z) = \frac{\sigma^2}{c}[1 - 2z[C_{22}T(1/z) + C_{11}T(1/\sigma z)] + 4z^2 cT(1/z)T(1/\sigma z)] ,$$

which is seen to be proportional to that of reference 14.

The eigenvectors corresponding to the eigenvalues v_i may be calculated by using Eq. (2-40) to be

$$\phi_i(s) = \begin{bmatrix} \sigma/(\sigma v_i - s) \\ -\Lambda_{21}(v_i)/\Lambda_{22}(v_i)(v_i - s) \end{bmatrix}$$

To identify the above vector with the result of reference 14, it is sufficient to multiply it by the constant factor $-v_i\Lambda_{12}(v_i)/\sigma^2$ and then use the property $\Omega(v_i) = 0$ to get

$$\phi_i(s) \propto \begin{bmatrix} v_i C_{12}/(\sigma v_i - s) \\ v_i(C_{22} - 2v_i cT(1/\sigma v_i))/(v_i - s) \end{bmatrix}$$

The last expression is identical to the result of the above mentioned reference.

To show that the continuum eigensolutions can also be matched, first consider the case $|t| < 1/\sigma$. Then one may use Eq. (2-59) to get

$$\phi(t,s) = \begin{bmatrix} \frac{t}{\sigma t - s} + \frac{1}{\sigma}\delta(\sigma t - s)\Lambda_{11}(t) & \frac{1}{\sigma}\delta(\sigma t - s)\Lambda_{12}(t) \\ \delta(t - s)\Lambda_{21}(t) & \frac{t}{t - s} + \delta(t - s)\Lambda_{22}(t) \end{bmatrix} \quad \text{(A-2)}$$

Utilizing Eq. (A-1), one observes that the first and the second columns of the matrix in Eq. (A-2) are proportional to $F_2^{(1)}$ and $F_1^{(1)}$ of reference 14.

For $\frac{1}{\sigma} < |t| < 1$ Eq. (2-59) reduces to

$$\Phi(t,s) = \begin{bmatrix} \dfrac{t\sigma}{\sigma t - s} & 0 \\ \delta(t - s)\Lambda_{21}(t) & \dfrac{t}{t - s} + \delta(t - s)\Lambda_{22}(t) \end{bmatrix}$$

In this case, one may show that the two expansion coefficients A_1 and A_2 become proportional to each other:

$$\frac{A_1(t)}{A_2(t)} = \frac{-\Lambda_{12}(t)}{\Lambda_{11}(t)}$$

so that Eq. (2-61) becomes

$$f_v(s) = \int_{-1}^{1} -\frac{A_2(t)}{\Lambda_{11}(t)} \begin{bmatrix} \dfrac{t\sigma\Lambda_{12}(t)}{\sigma t - s} \\ \delta(t - s)[\Lambda_{12}\Lambda_{21}(t) - \Lambda_{11}\Lambda_{22}(t)] - \Lambda_{11}(t)\dfrac{t}{t - s} \end{bmatrix} dt \; .$$

Thus, in this region the two degenerate eigensolutions collapse into one eigensolution. If one uses Eq. (A-1) in writing the dispersion matrix, one gets this eigensolution as

$$\psi(t,s) = \begin{bmatrix} \dfrac{t\sigma\Lambda_{12}(t)}{\sigma t - s} \\ -\Lambda_{11}(t)\dfrac{t}{t - s} + \delta(t - s)[\Lambda_{12}\Lambda_{21}(t) - \Lambda_{11}\Lambda_{22}(t)] \end{bmatrix}$$

which is seen to be proportional to $F^{(2)}$ of the above mentioned reference. Thus the correspondence of the eigensolutions with the results of reference 14 is completed.

APPENDIX B

Extension of the Case Formulas to L_p Spaces

In previous work dealing with the Case formulas, attention is restricted to Hölder-continuous (or piecewise Hölder-continuous) functions, since the explicit formulas involve principle value integrals and boundary values of Cauchy integrals. The purpose of the present appendix is to show how these results can be extended to a much larger class of functions; namely, the spaces $X_p = [f | sf(s) \varepsilon L_p, p > 1]$. This can be done by extending the results of reference 6 to these spaces. It is also shown that the expansion formulas of that reference can be used to construct the spectral family for the operator K.

Extension of the Case Formulas.

The first step is to quote a theorem which will guarantee that an integral operator A: $L_p \rightarrow L_p$ of the form

$$Af(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{t - x}dt = g(x) \tag{B-1}$$

is a bijection. This theorem is crucial for all the subsequent analysis.

THEOREM 0: Let f(x) $L_p(-\infty,\infty)$, p > 1. Then the formula

$$g(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{t - x}dt \tag{B-2}$$

defines almost everywhere a function g(x) also belonging to $L_p(-\infty,\infty)$.[23] The reciprocal formula

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(t)}{t - x}dt \tag{B-3}$$

also holds almost everywhere and

$$\int_{-\infty}^{\infty} |g(x)|^2 dx < (M_p)^p \int_{-\infty}^{\infty} |f(x)|^p dx \ . \tag{B-4}$$

where M_p depends on p only. If $p = 2$, then

$$\int_{-\infty}^{\infty} |g(x)|^2 dx = \int_{-\infty}^{\infty} |f(x)|^2 dx \ . \tag{B-5}$$

Now consider the formulas[6]

$$f'(s) = \int_{-1}^{1} A(t)\phi(t,s)dt + A(v_0)\phi(v_0,s) + A(-v_0)\phi(-v_0,s) \tag{B-6a}$$

$$A(t) = \frac{1}{N(t)}\int_{-1}^{1} sf'(s)\phi(t,s)ds \ , \tag{B-6b}$$

which hold for Hölder-continuous $sf'(s)$, and which are referred to as Case transforms. The notation will be simplified by defining

$$f(s) = f'(s) - A(v_0)\phi(v_0,s) - A(-v_0)\phi(-v_0,s) \ ; \tag{B-7}$$

then Eqs. (B-6a) and (B-6b) become

$$f(s) = \int_{-1}^{1} A(t)\phi(t,s)dt \ , \ -1 < s < 1 \ , \tag{B-8a}$$

$$A(t) = \frac{1}{N(t)}\int_{-1}^{1} sf(s)\phi(t,s)ds \ , \ -1 < t < 1 \ . \tag{B-8b}$$

The purpose is to show that Eqs. (B-6a) and (B-6b) are valid for $sf'(s)\epsilon L_p(-1,1)$, $p > 1$. Since the discrete parts $A(\pm v_0)\phi(\pm v_0,s)$ are in L_p, it suffices to show that Eqs. (B-8a) and (B-8b) hold for functions f (from which the contributions of the discrete modes has been subtracted out).

Lemma 1: For each

$$f\varepsilon X_p ; X_p = \{ f | \ ||f||_p = (\int_{-1}^{1} |sf(s)|^p ds)^{1/p} < \infty\} \ , \ p > 1 \ ,$$

there is a corresponding $A(t)\varepsilon X_p$ defined by Eqs. (B-8a), and (B-8b), and $A(t)$ depends continuously on f.

Proof: Using the definition

$$\phi(t,s) = \frac{ct}{2}\frac{1}{t-s} + \delta(t-s)\lambda(t) \ , \tag{B-9}$$

and the Case transform, one obtains

$$tA(t) = \frac{t}{N(t)}\{\frac{ct}{2}\int_{-1}^{1}\frac{sf(s)}{t-s}ds + \int_{-1}^{1} sf(s)\lambda(t)\delta(t-s)ds\} \tag{B-10}$$

which one may consider as a formal abbreviation for

$$tA(t) = \frac{\lambda(t)}{\Lambda^+(t)\Lambda^-(t)}tf(t) + \frac{c}{2}\frac{t}{\Lambda^+(t)\Lambda^-(t)}\int_{-1}^{1}\frac{sf(s)ds}{t-s} \ . \tag{B-11}$$

In these equations, the expression

$$N(t) = t\Lambda^+(t)\Lambda^-(t) \tag{B-12}$$

has been utilized.

The functions $\frac{\lambda(t)}{\Lambda^+(t)\Lambda^-(t)}$ and $\frac{t}{\Lambda^+(t)\Lambda^-(t)}$ are continuous on [-1,1].[1] Thus the first term in Eq. (B-11) is in $L_p(-1,1)$ and the second term is in L_p if

$$g(t) = \frac{1}{\pi}\int_{-1}^{1} \frac{sf(s)}{t - s} ds$$

is in $L_p(-1,1)$. But by the previous theorem,

$$\int_{-1}^{1} |g(t)|^p dt \leq \int_{-\infty}^{\infty} |g(t)|^p dt \leq (M_p)^p \int_{-1}^{1} |sf(s)|^p ds . \qquad \text{(B-12)}$$

Thus from Eq. (B-11), it is clear that

$$\int_{-1}^{1} |tA(t)|^p dt \leq N_p^p \int_{-1}^{1} |sf(s)|^p ds .$$

So

$$||A||_p \leq N_p ||f||_p . \qquad \text{(B-13)}$$

Define the map T: $X_p \to X_p$ by

$$A(t) = (Tf)(t).$$

Then

$$||T||_p \leq N_p . \qquad \text{(B-14)}$$

Note: One can show that if one multiplies Eq. (B-11) by λ, then

$$\int_{-1}^{1} |t\lambda(t)A(t)|^p dt \leq \hat{N}_p^p \int_{-1}^{1} |sf(s)|^p ds .$$

where $\hat{N}_p$ is a constant depending only on p.

Thus if

$$\lambda(t)A(t) = (\hat{T}f)(t) ,$$

then

$$||\hat{T}||_p \leq \hat{N}_p \ . \qquad \text{(B-15)}$$

Lemma 2: For each $A(t)$ such that $A(t)$ and $\lambda(t)A(t)\varepsilon X_p$, there exists a $f\varepsilon X_p$ defined by Eqs. (B-8a) and (B-8b).

Proof: By definition

$$\int_{-1}^{1} |tA(t)|^p dt < \infty$$

and

$$\int_{-1}^{1} |t\lambda(t)A(t)|^p dt < \infty \ .$$

By Eq. (B-8a) one defines the function f by

$$f(s) = \lambda(s)A(s) + \frac{c}{2}\int_{-1}^{1} \frac{tA(t)}{t - s} ds \ . \qquad \text{(B-16)}$$

Clearly the first term is in X_p and so is the second one by application of the previous theorem.

Also, if one has a sequence $\{A_n\}$ such that $||A_n - A||_p \to 0$ and $||\lambda(t)A_n(t) - \lambda(t)A(t)||_p \to 0$, then it is obvious that $||f_n - f||_p \to 0$.

It is know that for $sf(s)$ Hölder-continuous (and f of the form of Eq. (B-7)), Eqs. (B-8a) and (B-8b) hold simultaneously. Since the Hölder-continuous functions are dense in L_p, one may choose a sequence $\{f_n\}$ such that $sf_n(s)$ is Hölder-continuous and $f_n \to f\varepsilon X_p$. Then $A_n \to A$ and $\lambda(t)A_n(t) \to \lambda(t)A(t)$ by Eqs. (B-14) and (B-15). Thus by the above paragraph, Eq. (B-8b) holds in the limit.

The above result can be summarized by the following lemma:

Lemma 3: Eqs. (B-8a) and (B-8b) hold for any $f \varepsilon X_p$ and satisfying Eq. (B-7).

One may combine Lemmas 1, 2, and 3 to form the following theorem:

THEOREM A: The domain of the reduced transport operator K may be extended to the spaces X_p, $p > 1$ and the Case transform equations (B-6a) and (B-6b) hold for each f' such that $sf'(s) \varepsilon L_p(-1,1)$.

Resolution of the Identity of K.

For $-1 < w < 1$, one defines the operator E(w) as

$$E(w)f(s) = \int_{-1}^{w} A(t)\phi(t,s)dt \;,$$

$$E(w)f(s) = \begin{cases} A(s)\lambda(s) + \dfrac{c}{2}\displaystyle\int_{-1}^{w} \dfrac{tA(t)}{t - s}dv, & -1 < s \leq w \;, \\ \dfrac{c}{2}\displaystyle\int_{-1}^{w} \dfrac{tA(t)dv}{t - s}, & w < s < 1 \;. \end{cases} \tag{B-17}$$

From the above analysis, it is clear that the terms in Eq. (B-17) represent bounded operators in X_p, acting on f. Thus for $-1 < w < 1$, E(w) is a bounded operator. In this section it will be shown that the family E(w) forms part of the spectral family of K.

First, for $\varepsilon > 0$, one has

$$[E(w + \varepsilon) - E(w)]f(s) = \int_{w}^{w + \varepsilon} A(t)\phi(t,s)dt \;,$$

$$= \begin{cases} \lambda(s)A(s), & w \leq s \leq w + \varepsilon \\ 0 , & \text{otherwise} \end{cases} \quad + \frac{c}{2}\int_{w}^{w+\varepsilon} \frac{tA(t)}{t - s}dt \ .$$

The norm of the first term is just

$$\{\int_{w}^{w+\varepsilon} |\lambda(t)A(t)|^{p}dt\}^{1/p}$$

which tends to zero for a fixed A(i.e., fixed f) as $\varepsilon \to 0$. Call

$$g(s,\varepsilon) = \frac{1}{\pi}\int_{w}^{w+\varepsilon} \frac{tA(t)}{t - s}dt \ .$$

Then by theorem 0,

$$||g(s,\varepsilon)||_{p}^{p} = \int_{-1}^{1} |g(s,\varepsilon)|^{p}ds \leq \int_{-\infty}^{\infty} |g(s,\varepsilon)|^{p}ds \ ,$$
$$\leq M_{p}^{p}\int_{w}^{w+\varepsilon} |tA(t)|^{p}dt \ ,$$

and this term also approaches 0 for a fixed f as $\varepsilon \to 0$. One may now state the result as a lemma:

Lemma 4: For each f X_p and $-1 \leq w < 1$,

$$\lim_{\varepsilon \to 0} ||(E(w + \varepsilon) - E(w))f||_{p} = 0 \qquad \text{(B-18)}$$
$\varepsilon \to 0$

That is, E(w) is a continuous function of win the strong operator topology.

Next, one verifies the following lemma:

Lemma 5:

$$E(w_1)E(w_2) = E(w_2)E(w_1) = E(w) \text{ where } w = \min\{w_1,w_2\} \ . \tag{B-19}$$

Proof: For definiteness, one considers $w_1 \leq w_2$:

$$E(w_2)f(s) = \int_{-1}^{w_2} A(t)\phi(t,s)dt \ .$$

Then the expansion coefficients of $E(w_2)f(s)$ are:

$$B(t) = \begin{cases} A(t) & -1 \leq t \leq w_2 \\ 0 & w_2 < t \leq 1 \end{cases}$$

Thus,

$$E(w_1)E(w_2)f(s) = \int_{-1}^{w_1} B(t)\phi(t,s)dt \ ,$$

$$= \int_{-1}^{w_1} A(t)\phi(t,s)dt \ ,$$

$$= E(w_1)f(s) \ .$$

In a similar way, one obtains $E(w_2)E(w_1)f(s) = E(w_1)f(s)$.

Before proving lemma 7, the following lemma, which is essential in proving lemma 7, will be established.

Lemma 6: $Kf(s) = \int_{-1}^{1} zA(z)\phi(z,s)dz.$ (B-20)

Proof: Applying the same method as in the previous chapters, one gets a simple closed contour Γ containing the line segment [-1,1] and then uses

$$Kf(s) = \frac{1}{2\pi i}\int_\Gamma K(z - K)^{-1} f(s)dz ,$$

$$= \frac{1}{2\pi i}\int_\Gamma (z(z - K)^{-1} - I) f(s)dz ,$$

$$= \frac{1}{2\pi i}\int_\Gamma z(z - K)^{-1} f(s)dz .$$

One may now compute the contour integration exactly as before, with minor modifications, to get the following result:

$$Kf(s) = \int_{-1}^{1} zA(z)\phi(z,s)dz .$$

Lemma 7: For each $f\varepsilon X_p$ of the form (B-6),

$$Kf(s) = \int_{-1}^{1} wdE(w)f(s) . \qquad \text{(B-21)}$$

Proof: Let $U(w) = \int_{-1}^{1} (E(w)f(s))g(s)ds$ where $f\varepsilon X_p$ and $q\varepsilon X_q$ (the dual space of X_p: $1/p + 1/q = 1$.) Then

$$U(w) = \int_{-1}^{w} A(s)\lambda(s)g(s)ds + \int_{-1}^{1} ds\int_{-1}^{w} dt\frac{ct}{2}\,\frac{A(t)}{t - s}g(s) ,$$

$$= \int_{-1}^{w} A(s)\lambda(s)g(s)ds + \int_{-1}^{w} dt\frac{ct}{2}A(t)\int_{-1}^{1}\frac{g(s)}{t - s}ds ,$$

$$= \int_{-1}^{w} A(s)\lambda(s)g(s)ds + \int_{-1}^{w}\frac{cs}{2}A(s)Lg(s)ds,$$

$$= \int_{-1}^{w} A(s)[\lambda(s)g(s) + \frac{cs}{z}Lg(s)]ds .$$

Thus U is differentiable almost everywhere and

$$U'(w) = A(w)[\lambda(w)g(w) + \frac{cw}{2}Lg(w)] .$$

Now one may write

$$\int_{-1}^{1} wd\,U(w) = \int_{-1}^{1} wU'(w)dw \;,$$

$$= \int_{-1}^{1} wA(w)[\lambda(w)g(w) + \frac{cw}{2}Lg(w)]\,dw$$

$$= \int_{-1}^{1} wA(w)\lambda(w)g(w) + \int_{-1}^{1} wA(w)\frac{cw}{2}\int_{-1}^{1}\frac{g(s)}{w-s}dsdw \;,$$

$$= \int_{-1}^{1} [wA(w)\lambda(w) + \int_{-1}^{1} tA(t)\frac{ct}{2}\frac{dt}{t-w}]g(w)dw$$

and use the previous lemma,

$$\int_{-1}^{1} wd\,U(w) = \int_{-1}^{1} Kf(w)g(w)dw$$

to get the following expression:

$$\int_{-1}^{1} Kf(s)g(s)ds = \int_{-1}^{1} wd\,U(w) \;,$$

$$= \int_{w=-1}^{1} wd\int_{s=-1}^{1} [E(w)f](s)g(s)ds \;,$$

$$= \int_{-1}^{1} dsg(s)\int_{w=-1}^{1} wd(E(w)f)(s) \;,$$

where the interchange of limits is justified because w and $E(w)f(s)$ are continuous in w. The above equation implies

$$Kf(s) = \int_{-1}^{1} wd(E(w)f)(s) \;,$$

$$K = \int_{-1}^{1} wdE(w) \;.$$

Note: The following calculation gives the above result directly, but only formally:

$$\int_{-1}^{1} w dE(w) f(s) = \int_{-1}^{1} w \frac{dE(w)}{dw} f(s) dw ,$$

$$= \int_{-1}^{1} wA(w)\phi(w,s) dw ,$$

$$= Kf(s).$$

Lemma 8: $KE(w) = E(w)K$. (B-22)

Proof: By Lemma 6,

$$Kf(s) = \int_{-1}^{1} tA(t)\phi(t,s) dt .$$

Thus by Eq. (B-17),

$$E(w)Kf(s) = \int_{-1}^{w} tA(t)\phi(t,s) dt .$$

In a similar way, one obtains

$$KEf(s) = \int_{-1}^{w} tA(t)\phi(t,s) dt .$$

Equality of the last two equations proves the lemma.

Also the following identities hold by definition:

$$E(-1) = 0 ,$$

$$E(1) = 1 . \tag{B-23}$$

Lemmas 4, 5, 7 and 8, together with Eqs. (B-23) complete the proof of the following theorem:[24]

THEOREM B: For $sf'(s)L_p(-1,1)$, one defines

$$E(\pm v_0)f'(s) = \frac{1}{N(\pm v_0)}\int_{-1}^{1} tf'(t)\phi(\pm v_0,t)dt\phi(\pm v_0,s) .$$

For $t\varepsilon[-1,1]$, one lets $E(t)$ defined by Eq. (B-17) . Then

$$K^n f'(s) = v_0^n E(v_0)f'(s) + (-v_0)^n E(-v_0)f'(s) + \int_{-1}^{1} t^n dE(t)f(s) . \quad \text{(B-24)}$$

and $E(t)$ is the spectral family of projection operators for the operator $K, t\varepsilon\{\pm v_0 U[-1,1]\}$.

Half-Range Theory.

Let $sf_0(s)$ be defined and Hölder-continuous on $0 \leq s \leq 1$. Define[2]

$$f_e(s) = \begin{cases} f_0(s) & 0 < s \leq 1 \\ \int_0^1 J(s,t)f_0(t)dt & -1 \leq s < 0 , \end{cases} \quad \text{(B-25)}$$

where

$$J(s,t) = \frac{1}{X(s)X(-t)}\frac{ct}{2}\frac{1}{t-s}$$

and

$$X(z) = \Lambda^{1/2}(\infty)(z - v_0)\exp\frac{1}{2\pi i}\int_0^1 \ln\frac{\Lambda^+(s)}{\Lambda^-(s)}\frac{ds}{s-z} ,$$

$$\Lambda(z) = X(z)X(-z) .$$

Then the full range coefficients $A(v)$ of $f_e(s)$ are zero for $v < 0$;

$$f_e(s) = \int_0^1 A(t)\phi(t,s)dt + A(v_0)\phi(v_0,s) .$$

Extending f_0 onto X_p by continuity, one obtains the extension f_e of f_0, and $f_e \varepsilon X_p$, also.

By definition of E(w), one has for $-1 \leq w \leq 1$,

$$E(w)f_e(s) = \int_{-1}^{w} A(t)\phi(t,s)dt ,$$

$$= \begin{cases} \int_0^w A(t)\phi(t,s)dt , & 0 < w \leq 1 \\ 0 , & -1 \leq w \leq 0 . \end{cases}$$

Thus

$$f_e(s) = \int_0^1 dE(w)f_e(s) + E(v_0)\phi(v_0,s) .$$

For $0 < s < 1$, this reduces to

$$f_0(s) = \int_0^1 dE(w)f(s) + E(v_0)\phi(v_0,s) \qquad \text{(B-26)}$$

which is just the statement of the half-range completeness theorem.

Solutions of Transport Problems.

Consider the problem

$$\frac{\partial}{\partial x}f(x,s) + K^{-1}f(x,s) = 0 , \quad x > 0 ,$$

$$f(0,s) = f_0(s) .$$

Letting f_e be the extension of f_0 described in Eq. (B-25), one claims that the solution of the problem is

$$f(x,s) = \int_0^1 e^{-x/t} d(E(t) f_e(s)) \ .$$

This function satisfies the boundary conditions, and it also satisfies the transport equation, as can be seen by inspection.

Consider next the problem

$$\frac{\partial}{\partial x} f(x,s) + K^{-1} f(x,s) = q_0(x,s) \ , \ x_0 < x < x_1 \ , \tag{B-27}$$

$$q_0(x,s) = q(x,s)/s \ .$$

Let's look only for a particular solution. Boundary conditions can be met by using solutions of the homogenous equation.

One looks for a particular solution of the form

$$f(x,s) = \int_{-1}^{1} d[E(t) f(x,s,t)] \ . \tag{B-28}$$

One also has the identity

$$q_0(x,s) = \int_{-1}^{1} d[E(t) q_0(x,s)] \ . \tag{B-29}$$

Inserting Eqs. (B-28) and (B-29) in Eq. (B-27), one obtains

$$\int_{-1}^{1} d[E(w) q_0(x,s)] = \int_{-1}^{1} dE(t) [\frac{\partial}{\partial x} f(x,s,t) + \frac{1}{v} f(x,s,t)] \ .$$

This is solved by taking

$$q_0(x,s) = \frac{\partial}{\partial x} f(x,s,t) + \frac{1}{v} f(x,s,t)$$

or

$$\frac{\partial}{\partial x} e^{x/t} f(x,s,t) = e^{x/t} q_0(x,s) \ .$$

For $t > 0$, one integrates from x_0 to x to get

$$e^{x/t} f(x,s,t) - e^{x_0/t} f(x_0,s,t) = \int_{x_0}^{x} e^{y/t} q_0(y,s)\,dy \ .$$

So

$$f(x,s,t) = e^{(x_0 - x)/t} f(x_0,s,t) + \int_{x_0}^{x} e^{(y - x)/t} q_0(y,s)\,dy \ . \qquad \text{(B-30)}$$

For $t < 0$, one integrates from x_1 to x to get a similar equation:

$$f(x,s,t) = e^{(x_1 - x)/t} f(x_1,s,t) + \int_{x_1}^{x} e^{(y - x)/t} q_0(y,s)\,dy . \qquad \text{(B-31)}$$

The general solution of Eq. (B-27) which is bounded is then given by Eq. (B-28), where f is defined in Eqs. (B-31) and (B-32), and $f(x_i,s,t)$ is arbitrary, $i = 0,1$. A particular solution is obtained by setting $f(x_i,s,t) = 0$.

Discussion

The Case transform equations (B-6a) and (B-6b) were originally derived for Hölder-continuous functions f', and it is shown that they can be extended to functions $f' \varepsilon X_p$. Furthermore, the spectral family for K in each of the X_p spaces is constructed for $p > 1$. It is also shown how to use this spectral family to solve typical problems. Several aspects of these results seem worthy of further comment.

First, the condition that $sf'(s)$ be in L_p means that the Case formulas hold for functions f' which can be highly singular at $s = 0$. However, this feature was shown in the previous section to be essential in solving problems with sources, since the modified source $q_0 = q/s$ had to be written as a full-range expansion. Second, the removal of the unphysical Hölder-condition of f' constitutes an obvious generalization. It must be emphasized that for Hölder-continuous f', equations (B-6) hold pointwise for each s, while for $f' \varepsilon X_p$, these equations hold only in the integral norm of X_p.

The existence of a spectral family for K was established by Hangelbroek[26] for L_2 and $c < 1$ by showing that K is topologically equivalent to a self adjoint operator. The present results show that this family exists not only for L_2 and $c < 1$, but also for the much larger spaces X_p, $p > 1$, and for any value of $c > 0$.

The physically natural space for the transport operator is X_1, but in this space interesting mathematical difficulties occur. For $p = 1$, theorem 0 is no longer true; in fact, the principle value operator is definitely unbounded.[23] Also the projection operators $E(t)$ for $-1 \leq t \leq 1$ are unbounded, so formulas such as Eq. (B-21) cannot hold in the usual sense.

The problem of generalizing the Case formulas to X_1 is solved in reference 25 by methods different from those introduced here.

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A FUNCTIONAL ANALYTIC APPROACH TO

MULTIGROUP TRANSPORT THEORY

by

Selim Sancaktar

(Abstract)

A functional Analytic method which was first introduced by Larsen and Habetler for the one-speed isotropic case in 1973 is applied to full and half-space multigroup problems in one dimension with a constant and invertible transfer matrix. The Case-type eigenfunction expansion formulas for the solutions of these problems are explicitly obtained. For the half-space case, the formulas are expressed in terms of two matrices X and Y which provide the Wiener-Hopf factorization of the dispersion matrix. The method applied yields compact results avoiding the calculation of adjoint solutions and normalization integrals to determine the expansion coefficients. Since the method proves to be amenable to further generalization, the case of a degenerate transfer kernel is also considered along the same lines, yielding the expansion formulas for that problem in the full and half-space cases. The expansion formulas are shown to be valid at least for subcritical media, but an extension to critical problems is expected.