CIRCULARITY OF GRAPHS

by

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DEDICATION

This work is dedicated to

My parents: James and Barbara Blum

My brother, Jimmy; my cousin, Bobbi

My grandmothers: Agnes Thorne and Ethel Blum

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INTRODUCTION

The circularity of a finite connected graph was first defined in [2] by H. Bell, E. Brown, R. F. Dickman, Jr., and E. Green in conjunction with the circularity of locally connected, connected, normal topological spaces. These authors continued the study of the circularity of graphs in [1], using combinatorial techniques. This thesis is a continuation of [1].

The circularity of a graph is defined in terms of circular coverings. We say that a finite collection of distinct topologically closed connected subsets, $\{C_0, C_1, \ldots, C_{n-1}\}$, of a finite connected graph is a circular covering of the graph if and only if the union of the subsets is the graph; each element of the covering contains at least one vertex of the graph; and $C_i \cap C_j \neq \phi$ iff $|(i-j)(\text{mod } n)| \leq 1$. For any finite connected graph, G, the circularity of G, denoted by $\sigma(G)$, is defined as $\sigma(G) = \max\{n \in N \mid G \text{ has a circular covering with n elements}\}$.

Before proceeding further, we would like to review some of the basic definitions and terminology used in graph theory. These can be found in [4] and [5].

A <u>graph</u>, G, is a nonempty set, V(G), of elements called <u>vertices</u> together with a set, E(G), of two-element subsets of V(G) called <u>edges</u>. The set of vertices of the graph is called the <u>vertex set</u> of the graph and the set of edges of the graph is called the <u>edge set</u> of the graph. If $v,w \in V(G)$ are such that $\{v,w\} \in E(G)$, then we will write this edge as $e = vw$ and we say that the vertices v and w are <u>adjacent</u> in the graph G. The <u>order</u> or <u>valency</u> of a vertex, v, is equal to the number of vertices in G which are adjacent to v and will be denoted by $o(v)$.

It is sometimes convenient to allow a point x of a graph which is not a vertex to have order 2 in the sense that there is an open set about x whose frontier meets the graph in exactly two points. This notion can be found in [6].

If no vertex is adjacent to itself and if no pair of vertices of the graph determines more than one edge of the graph, then the graph will be called a simple graph. If $V(G)$ is a finite set, then G is called a finite graph. A subset H of G is called a subgraph of G if H is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

A walk in a graph G is a sequence of vertices: $v_1, v_2, \ldots, v_n$, not necessarily distinct, and a corresponding sequence of edges: $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$, not necessarily distinct. The length of a walk is equal to the number of edges in the walk. A walk in which no edge is repeated is called a simple walk. A walk in which no vertex is repeated is called a path. If v is the first vertex of a path and if w is the last vertex of the path, then the path is called a (v,w)-path. A walk is closed when its first and last vertices are equal. A cycle is a closed simple walk such that no vertices other than the first and last are equal. In other words, J is a cycle if and only if $V(J) = \{v_1, v_2, \ldots, v_n\}$ is a set of distinct vertices and $E(J) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$ is a set of distinct edges. A cycle on n vertices is called an n-cycle.

A graph G is connected if and only if for each pair of vertices $v,w \in V(G)$, there is a (v,w)-path in the graph. A connected graph G is nonseparable if and only if it is not the union of two connected graphs G_1 and G_2 such that $|V(G_1) \cap V(G_2)| = 1$ and $E(G_1) \cap E(G_2) = \phi$.

If G is a connected graph and if $v,w \in V(G)$, then the distance between v and w in the graph G is defined as $d_G(v,w) = \min\{m \mid \text{there is a } (v,w)\text{-path in } G \text{ of length } m\}$. The subscript G will be dropped if it is clear in which graph the distance is being considered. If G is a connected graph and if J is a subgraph of G which is a cycle, then J is called a proper cycle if and only if for each pair of vertices, $v,w \in V(J)$, $d_J(v,w) = d_G(v,w)$.

A tree is a simple connected graph which contains no cycle. The graph, K_n, with vertex set $V(K_n) = \{v_1,v_2,\ldots,v_n\}$ and edge set $E(K_n) = \{v_iv_j \mid 1 \leq i < j \leq n\}$, is called the complete graph on n vertices. If V_1 is a set of m vertices and if V_2 is a set of n vertices where $V_1 \cap V_2 = \phi$, then the graph, $K_{m,n}$, with vertex set $V(K_{m,n}) = V_1 \cup V_2$ and edge set $E(K_{m,n}) = \{vw \mid v \in V_1 \text{ and } w \in V_2\}$ is called the complete bipartite graph on m and n vertices. The graph, W, with vertex set $V(W) = \{v_1,\ldots,v_n,x\}$ and edge set $E(W) = \{v_1v_2,v_2v_3,\ldots,v_{n-1}v_n,v_nv_1\} \cup \{v_1x,v_2x,\ldots,v_nx\}$ is called a wheel.

The following results concerning the circularity of a finite connected graph have already been established in [1] and [2].

1. $\sigma(G) = 2$ if and only if G is a tree.
2. If G contains a cycle, then $\sigma(G) \geqq 6$.
3. If $p = \max\{n \in N \mid G \text{ has a proper cycle on } n \text{ vertices}\}$, then $\sigma(G) \leqq 2p$.
4. If G is a cycle on n vertices, then $\sigma(G) = 2n$.
5. If G is a complete graph on $n \geq 3$ vertices, then $\sigma(G) = 6$.
6. If G is a complete bipartite graph on $m \geq 3$ and $n \geq 3$ vertices, then $\sigma(G) = 6$.
7. If G is a planar graph, then $\sigma(G)$ is even.

8. If G is a separable graph with nonseparable components $G_1, G_2, \ldots, G_n$, then $\sigma(G) = \max\{\sigma(G_1), \sigma(G_2), \ldots, \sigma(G_n)\}$.

The combinatorial method for determining the circularity of a finite connected graph developed in [1] uses special functions called admissible maps. In order to define an admissible map, let $Z_r = \{\overline{a} \mid r \in Z \text{ and } 0 \leq a < r\} = \{\overline{0}, \overline{1}, \ldots, \overline{r-1}\}$ denote the cyclic additive group of order r. For each $\overline{a} \in Z_r$, let $T(\overline{a}) = \{(\overline{a-1}, \overline{a}), (\overline{a}, \overline{a}), (\overline{a}, \overline{a+1})\}$ and let $A(r) = \cup\{T(\overline{a}) \mid \overline{a} \in Z_r\}$.

<u>Definition</u>: If G is a finite connected graph with vertex set V(G) and edge set E(G), then a function $f: V(G) \longrightarrow A(r)$ is <u>admissible</u> if and only if

(A-1) For each edge $e = vw \in E(G)$, there exists $\overline{a} \in Z_r$, such that $f(v), f(w) \in T(\overline{a}) \cup T(\overline{a+1})$.

(A-2) If $v, w \in V(G)$ are such that $f(v), f(w) \in T(\overline{a})$ for some $\overline{a} \in Z_r$, then there is a (v,w)-path in G such that for each vertex u of the path, $f(u) \in T(\overline{a})$.

(A-3) For each $\overline{a} \in Z_r$, either there is a vertex $v \in V(G)$ with $f(v) = (\overline{a}, \overline{a+1})$ or there is an edge $e = vw \in E(G)$ with $f(v) \in T(\overline{a})$ and $f(w) \in T(\overline{a+1})$.

If the map $f: V(G) \longrightarrow A(r)$ satisfies only condition (A-1), then f is said to be <u>preadmissible</u>.

It was shown in [1] that a graph, G, has a circular covering with r elements if and only if there exists an admissible map $f: V(G) \longrightarrow A(r)$. Consequently, the circularity of a graph G can be described as $\sigma(G) = \max\{r \in N \mid \text{there is an admissible map } f: V(G) \longrightarrow A(r)\}$.

If $f:V(G) \longrightarrow A(r)$ is a function, then for each $\bar{a} \in Z_r$, f naturally induces the following subgraphs of G. The subgraph $G(\bar{a})$ has vertex set $V(\bar{a}) = f^{-1}(T(\bar{a}))$ and edge set $E(\bar{a}) = \{e = vw \in E(G) \mid v,w \in V(\bar{a})\}$. The subgraph $G(\bar{a},\overline{a+1})$ of G has vertex set $V(\bar{a},\overline{a+1}) = V(\bar{a}) \cup V(\overline{a+1})$ and edge set $E(\bar{a},\overline{a+1}) = \{e = vw \in E(G) \mid v,w \in V(\bar{a},\overline{a+1})\}$. These subgraphs provide the following alternate way of defining an admissible map. This result was established in [1].

<u>Proposition</u>: If G is a finite connected graph, then a preadmissible map $f:V(G) \longrightarrow A(r)$ is admissible if and only if for each $\bar{a} \in Z_r$, both $G(\bar{a})$ and $G(\bar{a},\overline{a+1})$ are connected subgraphs of G. Moreover, condition (A-2) is true if and only if $G(\bar{a})$ is connected for each $\bar{a} \in Z_r$.

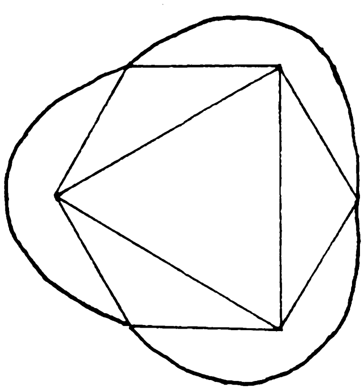
Two other results concerning admissible maps which have already been mentioned in [1] are:

(1) If $f:V(G) \longrightarrow A(r)$ is a preadmissible map, where G is a finite connected graph, then the restriction of f to the vertex set of any subgraph of G is also a preadmissible map.

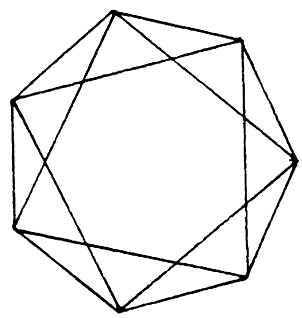
(2) If G is a finite connected graph and if $f:V(G) \longrightarrow A(r)$ is admissible, then for each r', $0 \leq r' \leq r$, there is an admissible map $f_{r'}:V(G) \longrightarrow A(r')$.

In [1], a class of graphs, $\{D(n) \mid n \in N,\ n \geq 6\}$ was introduced and it was shown that for each $n \geq 6$, $\sigma(D(n)) = n$. It was also shown that D(n) is planar when n is even and D(n) is nonplanar when n is odd. These graphs are defined as follows: If n is odd, $V(D(n)) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(D(n)) = \{v_i v_{i+1} \mid 0 \leq i \leq n-1\} \cup \{v_i v_{i+2} \mid 0 \leq i \leq n-1\}$ where this numbering is taken modulo n. If $n = 2r$ is even, $V(D(n)) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(D(n)) = \{v_i v_{i+1} \mid 0 \leq i \leq n-1\} \cup \{v_{2i} v_{2i+2} \mid 0 \leq i \leq r-1\} \cup \{v_{2i+1} v_{2i+3} \mid 0 \leq i \leq r-1\}$ where this numbering is also taken modulo n.

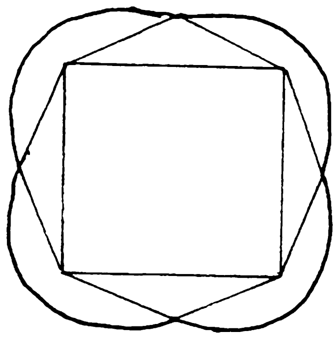
Examples of D(n) for n = 6,7,8,9 are shown in Figure 1. These examples, together with other examples, will be used in this thesis to illustrate various results.



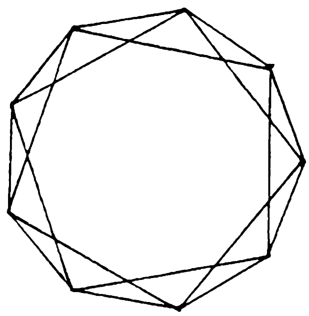
D(6)



D(7)



D(8)



D(9)

Figure 1

In this thesis, we only consider finite, connected, simple, nonseparable graphs. The reason for this restriction is clear, since the work done on the subject of the circularity of graphs in [1] and [2] has narrowed the questions down to these particular graphs.

Since the circularity of a graph is an invariant of the graph which has only recently been defined, one of the purposes of this thesis is to provide properties and techniques which lead to a better understanding of, and which facilitate the calculation of, the circularity of a graph. To this end, we first show, in Section 1, that when defining an admissible map $f:V(G) \longrightarrow A(r)$, the elements of the form $(\overline{a},\overline{a})$ in $A(r)$ are not necessary. We next show that when r is even, only the elements of the form $(\overline{2a},\overline{2a+1})$ in $A(r)$ are necessary. We next observe the one-to-one correspondence between the ordered pairs $(\overline{2a},\overline{2a+1})$ $(0 \leq a < \frac{r}{2})$ and the elements of $Z_{r/2}$. This observation leads to the definitions of a co-admissible map, $g:V(G) \longrightarrow Z_n$, and the co-circularity, $\eta(G)$, of a graph G, from which the result $\sigma(G) = 2\eta(G)$ or $2\eta(G)+1$ is obtained. We also show how the distance between two vertices in a graph is related to their images under a co-admissible map. We use this result to explain one reason why the graph $D(2r+1)$ has a proper cycle on $r+1$ vertices but does not have co-circularity $r+1$, whereas the graph $D(2r)$ does have a proper cycle on r vertices and does have co-circularity r. We conclude Section 1 by showing that if g is a co-admissible map on a graph G, then G contains a cycle J for which $g|_J$ is also a co-admissible map. We call such a cycle "co-admissible" and give an example which shows that a co-admissible cycle is not necessarily proper.

In Section 2 we give necessary and sufficient conditions for extending a co-admissible map on a cycle of a graph to the entire graph. This is a desirable result because it provides lower bounds for the co-circularity and hence the circularity of a graph. In order to achieve these conditions, we introduce "k-adequate cycles" and "k-spanning subgraphs". We show that when $k = 0, 1$, or 2, the graph contains a proper co-admissible cycle. As an example, we apply these ideas to the class of graphs, $\{D(n) \mid n \in N, n \geq 6\}$.

In Section 3 we consider graphs which have vertices of order two. We discuss the idea of a "reduced" graph and use the fact that every graph is homeomorphic to a reduced graph to show that $\sigma(G') \leq \sigma(G)$ if G' is reduced and homeomorphic to G. We also show that if $f: V(G) \longrightarrow A(\sigma(G))$ is an admissible map and if $v \in V(G)$ has order two, then either v induces two elements of $Z_{\sigma(G)}$ not induced by any other vertex of G or v only induces elements of $Z_{\sigma(G)}$ also induced by at least one of the two vertices adjacent to it. This result leads to a condition for which $\sigma(G) = \sigma(G') + 2k$ where G' is reduced and homeomorphic to G and k is a nonnegative integer.

In Section 4 we consider planar graphs. The principal result of this section shows that if $g: V(G) \longrightarrow Z_n$ $(n \geq 3)$ is any co-admissible map on a planar graph G, then in any planar representation of G, there are exactly two faces bounded by co-admissible cycles. This result provides an upper bound for the co-circularity involving only the faces of the planar graph.

Section 1

CO-ADMISSIBLE MAPS AND CYCLES

The circularity of a finite connected nonseparable simple graph, G, is known to be $\sigma(G) = \max\{r \in N \mid$ there is an admissible map $f:V(G) \longrightarrow A(r)\}$. Using this description, it is clear that calculating the circularity of a graph amounts to finding an admissible map on the vertex set of the graph for the largest possible r. Since any admissible map, $f:V(G) \longrightarrow A(r)$, can be used, it is reasonable to try to find an admissible map whose range has the fewest elements. In this section we first show that for any positive integer r, the elements of the form $(\overline{a},\overline{a})$ in A(r) are not necessary when defining admissible maps. Next we show that when r is even, only one-fourth of the elements of A(r) are necessary when defining an admissible map. Finally, we introduce co-admissible maps, the co-circularity of a graph, and show that for every graph and each co-admissible map on the graph, there is a cycle in the graph for which the restriction of the co-admissible map to the cycle is also a co-admissible map.

Let $Z_r = \{\overline{0},\overline{1},\ldots,\overline{r-1}\}$ denote the cyclic group of order $r \geq 1$. For each $\overline{a} \in Z_r$, let $T^*(\overline{a}) = \{(\overline{a-1},\overline{a}),(\overline{a},\overline{a+1})\}$ and define $A^*(r) = \cup\{T^*(\overline{a}) \mid \overline{a} \in Z_r\}$.

Theorem 1.1 Let G be a graph. If there is an admissible map $f:V(G) \longrightarrow A(r)$, then there is an admissible map $f^*:V(G) \longrightarrow A^*(r)$.

Proof: Let G be a graph and let $f:V(G) \longrightarrow A(r)$ be an admissible map. Define a sequence of maps, $\{f_i:V(G) \longrightarrow A(r)\}$, and a corresponding sequence of subsets of V(G), $\{V_i(G)\}$, as follows:

Let $f_0 = f:V(G) \longrightarrow A(r)$.

Let $V_i(G) = \{v \in V(G) \mid f_i(v) = (\bar{a},\bar{a}) \text{ for some } \bar{a} \in Z_r\}$.

Define $f_{i+1}:V(G) \longrightarrow A(r)$ inductively as follows:

If $v \in V(G) - V_i(G)$, then put $f_{i+1}(v) = f_i(v)$.

If $v \in V_i(G)$ and if $f_i(v) = (\bar{a},\bar{a})$, then put

1) $f_{i+1}(v) = (\bar{a},\overline{a+1})$ if v is adjacent to some vertex w with $f_i(w) \in T(\overline{a+1})$.

2) $f_{i+1}(v) = (\overline{a-1},\bar{a})$ if v is adjacent to some vertex u with $f_i(u) \in T(\overline{a-1})$ but v is not adjacent to any vertex w with $f_i(w) \in T(\overline{a+1})$.

3) $f_{i+1}(v) = f_i(v)$ if v is adjacent only to vertices z with $f_i(z) = (\bar{a},\bar{a})$.

Clearly, each $f_i:V(G) \longrightarrow A(r)$ is well-defined. Moreover, for each i, if $f_i(v) \in T(\bar{a})$, then $f_{i+1}(v) \in T(\bar{a})$. Using this fact and induction, we can prove that each $f_i:V(G) \longrightarrow A(r)$ is an admissible map.

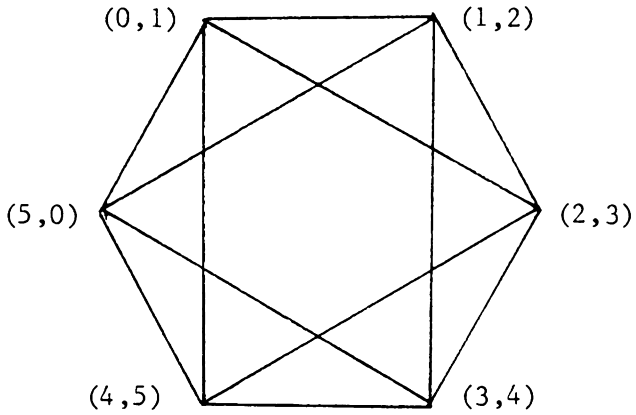
By hypothesis, $f:V(G) \longrightarrow A(r)$ is admissible, so since $f_0 = f$, $f_0:V(G) \longrightarrow A(r)$ is admissible. Suppose that $f_i:V(G) \longrightarrow A(r)$ is admissible. To show that $f_{i+1}:V(G) \longrightarrow A(r)$ is admissible, we verify the three conditions of admissibility. Let $e = vw \in E(G)$. Since f_i is admissible, there is $\bar{a} \in Z$ such that $f_i(v), f_i(w) \in T(\bar{a}) \cup T(\overline{a+1})$. Consequently, $f_{i+1}(v), f_{i+1}(w) \in T(\bar{a}) \cup T(\overline{a+1})$ so that (A-1) is satisfied. For each j and for each $\bar{a} \in Z_r$, let $G_j(\bar{a})$ be the subgraph of G with vertex set $V_j(\bar{a}) = f_j^{-1}(T(\bar{a}))$ and edge set $E_j(\bar{a}) = \{e = vw \in E(G) \mid v,w \in V_j(\bar{a})\}$. In order to show that f_{i+1} satisfies (A-2) it suffices to show that $G_{i+1}(\bar{a})$ is connected. By the induction hypothesis, f_i is admissible so that $G_i(\bar{a})$ is connected. Moreover, $G_i(\bar{a}) \subseteq G_{i+1}(\bar{a})$, so if $G_i(\bar{a}) = G_{i+1}(\bar{a})$, $G_{i+1}(\bar{a})$ is connected. If not, let $v \in V_{i+1}(\bar{a}) - V_i(\bar{a})$. The definition of f_{i+1} shows that v must be adjacent to some vertex w with $f_i(w) \in T(\bar{a})$; that is, $w \in V_i(\bar{a})$. Thus $G_{i+1}(\bar{a})$ is connected.

Let $\bar{a} \in Z_r$. Since f_i is admissible, either there is a vertex v such that $f_i(v) = (\bar{a},\overline{a+1})$, in which case $f_{i+1}(v) = (\bar{a},\overline{a+1})$, or there is an edge e = vw such that $f_i(v) \in T(\bar{a})$ and $f_i(w) \in T(\overline{a+1})$, in which case $f_{i+1}(v) \in T(\bar{a})$ and $f_{i+1}(w) \in T(\overline{a+1})$. Thus (A-3) is satisfied.

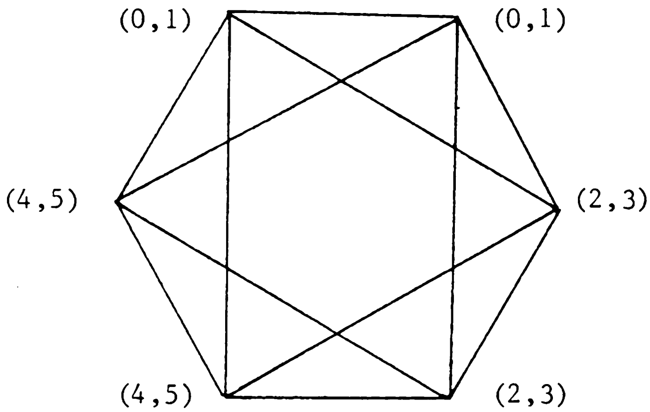
Consequently, each of the maps $f_i:V(G) \longrightarrow A(r)$ is admissible. In particular, the map $f_n:V(G) \longrightarrow A(r)$ where $n = \min\{i \mid V_i(G) = \phi\}$ is admissible. Furthermore, since $V_n(G) = \phi$, f_n maps into $A^*(r)$. By putting $f^* = f_n:V(G) \longrightarrow A^*(r)$, the proof is complete.

Consider the class of graphs $\{D(n) \mid n \in N,\ n \geq 6\}$. The map $f:V(D(n)) \longrightarrow A^*(n)$ defined by $f(v_i) = (\bar{i},\overline{i+1})$ where $V(D(n)) = \{v_0,v_1,\ldots,v_{n-1}\}$ is admissible. When n is odd, any admissible map $f:V(D(n)) \longrightarrow A^*(n)$ must be onto. However, when n is even, there is an admissible map $f^*:V(D(n)) \longrightarrow A^*(n)$ which maps to only half of the elements of $A^*(n)$. These two situations are illustrated in Figure 2 for D(6) and D(7). Each vertex is labeled by the element in $A^*(n)$ to which it maps under f or f^*.

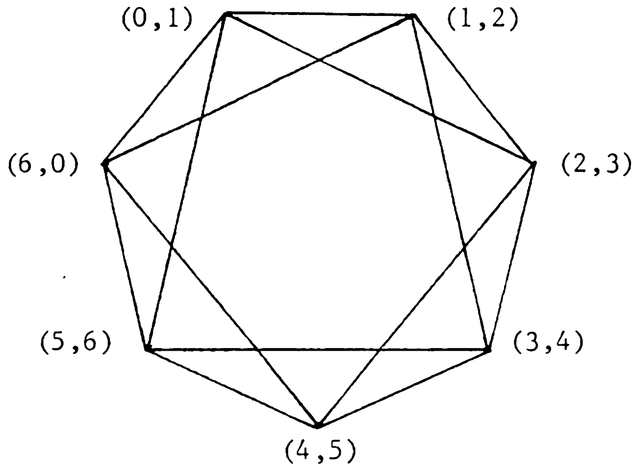
Moreover, it turns out that if G is any graph and if r is even, then if $f:V(G) \longrightarrow A^*(r)$ is admissible, there always exists an admissible map f^* on V(G) mapping to exactly half of the elements of $A^*(r)$. To be precise, let r be an even positive integer and let $Z_r = \{\bar{0},\bar{1},\ldots,\overline{r-1}\}$. We say that $\bar{a} \in Z_r$ is even if $a \in Z$ is even and $\bar{a} \in Z_r$ is odd if $a \in Z$ is odd. This convention is not ambiguous since r is even. Let $A^{**}(r) = \bigcup\{(\bar{a},\overline{a+1}) \mid \bar{a} \in Z_r \text{ is even}\}$.



f:V(D(6))⟶A*(6)



f*:V(D(6))⟶A*(6)



f:V(D(7))⟶A*(7)

Figure 2

Theorem 1.2 Let G be a graph and let r be a positive even integer. If there is an admissible map $f:V(G) \longrightarrow A^*(r)$, then there is an admissible map $f^*:V(G) \longrightarrow A^{**}(r)$.

Proof: Let G be a graph and let $f:V(G) \longrightarrow A^*(r)$ be admissible where r is even. Let $V'(G) = \{v \in V(G) \mid f(v) = (\overline{a},\overline{a+1})$ for some odd $\overline{a} \in Z_r\}$. Define $f^*:V(G) \longrightarrow A^{**}(r)$ as follows:

If $v \in V(G) - V'(G)$, then put $f^*(v) = f(v)$.

If $v \in V'(G)$ and $f(v) = (\overline{a},\overline{a+1})$ for some odd $\overline{a} \in Z_r$, then put $f^*(v) = (\overline{a-1},\overline{a})$.

The map $f^*:V(G) \longrightarrow A^{**}(r)$ is well-defined. Furthermore, f^* is onto, for if $(\overline{c},\overline{c+1}) \in A^{**}(r)$, then $\overline{c}$ is even. Since f is admissible, there is either a vertex v such that $f(v) = (\overline{c},\overline{c+1})$, in which case $f^*(v) = (\overline{c},\overline{c+1})$ or there is an edge $e = vw$ with $f(v) = (\overline{c-1},\overline{c})$ and $f(w) = (\overline{c+1},\overline{c+2})$. Since $\overline{c}$ is even, $\overline{c+1}$ is odd, so $f^*(w) = (\overline{c},\overline{c+1})$. Thus f^* is onto.

To show that f^* is admissible we verify the three conditions of admissibility. Let $e = vw \in E(G)$. Since f is admissible, there exists $\overline{a} \in Z_r$ such that $f(v),f(w) \in \{(\overline{a-1},\overline{a}),(\overline{a},\overline{a+1}),(\overline{a+1},\overline{a+2})\} \subseteq T(\overline{a}) \cup T(\overline{a+1})$. If $\overline{a}$ is odd, $f^*(v),f^*(w) \in \{(\overline{a-1},\overline{a}),(\overline{a+1},\overline{a+2})\}$ and if $\overline{a}$ is even, $f^*(v),f^*(w) \in \{(\overline{a-2},\overline{a-1}),(\overline{a},\overline{a+1})\}$. For every possibility, there exists $\overline{c} \in Z_r$ such that $f^*(v),f^*(w) \in T(\overline{c}) \cup T(\overline{c+1})$. Thus f^* satisfies (A-1).

For each $\overline{a} \in Z_r$, let $G^*(\overline{a})$ be the subgraph of G with vertex set $V^*(\overline{a}) = (f^*)^{-1}(T(\overline{a}))$ and edge set $E^*(\overline{a}) = \{e = vw \in E(G) \mid v,w \in V^*(\overline{a})\}$. To show that f^* satisfies (A-2) for each $\overline{a} \in Z_r$, it suffices to show that $G^*(\overline{a})$ is connected for each $\overline{a} \in Z_r$. First of all, if $\overline{a} \in Z_r$ is even, then $G^*(\overline{a}) = G^*(\overline{a+1})$. This follows immediately from the fact that $V^*(\bar{a}) = (f^*)^{-1}(T(\bar{a})) = (f^*)^{-1}(T(\overline{a+1})) = V^*(\overline{a+1})$ since $f^*(v) \in T(\bar{a})$ iff $f^*(v) = (\bar{a},\overline{a+1})$ and $f^*(v) \in T(\overline{a+1})$ iff $f^*(v) = (\bar{a},\overline{a+1})$. Furthermore, if $\bar{a}$ is even, $G^*(\overline{a+1}) = G(\overline{a+1})$. This follows from the fact that $f^*(v) = (\bar{a},\overline{a+1})$ iff $f(v) = (\bar{a},\overline{a+1})$ or $f(v) = (\overline{a+1},\overline{a+2})$, so that $V^*(\overline{a+1}) = V(\overline{a+1})$. Since $\overline{a+1} \in Z_r$ and f is admissible, $G(\overline{a+1})$ is connected. Consequently, both $G^*(\bar{a})$ and $G^*(\overline{a+1})$ are connected and hence for each $\bar{c} \in Z_r$, $G^*(\bar{c})$ is connected.

Let $\bar{a} \in Z_r$. If $\bar{a}$ is even, then since f^* is onto, there is a vertex $v \in V(G)$ such that $f^*(v) = (\bar{a},\overline{a+1})$. If $\bar{a}$ is odd, then there is an edge $e = vw \in E(G)$ such that $f^*(v) = (\overline{a-1},\bar{a})$ and $f^*(w) = (\overline{a+1},\overline{a+2})$. This follows from (A-3) of the admissibility of f, for if $\bar{a}$ is odd, either there is $v \in V(G)$ with $f(v) = (\bar{a},\overline{a+1})$ or there is $e = vw \in E(G)$ such that $f(v) = (\overline{a-1},\bar{a})$ and $f(w) = (\overline{a+1},\overline{a+2})$. If the latter, then since $\overline{a-1}$ and $\overline{a+1}$ are both even, $f^*(v) = f(v) \in T(\bar{a})$ and $f^*(w) = f(w) \in T(\overline{a+1})$ so e is the desired edge. If not, for $\overline{a+1}$, either there is $u \in V(G)$ with $f(u) = (\overline{a+1},\overline{a+2})$ or there is $e = ux \in E(G)$ such that $f(u) = (\bar{a},\overline{a+1})$ and $f(x) = (\overline{a+2},\overline{a+3})$. If the latter, then since $\bar{a}$ and $\overline{a+2}$ are both odd, $f^*(u) = (\overline{a-1},\bar{a}) \in T(\bar{a})$ and $f^*(x) = (\overline{a+1},\overline{a+2}) \in T(\overline{a+1})$ so e is the desired edge. If no such edge exists for either $\bar{a}$ or $\overline{a+1}$ under f, then the existence of the vertices v and u such that $f(v) = (\bar{a},\overline{a+1})$ and $f(u) = (\overline{a+1},\overline{a+2})$ and the connectedness of $G(\overline{a+1})$ guarantee a (v,u)-path, P, in G, all of whose vertices induce $\overline{a+1}$ under f. Since f maps into $A^*(r)$, the only possible images for the vertices of P are $(\bar{a},\overline{a+1})$ and $(\overline{a+1},\overline{a+2})$. Thus there is an edge $e = wx \in E(P) \subseteq E(G)$ with $f(w) = (\bar{a},\overline{a+1})$ and $f(x) = (\overline{a+1},\overline{a+2})$ so that $f^*(w) = (\overline{a-1},\bar{a}) \in T(\bar{a})$ and $f^*(x) = (\overline{a+1},\overline{a+2}) \in T(\overline{a+1})$ and $e = wx$ is the desired edge. Thus f^* satisfies (A-3).

If $r = 2n$, there is a one-to-one correspondence between the elements of $A^{**}(r)$ and the elements of Z_n. This observation leads to the following definition and theorem.

<u>Definition 1.3</u> Let G be a graph. A map, $g:V(G) \longrightarrow Z_n$, is <u>co-admissible</u> if and only if each of the following three conditions is satisfied:

(C-1) For each edge $e = vw \in E(G)$, there exists $\overline{a} \in Z_n$ such that $g(v), g(w) \in \{\overline{a}, \overline{a+1}\}$.

(C-2) If $v, w \in V(G)$ and $g(v) = g(w) = \overline{a}$, then there is a (v,w)-path in G such that for each vertex u in the path, $g(u) = \overline{a}$.

(C-3) For each $\overline{a} \in Z_n$, there is an edge $e = vw$ in G such that $g(v) = \overline{a}$ and $g(w) = \overline{a+1}$.

If $g:V(G) \longrightarrow Z_n$ satisfies only condition (C-1), then g is called a <u>pre-co-admissible</u> map.

<u>Theorem 1.4</u> Let G be a graph. If $r = 2n \geq 6$, then an admissible map $\overline{f}:V(G) \longrightarrow A(r)$ exists if and only if a co-admissible map $g:V(G) \longrightarrow Z_n$ exists.

<u>Proof</u>: <u>(Necessity)</u> Let G be a graph and let $r = 2n \geq 6$. If $\overline{f}:V(G) \longrightarrow A(r)$ is admissible, then by Theorem (1.1), there is an admissible map $f^*:V(G) \longrightarrow A^*(r)$. By Theorem (1.2), there is an admissible map $f:V(G) \longrightarrow A^{**}(r)$. Define $g:V(G) \longrightarrow Z_n$ by $g(v) = \overline{a}$ if and only if $f(v) = (\overline{c}, \overline{c+1})$ where $\overline{c} = \overline{2a}$ and $0 \leq a < n = \frac{r}{2}$. We assert that g is a co-admissible map.

Let $e = vw \in E(G)$. Since f is admissible, there exists $\overline{c} \in Z_r$ such that $f(v), f(w) \in T(\overline{c}) \cup T(\overline{c+1})$. Since f maps into $A^{**}(r)$, if $\overline{c}$ is even, $f(v), f(w) \in \{(\overline{c}, \overline{c+1})\}$ so that $g(v), g(w) \in \{\overline{a}\}$ where $\overline{c} = \overline{2a}$. If

$\overline{c}$ is odd, then $f(v),f(w) \in \{(\overline{c-1},\overline{c}),(\overline{c+1},\overline{c+2})\}$ so that $g(v),g(w) \in \{\overline{a},\overline{a+1}\}$ where $\overline{c-1} = \overline{2a}$ for some a $(0 \leq a < n)$. Thus g satisfies (C-1). If $v,w \in V(G)$ and $g(v) = g(w) = \overline{a}$, then $f(v),f(w) = (\overline{2a},\overline{2a+1}) \in A^{**}(r)$. Since f is admissible, there is a (v,w)-path in G, all of whose vertices must map to $(\overline{2a},\overline{2a+1})$ under f and hence each maps to $\overline{a}$ under g. Thus g satisfies (C-2). Let $\overline{a} \in Z_n$ so that $\overline{c} = \overline{2a} \in Z_r$ is even and $\overline{c+1} \in Z_r$ is odd. From the proof of Theorem (1.2), when $\overline{c+1}$ is odd, there is an edge $e = vw \in E(G)$ such that $f(v) = (\overline{c},\overline{c+1})$ and $f(w) = (\overline{c+2},\overline{c+3})$ and hence $g(v) = \overline{a}$ and $g(w) = \overline{a+1}$. Thus g satisfies (C-3).

(Sufficiency) Let G be a graph and let $n \geq 3$. Suppose $g:V(G) \longrightarrow Z_n$ is co-admissible. Let $r = 2n \geq 6$ and define $f:V(G) \longrightarrow A^{**}(r) \subseteq A(r)$ by $f(v) = (\overline{2a},\overline{2a+1})$ if and only if $g(v) = \overline{a}$. We assert that f is an admissible map.

Let $e = vw \in E(G)$. Since g is co-admissible, there exists $\overline{a} \in Z_n$ such that $g(v),g(w) \in \{\overline{a},\overline{a+1}\}$. Let $\overline{c} = \overline{2a} \in Z_r$. If $g(v) = g(w) = \overline{a}$, then $f(v) = f(w) = (\overline{c},\overline{c+1}) \in T(\overline{c}) \cup T(\overline{c+1})$. If $g(v) = \overline{a}$ and $g(w) = \overline{a+1}$, then $f(v) = (\overline{c},\overline{c+1})$ and $f(w) = (\overline{c+2},\overline{c+3})$ so that $f(v),f(w) \in T(\overline{d}) \cup T(\overline{d+1})$ where $\overline{d} = \overline{c+1}$. The other two possibilities are similar. Thus f satisfies (A-1). If $v,w \in V(G)$ and $f(v),f(w) \in T(\overline{c})$ for some $\overline{c} \in Z_r$, then $f(v) = f(w) = (\overline{c},\overline{c+1})$ if $\overline{c}$ is even and $f(v) = f(w) = (\overline{c-1},\overline{c})$ if $\overline{c}$ is odd. In the first case, $g(v) = g(w) = \overline{a}$ where $\overline{c} = \overline{2a}$, so there is a (v,w)-path in G all of whose vertices map to $\overline{a}$ under g and hence, they all map to $(\overline{c},\overline{c+1})$ under f. In the second case, $g(v) = g(w) = \overline{a}$ where $\overline{c-1} = \overline{2a}$, so there is a (v,w)-path in G all of whose vertices map to $\overline{a}$ under g and hence they all map to $(\overline{c-1},\overline{c})$ under f. Thus f satisfies (A-2). Let $\overline{c} \in Z_r$. If $\overline{c} = \overline{2a}$, then since g is co-admissible, there is an edge $e = vw$ in G such that $g(v) = \overline{a}$ and $g(w) = \overline{a+1}$ so that $f(v) = (\overline{c},\overline{c+1})$ for the vertex v. If $\overline{c}$ is odd, then $\overline{c-1}$ is even and hence $\overline{c-1} = \overline{2a}$ for some a. Since g is co-admissible, there is an edge $e = vw$ in G such that $g(v) = \overline{a}$ and $g(w) = \overline{a+1}$, so that $f(v) = (\overline{2a},\overline{2a+1}) = (\overline{c-1},\overline{c}) \in T(\overline{c})$ and $f(w) = (\overline{2a+2},\overline{2a+3}) = (\overline{c+1},\overline{c+2}) \in T(\overline{c+1})$. Thus f satisfies (A-3).

Figure 3 shows a graph G with an admissible map $f:V(G) \longrightarrow A^{**}(10)$ and its corresponding co-admissible map $g:V(G) \longrightarrow Z_5$.

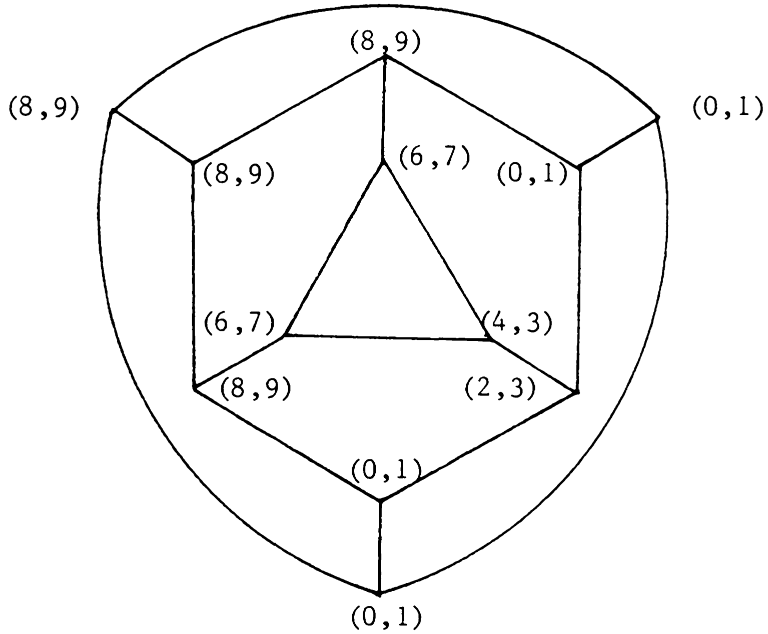
<u>Definition 1.5</u> Let G be a graph. The <u>co-circularity</u> of G, denoted by $\eta(G)$ (read eta of G), is defined as:

$\eta(G) = \max\{n \in N \mid$ there is a co-admissible map $g:V(G) \longrightarrow Z_n\}$.

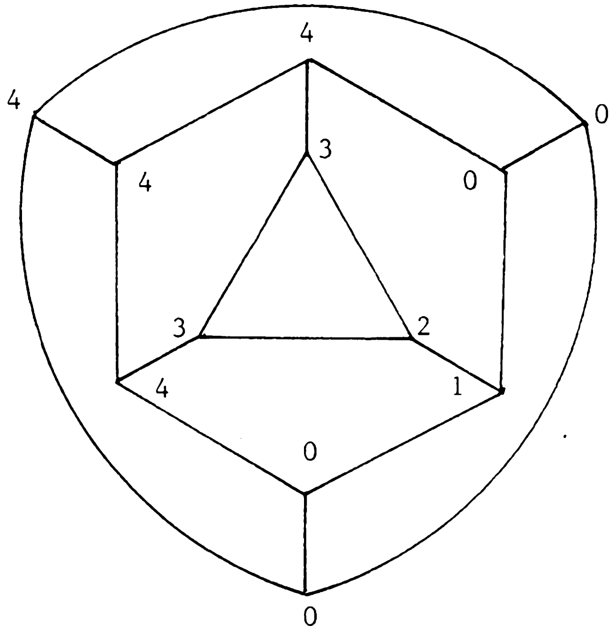
The following theorem shows that co-admissible maps can be used to narrow down the circularity of a graph to at most two integers.

<u>Theorem 1.6</u> If G is a graph with circularity $\sigma(G)$ and co-circularity $\eta(G)$, then $\sigma(G) = 2\eta(G)$ or $2\eta(G)+1$.

<u>Proof</u>: Let G be a graph with $\sigma(G) = r \geq 6$ so that there is an admissible map $f:V(G) \longrightarrow A^*(r)$. If $r = 2n$, by Theorem (1.4), there is a co-admissible map $g:V(G) \longrightarrow Z_n$. If $\eta(G) > n$, there would be a co-admissible map $\overline{g}:V(G) \longrightarrow Z_{n+1}$ and hence an admissible map $\overline{f}:V(G) \longrightarrow A^*(r+2)$, contradicting the circularity of G. Thus $\sigma(G) = 2\eta(G)$, if $\sigma(G)$ is even. If $r = 2n+1$, there is an admissible map $f:V(G) \longrightarrow A^*(r)$ and hence an admissible map $\overline{f}:V(G) \longrightarrow A^*(r-1)$. By Theorem (1.4), there is a



$f: V(G) \longrightarrow A^{**}(10)$



$g: V(G) \longrightarrow Z_5$

Figure 3

co-admissible map $g:V(G) \longrightarrow Z_n$. If $\eta(G) > n$, as above, there would be a co-admissible map $\bar{g}:V(G) \longrightarrow Z_{n+1}$ and hence an admissible map $f':V(G) \longrightarrow A^*(r+1)$, again contradicting the circularity of G. Thus if $\sigma(G)$ is odd, $\sigma(G) = 2\eta(G)+1$.

Theorem (1.6) shows that co-admissible maps provide a relatively uncomplicated method of approximating the circularity of a graph. Moreover, since it is already known that the circularity of any planar graph is even, we need only consider co-admissible maps when dealing with planar graphs. In view of this, it is desirable to study some properties of co-admissible maps. Many of these properties are very similar to properties of admissible maps.

Let $g:V(G) \longrightarrow Z_n$ be a function where G is a graph. We define the following subgraphs of G for each $\bar{a} \in Z_n$. The subgraph G[a] has vertex set $V[a] = g^{-1}(\bar{a}) = \{v \in V(G) \mid g(v) = \bar{a}\}$ and edge set $E[a] = \{e = vw \in E(G) \mid v,w \in V[a]\}$. The subgraph G[a,a+1] has vertex set $V[a,a+1] = V[a] \cup V[a+1]$ and edge set $E[a,a+1] = \{e = vw \in E(G) \mid v,w \in V[a,a+1]\}$. These subgraphs have the following properties: (1) $G[a] \cup G[a+1] \subsetneq G[a,a+1]$ for each $\bar{a} \in Z_n$; (2) $G[a] \cap G[b] = \phi$ iff $\bar{a} \neq \bar{b}$; (3) $G - G[a]$ is connected for each $\bar{a} \in Z_n$; (4) $G - (G[a] \cup G[a+1])$ is disconnected for each $\bar{a} \in Z_n$; (5) Each component of $G - \cup\{G[a] \mid \bar{a} \in Z_n\}$ is an edge of G. We note that properties (2)-(5) show that the collection $\{G[a] \mid \bar{a} \in Z_n\}$ satisfies conditions defined in [3] by R. F. Dickman, Jr. concerning multicoherent topological spaces.

<u>Proposition 1.7</u> Let G be a graph. A pre-co-admissible map $g:V(G) \longrightarrow Z_n$ is co-admissible if and only if both G[a] and G[a,a+1] are connected subgraphs of G for each $\bar{a} \in Z_n$.

<u>Proof</u>: (<u>Necessity</u>) Suppose that $g:V(G) \longrightarrow Z_n$ is co-admissible. To show that $G[a]$ is connected, let $v,w \in V[a]$. By (C-2), since $g(v) = g(w) = \bar{a}$, there is a (v,w)-path in G such that each vertex u of the path maps to $\bar{a}$ under g. Since $g(u) = \bar{a}$, $u \in V[a]$, so this path is a (v,w)-path in $G[a]$. Thus $G[a]$ is connected. To show that $G[a,a+1]$ is connected, let $u,x \in V[a,a+1]$. Since $G[a]$ and $G[a+1]$ are both connected, we need only consider the case where $u \in V[a]$ and $x \in V[a+1]$. By (C-3), there is an edge $e = vw \in E(G)$ such that $g(v) = \bar{a}$ and $g(w) = \overline{a+1}$. Let P be a (u,v)-path in $G[a]$ and let Q be a (w,x)-path in $G[a+1]$. The path $P \cup \{e\} \cup Q$ is thus a (u,x)-path in $G[a,a+1]$, showing that $G[a,a+1]$ is connected. (Note that if either $v = u$ or $w = x$, the vertex can be considered to be a degenerate (u,v) or (w,x)-path).

(<u>Sufficiency</u>) Suppose $g:V(G) \longrightarrow Z_n$ is a pre-co-admissible map such that for each $\bar{a} \in Z_n$, both $G[a]$ and $G[a,a+1]$ are connected subgraphs of G. Clearly g satisfies (C-1) since g is pre-co-admissible. To show that g satisfies (C-2), let $v,w \in V(G)$ be such that $g(v) = g(w) = \bar{a}$ so that $v,w \in V[a]$. Since $G[a]$ is connected, there is a (v,w)-path in $G[a]$ and every vertex of this path maps to $\bar{a}$ under g. To show that (C-3) is satisfied by g, let $e = vw \in E[a,a+1] \subseteq E(G)$ be such that $e \notin E[a] \cup E[a+1]$. Such an edge exists since $G[a] \cap G[a+1] = \phi$, $G[a,a+1]$ is connected, and $G[a] \cup G[a+1] \subsetneq G[a,a+1]$. Since $v,w \in V[a] \cup V[a+1]$, without loss of generality, it must be that $v \in V[a]$ and $w \in V[a+1]$ so that $g(v) = \bar{a}$ and $g(w) = \overline{a+1}$. Consequently, g is a co-admissible map.

We note that this proof shows that $G[a]$ is connected for each $\bar{a} \in Z_n$ if and only if $g:V(G) \longrightarrow Z_n$ satisfies (C-2).

<u>Proposition 1.8</u> If H is a subgraph of the graph G and if $g:V(G) \longrightarrow Z_n$ is pre-co-admissible, then $g:V(H) \longrightarrow Z_n$ is also pre-co-admissible.

<u>Proof</u>: Let $e = vw \in E(H) \subseteq E(G)$. Since $g:V(G) \longrightarrow Z_n$ is pre-co-admissible, there exists $\overline{a} \in Z_n$ such that $g(v), g(w) \in \{\overline{a}, \overline{a+1}\}$. Now, $v, w \in V(H)$, so $g:V(H) \longrightarrow Z_n$ is pre-co-admissible.

<u>Proposition 1.9</u> If G is a graph and if $g:V(G) \longrightarrow Z_n$ is co-admissible, then for each positive integer $m \leq n$, there is a co-admissible map $g_m:V(G) \longrightarrow Z_m$.

<u>Proof</u>: If $g:V(G) \longrightarrow Z_n$ is co-admissible, define $g_m:V(G) \longrightarrow Z_m$ where $m \leq n$ by $g_m(v) = g(v)$ if $g(v) \in \{\overline{0}, \overline{1}, \ldots, \overline{m-1}\}$ and $g_m(v) = \overline{m-1}$ if $g(v) \in \{\overline{m}, \overline{m+1}, \ldots, \overline{n-1}\}$. Clearly $g_m:V(G) \longrightarrow Z_m$ is co-admissible.

Counterparts to the following definitions and results were established in [1] for admissible maps.

Let n be a positive integer and let $a, b \in Z$. We recall from [1] that $d_n(a,b) = \min\{x \in N \mid a+x \equiv b \pmod{n}\}$. We note that for $\overline{a}, \overline{b} \in Z_n$, if $A = \{\overline{a+x} \mid 0 \leq x \leq d_n(a,b)\}$ and $B = \{\overline{b+x} \mid 0 \leq x \leq d_n(b,a)\}$, then $|A| = d_n(a,b)+1$, $|B| = d_n(b,a)+1$, and $d_n(a,b) + d_n(b,a) = n$. If $U \subseteq Z_n$, then U is said to be <u>pseudoconnected</u> if and only if for each $\overline{a}, \overline{b} \in U$, either $A \subseteq U$ or $B \subseteq U$. We also recall that if $U_1, U_2 \subseteq Z_n$ are pseudoconnected with $U_1 \cap U_2 \neq \phi$, then $U_1 \cup U_2$ is pseudoconnected.

Let X be a nonempty set and let $g:X \longrightarrow Z_n$ be a function. If $x \in X$ and $g(x) = \overline{a}$, then we say that x <u>induces</u> $\overline{a}$ under g. If Y is a nonempty subset of X, then $g(Y) = \{\overline{a} \in Z_n \mid g(y) = \overline{a}$ for some $y \in Y\}$ will be called the set of elements of Z_n induced by Y under the map g. Y is said to be pseudoconnected (relative to g) if and only if $g(Y) \subseteq Z_n$ is pseudoconnected. If G is a finite connected graph and if $g:V(G) \longrightarrow Z_n$

is pre-co-admissible, then G is pseudoconnected (relative to g) as is every connected subgraph of G.

The next two results relate the distance between two vertices in a connected graph with their images under a co-admissible map.

<u>Proposition 1.10</u> Let G be a graph. If $g:V(G)\longrightarrow Z_n$ is co-admissible, then for any pair $v,w \in V(G)$, if $g(v) = \overline{a}$ and $g(w) = \overline{b}$, then $d(v,w) \geq \min\{d_n(a,b), d_n(b,a)\}$.

<u>Proof</u>: Let P be any (v,w)-path in G. If ℓ denotes the length of P, then $d(v,w) \leq \ell$. Since P is a connected subgraph of G, P is pseudoconnected. Thus either $A \subseteq g(P)$ or $B \subseteq g(P)$ where $A = \{\overline{a+x} \mid 0 \leq x \leq d_n(a,b)\}$ and $B = \{\overline{b+x} \mid 0 \leq x \leq d_n(b,a)\}$. If $r = \min\{|A|,|B|\}$, then $r \leq |g(P)| \leq |V(P)|$. Since $|A| = d_n(a,b)+1$ and $|B| = d_n(b,a)+1$, it follows that $r-1 = \min\{d_n(a,b), d_n(b,a)\}$. Now, $r-1 \leq |V(P)|-1 = \ell$. Since P was any (v,w)-path in G, $r-1 \leq d(v,w)$; that is, $d(v,w) \geq \min\{d_n(a,b), d_n(b,a)\}$.

<u>Corollary 1.11</u> Let G be a graph and let $g:V(G)\longrightarrow Z_n$ be a co-admissible map. Let $v,w \in V(G)$ with $g(v) = \overline{a}$ and $g(w) = \overline{b}$. If there is a (v,w)-path, P, in G with $|V(P)|-1 = \min\{d_n(a,b), d_n(b,a)\}$, then $d(v,w) = \min\{d_n(a,b), d_n(b,a)\}$.

<u>Proposition 1.12</u> Let G be a graph and let $g:V(G)\longrightarrow Z_n$ be a co-admissible map. Suppose J is a proper cycle in G on $n \geq 4$ vertices so that $V(J) = \{v_0, v_1, \ldots, v_{n-1}\}$. Let $v_i, v_j \in V(J)$ and let P be a (v_i,v_j)-path in G such that $V(P) \cap V(J) = \{v_i, v_j\}$. If there exists $p \in V(P) - \{v_i, v_j\}$ and $v_k \in \{v_{i+1}, \ldots, v_{j-1}\}$ such that $g(p) = g(v_k)$ and if there exists $v_h \in \{v_{j+1}, \ldots, v_{i-1}\}$ with $d(p,v_h) < d(v_h,v_k)$, then $|g(J)| < n$.

Proof: Suppose $|g(J)| = n$ so that the pre-co-admissible map $g:V(J) \longrightarrow Z_n$ is also one-to-one. Let $m = d(v_h, v_k)$ so that if $g(v_k) = \bar{c}$, then $g(v_h) = \overline{c+m}$ or $\overline{c-m}$. Since $g(p) = \bar{c}$, any (p,v_h)-path in G of length $\ell = d(p,v_h) < m$ must induce all of the elements in either $\{\bar{c},\overline{c+1},\ldots,\overline{c+m}\}$ or $\{\overline{c-m},\ldots,\overline{c-1},\bar{c}\}$ since it would be pseudoconnected. However, $\ell+1$ vertices can induce at most $\ell+1 < m+1$ elements of Z_n. Thus $|g(J)| < n$.

It is interesting to note that Proposition (1.12) describes the situation which occurs in the class of graphs, $\{D(n) \mid n \in N \text{ is odd}\}$. If J is any proper cycle on $\left[\frac{n}{2}\right]+1$ vertices in D(n), then for any pair of nonadjacent vertices, $v_i, v_j \in V(J)$, there is always a (v_i,v_j)-path, P, in D(n) with $V(P) \cap V(J) = \{v_i, v_j\}$ and vertices $p \in V(P)-\{v_i,v_j\}$, $v_k \in \{v_{i+1},\ldots,v_{j-1}\}$, $v_h \in \{v_{j+1},\ldots,v_{i-1}\}$ such that $d(p,v_h) = 1 < d(v_h,v_k)$. Consequently, J can never induce as many elements as it has vertices.

The next two definitions are needed in the proofs of Proposition (1.15) and Theorem (1.17).

Definition 1.13 Let G be a graph and let $g:V(G) \longrightarrow Z_n$ be a co-admissible map. For each $v \in V(G)$, define $Q(v) = \{(v,w)\text{-paths } P \text{ in } G \mid$ if $V(P) = \{v,u_1,\ldots,u_m,w\}$, then for some $\bar{c} \in Z_n$, $g(v) = g(u_i) = \bar{c}$ for all i $(1 \leq i \leq m)$ and $g(w) = \overline{c+1}\}$.

Note that for each $v \in V(G)$, $Q(v) \neq \phi$. To see this, let $v \in V(G)$ and let $e = uw$ be an edge in G with $g(u) = g(v) = \bar{c}$ and $g(w) = \overline{c+1}$. Since G[c] is connected, there is a (v,u)-path, P, in G[c]. Thus $Q = P \cup \{e\} \in Q(v)$.

<u>Definition 1.14</u> Let P be a (v,w)-path in a graph G with $V(P) = \{v = v_0, v_1, \ldots, v_{r-1}, v_r = w\}$. Let $g:V(G) \longrightarrow Z_n$ be a co-admissible map. Suppose $g(P) = \{\overline{a}, \overline{a+1}, \ldots, \overline{a+m}\}$. P is said to be <u>monotone</u> if and only if there are m+2 integers $k_0, k_1, \ldots, k_{m+1}$ with $0 = k_0 < k_1 < \ldots < k_m \leq n = k_{m+1}$ such that $g(v_j) = \overline{a+i}$ iff $k_i \leq j < k_{i+1}$ $(0 \leq i \leq m)$.

<u>Proposition 1.15</u> If G is a graph and if $g:V(G) \longrightarrow Z_n$ is co-admissible, then for every pair of vertices v,w in G, there exists a monotone (v,w)-path in G.

<u>Proof</u>: Let $g:V(G) \longrightarrow Z_n$ be co-admissible and let $v,w \in V(G)$ with $g(v) = \overline{a}$ and $g(w) = \overline{b}$. If $\overline{a} = \overline{b}$, then $v,w \in V[a]$ and since G[a] is connected, there is a (v,w)-path in G[a] which is necessarily monotone. Suppose that $\overline{a} \neq \overline{b}$ so that for some positive integer m, $\overline{b} = \overline{a+m}$. Without loss of generality, we can assume that $m = d_n(a,b) \leq d_n(b,a)$. Put $v = v_{k_0}$ where $k_0 = 0$. Define a sequence of paths as follows for i $(0 \leq i \leq m-1)$: Let $Q_i \in Q(v_{k_i})$ be a $(v_{k_i}, v_{k_{i+1}})$-path where $g(v_{k_{i+1}}) = \overline{a+i+1}$, length of $Q_i = \ell_i$, and $k_{i+1} = \ell_i + k_i$. If $v_{k_m} = w$, then $P = Q_0 \cup Q_1 \cup \ldots \cup Q_{m-1}$ is a monotone (v,w)-path. If $v_{k_m} \neq w$, then since G[a+m] is connected, there is a (v_{k_m}, w)-path, Q_m, in G[a+m] so that $P = Q_0 \cup Q_1 \cup \ldots \cup Q_{m-1} \cup Q_m$ is the required path.

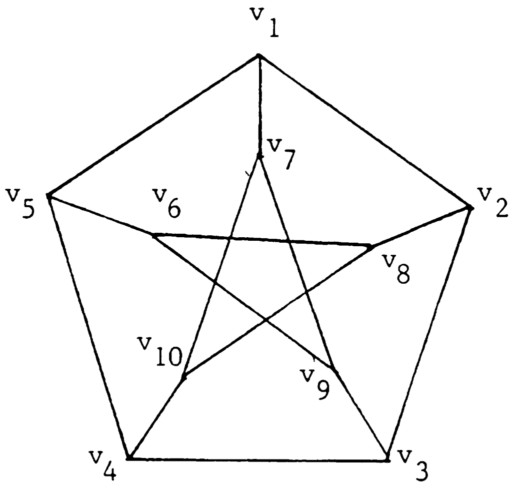
It has previously been shown that the restriction of a pre-co-admissible map on a graph to a subgraph is pre-co-admissible. However, the restriction of a co-admissible map on a graph to a subgraph is not, in general, co-admissible. We do have the following result: If g is any co-admissible map on a graph G, then G contains a cycle J such that $g|_J$ is co-admissible.

Definition 1.16 Let G be a graph and let $g:V(G)\longrightarrow Z_n$ be a co-admissible map. A subgraph H of G is co-admissible (relative to n and g) if and only if $g:V(H)\longrightarrow Z_n$ is co-admissible.

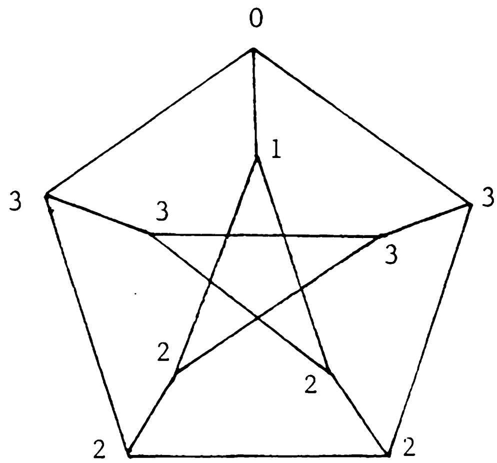
Figure 4 shows the Peterson graph and a co-admissible map into Z_4. The vertices $\{v_1,v_7,v_{10},v_4,v_5\}$ form a co-admissible subgraph which is a cycle as do the vertices $\{v_1,v_7,v_9,v_3,v_2\}$.

Theorem 1.17 If G is a graph and if $g:V(G)\longrightarrow Z_n$ is co-admissible. then G contains a co-admissible cycle (relative to n and g).

Proof: Let $v \in V(G)$ be such that $g(v) = \overline{0}$ and put $v = v_{k_0}$ where $k_0 = 0$. As in Proposition (1.15), we define a sequence of paths for i $(0 \leq i \leq n-1)$. Let $Q_i \in Q(v_{k_i})$ be a $(v_{k_i},v_{k_{i+1}})$-path of length ℓ_i where $g(v_{k_{i+1}}) = \overline{i+1}$ and $k_{i+1} = \ell_i + k_i$. If $v_{k_n} = v$, let $J = \cup\{Q_i |\ 0 \leq i \leq n-1\}$. If $v_{k_n} \neq v$, let $J = \cup\{Q_i|\ 0 \leq i \leq n\}$ where Q_n is a (v_{k_n},v)-path in the connected set G[0]. In either case, we assert that J is the desired co-admissible cycle.



The Peterson graph, G

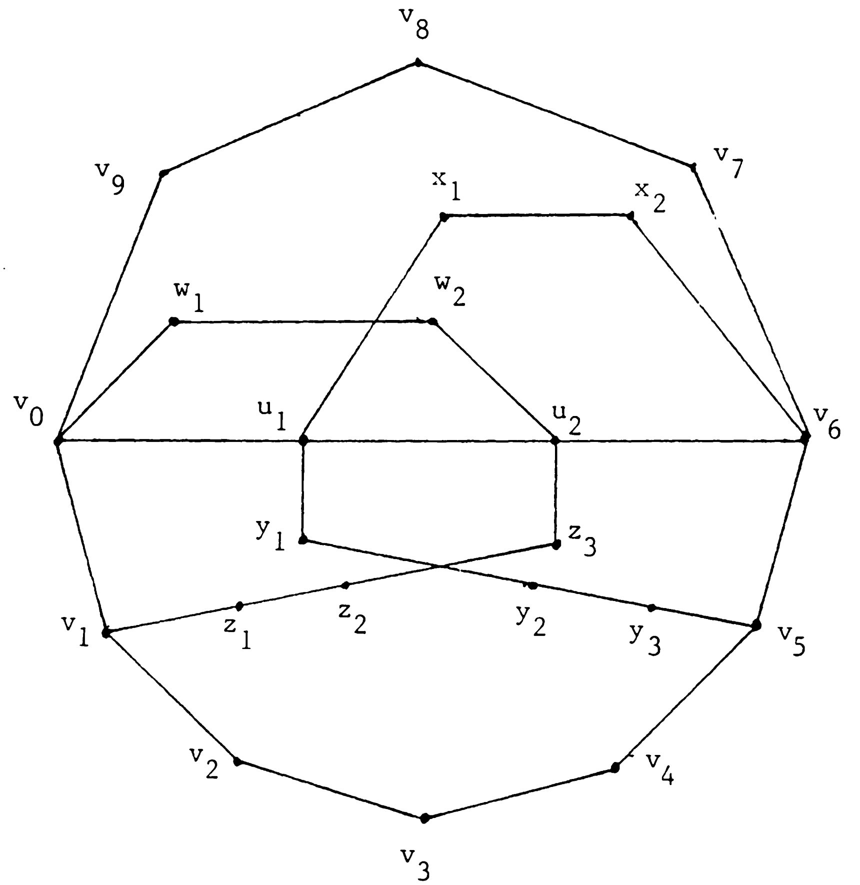


$g: V(G) \longrightarrow Z_4$

Figure 4

Clearly $g:V(J)\longrightarrow Z_n$ satisfies (C-1) since J is a subgraph of G. To prove (C-2) for $g:V(J)\longrightarrow Z_n$ we note that, by construction, for each $\bar{a} \varepsilon Z_n-\{\bar{0}\}$, $J[a] = Q_a-\{v_{k_{a+1}}\}$ is connected. For $\bar{a} = \bar{0}$, $J[0] = Q_n \cup Q_0-\{v_{k_1}\}$ is connected. To prove (C-3) for $g:V(J)\longrightarrow Z_n$, let $\bar{a} \varepsilon Z_n$. The construction of J shows that the last edge in the path Q_a, $e = wv_{k_{a+1}}$, is such that $g(w) = \bar{a}$ and $g(v_{k_{a+1}}) = \overline{a+1}$. This completes the proof.

The existence of a co-admissible cycle in any graph G for any co-admissible map, $g:V(G)\longrightarrow Z_n$, naturally leads to the question: Given a graph G, does G contain a proper co-admissible cycle for some co-admissible map $g:V(G)\longrightarrow Z_{\eta(G)}$? The answer to this question is, unfortunately, no. Figure 5 shows a graph, Γ, with co-circularity 7, which does not contain a proper co-admissible cycle for any co-admissible map $g:V(\Gamma)\longrightarrow Z_7$, though, of course, it has several co-admissible non-proper cycles. Any co-admissible map $g:V(\Gamma)\longrightarrow Z_7$ must map the vertices $v_0,v_1,\ldots,v_6$ to 7 distinct elements. Subsequently, g must be such that $g(u_1) = g(x_1) = g(x_2) = g(v_6)$; $g(u_2) = g(w_1) = g(w_2) = g(v_0)$; for $i = 1,2,3$, $g(z_i) \varepsilon \{g(v_0),g(v_1)\}$; $g(y_i) \varepsilon \{g(v_5),g(v_6)\}$; and for $i = 7,8,9$, $g(v_i) \varepsilon \{g(v_0),g(v_6)\}$. None of the six proper cycles in Γ can ever be co-admissible (relative to 7 and g) under these conditions.



The graph Γ

Figure 5

Section 2

ADEQUATE AND k-ADEQUATE CYCLES

If G is a finite, connected, nonseparable, simple graph with co-circularity $\eta(G)$, then by Proposition (1.9), for each $n \leq \eta(G)$, there is a co-admissible map $g_n : V(G) \longrightarrow Z_n$. Conversely, if there is a co-admissible map $g : V(G) \longrightarrow Z_n$, then $n \leq \eta(G)$. Furthermore, we know from Theorem (1.17) that any co-admissible map induces a co-admissible cycle in the graph. In this section we investigate the converse: When does a co-admissible map on a cycle extend to a co-admissible map on the graph? The answer to this question provides lower bounds for the co-circularity of the graph.

Definition 2.1 A cycle J on n vertices in a graph G is adequate if and only if there is a co-admissible map $g : V(J) \longrightarrow Z_n$ which has a co-admissible extension on all of V(G).

Proposition 2.2 An adequate cycle in a graph is both proper and co-admissible.

Proof: Let G be a graph and let J be an adequate cycle on n vertices in G. By definition there is a co-admissible map $g : V(J) \longrightarrow Z_n$ which can be extended to a co-admissible map $\overline{g} : V(G) \longrightarrow Z_n$. Thus $\overline{g}|_J = g$ is co-admissible and hence J is a co-admissible cycle.

To show that J is proper, let $v, w \in V(J)$. Since $|V(J)| = n$ and $g : V(J) \longrightarrow Z_n$ is co-admissible, there is a one-to-one correspondence between the vertices of J and the elements of Z_n under the map g. Thus there are distinct elements $\overline{a}, \overline{b} \in Z_n$ such that $g(v) = \overline{a}$ and $g(w) = \overline{b}$. Since J is a cycle, it has exactly two (v,w)-paths, L_1 and L_2.

Each of these paths is pseudoconnected so since $L_1 \cup L_2 = J$ and $|g(J)| = n$, then, without loss of generality, $A = g(L_1)$ and $B = g(L_2)$ where $A = \{\overline{a+x} \mid 0 \leq x \leq d_n(a,b)\}$ and $B = \{\overline{b+x} \mid 0 \leq x \leq d_n(b,a)\}$. Thus either L_1 or L_2 is a (v,w)-path in J whose length equals $\min\{d_n(a,b), d_n(b,a)\}$. By Corollary (1.11), $d_G(v,w) = \min\{d_n(a,b), d_n(b,a)\} = d_J(v,w)$, showing that J is a proper cycle in G.

Definition 2.3 Let G be a graph and let J be a cycle in G on n vertices. A spanning subgraph generated by J (or a spanning J-subgraph), S, is a connected subgraph of G satisfying each of the following conditions:

(i) $V(S) = V(G)$ and J is a subgraph of S.

(ii) For each $v \in V(J)$, there is a connected subgraph, T_v, of S such that $T_v \cap J = \{v\}$ and $T_v \cap T_w = \phi$ iff $v \neq w$.

(iii) If $V(J) = \{v_0, v_1, \ldots, v_{n-1}\}$, then $e = vw \in E(J) \cup [E(G)-E(S)]$ iff there exists i such that $v \in T_{v_i}$ and $w \in T_{v_{i+1}}$ (where $n \equiv 0$).

In this case, we write $S = J \cup (\cup\{T_v \mid v \in V(J)\})$.

Theorem 2.4 A cycle is adequate if and only if it generates a spanning subgraph.

Proof: (Necessity) Let G be a graph and let J be an adequate cycle in G on n vertices, so that there exists a co-admissible map $g: V(J) \longrightarrow Z_n$ which can be co-admissibly extended to $g: V(G) \longrightarrow Z_n$. Without loss of generality, we can assume that $g(v_i) = \overline{i}$ for each $v_i \in V(J) = \{v_0, v_1, \ldots, v_{n-1}\}$. We assert that the required spanning subgraph generated by J is $S = J \cup (\cup\{G[a] \mid \overline{a} \in Z_n\})$.

Clearly J is a subgraph of S and $V(S) \subseteq V(G)$. To show that $V(G) \subseteq V(S)$, let $v \in V(G)$. Since g is a function, there exists

$\overline{a} \in Z_n$ such that $g(v) = \overline{a}$. Thus $v \in V[a] \subseteq G[a]$ so that $v \in V(S)$. Consequently, $V(S) = V(G)$. For each $v_i \in V(J)$, let $T_{v_i} = G[i]$ so that each T_{v_i} is a connected subgraph of S such that $T_{v_i} \cap J = \{v_i\}$. Furthermore, $G[i] \cap G[j] = \phi$ iff $\overline{i} \neq \overline{j}$ so $T_{v_i} \cap T_{v_j} = \phi$ iff $v_i \neq v_j$. Since J is connected and each $G[i]$ is connected with $J \cap G[i] \neq \phi$, it follows that $S = J \cup (\cup\{G[i] \mid \overline{i} \in Z_n\})$ is a connected subgraph of G. Finally, let $e = vw \in E(G)$. Since g is co-admissible on $V(G)$, there exists $\overline{a} \in Z_n$ such that $g(v), g(w) \in \{\overline{a}, \overline{a+1}\}$. If $g(v) = \overline{a}$ and $g(w) = \overline{a+1}$, then $v \in T_{v_a} = G[a]$ and $w \in T_{v_{a+1}} = G[a+1]$ so that $e \notin E[c]$ for any $\overline{c} \in Z_n$. Consequently, $e \in E(J) \cup [E(G)-E(S)]$. Conversely, if $g(v) = g(w) = \overline{a}$, then $e \in E[a]$ so that $e \notin E(J) \cup [E(G)-E(S)]$.

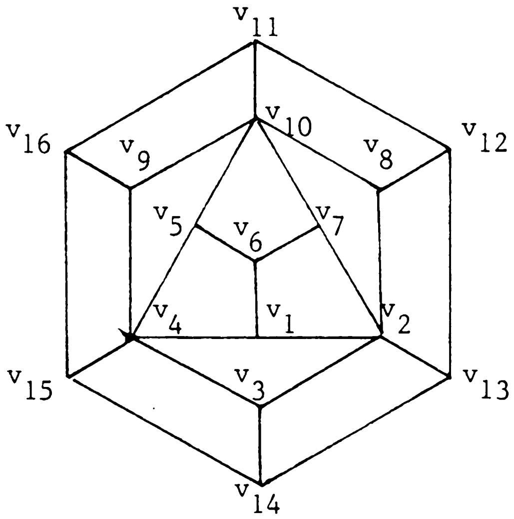
(Sufficiency) Let G be a graph and let J be a cycle in G on n vertices which generates a spanning subgraph $S = J \cup (\cup\{T_v \mid v \in V(J)\})$. Define $g: V(J) \longrightarrow Z_n$ as follows: If $v_i \in V(J) = \{v_0, v_1, \ldots, v_{n-1}\}$, put $g(v_i) = \overline{i}$. Clearly g is a co-admissible map on $V(J)$. Extend $g: V(J) \longrightarrow Z_n$ to $V(G)$ as follows: If $v \in V(G)-V(J)$, then since $V(S) = V(G)$, $v \in V(T_{v_i}) \subseteq V(S)$ for some unique $v_i \in V(J)$. In this case, put $g(v) = g(v_i) = \overline{i}$. We assert that the map $g: V(G) \longrightarrow Z_n$ is co-admissible.

Let $e \in E(G) = E(J) \cup [E(S)-E(J)] \cup [E(G)-E(S)]$. If $e \in E(J)$, then $e = v_i v_{i+1}$ for some i so that $g(v_i) = \overline{i}$ and $g(v_{i+1}) = \overline{i+1}$. If $e = vw \in E(S)-E(J)$, then $v, w \in T_{v_i}$ for some i so that

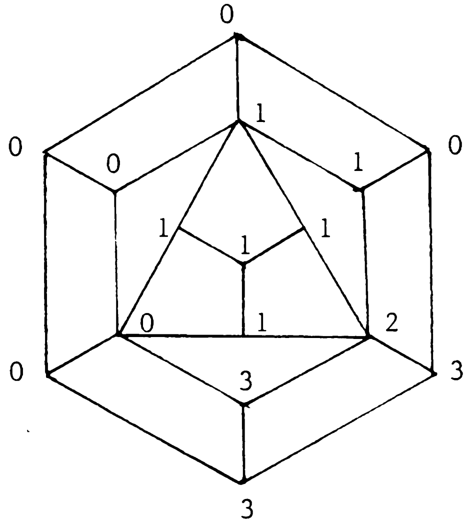
$g(v) = g(w) = g(v_i) = \overline{i}$. If $e = vw \in E(G)-E(S)$, then there exists i such that $v \in T_{v_i}$ and $w \in T_{v_{i+1}}$ so that $g(v) = \overline{i}$ and $g(w) = \overline{i+1}$. Thus g satisfies (C-1). Let $v,w \in V(G)$ be such that $g(v) = g(w) = \overline{i}$ so that $v,w \in T_{v_i}$. Since T_{v_i} is a connected subgraph, there is a (v,w)-path in T_{v_i} and every vertex in this path maps to $\overline{i}$ under g. Thus g satisfies (C-2). If $\overline{i} \in Z_n$, then $e = v_iv_{i+1} \in E(J) \subseteq E(G)$ is such that $g(v_i) = \overline{i}$ and $g(v_{i+1}) = \overline{i+1}$. Thus g satisfies (C-3).

Corollary 2.5 Let G be a graph. If $M = \max\{m \in N \mid G$ has an adequate cycle on m vertices$\} = \max\{m \in N \mid G$ has a spanning subgraph generated by a cycle on m vertices$\}$, then $M \leq \eta(G)$.

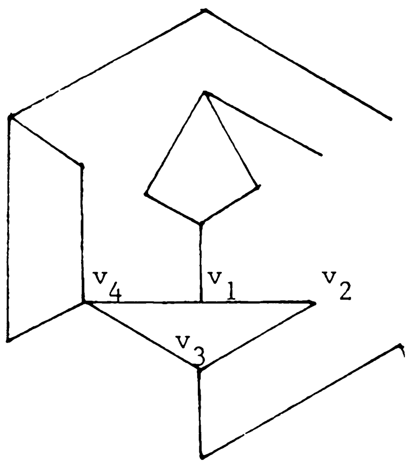
Figure 6 shows a graph G, a co-admissible map $g:V(G) \longrightarrow Z_4$, and a spanning J-subgraph for the adequate cycle J with vertex set $V(J) = \{v_1,v_2,v_3,v_4\}$.



A graph G



$g: V(G) \longrightarrow Z_4$



A spanning J-subgraph

Figure 6

Theorem (2.4) and Corollary (2.5) provide a method for finding lower bounds for the co-circularity, and hence the circularity, of a graph. However, even the largest adequate cycle in a graph, G, might not have enough vertices to induce all of $Z_{\eta(G)}$. Since we already know that every graph has a co-admissible cycle relative to its co-circularity, it must be that some graphs have cycles which are not adequate, but which can induce all of the elements of $Z_{\eta(G)}$, whereas the largest adequate cycle of the graph cannot. The following definitions and theorem formalize this situation and generalize the ideas of adequate cycles and spanning subgraphs generated by a cycle.

Definition 2.6 Let k be a nonnegative integer. A cycle J on m vertices of a graph G is k-adequate if and only if there is a co-admissible map $g:V(J) \longrightarrow Z_n$, where $n = m-k \geq 3$, which has a co-admissible extension on all of V(G) and no co-admissible map $g:V(J) \longrightarrow Z_{n+1}$ has a co-admissible extension on V(G).

Definition 2.7 Let G be a graph and let J be a cycle in G on m vertices. A k-spanning subgraph generated by J (or a k-spanning J-subgraph), S, is a connected subgraph of G of the form $S = J \cup B$ which satisfies each of the following conditions:

(i) $V(S) = V(G)$ and $V(J) \subseteq V(B)$.

(ii) There is a maximal subset, $V(k) = \{v_0, v_1, \ldots, v_{n-1}\}$ of V(J), called a base for S, where $n = m-k \geq 3$, such that for each $v_i \in V(k)$, there is a connected subgraph, T_{v_i}, of S such that $v_i \in V(T_{v_i})$; $T_{v_i} \cap J$ is a connected subgraph of G; $T_{v_i} \cap T_{v_j} = \phi$ iff $v_i \neq v_j$; and $B = \cup\{T_{v_i} \mid 0 \leq i \leq n-1\}$.

(iii) If $e = vw \in E(G)-E(B)$, then there exists i such that $v \in T_{v_i}$ and $w \in T_{v_{i+1}}$.

In Definition (2.6), k "measures" how far away the cycle J is from being adequate. Furthermore, when $k = 0$, Definitions (2.6) and (2.7) become Definitions (2.1) and (2.3) respectively. Thus the notions of k-adequate cycles and k-spanning J-cycles are generalizations of the notions of adequate cycles and spanning J-subgraphs. The next theorem generalizes Theorem (2.4).

Theorem 2.8 A cycle is k-adequate if and only if it generates a k-spanning subgraph.

Proof: (Necessity) Let G be a graph and let J be a k-adequate cycle in G on m vertices so that there exists a co-admissible map $g:V(J)\longrightarrow Z_n$, where $n = m-k \geq 3$, which can be co-admissibly extended to $g:V(G)\longrightarrow Z_n$, but there is no co-admissible map $g:V(J)\longrightarrow Z_{n+1}$ which can be co-admissibly extended to $V(G)$.

Let $S = J\cup B$ where $B = \cup\{G[a] \mid \bar{a} \in Z_n\}$. We assert that S is a k-spanning subgraph generated by J. If $v \in V(G)$, then there exists $\bar{a} \in Z_n$ such that $g(v) = \bar{a}$ so that $v \in V[a] \subseteq G[a]$. Thus $v \in V(S)$ so $V(G) \subseteq V(S)$. Since S is a subgraph of G, $V(S) \subseteq V(G)$. Therefore, $V(S) = V(G)$. Similarly, if $v \in V(J)$, $g(v) = \bar{a}$ for some $\bar{a} \in Z_n$ so that $v \in V[a] \subseteq G[a]$ and thus $V(J) \subseteq V(B)$. Since $g:V(J)\longrightarrow Z_n$ is a co-admissible map, for each $\bar{a} \in Z_n$, $V(J)\cap V[a] \neq \phi$. Form a base, $V(k)$, for S by choosing one element from each set $V(J)\cap V[a]$ and calling it v_a so that $V(k) = \{v_0, v_1, \ldots, v_{n-1}\} \subseteq V(J)$ and $|V(k)| = n = m-k \geq 3$. Now, $g:V(G)\longrightarrow Z_n$ is co-admissible, so for each $v_a \in V(k)$, if $G[a] = T_{v_a}$, then T_{v_a} is a connected subgraph of S with $v_a \in T_{v_a}$ and such that $T_{v_a} \cap T_{v_b} = \phi$ iff $v_a \neq v_b$. Moreover, since $g: V(J) \longrightarrow Z_n$ is co-admissible, $G[a] \cap J = T_{v_a} \cap J = J[a]$ is a connected subgraph of G. If $e = vw \in E(G)-E(B)$, then there exists $\overline{a} \in Z_n$ such that $g(v), g(w) \in \{\overline{a}, \overline{a+1}\}$ since $g: V(G) \longrightarrow Z_n$ is co-admissible. However, $g(v) \neq g(w)$ since $e \notin E(B) = \cup\{E[a] \mid \overline{a} \in Z_n\}$. Thus, without loss of generality, $g(v) = \overline{a}$ and $g(w) = \overline{a+1}$ so that $v \in T_{v_a}$ and $w \in T_{v_{a+1}}$. Finally, S is a connected subgraph of G because J is connected, each G[a] is connected, and for each $\overline{a} \in Z_n$, $J \cap G[a] \neq \phi$. The maximality of the base, V(k), comes from the fact that no co-admissible map $g: V(J) \longrightarrow Z_{n+1}$ exists which can be co-admissibly extended to V(G).

(Sufficiency) Let G be a graph and let J be a cycle in G on m vertices which generates a k-spanning subgraph, $S = J \cup B$, where $B = \cup\{T_{v_i} \mid v_i \in V(k)\}$ and $V(k) = \{v_0, v_1, \ldots, v_{n-1}\} \subseteq V(J)$ is a maximal base for S.

Define $g: V(J) \longrightarrow Z_n$ by $g(v_i) = \overline{i}$ if $v_i \in V(k)$. If $v \in V(J)-V(k)$, then since $V(J) \subseteq V(B)$, there exists $v_i \in V(k)$ such that $v \in V(T_{v_i})$. In this case, put $g(v) = g(v_i) = \overline{i}$. Since $T_{v_i} \cap T_{v_j} = \phi$ iff $v_i \neq v_j$, the map g is well-defined. It is also co-admissible: Let $e = vw \in E(J) \subseteq E(G)$. If $e \in E(B)$, then there exists i such that $v, w \in T_{v_i}$ so that $g(v) = g(w) = \overline{i}$. If $e \notin E(B)$, then there exists i such that $v \in T_{v_i}$ and $w \in T_{v_{i+1}}$ so that $g(v) = \overline{i}$ and $g(w) = \overline{i+1}$. In either case, there exists $\overline{i} \in Z_n$ such that $g(v), g(w) \in \{\overline{i}, \overline{i+1}\}$, showing that g satisfies (C-1). Let $v,w \in V(J)$ be such that $g(v) = g(w) = \bar{i}$. In this case, $v,w \in T_{v_i} \cap J$, a connected subgraph, so there is a (v,w)-path in $T_{v_i} \cap J$ and each of its vertices maps to $\bar{i}$ under g. Thus g satisfies (C-2).

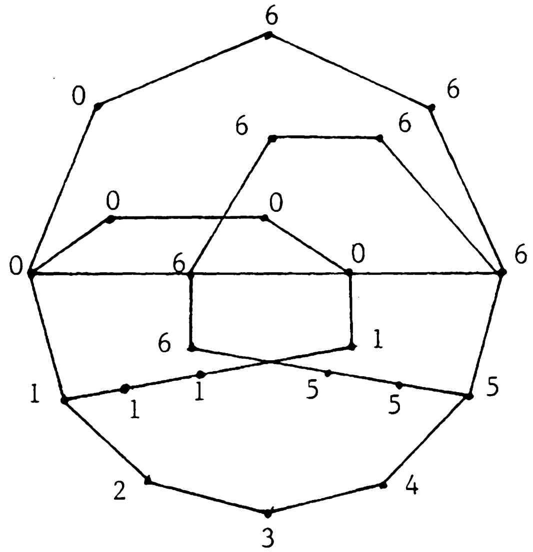
Let $\bar{i} \in Z_n$. Since J is a cycle and $T_{v_i} \cap J$ is a connected subgraph, either $J \cap T_{v_i} = \{v_i\}$ or $J \cap T_{v_i}$ is a (v,w)-path for some $v,w \in V(J)$. If $J \cap T_{v_i} = \{v_i\}$, let $u,x \in V(J)$ be adjacent to v_i so that $uv_i, v_ix \in E(J)-E(B)$ and $u,x \in V(T_{v_{i-1}}) \cup V(T_{v_{i+1}})$. If $u,x \in T_{v_{i-1}}$, then since $J \cap T_{v_{i-1}}$ is connected and $T_{v_{i-1}} \cap T_{v_i} = \phi$, it follows that $n = 2$, a contradiction since $n \geq 3$. Thus either $u \in T_{v_{i+1}}$ or $x \in T_{v_{i+1}}$ so that either uv_i or v_ix is an edge in J with one vertex mapping to $\bar{i}$ under g and the other vertex mapping to $\overline{i+1}$ under g. In the case where $J \cap T_{v_i}$ is a (v,w)-path, let $u \in V(J)$ be adjacent to v and let $x \in V(J)$ be adjacent to w so that $uv, wx \in E(J)-E(B)$. As above, either $u \in T_{v_{i+1}}$ or $x \in T_{v_{i+1}}$ (but not both) so that either uv or wx is an edge in J with one vertex mapping to $\bar{i}$ under g and the other vertex mapping to $\overline{i+1}$ under g. Thus g satisfies (C-3).

To show that J is k-adequate, we must extend the co-admissible map $g:V(J) \longrightarrow Z_n$ co-admissibly to all of V(G). Since $V(S) = V(G)$, any $v \in V(G)-V(J)$ belongs to $V(T_{v_i})$ for some $v_i \in V(k)$. Consequently, for each $v \in V(G)-V(J)$, define $g(v) = g(v_i) = \bar{i}$ where $v \in T_{v_i}$. Since $T_{v_i} \cap T_{v_j} = \phi$ for $v_i \neq v_j$, g is well-defined. To show that g has been co-admissibly extended, we verify the three conditions of co-admissibility.

Let $e = vw \in E(G)$. If $e \in E(B)$, then for some $v_i \in V(k)$, $v,w \in T_{v_i}$ so that $g(v) = g(w) = \bar{i}$. If $e \in E(G)-E(B)$, there exists $\bar{i}$ such that $v \in T_{v_i}$ and $w \in T_{v_{i+1}}$ so that $g(v) = \bar{i}$ and $g(w) = \overline{i+1}$. In either case, g satisfies (C-1). Let $v,w \in V(G)$ with $g(v) = g(w) = \bar{i}$. By the definition of g, $v,w \in T_{v_i}$. Since T_{v_i} is connected, there is a (v,w)-path in T_{v_i}, each of whose vertices maps to $\bar{i}$ under g. Thus g satisfies (C-2). If $\bar{i} \in Z_n$, the required edge exists in E(J) and hence in E(G), so g satisfies (C-3). Finally, since V(k) is a maximal subset of V(J) satisfying the properties of a k-spanning J-subgraph, these properties cannot be all satisfied by a subset of V(J) with n+1 elements. Therefore, there can be no co-admissible map $g:V(J) \longrightarrow Z_{n+1}$ which can be co-admissibly extended to V(G).

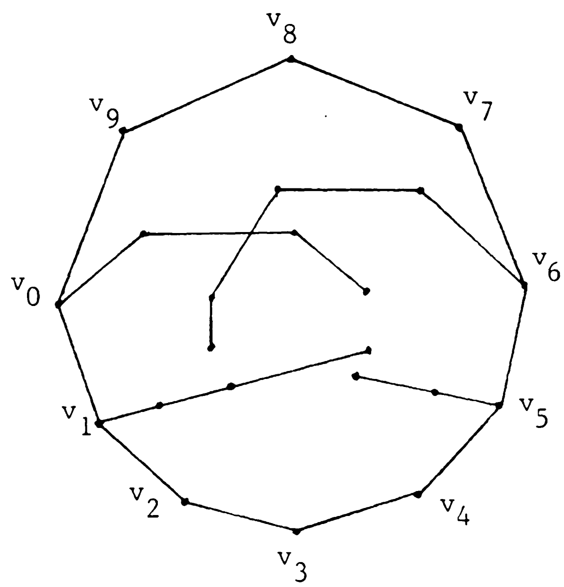
Corollary 2.9 If G is a graph, then the co-circularity of G, $\eta(G) = \max\{n = m-k \mid$ G has a k-adequate cycle on m vertices$\}$ = $\max\{n = m-k \mid$ G has a k-spanning subgraph generated by a cycle on m vertices$\}$.

Figure 7 shows the graph, Γ, of Figure 5 with its vertices labeled according to a co-admissible map $g:V(\Gamma) \longrightarrow Z_7$. Figure 8 shows a 3-spanning subgraph of Γ generated by the cycle, J, with base $V(3) = \{v_0, v_1, \ldots, v_6\} \subseteq \{v_0, v_1, \ldots, v_9\} = V(J)$.



$g: V(\Gamma) \longrightarrow Z_7$

Figure 7



3-spanning J-subgraph

Figure 8

We have previously shown that a graph does not necessarily contain a proper co-admissible cycle. We did show, however, that every adequate cycle is proper. In fact, we have the following result concerning proper cycles.

Proposition 2.10 If an m-cycle in a graph G is 1-adequate, then it is a proper co-admissible cycle relative to $n = m-1$ or the graph G contains a proper co-admissible cycle relative to n. If an m-cycle in a graph G is 2-adequate, then it is a proper co-admissible cycle relative to $s = m-2$ or the graph G contains a proper co-admissible cycle relative to s.

Proof: Let G be a graph and let J be a 1-adequate m-cycle which is not proper. Let $g:V(G) \longrightarrow Z_n$, where $n = m-1$, be a co-admissible map such that $g:V(J) \longrightarrow Z_n$ is also co-admissible. Since J is 1-adequate, every vertex of J induces a distinct element of Z_n except for two adjacent vertices which induce the same element of Z_n.

By assumption, J is not proper, so there exists a pair of vertices v,w in J such that $\ell = d_G(v,w) < d_J(v,w) = \ell_1$. Let $\overline{a},\overline{b} \in Z_n$ be such that $g(v) = \overline{a}$ and $g(w) = \overline{b}$ and let $r = \min\{d_n(a,b), d_n(b,a)\}$. We claim that $\ell_1 = r+1$. To prove this claim, we note that since J is a cycle, there are only two (v,w)-paths in J. Call them L_1 and L_2 where length of $L_1 = \ell_1 \leq \ell_2 =$ length of L_2. Now each of these paths is pseudoconnected, so if $A = \{\overline{a+x} \mid 0 \leq x \leq d_n(a,b)\}$ and $B = \{\overline{b+x} \mid 0 \leq x \leq d_n(b,a)\}$, then either $A = g(L_1)$ and $B = g(L_2)$ or $B = g(L_1)$ and $A = g(L_2)$. It must be one of these two choices since $J = L_1 \cup L_2$; $Z_n = g(J) = g(L_1) \cup g(L_2) = A \cup B$; $A \cap B = \{\overline{a},\overline{b}\}$; and $V(L_1) \cap V(L_2) = \{v,w\}$.

Now $|A| = d_n(a,b)+1$ and $|B| = d_n(b,a)+1$ so either $|A| = r+1$ or $|B| = r+1$. If $r+1 < \ell_1$, then since $n-r \leq \ell_2$, we would have $r+1 + n-r = n+1 < \ell_1+\ell_2 = m$, contradicting the fact that $n = m-1$. Thus $\ell_1 \leq r+1$ so $\ell = r$. If P is any (v,w)-path of length r, then since P is pseudoconnected, either $A = g(P)$ or $B = g(P)$, depending on which set has cardinality r+1. Thus one of the two cycles, $J_1 = L_1 \cup P$ or $J_2 = L_2 \cup P$, has n vertices and induces all of Z_n so must clearly be adequate and hence proper and co-admissible by Proposition (2.2).

Next suppose that J is a 2-adequate m-cycle which is not proper. Let $g:V(G) \longrightarrow Z_s$, where $s = m-2$, be a co-admissible map such that $g:V(J) \longrightarrow Z_s$ is also co-admissible. Since J is 2-adequate there is either one element of Z_s which is induced by three consecutive vertices of J or there are two elements of Z_s which are each induced by two consecutive vertices of J (that is, two adjacent vertices induce one of the elements and two other adjacent vertices induce the other element).

Since J is not proper, there exists a pair of vertices v,w in J such that $\ell = d_G(v,w) < d_J(v,w) = \ell_1$. Let $\overline{a},\overline{b} \in Z_s$ be such that $g(v) = \overline{a}$ and $g(w) = \overline{b}$. Let r, A, B, L_1, and L_2 be as before. We claim that $\ell_1 \leq r+2$ so that $\ell = r$ or $r+1$. If $r+2 < \ell_1$, then $s-r \leq \ell_2$ implies that $r+2 + s-r = s+2 < \ell_1+\ell_2 = m$, contradicting the fact that $s = m-2$. Let P be any (v,w)-path in G with length ℓ. As before, $J_1 = L_1 \cup P$ and $J_2 = L_2 \cup P$ are two cycles in G, one of which induces all of Z_s. If $\ell = r$, it is adequate, hence proper and co-admissible. If $\ell = r+1$, it is 1-adequate, so the first part of this proof applies, yielding a proper co-admissible cycle.

In the graph, Γ, of Figures 5, 7, and 8, we see that if a cycle is k-adequate for $k \geq 3$, the cycle is not proper. On the other hand, each proper cycle of this graph is k-adequate for some $k \in \{1,2\}$. However, if m equals the number of vertices in any one of these proper cycles, then $m-k < 7 = \eta(G)$.

We conclude this section by showing how k-adequate cycles can be used in determining the co-circularity of a graph. As our example, we use the class of graphs, $\{D(n) \mid n \in N, n \geq 6\}$, previously discussed. It is already known that $\sigma(D(n)) = n$, so $\eta(D(n)) = \left[\frac{n}{2}\right]$. Consequently, this example provides an alternate method of calculating $\eta(D(n))$. Figure 9 will illustrate the next proposition using D(8) and D(9).

Proposition 2.11 If $G = D(2r)$ or if $G = D(2r+1)$, where $r \geq 3$, then $\eta(G) = r$.

Proof: Suppose first that $n = 2r \geq 6$. Recall the vertex set and edge set for D(n) as given in the Introduction. $G = D(n)$ has proper cycles on 3 and r vertices and non-proper cycles on $r+1, r+2, \ldots, n$ vertices. Clearly any 3-cycle is adequate, so $\eta(G) \geq 3$. Let J be a cycle in G on r vertices. Since there are only two, by the symmetry of G, we can take $V(J) = \{v_{2i} \mid 0 \leq i \leq r-1\}$ and $E(J) = \{v_{2i}v_{2i+2} \mid 0 \leq i \leq r-1\}$. For each i $(0 \leq i \leq r-1)$, let $T_{v_{2i}} = \{e_i = v_{2i}v_{2i+1}\}$. Let $B = \cup\{T_{v_{2i}} \mid 0 \leq i \leq r-1\}$. If $S = J \cup B$, then S is a spanning subgraph generated by J, so J is adequate. Hence $\eta(G) \geq r$.

Consider the non-proper cycles of G. Each has $r+\ell$ vertices for some ℓ $(1 \leq \ell \leq r)$. We claim that if J is any cycle in G on $r+\ell > r$ vertices, then J is ℓ-adequate. Consequently, $m-k = r+\ell-\ell = r$, so $\eta(G)$ will equal r. To prove the claim, let J be any cycle on $r+\ell > r$ vertices in G, so there are $n-(r+\ell)$ vertices of G not in $V(J)$. Note that if $v \in V(G)-V(J)$ and $v = v_{2i}$, then $v_{2i+1} \in V(J)$ and $e_i = v_{2i}v_{2i+1} \in E(G)-E(J)$. If $v = v_{2i+1}$, then $v_{2i} \in V(J)$ and $e_i = v_{2i}v_{2i+1} \in E(G)-E(J)$. This is clear since $|V(J)| \geq r+1$. The maximal subset, $V(\ell)$, of $V(J)$ required for an ℓ-spanning J-subgraph is described as follows: If $v_{2i} \in V(J)$, then put $v_{2i} \in V(\ell)$. If $v_{2i} \notin V(J)$, then put $v_{2i+1} \in V(\ell)$. Thus $|V(\ell)| = r$ and for each $v \in V(\ell)$, there exists i ($1 \leq i \leq r$) such that $v = v_{2i}$ or $v = v_{2i+1}$. Let $T_v = \{e_i = v_{2i}v_{2i+1}\}$, $B = \cup\{T_v | v \in V(\ell)\}$ and $S = J \cup B$, so that S is an ℓ-spanning J-subgraph. Only the maximality of $|V(\ell)|$ needs to be verified since everything else is clear.

Suppose $V(J)$ has a subset, $V(\ell-1)$, of $r+1$ elements so that J generates an $(\ell-1)$-spanning subgraph. If $V(\ell-1) = \{v_0, v_1, \ldots, v_r\}$, then there exists a co-admissible map $g: V(J) \longrightarrow Z_{r+1}$ such that $g(v_i) = \overline{i}$ for each $v_i \in V(\ell-1)$ which can be co-admissibly extended to all of $V(G)$. Note that if any subset of $V(G)$ has exactly four consecutive vertices, of which three are from $V(\ell-1)$, then g cannot be co-admissibly extended to $V(G)$. This is clear since the three vertices from $V(\ell-1)$ induce three distinct elements of Z_{r+1} and among the four consecutive vertices, there would be two, say v and w, such that $e = vw \in E(G)$, but for which no $\overline{c} \in Z_{r+1}$ exists with $g(v), g(w) \in \{\overline{c}, \overline{c+1}\}$. Now, any distribution of the $r+1$ elements of $V(\ell-1)$ among the $2r$ elements of $V(G)$ results in at least one subset of $V(G)$ with exactly four consecutive vertices, three of which are also in $V(\ell-1)$. Thus $V(\ell)$ is a maximal set so S is an ℓ-spanning J-subgraph and J is ℓ-adequate if J has $r+\ell > r$ vertices. Thus if $n = 2r \geq 6$, then $\eta(G) = r$.

Next suppose that $n = 2r+1 \geq 6$. Recall the vertex set and edge set of $D(n)$ as given in the Introduction. $G = D(n)$ has proper cycles on 3 and $r+1$ vertices and non-proper cycles on $r+2, r+3, \ldots, n$ vertices. Clearly any cycle on 3 vertices is adequate, so $\eta(G) \geq 3$.

Let J be a cycle on $r+1$ vertices. There are n such cycles in G, but because of the symmetry of G, we need only consider one of these. Suppose $V(J) = \{v_{2i-1} \mid 1 \leq i \leq r+1\}$ and $E(J) = \{v_1v_3, v_3v_5, \ldots, v_nv_1\}$. Let $V(1) = \{v_1, v_3, \ldots, v_{n-2}\}$ so $|V(1)| = r$. Let $T_{v_1} = \{v_nv_1, v_nv_2, v_1v_2\}$ and for $2 \leq i \leq r$, let $T_{v_{2i-1}} = \{e_i = v_{2i-1}v_{2i}\}$.

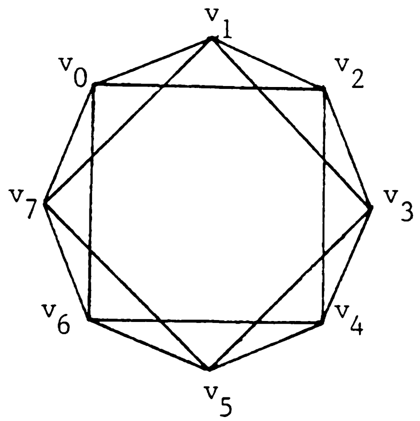
If $B = \cup\{T_{v_{2i-1}} \mid 1 \leq i \leq r\}$, then $S = J \cup B$ is a 1-spanning subgraph generated by J. Again, only the maximality of $|V(1)|$ needs to be verified, so we show that J cannot be adequate. If it were, there would be, without loss of generality, a co-admissible map $g: V(J) \longrightarrow Z_{r+1}$ such that $g(v_1) = \overline{0}$, $g(v_3) = \overline{1}, \ldots, g(v_{n-2}) = \overline{r-1}$, $g(v_n) = \overline{r}$ and such that g could be co-admissibly extended to all of $V(G)$. Consider $v_{n-1} \in V(G)-V(J)$ which is adjacent to v_{n-2}, v_n, v_1, each of which induces a distinct element of Z_{r+1}. If $e = vw \in \{v_{n-2}v_{n-1}, v_{n-1}v_n, v_{n-1}v_1\} \subseteq E(G)$, then there is no $\overline{c} \in Z_{r+1}$ such that $g(v), g(w) \in \{\overline{c}, \overline{c+1}\}$, contradicting the co-admissibility of $g: V(G) \longrightarrow Z_{r+1}$. Thus J is 1-adequate since it cannot be adequate and thus $\eta(G) \geq r$.

Consider the non-proper cycles of G. Each has $(r+1)+\ell$ vertices for some ℓ $(1 \leq \ell \leq r)$. We claim that if J is a cycle in G on $(r+1)+\ell > r+1$ vertices, then J is $(\ell+1)$-adequate. This would show that $\eta(G) = r$. To prove this claim, we show that J generates an $(\ell+1)$-spanning subgraph.

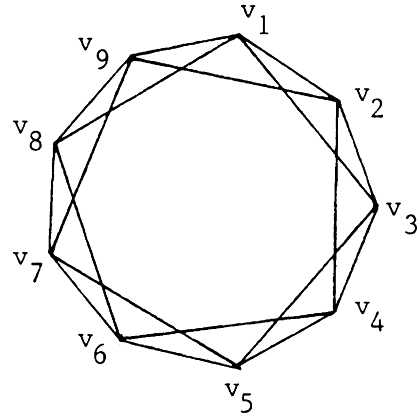
Since $|V(J)| > r+1$, for each $v \in V(G)-V(J)$, if $v = v_{2i-1}$, then $v_{2i} \in V(J)$ and if $v = v_{2i}$, then $v_{2i-1} \in V(J)$. Form $V(\ell+1) \subseteq V(J)$ as follows: For each i $(1 \leq i \leq r)$, if $v_{2i} \in V(J)$, put $v_{2i} \in V(\ell+1)$ and if $v_{2i} \notin V(J)$, put $v_{2i-1} \in V(\ell+1)$. Now $|V(\ell+1)| = r$. For each $v \in V(\ell+1)$, define T_v as follows. If $v = v_1$ or v_2, then $T_v = \{v_nv_1, v_nv_2, v_1v_2\}$. If $v = v_{2i}$ $(i \neq 1)$, then $T_v = \{e_i = v_{2i-1}v_{2i}\}$. If $v = v_{2i-1}$ $(i \neq 1, r+1)$, then $T_v = \{e_i = v_{2i-1}v_{2i}\}$. If $B = \cup\{T_v | v \in V(\ell+1)\}$, then $S = J \cup B$ is an $(\ell+1)$-spanning J-subgraph. It suffices to show the maximality of $V(\ell+1)$, since all of the other properties are clearly satisfied.

Suppose J has a set of r+1 vertices, $V(\ell) \subseteq V(J)$, so that J generates an ℓ-spanning subgraph and there exists a co-admissible map $g: V(J) \longrightarrow Z_{r+1}$ which can be co-admissibly extended to all of V(G) and such that, without loss of generality, for each $v_i \in V(\ell) = \{v_1, v_2, \ldots, v_{r+1}\}$, $g(v_i) = \overline{i-1}$. As in the even case, any distribution of the r+1 vertices of $V(\ell)$ among the 2r+1 vertices of V(G) results in a subset of V(G) consisting of four consecutive vertices, three of which are in $V(\ell)$. This situation, as in the even case, violates the co-admissibility of g in the same way. Thus any cycle in G on $(r+1)+\ell$ $(1 \leq \ell \leq r)$ vertices is $(\ell+1)$-adequate.
Thus if $n = 2r+1 \geq 6$, then $\eta(G) = r$.

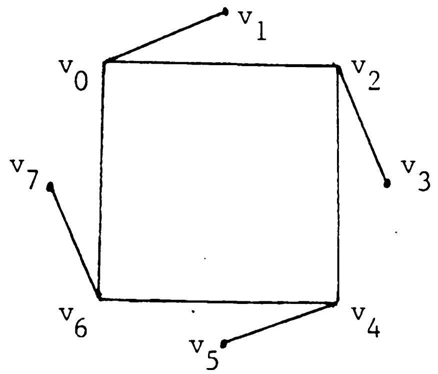
The graphs D(8) and D(9) are depicted in Figure 9 with their respective spanning and 1-spanning subgraphs as described in Proposition (2.11).



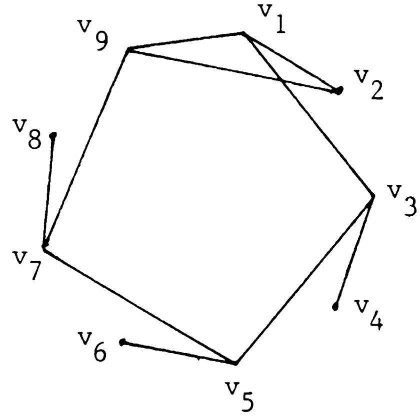
D(8)



D(9)



A spanning subgraph generated by the cycle with vertex set $\{v_0, v_2, v_4, v_6\}$



A 1-spanning subgraph generated by the cycle with vertex set $\{v_1, v_3, v_5, v_7, v_9\}$

Figure 9

Section 3

SOME RESULTS ON ADMISSIBLE MAPS

In this section we investigate the effect of vertices of order 2 on the circularity of a finite, connected, nonseparable, simple graph. We discuss the notion of a reduced graph and compare the circularities of a graph and the reduced graph to which it is homeomorphic. We also show that if G is a graph with $\sigma(G) = r$ and if $f:V(G) \longrightarrow A^*(r)$ is an admissible map, then any vertex v of G with order 2 either induces exactly two elements of Z_r not induced by any other vertex of G or it induces only elements of Z_r induced by at least one of the vertices to which it is adjacent. The first two definitions in this section are adapted from [4] and [5] respectively.

<u>Definition 3.1</u> Let G be a graph. A (v,w)-path P of length $\ell \geq 2$ is called a <u>suspended</u> (v,w)-path if and only if $o(v), o(w) \geq 3$ and the order of every other vertex of P is 2.

<u>Definition 3.2</u> Let G be a graph. Suppose G has a vertex, x, of order 2 which is adjacent to $v,w \in V(G)$ so that $vw \notin E(G)$. If G' is the graph with vertex set $V(G') = V(G)-\{x\}$ and edge set $E(G') = \{vw\} \cup [E(G)-\{vx,xw\}]$, then G is called an <u>expansion</u> of G'.

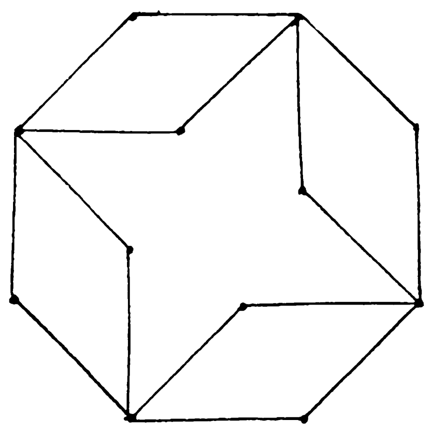
<u>Definition 3.3</u> A graph G' is said to be <u>reduced</u> if and only if it is not an expansion of any graph G. In other words, a graph G' is reduced if and only if its vertices of order 2 occur only in 3-cycles.

The following proposition is already known, but we prove it by exhibiting a specific homeomorphism which will be needed in Proposition (3.5).

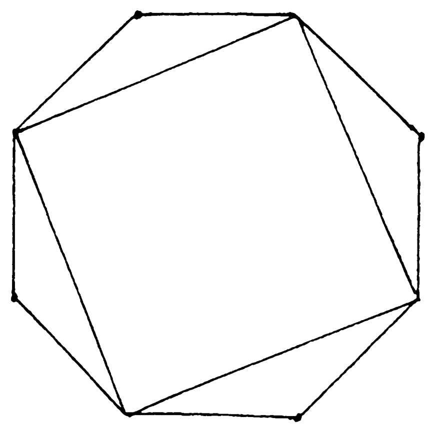
<u>Proposition 3.4</u> Every finite, connected, nonseparable, simple graph, G, is homeomorphic to a reduced, finite, connected, nonseparable, simple graph, G'.

<u>Proof</u>: Let G be a graph. If G is reduced, then G is homeomorphic to itself. If G is not reduced, then define a map $h:G \longrightarrow G'$ as follows. If $v \in V(G)$ and $o(v) \geq 3$, put $h(v) = v \in V(G')$. If $e = vw \in E(G)$ and $o(v), o(w) \geq 3$, put $h(x) = x \in G'$ for each point x in e. Next consider all pairs of vertices $v, w \in V(G)$ which determine suspended (v,w)-paths. If $vw \in E(G)$, then for each suspended (v,w)-path, P_i, there is a homeomorphism $\psi_i : P_i \longrightarrow Q_i$ where Q_i is a suspended (v,w)-path of length 2 in G'. In this case, for each P_i, put $h(x) = \psi_i(x)$ for each point x in P_i. If $vw \notin E(G)$, then for one suspended (v,w)-path, P_1, there is a homeomorphism $\psi_1 : P_1 \longrightarrow e = vw \in E(G')$, so put $h(x) = \psi_1(x)$ for each point x in P_1. For any other suspended (v,w)-path, P_i, there is a homeomorphism $\psi_i : P_i \longrightarrow Q_i$ where Q_i is a suspended (v,w)-path of length 2 in G'. In this case, for each P_i, put $h(x) = \psi_i(x)$ for each point x in P_i. Clearly $h:G \longrightarrow G'$ is a homeomorphism and G' is a reduced, finite, connected, nonseparable, simple graph.

If G' is a reduced graph homeomorphic to the graph G, then G' is unique up to graph isomorphism. Thus we can simply say G is a graph with reduced graph G'. The following observation will be used later in this section: If $V'(G) = \{v \in V(G) \mid o(v) \geq 3\}$ and if $V'(G') = \{v \in V(G') \mid o(v) \geq 3\}$, then $V'(G) = V'(G')$. Figure 10 shows a graph G and its reduced graph G'.



G, a graph which is not reduced



G', the reduced graph homeomorphic to G

Figure 10

Proposition 3.5 If G is a graph with reduced graph G', then $\sigma(G') \leq \sigma(G)$.

Proof: Let $h:G \longrightarrow G'$ be the homeomorphism of Proposition (3.4). Let $f:V(G') \longrightarrow A^*(r)$ be an admissible map where $\sigma(G') = r$. Define $g:V(G) \longrightarrow A^*(r)$ as follows: If $v \in V(G)$ and $o(v) \geq 3$, then put $g(v) = f(v) = f(h(v))$. If $x \in V(G)$ and $o(x) = 2$, then x belongs to a suspended (v,w)-path, P, in G. If $h(P) = Q$ where Q is a suspended (v,w)-path of length 2 in G', then put $g(x) = f(v')$ for every $x \in V(P)$ with order 2 where $v' \in V(Q)$ has order 2. If $h(P) = e = vw \in E(G')$, then put $g(x) = f(h(v)) = f(v)$ for every $x \in V(P)$ with order 2, where v is chosen over w. We claim that $g:V(G) \longrightarrow A^*(r)$ is an admissible map.

Let $e = vw \in E(G)$. If $o(v), o(w) \geq 3$, then $g(v) = f(v)$, $g(w) = f(w) \in T(\overline{a}) \cup T(\overline{a+1})$ for some $\overline{a} \in Z_r$ since f is an admissible map. If $o(v), o(w) = 2$, then v and w belong to the same suspended (u,x)-path in G, so there exists $z \in V(G')$ such that $g(v) = g(w) = f(z) \in T(\overline{a})$ for some $\overline{a} \in Z_r$. If $o(v) \geq 3$ and $o(w) = 2$, then $g(v) = f(v)$ and w belongs to a suspended (v,x)-path in G. Thus there is $z \in V(G')$ such that $g(w) = f(z)$ and $vz \in E(G')$ so there exists $\overline{a} \in Z_r$ such that $g(v) = f(v)$, $g(w) = f(z) \in T(\overline{a}) \cup T(\overline{a+1})$. Thus g satisfies (A-1).

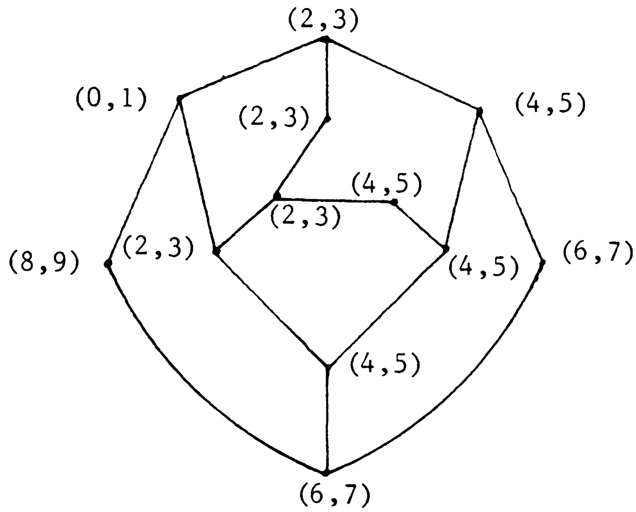
To show that g satisfies (A-2) we show that for each $\overline{a} \in Z_r$, $G(\overline{a})$ is connected. Since f is admissible, $G'(\overline{a})$, the usual subgraph of G' with vertex set $V'(\overline{a}) = f^{-1}(T(\overline{a}))$, is connected for each $\overline{a} \in Z_r$. Let $x \in V(\overline{a})$. If $o(x) \geq 3$, then $h(x) = x$ so $g(x) = f(x) \in T(\overline{a})$. If $o(x) = 2$, then x belongs to a suspended (v,w)-path in G, so either $g(x) = f(v) \in T(\overline{a})$ for all vertices of order 2 in that path or

$g(x) = f(v') \varepsilon T(\bar{a})$ where $v' \varepsilon V(G')$ has order 2. Now v' is either the only vertex of G' inducing $\bar{a}$ (and this can happen only if $\sigma(G') = 6$) or v' is adjacent to some $v \varepsilon V(G')$ with $o(v) \geq 3$ and $f(v) \varepsilon T(\bar{a})$. Thus every $x \varepsilon V(\bar{a})$ is either a vertex of order ≥ 3; adjacent to a vertex of order ≥ 3 belonging to $V(\bar{a})$; or connected to a vertex of order ≥ 3 belonging to $V(\bar{a})$ by a path in $G(\bar{a})$ of length ≥ 2. This shows that $G(\bar{a})$ is connected since $G'(\bar{a})$ is connected.

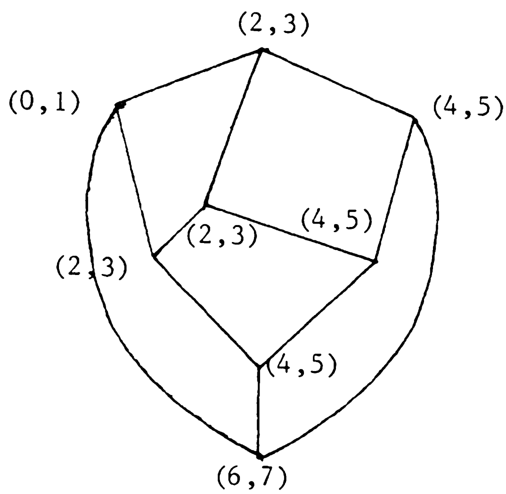
Let $\bar{a} \varepsilon Z_r$. Since f is admissible, either there is $v \varepsilon V(G')$ such that $f(v) = (\bar{a},\overline{a+1})$ or there is $e = vw \varepsilon E(G')$ such that $f(v) = (\overline{a-1},\bar{a})$ and $f(w) = (\overline{a+1},\overline{a+2})$. In the first case, if $o(v) \geq 3$, then $f(v) = g(v) = (\bar{a},\overline{a+1})$. If $o(v) = 2$, then v belongs to a suspended (u,x)-path, Q, of length 2 in G'. Thus there is $z \varepsilon V(G)$ of order 2 which belongs to the suspended (u,x)-path $h^{-1}(Q)$ in G and such that $g(z) = f(v) = (\bar{a},\overline{a+1})$. In the second case, if $o(v), o(w) \geq 3$, then either $h^{-1}(e) = e \varepsilon E(G)$, in which case $g(v) = f(v) = (\overline{a-1},\bar{a})$ and $g(w) = f(w) = (\overline{a+1},\overline{a+2})$ or $h^{-1}(e)$ is a suspended (v,w)-path in G with an edge uw where $o(u) = 2$ and $g(u) = g(v) = f(v) = (\overline{a-1},\bar{a})$ and $g(w) = f(w) = (\overline{a+1},\overline{a+2})$. Thus g satisfies (A-3). Consequently, $g:V(G) \longrightarrow A^*(r)$ is admissible and hence, $\sigma(G') = r \leq \sigma(G)$.

Corollary 3.6 If G is a graph with reduced graph G', then $\eta(G') \leq \eta(G)$.

An example of a nonreduced graph, G, with $\sigma(G) = 10$ and its reduced graph, G', with $\sigma(G') = 8$, is given in Figure 11. In Figure 12, the nonreduced graph, Γ, is given, together with its reduced graph Γ'. Note that $\eta(\Gamma') = 4 < 7 = \eta(\Gamma)$.

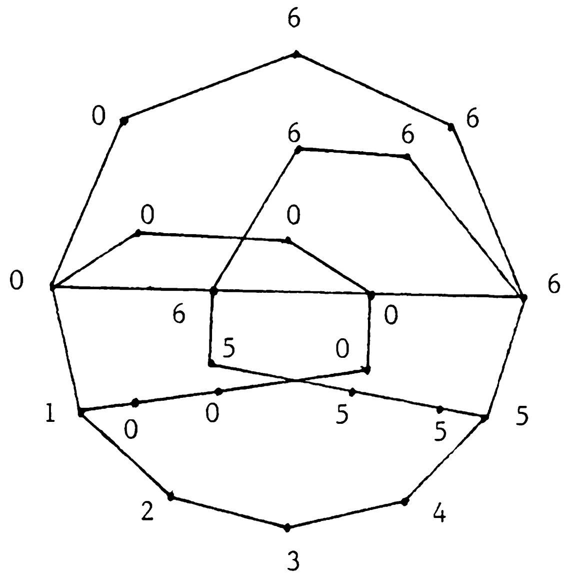


f:V(G)—>A*(10)

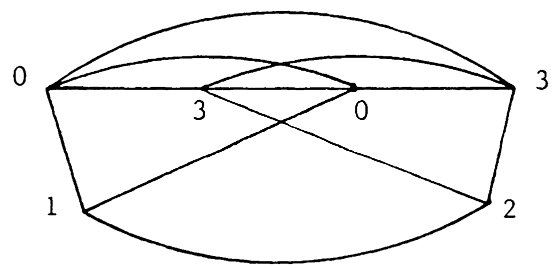


f':V(G')—>A*(8)

Figure 11



$g: V(\Gamma) \longrightarrow Z_7$



$g: V(\Gamma') \longrightarrow Z_4$

Figure 12

The following observation will be useful in the next two propositions. If $f:V(G) \longrightarrow A^*(r)$ is admissible, then so is $f^*:V(G) \longrightarrow A^*(r)$ where $f^*(v) = (\overline{a+x}, \overline{a+1+x})$ whenever $f(v) = (\overline{a}, \overline{a+1})$ and $x \in Z$.

We also need to review some notation and terminology from [1]. Let G be a graph and let $f:V(G) \longrightarrow A^*(r)$ be a preadmissible map. If $v \in V(G)$ and if $f(v) \in T(\overline{a})$, then we say that v induces $\overline{a}$ under the map f. If H is a subgraph of G, then $U(H) = \{\overline{a} \in Z_r \mid f(x) \in T(\overline{a})$ for some $x \in V(H)\}$ is called the set of elements of Z_r induced by H under f. If, for all $\overline{a}, \overline{b} \in U(H)$, either $A \subseteq U(H)$ or $B \subseteq U(H)$ where $A = \{\overline{a+x} \mid 0 \leq x \leq d_r(a,b)\}$ and $B = \{\overline{b+x} \mid 0 \leq x \leq d_r(b,a)\}$, then $U(H)$ is said to be pseudoconnected. If H is any connected subgraph of G, then $U(H)$ is pseudoconnected.

Proposition 3.7 Let G be a graph and let $f:V(G) \longrightarrow A^*(r)$ be an admissible map. If $x \in V(G)$ has order 2 and if x induces exactly one element of Z_r not induced by any other vertex of G, then there is an admissible map $f^*:V(G) \longrightarrow A^*(r+1)$.

Proof: Let $f:V(G) \longrightarrow A^*(r)$ be admissible. From the above observation, we can assume that $f(x) = (\overline{0}, \overline{1})$. There are two cases to consider. First suppose that $\overline{0}$ is the element of Z_r only induced by x. Thus if v and w are adjacent to x, we can assume that v induces $\overline{r-1}$ and w induces $\overline{1}$. Define $f^*:V(G) \longrightarrow A^*(r+1)$ as follows. If $v \in V(G)-\{x\}$, put $f^*(v) = (\overline{a+1}, \overline{a+2})$ if $f(v) = (\overline{a}, \overline{a+1})$. For $x \in V(G)$, put $f^*(x) = f(x) = (\overline{0}, \overline{1})$. We claim that f^* is admissible.

Let $e = vw \in E(G)$. If neither v nor w is x, then since f is preadmissible, there exists $\overline{a} \in Z_r$ such that $f(v), f(w) \in T(\overline{a}) \cup T(\overline{a+1})$. Consequently, $f^*(v), f^*(w) \in T(\overline{a+1}) \cup T(\overline{a+2})$. If either v or w is x, then the other vertex maps to either $(\overline{r-2},\overline{r-1})$ or $(\overline{1},\overline{2})$ under f and hence it maps to $(\overline{r-1},\overline{r})$ or $(\overline{2},\overline{3})$ under f* while $f^*(x) = (\overline{0},\overline{1})$. Thus f* satisfies (A-1). To prove (A-2) we show that $G^*(\overline{a})$ is connected for each $\overline{a} \varepsilon Z_{r+1}$ where $G^*(\overline{a})$ is the usual subgraph of G with vertex set $(f^*)^{-1}(T(\overline{a}))$. If $\overline{a} = \overline{0}$ or $\overline{1}$, then $G^*(\overline{a}) = \{x\}$ is connected. If $\overline{a} = \overline{2}$, then $G^*(\overline{2}) = G(\overline{1})-\{x\}$ is connected since $G(\overline{1})$ is connected and x is not a cut point of $G(\overline{1})$. For every other $\overline{a} \varepsilon Z_{r+1}$, $G^*(\overline{a}) = G(\overline{a-1})$ is connected since f is admissible. To prove (A-3) let $\overline{a} \varepsilon Z_{r+1}$. If $\overline{a} = \overline{0}$, then $f^*(x) = (\overline{0},\overline{1})$. If $\overline{a} = \overline{1}$, then e = xw is such that $f^*(x) \varepsilon T(\overline{1})$ and $f^*(w) \varepsilon T(\overline{2})$. For every other $\overline{a} \varepsilon Z_{r+1}$, $\overline{a-1} \varepsilon Z_r$ so either there is a vertex v with $f(v) = (\overline{a-1},\overline{a})$, in which case $f^*(v) = (\overline{a},\overline{a+1})$ or there is an edge e = vw with $f(v) = (\overline{a-2},\overline{a-1})$ and $f(w) = (\overline{a},\overline{a+1})$, in which case $f^*(v) = (\overline{a-1},\overline{a})$ and $f^*(w) = (\overline{a+1},\overline{a+2})$.

Next suppose that $f(x) = (\overline{0},\overline{1})$ where $\overline{1}$ is the element of Z_r induced only by x. In this case, there is a vertex v adjacent to x such that $f(v) = (\overline{r-1},\overline{0})$ and a vertex w adjacent to x such that $f(w) = (\overline{2},\overline{3})$. Define $f^*:V(G) \longrightarrow A^*(r+1)$ by $f^*(v) = f(v)$ if $f(v) \neq (\overline{r-1},\overline{0})$. If $f(v) = (\overline{r-1},\overline{0})$, put $f^*(v) = (\overline{r-1},\overline{r})$. To show that f* is admissible, we need only verify the conditions of admissibility as they pertain to $\overline{r}$.

If e = vw ε E(G) is such that $f(v),f(w) \varepsilon T(\overline{r-1}) \cup T(\overline{0})$, then if $f(v),f(w) \varepsilon T(\overline{r-1})$, $f^*(v),f^*(w) \varepsilon T(\overline{r-1})$. If $f(v) = (\overline{r-1},\overline{0})$ and $f(w) = (\overline{0},\overline{1})$, then $f^*(v) = (\overline{r-1},\overline{r})$ and $f^*(w) = (\overline{0},\overline{1})$. These are the only possibilities, so (A-1) holds for f*. Let v,w ε V(G) be such that v,w ε $V^*(\overline{r})$. By definition of f*, $f^*(v) = f^*(w) = (\overline{r-1},\overline{r})$ so that $f(v) = f(w) = (\overline{r-1},\overline{0})$. Since $G(\overline{0})$ is connected, there is a

(v,w)-path in $G(\overline{0})$, each vertex of which must map to $(\overline{r-1},\overline{0})$ since only x maps to $(\overline{0},\overline{1})$ under f. Thus each vertex of this path maps to $(\overline{r-1},\overline{r})$ under f^*, showing that $G^*(\overline{r})$ is connected. Finally, for $\overline{r} \varepsilon Z_{r+1}$, there is an edge, vx, in G such that $f^*(v) = (\overline{r-1},\overline{r}) \varepsilon T(\overline{r})$ and $f^*(w) = (\overline{0},\overline{1}) \varepsilon T(\overline{0})$. This completes the proof.

Proposition 3.8 Let G be a graph and let $f:V(G) \longrightarrow A^*(r)$ be admissible. Let $xy \varepsilon E(G)$ be such that $x,y \varepsilon V(G)$ each have order 2.

a) If $f(x) \neq f(y)$, but there is an element of Z_r induced by both x and y which is not induced by any other vertex of G, then there is an admissible map $f^*:V(G) \longrightarrow A^*(r+1)$.

b) If $f(x) = f(y)$ and x induces two elements of Z_r not induced by any other vertex of G (except y), then there is an admissible map $f^*:V(G) \longrightarrow A^*(r+2)$.

c) If $f(x) = f(y)$ and x induces exactly one element of Z_r not induced by any other vertex of G (except y), then there is an admissible map $f^{**}:V(G) \longrightarrow A^*(r+3)$.

Proof: For (a) we can assume that $f(x) = (\overline{0},\overline{1})$ and $f(y) = (\overline{1},\overline{2})$. Define $f^*:V(G) \longrightarrow A^*(r+1)$ by $f^*(v) = f(x)$ if $v = x$ and $f^*(v) = (\overline{a+1},\overline{a+2})$ if $f(v) = (\overline{a},\overline{a+1})$ and $v \neq x$. The verification of the admissibility of f^* is identical to that of case 1' in Proposition (3.7).

For (b) we can assume that $f(x) = (\overline{0},\overline{1}) = f(y)$. Define $f^*:V(G) \longrightarrow A^*(r+2)$ by $f^*(v) = f(x)$ if $v = x$ and $f^*(v) = (\overline{a+2},\overline{a+3})$ if $f(v) = (\overline{a},\overline{a+1})$ and $v \neq x$. To verify that f^* is admissible, let $e = vw \varepsilon E(G)$. If neither v nor w is x, then since f is admissible, there exists $\overline{a} \varepsilon Z_r$ such that $f(v),f(w) \varepsilon T(\overline{a}) \cup T(\overline{a+1})$. Thus for f^*, $f^*(v),f^*(w) \varepsilon T(\overline{a+2}) \cup T(\overline{a+3})$. If either v or w is x, then the other vertex maps to either $(\overline{2},\overline{3})$ or $(\overline{r},\overline{r+1})$ under f^*. Since $f^*(x) = (\overline{0},\overline{1})$, (A-1) is satisfied. Let $v,w \in V(G)$ be such that $f^*(v)$, $f^*(w) \in T(\overline{a})$. If $\overline{a} = \overline{0}$ or $\overline{1}$, then $G^*(\overline{a}) = \{x\}$ is connected. If $\overline{a} = \overline{2}$ or $\overline{3}$, then $G^*(\overline{a}) = G(\overline{a-2})-\{x\}$ is connected since $G(\overline{a-2})$ is connected by the admissibility of f and x is not a cut point of $G(\overline{a-2})$. For every other $\overline{a} \in Z_{r+2}$, $G^*(\overline{a}) = G(\overline{a-2})$ is connected since f is admissible and $\overline{a-2} \in Z_r$. Thus (A-2) is satisfied. Let $\overline{a} \in Z_r$. If $\overline{a} = \overline{0}$, the vertex x is such that $f^*(x) = (\overline{0},\overline{1})$. If $\overline{a} = \overline{1}$, the edge $e = xy$ is such that $f^*(x) = (\overline{0},\overline{1}) \in T(\overline{1})$ and $f^*(y) = (\overline{2},\overline{3}) \in T(\overline{2})$. For every other $\overline{a} \in Z_{r+2}$, $\overline{a-2} \in Z_r$, so either there is a vertex v with $f(v) = (\overline{a-2},\overline{a-1})$ so that $f^*(v) = (\overline{a},\overline{a+1})$ or there is an edge $e = vw$ with $f(v) = (\overline{a-3},\overline{a-2})$ and $f(w) = (\overline{a-1},\overline{a})$ so that $f^*(v) = (\overline{a-1},\overline{a}) \in T(\overline{a})$ and $f^*(w) = (\overline{a+1},\overline{a+2}) \in T(\overline{a+1})$. Thus (A-3) is satisfied.

For (c) we can assume that $f(x) = (\overline{0},\overline{1}) = f(y)$ where either $\overline{0}$ is induced only by x and y or $\overline{1}$ is induced only by x and y. In each case, define an admissible map $f^*:V(G) \longrightarrow A^*(r+1)$ as in Proposition (3.7) with the exception that $f^*(y) = f(y) = (\overline{0},\overline{1}) = f(x) = f^*(x)$. Next define an admissible map $f^{**}:V(G) \longrightarrow A^*(r+3)$ as f^* was defined in part (b) of this proposition.

Proposition 3.9 Let G be a graph and let $f:V(G) \longrightarrow A^*(r)$ be an admissible map. Let P be a suspended (v,w)-path in G with $V(P) = \{v,x_1,\ldots,x_n,w\}$. If there exists $k < \ell$ such that for all i ($k \leq i \leq \ell$), there is an element of Z_r induced by each x_i but not induced by both v and w or, if $k = \ell$, there is an element of Z_r induced by x_k and either v or w but not both, then there exists an admissible map $f^*:V(G) \longrightarrow A^*(r+1)$.

Proof: Let $f:V(G) \longrightarrow A^*(r)$ be an admissible map. Let P be a suspended (v,w)-path in the graph G with $V(P) = \{v,x_1,\ldots,x_n,w\}$. Let k be the smallest subscript which satisfies the hypotheses.

Suppose first that v does not induce the common element induced by the x_i's. Without loss of generality, suppose $f(x_k) = (\overline{0},\overline{1})$ so that $f(x_i) \in T(\overline{1})$ for all i $(k \leq i \leq \ell)$ and either $f(x_{k-1}) = (\overline{r-2},\overline{r-1})$ or $f(v) = (\overline{r-2},\overline{r-1})$. Define $f^*:V(G) \longrightarrow A^*(r+1)$ by $f^*(x_k) = f(x_k) = (\overline{0},\overline{1})$; if $u \in V(G)-\{x_k\}$ and $f(u) = (\overline{r-2},\overline{r-1})$, put $f^*(u) = (\overline{r-1},\overline{r})$; for every other $u \in V(G)$, put $f^*(u) = (\overline{a+1},\overline{a+2})$ where $f(u) = (\overline{a},\overline{a+1})$. The admissibility of the map $f^*:V(G) \longrightarrow A^*(r+1)$ can be routinely verified.

Next suppose w does not induce the common element induced by the x_i's, but v does. In this case, k = 1, so without loss of generality, either $f(v) = (\overline{r-1},\overline{0})$ and $f(x_1) = (\overline{0},\overline{1})$ or $f(v) = f(x_1) = (\overline{0},\overline{1})$. In the first case, define $f^*:V(G) \longrightarrow A^*(r+1)$ by $f^*(x_1) = f(x_1) = (\overline{0},\overline{1})$; $f^*(u) = (\overline{r},\overline{0})$ if $f(u) = (\overline{r-1},\overline{0})$; for every other vertex u in G, put $f^*(u) = (\overline{a+1},\overline{a+2})$ where $f(u) = (\overline{a},\overline{a+1})$. Again the admissibility of f^* can be routinely verified. In the second case, define $f^*:V(G) \longrightarrow A^*(r+1)$ as follows: If $f(u) = (\overline{r-2},\overline{r-1})$, put $f^*(u) = (\overline{r-1},\overline{r})$. If $f(u) = (\overline{r-1},\overline{0})$, put $f^*(u) = (\overline{r},\overline{0})$. If $u \in V(G)-V(P)$ and $f(u) = (\overline{0},\overline{1})$, put $f^*(u) = f(u) = (\overline{0},\overline{1})$. For v, put $f^*(v) = f(v) = (\overline{0},\overline{1})$. For every other vertex u of G, including the x_i's, put $f^*(u) = (\overline{a+1},\overline{a+2})$ where $f(u) = (\overline{a},\overline{a+1})$. The admissibility of f^* is, again, routinely verifiable.

Proposition 3.10 Let G be a graph with $\sigma(G) = r$. Let $f:V(G) \longrightarrow A^*(r)$ be an admissible map. Let P be a suspended (v,w)-path of length $\ell \geq 2$ in G such that for some $\overline{a},\overline{b} \in Z_r$, $f(v) = (\overline{a-1},\overline{a})$ and $f(w) = (\overline{b},\overline{b+1})$.

If $|A| \geq 3$ when $A = \{\overline{a+x} \mid 0 \leq x \leq d_r(a,b)\} \subseteq U(P)$ or if $|B| \geq 5$, when $B = \{\overline{b+x} \mid 0 \leq x \leq d_r(b,a)\} \subseteq U(P)$, then each vertex of order 2 in $V(P)$ must induce two elements of Z_r not induced by any other vertices of G.

<u>Proof</u>: The suspended (v,w)-path P is connected so that U(P) is pseudoconnected. Thus either $A \subseteq U(P)$ or $B \subseteq U(P)$. Since $|A| \geq 3$ if $A \subseteq U(P)$ and $|B| \geq 5$ if $B \subseteq U(P)$, there exists $\overline{c} \in U(P)$ which is not induced by either v or w and hence is induced only by vertices of order 2 belonging to the path P. If some vertex u of P with order 2 does not induce two elements of Z_r not induced by any other vertices of G, then at least one of the situations described in Propositions 3.7, 3.8, and 3.9 exists, so that there is an admissible map $f^*:V(G) \longrightarrow A^*(r+1)$, contradicting the fact that $\sigma(G) = r$. Thus every vertex of order 2 in V(P) induces two elements of Z_r not induced by any other vertices of G.

<u>Lemma 3.11</u> Let G' be a reduced graph and let $V'(G') = \{v \in V(G') \mid o(v) \geq 3\}$. If $\sigma(G') \geq 7$, then $\sigma(G') = |U(V'(G'))|$. If $\sigma(G') \neq |U(V'(G'))|$, then $\sigma(G') = 6$ and $|U(V'(G'))| = 4$.

<u>Proof</u>: Let G' be a reduced graph with $\sigma(G') = r$ and let $f:V(G) \longrightarrow A^*(r)$ be an admissible map. If G' has no vertices or order 2, then every element of Z_r is induced by a vertex with order ≥ 3, so $r = |U(V'(G'))|$. Suppose G' has at least one vertex of order 2. By the definition of reduced graph, any vertex of order 2 must occur in a 3-cycle. If $r \geq 7$, as a consequence of Corollary (4.3) in [1], any 3-cycle can induce at most four elements of Z_r. Thus no vertex of order 2 in the reduced graph G' can induce an element of Z_r which is not induced by some vertex of order ≥ 3 which is adjacent to it. Thus $r = |U(V'(G'))|$ if $r \geq 7$. If $r \neq |U(V'(G'))|$, then there

exists a vertex of order 2 which induces exactly two elements of Z_r not induced by any other vertex of G'. Thus $\sigma(G') = 6$ and $|U(V'(G'))| = |Z_6| - 2 = 6 - 2 = 4$.

<u>Theorem 3.12</u> Let G be a graph with reduced graph G'. Let $\sigma(G) = r$. If there exists an admissible map $f:V(G) \longrightarrow A^*(r)$ for which $|U(V'(G))| = |U(V'(G'))|$ where $V'(G) = \{v \in V(G) | o(v) \geq 3\}$ and $V'(G') = \{v \in V(G') | o(v) \geq 3\}$, then for some nonnegative integer, k, $\sigma(G) = \sigma(G') + 2k$.

<u>Proof</u>: By Proposition (3.5) we know that $\sigma(G') \leq \sigma(G)$. If $\sigma(G') = \sigma(G)$, let $k = 0$. If $r' = \sigma(G') < \sigma(G) = r$, let $f:V(G) \longrightarrow A^*(r)$ be an admissible map for which $|U(V'(G))| = |U(V'(G'))|$.

Suppose first that $|U(V'(G'))| = r'$. Any element of $Z_r - U(V'(G))$ must be induced by a vertex of order 2 which belongs to some suspended (v,w)-path in G. By Proposition (3.10), every vertex of this path must induce two elements of Z_r not induced by any other vertex of G. Let $P_1, P_2, \ldots, P_n$ be the suspended (v,w)-paths in G which have a vertex of order 2 inducing an element of Z_r induced by no other vertex of G. Let k_i be the number of vertices of order 2 in P_i. If $k = k_1 + k_2 + \ldots k_n$, then $\sigma(G) = |U(V'(G))| + 2k = |U(V'(G'))| + 2k = r' + 2k = \sigma(G') + 2k$.

If $|U(V'(G'))| \neq r'$, then by Lemma (3.11), $\sigma(G') = 6$ and $|U(V'(G'))| = 4$. In this case there is a vertex x of G' with order 2 which induces two elements of Z_r not induced by any other vertex of G'. Now x belongs to a suspended (v,w)-path, Q, of length 2 in G'. In G, Q maps to a suspended (v,w)-path, P, of length ≥ 2. Since $|U(V'(G))| = |U(V'(G'))| = 4$, then every vertex of order 2 in P must induce two elements of Z_r not induced by any other vertex of G. If $2 \leq k' =$ length of P, then

$\sigma(G) = |U(V'(G))| + 2(k'-1) = 4 + 2 + 2(k'-2) = 6 + 2(k'-2) = \sigma(G') + 2k$ where $k = k'-2 \geq 0$. This completes the proof.

The graphs of Figure 11 and Figure 12 illustrate Theorem (3.12). In Figure 11, $\sigma(G) = 10 = 8 + 2 = \sigma(G') + 2$ and in Figure 12, $\sigma(\Gamma) = 14 = 8 + 6 = \sigma(\Gamma') + 2(3)$. We note that the condition $|U(V'(G))| = |U(V'(G'))|$ is necessary in Theorem (3.12). For example, let $G' = D(n)$ where $n = 2r+1 \geq 6$. Now, G' is a reduced graph. Let G be an expansion of G' formed by adding k vertices of order 2 to exactly one edge of G' where $6+2k = n+1$. Clearly $\sigma(G) = n+1$ while $\sigma(G') = n$. Moreover, $|U(V'(G))| = 6 < |U(V'(G'))| = n$. This is illustrated in Figure 13 for $D(7)$ and $k = 1$.

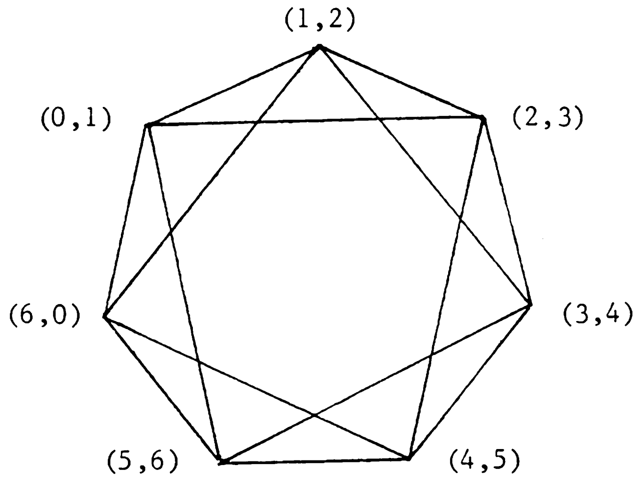
We conclude this section with a third way of showing that the circularity of a complete graph on $n \geq 3$ vertices is 6. The other two proofs are in [1] and [2].

Proposition 3.13 Let G be a graph with $\sigma(G) = r \geq 6$. If there exists a vertex, v, of G with valency $|V(G)| - 1$, then $\sigma(G) = 6$.

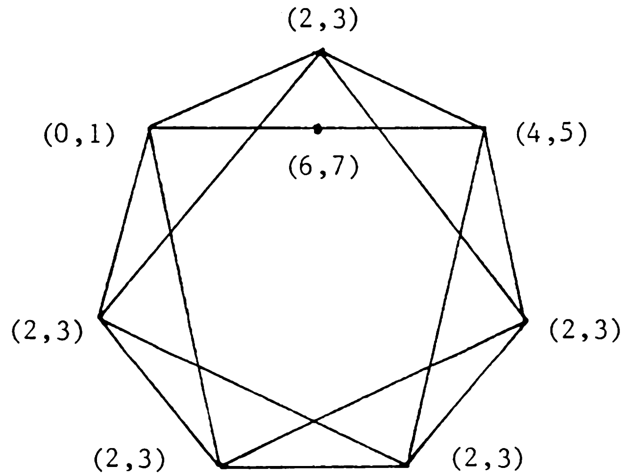
Proof: Let $f: V(G) \longrightarrow A^*(r)$ be an admissible map. Let $v \in V(G)$ have valency $|V(G)| - 1$. Let $\overline{a} \in Z_r$ be such that $f(v) = (\overline{a}, \overline{a+1})$. If $w \in V(G)-\{v\}$, then w is adjacent to v so by the preadmissibility of f, $f(w) \in \{\overline{a-2}, \overline{a-1}, \overline{a}, \overline{a+1}, \overline{a+2}, \overline{a+3}\}$. Thus $6 \leq \sigma(G) = |U(G)| \leq 6$, so $\sigma(G) = 6$.

Corollary 3.14 The circularity of any complete graph, K_n, $n \geq 3$, and the circularity of any wheel, W_n, $n \geq 4$, is 6.

Proof: Both types of graphs have at least one vertex which is adjacent to every other vertex of the graph.



$\sigma(D(7)) = 7$



$\sigma(G) = 8$

Figure 13

Section 4

SOME RESULTS ON PLANAR GRAPHS

In this section we examine some properties of admissible and co-admissible maps on planar graphs. We recall that a graph is planar if and only if it does not contain, as a subgraph, either K_5, the complete graph on n = 5 vertices, or $K_{(3,3)}$, the complete bipartite graph on m = 3 and n = 3 vertices, or any expansion of either of these graphs. A planar representation of a planar graph is a representation of the graph in the plane in which no edges cross.

As all graphs, planar graphs have vertices and edges. However, planar graphs also have faces. A face in a planar representation of a graph is a region bounded by a cycle in the graph which contains no other region bounded by any other cycle of the graph. The number of vertices, v, edges, e, and faces, f, of a planar graph are related by Euler's Formula: $v - e + f = 2$.

Any cycle J in a planar representation of a planar graph is a simple closed curve and hence, by the Jordan Curve Theorem, divides the plane into exactly two regions, each bounded by J. One of these regions is bounded and will be referred to as the "inside" of J for that particular representation of J. The other region is unbounded and will be referred to as the "outside" of J for that particular representation of J. When J bounds a face, the face will be called finite if the region comprising the face is bounded. Otherwise, the face will be called infinite. A planar graph has exactly one infinite face in any planar representation.

It has already been shown in [2] that the circularity of any finite connected planar graph is even. Consequently, we need only consider co-admissible maps when dealing with planar graphs. However, the first result in this section is established using admissible maps.

Theorem 4.1 Let G be a planar graph and let $f:V(G) \longrightarrow A^*(r)$ be an admissible map. Any face in any planar representation of G induces either all of Z_r or at most 4 elements of Z_r.

Proof: Let G be a planar graph and fix a planar representation of G. Let $f:V(G) \longrightarrow A^*(r)$ be an admissible map. Let F be the boundary of a face which does not induce all of Z_r. Let $V(F) = \{u_0, u_1, \ldots, u_{m-1}\}$ where this numbering is taken modulo m. Since F is a cycle and hence a simple closed curve, by the Jordan Curve Theorem, F separates the plane into two regions: the inside of F and the outside of F.

Since f is admissible and F is connected, U(F) is pseudoconnected, so without loss of generality, suppose $U(F) = \{\overline{0}, \overline{1}, \ldots, \overline{s}\}$ where $4 \leq s < r-1$. Choose subscripts i and j such that $u_i, u_j \in V(F)$ where $u_i \in V(F)$ induces $\overline{0}$ but u_{i-1} does not and such that u_j induces $\overline{s}$ while u_{j+1} does not. Let F_1 and F_2 denote the two (u_i, u_j)-paths in F. Now $F_1 \cup F_2 = F$ and $U(F_1) = U(F_2) = U(F)$ by pseudoconnectivity. Since $|U(F)| \geq 5$, there exists $\overline{a} \notin \{\overline{s-1}, \overline{s}, \overline{0}, \overline{1}\}$ and $u_\ell \in V(F_1) - \{u_i, u_j\}$ and $u_k \in V(F_2) - \{u_i, u_j\}$ such that both u_ℓ and u_k induce $\overline{a}$. By the admissibility of f, there is a (u_ℓ, u_k)-path P in G, all of whose vertices induce $\overline{a}$. If F bounds a finite face, then P must lie on the outside of F. If F bounds the infinite face, then P must be on the inside of F.

Let $v \in V(G)$ induce $\overline{r-1}$ so that $v \notin V(F)$. Since G is a connected graph, there is a (v,u_i)-path Q_1 in G and a (v,u_j)-path Q_2 in G. These two paths can be chosen such that neither contains any vertex of F which does not induce either $\overline{0}$ or $\overline{s}$. This is by the preadmissibility of f since Q_1 can be chosen such that $U(Q_1) \subseteq \{\overline{r-2},\overline{r-1},\overline{0},\overline{1}\}$ and Q_2 can be chosen such that $U(Q_2) \subseteq \{\overline{s-1},\overline{s},\ldots,\overline{r-2},\overline{r-1},\overline{0},\overline{1}\}$.

Let $Q = Q_1 \cup Q_2$. If F bounds a finite face of G, then Q is a (u_i,u_j)-path on the outside of F. If F bounds the infinite face of G, then Q is a (u_i,u_j)-path on the inside of F. In either case, Q must meet P so that either $\overline{a} \in U(Q) \cap U(P)$, a contradiction, or Q must cross P, contradicting the planarity of G. Thus $|U(F)| \leq 4$ if $U(F) \neq Z_r$.

<u>Corollary 4.2</u> If G is a planar graph and $g:V(G) \longrightarrow Z_n$ is a co-admissible map, then any face in any planar representation of G induces either all of Z_n or at most two elements of Z_n.

<u>Proof</u>: Let G be a planar graph and let $g:V(G) \longrightarrow Z_n$ be a co-admissible map. Let $f:V(G) \longrightarrow A^{**}(2n)$ be the admissible map corresponding to g in the sense of Theorem (1.4). Let F be a face in any planar representation of G. By Theorem (4.1), $|U(F)| = 2n$ or $|U(F)| \leq 4$. If F induces all of Z_{2n} under f, then under g, F induces all of Z_n. If F induces at most 4 elements of Z_{2n}, then since f maps into $A^{**}(2n)$, F can induce either 2 or 4 elements of Z_{2n} under f. Thus, under g, F can induce either 1 or 2 elements of Z_n.

One interesting observation which can be made in view of Corollary (4.2) is the following. Let G be a planar graph and let J be a cycle on $n \geq 4$ vertices which can never occur as a face in any planar representation of G. If J meets at least one face of G on fewer than n vertices in more than one edge, then J cannot be an adequate cycle. This is clear, for if J were adequate, there would be a co-admissible map $g:V(G) \longrightarrow Z_n$ such that $g(J) = Z_n$. Consequently, any face F on $m < n$ vertices meeting J in at least 3 vertices would be such that $3 \leq |g(F)| \leq m < n$, contradicting Corollary (4.2).

<u>Proposition 4.3</u> Let G be a planar graph and let $g:V(G) \longrightarrow Z_n$ be a co-admissible map where $n \geq 3$. If a cycle F, bounding a face of G, induces all of Z_n, then F is a co-admissible cycle.

<u>Proof</u>: Let F be a face in a fixed planar representation of the planar graph G which induces all of Z_n under the co-admissible map $g:V(G) \longrightarrow Z_n$. Let $V(F) = \{v_0, v_1, \ldots, v_m\}$. If F is not co-admissible, then there exist vertices v_i, v_j in F such that for some $\overline{c} \in Z_n$, $g(v_i) = g(v_j) = \overline{c}$ where $j \geq i+1$ and such that for each $v_k \in V(F)$ $(i < k < j)$, $g(v_k) = \overline{c+1}$ (respectively $g(v_k) = \overline{c-1}$). Now $G[c]$ is connected so there is a (v_i, v_j)-path, P, in $G[c]$. Since F bounds a face, this path must lie either on the outside of F or the inside of F depending on whether F bounds a finite or infinite face, respectively. Since $n \geq 3$ and g is co-admissible, there exists $v_\ell \in V(F) - \{v_i, v_{i+1}, \ldots, v_{j-1}, v_j\}$ such that $g(v_\ell) \in \{\overline{c+1}, \overline{c+2}\}$ (respectively, $g(v_\ell) \in \{\overline{c-2}, \overline{c-1}\}$). By Proposition (1.15), there is a monotone (v_k, v_ℓ)-path in $G[c+1, c+2]$ (respectively $G[c-2, c-1]$) on the same side of F as P. This path must thus meet P, a contradiction since G is a planar graph and both $G[c] \cap G[c+1, c+2]$ and $G[c] \cap G[c-2, c-1]$ are empty. Thus any cycle inducing all of Z_n and bounding a face is a co-admissible cycle.

We will call a face of a planar graph which is bounded by a co-admissible cycle, a <u>co-admissible face</u>.

<u>Proposition 4.4</u> If G is a planar graph and if $g:V(G) \longrightarrow Z_n$ is a co-admissible map where $n \geq 3$, then G contains at least two co-admissible faces.

<u>Proof</u>: Fix a planar representation of the graph G and let $g:V(G) \longrightarrow Z_n$ be a co-admissible map where $n \geq 3$. By Theorem (1.17), G contains a co-admissible cycle J. If G = J, then G has exactly two faces, each of which is co-admissible. If $G \neq J$, then J can bound a finite face, the infinite face, or neither.

If J does not bound a finite face, then there exist vertices v,w in J and a monotone (v,w)-path, P_1, which lies on the inside of J. Let L_{11} and L_{21} be the two (v,w)-paths in J and let $J_{11} = L_{11} \cup P_1$ and $J_{21} = L_{21} \cup P_1$ so that J_{11} and J_{21} are both cycles in G. Since g is co-admissible and P_1 is monotone, either $g(L_{11}) = g(P_1)$ or $g(L_{21}) = g(P_1)$ so that either $g(J_{11}) = Z_n$ or $g(J_{21}) = Z_n$. Thus at least one of J_{11} and J_{21} is a co-admissible cycle in G. If either is a co-admissible face, the proof can stop. If not, choose one which is co-admissible and call it J_1. As before, there are two vertices v,w in J_1 and a monotone (v,w)-path, P_2, on the inside of J_1 (hence on the inside of J). Let L_{12} and L_{22} be the two (v,w)-paths in J_1 so that $J_{12} = L_{12} \cup P_2$ and $J_{22} = L_{22} \cup P_2$ are both cycles in G, at least one of which is co-admissible. If either is a co-admissible face, the proof can stop. If not, choose one which is co-admissible and call it J_2. Repeat this process a finite number of times until a co-admissible face is obtained on the inside of J. It will be obtained because G is a finite graph and each application of the process yields a co-admissible cycle inside J.

If J does not bound the infinite face, then reasoning similar to that in the previous case yields a co-admissible face on the outside of J. This completes the proof, for if J bounds a finite face and not the infinite face or if J bounds the infinite face and not a finite face, then one of the above processes yields another co-admissible face. If J bounds neither kind of co-admissible face, then both processes together yield co-admissible faces on the outside and inside of J. Thus G contains at least two co-admissible faces.

Lemma 4.5 If G is a planar graph and if $g:V(G) \longrightarrow Z_3$ is a co-admissible map, then G has exactly two co-admissible faces.

Proof: (By induction on n, the number of faces of G). Fix a planar representation of the graph G and let $g:V(G) \longrightarrow Z_3$ be a co-admissible map. If G has n = 2 faces, then G is a cycle and hence each face is co-admissible. Suppose that any planar graph with $n \geq 2$ faces has exactly two co-admissible faces (relative to any co-admissible map g and 3). Let G be a planar graph with $n + 1 \geq 3$ faces. Since G has at least three faces, there exist $\overline{a} \in Z_3$ and two edges, $e = vw$ and $e' = v'w'$, in G such that $g(v) = g(v') = \overline{a}$ and $g(w) = g(w') = \overline{a+1}$. Let G' be the subgraph of G with vertex set $V(G') = V(G)$ and edge set $E(G') = E(G) - \{e'\}$. Clearly $g:V(G') \longrightarrow Z_3$ is a co-admissible map. Moreover, the Euler formula shows that G' has n faces, so by the induction hypothesis, G' has exactly two co-admissible faces. Theorem 1 of [4] shows that $e' \in E(F_1) \cap E(F_2)$ for exactly two faces, F_1 and F_2, of G, so in G', the face $F' = [F_1 \cup F_2 - \{e'\}] \cup \{v', w'\}$

replaces the two faces F_1 and F_2. Furthermore, if F is any face in G (respectively G') other than F_1 or F_2 (respectively F'), then F is also a face in G' (respectively G).

In G', F' is either a co-admissible face or it isn't. If F' is not co-admissible, then $g(F') = \{\overline{a},\overline{a+1}\}$ and hence $g(F_1) = g(F_2) = \{\overline{a},\overline{a+1}\}$, showing that neither F_1 nor F_2 is co-admissible. Consequently, the only co-admissible faces in G are the two co-admissible faces in G'. If F' is a co-admissible face, then $g(F') = Z_3$. Let $L_1 = (F_1-\{e'\}) \cup \{v',w'\}$ and $L_2 = (F_2-\{e'\}) \cup \{v',w'\}$, so that L_1 and L_2 are the two (v',w')-paths in F'. Since F'[c] is connected for each $\overline{c} \in Z_3$, it follows that either $\overline{a-1} \in g(L_1)$ or $\overline{a-1} \in g(L_2)$, but not both. Consequently, either F_1 or F_2 is a co-admissible face in G, but not both. The second co-admissible face of G is thus the co-admissible face of G' which is not F'. This completes the proof.

Theorem 4.6 If G is a planar graph and if $g:V(G) \longrightarrow Z_n$ is a co-admissible map where $n \geq 3$, then G has exactly two co-admissible faces.

Proof: Fix a planar representation of the graph G and let $g:V(G) \longrightarrow Z_n$ be a co-admissible map where $n \geq 3$. Proposition (4.4) guarantees the existence of at least two co-admissible faces in G. If G is a cycle, then G has exactly two co-admissible faces. If G is not a cycle, suppose that G has three co-admissible faces. By Proposition (1.9), there exists a co-admissible map $g^*:V(G) \longrightarrow Z_3$ such that $g^*(v) = g(v)$ if $g(v) \in \{\overline{0},\overline{1},\overline{2}\}$ and such that $g^*(v) = \overline{2}$ if $g(v) \in \{\overline{3},\ldots,\overline{n-1}\}$. Consequently, each co-admissible face (relative to g and n) would also be a co-admissible face (relative to g* and 3), contradicting Lemma (4.5). Thus G must have exactly two co-admissible faces.

<u>Corollary 4.7</u> If G is a planar graph, then $\eta(G) \leq \max\{n \in N \mid G$ has at least two faces on $m \geq n$ vertices in any planar representation of $G\}$.

<u>Proof:</u> Fix a planar representation of the graph G with co-circularity $\eta(G)$. Let $g: V(G) \longrightarrow Z_{\eta(G)}$ be a co-admissible map. Let $M = \max\{n \in N \mid G$ has at least two faces on $m \geq n$ vertices$\}$. By Theorem (4.6), G has exactly two faces inducing all of $Z_{\eta(G)}$, say F_1 and F_2. If $|V(F_1)| \leq |V(F_2)|$, then $|V(F_1)| \leq M$. Since $|U(F_1)| = \eta(G)$ and $|U(F_1)| \leq |V(F_1)|$, it follows that $\eta(G) \leq M$.

Figure 14 shows a planar representation of the reduced graph G' in Figure 11 with a co-admissible map into Z_4 where the two co-admissible faces of G' are the infinite face and a finite face not meeting the infinite face. For this graph, $\eta(G') = 4 = M$. An example of a graph for which $\eta(G) < M$ is the nonreduced graph G in Figure 11. For this graph $\eta(G) = 5 < 6 = M$.

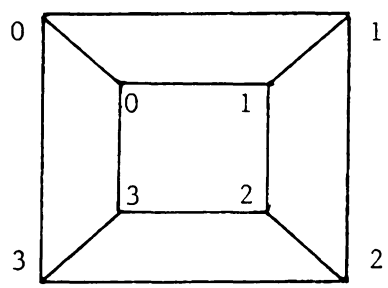


Figure 14

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CIRCULARITY OF GRAPHS

by

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(ABSTRACT)

Let G be a finite connected graph. The circularity of G has been previously defined as $\sigma(G) = \max\{r \in N |$ G has a circular covering of r elements, each element being a closed, connected subset of G containing at least one vertex of G$\}$. This definition is known to be equivalent to the combinatorial description, $\sigma(G) = \max\{r \in N |$ there is an admissible map $f:V(G) \longrightarrow A(r)\}$. In this thesis, co-admissible maps are introduced and the co-circularity of a graph, G, is defined as $\eta(G) = \max\{n \in N |$ there is a co-admissible map $g:V(G) \longrightarrow Z_n\}$. It is shown that $\sigma(G) = 2\eta(G)$ or $2\eta(G) + 1$. It is also shown that if G is a graph and $g:V(G) \longrightarrow Z_n$ is a co-admissible map, then G contains a cycle, J, called a co-admissible cycle, for which $g:V(J) \longrightarrow Z_n$ is also co-admissible. Necessary and sufficient conditions are given for extending a co-admissible map on a cycle of a graph to the entire graph. If G is a graph with $\sigma(G) = r$, it is shown that any suspended (v,w)-path P in G induces, under any admissible map $f:V(G) \longrightarrow A(r)$, either at most four elements of Z_r or every vertex of P with valency two induces exactly two elements of Z_r not induced by any other vertex of G. Finally it is shown that if G is a planar graph and if $g:V(G) \longrightarrow Z_n$ is a co-admissible map, then any planar representation of G has exactly two faces bounded by co-admissible cycles.