Double Affine Bruhat Order

Amanda Renee Welch

Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

Daniel Orr, Chair
Nicholas Loehr
Leonardo Mihalcea
Mark Shimozono

March 21, 2019
Blacksburg, Virginia

Keywords: Weyl groups, Bruhat order, double affine, coverings, cocoverings

Double Affine Bruhat Order

Amanda Renee Welch

(ABSTRACT)

Given a finite Weyl group W_{fin} with root system Φ_{fin}, one can create the affine Weyl group W_{aff} by taking the semidirect product of the translation group associated to $Q^\vee$, the coroot lattice for Φ_{fin}, with W_{fin}. The double affine Weyl semigroup W can be created by using a similar semidirect product where one replaces W_{fin} with W_{aff} and $Q^\vee$ with the Tits cone of W_{aff}. We classify cocovers and covers of a given element of W with respect to the Bruhat order, specifically when W is associated to a finite root system that is irreducible and simply laced. We show two approaches: one extending the work of Lam and Shimozono, and its strengthening by Milićević, where cocovers are characterized in the affine case using the quantum Bruhat graph of W_{fin}, and another, which takes a more geometrical approach by using the length difference set defined by Muthiah and Orr.

Double Affine Bruhat Order

Amanda Renee Welch

(GENERAL AUDIENCE ABSTRACT)

The Bruhat order is a way of organizing elements of the double affine Weyl semigroup so that we have a better understanding of how the elements interact. In this dissertation, we study the Bruhat order, specifically looking for when two elements are separated by exactly one step in the order. We classify these elements and show that there are only finitely many of them.

Contents

Overview

The first two chapters of this dissertation will introduce the topic of Weyl groups and affine Weyl groups. A Weyl group W_{fin} is a reflection group associated to a crystallographic root system Φ_{fin}. An example of this is the symmetric group $W_{\text{fin}} = S_n$ where $\Phi_{\text{fin}} = \{\epsilon_i - \epsilon_j : 1 \leq i \neq j \leq n\}$. Weyl groups and their associated root systems play a central role in the study of Lie algebras. In fact, there is a one-to-one correspondence between complex, simple Lie algebras and irreducible, crystallographic root systems.

Every element $w \in W_{\text{fin}}$ can be written as a product of simple reflections. For the symmetric group, this is the same idea as every permutation being generated by transpositions of the form $(i\ i+1)$. A reduced expression for w is a product of simple reflections that equals w and uses the smallest number of simple reflections. Every w has a reduced expression, so we can define a function ℓ on W_{fin} such that $\ell(w)$ equals the number of simple reflections in a reduced expression of w.

One can put a partial ordering on the Weyl group W_{fin} that relates to the length function. An element $u \in W_{\text{fin}}$ is said to be greater than $w = s_\alpha u$ if α is a positive root and $u^{-1}(\alpha) < 0$. The Bruhat order is the order generated by these relations. A Bruhat interval $[w, u]$ consists of all $z \in W_{\text{fin}}$ such that $w \leq z \leq u$ with respect to the Bruhat order. When this interval consists of only u and w, w is said to be a cocover of u and $\ell(u) = \ell(w) + 1$.

The affine Weyl group W_{aff} can be defined by taking the semidirect product of the translation group associated to the coroot lattice $Q^\vee = \mathbb{Z}\Phi_{\text{fin}}^\vee$ with W_{fin}. As finite Weyl groups are fundamental to Lie algebras, affine Weyl groups play a similar role in the study of affine Lie algebras.

Chapter three will introduce double affine Weyl semigroups and will include some results that will be used later in the paper. The double affine Weyl semigroup W can be created by using a similar semidirect product where one replaces W_{fin} with W_{aff} and $Q^\vee$ with the Tits cone of W_{aff}. Braverman, Kazhdan, and Patnaik introduced a pre-order on W in [BKP] while examining Iwahori-Hecke algebras for affine Kac-Moody groups. Later, Muthiah studied the Bruhat order of W in an effort to develop Kazhdan-Lusztig theory and understand the double affine flag variety.

When considering the double affine Weyl semigroup, we do not have that every element is generated by simple reflections. The length function can still be defined, but instead of defining it by reduced expressions, we follow the work of Muthiah [M] and define it in an alternative fashion that still corresponds with the Bruhat order.

Given $x \in W$ and α a positive double affine root, [BKP] defined a preorder with generating relations: $x \geq s_\alpha x$ if and only if $x^{-1}(\alpha) < 0$. They called this preorder the Bruhat preorder and conjectured that it was an order (it was known that in the finite and affine case it is an order). In [M] it was shown that the preorder is in fact an order and in [MO] it was shown that the order coincides with the order generated by the relations: $x \geq xs_\alpha$ if and only if $\ell(x) \geq \ell(xs_\alpha)$.

Further, Muthiah and Orr [MO] related the cocover and cover relationships to a difference in lengths by proving: For α a positive double affine root, $s_\alpha x$ is a cocover of x if and only if $\ell(x) = \ell(s_\alpha x) + 1$.

The final chapters of this dissertation focus on classifying cocovers and covers of a given element of W with respect to the Bruhat order, specifically when W is associated to a finite root system that is irreducible and simply laced. Classifying the cocovers and covers of given elements helps us describe the Bruhat intervals and move forward in our goal of fully understanding the Bruhat order.

In chapter four, we classify cocovers by extending the work done by Lam and Shimozono and further strengthened by Milićević, where cocovers were classified in the affine case. This approach allows us to classify cocovers by using the quantum Bruhat graph of W_{aff}, but in doing so, we must impose length bounds on the translation parts of the elements under consideration.

In chapter five, we use the length difference set defined by Muthiah and Orr [MO] to better our classification. This approach allows us to show that there are finitely many cocovers and covers for a given element $x \in W$, which ensures that intervals are finite in the double affine Bruhat order.

List of Notation

- Φ_{fin} - simply laced, irreducible root system
- Δ_{fin} - the set of simple roots
- W_{fin} - finite Weyl group
- Q_{fin} or Q - the root lattice
- ω_i for $i = 1, 2, \ldots, n$ - the finite fundamental weights
- $\rho_{\text{fin}} = \sum_{i=1}^{n} \omega_i$
- $W_{\text{aff}} = Q \rtimes W_{\text{fin}}$ - the affine Weyl group
- $\widetilde{w} = Y^{\lambda}w$ - general element of W_{aff}
- Φ_{aff} - the set of affine roots
- $\widetilde{\alpha} = \nu + r\delta$ - general affine root
- Δ_{aff} - the set of simple affine roots
- Λ_i for $i = 0, 1, 2, \ldots, n$ - the affine fundamental weights
- $\rho = \rho_{\text{aff}} = \sum_{i=0}^{n} \Lambda_i$
- $\mathcal{T}$ - Tits cone
- W - double affine Weyl semigroup
- $X^{\zeta}Y^{\lambda}w$ - general element of W
- Φ - the set of double affine roots
- $\alpha = \nu + r\delta + j\pi$ - general double affine root
- $L_{x,\alpha}$ - length difference set
- $\Gamma_{x,\nu}$ - lower graph of x corresponding to ν
- $\Gamma'_{x,\nu}$ - upper graph of x corresponding to ν

CHAPTER 1

Finite Weyl Groups

Given a collection of n distinct objects, how many ways can we arrange them? Let us look at the small example where $n = 3$. We will give each item a number $1, 2$, or 3 and we will use notation $[2, 1, 3]$ to mean item 2 is placed first, item 1 is placed second, and item 3 is placed third.

It is easy to see that there are only six ways to form such an arrangement: $[1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1]$. These six elements form a group with group operation composition, and we call this group S_3, the symmetric group with $n = 3$ (or the permutation group of $[3] = \{1, 2, 3\}$). Note that the order of S_3 is $6 = 3 \times 2 \times 1$ since there are three choices for the first position, two choices for the second position, and one for the third position.

In general, we can look at the symmetric group S_n as the group whose elements represent all the ways you could permute n distinct objects. The order of $S_n = n! = n(n-1)\cdots 1$. The notation we used above when writing elements of S_3 is called window notation. It is nice to use because it gives all of the elements a uniform look and it is easy to read, but it is also useful to view the elements in another form. The element $[2, 1, 3]$ in window notation becomes $(1\ 2)(3)$ or $(1\ 2)$ in cycle notation because the first and second objects switch positions and the third object remains unchanged.

2-cycles (cycles with only two values) are given the special name transposition and play an important role in the symmetric group. Any element of the symmetric group can be written as a product of transpositions of the form $(i\ i+1)$, so these transpositions generate the group. Another way to view the transpositions of S_n is by thinking of them as reflections.

1. Reflection Groups

In our introduction to Weyl groups and root systems, we will cover some definitions and results from the general theory, and in doing so, we will follow the work of Humphreys [H].

Definition 1.1.1. Given V some Euclidean space, a **reflection** is a linear map that sends some nonzero $\alpha \in V$ to $-\alpha \in V$ while fixing all vectors in the hyperplane orthogonal to α that passes through the origin. We denote this hyperplane by H_α.

Example 1.1.2. Consider $\mathbb{R}^n$ with $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ the usual basis vectors. Let V be the $(n-1)-$dimensional subspace of $\mathbb{R}^n$ consisting of all vectors whose components sum to 0. Let $\langle \cdot, \cdot \rangle$ be the usual dot product.

Then each transposition $(i \ i+1) \in S_n$ can be thought of as a map sending $\alpha = \epsilon_i - \epsilon_{i+1}$ to $-\alpha = \epsilon_{i+1} - \epsilon_i$ and fixing all vectors v such that $\langle v, \epsilon_i - \epsilon_{i+1} \rangle = 0$. Then

$$H_\alpha := \{v \in \mathbb{R}^n : \langle v, \ \epsilon_i - \epsilon_{i+1} \rangle = 0\}$$

is the hyperplane fixed by $(i \ i+1)$.

Note that if we consider S_n acting on all of $\mathbb{R}^n$, then $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$ is fixed under S_n and so is any multiple of it. By restricting V to the $(n-1)-$dimensional subspace of $\mathbb{R}^n$ consisting of all vectors whose components sum to 0, S_n will only fix the origin when acting on V.

For the remainder of the paper we will fix V to be a Euclidean space with inner product $\langle \cdot, \cdot \rangle$.

Definition 1.1.3. A **reflection group** is a subgroup of O(V) (the orthogonal group of V consisting of orthogonal transformations of V) that is generated by reflections.

Example 1.1.4. S_n is generated by transpositions, which we have just shown to be reflections, so it is isomorphic to a reflection group.

DEFINITION 1.1.5. Let $\alpha \in V$. Then H_α is the hyperplane orthogonal to α that passes through the origin and we define s_α to be the **reflection** that fixes H_α and sends α to $-\alpha$.

EXAMPLE 1.1.6. Let $W_{\text{fin}} = S_n$. Then W_{fin} is generated by $s_1, s_2, \ldots, s_{n-1}$ where $s_i = s_{\epsilon_i - \epsilon_{i+1}}$ represents the transposition $(i\ i+1)$.

To make calculations easier, it is nice to have a formula for s_α. This can be given by

$$s_\alpha(v) = v - 2\frac{\langle v, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha.$$

2. Root Systems

DEFINITION 1.2.1. Let Φ_{fin} be a finite set of non-zero vectors in V such that

(1) $\Phi_{\text{fin}} \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi_{\text{fin}}$

(2) $s_\alpha(\Phi_{\text{fin}}) = \Phi_{\text{fin}}$ for all $\alpha \in \Phi_{\text{fin}}$.

Then Φ_{fin} is a **root system**. If the set $\{s_\alpha : \alpha \in \Phi_{\text{fin}}\}$ is a generating set for W_{fin}, then Φ_{fin} is the **root system associated with the reflection group** W_{fin}.

REMARK 1.2.2. Some authors specify the first condition as necessary only for reduced root systems.

DEFINITION 1.2.3. Let $\alpha \in \Phi_{\text{fin}}$. We call $\frac{2\alpha}{\langle \alpha,\alpha\rangle}$ the **coroot** of α and denote it by $\alpha^\vee$. The set of coroots, $\Phi^\vee_{\text{fin}} = \{\alpha^\vee : \alpha \in \Phi_{\text{fin}}\}$, is a root system.

DEFINITION 1.2.4. A root system is said to be **crystallographic** if $\langle \beta, \alpha^\vee\rangle = 2\frac{\langle \beta,\alpha\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi_{\text{fin}}$.

REMARK 1.2.5. Some authors require $2\frac{\langle \beta,\alpha\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi_{\text{fin}}$ as a condition for a root system.

EXAMPLE 1.2.6. Let $W_{\text{fin}} = S_n$. Then for any $\alpha_i \in \Delta_{\text{fin}}$, $\langle \alpha_i, \alpha_i\rangle = 2$ so $\alpha_i^\vee = \alpha_i$. In fact, for all $\alpha \in \Phi_{\text{fin}}$, $\alpha^\vee = \alpha$.

DEFINITION 1.2.7. A root system Φ_{fin} is said to be **simply laced** if every root has the same length (i.e. $\langle \alpha, \alpha\rangle^{1/2} = \langle \beta, \beta\rangle^{1/2}$ for all $\alpha, \beta \in \Phi_{\text{fin}}$).

For the the remainder of the paper we will only consider W_{fin} for a simply laced, crystallographic root system. We will normalize the pairing so $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in \Phi_{\text{fin}}$. This allows us to identify $\alpha^\vee$ with α, so from now on we will identify $\Phi_{\text{fin}}^\vee$ with Φ_{fin} and stop using coroot notation.

Definition 1.2.8. Pick an element $v \in V$ such that $\langle \alpha, v \rangle \neq 0$ for all $\alpha \in \Phi_{\text{fin}}$. The hyperplane orthogonal to v splits V into two half spaces. Let Φ_{fin}^+ be the set of roots $\alpha \in \Phi_{\text{fin}}$ such that $\langle \alpha, v \rangle > 0$. We call these the **positive roots** and denote this by $\alpha > 0$. Let Φ_{fin}^- be the set of roots $\alpha \in \Phi_{\text{fin}}$ such that $\langle \alpha, v \rangle < 0$. We call these the **negative roots** and denote this by $\alpha < 0$.

For every $\alpha \in \Phi_{\text{fin}}$ there is $-\alpha \in \Phi_{\text{fin}}$, so it is clear that $\Phi_{\text{fin}} = \Phi_{\text{fin}}^+ \sqcup \Phi_{\text{fin}}^-$.

Definition 1.2.9. Let $\Delta_{\text{fin}} = \{\alpha_i : i = 1, 2, \ldots, n\}$ be a subset of Φ_{fin} that is a vector space basis for the $\mathbb{Z}$-span of Φ_{fin} in V (we use $\mathbb{Z}$ here instead of $\mathbb{R}$ because our root system is crystallographic). We call Δ_{fin} a **simple system** and say that the roots α_i are **simple roots** if any α in Φ_{fin} can be written as $\alpha = \sum_{i=1}^n a_i \alpha_i$ where $a_i \in \mathbb{Z}$ and either all $a_i \geq 0$ or all $a_i \leq 0$.

Proposition 1.2.10. *Given $\Delta_{\text{fin}} \in \Phi_{\text{fin}}$, we can find a unique Φ_{fin}^+ and Φ_{fin}^- containing Δ_{fin}. We define Φ_{fin}^+ to be the set consisting of all roots written as $\sum_{i=1}^n a_i \alpha_i$ where α_i are the simple roots and $a_i \in \mathbb{Z}$ are positive. Similarly, we define Φ_{fin}^- to be the set consisting of all roots written as $\sum_{i=1}^n a_i \alpha_i$ where α_i are the simple roots and $a_i \in \mathbb{Z}$ are negative.*

Example 1.2.11. Let $W_{\text{fin}} = S_n$, the reflection group with root system type A_{n-1}. Define $\alpha_i = \epsilon_i - \epsilon_{i+1}$. Then

$$\Phi_{\text{fin}} = \{\epsilon_i - \epsilon_j : 1 \leq i \neq j \leq n\},$$

$$\Delta_{\text{fin}} = \{\epsilon_i - \epsilon_{i+1} : 1 \leq i \leq n-1\},$$

and the standard choices for Φ_{fin}^+ and Φ_{fin}^- are

$$\Phi_{\text{fin}}^+ = \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq n\},$$

$$\Phi_{\text{fin}}^- = -\Phi_{\text{fin}}^+ = \{\epsilon_j - \epsilon_i : 1 \leq i < j \leq n\}.$$

Definition 1.2.12. We define a partial order on the roots and say $\alpha \geq \beta$ if $\alpha - \beta = \sum_{i=1}^{n} a_i\alpha_i$ where a_i are nonnegative integers.

There is a unique maximal root with respect to this order, called the **highest root**. We denote this root by θ.

Remark 1.2.13. Because $\theta \geq \alpha$ for all roots $\alpha \in \Phi_{\text{fin}}$, $\theta - \alpha_i = \sum_{i=1}^{n} a_i\alpha_i$, where a_i are nonnegative integers for $i = 1, 2, \ldots, n$. So $\theta = \sum_{i=1}^{n} c_i\alpha_i$ where c_i are positive integers for $i = 1, 2, \ldots, n$.

Definition 1.2.14. Given $\alpha \in \Phi_{\text{fin}}$ such that $\alpha = \sum_{i=1}^{n-1} a_i\alpha_i$, we define the height function by $\text{ht}(\alpha) := \sum_{i=1}^{n} a_i$.

Note that $\text{ht}(\alpha) < \text{ht}(\theta)$ for all $\alpha \neq \theta$ in Φ_{fin}.

Example 1.2.15. Let $W_{\text{fin}} = S_n$, the reflection group with root system type A_{n-1}. Then $\theta = \sum_{i=1}^{n-1} \alpha_i$ and $\text{ht}(\alpha) = n - 1$.

We define ρ_{fin} such that $2\rho_{\text{fin}} = \sum_{\alpha\in\Phi_+} \alpha$, the sum of all the positive roots. Then $\text{ht}(\alpha) = \langle \alpha, \rho_{\text{fin}} \rangle$ for all $\alpha \in \Phi_{\text{fin}}$.

3. Weyl Groups

Definition 1.3.1. A **Weyl group** W_{fin} is a reflection group generated by s_α where $\alpha \in \Phi_{\text{fin}}$ and Φ_{fin} is a crystallographic root system.

Example 1.3.2. Let $W_{\text{fin}} = S_n$. Then for all $\beta, \alpha \in \Phi_{\text{fin}}, 2\frac{\langle \beta,\alpha \rangle}{\langle \alpha,\alpha \rangle} \in \mathbb{Z}$ since $\langle \alpha, \alpha \rangle = 2$ and $\langle \beta, \alpha \rangle \in \mathbb{Z}$. So S_n is a Weyl group.

Definition 1.3.3. Let $s_i = s_{\alpha_i}$ for $i = 1, 2, \ldots, n$. We call s_i the **ith simple reflection**.

Proposition 1.3.4. *A Weyl group W_{fin} is generated by the simple reflections s_i where $i = 1, 2, \ldots, n$.*

Let W_{fin} be a Weyl group and w some element. Since W_{fin} is a reflection group, $w = s_{i_1}s_{i_2}\cdots s_{i_m}$ where each $s_{i_k} = s_{\alpha_i}$ for some $\alpha_i \in \Delta_{\text{fin}}$ (recall that in the Symmetric group this is equivalent to saying that each element of S_n can be decomposed into a product of transpositions of the form $(i, i+1)$). There are infinitely many ways to write w as such a product. Each reflection is its own inverse, so $s_{i_1} s_{i_2} \cdots s_{i_m} = s_{i_1} s_{i_2} \cdots s_{i_m} s_{i_j} s_{i_j} = s_{i_1} s_{i_2} \cdots s_{i_m} s_{i_j} s_{i_j} s_{i_j} s_{i_j}$ and so on.

Definition 1.3.5. Let w be an element of the Weyl group W_{fin}. Then $w = s_{i_1} s_{i_2} \cdots s_{i_m}$ where each $s_{i_k} = s_{\alpha_i}$ for some $\alpha_i \in \Delta_{\text{fin}}$. We say $s_{i_1} s_{i_2} \cdots s_{i_m}$ is a **reduced expression** if m is the smallest number of simple reflections needed such that $w = s_{i_1} s_{i_2} \cdots s_{i_m}$. We define m to be the **length** of w and write $\ell(w) = m$.

Proposition 1.3.6. *Reduced expressions are not unique in general, but there are only finitely many ways of writing w as a reduced expression.*

Example 1.3.7. Let $W_{\text{fin}} = S_4$ and let $w = [4, 2, 3, 1]$ in window notation. In cycle notation $w = (1, 4)$, which has $(3, 4)(2, 3)(1, 2)(2, 3)(3, 4)$ as a reduced expression, so $\ell(w) = 5$. We can also write $w = (1, 2)(2, 3)(3, 4)(2, 3)(1, 2)$, so we see that reduced expressions are not unique.

Proposition 1.3.8. *Let W_{fin} be a Weyl group and $w \in W_{\text{fin}}$. Then*

$$\ell(ws_i) = \begin{cases} \ell(w) + 1 & w(\alpha_i) > 0 \\ \ell(w) - 1 & w(\alpha_i) < 0. \end{cases}$$

Definition 1.3.9. Let $w \in W_{\text{fin}}$. Then $\alpha \in V$ is an **inversion** of w if $\alpha \in \Phi^+_{\text{fin}}$ and $w(\alpha) \in \Phi^-_{\text{fin}}$. We denote the set of inversions of w by $\text{Inv}(w)$.

For $W_{\text{fin}} = S_n$, we have a bijection between Φ_{fin} and $\{(i, j) \in \mathbb{Z}^2 : 1 \leq i \neq j \leq n\}$ by identifying $\epsilon_i - \epsilon_j$ with (i, j). Using window notation, we write $w = [w_1, w_2, \ldots, w_n]$ where $w_i \in [n]$ represents what i gets sent to under w. Then $w(\epsilon_i - \epsilon_j) = \epsilon_{w(i)} - \epsilon_{w(j)}$ and $\text{Inv}(w) = \{\epsilon_i - \epsilon_j : i < j,\ w_i > w_j\}$, which we identify with the set $\{(i, j) : i < j,\ w_i > w_j\}$.

Example 1.3.10. Let $W_{\text{fin}} = S_4$ and $w = [4, 2, 3, 1]$. Then

$$\text{Inv}(w) = \{(1, 2), (1, 3), (1, 4), (2, 4), (3, 4)\}$$

and $|\operatorname{Inv}(w)| = 5$.

In Example 1.3.7, we saw that $\ell([4,2,3,1]) = 5$, so $\ell([4,2,3,1]) = |\operatorname{Inv}([4,2,3,1])|$. This is not a coincidence. One can show that the smallest number of transpositions of the form $(i\ i{+}1)$ needed to construct $w \in S_n$ is equal to the number of inversions of w. In fact, this is true for Weyl groups in general.

Proposition 1.3.11. *Let W_{fin} be some Weyl group and $w \in W_{\text{fin}}$. Then $\ell(w) = |\operatorname{Inv}(w)|$.*

To conclude this section, we will look at a nice property of the length function that will prove useful in our later work.

Proposition 1.3.12. *Let W_{fin} be some Weyl group and $x, y \in W_{\text{fin}}$. Then*

$$\begin{aligned}\ell(xy) &= \ell(x) + \ell(y) - 2|\{\operatorname{Inv}(y) \cap -y^{-1}\operatorname{Inv}(x)\}|.\\ &= \ell(x) + \ell(y) - 2|\{\alpha \in \operatorname{Inv}(y) : \alpha \notin \operatorname{Inv}(xy)\}|\end{aligned}$$

Proof. Let $\alpha \in \operatorname{Inv}(xy)$. There are two possibilities:

(1) $\alpha > 0,\ y(\alpha) < 0$, and $xy(\alpha) < 0$
(2) $\alpha > 0,\ y(\alpha) > 0$, and $xy(\alpha) < 0$.

So $\operatorname{Inv}(xy) \subset y^{-1}\operatorname{Inv}(x) \sqcup \operatorname{Inv}(y)$ (this is a disjoint union because if $\alpha \in y^{-1}\operatorname{Inv}(x)$, then $y(\alpha) > 0$ and so $\alpha \notin \operatorname{Inv}(y)$). In general, this is a proper subset because there could be $\alpha \in \operatorname{Inv}(y)$ such that $-y(\alpha) \in \operatorname{Inv}(x)$ (so $\alpha \notin \operatorname{Inv}(xy)$), or there could be $\alpha < 0$ such that $y(\alpha) \in \operatorname{Inv}(x)$ (so $\alpha \in y^{-1}\operatorname{Inv}(x)$ but $\alpha \notin \operatorname{Inv}(xy)$).

So $|\operatorname{Inv}(xy)| \leq |y^{-1}\operatorname{Inv}(x)| + |\operatorname{Inv}(y)|$, and to find an exact representation of $|\operatorname{Inv}(xy)|$, we must subtract $|\{\alpha \in \operatorname{Inv}(y) : -y(\alpha) \in \operatorname{Inv}(x)\}| = |\operatorname{Inv}(y) \cap -y^{-1}\operatorname{Inv}(x)|$ and $|\{\alpha < 0 : y(\alpha) \in \operatorname{Inv}(x)\}| = |\{\beta > 0 : -y(\beta) \in \operatorname{Inv}(x)\}| = |\operatorname{Inv}(y) \cap -y^{-1}\operatorname{Inv}(x)|$.

Using $|\operatorname{Inv}(x)| = |y^{-1}\operatorname{Inv}(x)|$, we have $|\operatorname{Inv}(xy)| = |\operatorname{Inv}(x)| + |\operatorname{Inv}(y)| - 2|\operatorname{Inv}(y) \cap -y^{-1}\operatorname{Inv}(x)| = 2|\{\alpha \in \operatorname{Inv}(y) : \alpha \notin \operatorname{Inv}(xy)\}|$. □

4. Coxeter Groups

We have described how W_{fin} is generated by s_α with $\alpha \in \Delta$. Using these generators (and some relations we have yet to discuss), we can give a presentation for W_{fin}.

Example 1.4.1. First we will look at the specific case $W_{\text{fin}} = S_n$. Every reflection s_i has order two, so we have the relation $s_i^2 = 1$ for $i = 1, 2, ..., n-1$. We know that disjoint transpositions commute so $(s_i s_j)^2 = 1$ for $|i-j| > 1$. Lastly, $(i\ i+1)(i+1\ i+2) = (i\ i+1\ i+2)$, which has order 3 (as it is a 3-cycle), so $(s_i s_{i+1})^3 = 1$ (this is referred to as a braid relation). Therefore,

$$S_n = \langle s_i\ i = 1, 2, ..., n-1 : (s_i s_i)^1\ \forall\ i,\ (s_i s_{i+1})^3\ i < n-1,\ (s_i s_j)^2\ |i-j| > 1 \rangle.$$

Theorem 1.4.2. *Fix a simple system Δ_{fin} for Φ_{fin}. Then the reflection group W_{fin} is generated by $S := \{s_\alpha, \alpha \in \Delta_{\text{fin}}\}$ subject only to the relations:*

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1\ (\alpha, \beta \in \Delta_{\text{fin}})$$

where $m(\alpha, \beta)$ is the order of $s_\alpha s_\beta \in W_{\text{fin}}$.

Definition 1.4.3. A **Coxeter group** is a group W_{fin} with some generating set $S = \{s_1, s_2, ..., s_k\}$ and relations $(s_i s_j)^{m_{ij}} = 1$ where $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$ (note that m_{ij} is not necessarily finite). The pair $(W_{\text{fin}}, S_{\text{fin}})$ is called a Coxeter System.

It is clear that any Weyl group is a Coxeter group. In fact, any reflection group is a Coxeter group, and the finite reflection groups are exactly the finite Coxeter groups. However, there are Coxeter groups (type H_3, H_4, and $I_2(m)$ for $m \neq 2, 3, 4, 6$) that are not Weyl groups (they are reflection groups, but not reflection groups of crystallographic root systems).

One way to organize the information in a Weyl group is to create a **Dynkin diagram**. This is a graph where the vertices are made up of the simple roots of Φ_{fin} and the edges are labeled by the following rules:

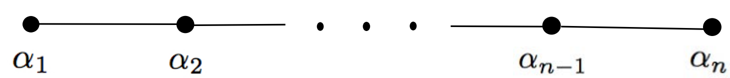
For $\alpha_i \neq \alpha_j$,

- If $m(\alpha_i \alpha_j) = 2$, then there is no edge connecting α_i and α_j.

- If $m(\alpha_i\alpha_j) = 3$, then there is a single edge connecting α_i and α_j.
- If $m(\alpha_i\alpha_j) = 4$, then there is a double edge connecting α_i and α_j.
- If $m(\alpha_i\alpha_j) = 6$, then there is a triple edge connecting α_i and α_j.
- If the root system is not simply laced (if there are two different root lengths), then the edge (be it single, double, or triple) is directed from the long root to the short root.

Without edge directions (which are unnecessary here as we are working with a simply laced root system), this is also called the **Coxeter graph**.

Example 1.4.4. Below is the Dynkin diagram for a root system of type A_n.



5. Bruhat Order

Definition 1.5.1. Using the length function, we can put a partial ordering on the elements of W_{fin}. We say that $w \to u$ if $\ell(w) \leq \ell(u)$ and there is some $\alpha \in \Phi_{\text{fin}}$ such that $u = ws_\alpha$.

We say that $w \leq u$ if there is some chain $w \to w_1 \to \cdots \to w_{k-1} \to u$. Such a partial order exists for any Coxeter system $(W_{\text{fin}}, S_{\text{fin}})$ and is called the **Bruhat Order**.

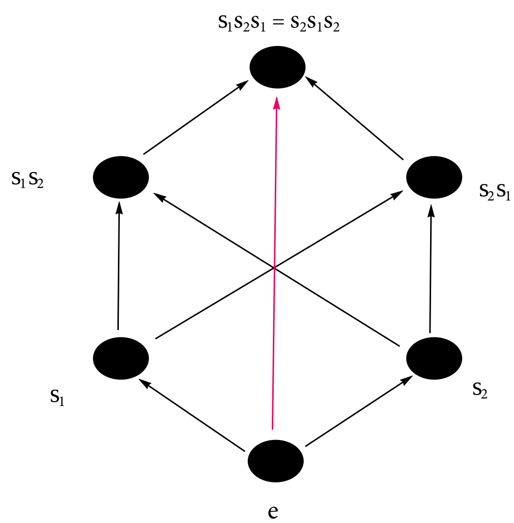
We can get the same order when multiplying on the left. We say that $w \to u$ if $\ell(w) \leq \ell(u)$ and there is some $\alpha \in \Phi_{\text{fin}}$ such that $u = s_\alpha w$. Then we say $w \leq u$ if there is some chain $w \to w_1 \to \cdots \to w_{k-1} \to u$.

Equivalently, we can say $w \leq u$ if $u = ws_\alpha$ where $\alpha > 0$ and $w(\alpha) > 0$. When multiplying on the left, this becomes $w \leq u$ if $u = s_\alpha u$ where $\alpha > 0$ and $w^{-1}(\alpha) > 0$.

Example 1.5.2. Let W_{fin} be a Coxeter group of type A_2. Then $s_1 \leq s_1 s_2$ because $\alpha_2 > 0$ and $s_1(\alpha_2) = e_1 - e_3 > 0$.

We can graph the partial order by making the elements of W_{fin} the vertices of the graph and by creating a directed edge from w to u when $w \to u$. Such a graph is called the **Bruhat graph** of W_{fin}.

Example 1.5.3. Let $W_{\text{fin}} = S_3$ (a Coxeter group of type A_2). Then the graph below represents the Bruhat graph of W_{fin}.



Note that the black edges represent a length increase of 1 and the red edge represents a length increase of 3.

Definition 1.5.4. We define the **Bruhat interval** $[w, u]$ to be the set of all z such that $w \leq z \leq u$ under the Bruhat order.

Definition 1.5.5. Let W_{fin} be a Coxeter group and w, u be elements of W_{fin} such that $w \leq u$. If $[w, u]$ contains only w and u (if there is no $z \in W_{\text{fin}}$ such that $w < z < u$), then u is said to be a **cover** of w (or u covers w), w is said to be a **cocover** of u, and we denote this by $w \lessdot u$.

Definition 1.5.6. Let $w \in W_{\text{fin}}$ and let $w = s_1 s_2 \cdots s_k$ be a reduced expression for w (here we let s_i be a general simple reflection and not necessarily s_{α_i}). Then a **subexpression** is of the form $s_{i_1} s_{i_2} \cdots s_{i_k}$ where s_{i_j} is either s_j or $e = id$.

Proposition 1.5.7. *Let W_{fin} be a Coxeter group and let w, u be elements of W_{fin}. Then $w \leq u$ if and only if w can be written as a subexpression of u. In fact, given any reduced expression for u, $w \leq u$ if and only if w can be written as a subexpression of that specific reduced expression.*

From Proposition 1.5.7, we see that for any $w, v \in W_{\text{fin}}$, the Bruhat interval $[w, v]$ must be finite, because $[w, v] \leq [e, v]$, and $[e, v]$ is finite because if $u \in [e, v]$, then u is a subexpression of v.

Definition 1.5.8. Let W be a Coxeter group and let w_i be elements of W_{fin} for $i = 1, 2, \ldots, k$. Then

$$w_1 \leq w_2 \leq \cdots \leq w_k$$

is called a **saturated chain** if each $\leq$ is really a covering relation. This requirement is equivalent to saying $\ell(w_i) = \ell(w_{i+1}) - 1$ for $i = 1, 2, \ldots, k-1$.

Proposition 1.5.9. *Let W be a Coxeter group and w, u elements of W_{fin}. If $w \leq u$, then there is a saturated chain from w to u.*

CHAPTER 2

Affine Weyl Groups

We began by looking at reflections, which reflect over a hyperplane that is passing through the origin. Now we will look at affine reflections, which reflect over any hyperplane. As in chapter one, our definitions will follow the work of Humphreys [H].

1. Affine Reflection Groups

Definition 2.1.1. For $\alpha \in \Phi_{\text{fin}}$, a crystallographic root system, and $k \in \mathbb{Z}$, we define

$$H_{\alpha,k} = \{v \in V : \langle v, \alpha \rangle = k\}.$$

Then $H_{\alpha,k}$ is a hyperplane (not necessarily through the origin) and $H_{\alpha,0}$ corresponds to the hyperplane H_α we have seen previously.

We define $s_{\alpha,k}$ to be the **affine reflection** that fixes $H_{\alpha,k}$. Then $s_{\alpha,0} = s_\alpha$.

Definition 2.1.2. The **affine Weyl group**, W_{aff}, is the subgroup of $\text{Aff}(V)$ that is generated by $\{s_{\alpha,k} : \alpha \in \Phi_{\text{fin}}, k \in \mathbb{Z}\}$.

Proposition 2.1.3. *Let $s_0 = s_{\theta,-1}$ and $s_i = s_{\alpha_i}$ for $i = 1, 2, \dots, n$. Then W_{aff} is generated by $S_{\text{aff}} = \{s_0,\ s_1,\ \dots,\ s_n\}$, the set of simple affine reflections, and $(W_{\text{aff}}, S_{\text{aff}})$ is a Coxeter system.*

Example 2.1.4. When considering the affine symmetric group, the set of relations for the Coxeter system is defined by

$$R = \{s_i s_i = 1\ \forall i,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}\ i \in [0, n-1],\ s_i s_j = s_j s_i \text{ for } |i-j| > 1\}.$$

2. Affine Permutations

The elements of the affine symmetric group $\widetilde{S}_n$ are called affine permutations.

Definition 2.2.1. An element w is said to be an **affine permutation** if w is a bijection from $\mathbb{Z}$ to $\mathbb{Z}$ of the form $[w_1, w_2, \ldots, w_n]$ where

(1) w_i represents what i is being mapped to
(2) $\sum_{i=1}^{n} w_i = \binom{n+1}{2}$
(3) $w_{i+n} = w_i + n$

Unlike a finite permutation, which acts only on the finite set $[n]$, an affine permutation acts on all integers. The element is written as $[w_1, w_2, \ldots, w_n]$ to represent the main window of the permutation. To see how all of $\mathbb{Z}$ will be affected, one can expand the permutation:

$$\cdots \ |w_1 - n \ w_2 - n \ \cdots \ w_n - n \ |w_1 \ w_2 \ \cdots \ w_n \ |w_1 + n \ w_2 + n \ \cdots \ w_n + n \ | \ \cdots .$$

Remark 2.2.2. Because w is a bijection from $\mathbb{Z}$ to $\mathbb{Z}$, no integer can be repeated in the permutation (if $w_i = w_j$ then $i = j$). Therefore, $w_i \equiv w_j \mod n$ if and only if $i \equiv j \mod n$.

Example 2.2.3. The affine permutation $[0,\ 2,\ 4] \in \widetilde{S}_3$ would send 1 to 0, 2 to itself, and 3 to 4, but it would also send 4 to 3, 5 to 5, and 6 to 7. That pattern would continue through all the positive integers (and it would also continue backwards through 0 and all the negative integers). Expanding past the main window, we can write $[0,\ 2,\ 4]$ as

$$\cdots \ | \ -6 \ \ -4 \ \ -2 \ | \ -3 \ \ -1 \ 1 \ | \ \mathbf{0} \ \ \mathbf{2} \ \ \mathbf{4} \ \ | \ 3 \ 5 \ 7 \ | \ 6 \ 8 \ 10 | \ \cdots .$$

Example 2.2.4. The element $[-110,\ -6,\ 122]$ is an element of $\widetilde{S}_3$. It seems odd to have negatives and such large numbers in the main window of the permutation, but we can quickly check that the sum of the elements in the main window is 6, and each of them differ from the other when considered mod(3), so there will be no repetition of integers.

We can extend the isomorphism between the symmetric group and the finite Weyl group of type A_{n-1} to an isomorphism between the affine symmetric group $\widetilde{S_n}$ and affine Weyl groups of type A_{n-1} by identifying the affine permutation $[0,\ 2,\ 3,\ \ldots,\ n+1]$ with the affine reflection $s_{\theta,-1}$.

3. Affine Roots

Definition 2.3.1. We define the **affine root system** to be

$$\Phi_{\text{aff}} = \{\nu + r\delta : \nu \in \Phi_{\text{fin}}, r \in \mathbb{Z}\}$$

where $\langle \alpha_i, \delta \rangle = 0$ for $i = 1, \ldots, n$. Given a root $\nu + r\delta \in \Phi_{\text{aff}}$, we define the **affine reflection** associated to that root to be

$$s_{\nu + r\delta} := s_{\nu, -r}.$$

We define the action of W_{aff} on V by

$$s_{\nu+r\delta}(\lambda) = \lambda - \langle \lambda, \nu + r\delta \rangle (\nu + r\delta).$$

When considering W_{aff} acting on Φ_{aff}, this can be written as

$$s_{\nu+r\delta}(\gamma + p\delta) = s_\nu(\gamma) + (p - \langle s_\nu(\gamma), -r\nu \rangle)\delta.$$

As in the finite case, Φ_{aff} can be written as the disjoint union of the positive affine roots and the negative affine roots:

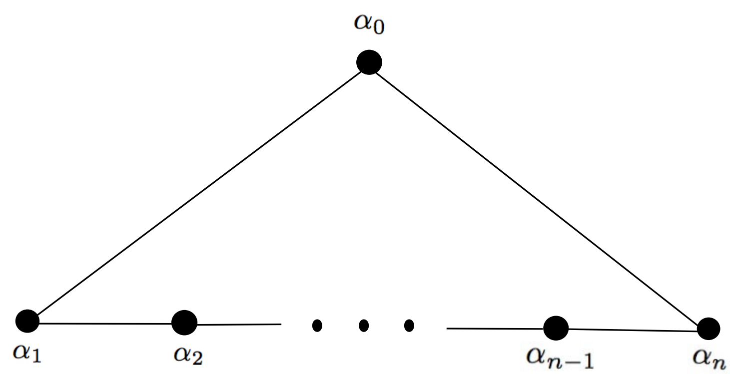
$$\Phi^+_{\text{aff}} = \{\nu + r\delta : \nu > 0,\ r \in \mathbb{Z}_{\geq 0}\} \cup \{\nu + r\delta :\ \nu < 0,\ r \in \mathbb{Z}_{>0}\}$$

$$\Phi^-_{\text{aff}} = \{\nu + r\delta :\ \nu < 0,\ r \in \mathbb{Z}_{\leq 0}\} \cup \{\nu + r\delta :\ \nu > 0,\ r \in \mathbb{Z}_{<0}\}.$$

Example 2.3.2. Let $W_{\text{aff}} = \widetilde{S}_n$. Then $\alpha_1 - 2\delta$ is a negative affine root and $-\theta + 4\delta$ is a positive affine root.

The simple affine roots are given by $\Delta_{\text{aff}} = \Delta_{\text{fin}} \cup \{\alpha_0\}$ where $\alpha_0 = -\theta + \delta$, so the Dynkin diagrams for the affine Weyl groups differ from the Dynkin diagrams for the finite Weyl groups by only an addition of α_0 and the edges connecting it to the finite Dynkin diagram.

Example 2.3.3. Below is the Dynkin diagram for a root system of type $\widetilde{A}_n$ (affine type A_n).



4. Affine Length Function

As in the finite case, we can define the length of $w \in W_{\text{aff}}$ to be the number of simple affine reflections needed in a reduced expression for w. It remains true that $\ell(w) = |\operatorname{Inv}(w)|$ for all $w \in W_{\text{aff}}$; however, it is not always easy to count inversions or express w as a reduced word. To find an easier way of calculating an element's length, we discuss an alternative way of viewing W_{aff}.

Besides defining W_{aff} as a group generated by affine reflections, we can see it as a semi-direct product involving W_{fin}. We will need some preliminary definitions to do so.

Definition 2.4.1. Let $Q_{\text{fin}} = \mathbb{Z}\Phi_{\text{fin}}$. Then Q_{fin} is called the **finite root lattice**.

The affine Weyl group, W_{aff}, is the semidirect product of the translation group corresponding to Q_{fin} with W_{fin}. That is,

$$W_{\text{aff}} \cong Q_{\text{fin}} \rtimes W_{\text{fin}}.$$

For $\mu \in V$, we define Y^{μ} to be the translation that sends $\lambda \in V$ to $\lambda + \mu$. Then $s_{\alpha,k} = s_{\alpha - k\delta} = Y^{k\alpha} s_{\alpha}$.

When considering $\alpha = \nu + 0\delta \in \Phi_{\text{aff}}$ the length of $s_{\alpha} \in W_{\text{aff}}$ remains the same as the length of $s_{\alpha} \in W_{\text{fin}}$ (it is still generated by the same number of simple reflections). To determine the length of a translation element, one can write the translation in terms of simple reflections, or we can rely on dominant weights (in the root lattice).

Definition 2.4.2. A weight $\lambda \in Q_{\text{fin}}$ is said to be **dominant** if $\langle \alpha_i, \lambda \rangle \geq 0$ for $i = 1, \ldots, n$.

Remark 2.4.3. Given any element $\lambda \in Q_{\text{fin}}$ there is some $w \in W_{\text{fin}}$ such that $\lambda_+ = w(\lambda)$ is dominant We call λ_+ the dominant weight associated to λ.

Definition 2.4.4. [M] Let Y^λ be a translation element in W_{aff}. Then

$$\ell(Y^\lambda) = \langle \lambda_+, 2\rho_{\text{fin}} \rangle.$$

Recall that ρ_{fin} is defined as the half sum of all the positive finite roots.

Before we can define our alternative definition of the length function, we need the following proposition.

Proposition 2.4.5. *Let $x, y \in W_{\text{aff}}$. Then*

$$\begin{aligned} \ell(xy) &= \ell(x) + \ell(y) - 2|\{\operatorname{Inv}(y) \cap -y^{-1} \operatorname{Inv}(x)\}| \\ &= \ell(x) + \ell(y) - 2|\{\alpha \in \operatorname{Inv}(y) : \alpha \notin \operatorname{Inv}(xy)\}|. \end{aligned}$$

As $\ell(w) = |\operatorname{Inv}(w)|$ for $w \in W_{\text{aff}}$, the proof for this proposition follows exactly as the proof in the finite case.

Proposition 2.4.6. *Let x, y be elements of W. Then*

$$\ell(xy) = \ell(x) + \ell(y) - 2|\operatorname{Inv}(x) \cap \operatorname{Inv}(y^{-1})|.$$

Proof. Using Proposition 2.4.5, this is equivalent to showing $|\{\gamma \in \operatorname{Inv}(x) \cap \operatorname{Inv}(y^{-1})\}| = |\{\gamma \in \operatorname{Inv}(y) : -y(\gamma) \in \operatorname{Inv}(x)\}|$.

We create a bijection by mapping $\gamma \in \{\gamma \in \operatorname{Inv}(y) : -y(\gamma) \in \operatorname{Inv}(x)\}$ to $-y(\gamma) \in \{\gamma \in \operatorname{Inv}(x) \cap \operatorname{Inv}(y^{-1})\}$, so the sets have the same size. $\square$

Now we can define an alternative definition for the length function and prove that it matches the original.

Definition 2.4.7. [M] Let $Y^\lambda w \in W_{\text{aff}}$. Then

$$\ell_{\text{aff}}(Y^\lambda w) = \langle \lambda_+, 2\rho_{\text{fin}}\rangle + |\{\beta \in \text{Inv}(w^{-1}) : \langle \beta, \lambda\rangle \leq 0\}| - |\{\beta \in \text{Inv}(w^{-1}) : \langle \beta, \lambda\rangle > 0\}|.$$

Proposition 2.4.8. *The two length functions ℓ and ℓ_{aff} are the same.*

Proof. Using Proposition 2.4.5 and Definition 2.4.4 we can re-write our original length function:

$$\begin{aligned}\ell(Y^\lambda w) &= \ell(Y^\lambda) + \ell(w) - 2|\{\alpha \in \text{Inv}(w) : \alpha \notin \text{Inv}(Y^\lambda w)\}| \\ &= \langle \lambda_+, 2\rho_{\text{fin}}\rangle + \ell(w) - 2|\{\alpha \in \text{Inv}(w) : \alpha \notin \text{Inv}(Y^\lambda w)\}|.\end{aligned}$$

Using the fact that $\ell(w) = |\operatorname{Inv}(w)|$ we can re-write our alternative length:

$$\begin{aligned}\ell_{\text{aff}}(Y^\lambda w) = \langle \lambda_+, 2\rho_{\text{fin}}\rangle &+ |\{\beta \in \text{Inv}(w^{-1}) : \langle \beta, \lambda\rangle \leq 0\}| \\ &- |\{\beta \in \text{Inv}(w^{-1}) : \langle \beta, \lambda\rangle > 0\}| \\ =&\langle \lambda_+, 2\rho_{\text{fin}}\rangle + \ell(w) - 2|\{\beta \in \text{Inv}(w^{-1}) : \langle \beta, \lambda\rangle > 0\}|.\end{aligned}$$

Now we only need to show that

$$|\{\alpha \in \text{Inv}(w) : \alpha \notin \text{Inv}(Y^\lambda w)\}| = |\{\beta \in \text{Inv}(w^{-1}) : \langle \beta, \lambda\rangle > 0\}|.$$

But $|\{\alpha \in \text{Inv}(w) : \alpha \notin \text{Inv}(Y^\lambda w)\}| = |\{\alpha \in \text{Inv}(w) : \langle w(\alpha), \lambda\rangle < 0\}|$ since $\alpha \in \text{Inv}(w)$ and $\alpha \notin \text{Inv}(Y^\lambda w)$ if and only if $\alpha > 0, w(\alpha) < 0,$ and $Y^\lambda w(\alpha) > 0$ if and only if $\alpha \in \text{Inv}(w)$ and $\langle w(\alpha), \lambda\rangle < 0$.

So we only need to show that

$$|\{\alpha \in \text{Inv}(w) : \langle w(\alpha), \lambda\rangle < 0\}| = |\{\beta \in \text{Inv}(w^{-1}) : \langle \beta, \lambda\rangle > 0\}|.$$

We do so by showing a bijection between $\text{Inv}(w)$ and $\text{Inv}(w^{-1})$. Let $\alpha \in \text{Inv}(w)$ and define $\beta = -w(\alpha)$. Then $\beta \in \text{Inv}(w^{-1})$. Similarly, we can map $\beta \in \text{Inv}(w^{-1})$ to $\alpha = -w^{-1}(\beta) \in \text{Inv}(w)$.

Additionally, if $\langle \beta, \lambda\rangle > 0$, then $\langle -w(\alpha), \lambda\rangle > 0$, so $\langle w(\alpha), \lambda\rangle < 0$. Therefore $\{\alpha \in \text{Inv}(w) : \langle w(\alpha), \lambda\rangle < 0\}$ and $\{\beta \in \text{Inv}(w^{-1}) : \langle \beta, \lambda\rangle > 0\}$ are bijective and so they must have the same order. □

Proposition 2.4.9. *Let $\lambda \in Q$ be regular and dominant, and let $\widetilde{w} = Y^{v\lambda}w \in W_{\text{aff}}$ with $v, w \in W_{\text{fin}}$. Then*

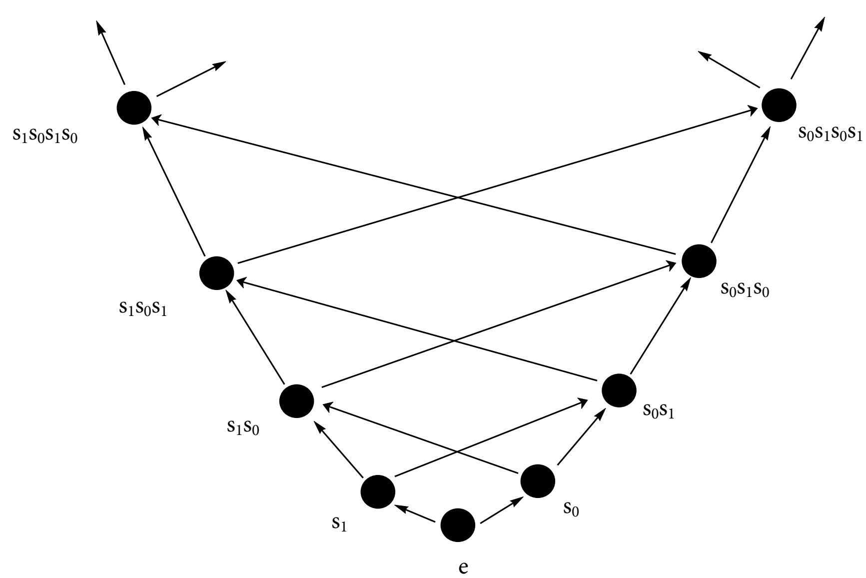
$$\ell(Y^{v\lambda}w) = \ell(Y^{\lambda}) + \ell(v) - \ell(w^{-1}v).$$

5. Bruhat Order

The Bruhat order remains the same in the affine case, and even though the affine Weyl group is infinite, the Bruhat intervals remain finite.

Definition 2.5.1. Recall that the Bruhat graph of a Weyl group uses the elements of the group as vertices and has directed edges from x to xs_α for $\alpha > 0$ when $\ell(x) < \ell(xs_\alpha)$. If we restrict to only including directed edges from x to xs_α when $\ell(xs_\alpha) = \ell(x) + 1$, we get the **Hasse diagram**.

Example 2.5.2. Below we show part of the Hasse diagram for W_{aff} of type $\widetilde{A}_1$.



6. The Height Function

Definition 2.6.1. Fix $\widetilde{\alpha} \in \Phi_{\text{aff}}$. Then there are $a_i \in \mathbb{Z}$ such that $\widetilde{\alpha} = \sum_{i=0}^{n} a_i\alpha_i$. As in the finite case, we define the **height** of $\widetilde{\alpha}$ to be $\text{ht}(\widetilde{\alpha}) = \sum_{i=0}^{n} a_i$.

The following proposition by Mare and Mihalcea will be useful when considering the double affine case.

Proposition 2.6.2. [MM, Prop. 6.5] *Let $\widetilde{\alpha}$ be a positive affine root. Then* $\ell(s_{\widetilde{\alpha}}) \leq 2\mathrm{ht}(\widetilde{\alpha})-1$. *If the equality* $\ell(s_{\widetilde{\alpha}}) = 2\mathrm{ht}(\widetilde{\alpha})-1$ *holds, then* $\delta-\widetilde{\alpha} = \sum_{i=0}^{n} a_i\alpha_i$ *where* $a_i \geq 0$ *and not all* $a_i = 0$.

We would like to relate $\mathrm{ht}(\widetilde{\alpha})$ to ρ_{aff} as was done in the finite case, but to do so we must first define ρ_{aff}. In the finite case, ρ_{fin} was defined by summing the positive finite roots; however, when summing all the positive affine roots, we run into a problem since there are infinitely many of them. Instead, we will look at another way of defining ρ_{fin} and extend it to the affine case to define ρ_{aff}.

Definition 2.6.3. Let $P_{\mathrm{fin}} = \{\lambda \in V : \langle\lambda,\alpha\rangle \in \mathbb{Z}\ \forall\ \alpha \in \Phi_{\mathrm{fin}}\}$. We call this the **finite weight lattice**. The **finite fundamental weights**, ω_i for $i = 1, 2, \ldots n$, are the elements of P such that

$$\langle\omega_i, \alpha_j\rangle = \delta_{i,j}$$

for $j = 1, 2, \ldots, n$ (where $\delta_{i,j}$ is the Kronecker delta).

Definition 2.6.4. [K] We define $P_{\mathrm{aff}} = P_{\mathrm{fin}} \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\Lambda_0$ to be the **affine weight lattice**. Let a_i for $i = 0, 1, \ldots, n$ represent the vertex labels of the affine Dynkin diagram. Choose the weight Λ_0 so that $\langle\alpha_i, \Lambda_0\rangle = 0$ for $i = 1, 2, \ldots, n$ and $\langle\delta, \Lambda_0\rangle = 1$.

The **affine fundamental weights**, Λ_i for $i = 0, 1\ldots, n$, are the elements of P_{aff} such that $\Lambda_i = \omega_i + a_i\Lambda_0$ where $\omega_0 = 0$.

With this definition,

$$\langle\Lambda_i, \alpha_j\rangle = \delta_{i,j}$$

for $j = 0, 1, \ldots, n$ and $i = 0, 1, \ldots, n$.

In the finite case, $\rho_{\mathrm{fin}} = \frac{1}{2}\sum_{\alpha\in\Phi^+_{\mathrm{fin}}} \alpha = \sum_{i=1}^{n} \omega_i$. In the affine case, we can still use the fundamental weight definition and get $\rho_{\mathrm{aff}} = \sum_{i=0}^{n} \Lambda_i = \rho_{\mathrm{fin}} + \sum_{i=0}^{n} a_i\Lambda_0$.

Proposition 2.6.5. *Let* $\widetilde{\alpha} \in \Phi_{\mathrm{aff}}$. *Then* $\mathrm{ht}(\widetilde{\alpha}) = \langle\widetilde{\alpha}, \rho_{\mathrm{aff}}\rangle$.

CHAPTER 3

Double Affine Weyl Semigroup

Now we will introduce the double affine case, which will be the setting for our results in the final chapters. For the the remainder of the paper we will use Q to represent Q_{fin}, the finite root lattice. Define $Q_{\text{aff}} = Q \oplus \mathbb{Z}\delta$, the affine root lattice, and $X = Q \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\Lambda_0$. We take the pairing $\langle \cdot, \cdot \rangle : Q \times Q \to \mathbb{Z}$ to be the usual inner product on V.

Extend the symmetric bilinear form $\langle Q, Q \rangle \to \mathbb{Z}$ to $\langle X, Q_{\text{aff}} \rangle \to \mathbb{Z}$ by

$$\langle \delta, \delta \rangle = \langle Q, \delta \rangle = \langle \delta, Q \rangle = \langle \Lambda_0,\ Q \rangle = 0,\ \langle \Lambda_0, \delta \rangle = 1.$$

We call l the **level** of the element $\zeta = \mu + m\delta + l\Lambda_0 \in X$ and use the notation $\text{lev}(\zeta) = l$.

PROPOSITION 3.0.1. *Let Q_{dom} be the set of dominant elements of Q (i.e., those $\mu \in Q$ such that $\langle \mu, \alpha_i \rangle \geq 0$ for all $0 < i \leq n$) and let X_{dom} be the set of dominant elements of X (i.e., those $\zeta \in X$ such that $\langle \mu, \alpha_i \rangle \geq 0$ for all $0 \leq i \leq n$). Then $\zeta = \mu + m\delta + l\Lambda_0 \in X_{\text{dom}}$ if and only if $\mu \in Q_{\text{dom}}$ and $l \geq \langle \mu, \theta \rangle$.*

PROOF. Let $\zeta = \mu + m\delta + l\Lambda_0 \in X$. Then $\zeta \in X_{\text{dom}}$ if and only if $\langle \zeta, \alpha_i \rangle \geq 0$ for $0 \leq i \leq n$. We have

$$\langle \zeta, \alpha_i \rangle = \begin{cases} \langle \mu, \alpha_i \rangle & 0 < i \leq n \\ \langle \mu, -\theta \rangle + l & i = 0. \end{cases}$$

So $\zeta \in X_{\text{dom}}$ if and only if $\langle \mu, \alpha_i \rangle \geq 0$ for $0 < i \leq n$ and $l \geq \langle \mu, \theta \rangle$. So $\zeta \in X_{\text{dom}}$ if and only if $\mu \in Q_{\text{dom}}$ and $l \geq \langle \mu, \theta \rangle$. $\square$

DEFINITION 3.0.2. Let X_{dom} be the set of all dominant elements of X. Then the **Tits cone** $\mathcal{T}$ is given by

$$\mathcal{T} = \cup_{w \in W_{\text{aff}}} w(X_{\text{dom}}).$$

Another way to think of the Tits cone is that it is the subset of X containing all elements that can be made dominant by some element of W_{aff}.

REMARK 3.0.3. Let $\zeta \in \mathcal{T}$. Then there is some $\widetilde{w} \in W_{\text{aff}}$ such that $\widetilde{w}(\zeta) \in X_{\text{dom}}$, and for any other $\widetilde{v} \in W_{\text{aff}}$ such that $\widetilde{v}(\zeta) \in X_{\text{dom}}, \widetilde{v}(\zeta) = \widetilde{w}(\zeta)$. We call $\widetilde{w}(\zeta)$ the dominant weight associated to ζ and denote it by ζ_+. To distinguish this from a weight being finite dominant, we will use $\lambda_{+_{\text{fin}}}$ to denote a weight that is finite dominant (i.e., $\lambda \in Q$ and $\langle \lambda_{+_{\text{fin}}}, \alpha_i \rangle \geq 0$ for all $0 < i \leq n$).

PROPOSITION 3.0.4. [**K**, (6.5, 2)] *Let $\lambda \in Q$ and $w \in W_{\text{fin}}$. The action of W_{aff} on X is defined by*

$$Y^\lambda w(\mu + m\delta + l\Lambda_0) = w(\mu) + l\lambda + (m - \langle w(\mu), \lambda \rangle - l\frac{\langle \lambda, \lambda \rangle}{2})\delta + l\Lambda_0.$$

REMARK 3.0.5. [**K**, Prop. 3.12] We can also define the Tits cone by $\mathcal{T} = \{\zeta \in X : \langle \zeta, \widetilde{\alpha} \rangle < 0$ for finitely many $\widetilde{\alpha} \in \Phi^+_{\text{aff}}\}$.

PROPOSITION 3.0.6. *We define $\mathcal{T}_l$ to be the subset of $\mathcal{T}$ containing elements of level l. Then*

$$\mathcal{T} = \cup_{l \in \mathbb{Z}} \mathcal{T}_l.$$

PROPOSITION 3.0.7.

$$\mathcal{T} = \{m\delta : m \in \mathbb{Z}\} \cup \{\mu + m\delta + l\Lambda_0 : \mu \in Q,\ m \in \mathbb{Z},\ l \in \mathbb{Z}_{>0}\}.$$

PROOF. Let $\mu \in Q$ be nonzero and let $m \in \mathbb{Z}$. Then $\mu + m\delta \in X$. Suppose by way of contradiction that $\mu + m\delta \in \mathcal{T}$. Then there exist $w \in W, \lambda \in Q$ and $\zeta \in X_{\text{dom}}$ such that $\mu + m\delta = Y^\lambda w(\zeta)$.

So $\zeta = w^{-1}Y^{-\lambda}(\mu + m\delta) = w^{-1}(\mu) + (m + \langle \mu, \lambda \rangle)\delta$. From Proposition 3.0.1 we know $w^{-1}(\mu) \in Q_{\text{dom}}$ and $\langle w^{-1}(\mu), \theta \rangle \leq l = 0$ since $\zeta \in X_{\text{dom}}$.

So $\langle w^{-1}(\mu), \alpha_i \rangle \geq 0$ for $0 < i \leq n$ and $0 \geq \langle w^{-1}(\mu), \theta \rangle = \sum_{i=1}^n a_i \langle w^{-1}(\mu), \alpha_i \rangle \geq 0$ where $a_i > 0$ (this uses 1.2.13). Hence $\langle w^{-1}(\mu), \alpha_i \rangle = 0$ for $0 < i \leq n$ and $\mu = 0$.

So the only elements of $\mathcal{T}$ with level 0 are the elements of the form $m\delta$ with $m \in \mathbb{Z}$. Therefore, $\mathcal{T}_o \subset \{m\delta : m \in \mathbb{Z}\}$.

Consider $\zeta = m\delta$. Then $\mathrm{lev}(\zeta) = 0$ and $\langle m\delta, \alpha_i \rangle = 0$ for $0 \leq i \leq n$. So $m\delta \in X_{\mathrm{dom}}$ and $\zeta = m\delta \in \mathcal{T}$. So $\{m\delta : m \in \mathbb{Z}\} \subset \mathcal{T}_0$. Therefore, $\{m\delta : m \in \mathbb{Z}\} = \mathcal{T}_0$.

Now let $\zeta \in \mathcal{T}$ such that $\mathrm{lev}(\mu) > 0$. Then $\zeta = \mu + m\delta + l\Lambda_0$ where $\mu \in Q$ and $m \in \mathbb{Z}$. So $\mathcal{T}_l \subset Q \oplus \mathbb{Z}\delta + ld$ for $l > 0$.

Consider $\zeta = \mu + m\delta + l\Lambda_0$ such that $\mu \in Q,\ l > 0$, and $m \in \mathbb{Z}$. We will show that ζ is in $\mathcal{T}$ by showing there are finitely many $\widetilde{\alpha} \in \Phi^+_{\mathrm{aff}}$ such that $\langle \zeta, \widetilde{\alpha} \rangle < 0$.

Let $\widetilde{\alpha} \in \Phi^+_{\mathrm{aff}}$, then $\widetilde{\alpha} = \nu + r\delta$ where $\nu \in \Phi_{\mathrm{fin}}$ and $r \in \mathbb{Z}_{\geq 0}$. Then we have

$$\langle \zeta, \widetilde{\alpha} \rangle = \langle \mu, \nu \rangle + rl.$$

If $\langle \zeta, \widetilde{\alpha} \rangle < 0$, then $rl < -\langle \mu, \nu \rangle$, so $r < -\frac{1}{l}\langle \mu, \nu \rangle$. So if $\widetilde{\alpha} \in \Phi^+_{\mathrm{aff}}$ and $\langle \zeta, \widetilde{\alpha} \rangle < 0$, then $0 \leq r < -\frac{1}{l}\langle \mu, \nu \rangle$. Therefore, for a fixed $\nu \in \Phi_{\mathrm{fin}}$ there are finitely many possibilities for $\nu + r\delta$. And since Φ_{fin} is finite, there are finitely many possibilities for $\widetilde{\alpha}$. So $\{\mu + m\delta + l\Lambda_0 : \mu \in Q,\ m \in \mathbb{Z},\ l \in \mathbb{Z}_{>0}\} \subset \mathcal{T}_l$. Therefore, $\mathcal{T}_l = \{\mu + m\delta + l\Lambda_0 : \mu \in Q,\ m \in \mathbb{Z},\ l \in \mathbb{Z}_{>0}\}$. □

Note that $\mathcal{T}$ contains all the imaginary roots (roots of the form $m\delta$) and all the roots with $l > 0$. Specifically, it contains no elements with a negative level.

Definition 3.0.8. We define the **double affine Weyl semigroup** W to be the semidirect product of the the translation semigroup associated to $\mathcal{T}$ with W_{aff}:

$$\begin{aligned} W &= \mathcal{T} \rtimes W_{\mathrm{aff}} \\ &= \{X^\zeta \widetilde{w} : \zeta \in \mathcal{T}, \widetilde{w} \in W_{\mathrm{aff}}\} \\ &= \{X^\zeta Y^\lambda w : \zeta \in \mathcal{T}, \lambda \in Q, w \in W_{\mathrm{fin}}\}. \end{aligned}$$

Remark 3.0.9. This is a semigroup, but not a group, as it is not closed under inverses.

For simplicity, we will use $\mathrm{lev}(x)$ to denote the level of the X-weight of $x \in W$ (i.e. if $x = X^\zeta Y^\lambda w \in W$ then $\mathrm{lev}(x) = \mathrm{lev}(\zeta)$).

1. Roots and Reflections

DEFINITION 3.1.1. Define $Q_{\text{daff}} = Q_{\text{aff}} \oplus \mathbb{Z}\pi$. The set of **double affine roots** is given by

$$\Phi = \{\widetilde{\alpha} + j\pi \in Q_{\text{daff}} : \widetilde{\alpha} \in \Phi_{\text{aff}}, j \in \mathbb{Z}\} = \{\nu + r\delta + j\pi : \nu \in \Phi_{\text{fin}}, r, j \in \mathbb{Z}\}.$$

Let $\widetilde{\alpha} = \nu + r\delta$ be an affine root. We say that a double affine root $\alpha = \widetilde{\alpha} + j\pi$ is **positive** if $\widetilde{\alpha} > 0$ and $j \geq 0$ or $\widetilde{\alpha} < 0$ and $j > 0$. For our purposes, we will consider π to be a placeholder like δ for the affine root.

NOTE 3.1.2. ***Where does π come from?*** *Let G_{aff} be the affine Kac-Moody group associated with W_{aff}, and let $F = k((\pi))$. Then Φ is the root system for $G_{\text{aff}}(F)$.*

Each double affine root $\alpha = \nu + r\delta + j\pi$, can be associated to a reflection s_α. Let $\widetilde{\alpha} = \nu + r\delta$. Then define:

$$\begin{aligned} s_\alpha &= s_{\widetilde{\alpha}+j\pi} \\ &= X^{-j\widetilde{\alpha}} s_{\nu+r\delta} \\ &= X^{-j\widetilde{\alpha}} Y^{-r\nu} s_\nu. \end{aligned}$$

Note that s_α is an element of $Q_{\text{aff}} \rtimes W_{\text{aff}}$.

REMARK 3.1.3. If $\alpha = \nu + r\delta + j\pi$ is a double affine root, and $j \neq 0$, then s_α is not an element of W.

Consider

$$s_{\nu+r\delta+j\pi} = X^{-j(\nu+r\delta)} Y^{-r\nu} s_\alpha$$

with $j \neq 0$. Then $s_{\nu+r\delta+j\pi}$ is not an element of W because $-j(\nu + rd)$ is not in $\mathcal{T}$; however, when we consider $x = X^\zeta \widetilde{w} \in W$with $\text{lev}(x) > 0$, $xs_{\nu+r\delta+j\pi}$ is an element of the double affine Weyl semigroup.

REMARK 3.1.4. The semigroup W is not generated by reflections.

Consider $x = X^{\mu+m\delta+l\Lambda_0} \in W$ with $\mathrm{lev}(x) > 0$. Then x cannot be written as a product of reflections because the reflections contain no $X^{l\Lambda_0}$ part.

So even though we call W the double affine Weyl semigroup, it does not contain a single double affine reflection and it is not generated by reflections.

PROPOSITION 3.1.5. *Let $\zeta \in X$ and $\widetilde{w} \in W_{\mathrm{aff}}$. W acts on Φ by*

$$X^{\zeta}\widetilde{w}(\widetilde{\alpha} + j\pi) = \widetilde{w}(\widetilde{\alpha}) + (j - \langle \zeta, \widetilde{w}(\widetilde{\alpha})\rangle)\pi.$$

This is similar to the action we defined for W_{aff} on Φ_{aff}. Letting $\zeta = \mu + m\delta + l\Lambda_0$ and $\widetilde{w} = Y^{\lambda}w$ we can expand this to

$$\begin{aligned}
X^{\zeta}Y^{\lambda}w(\alpha + r\delta + j\pi) &= Y^{\lambda}w(\alpha + r\delta) + (j - \langle \zeta, Y^{\lambda}w(\alpha + r\delta)\rangle)\pi \\
&= Y^{\lambda}w(\alpha + r\delta) + (j - \langle \mu + m\delta + l\Lambda_0, Y^{\lambda}w(\alpha + r\delta)\rangle)\pi \\
&= w(\alpha) + (r - \langle \lambda, w(\alpha)\rangle)\delta + (j - \langle \mu + m\delta + l\Lambda_0, w(\alpha) + (r - \langle \lambda, w(\alpha)\rangle)\delta\rangle)\pi \\
&= w(\alpha) + (r - \langle \lambda, w(\alpha)\rangle)\delta + (j - \langle \mu, w(\alpha)\rangle - l(r - \langle \lambda, w(\alpha)\rangle))\pi \\
&= Y^{\lambda}w(\alpha + r\delta) + (j - \langle \mu, w(\alpha)\rangle - l(r - \langle \lambda, w(\alpha)\rangle))\pi.
\end{aligned}$$

PROPOSITION 3.1.6. *Let α and β be double affine roots. Then $s_\alpha(\beta)$ as defined in Proposition 3.1.5 is the same as*

$$s_\alpha(\beta) = \beta - \langle \alpha, \beta\rangle\alpha.$$

PROOF. When we expand the action defined in Proposition 3.1.5, we can see that the two actions are the same:

$$\begin{aligned}
s_\alpha(\beta) &= X^{-j(\nu+r\delta)}Y^{-r\nu}s_\nu(\gamma + p\delta + q\pi) \\
&= s_\nu(\gamma) + (p + \langle r\nu, s_\nu(\gamma)\rangle)\delta + (q + \langle j(\nu + r\delta), s_\nu(\gamma) + (p + \langle r\nu, s_\nu(\gamma)\rangle)\delta\rangle)\pi \\
&= s_\nu(\gamma) + (p - r\langle \nu, \gamma\rangle)\delta + (q + j\langle \nu + r\delta, s_\nu(\gamma)\rangle)\pi \\
&= (\gamma - \langle \nu, \gamma\rangle\nu) + (p - r\langle \nu, \gamma\rangle)\delta + (q - j\langle \nu, \gamma\rangle)\pi \\
&= \gamma + p\delta + q\pi - \langle \nu, \gamma\rangle(\nu + r\delta + j\pi) \\
&= \beta - \langle \alpha, \beta\rangle\alpha
\end{aligned}$$

□

2. Length Function and Bruhat Order

In Definition 2.4.7, we defined an alternate length function ℓ_{aff} for elements of W_{aff}, and we showed that it was the same as the original length function that we defined by counting the number of simple reflections in reduced expressions of an element. When considering elements of W, it no longer makes sense to define a length function based on reduced words because not every element of w can be expressed as a product of simple reflections. Instead, we use the length function defined in [M] that has a similar form to ℓ_{aff}. For the rest of this dissertation, we will only refer to the length function on W_{aff} as ℓ_{aff}.

Definition 3.2.1. [M] Let $x = X^{\zeta}\widetilde{w}$ be an element of W. Then the **length** of x is defined to be

$$\ell(x) = \langle \zeta_+, 2\rho_{\text{aff}}\rangle + |\{\widetilde{\alpha} \in \text{Inv}(\widetilde{w}^{-1}) : \langle \zeta, \widetilde{\alpha}\rangle \leq 0\}| - |\{\widetilde{\alpha} \in \text{Inv}(\widetilde{w}^{-1}) : \langle \zeta, \widetilde{\alpha}\rangle > 0\}|,$$

where ζ_+ is the dominant element associated to ζ and $\widetilde{\alpha} = \nu + r\delta$ is an affine root. We break this into a big and small part by defining the **big length** as

$$\ell_{\text{big}}(x) = \langle \zeta_+, 2\rho_{\text{aff}}\rangle$$

and the **small length** as

$$\ell_{\text{small}}(x) = |\{\widetilde{\alpha} \in \text{Inv}(\widetilde{w}^{-1}) : \langle \zeta, \widetilde{\alpha}\rangle \leq 0\}| - |\{\widetilde{\alpha} \in \text{Inv}(\widetilde{w}^{-1}) : \langle \zeta, \widetilde{\alpha}\rangle > 0\}|.$$

From our definition of ℓ, we can see why we must use $\mathcal{T}$ and not all of X. Recall that $\mathcal{T}$ contains all elements that can be made dominant. We need the X-weight to be made dominant since we use $\ell_{\text{big}}(X^{\zeta}\widetilde{w}) = \langle \zeta_+, 2\rho_{\text{aff}}\rangle$ where ζ_+ is the dominant element of X associated to ζ. In appendix A we will show that using a subset of the level zero elements of X would be problematic when extending to the double affine case.

Remark 3.2.2. For $\widetilde{w} = Y^{\lambda}w \in W$, $\ell(\widetilde{w}) = \ell(X^0\widetilde{w}) = \ell_{\text{aff}}(\widetilde{w})$.

Proposition 3.2.3. *Let $\zeta \in \mathcal{T}$ be regular and dominant and let $x = X^{\widetilde{v}\zeta}\widetilde{w}$ where $\widetilde{w}, \widetilde{v} \in W_{\text{aff}}$. Then*

$$\ell(x) = \ell(X^{\zeta}) - \ell(\widetilde{v}^{-1}\widetilde{w}) + \ell(\widetilde{v})$$

$$= \langle \zeta, 2\rho_{\mathrm{aff}} \rangle - \ell(\widetilde{w}^{-1}\widetilde{v}) + \ell(\widetilde{v}).$$

Proof. We have

$$\ell(X^{\widetilde{v}\zeta}\widetilde{w}) = \ell_{\mathrm{big}}(X^{\widetilde{v}\zeta}\widetilde{w}) + \ell_{\mathrm{small}}(X^{\widetilde{v}\zeta}\widetilde{w})$$
$$= \langle \zeta, 2\rho_{\mathrm{aff}} \rangle + |\{\gamma \in \mathrm{Inv}(\widetilde{w}^{-1}) : \langle \widetilde{v}\zeta, \gamma \rangle \leq 0\}| - |\{\gamma \in \mathrm{Inv}(\widetilde{w}^{-1}) : \langle \widetilde{v}\zeta, \gamma \rangle > 0\}|.$$

We need to show that $\ell(\widetilde{v}) - \ell(\widetilde{w}^{-1}\widetilde{v}) = \ell_{\mathrm{small}}(X^{\widetilde{v}\zeta}\widetilde{w})$.

Note that $\langle \widetilde{v}\zeta, \gamma \rangle = \langle \zeta, \widetilde{v}^{-1}(\gamma) \rangle$ and since ζ is dominant and regular, $\langle \zeta, \widetilde{v}^{-1}(\gamma) \rangle > 0$ if and only if $\widetilde{v}^{-1}(\gamma) > 0$. Similarly, $\langle \zeta, \widetilde{v}^{-1}(\gamma) \rangle < 0$ if and only if $\widetilde{v}^{-1}(\gamma) < 0$ (since ζ is regular, we know $\langle \zeta, \widetilde{v}^{-1}(\gamma) \rangle \neq 0$).

So $\{\gamma \in \mathrm{Inv}(\widetilde{w}^{-1}) : \langle \widetilde{v}\zeta, \gamma \rangle \leq 0\} = \mathrm{Inv}(\widetilde{w}^{-1}) \cap \mathrm{Inv}(\widetilde{v}^{-1})$ and $\{\gamma \in \mathrm{Inv}(\widetilde{w}^{-1}) : \langle \widetilde{v}\zeta, \gamma \rangle > 0\} = \{\gamma \in \mathrm{Inv}(\widetilde{w}^{-1}) : \widetilde{v}^{-1}(\gamma) > 0\}$.

By Proposition 2.4.6, $\ell(\widetilde{w}^{-1}\widetilde{v}) = \ell(\widetilde{w}^{-1}) + \ell(\widetilde{v}) - 2|\mathrm{Inv}(\widetilde{w}^{-1}) \cap \mathrm{Inv}(\widetilde{v}^{-1})|$ so $2|\mathrm{Inv}(\widetilde{w}^{-1}) \cap \mathrm{Inv}(\widetilde{v}^{-1})| - \ell(\widetilde{w}^{-1}) = \ell(\widetilde{v}) - \ell(\widetilde{w}^{-1}\widetilde{v})$. Therefore,

$$\ell_{\mathrm{small}}(X^{\widetilde{v}\zeta}\widetilde{w}) = |\{\gamma \in \mathrm{Inv}(\widetilde{w}^{-1}) : \langle \widetilde{v}\zeta, \gamma \rangle \leq 0\}| - |\{\gamma \in \mathrm{Inv}(\widetilde{w}^{-1}) : \langle \widetilde{v}\zeta, \gamma \rangle > 0\}|$$
$$= |\mathrm{Inv}(\widetilde{w}^{-1}) \cap \mathrm{Inv}(\widetilde{v}^{-1})| - |\{\gamma \in \mathrm{Inv}(\widetilde{w}^{-1}) : \widetilde{v}^{-1}(\gamma) > 0\}|$$
$$= |\mathrm{Inv}(\widetilde{w}^{-1}) \cap \mathrm{Inv}(\widetilde{v}^{-1})| - (|\mathrm{Inv}(\widetilde{w}^{-1})| - |\mathrm{Inv}(\widetilde{w}^{-1}) \cap \mathrm{Inv}(\widetilde{v}^{-1})|)$$
$$= 2|\mathrm{Inv}(\widetilde{w}^{-1}) \cap \mathrm{Inv}(\widetilde{v}^{-1})| - \ell(\widetilde{w}^{-1})$$
$$= \ell(\widetilde{v}) - \ell(\widetilde{w}^{-1}\widetilde{v}).$$

□

2.1. Bruhat Order. Given $x \in W$ with $\mathrm{lev}(x) > 0$ and α a positive double affine root, [**BKP**, 5, Section B.2] defined $x \to xs_\alpha$ if $x(\alpha) > 0$ (we exclude $x \in W$ with $\mathrm{lev}(x) = 0$ because in that case, xs_α is not always in W). They defined the **double affine Bruhat preorder** to be the preorder generated by these relations, (that is, $x \leq y$ if there is some chain $x \to xs_{\alpha_1} \to \cdots \to y$), and they conjectured that it was an order. In [**M**] it was shown that the preorder is in fact an order, and in [**MO**] it was shown that this order coincides with the order generated by the relations: $x \to xs_\alpha$ if $\ell(x) \leq \ell(xs_\alpha)$. When multiplying on the left, we use the relation $x \to s_\alpha x$ if $x^{-1}(\alpha) > 0$.

Let $x, y \in W$. Recall that y is said to be a cover of x if $x < y$ and there is no $z \in W$ such that $x < z < y$. Similarly, y is said to be a cocover of x if $y < x$ and there is no $z \in W$ such that $y < z < x$.

We are interested in classifying covers and cocovers for a fixed $x \in W$. Muthiah and Orr [MO] proved the following theorem that will allow us to identify cocovers and covers by a difference in length.

THEOREM 3.2.4. [MO, Thm 1.6] *For α a positive double affine root and $x \in W$ with $lev(x) > 0$, xs_α is a cover of x if and only if $\ell(x) = \ell(xs_\alpha) - 1$.*

We can similarly say that xs_α is a cocover of x if and only if $\ell(x) = \ell(xs_\alpha) + 1$.

REMARK 3.2.5. For any $x \leq y$ with respect to the Bruhat order on W, we have $\text{lev(x)} = \text{lev(y)}$.

Let $x = X^{\mu_1} Y^{\lambda_1} w_1$. If $x \leq y$, then there exists a chain $x \to xs_{\alpha_i} \to \cdots \to xs_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k} = y$. We reduce to the case where $k = 1$. Then there is a positive double affine root $\nu + r\delta + j\pi$ such that $y = xs_{\nu+r\delta+j\pi}$. So $y = X^{\mu_1} Y^{\lambda_1} w_1 s_{\nu+r\delta+j\pi}$ and since there is no X^{Λ_0} part of $s_{\nu+r\delta+j\pi}$, $\text{lev}(y) = \text{lev}(\mu_1) = \text{lev}(x)$.

3. Examples

Consider W with finite root system of type A_1 (so $n = 2$). We will look at different examples of $x, y \in W$ such that $y = xs_\alpha$ and $x \leq y$ with respect to the Bruhat order. With this setup,

$$\mathcal{T} = \{k\theta + m\delta + l\Lambda_0 : k, m \in \mathbb{Z}, l \in \mathbb{Z}_+\} \cup \{m\delta : m \in \mathbb{Z}\}.$$

Consider $x = X^{\theta+\Lambda_0} Y^{-\theta} s_\theta$ and the double affine root $\alpha = \theta + r\delta + j\pi$. We want α to be a positive root, so either $r \geq 0$ and $j = 0$ or $j > 0$.

$$\begin{aligned} X^{\theta+\Lambda_0} Y^{-\theta} s_\theta(\theta + r\delta + j\pi) &= -\theta + (r - \langle -\theta, -\theta \rangle)\delta + (j - \langle \theta, -\theta \rangle - r + \langle -\theta, -\theta \rangle)\pi \\ &= -\theta + (r-2)\delta + (j + 4 - r)\pi. \end{aligned}$$

For $x(\alpha) > 0$, we need either $j+4 > r$ or $j+4 = r$ and $r > 2$. So by choosing finite part θ, we've reduced the possibilities for α to

$$\{\theta+r\delta+j\pi : r \geq 0 \text{ and } j = 0 \text{ or } j > 0\} \cap \{\theta+r\delta+j\pi : r > 2 \text{ and } j = r-4 \text{ or } j > r-4\}.$$

Set $y = xs_\alpha = X^{\theta+\Lambda_0}Y^{-\theta}s_\theta s_{\theta+r\delta+j\pi}$. Then

$$\begin{aligned} y &= X^{\theta+\Lambda_0}Y^{-\theta}s_\theta X^{-j(\theta+r\delta)}Y^{-r\theta}s_\theta \\ &= X^{\theta+\Lambda_0}Y^{-\theta}X^{-js_\theta(\theta+r\delta)}Y^{-rs_\theta(\theta)}s_\theta s_\theta \\ &= X^{\theta+\Lambda_0+j\theta-(jr-2j)\delta}Y^{-\theta+r\theta} \\ &= X^{(j+1)\theta-j(r-2)\delta+\Lambda_0}Y^{\theta(r-1)}. \end{aligned}$$

For $y \geq x$ we need $x(\alpha) > 0$, so either $r > 2$ and $j = r-4$ or $j > r-4$.

To determine when x is covered by y, we will calculate the lengths.

The big length of x is $\ell_{\text{big}}(x) = \ell_{\text{big}}(X^{\theta+\Lambda_0}) = \langle \delta+\Lambda_0, 2\rho_{\text{aff}}\rangle = 4(1) = 4$. We use $\delta+\Lambda_0$ because $\theta+\Lambda_0$ is not affine dominant but $Y^{-\theta}(\theta+\Lambda_0) = \delta+\Lambda_0$ is affine dominant (so if $\zeta = \theta+\Lambda_0$, then $\zeta_+ = \delta+\Lambda_0$).

To determine $\ell_{\text{small}}(x)$ we will first look at

$$s_\theta Y^\theta(\alpha+r\delta) = s_\theta(\alpha) + (r - \langle\alpha,\theta\rangle)\delta \text{ and } \operatorname{Inv}(s_\theta Y^\theta).$$

If $\alpha = \theta$, then $\alpha + r\delta \in \operatorname{Inv}(s_\theta Y^\theta)$ if and only if $r \geq 0$ and $r \leq 2$. If $\alpha = -\theta$, then $\alpha+r\delta \in \operatorname{Inv}(s_\theta Y^\theta)$ if and only if $r > 0$ and $r < -2$. So

$$\operatorname{Inv}(s_\theta Y^\theta) = \{\theta + r\delta : 0 \leq r \leq 2\}.$$

Since $\langle\theta+\Lambda_0, \theta+r\delta\rangle = 2+r > 0$ when $0 \leq r \leq 2$, $\ell_{\text{small}}(x) = -3$ and $\ell(x) = 4-3 = 1$.

To determine $\ell_{\text{small}}(y)$, we will first look at

$$Y^{-\theta(r-1)}(\alpha+t\delta) = \alpha + (t + (r-1)\langle\theta,\alpha\rangle)\delta \text{ and } \operatorname{Inv}(Y^{-\theta(r-1)}).$$

If $\alpha = \theta$, then $\alpha+t\delta \in \operatorname{Inv}(Y^{-\theta(r-1)})$ if and only if $t \geq 0$ and $t + 2(r-1) < 0$. If $\alpha = -\theta$, then $\alpha+t\delta \in \operatorname{Inv}(Y^{-\theta(r-1)})$ if and only if $t > 0$ and $t - 2(r-1) \leq 0$. So

$$\operatorname{Inv}(Y^{-\theta(r-1)}) = \{\theta+t\delta : 0 \leq t < 2(1-r)\} \cup \{-\theta+t\delta : 0 < t \leq 2(r-1)\}.$$

Up to this point we have been working with the general α given by $\alpha = \theta + r\delta + j\pi$. With this set-up, we will now examine specific examples where we pick values for r and j.

EXAMPLE 3.3.1. First we will look at an example where y is not a covering of x. Fix $j = 3$. Then we need $r \leq 7$ for $x \leq y$, and we'll choose $r = 3$. Then

$$y = X^{4\theta-3\delta+\Lambda_0}Y^{2\theta},$$

$$\mathrm{Inv}(Y^{-\theta(r-1)}) = \emptyset \cup \{-\theta + t\delta : 0 < t \leq 4\},$$

and $\langle -\theta + t\delta, 4\theta - 3\delta + \Lambda_0\rangle = -8 + t \leq 0$ for $0 < t \leq 4$, so $\ell_{\text{small}}(y) = 4$.

The big length of y is $\ell_{\text{big}}(y) = \ell_{\text{big}}(X^{4\theta-3\delta+\Lambda_0}) = \ell_{\text{big}}(X^{13\delta+\Lambda_0}) = 4(13) = 52$. We use $X^{13\delta+\Lambda_0}$ because $4\theta - 3\delta + \Lambda_0$ is not dominant but $Y^{-4\theta}(4\theta - 3\delta + \Lambda_0) = 4\theta - 4\theta + (-3 + 32 - 16)\delta + \Lambda_0 = 13\delta + \Lambda_0$ is dominant.

So $\ell(y) = 52+4 = 56$, and $y = X^{4\theta-3\delta+\Lambda_0}Y^{2\theta}$ does not cover $x = X^{\theta+\Lambda_0}Y^{-\theta}s_\theta$ since $\ell(y) \neq \ell(x) + 1$.

EXAMPLE 3.3.2. Next, let us look at an example where y is a covering of x. Fix $j = 0$. We need to choose $0 \leq r \leq 4$ for $x \leq y$, and we select $r = 4$. Then $y = X^{\theta+\Lambda_0}Y^{3\theta}$ and $\mathrm{Inv}((Y^{3\theta})^{-1}) = \mathrm{Inv}(Y^{-3\theta}) = \{-\theta + t\delta : 0 < t \leq 6\}$.

Since $\langle \theta + \Lambda_0, -\theta + t\delta\rangle = \langle \theta, -\theta\rangle + \langle \Lambda_0, t\delta\rangle = -2 + t > 0$ when $2 < t \leq 6$ and $\langle \theta + \Lambda_0, -\theta + t\delta\rangle \leq 0$ when $0 < t \leq 2$, $\ell_{\text{small}}(y) = 2 - 4 = -2$. Since the X weight for y is the same as the X weight for x, $\ell_{\text{big}}(y) = \ell_{\text{big}}(x) = 4$.

So $\ell(y) = 4 - 2 = 2 = \ell(x) + 1$ and y covers x.

EXAMPLE 3.3.3. Again we'll look at $j = 0$, but this time we'll pick $r = 0$. Then $y = xs_\theta = X^{\theta+\Lambda_0}Y^{-\theta}$ and, as before, $\ell_{\text{big}}(y) = \ell_{\text{big}}(x) = 4$. Then

$$\mathrm{Inv}((Y^{-\theta})^{-1}) = \mathrm{Inv}(Y^{\theta}) = \{\theta + t\delta : 0 \leq t < 2\},$$

and $\langle \theta+\Lambda_0, \theta+t\delta\rangle = \langle \theta, \theta\rangle + \langle \Lambda_0, t\delta\rangle = 2+t > 0$ for $0 \leq t < 2$, so $\ell_{\text{small}}(y) = -2$.

So $\ell(y) = 4 - 2 = 2 = \ell(x) + 1$ and y covers x.

Note that the two examples of covers occurred when we picked r to be either as small as possible or as large as possible within the given bounds. We will see later that these extreme cases are important when classifying covers.

4. Why this Works

First, it is important to recall that with this definition of the double affine Weyl semigroup we do not have that the semigroup is generated by reflections. In fact, we've seen that a reflection s_α with $\alpha \in \Phi$ is not generally an element of W. While this seems a bit unsettling, it allows us to avoid problems that arise if we attempt to define W as a group generated by s_α (which we examine in appendix A).

Specifically, this definition for W gives finite Bruhat intervals. To see this we will need the following remarks.

Remark 3.4.1. For $u, v \in W$, any saturated chain from u to v (meaning any chain where each step up is a covering relation) must have finite length.

This is clear because each step up in the chain means the length increases by one (because it is a cover), so the length of the chain is equal to $\ell(v) - \ell(u) + 1$, which is finite.

Remark 3.4.2. For $u, v \in W$ such that $u < v$, there exists a saturated chain from u to v.

Remark 3.4.3. For $u \in W$, there are finitely many covers of u and finitely many cocovers of u (this will be proven later).

Theorem 3.4.4. *Let $u, v \in W$ with $lev(u) > 0$. Then $[u, v]$ is finite.*

Proof. Consider a path from u to v. The path has at most $\ell(v) - \ell(u)$ steps. Assume the path is a saturated chain. Then there are finitely many options for each step (because there are finitely many covers for any element of W), so there are finitely many options for a saturated chain. And since every z such that $u < z < v$ is part of a saturated chain, we see there are finitely many options for $z \in [u, v]$ and $[u, v]$ must be finite. □

CHAPTER 4

Length Conditions

We wish to determine the cocovers of a given element x in W. To do this we extend [LS, Prop 4.1] of Lam and Shimozono and the further strengthening [Mi, Prop 4.2] by Milićević that classifies cocovers of an element x in W_{aff} by using the quantum Bruhat graph of W_{fin}. For the remaining two chapters, when considering elements $x \in W$, we will assume $\text{lev}(x) > 0$, and for simplicity we will allow ρ to denote ρ_{aff}.

1. Main Result

THEOREM 4.1.1. *Let $x = X^{\widetilde{v}\zeta}\widetilde{w} \in W$ where ζ is dominant and $\widetilde{v}, \widetilde{w} \in W_{\text{aff}}$. Let $y = s_\alpha x$ where $\alpha = -\widetilde{v}\widetilde{\alpha} + j\pi$ is a positive double affine root. Choose M so that $\ell(\widetilde{w})$, $\ell(s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}) \leq M$, and assume that $\langle \zeta, \alpha_i \rangle \geq 2(M+1)$ for $i = 0, 1, \ldots, n$. Then y is a cocover of x if and only if one of the following holds:*

(1) $\ell(\widetilde{v}) = \ell(\widetilde{v}s_{\widetilde{\alpha}}) + 1$ *and* $j = 0$ *so* $y = X^{\widetilde{v}s_{\widetilde{\alpha}}\zeta}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}$.
(2) $\ell(\widetilde{v}) = \ell(\widetilde{v}s_{\widetilde{\alpha}}) + 1 - \langle \widetilde{\alpha}, 2\rho \rangle$ *and* $j = 1$ *so* $y = X^{\widetilde{v}s_{\widetilde{\alpha}}(\zeta - \widetilde{\alpha})}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}$.
(3) $\ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}) = \ell(\widetilde{w}^{-1}\widetilde{v}) + 1$ *and* $j = \langle \zeta, \widetilde{\alpha} \rangle$ *so* $y = X^{\widetilde{v}\zeta}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}$.
(4) $\ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}) = \ell(\widetilde{w}^{-1}\widetilde{v}) + 1 - \langle \widetilde{\alpha}, 2\rho \rangle$ *and* $j = \langle \zeta, \widetilde{\alpha} \rangle - 1$ *so* $y = X^{\widetilde{v}(\zeta - \widetilde{\alpha})}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}$.

Before we can prove this theorem we will need the following lemmas, which are inspired by the proofs of [LS, Prop 4.1] and [Mi, Prop 4.2].

LEMMA 4.1.2. *Define $f(j) = \ell(X^{\widetilde{v}(\zeta - j\widetilde{\beta})})$ where $j \in \mathbb{Z}$, $\zeta \in \mathcal{T}$, $\widetilde{v} \in W_{\text{aff}}$ and $\widetilde{\beta}$ is an affine root. Then $f(j)$ is a convex function.*

PROOF. Define $S(t) = \{\gamma \in \Phi^+ : \langle \zeta, \widetilde{v}^{-1}(\gamma) \rangle - t\langle \widetilde{\beta}, \widetilde{v}^{-1}(\gamma) \rangle < 0\}$ where $t \in \mathbb{R}$. Let $\widetilde{w} \in W_{\text{aff}}$ such that $\widetilde{w}(\widetilde{v}(\zeta - j\widetilde{\beta}))$ is dominant. Then $S(t)$ is a finite set contained in $\mathrm{Inv}(\widetilde{w})$. Using [**MO**, Prop 3.10], we have

$$f(j) = \langle \zeta - j\widetilde{\beta}, \widetilde{v}^{-1}(2\rho)\rangle - \sum_{\gamma \in S_{(j)}} \langle \zeta - j\widetilde{\beta}, \widetilde{v}^{-1}(2\gamma)\rangle.$$

Define $g(t) = -\sum_{\gamma\in S(t)}\langle \zeta, \widetilde{v}^{-1}(2\gamma)\rangle - t\langle \widetilde{\beta}, \widetilde{v}^{-1}(2\gamma)\rangle$ for $t \in \mathbb{R}$. We show $g(t)$ is convex by showing $g(tj_1 + (1-t)j_2) \le tg(j_1) + (1-t)g(j_2)$ for $t \in [0,1], j_i \in \mathbb{Z}$. This is trivially true for $t = 0, 1$. We consider $t \in (0,1)$. Note that t and $(1-t)$ are positive for these cases. Let

$$\begin{aligned}
T &= S(tj_1 + (1-t)j_2) \\
&= \{\gamma \in \Phi^+ :\ t\langle \zeta - j_1\widetilde{\beta}, \widetilde{v}^{-1}(\gamma)\rangle + (1-t)\langle \zeta - j_2\widetilde{\beta}, \widetilde{v}^{-1}(\gamma)\rangle < 0\} \\
A_1 &= \{\gamma \in \Phi^+ :\ \langle \zeta - j_1\widetilde{\beta}, \widetilde{v}^{-1}(\gamma)\rangle < 0,\ \langle \zeta - j_2\widetilde{\beta}, \widetilde{v}^{-1}(\gamma)\rangle \ge 0\} \subseteq S(j_1) \\
A_2 &= \{\gamma \in \Phi^+ :\ \langle \zeta - j_1\widetilde{\beta}, \widetilde{v}^{-1}(\gamma)\rangle \ge 0,\ \langle \zeta - j_2\widetilde{\beta}, \widetilde{v}^{-1}(\gamma)\rangle < 0\} \subseteq S(j_2) \\
B &= \{\gamma \in \Phi^+ :\ \langle \zeta - j_1\widetilde{\beta}, \widetilde{v}^{-1}(\gamma)\rangle < 0,\ \langle \zeta - j_2\widetilde{\beta}, \widetilde{v}^{-1}(\gamma)\rangle < 0\} \subseteq S(j_1).
\end{aligned}$$

All of these sets are finite because they can be contained in $S(j)$ for some j. Note $B = B \cap T$ and $T = (A_1 \cap T) \sqcup (A_2 \cap T) \sqcup B$. Also note $-\sum_{\gamma\in A_i\cap T}\langle \zeta - j_i\widetilde{\beta}, \widetilde{v}^{-1}(2\gamma)\rangle \le -\sum_{\gamma\in A_i}\langle \zeta - j_i\widetilde{\beta}, \widetilde{v}^{-1}(2\gamma)\rangle$ for $i = 1, 2$.

For j an integer and S some set, define $g(j, S) = \sum_{\gamma\in S}\langle \zeta - j\widetilde{\beta}, \widetilde{v}^{-1}(2\gamma)\rangle$. Then

$$\begin{aligned}
g(tj_1 + (1-t)j_2) &= -\sum_{\gamma\in T}\Big(t\langle \zeta - j_1\widetilde{\beta}, \widetilde{v}^{-1}(2\gamma)\rangle + (1-t)\langle \zeta - j_2\widetilde{\beta}, \widetilde{v}^{-1}(2\gamma)\rangle\Big) \\
&= -tg(j_1, T) - (1-t)g(j_2, T) \\
&= -t(g(j_1, A_1 \cap T) + g(j_1, B) + g(j_1, A_2 \cap T)) \\
&\quad - (1-t)(g(j_2, A_1 \cap T) + g(j_2, B) + g(j_2, A_2 \cap T)) \\
&\le -t(g(j_1, A_1) + g(j_1, B)) - (1-t)(g(j_2, A_2) + g(j_2, B)) \\
&= -tg(j_1, A_1 \cup B) - (1-t)g(j_2, A_2 \cup B) \\
&= -tg(j_1, S(j_1)) - (1-t)g(j_2, S(j_2)) \\
&= tg(j_1) + (1-t)g(j_2).
\end{aligned}$$

So $f(j) = \langle \zeta - j\widetilde{\beta}, \widetilde{v}^{-1}(2\rho)\rangle + g(j)$ is convex as it is the sum of two convex functions ($\langle \zeta - j\widetilde{\beta}, \widetilde{v}^{-1}(2\rho)\rangle$ is a function of the form $j \mapsto a + bj$ and so is convex). $\square$

LEMMA 4.1.3. *Let* $x = X^{\widetilde{v}\zeta}\widetilde{w} \in W$ *where* ζ *is dominant. Let* $y = s_\alpha x$ *where* $\alpha = -\widetilde{v}\widetilde{\alpha}+j\pi$ *is a positive double affine root. Choose* M *so that* $\ell(\widetilde{w}),\ \ell(s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}) \leq M$*. If* $\langle \zeta, \alpha_i\rangle \geq 2(M+1)$ *for* $i = 0, 1, \ldots, n$ *and if* y *is a cocover of* x*, then* $0 \leq j \leq M$ *or* $\langle \zeta, \widetilde{\alpha}\rangle - M \leq j \leq \langle \zeta, \widetilde{\alpha}\rangle$.

PROOF. By assumption, y is a cocover of x and $-\widetilde{v}\widetilde{\alpha} + j\pi > 0$ so $x^{-1}(-\widetilde{v}\widetilde{\alpha} + j\pi) = -\widetilde{w}\widetilde{v}\widetilde{\alpha} + (j - \langle \zeta, \widetilde{\alpha}\rangle)\pi < 0$. This tells us $0 \leq j \leq \langle \zeta, \widetilde{\alpha}\rangle$ hence $\widetilde{\alpha}$ is positive since ζ is dominant and regular.

Consider α_i such that $\langle \widetilde{\alpha}, \alpha_i\rangle < 0$. Then $\langle \zeta - j\widetilde{\alpha}, \alpha_i\rangle \geq \langle \zeta, \alpha_i\rangle > 0$ since ζ is dominant and regular. Now consider the remaining α_i. By assumption, $\langle \zeta, \alpha_i\rangle \geq 2(M+1) \geq \langle \widetilde{\alpha}, \alpha_i\rangle(M+1)$ (we are using the fact that $\langle \widetilde{\alpha}, \widetilde{\beta}\rangle \leq 2$ for all $\widetilde{\alpha}, \widetilde{\beta} \in \Phi_{\text{aff}}$ as seen in [B]), so $\langle \zeta - (M+1)\widetilde{\alpha}, \alpha_i\rangle \geq 0$. If $j \leq M+1$ then $\zeta - j\widetilde{\alpha}$ is dominant since $\langle \zeta - j\widetilde{\alpha}, \alpha_i\rangle \geq \langle \zeta - (M+1)\widetilde{\alpha}, \alpha_i\rangle \geq 0$.

Let $j' = \langle \zeta, \widetilde{\alpha}\rangle - j$. If $j' \leq M+1$, then $\zeta - j'\widetilde{\alpha}$ is dominant. So if $j \geq \langle \zeta, \widetilde{\alpha}\rangle - (M+1)$, then $\zeta - (\langle \zeta, \widetilde{\alpha}\rangle - j)\widetilde{\alpha}$ is dominant.

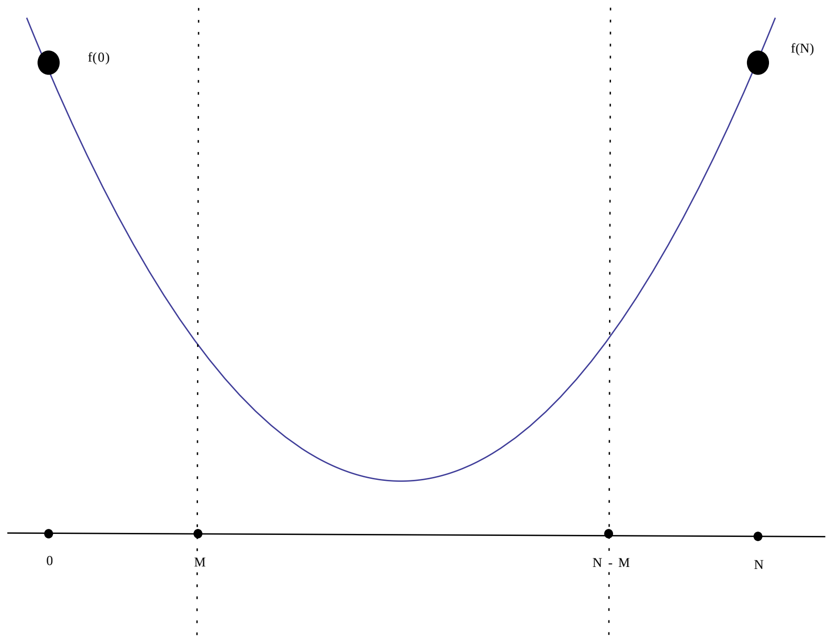
Following Milićević [Mi, Proof of Prop 4.2], we re-write y as

$$\begin{aligned} y = s_{-\widetilde{v}\widetilde{\alpha}+j\pi}x &= X^{j\widetilde{v}\widetilde{\alpha}} s_{\widetilde{v}\widetilde{\alpha}} X^{\widetilde{v}\zeta}\widetilde{w} \\ &= X^{j\widetilde{v}\widetilde{\alpha}+s_{\widetilde{v}\widetilde{\alpha}}\widetilde{v}\zeta} s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w} \\ &= X^{\widetilde{v}(s_{\widetilde{\alpha}}\zeta+j\widetilde{\alpha})} s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w} \\ &= X^{\widetilde{v}s_{\widetilde{\alpha}}(\zeta-j\widetilde{\alpha})} s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w} \\ &= X^{\widetilde{v}(\zeta-(\langle \zeta,\widetilde{\alpha}\rangle-j)\widetilde{\alpha})} s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}. \end{aligned}$$

Define $f(j) = \ell(X^{\widetilde{v}(\zeta-j\widetilde{\alpha})}) = \ell(X^{\widetilde{v}s_{\widetilde{\alpha}}(\zeta-j\widetilde{\alpha})})$ as in Lemma 4.1.2. Then $f(j)$ is a convex function. Note also that $f(0) = \ell(X^{\widetilde{v}\zeta}) = \langle \zeta, 2\rho\rangle = \ell(X^{\widetilde{v}s_{\widetilde{\alpha}}(\zeta)}) = f(\langle \zeta, \widetilde{\alpha}\rangle)$, and for $j \in [0, \langle \zeta, \widetilde{\alpha}\rangle]$, $f(j) = f(\langle \zeta, \widetilde{\alpha}\rangle - j)$ since $\widetilde{v}s_{\widetilde{\alpha}}(\zeta - j\widetilde{\alpha}) = \widetilde{v}(\zeta - \langle \zeta, \widetilde{\alpha}\rangle\widetilde{\alpha} + j\widetilde{\alpha}) = \widetilde{v}(\zeta - (\langle \zeta, \widetilde{\alpha}\rangle - j)\widetilde{\alpha})$.

The idea behind the proof, which comes from [LS, Proof of Prop 4.1], is that $\ell(x) \approx f(0) = f(\langle \zeta, \widetilde{\alpha}\rangle)$ and $\ell(y) \approx f(j)$, so for y to be a cocover of x, either $f(j)$

is close to $f(0)$ (and so j is close to 0) or $f(j)$ is close to $f(\langle\zeta,\widetilde{\alpha}\rangle)$ (and so j is close to $\langle\zeta,\widetilde{\alpha}\rangle$). We illustrate with a picture below, using $N=\langle\zeta,\widetilde{\alpha}\rangle$ for simplicity.



We use the fact that $f(j)$ is convex and symmetric to approximate the shape of the graph. We will show that when j moves beyond the dashed lines so that $M<j<N-M$, $f(j)$ becomes too far from $f(0)$ or $f(N)$ to allow y to be a cocover of x.

Fix j such that $M<j<N-M$. First we will make precise what we mean by $\ell(x)\approx f(0)=f(\langle\zeta,\widetilde{\alpha}\rangle)$ and $f(y)\approx f(j)$:

We have $|\ell(x)-f(0)|=|\ell(x)-f(\langle\zeta,\widetilde{\alpha}\rangle)|=|\ell(X^{\widetilde{v}\zeta}\widetilde{w})-\ell(X^{\widetilde{v}\zeta})|\leq\ell(\widetilde{w})\leq M$.

For the given j, $|\ell(y)-f(j)|=|\ell(X^{\widetilde{v}s_{\widetilde{\alpha}}(\zeta-j\widetilde{\alpha})}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w})-\ell(X^{\widetilde{v}s_{\widetilde{\alpha}}(\zeta-j\widetilde{\alpha})})|\leq\ell(s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w})\leq M$.

Additionally, if $m\leq M+1$, then $\zeta-m\widetilde{\alpha}$ is dominant, so $f(m)=\langle\zeta-m\widetilde{\alpha},2\rho\rangle=f(0)-m\langle\widetilde{\alpha},2\rho\rangle\leq f(0)-2m$ (we are using Proposition 2.6.5 that says $\langle\widetilde{\alpha},2\rho\rangle=2\mathrm{ht}(\widetilde{\alpha})\geq 2$).

Lastly, note that $f(j) \leq f(M+1)$ because of f's symmetry and convexity. By putting these three things together, we have

$$\begin{aligned}\ell(y) &\leq f(j) + M \\ &\leq f(M+1) + M \\ &\leq f(0) - 2(M+1) + M \\ &= f(0) - M - 2 \\ &\leq \ell(x) - 2.\end{aligned}$$

So if $M < j < N - M$, y cannot be covered by x. □

Now we can prove Theorem 4.1.1.

Proof of Theorem 4.1.1. We follow the proof of [Mi, Prop 4.2]. Assume y is covered by x. Recall from our proof of Lemma 4.1.3 that

$$\begin{aligned}y &= s_{-\widetilde{v}\widetilde{\alpha}+j\pi}x \\ &= X^{\widetilde{v}s_{\widetilde{\alpha}}(\zeta-j\widetilde{\alpha})}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w} \\ &= X^{\widetilde{v}(\zeta-(\langle\zeta,\widetilde{\alpha}\rangle-j)\widetilde{\alpha})}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}.\end{aligned}$$

By Lemma 4.1.3, we know that if $s_\alpha x$ is covered by x, then $0 \leq j \leq M$ or $\langle\zeta,\widetilde{\alpha}\rangle - M \leq j \leq \langle\zeta,\widetilde{\alpha}\rangle$. We will show that under these conditions either $\zeta - j\widetilde{\alpha}$ or $\zeta - (\langle\zeta,\widetilde{\alpha}\rangle - j)\widetilde{\alpha}$ must be dominant and regular.

By assumption, $\langle\zeta,\alpha_i\rangle \geq 2(M+1) \geq \langle\widetilde{\alpha},\alpha_i\rangle(M+1)$, so $\langle\zeta-(M+1)\widetilde{\alpha},\alpha_i\rangle \geq 0$. If $\langle\widetilde{\alpha},\alpha_i\rangle < 0$, then $\langle\zeta - j\widetilde{\alpha},\alpha_i\rangle > \langle\zeta,\alpha_i\rangle > 0$. Otherwise, if $j \leq M$, $\langle\zeta - j\widetilde{\alpha},\alpha_i\rangle > \langle\zeta-(M+1)\widetilde{\alpha},\alpha_i\rangle \geq 0$, so $\zeta - j\widetilde{\alpha}$ is dominant and regular. As in the proof of Lemma 4.1.3, we set $j' = \zeta - j\widetilde{\alpha}$ to show that $\zeta - (\langle\zeta,\widetilde{\alpha}\rangle - j)\widetilde{\alpha}$ is dominant and regular if $j \geq \langle\zeta,\widetilde{\alpha}\rangle - M$.

First, suppose $\zeta - j\widetilde{\alpha}$ is dominant and regular. Using Proposition 3.2.3 and $\ell(y) = \ell(X^{\widetilde{v}s_{\widetilde{\alpha}}(\zeta-j\widetilde{\alpha})}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w})$:

$$\ell(y) = \ell(X^{\zeta-j\widetilde{\alpha}}) - \ell(\widetilde{w}^{-1}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{v}s_{\widetilde{\alpha}}) + \ell(\widetilde{v}s_{\widetilde{\alpha}})$$

$$= \langle \zeta - j\widetilde{\alpha}, 2\rho\rangle - \ell(\widetilde{w}^{-1}\widetilde{v}) + \ell(\widetilde{v}s_{\widetilde{\alpha}})$$
$$= \langle \zeta - j\widetilde{\alpha}, 2\rho\rangle - \ell(\widetilde{v}^{-1}\widetilde{w}) + \ell(\widetilde{v}s_{\widetilde{\alpha}}).$$

And since $\ell(x) = \langle \zeta, 2\rho\rangle - \ell(\widetilde{v}^{-1}\widetilde{w}) + \ell(\widetilde{v})$, we have $\ell(x) - \ell(s_\alpha x) = j\langle\widetilde{\alpha}, 2\rho\rangle + \ell(\widetilde{v}) - \ell(\widetilde{v}s_{\widetilde{\alpha}})$. So $\ell(x) - \ell(s_\alpha x) = 1$ if and only if $1 = j\langle\widetilde{\alpha}, 2\rho\rangle + \ell(\widetilde{v}) - \ell(\widetilde{v}s_{\widetilde{\alpha}})$.

Using [MM, Prop 6.5], which says $\ell(s_{\widetilde{\alpha}}) \leq \langle\widetilde{\alpha}, 2\rho\rangle - 1$, and using the fact that $\ell(\widetilde{v}) - \ell(\widetilde{v}s_{\widetilde{\alpha}}) \geq -\ell(s_{\widetilde{\alpha}})$, we have $1 - j\langle\widetilde{\alpha}, 2\rho\rangle = \ell(\widetilde{v}) - \ell(\widetilde{v}s_{\widetilde{\alpha}}) \geq 1 - \langle\widetilde{\alpha}, 2\rho\rangle$. This gives $(1-j)\langle\widetilde{\alpha}, 2\rho\rangle \geq 0$, and since $\widetilde{\alpha} > 0$ and $j \geq 0$, we have two possibilities. Either $j = 0$ and $\ell(\widetilde{v}) = \ell(\widetilde{v}s_{\widetilde{\alpha}}) + 1$, or $j = 1$ and $\ell(\widetilde{v}) = \ell(\widetilde{v}s_{\widetilde{\alpha}}) + 1 - \langle\widetilde{\alpha}, 2\rho\rangle$. Then the form of y is determined by these values of j.

Next, suppose $\zeta - (\langle\zeta, \widetilde{\alpha}\rangle - j)\widetilde{\alpha}$ is dominant and regular. Using Proposition 3.2.3 and $\ell(s_\alpha x) = \ell(X^{\widetilde{v}(\zeta - (\langle\zeta,\widetilde{\alpha}\rangle - j)\widetilde{\alpha})} s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w})$:

$$\ell(s_\alpha x) = \ell(X^{\zeta - (\langle\zeta,\widetilde{\alpha}\rangle - j)\widetilde{\alpha}}) - \ell(\widetilde{v}^{-1}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}) + \ell(\widetilde{v}^{-1})$$
$$= \langle \zeta - (\langle\zeta, \widetilde{\alpha}\rangle - j)\widetilde{\alpha}, 2\rho\rangle - \ell(s_{\widetilde{\alpha}}\widetilde{v}^{-1}\widetilde{w}) + \ell(\widetilde{v}).$$

So $1 = \ell(x) - \ell(s_\alpha x) = \ell(s_{\widetilde{\alpha}}\widetilde{v}^{-1}\widetilde{w}) - \ell(\widetilde{v}^{-1}\widetilde{w}) - \langle (j - \langle\zeta, \widetilde{\alpha}\rangle)\widetilde{\alpha}, 2\rho\rangle$.

Using [MM, Prop 6.5] and the fact that $\ell(s_{\widetilde{\alpha}}\widetilde{v}^{-1}\widetilde{w}) - \ell(\widetilde{v}^{-1}\widetilde{w}) \geq -\ell(s_{\widetilde{\alpha}})$, we have $1 - \langle\widetilde{\alpha}, 2\rho\rangle \leq \ell(s_{\widetilde{\alpha}}\widetilde{v}^{-1}\widetilde{w}) - \ell(\widetilde{v}^{-1}\widetilde{w}) = 1 + (j - \langle\zeta, \widetilde{\alpha}\rangle)\langle\widetilde{\alpha}, 2\rho\rangle$, and $0 \leq (j - \langle\zeta, \widetilde{\alpha}\rangle + 1)\langle\widetilde{\alpha}, 2\rho\rangle$.

Using $\langle\widetilde{\alpha}, 2\rho\rangle > 0$ and $0 \leq j \leq \langle\zeta, \widetilde{\alpha}\rangle$, we have two possibilities. Either $j = \langle\zeta, \widetilde{\alpha}\rangle$ and $\ell(s_{\widetilde{\alpha}}\widetilde{v}^{-1}\widetilde{w}) - \ell(\widetilde{v}^{-1}\widetilde{w}) = 1$, or $j = \langle\zeta, \widetilde{\alpha}\rangle - 1$ and $\ell(s_{\widetilde{\alpha}}\widetilde{v}^{-1}\widetilde{w}) - \ell(\widetilde{v}^{-1}\widetilde{w}) = 1 - \langle\widetilde{\alpha}, 2\rho\rangle$. Then the form of y is determined by these values of j. □

2. Quantum Bruhat Graphs and Paths

Definition 4.2.1. We define the **quantum Bruhat graph** (QBG) of W_{aff} to be the graph whose set of vertices are the elements of W_{aff} and whose edge set is created by making a directed edge from $\widetilde{v}s_{\widetilde{\alpha}}$ to $\widetilde{v}$ for $\widetilde{\alpha}$ a positive affine root if one of the following holds:

(1) $\ell(\widetilde{v}) = \ell(\widetilde{v}s_{\widetilde{\alpha}}) + 1$

(2) $\ell(\widetilde{v}) = \ell(\widetilde{v}s_{\widetilde{\alpha}}) - \langle\widetilde{\alpha}, 2\rho\rangle + 1$.

The edges are labeled by $\widetilde{\alpha}$.

Proposition 4.2.2. *The quantum Bruhat graph of W_{aff} is path connected. That is, for any $\widetilde{w}, \widetilde{v} \in W_{\text{aff}}$, there is some path connecting $\widetilde{w}$ to $\widetilde{v}$.*

Proof. Let $\widetilde{\alpha} \in \Delta_{\text{aff}}$. Then $\ell(\widetilde{v}) = \ell(\widetilde{v}s_{\widetilde{\alpha}}) + 1$ or $\ell(\widetilde{v}) = \ell(\widetilde{v}s_{\widetilde{\alpha}}) - 1$. In the first case, $\widetilde{v}s_{\widetilde{\alpha}}$ to $\widetilde{v}$ has an edge in the QBG of type 1. In the second case, $\ell(\widetilde{v}) = \ell(\widetilde{v}s_{\widetilde{\alpha}}) - 1 = \ell(\widetilde{v}s_{\widetilde{\alpha}}) - 2\text{ht}(\widetilde{\alpha}) + 1 = \ell(\widetilde{v}s_{\widetilde{\alpha}}) - \langle\widetilde{\alpha}, 2\rho\rangle + 1$ (using the fact that $\text{ht}(\widetilde{\alpha}) = 1$ and $\text{ht}(\widetilde{\alpha}) = \langle\widetilde{\alpha}, \rho\rangle$), so $\widetilde{v}s_{\widetilde{\alpha}}$ to $\widetilde{v}$ has an edge in the QBG of type 2. So for any $\widetilde{\alpha} \in \Delta_{\text{aff}}$, there's an edge from $\widetilde{v}s_{\widetilde{\alpha}}$ to $\widetilde{v}$. And since $\widetilde{v}s_{\widetilde{\alpha}}s_{\widetilde{\alpha}} = \widetilde{v}$, there is also an edge from $\widetilde{v}$ to $\widetilde{v}s_{\widetilde{\alpha}}$. There is a two way path from any $\widetilde{w} \in W_{\text{aff}}$ to e, the identity, so for any $\widetilde{w}, \widetilde{v} \in W_{\text{aff}}$, there is a path from $\widetilde{w}$ to $\widetilde{v}$. □

Remark 4.2.3. The edges in the QBG of W_{aff} that meet the first length requirement represent covers in the affine Bruhat order of the form $\widetilde{v}s_{\widetilde{\alpha}} \lessdot \widetilde{v}$. They are the edges that appear in the Hasse diagram for W_{aff}.

Remark 4.2.4. There is a correspondence from the length conditions required in Theorem 4.1.1 to the edges in the quantum Bruhat graph of W_{aff}.

- Length condition (1) corresponds to an upward edge in the QBG of the form $\widetilde{v}s_{\widetilde{\alpha}} \to \widetilde{v}$ with length change $+1$.
- Length condition (2) corresponds to a downward edge in the QBG of the form $\widetilde{v}s_{\widetilde{\alpha}} \to \widetilde{v}$ with length change $-(\langle\widetilde{\alpha}, 2\rho\rangle - 1)$.
- Length condition (3) corresponds to an upward edge in the QBG of the form $\widetilde{w}^{-1}\widetilde{v} \to \widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}$ with length change $+1$.
- Length condition (4) corresponds to a downward edge in the QBG of the form $\widetilde{w}^{-1}\widetilde{v} \to \widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}$ with length change $-(\langle\widetilde{\alpha}, 2\rho\rangle - 1)$.

Remark 4.2.5. If $\widetilde{v}s_{\widetilde{\alpha}} \to \widetilde{v}$ is a quantum edge (this corresponds to type (2) in Theorem 4.1.1), then $\ell(\widetilde{v}) - \ell(\widetilde{v}s_{\widetilde{\alpha}}) = 1 - \langle\widetilde{\alpha}, 2\rho\rangle$, so $-\ell(s_{\widetilde{\alpha}}) \leq 1 - \langle\widetilde{\alpha}, 2\rho\rangle$. And using [MM, Prop 6.5], which says $\ell(s_{\widetilde{\alpha}}) \leq \langle\widetilde{\alpha}, 2\rho\rangle - 1$, we have $\ell(s_{\widetilde{\alpha}}) = \langle\widetilde{\alpha}, 2\rho\rangle - 1 = 2\text{ht}(\widetilde{\alpha}) - 1$.

Similarly, if $\widetilde{w}^{-1}\widetilde{v} \to \widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}$ is a quantum edge (this corresponds to type (4) in Theorem 4.1.1), then $\ell(s_{\widetilde{\alpha}}) = \langle\widetilde{\alpha}, 2\rho\rangle - 1 = 2\text{ht}(\widetilde{\alpha}) - 1$.

Therefore, if $\widetilde{\alpha}$ is the edge labeling for a quantum edge in the QBG, then $\ell(s_{\widetilde{\alpha}}) = 2\mathrm{ht}(\widetilde{\alpha}) - 1$.

The QBG for W_{aff} of type $\widetilde{A}_1$.

LEMMA 4.2.6. *Let W_{aff} be of type $\widetilde{A}_1$ and let k be a nonnegative integer. Then $(s_1s_0)^k(\alpha_1) = (2k+1)\alpha_1 + (2k)\alpha_0$ and $(s_0s_1)^k(\alpha_0) = (2k+1)\alpha_0 + (2k)\alpha_1$.*

PROOF. First, note that

$$\begin{aligned} s_0(\alpha_1) &= \alpha_1 - \langle \alpha_1, \alpha_0 \rangle \alpha_0 \\ &= \alpha_1 + 2\alpha_0 \end{aligned}$$

and

$$\begin{aligned} s_1(\alpha_0) &= \alpha_0 - \langle \alpha_0, \alpha_1 \rangle \alpha_1 \\ &= \alpha_0 + 2\alpha_1. \end{aligned}$$

Now we will use induction on k to prove the claim. If $k = 0$ then $(s_1s_0)^k = e = (s_0s_1)^k$, so the result holds. Assume $k > 0$. Then

$$\begin{aligned} (s_1s_0)^k(\alpha_1) &= (s_1s_0)(s_1s_0)^{k-1}(\alpha_1) \\ &= (s_1s_0)((2(k-1)+1)\alpha_1 + (2(k-1))\alpha_0) \\ &= s_1((2(k-1)+1)\alpha_1 + 2(2(k-1)+1)\alpha_0 - 2(k-1)\alpha_0) \\ &= s_1((2(k-1)+1)\alpha_1 + (2(k-1)+2)\alpha_0) \\ &= -(2(k-1)+1)\alpha_1 + (2(k-1)+2)\alpha_0 + 2(2(k-1)+2)\alpha_1 \\ &= (2(k-1)+3)\alpha_1 + (2k)\alpha_0 \\ &= (2k+1)\alpha_1 + (2k)\alpha_0 \end{aligned}$$

and

$$\begin{aligned} (s_0s_1)^k(\alpha_0) &= (s_0s_1)(s_0s_1)^{k-1}(\alpha_0) \\ &= (s_0s_1)((2(k-1)+1)\alpha_0 + (2(k-1))\alpha_1) \\ &= s_0((2(k-1)+1)\alpha_0 + 2(2(k-1)+1)\alpha_1 - 2(k-1)\alpha_1) \end{aligned}$$

$$
\begin{aligned}
&= s_0((2(k-1)+1)\alpha_0 + (2(k-1)+2)\alpha_1) \\
&= -(2(k-1)+1)\alpha_0 + (2(k-1)+2)\alpha_1 + 2(2(k-1)+2)\alpha_0 \\
&= (2(k-1)+3)\alpha_0 + (2k)\alpha_1 \\
&= (2k+1)\alpha_0 + (2k)\alpha_1.
\end{aligned}
$$

$\square$

With this lemma, we can classify the quantum edges for the QBG for W_{aff} of type $\widetilde{A}_1$.

PROPOSITION 4.2.7. *Let W_{aff} be of type $\widetilde{A}_1$ and let $s_{\widetilde{\alpha}}$ be a reflection of W_{aff}. Then $\ell(s_{\widetilde{\alpha}}) = \text{ht}(\widetilde{\alpha})$. Therefore, $\ell(s_{\widetilde{\alpha}}) = 2\text{ht}(\widetilde{\alpha}) - 1$ if and only if $\ell(s_{\widetilde{\alpha}}) = 1$, and the only quantum edges in the QBG are edges with label $\widetilde{\alpha} \in \Delta_{\text{aff}}$.*

PROOF. Let $\widetilde{\alpha} \in \Phi_{\text{aff}}$. Then $s_{\widetilde{\alpha}}$ is generated by s_0 and s_1 and there are four possible forms it could take:

(1) $s_{\widetilde{\alpha}} = (s_0 s_1)^k s_0 (s_1 s_0)^k$
(2) $s_{\widetilde{\alpha}} = s_1 (s_0 s_1)^k s_0 (s_1 s_0)^k s_1$
(3) $s_{\widetilde{\alpha}} = (s_1 s_0)^k s_1 (s_0 s_1)^k$
(4) $s_{\widetilde{\alpha}} = s_0 (s_1 s_0)^k s_1 (s_0 s_1)^k s_0$

Elements of the form $s_0 s_1 \cdots s_0 s_1$ or $s_1 s_0 \cdots s_1 s_0$ are pure translations (hence not reflections) and so are not considered.

Case 1: Assume $s_{\widetilde{\alpha}} = (s_0 s_1)^k s_0 (s_1 s_0)^k$. Then $\ell(s_{\widetilde{\alpha}}) = 4k + 1$, and we wish to show that $\text{ht}(\widetilde{\alpha}) = 4k + 1$. Using Lemma 4.2.6, we have

$$\widetilde{\alpha} = (s_0 s_1)^k (\alpha_0) = 2k\alpha_1 + (2k+1)\alpha_0$$

so $\text{ht}(\widetilde{\alpha}) = 4k + 1$.

Case 2: Assume $s_{\widetilde{\alpha}} = s_1 (s_0 s_1)^k s_0 (s_1 s_0)^k s_1$. Then $\ell(s_{\widetilde{\alpha}}) = 4k+3$, and we wish to show that $\text{ht}(\widetilde{\alpha}) = 4k + 3$. Using Lemma 4.2.6, we have

$$
\begin{aligned}
\widetilde{\alpha} &= s_1 (s_0 s_1)^k (\alpha_0) \\
&= s_1 (2k\alpha_1 + (2k+1)\alpha_0)
\end{aligned}
$$

$$= -2k\alpha_1 + (2k+1)\alpha_0 + 2(2k+1)\alpha_1$$

$$= (2k+2)\alpha_1 + (2k+1)\alpha_0$$

so $\text{ht}(\widetilde{\alpha}) = 4k+3$.

Case 3: Assume $s_{\widetilde{\alpha}} = (s_1 s_0)^k s_1 (s_0 s_1)^k$. Then $\ell(s_{\widetilde{\alpha}}) = 4k+1$, and we wish to show that $\text{ht}(\widetilde{\alpha}) = 4k+1$.

Using Lemma 4.2.6, we have

$$\widetilde{\alpha} = (s_1 s_0)^k(\alpha_1) = 2k\alpha_0 + (2k+1)\alpha_1$$

so $\text{ht}(\widetilde{\alpha}) = 4k+1$.

Case 4: Assume $s_{\widetilde{\alpha}} = s_0(s_1 s_0)^k s_1 (s_0 s_1)^k s_0$. Then $\ell(s_{\widetilde{\alpha}}) = 4k+3$, and we wish to show that $\text{ht}(\widetilde{\alpha}) = 4k+3$. Using Lemma 4.2.6, we have

$$\widetilde{\alpha} = s_0(s_1 s_0)^k(\alpha_1)$$

$$= s_0(2k\alpha_0 + (2k+1)\alpha_1)$$

$$= -2k\alpha_0 + (2k+1)\alpha_1 + 2(2k+1)\alpha_0$$

$$= (2k+2)\alpha_0 + (2k+1)\alpha_1$$

so $\text{ht}(\widetilde{\alpha}) = 4k+3$. □

This tells us that the quantum edges (downward edges) of the QBG for W_{aff} of type $\widetilde{A}_1$ are labeled by $\widetilde{\alpha} = \alpha_i \in \Delta_{\text{aff}}$.

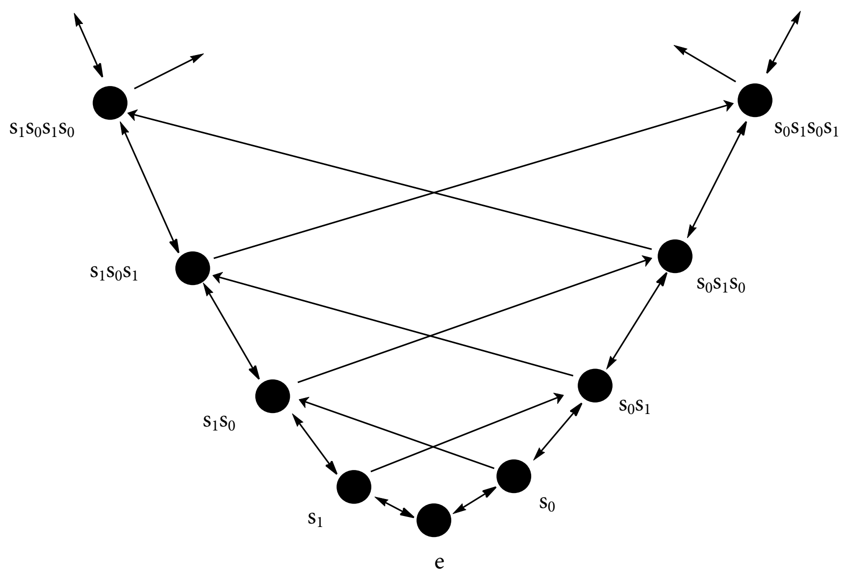
Remark 4.2.8. The upward edges of the QBG are of the form

$$\widetilde{v} s_{\widetilde{\alpha}} \to \widetilde{v} \text{ where } \ell(\widetilde{v}) = \ell(\widetilde{v} s_{\widetilde{\alpha}}) + 1.$$

These are the edges of the Hasse diagram for W_{aff} of type $\widetilde{A}_1$. Because there are four forms that $s_{\widetilde{\alpha}}$ can take, as seen in the proof of Proposition 4.2.7, there are four possible edge labels for the upward edges:

(1) If $s_{\widetilde{\alpha}} = (s_0 s_1)^k s_0 (s_1 s_0)^k$, then $\widetilde{\alpha} = 2k\alpha_1 + (2k+1)\alpha_0$.
(2) If $s_{\widetilde{\alpha}} = s_1 (s_0 s_1)^k s_0 (s_1 s_0)^k s_1$, then $\widetilde{\alpha} = (2k+2)\alpha_1 + (2k+1)\alpha_0$.
(3) If $s_{\widetilde{\alpha}} = (s_1 s_0)^k s_1 (s_0 s_1)^k$, then $\widetilde{\alpha} = 2k\alpha_0 + (2k+1)\alpha_1$.
(4) If $s_{\widetilde{\alpha}} = s_0 (s_1 s_0)^k s_1 (s_0 s_1)^k s_0$, then $\widetilde{\alpha} = (2k+2)\alpha_0 + (2k+1)\alpha_1$.

Example 4.2.9. Let W_{aff} be of type $\widetilde{A}_1$. Then the QBG of W_{aff} is given below.



Because of the correspondence in Theorem 4.1.1 between the length conditions and the QBG, we can find cocovers in W by considering edges in the QBG of W_{aff}.

Example 4.2.10. The QBG with W_{aff} of type $\widetilde{A}_1$ has the upward edge $s_1s_0 \to s_0s_1s_0$. If we pick $\widetilde{v} = s_0s_1s_0$ and $\widetilde{v}s_{\widetilde{\alpha}} = s_1s_0$, then the edge corresponds to the first cocover type in Theorem 4.1.1 and $j = 0$. The reflection we are extending by is $s_{\widetilde{\alpha}} = s_0s_1s_0s_1s_0$, so the edge is labeled with $\widetilde{\alpha} = s_0s_1(\alpha_0) = 3\alpha_0 + 2\alpha_1 = -\alpha_1 + 3\delta$. So $\alpha = -\widetilde{v}\widetilde{\alpha} + j\pi = -\alpha_1 + \delta = \alpha_0$. To make the length bound smaller, we pick $\widetilde{w} = id$. Then $\ell(\widetilde{w}) = 0$ and $\ell(s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}) = 1$ so we can take $M = 1$.

We pick $\zeta = 2\alpha_1 + \delta + 8\Lambda_0$ and check $\langle \zeta, \alpha_i \rangle \geq 2(M+1) = 4$ for $i = 0, 1$. With these choices,

$$x = X^{s_0s_1s_0(\zeta)} = X^{14\alpha_1 - 23\delta + 8\Lambda_0}, \quad y = X^{s_1s_0(\zeta)}Y^{\alpha_1}s_1 = X^{-6\alpha_1 - 3\delta + 8\Lambda_0}Y^{\alpha_1}s_1,$$

and y is a cocover of x. Further, we can confirm this by using Sage to check the lengths. Indeed, $\ell(x) - \ell(y) = 8 - 7 = 1$.

3. Orderings of Reflection Groups

Definition 4.3.1. [**D**, Def 2.1] A total order $\prec$ on the reflections of a Weyl group is called a **reflection order** if for any dihedral subgroup of the Weyl group, either $r \prec rsr \prec \cdots \prec srs \prec s$ or $s \prec srs \prec \cdots \prec rsr \prec r$, where r and s are the generators for the dihedral subgroup.

We can place a reflection order on the reflections of W_{aff} of type $\widetilde{A}_1$:

$$s_0 \prec s_0s_1s_0 \prec s_0s_1s_0s_1s_0 \prec \ldots \prec s_1s_0s_1s_0s_1 \prec s_1s_0s_1 \prec s_1.$$

Let w_0, w_1 be reflections of W_{aff} with more s_0 than s_1 in their reduced expression. Let v_0, v_1 be reflections of W_{aff} with more s_1 than s_0 in their reduced expression.

Then the following rules describe the ordering.

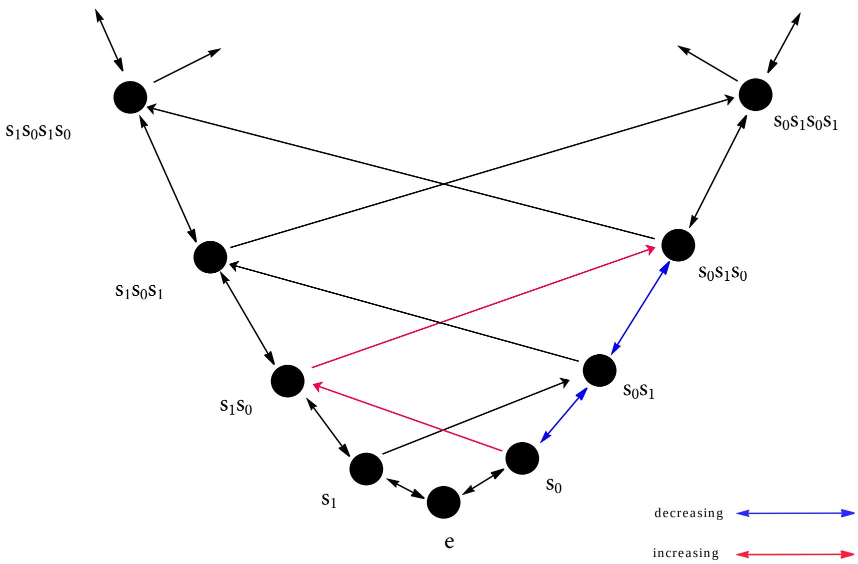
(1) $w_0 \prec w_1$ if and only if $\ell(w_0) < \ell(w_1)$
(2) $v_0 \prec v_1$ if and only if $\ell(v_0) > \ell(v_1)$
(3) $w_i \prec v_i$

In the finite case, one can use this ordering to find unique paths in the QBG by labeling the edges of the QBG with the reflection used to extend (i.e., label the edge $s_{\widetilde{\alpha}}$ if it would normally be labeled $\widetilde{\alpha}$).

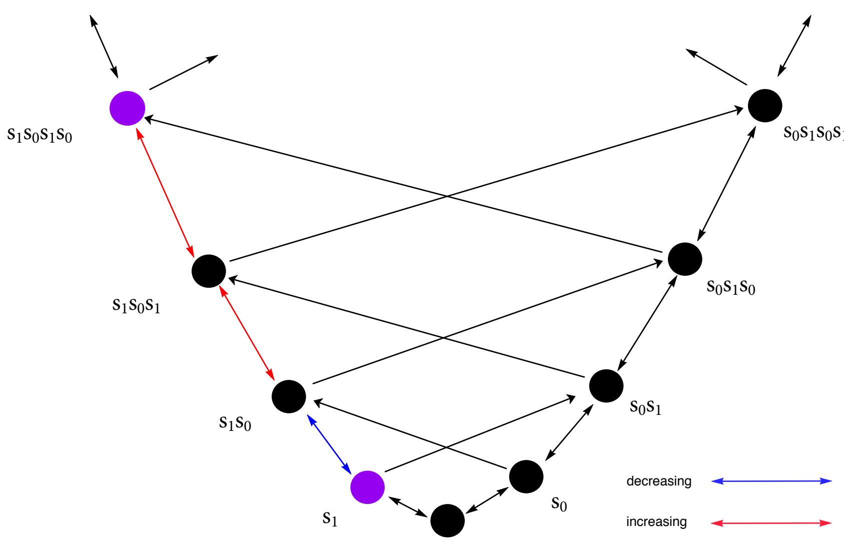
Theorem 4.3.2. [**BFP**, Thm. 6.4] *For any pair of elements $u, v \in W_{\text{fin}}$, there is a unique path from u to v such that the sequence of edge labels is strictly increasing (resp. strictly decreasing) with respect to the reflection ordering.*

This is not the case when considering the QBG of W_{aff}, as we now demonstrate. For some u and v in W_{aff} of type $\widetilde{A}_1$ we can find a strictly increasing (resp. strictly decreasing) path, but we cannot find such paths for all elements.

Example 4.3.3. Consider the QBG for W_{aff} of type $\widetilde{A}_1$.



Above, we mark a strictly increasing path from $u = s_0$ to $v = s_0s_1s_0$ in red and a strictly decreasing path from $u = s_0$ to $v = s_0s_1s_0$ in blue.



But when considering elements u, v such that $\ell(u) > \ell(v) + 2$, we cannot find such paths from u to v. For example, when considering $u = s_1s_0s_1s_0$ and $v = s_1$, any path from u to v will need at least three downward edges since $\ell(u) - \ell(v) = 3$.

All the downward edges are labeled with either s_0 (if normally the edge labeling would be α_0) or s_1 (if normally the edge labeling would be α_1), so any path from u to v will include two edges labeled s_0 or two edges labeled s_1. Either option would result in a path that is neither strictly increasing nor strictly decreasing in edge labels.

CHAPTER 5

Cocovers and Corners

To rid ourselves of the bounds needed on $\ell(\widetilde{w})$ and $\ell(s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w})$ in Theorem 4.1.1, we look at cocovers of $x \in W$ in a different way, taking a more geometrical approach by using the length difference set defined by Muthiah and Orr [**MO**].

Recall that we are assuming $\mathrm{lev}(x) > 0$ whenever considering $x \in W$.

THEOREM 5.0.1 (MO). *Let $x = X^{\zeta}\widetilde{w}$ with $\zeta \in \mathcal{T}$ and $\widetilde{w} \in W_{\mathrm{aff}}$. Let α be a positive double affine root such that $x^{-1}(\alpha) < 0$. Then $y = s_\alpha x \leq x$ with respect to the Bruhat order by definition, and*

$$\ell(y) = \ell(x) - |\{\beta \in \Phi^+ \ : \ x^{-1}(\beta) < 0, \quad s_\alpha(\beta) < 0, \quad x^{-1}s_\alpha(\beta) > 0\}|.$$

In particular, $L_{x,\alpha} := \{\beta \in \Phi^+ \ : \ x^{-1}(\beta) < 0, \quad s_\alpha(\beta) < 0, \quad x^{-1}s_\alpha(\beta) > 0\}$ is finite.

We call $L_{x,\alpha}$ the length difference set for x and $y = s_\alpha x$, and note that y is a cocover of x if and only if $L_{x,\alpha} = \{\alpha\}$. This is because y is a cocover if and only if the length difference is 1, and α is always in $L_{x,\alpha}$ if $y = s_\alpha x \leq x$.

EXAMPLE 5.0.2. Let W_{aff} be of type $\widetilde{A}_2$, $x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0}Y^{\alpha_2}$, and $\alpha = \alpha_1 - 2\delta + \pi$. With this setup,

$$L_{x,\alpha} = \{\alpha, \theta - 3\delta + \pi, -\alpha_2 + \delta\}.$$

At this point we will not go into detail checking that these elements do in fact belong to the length difference set (and are the only elements that do belong), but we can do a quick check of the order. By using Sage to check the lengths, we find $\ell(x) = 12$ and $\ell(s_\alpha x) = 9$, so the length difference set must indeed contain 3 elements.

Note that $-\alpha_2 + \delta = -s_\alpha(\theta - 3\delta + \pi)$. In general, the elements of the length difference set that are not equal to α will come in pairs. If $\beta \in L_{x,\alpha}$ and $\beta \neq \alpha$ then $-s_\alpha(\beta) \in L_{x,\alpha}$, which is clear from the definition.

1. Graphs

To take a geometrical approach, we need a way to envision the elements of the length difference set. We will begin by graphing the positive double affine roots α such that $y = s_\alpha x$ is less than x with respect to the Bruhat order.

DEFINITION 5.1.1. Let $\nu \in \Phi_{\text{fin}}$ and let $\Gamma_{x,\nu}$ denote the points $(r, j) \in \mathbb{Z}^2$ such that $\alpha = \nu + r\delta + j\pi > 0$ and $x^{-1}(\alpha) < 0$. We call this the **lower graph of** x **corresponding to** ν and say α corresponds to a point in $\Gamma_{x,\nu}$ if $\alpha = \nu + r\delta + j\pi$ such that $(r, j) \in \Gamma_{x,\nu}$.

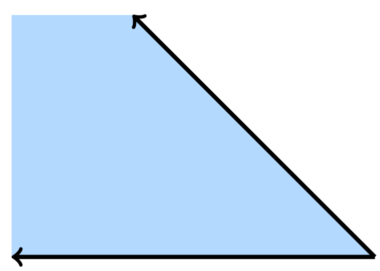


FIGURE 1. A general $\Gamma_{x,\nu}$

It is important to note that the two outer edges appearing in the graph above may or may not be included (and it is very possible only part of an edge will be included) depending on the choice of x and ν.

Because the graph shows all α such that $x \geq s_\alpha x$, it is clear that the order of $L_{x,\alpha}$ will be greater than or equal to one if α corresponds to a point in $\Gamma_{x,\nu}$.

PROPOSITION 5.1.2. *Let* $\alpha = \nu + r\delta + j\pi$ *and* $\beta = \gamma + p\delta + q\pi$. *The double affine root* β *is in* $L_{x,\alpha}$ *if and only if* $\beta \in \Gamma_{x,\gamma}$ *and* $-s_\alpha\beta \in \Gamma_{x,-s_\nu(\gamma)}$.

PROOF. Let $\beta \in L_{x,\alpha}$. Then $\beta > 0$ and $x^{-1}(\beta) < 0$, so $\beta \in \Gamma_{x,\gamma}$. Additionally, $s_\alpha(\beta) < 0$ and $x^{-1}(s_\alpha\beta) > 0$, so $-s_\alpha(\beta) \in \Gamma_{x,-s_\nu(\gamma)}$.

Let $\beta \in \Gamma_{x,\nu}$ and $-s_\alpha(\beta) \in \Gamma_{x,-s_\nu(\gamma)}$. Then $\beta > 0$, $x^{-1}(\beta) < 0$, $-s_\alpha(\beta) > 0$, and $-x^{-1}(s_\alpha(\beta)) < 0$. So $\beta \in L_{x,\alpha}$. □

EXAMPLE 5.1.3. Consider W_{aff} of type $\widetilde{A}_2$ and $x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0}Y^{\alpha_2}$ (the same choices from Example 5.0.2). The lower graph of x corresponding to α_1 is given below.

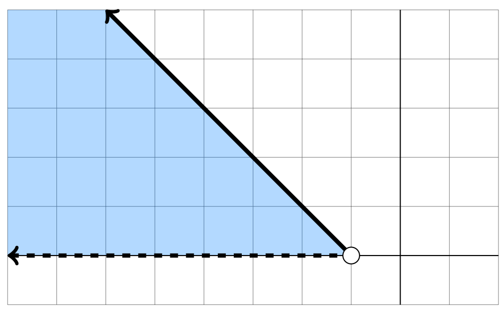


FIGURE 2. Γ_{x,α_1}

To see why this is the graph for Γ_{x,α_1}, we need to examine when $\alpha > 0$ and $x^{-1}(\alpha) < 0$.

The double affine root $\alpha = \alpha_1 + r\delta + j\pi$ is positive if and only if one of the following holds:

(1) $j > 0$
(2) $j = 0$ and $r > 0$.

To determine when $x^{-1}(\alpha) < 0$, it will help to expand $x^{-1}(\alpha)$:

$$\begin{aligned} x^{-1}(\alpha) &= Y^{-\alpha_2}X^{-\alpha_1-\alpha_2-\delta-\Lambda_0}(\alpha_1 + r\delta + j\pi) \\ &= \alpha_1 + (r + \langle\alpha_1, \alpha_2\rangle)\delta + (j + \langle\alpha_1 + r\delta, -\alpha_1 - \alpha_2 - \delta - \Lambda_0\rangle)\pi \\ &= \alpha_1 + (r-1)\delta + (j + r + 1)\pi. \end{aligned}$$

Now we can see that $x^{-1}(\alpha) < 0$ if and only if one of the following holds:

(1) $j < -r - 1$
(2) $j = -r - 1$ and $r < 1$.

Combining these restrictions results in the graph shown above.

Proposition 5.1.4. *Fix $x = X^{\zeta}\widetilde{w} \in W$ with $\widetilde{w} = Y^{\lambda}w \in W_{\text{aff}}$ and fix $\nu \in \Phi_{\text{fin}}$. The point $(r, j) \in \Gamma_{x,\nu}$ if and only if one of the following holds:*

(1) $0 < j < \langle -\zeta, \widetilde{\alpha}\rangle = -\langle \zeta, \nu + r\delta\rangle$

(2) $(r, j) = (r, 0)$ *with* $0 \leq r \leq \frac{\langle \nu,\mu\rangle}{-l}$*, and if* $r = 0$*, then* $\nu > 0$

(3) $(r, j) = (r, \langle -\zeta, \widetilde{\alpha}\rangle) = (r, -\langle \nu, \mu\rangle - lr)$ *with* $r \leq min\{\frac{\langle \nu,\mu\rangle}{-l}, -\langle \lambda, \nu\rangle\}$*, and if* $r = -\langle \lambda, \nu\rangle$*, then* $w^{-1}(\nu) < 0$.

Proof. For $\alpha = \nu + r\delta + j\pi \in \Phi$ to correspond to a point in $\Gamma_{x,\nu}$, we need both $\alpha > 0$ and $x^{-1}(\alpha) < 0$.

For $\alpha = \nu + r\delta + j\pi > 0$ we need one of the following:

(1) $j > 0$

(2) $j = 0,\ r > 0$

(3) $j = 0,\ r = 0, \nu > 0$.

For $x^{-1}(\alpha) < 0$ we need

$$\begin{aligned} x^{-1}(\alpha) &= \widetilde{w}^{-1}(\widetilde{\alpha}) + (j - \langle -\zeta, \widetilde{\alpha}\rangle)\pi \\ &= w^{-1}(\nu) + (r + \langle \lambda, \nu\rangle)\delta + (j + \langle \mu, \nu\rangle + lr)\pi < 0, \end{aligned}$$

so we need one of the following:

(1) $j < \langle -\zeta, \widetilde{\alpha}\rangle = -\langle \mu, \nu\rangle - lr$

(2) $j = \langle -\zeta, \widetilde{\alpha}\rangle,\ r < -\langle \lambda, \nu\rangle$

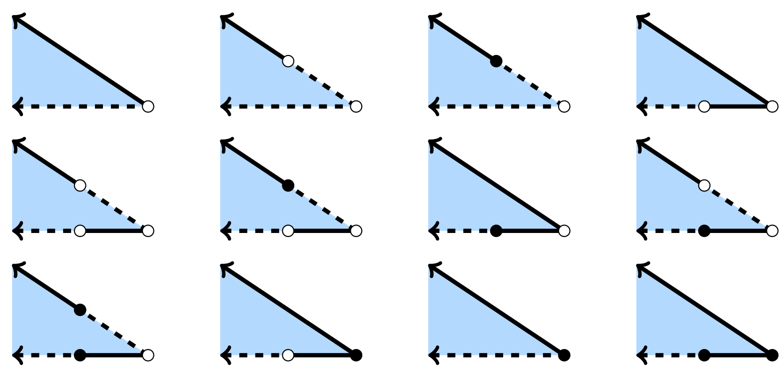
(3) $j = \langle -\zeta, \widetilde{\alpha}\rangle,\ r = -\langle \lambda, \nu\rangle, w^{-1}(\nu) < 0$.

Combining these results, we see that if (r, j) is in the graph, then $0 \leq j \leq \langle -\zeta, \widetilde{\alpha}\rangle$. This tells us that $-\langle \zeta, \widetilde{\alpha}\rangle \geq 0$ and since $-\langle \zeta, \widetilde{\alpha}\rangle = \langle -\mu - m\delta - l\Lambda_0, \nu + r\delta\rangle = -\langle \mu, \nu\rangle - lr$, we can solve for r and get $r \leq \frac{\langle \nu,\mu\rangle}{-l}$. The point $(r, j) = (\frac{\langle \nu,\mu\rangle}{-l}, 0)$ is the intersection point of $j = 0$ and $j = \langle -\zeta, \widetilde{\alpha}\rangle$. The only time $(r, j) = (\frac{\langle \nu,\mu\rangle}{-l}, j)$ is in the graph is when $j = 0$ because if $r = \frac{\langle \nu,\mu\rangle}{-l}$ and $0 \leq j \leq \langle -\zeta, \widetilde{\alpha}\rangle$, then $j = 0$.

So when $j = 0$, we can restrict r to $0 \leq r \leq \frac{\langle \nu,\mu\rangle}{-l}$. And when $j = \langle -\zeta, \widetilde{\alpha}\rangle$, we can restrict to $r \leq \min\{\frac{\langle \nu,\mu\rangle}{-l}, -\langle \lambda, \nu\rangle\}$. □

Definition 5.1.5. For simplicity we will refer to the line segment of $j = 0$ that is included in the graph and the ray of $j = -\langle \zeta, \widetilde{\alpha}\rangle = -\langle \zeta, \nu + r\delta\rangle$ that is included in the graph as the **lower and upper outer edges** respectively. We will refer to the ray of $j = 1$ that is included in the graph and the ray of $j = -\langle \zeta, \nu + r\delta \rangle - 1$ that is included in the graph as the **lower and upper inner edges** respectively.

PROPOSITION 5.1.6. *For a fixed $x \in W$ and $\nu \in \Phi_{\text{fin}}$, there are 12 possible forms for $\Gamma_{x,\nu}$, and they are represented by the graphs below.*



PROOF. Because the lower outer edge will be the line segment $j = 0$ with $0 \le r \le \frac{\langle \nu, \mu \rangle}{-l}$ and endpoints possibly not included, there are four possibilities:

L1 L2 L3 L4

Note that the first two types do not include the intersection point of $j = 0$ and $j = -\langle \zeta, \widetilde{\alpha} \rangle$ (represented by the right endpoint), but the last two types do. Also note that type L1 and type L4 may or may not include the line segment between the endpoints. If the line segment is not included, we will refer to these as type L1* or L4* respectively.

Now we will look at the possibilities for the upper outer edge given by $j = -\langle \zeta, \widetilde{\alpha} \rangle$ with $r \le \min\{\frac{\langle \nu, \mu \rangle}{-l}, -\langle \lambda, \nu \rangle\}$ and endpoint possibly not included:

U1 U2 U3 U4

Note that the first three types do not include the intersection point of $j = 0$ and $j = -\langle \zeta, \widetilde{\alpha} \rangle$ (represented by the right point). So these upper outer edges will match with the lower outer edges of type L1, L1*, and L2. The only upper outer edge containing the intersection point is of type U4, so this will match with the lower outer edges of type L3, L4, and L4*. In total, this gives 12 possibilities for the graph. □

2. Corners

Recall that we are interested in determining which $\alpha = \nu + r\delta + j\pi$ of $\Gamma_{x,\nu}$ correspond to cocovers (meaning $y = s_\alpha x$ is a cocover of x). To do this, we must examine specific $(r, j) \in \Gamma_{x,\nu}$.

DEFINITION 5.2.1. Let $\alpha = \nu + r\delta + j\pi$ be a double affine root. Then we say ν is the **finite part of** α because $\nu \in \Phi_{\text{fin}}$. We denote this by $\text{fin}(\alpha) = \nu$.

DEFINITION 5.2.2. For double affine roots $\alpha = \nu + r\delta + j\pi$ and $\beta = \nu + p\delta + q\pi$, define β_α^- to be the root found by rotating (p, q) 180 degrees about (r, j).

PROPOSITION 5.2.3. *If β and α are double affine roots such that* $\text{fin}(\alpha) = \text{fin}(\beta)$*, then* $\beta_\alpha^- = -s_\alpha\beta$.

PROOF. Let $\alpha = \nu + r\delta + j\pi$ and $\beta = \nu + p\delta + j\pi$. Then

$$\begin{aligned} -s_\alpha(\beta) &= -\beta + \langle \beta, \alpha \rangle \alpha \\ &= -\beta + \langle \nu, \nu \rangle \alpha \\ &= -\beta + 2\alpha \\ &= \nu + (2r - p)\delta + (2j - q)\pi. \end{aligned}$$

The root β_α^- is equal to $\nu + p'\delta + q'\pi$ where (p', q') is the result of rotating (p, q) 180 degrees about (r, j). To determine (p', q'), first shift so that we are rotating about the center: $(p, q) \to (p-r, q-j)$ and $(r, j) \to (0, 0)$, then reflect over the x and y axes: $(p - r, q - j) \to (-p + r, -q + j)$, and now shift back to original orientation: $(0, 0) \to (r, j)$ and $(-p + r, -q + j) \to (-p + 2r, -q + 2j) = (2r - p, 2j - q)$.

So $(p', q') = (2r - p, 2j - q)$ and $\beta_\alpha^- = \nu + (2r - p)\delta + (2j - q)\pi = -s_\alpha(\beta)$. □

Definition 5.2.4. We say that α is a **corner of the graph** $\Gamma_{x,\nu}$, or a **corner relative to** x, if α corresponds to a point in $\Gamma_{x,\nu}$, and if for any $\beta = \nu + p\delta + q\pi$ corresponding to a point in $\Gamma_{x,\nu}$, β_α^- is not in the graph.

Example 5.2.5. Consider W_{aff} of type $\widetilde{A}_2$, $x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0} Y^{\alpha_2}$ (still the same example from Example 5.0.2).

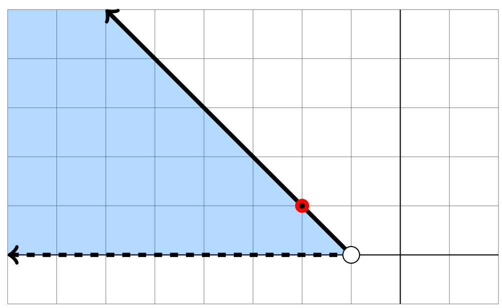


Figure 3. Γ_{x,α_1}

With this setup, $\alpha = \alpha_1 - 2\delta + \pi$ corresponds to a corner of Γ_{x,α_1}.

Proposition 5.2.6. *If $y = s_\alpha x$ is a cocover of x, then α must correspond to a corner in the graph $\Gamma_{x,\mathrm{fin}(\alpha)}$.*

Proof. Suppose $\alpha = \nu + r\delta + j\pi$ is not a corner of $\Gamma_{x,\nu}$. Then there is some $\beta = \nu + p\delta + q\pi$ such that $\beta \neq \alpha$, $\beta \in \Gamma_{x,\nu}$, and $\beta_\alpha^- \in \Gamma_{x,\nu}$. But $\beta_\alpha^- = -s_\alpha(\beta)$, so by Proposition 5.1.2, $\beta \in L_{x,\alpha}$. So $|L_{x,\alpha}| > 1$, and y is not a cocover of x. □

Remark 5.2.7. In general, the set of corners will be larger than the set of roots corresponding to cocovers of a fixed $x \in W$. Consider W_{aff} of type $\widetilde{A}_2$, $x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0} Y^{\alpha_2}$, $\alpha = \alpha_1 - 2\delta + \pi$. Then α corresponds to a corner of Γ_{x,α_1}, as show in Example 5.2.5, but $s_\alpha x$ is not a cocover of x (we saw in Example 5.0.2 the length difference set contains 3 elements).

We have seen that for $y = s_\alpha x$ to be a cocover, α must be a corner relative to x. To determine which corners give rise to cocovers, one could directly check by calculating the length difference set $L_{x,\alpha}$. This is not always easy, so instead we detail a method using the graphs $\Gamma_{x,\mathrm{fin}(\alpha)}$.

(1) Fix $x \in W$ and ν a finite root.

(2) Let $\alpha = \nu + r\delta + j\pi$ be a corner of $\Gamma_{x,\nu}$.

(3) For any $\beta = \gamma + p\delta + q\pi$ such that $\langle \gamma, \nu \rangle = 1$, check if $(p, q) \in \Gamma_{x,\gamma}$ and $(r - p, j - q) \in \Gamma_{x,\nu-\gamma}$ (this is equivalent to checking that $-s_\alpha(\beta) \in \Gamma_{x,\text{fin}(-s_\alpha(\beta))}$).

(4) If there is such a β, then it is in the length difference set and α is not a cocover. Otherwise, α is a cocover.

Note that if α is a corner, then there are no $\beta \in L_{x,\alpha}$ such that β shares the same finite part as α. In other words, there is no $\beta \in L_{x,\alpha}$ such that $\langle \beta, \alpha \rangle = 2$. Here we are relying on the fact that our finite root system Φ_{fin} is irreducible and simply laced. In [**B**, VI 1.3] it was shown that for $\nu, \gamma \in \Phi_{\text{fin}}$ where Φ_{fin} is irreducible and simply laced, $\langle \nu, \gamma \rangle = 2$ if and only if $\nu = \gamma$.

In part three, we say to only check β that pair with α to equal 1. The reason for this is that if β is an element of the length difference set, then $\beta > 0$ by definition, and so $\langle \alpha, \beta \rangle$ must be positive. If not, then $s_\alpha(\beta) = \beta - \langle \alpha, \beta \rangle \alpha$ will be positive. So $\langle \beta, \alpha \rangle$ must be 1 or 2, and 2 is already verified by choosing α to be a corner.

When checking $(p, q) \in \Gamma_{x,\gamma}$, we are checking that β is an element of $\Gamma_{x,\gamma}$. When checking $(r - p, j - q) \in \Gamma_{x,\nu-\gamma}$, we are checking that $-s_\alpha(\beta) = \alpha - \beta$ is an element of $\Gamma_{x,\nu-\gamma}$. By Proposition 5.1.2 this is equivalent to checking that $\beta \in L_{x,\alpha}$.

Requiring $\langle \gamma, \nu \rangle = 1$ reduces the possible γ to check, but we are still left with infinitely many possibilities for (p, q). However, step (3) can be done by examining the intersection of $\Gamma_{x,\gamma}$ and a translated version of $\Gamma_{x,\text{fin}(-s_\alpha(\beta))}$ so that once $\text{fin}(\beta) = \gamma$ is fixed, β can be checked for all (p, q).

Definition 5.2.8. Let α and β be double affine roots such that $\langle \beta, \alpha \rangle = 1$. Let $(-s_\alpha)\Gamma_{x,\text{fin}(-s_\alpha(\beta))}$ be the translated graph of $\Gamma_{x,\text{fin}(-s_\alpha(\beta))}$ such that if β corresponds to $(p, q) \in \Gamma_{x,\text{fin}(-s_\alpha(\beta))}$, then $-s_\alpha(\beta)$ corresponds to the point $(r-p, j-q) \in (-s_\alpha)\Gamma_{x,\text{fin}(-s_\alpha(\beta))}$.

The graph $(-s_\alpha)\Gamma_{x,\text{fin}(-s_\alpha(\beta))}$ is found by shifting to the right r units (turning (p, q) to $(-r+p, q)$), reflecting over the x and y axis (turning $(-r+p, q)$ to $(r-p, -q)$) and then shifting up j units (turning $(r - p, -q)$ to $(r - p, j - q)$).

We will illustrate how this simplifies step (3) with an example.

Example 5.2.9. Consider W_{aff} of type $\widetilde{A}_2$, $x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0}Y^{\alpha_2}$, $\alpha = \alpha_1 - 2\delta + \pi$.

Recall from Example 5.0.2, $L_{x,\alpha} = \{\alpha, \theta - 3\delta + \pi, -\alpha_2 + \delta\}$. We can check that these elements are in fact part of the length difference set by checking that $\beta = \theta - 3\delta + \pi \in \Gamma_{x,\theta}$ and $-s_\alpha(\beta) = -\alpha_2 + \delta \in \Gamma_{x,-\alpha_2}$.

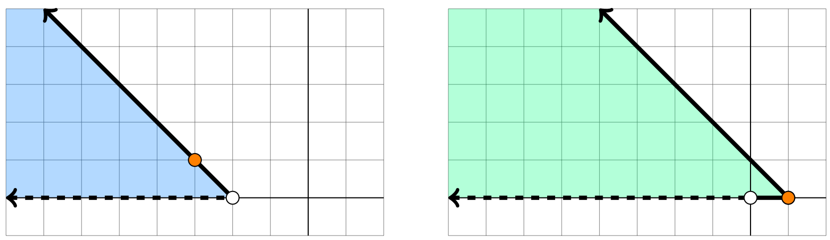
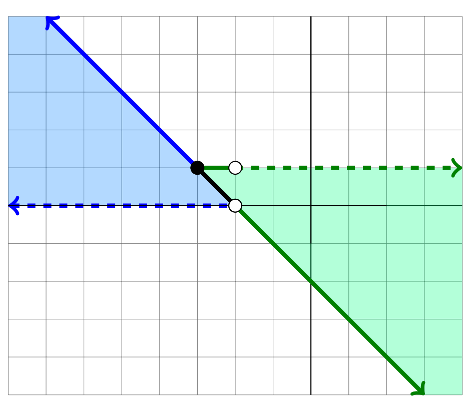


Figure 4. Left: $\Gamma_{x,\theta}$ Right: $\Gamma_{x,-\alpha_2}$

This verifies that the elements are in fact part of the length difference set, and since we know the length difference set contains only three elements (by using Sage to check the lengths), this is enough to verify our statement about $L_{x,\alpha}$; however, there remains the question of how we found these elements in the first place.

First note that since $\nu = \alpha_1$, we only need to check $\beta = \gamma + p\delta + q\pi$ such that $\gamma = \theta$. The roots θ and $-\alpha_2$ are the only finite roots that pair with ν to give 1, and checking $\gamma = \theta$ automatically checks $-\alpha_2$ since $-\alpha_2 = -s_{\alpha_1}(\theta)$.

To complete step (3) for $\beta = \theta + p\delta + q\pi$, we use the following graph.



This graph shows the intersection of $\Gamma_{x,\theta}$ and $(-s_\alpha)\Gamma_{x,\mathrm{fin}(-s_\alpha(\beta))}$, the translated version of $\Gamma_{x,\mathrm{fin}(-s_\alpha(\beta))} = \Gamma_{x,-\alpha_2}$. The intersection includes all (p,q) such that $(p,q) \in \Gamma_{x,\theta}$ and $(r-p, j-q) = (-2-p, 1-q) \in \Gamma_{x,-\alpha_2}$ (so this checks step (3) for $\mathrm{fin}(\beta) = \theta$). This tells us all β with finite part θ that are in the length difference set, and by using $-s_\alpha(\beta)$, we can determine all elements of the length difference set with finite part $-\alpha_2$. Therefore the only elements of $L_{x,\alpha}$ are $\alpha, \beta = \theta - 3\delta + \pi$, and $-s_\alpha(\beta) = -\alpha_2 + \delta$.

This process allows us to reduce step (3) to checking the intersection of $\Gamma_{x,\mathrm{fin}(\beta)}$ and $(-s_\alpha)\Gamma_{x,\mathrm{fin}(-s_\alpha(\beta))}$ for any $\mathrm{fin}(\beta)$ such that $\langle \mathrm{fin}(\beta), \mathrm{fin}(\alpha)\rangle = 1$. This intersection will always be finite because if (p,q) is in the intersection, then $\mathrm{fin}(\beta)+p\delta+q\pi \in L_{x,\alpha}$, which we know to be a finite set.

Now we would like to show that there are finitely many α that are corners relative to a fixed x, but before we do, we need to make some observations about the graphs:

- If (r,j) is a point of the graph and $j \neq 0$, then (p,j) is a point of the graph for all $p < r$ because when $j \neq 0$, the only bound on r is $r \leq \min\{\frac{\langle \nu,\mu\rangle}{-l}, -\langle \lambda,\nu\rangle\}$.
- The upper outer edge is given by $j = -\langle \zeta, \nu + r\delta\rangle = -\langle \mu,\nu\rangle - rl$. The slope is $-l$, which is an integer (it comes from the level of $x's$ X weight). Additionally, for any integer r, $j = -\langle \zeta, \nu + r\delta\rangle$ is also an integer because $-\langle \zeta, \nu + r\delta\rangle = -\langle \mu,\nu\rangle - rl$ where $r, l, \langle \mu,\nu\rangle \in \mathbb{Z}$.
- If (r,j) is a point on $y = -\langle \zeta, \nu + x\delta\rangle$ and (r,j) is in the graph, and if (p,q) is a point on $y = -\langle \zeta, \nu + x\delta\rangle$ such that $q > j$, then (p,q) is in the graph. This is because if $q > j$, then $p < r$ (because the slope of $y = -\langle \zeta, \nu + x\delta\rangle$ is negative) and when (r,j) is on $y = -\langle \zeta, \nu + x\delta\rangle$ and in the graph, then $r \leq \min\{\frac{\langle \nu,\mu\rangle}{-l}, -\langle \lambda,\nu\rangle\}$. So if (p,q) is on the same line and $p < r \leq \min\{\frac{\langle \nu,\mu\rangle}{-l}, -\langle \lambda,\nu\rangle\}$, then (p,q) is also in the graph.
- If (r,j) is on $y = -\langle \zeta, \nu + x\delta\rangle - k$ with k a positive integer, and if (r,j) is in the graph, then any (p,q) on $y = -\langle \zeta, \nu + x\delta\rangle - k$ with $q > j$ is also in the graph. Since (p,q) is on $y = -\langle \zeta, \nu + x\delta\rangle - k$, it is clear that

$q < -\langle \zeta, \nu + p\delta \rangle$. And since $q > j \geq 0$, we have $0 < q < -\langle \zeta, \nu + x\delta \rangle$, so (p, q) must be a point in the graph.

PROPOSITION 5.2.10. *Fix $x \in W$ and $\nu \in \Phi_{\text{fin}}$. The number of corners of $\Gamma_{x,\nu}$ is finite.*

Idea: We show that if $\alpha = \nu + r\delta + j\pi$ corresponds to a corner relative to x, then α must fall on one of the two outer edges or one of the two inner edges of $\Gamma_{x,\nu}$. But on these edges, only the $(r, j) \in \mathbb{Z}^2$ closest to endpoints can be corners.

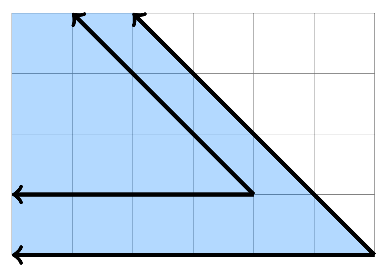


FIGURE 5. Inner and Outer Edges of a General $\Gamma_{x,\nu}$

PROOF. We break the proof into several cases. Let (r', j') represent a corner.

Case 1: Assume $j' = 0$. Then (r', j') falls along the lower outer edge, which is either a line segment or a single point. In either case, there are finitely many possibilities for (r', j').

Case 2: Assume (r', j') falls along the upper outer edge (the diagonal $j = -\langle \zeta, \nu + r\delta \rangle$). Then any other $(r, j) \in \Gamma_{x,\nu}$ on the outer upper edge must have $j > j'$. If there exists some (r, j) on the graph's upper outer edge such that $j < j'$, then it can be rotated 180 degrees about (r', j') and end up in the graph (because it will land on the diagonal and be higher up than (r', j')), which contradicts the fact that (r', j') is a corner. So there is only one possibility for (r', j').

Case 3: Assume (r', j') falls along the upper inner edge (the diagonal given by $j = -\langle \zeta, \nu + r\delta \rangle - 1$). Then using the same logic from above, (r', j') must have smallest possible j' and so there is only one possibility.

Case 4: Assume $j' = 1$. Suppose $(r, 1)$ is another point of the graph such that $r > r'$. Then $(r, 1)$ rotated 180 degrees about $(r', 1)$ results in some $(p, 1)$ with

$p < r'$. So $(p, 1)$ is in the graph, but this contradicts the fact that (r', j') is a corner. So again there is only one possibility.

Case 5: Assume (r', j') does not lie on any of the outer or inner edges. Then $1 < j' < \langle -\mu, \nu \rangle - r'l$, so $1 \leq j' - 1 < j' < \langle -\mu, \nu \rangle - r'l$, and $(r', j' - 1)$ is a point of the graph. And $1 < j' < j' + 1 \leq \langle -\mu, \nu \rangle - r'l$, so $(r', j' + 1)$ is also a point on the graph. Thus (r', j') cannot be a corner.

So to be a corner, (r', j') must fall along one of the two outer edges or one of the two inner edges. On those edges there are finitely many possibilities for corners. Thus for any given x and ν, the corresponding graph $\Gamma_{x,\nu}$ contains finitely many corners. $\square$

COROLLARY 5.2.11. *The number of cocovers of x is finite.*

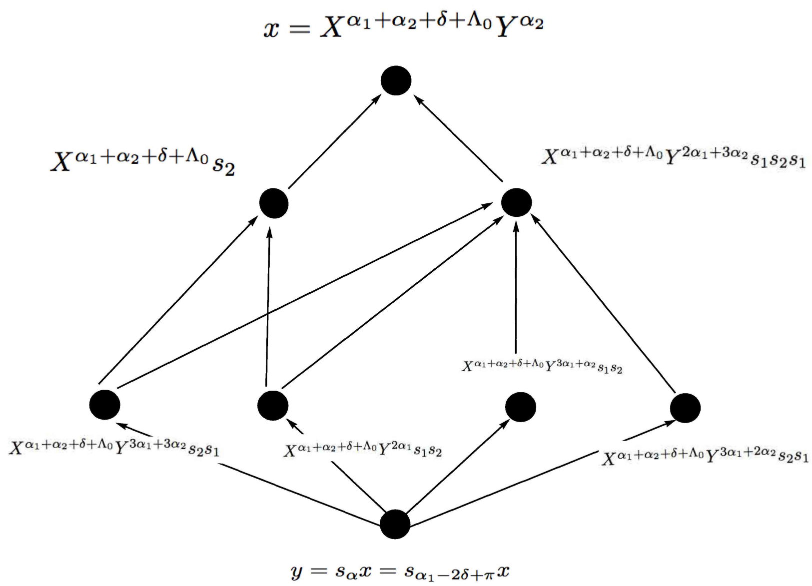
PROOF. Fix x. For any $\nu \in \Phi_{\text{fin}}$, the graph $\Gamma_{x,\nu}$ has finitely many corners. So there are finitely many $\alpha = \nu + r\delta + j\pi$ such that $y = s_\alpha x$ is a cocover of x. Since ν is a finite root, there are finitely many possibilities for ν. So there are finitely many cocovers for a given x. $\square$

COROLLARY 5.2.12. *Let $x, y \in W$ such that $y \leq x$. Then the double affine Bruhat interval $[y, x]$ will be finite.*

We proved this in Chapter Three, Theorem 3.4.4.

EXAMPLE 5.2.13. Consider W_{aff} of type $\widetilde{A}_2$, and let $x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0} Y^{\alpha_2}$ and $\alpha = \alpha_1 - 2\delta + \pi$. Then $[s_\alpha x, x]$ contains 8 elements:

(1) $x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0} Y^{\alpha_2}$
(2) $s_{\theta-3\delta+\pi} x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0} Y^{2\alpha_1+3\alpha_2} s_1 s_2 s_1$, a cocover of x
(3) $s_{-\alpha_2+\delta} x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0} s_2$, a cocover of x
(4) $s_{\alpha_1+\alpha_2-4\delta+2\pi} s_\alpha x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0} Y^{3\alpha_1+\alpha_2} s_1 s_2$, a cover of $s_\alpha x$
(5) $s_{\theta-3\delta+\pi} s_\alpha x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0} Y^{2\alpha_1} s_1 s_2$, a cover of $s_\alpha x$
(6) $s_{-\alpha_2+\delta} s_\alpha x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0} Y^{3\alpha_1+3\alpha_2} s_2 s_1$, a cover of $s_\alpha x$
(7) $s_{-\alpha_2+\pi} s_\alpha x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0} Y^{3\alpha_1+2\alpha_2} s_2 s_1$, a cover of $s_\alpha x$
(8) $s_\alpha x = s_{\alpha_1-2\delta+\pi} x$.



Corollary 5.2.14. *Let $x = X^{\zeta}\widetilde{w}$ with $\zeta \in \mathcal{T}$ and $\widetilde{w} \in W_{\mathrm{aff}}$. If $\alpha = \nu + r\delta + j\pi$ corresponds to a corner of the graph $\Gamma_{x,\nu}$ then one of the following must hold:*

(1) $j = 0$

(2) $j = 1$

(3) $j = -\langle \zeta, \widetilde{\alpha} \rangle$

(4) $j = -\langle \zeta, \widetilde{\alpha} \rangle - 1$.

3. Main Result

Using our new approach, we can bypass the bounds we needed on $\ell(\widetilde{w})$ and $\ell(s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w})$ in Theorem 4.1.1, and we can reduce the bound we needed on $\langle \zeta, \alpha_i \rangle$.

Theorem 5.3.1. *Let $x = X^{\widetilde{v}\zeta}\widetilde{w}$ and $y = s_\alpha x$ where $\alpha = -\widetilde{v}\widetilde{\alpha} + j\pi$ is a positive double affine root and $\langle \zeta, \alpha_i \rangle > 2$ for $i = 0, 1, \ldots, n$. Then y is a cocover of x if and only if one of the following holds:*

(1) $j = 0$ *and* $\ell(\widetilde{v}) = \ell(\widetilde{v}s_{\widetilde{\alpha}}) + 1$.

(2) $j = 1$ *and* $\ell(\widetilde{v}) = \ell(\widetilde{v}s_{\widetilde{\alpha}}) + 1 - \langle \widetilde{\alpha}, 2\rho \rangle$.

(3) $j = \langle \zeta, \widetilde{\alpha} \rangle$ *and* $\ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}) = \ell(\widetilde{w}^{-1}\widetilde{v}) + 1$.

(4) $j = \langle \zeta, \widetilde{\alpha} \rangle - 1$ *and* $\ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}) = \ell(\widetilde{w}^{-1}\widetilde{v}) + 1 - \langle \widetilde{\alpha}, 2\rho \rangle$.

PROOF. Following the proof to Theorem 4.1.1 we write y in two different forms.

$$
\begin{aligned}
y &= s_\alpha x \\
&= X^{\widetilde{v}s_{\widetilde{\alpha}}(\zeta - j\widetilde{\alpha})} s_{\widetilde{v}\widetilde{\alpha}} \widetilde{w} \\
&= X^{\widetilde{v}(\zeta - (\langle \zeta, \widetilde{\alpha} \rangle - j)\widetilde{\alpha})} s_{\widetilde{v}\widetilde{\alpha}} \widetilde{w}
\end{aligned}
$$

Using Proposition 3.2.3 and the fact that ζ is dominant and regular, we have

$$\ell(x) = \langle \zeta, 2\rho \rangle - \ell(\widetilde{w}^{-1}\widetilde{v}) + \ell(\widetilde{v}).$$

If y is a cocover of x, then α is a corner relative to x, and by Corollary 5.2.14, there are four possibilities for j:

(1) $j = 0$ and $y = X^{\widetilde{v}s_{\widetilde{\alpha}}\zeta} s_{\widetilde{v}\widetilde{\alpha}} \widetilde{w}$
(2) $j = 1$ and $y = X^{\widetilde{v}s_{\widetilde{\alpha}}(\zeta - \widetilde{\alpha})} s_{\widetilde{v}\widetilde{\alpha}} \widetilde{w}$
(3) $j = -\langle \widetilde{v}\zeta, -\widetilde{v}\widetilde{\alpha} \rangle = \langle \zeta, \widetilde{\alpha} \rangle$ and $y = X^{\widetilde{v}\zeta} s_{\widetilde{v}\widetilde{\alpha}} \widetilde{w}$
(4) $j = -\langle \widetilde{v}\zeta, -\widetilde{v}\widetilde{\alpha} \rangle - 1 = \langle \zeta, \widetilde{\alpha} \rangle - 1$ and $y = X^{\widetilde{v}(\zeta - \widetilde{\alpha})} s_{\widetilde{v}\widetilde{\alpha}} \widetilde{w}$.

So no matter which direction we are proving, we may reduce to these four cases. Using $\langle \widetilde{\alpha}, \widetilde{\beta} \rangle \leq 2$ for all $\widetilde{\alpha}, \widetilde{\beta} \in \Phi_{\text{aff}}$ [B, VI 1.3] and the assumption that $\langle \zeta, \alpha_i \rangle > 2$ for $i = 0, 1, \ldots, n$, we have that $\zeta - \widetilde{\alpha}$ is dominant and regular.

Case (1): Let $j = 0$. Then $y = X^{\widetilde{v}s_{\widetilde{\alpha}}\zeta} s_{\widetilde{v}\widetilde{\alpha}} \widetilde{w}$, and by using Proposition 3.2.3 we have

$$
\begin{aligned}
\ell(y) =& \langle \zeta, 2\rho \rangle - \ell(\widetilde{w}^{-1} s_{\widetilde{v}\widetilde{\alpha}} \widetilde{v} s_{\widetilde{\alpha}}) + \ell(\widetilde{v} s_{\widetilde{\alpha}}) \\
=& \langle \zeta, 2\rho \rangle - \ell(\widetilde{w}^{-1} \widetilde{v} s_{\widetilde{\alpha}} \widetilde{v}^{-1} \widetilde{v} s_{\widetilde{\alpha}}) + \ell(\widetilde{v} s_{\widetilde{\alpha}}) \\
=& \langle \zeta, 2\rho \rangle - \ell(\widetilde{w}^{-1}\widetilde{v}) + \ell(\widetilde{v} s_{\widetilde{\alpha}}).
\end{aligned}
$$

So $\ell(x) - \ell(y) = \ell(\widetilde{v}) - \ell(\widetilde{v}s_{\widetilde{\alpha}})$, and y is a cocover of x if and only if $\ell(\widetilde{v}) - \ell(\widetilde{v}s_{\widetilde{\alpha}}) = 1$.

Case (2): Let $j = 1$. Then $y = X^{\widetilde{v}s_{\widetilde{\alpha}}(\zeta - \widetilde{\alpha})} s_{\widetilde{v}\widetilde{\alpha}} \widetilde{w}$, and by using Proposition 3.2.3 we have

$$\ell(y) = \langle \zeta - \widetilde{\alpha}, 2\rho \rangle - \ell(\widetilde{w}^{-1} s_{\widetilde{v}\widetilde{\alpha}} \widetilde{v} s_{\widetilde{\alpha}}) + \ell(\widetilde{v} s_{\widetilde{\alpha}})$$

$$=\langle\zeta,2\rho\rangle-\langle\widetilde{\alpha},2\rho\rangle-\ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}\widetilde{v}^{-1}\widetilde{v}s_{\widetilde{\alpha}})+\ell(\widetilde{v}s_{\widetilde{\alpha}})$$
$$=\langle\zeta,2\rho\rangle-\langle\widetilde{\alpha},2\rho\rangle-\ell(\widetilde{w}^{-1}\widetilde{v})+\ell(\widetilde{v}s_{\widetilde{\alpha}}).$$

So $\ell(x)-\ell(y)=\ell(\widetilde{v})-\ell(\widetilde{v}s_{\widetilde{\alpha}})+\langle\widetilde{\alpha},2\rho\rangle$, and y is a cocover of x if and only if $\ell(\widetilde{v})-\ell(\widetilde{v}s_{\widetilde{\alpha}})+\langle\widetilde{\alpha},2\rho\rangle=1$.

Case (3): Let $j=\langle\zeta,\widetilde{\alpha}\rangle$. Then $y=X^{\widetilde{v}\zeta}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}$, and by using Proposition 3.2.3 we have

$$\ell(y)=\langle\zeta,2\rho\rangle-\ell(\widetilde{w}^{-1}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{v})+\ell(\widetilde{v})$$
$$=\langle\zeta,2\rho\rangle-\ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}})+\ell(\widetilde{v}).$$

So $\ell(x)-\ell(y)=\ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}})-\ell(\widetilde{w}^{-1}\widetilde{v})$, and y is a cocover of x if and only if $\ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}})-\ell(\widetilde{w}^{-1}\widetilde{v})=1$.

Case (4): Let $j=\langle\zeta,\widetilde{\alpha}\rangle-1$. Then $y=X^{\widetilde{v}(\zeta-\widetilde{\alpha})}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}$, and by using Proposition 3.2.3 we have

$$\ell(y)=\langle\zeta,2\rho\rangle-\langle\widetilde{\alpha},2\rho\rangle-\ell(\widetilde{w}^{-1}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{v})+\ell(\widetilde{v})$$
$$=\langle\zeta,2\rho\rangle-\langle\widetilde{\alpha},2\rho\rangle-\ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}})+\ell(\widetilde{v}).$$

So $\ell(x)-\ell(y)=\ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}})-\ell(\widetilde{w}^{-1}\widetilde{v})+\langle\widetilde{\alpha},2\rho\rangle$, and y is a cocover of x if and only if $\ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}})-\ell(\widetilde{w}^{-1}\widetilde{v})+\langle\widetilde{\alpha},2\rho\rangle=1$. □

4. Covers and Corners

We can prove a similar result for covers using the same process.

DEFINITION 5.4.1. Let $x\in W$ and $\nu\in\Phi_{\text{fin}}$. We define $\Gamma'_{x,\nu}$ to be graph containing the points $(r,j)\in\mathbb{Z}^2$ such that $\alpha=\nu+r\delta+j\pi>0$ and $x^{-1}(\alpha)>0$. We call this the **upper graph of x corresponding to ν** and say α corresponds to a point in $\Gamma'_{x,\nu}$ if $\alpha=\nu+r\delta+j\pi$ such that $(r,j)\in\Gamma'_{x,\nu}$.

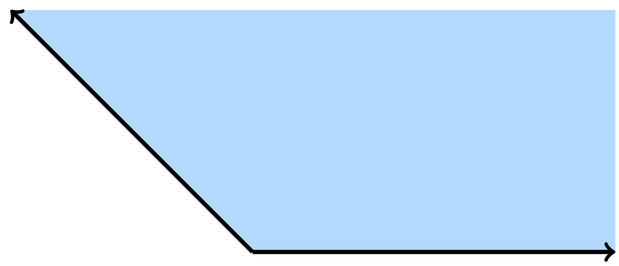


Figure 6. A general $\Gamma'_{x,\nu}$

As was the case of $\Gamma_{x,\nu}$, the edges shown above may or may not be included (and may only be partly included).

Remark 5.4.2. The point (r, j) is in the graph $\Gamma'_{x,\nu}$ if and only if $x \leq s_\alpha x$.

Proposition 5.4.3. *Fix $x \in W$ and $\nu \in \Phi_{\text{fin}}$. The point (r, j) is in the graph $\Gamma'_{x,\nu}$ if and only if one of the following holds:*

(1) $j >$ *max* $\{0, \langle -\zeta, \widetilde{\alpha} \rangle\}$
(2) $j = 0$ *and* $r \geq$ *max*$\{\frac{\langle -\mu,\nu \rangle}{l}, 0\}$, *and if* $r = 0$, *then* $\nu > 0$
(3) $j = \langle -\zeta, \widetilde{\alpha} \rangle = -\langle \nu, \mu \rangle - lr$ *such that* $-\langle \lambda, \nu \rangle \leq r \leq \frac{\langle -\mu,\nu \rangle}{l}$, *and if* $r = -\langle \lambda, \nu \rangle$, *then* $w^{-1}(\nu) > 0$.

Proof. The point (r, j) is in $\Gamma'_{x,\nu}$ if and only if $\alpha > 0$ and $x^{-1}(\alpha) > 0$.

For $\alpha = \nu + r\delta + j\pi > 0$, we need one of the following:

(1) $j > 0$
(2) $j = 0,\ r > 0$
(3) $j = 0,\ r = 0, \nu > 0$.

To determine when $x^{-1}(\alpha) > 0$, it will help to first expand $x^{-1}(\alpha)$:

$$\begin{aligned} x^{-1}(\alpha) &= \widetilde{w}^{-1}(\widetilde{\alpha}) + (j - \langle -\zeta, \widetilde{\alpha} \rangle)\pi \\ &= w^{-1}(\nu) + (r + \langle \lambda, \nu \rangle)\delta + (j + \langle \mu, \nu \rangle + lr)\pi > 0. \end{aligned}$$

Now we can see that for $x^{-1}(\alpha) > 0$, we need one of the following:

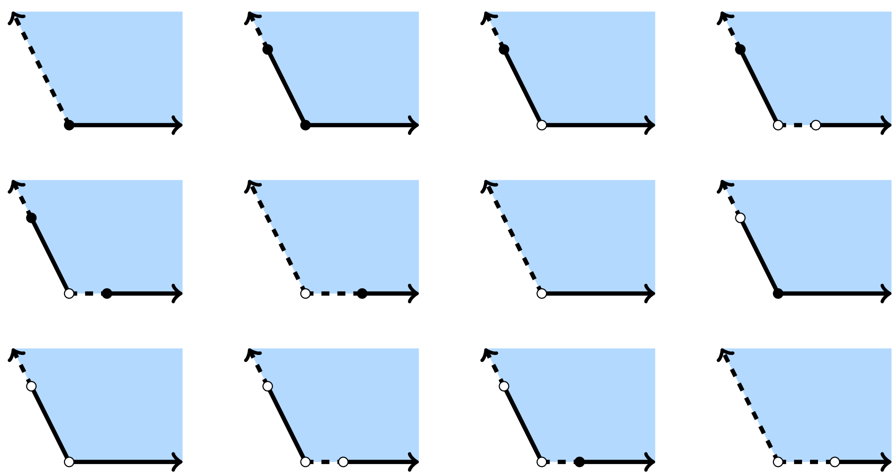
(1) $j > \langle -\zeta, \widetilde{\alpha} \rangle = -\langle \mu, \nu \rangle - lr$
(2) $j = \langle -\zeta, \widetilde{\alpha} \rangle,\ r > -\langle \lambda, \nu \rangle$

(3) $j = \langle -\zeta, \widetilde{\alpha} \rangle,\ r = -\langle \lambda, \nu \rangle, w^{-1}(\nu) > 0$.

Combining our results, we see that if $j = \langle -\zeta, \widetilde{\alpha} \rangle = -\langle \nu, \mu \rangle - lr = 0$, then $r = \frac{\langle -\mu, \nu \rangle}{l}$. So when $j = \langle -\zeta, \widetilde{\alpha} \rangle \geq 0$, we restrict r to $-\langle \lambda, \nu \rangle \leq r \leq \frac{\langle -\mu, \nu \rangle}{l}$, and when $j = 0 \geq \langle -\zeta, \widetilde{\alpha} \rangle$, we restrict r to $r \geq \max\{\frac{\langle -\mu, \nu \rangle}{l}, 0\}$.

□

We categorize the graphs into forms similar to the cocover case and show them below.



We define corners and β_α^- exactly as we did when considering cocovers.

Proposition 5.4.4. *If $y = s_\alpha x$ is a cover of x, then $\alpha = \nu + r\delta + j\pi$ must correspond to a corner in the graph $\Gamma'_{x,\nu}$.*

Now we would like to show that there are finitely many α that are corners relative to a fixed x, but before we do, we need to make some observations about the graphs:

- If (r, j) is a point of the graph, then (p, j) is a point of the graph for all $p > r$. Since (p, j) lies to the right of (r, j), either (p, j) will fall on the lower outer edge given by $j = 0$ or it will not fall on an outer edge. If $j = 0$, then $p > r \geq \max\{\frac{\langle -\mu, \nu \rangle}{l}, 0\}$, so (p, j) is in the graph. If $j \neq 0$, then $j > \max\{0, \langle -\zeta, \nu + p\delta \rangle\}$, so (p, j) is in the graph.

- If (r,j) is a point of the graph, then (r,q) is a point of the graph for all $q>j$. Since (r,j) is in the graph, we know $j \geq \max\{0, \langle -\zeta, \widetilde{\alpha}\rangle\}$, so $q>j\geq \max\{0,\langle -\zeta,\widetilde{\alpha}\rangle\}$ and (r,q) is in the graph.
- The outer diagonal edge is given by $j=-\langle\zeta,\nu+r\delta\rangle = -\langle\mu,\nu\rangle - rl$. The slope is $-l$, which is an integer (it comes from the level of $x's$ X weight). Additionally, for any integer r, $j=-\langle\zeta,\nu+r\delta\rangle$ is also an integer because $-\langle\zeta,\nu+r\delta\rangle = -\langle\mu,\nu\rangle - lr$ and $l, r, \langle\mu,\nu\rangle\in\mathbb{Z}$.
- If (r,j) is a point of the graph such that $j=-\langle\zeta,\nu+r\delta\rangle + k$ where k is some positive integer, and if (p,q) is a point such that $q=-\langle\zeta,\nu+p\delta\rangle+k$ and $q>j$, then (p,q) is in the graph. The point (p,q) cannot fall on any outer edges because $k\neq 0$ and $q>j\geq 0$. So $q\geq\max\{0,\langle-\zeta,\widetilde{\alpha}\rangle\}$ and (p,q) is in the graph.

PROPOSITION 5.4.5. *The number of corners of* $\Gamma'_{x,\nu}$ *is finite.*

PROOF. We break the proof into several cases. Let (r',j') represent a corner.

Case 1: Let $j'=0$. Then (r',j') falls along the lower outer edge and r' must be minimal. If $(p,0)$ is another point of the graph such that $p<r$ then $(p,0)$ rotated 180 degrees about $(r',0)$ would land in the graph (because it would be on $j=0$ and be further to the right than $(r',0)$), which would contradict the fact that $(r',0)$ is a corner.

Case 2: Let $j'=1$. Then (r',j') falls along the lower inner edge and r' must be minimal. If $(p,1)$ is another point of the graph such that $p<r$ then $(p,1)$ rotated 180 degrees about $(r',1)$ would land in the graph (similarly to the reasoning Case (1)), which would mean $(r',1)$ wouldn't be a corner.

Case 3: Let $j'=-\langle\zeta,\nu+r'\delta\rangle$. Then (r',j') lies on a line segment, so there are finitely many possibilities.

Case 4: Let $j'=-\langle\zeta,\nu+r'\delta\rangle+1$. Then (r',j') lies on the upper inner edge and j' must be minimal. If not, then there is another $(p,q)\in\Gamma'_{x,\nu}$ on the upper inner edge such that $q<j'$. In that case (p,q) could be rotated 180 degrees and land in the graph (along the same diagonal but higher on it), which means (r',j') wouldn't be a corner.

Case 5: Assume (r', j') is a point of $\Gamma'_{x,\nu}$ that is not on either of the two outer edges or the two inner edges. Then either $(r', -\langle \zeta, \nu + r'\delta\rangle + 1)$ is lower than (r', j') and is in the graph, or $(r', 1)$ is lower than (r', j') and is in the graph. In either case, this lower point can be rotated 180 degrees about (r', j') and land in the graph (along the same vertical line but higher), so (r', j') wouldn't be a corner.

So to be a corner, (r', j') must fall along one of the two outer edges or one of the two inner edges. On those edges there are finitely many possibilities for corners. Thus for any given x and ν, the corresponding graph $\Gamma'_{x,\nu}$ contains finitely many corners. $\square$

COROLLARY 5.4.6. *There are finitely many covers of a fixed $x \in W$.*

REMARK 5.4.7. Let $x = X^{\widetilde{v}\zeta}\widetilde{w} \in W$ with $\widetilde{w}, \widetilde{v} \in W_{\text{aff}}$ and $\zeta \in \mathcal{T}$, and let $y = s_{-\widetilde{v}\widetilde{\alpha}+j\pi}x$ where $\alpha = -\widetilde{v}\widetilde{\alpha} + j\pi$ is a positive double affine root and $\nu \in \Phi_{\text{fin}}$. If y is a cover of x, then one of the following must hold:

(1) $j = 0$ and $y = X^{\widetilde{v}s_{\widetilde{\alpha}}\zeta}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}$.
(2) $j = 1$ and $y = X^{\widetilde{v}s_{\widetilde{\alpha}}(\zeta-\widetilde{\alpha})}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}$.
(3) $j = -\langle \widetilde{v}\zeta, -\widetilde{v}\widetilde{\alpha}\rangle = \langle \zeta, \widetilde{\alpha}\rangle$ and $y = X^{\widetilde{v}\zeta}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}$.
(4) $j = -\langle \widetilde{v}\zeta, -\widetilde{v}\widetilde{\alpha}\rangle = \langle \zeta, \widetilde{\alpha}\rangle + 1$ and $y = X^{\widetilde{v}(\zeta+\widetilde{\alpha})}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}$.

THEOREM 5.4.8. *Let $x = X^{\widetilde{v}\zeta}\widetilde{w}$ and $y = s_{-\widetilde{v}\widetilde{\alpha}+j\pi}x$ where $\alpha = -\widetilde{v}\widetilde{\alpha} + j\pi$ is a positive double affine root and $\langle \zeta, \alpha_i\rangle > 2$ for $i = 0, 1, \ldots, n$. Then y is a cover of x if and only if one of the following holds:*

(1) $j = 0$ *and* $\ell(\widetilde{v}) = \ell(\widetilde{v}s_{\widetilde{\alpha}}) - 1$.
(2) $j = 1$ *and* $\ell(\widetilde{v}) = \ell(\widetilde{v}s_{\widetilde{\alpha}}) - 1 - \langle \widetilde{\alpha}, 2\rho\rangle$.
(3) $j = \langle \zeta, \widetilde{\alpha}\rangle$ *and* $\ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}) = \ell(\widetilde{w}^{-1}\widetilde{v}) - 1$.
(4) $j = \langle \zeta, \widetilde{\alpha}\rangle + 1$ *and* $\ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}) = \ell(\widetilde{w}^{-1}\widetilde{v}) - 1 + \langle \widetilde{\alpha}, 2\rho\rangle$.

5. The Affine Case

This method also works when looking for covers and cocovers for elements of W_{aff}.

Let $x = Y^{v\lambda}w \in W_{\text{aff}}$ with $\lambda \in Q$ and $w, v \in W_{\text{fin}}$, and let $\alpha = -v\nu + r\delta$ be a positive affine root such that $x^{-1}(\alpha) < 0$. Then $x \geq s_\alpha x$ by definition, and the length difference set

$$\widetilde{L}_{x,\alpha} = \{\beta > 0 : \ x^{-1}(\beta) < 0, \ s_\alpha(\beta) < 0, \ x^{-1}s_\alpha(\beta) > 0\}$$

contains only one element (namely α) if and only if $s_\alpha x$ is a cocover of x.

Definition 5.5.1. Let $\widetilde{\Gamma}_{x,\gamma}$ be the graph containing all $(p,0)$ such that $\gamma + p\delta > 0$ and $x^{-1}(\gamma + p\delta) < 0$.

Note that $(p,0)$ is a point of the graph if and only if $s_\beta x \le x$ where $\beta = \gamma + p\delta$. In this case we say β corresponds to a point of $\widetilde{\Gamma}_{x,\gamma}$.

Note that $(r,0)$ is a point of $\widetilde{\Gamma}_{x,-v\nu}$ if and only if the following hold

(1) $r > 0$ or $r = 0$ and $-v\nu > 0$

(2) $r < \langle \lambda, \nu \rangle$ or $r = \langle \lambda, \nu \rangle$ and $-w^{-1}v\nu < 0$.

This gives four types of graphs:

Definition 5.5.2. Define the **corners** of $\widetilde{\Gamma}_{x,\gamma}$ to be the points $(p,0)$ such that either there is no point of $\widetilde{\Gamma}_{x,\gamma} \cap (\mathbb{Z} \oplus \mathbb{Z})$ to the left of $(p,0)$ or there is no point of $\widetilde{\Gamma}_{x,\gamma} \cap (\mathbb{Z} \oplus \mathbb{Z})$ to the right of $(p,0)$. Another way of saying this would be to say no point of the graph can be rotated 180 degrees about $(p,0)$ and remain in the graph.

Proposition 5.5.3. *Let $x = Y^{v\lambda}w \in W_{\text{aff}}$ where $\lambda \in Q$ and $v, w \in W_{fin}$, and let $\alpha = -v\nu + r\delta$. If α is a corner of $\widetilde{\Gamma}_{x,-v\nu}$, then there are four possibilities for r:*

(1) $r = 0$,

(2) $r = 1$,

(3) $r = \langle \lambda, \nu \rangle$,

(4) $r = \langle \lambda, \nu \rangle - 1$.

Proposition 5.5.4. *Let $x, y \in W_{\text{aff}}$ and let $\alpha = -v\nu + r\delta \in \Phi_{\text{aff}}$ with $\nu \in \Phi_{\text{fin}}$. If $y = s_\alpha x$ is a cocover of x, then α must correspond to a corner of $\widetilde{\Gamma}_{x,-v\nu}$.*

The proof mirrors that of the double affine case and will not be included.

Proposition 5.5.5. *Let $x = Y^{v\lambda}w \in W_{\text{aff}}$ with $\lambda \in Q$ and $v, w \in W_{\text{fin}}$ and let $\alpha = -v\nu + r\delta$ be a positive affine root with $\nu \in \Phi_{\text{fin}}$ such that $x^{-1}(\alpha) < 0$. If $y = s_\alpha x$ is a cocover of x, then one of the following must hold:*

(1) $r = 0$ *and* $y = Y^{vs_\nu\lambda}s_{v\nu}w$,
(2) $r = 1$ *and* $y = Y^{vs_\nu(\lambda-\nu)}s_{v\nu}w$,
(3) $r = \langle\lambda, \nu\rangle$ *and* $y = Y^{v\lambda}s_{v\nu}w$,
(4) $r = \langle\lambda, \nu\rangle - 1$ *and* $y = Y^{v(\lambda-\nu)}s_{v\nu}w$.

Proof. We use the fact that α must correspond to a corner of $\widetilde{\Gamma}_{x,-v\nu}$ (and so has one of the four forms given by 5.5.3). The result becomes clear when we re-write y as

$$\begin{aligned} y &= s_{-v\nu+r\delta}Y^{v\lambda}w \\ &= Y^{rv\nu}s_{v\nu}Y^{v\lambda}w \\ &= Y^{rv\nu+s_{v\nu}v\lambda}s_{v\nu}w \\ &= Y^{vs_\nu(\lambda-r\nu)}s_{v\nu}w \\ &= Y^{v(\lambda-(\langle\lambda,\nu\rangle-r)\nu)}s_{v\nu}w. \end{aligned}$$

□

Proposition 5.5.6. *Let $x = Y^{v\lambda}w \in W_{\text{aff}}$ where $\lambda \in Q$ is dominant, $v, w \in W_{\text{fin}}$, and $\langle\lambda, \alpha_i\rangle > 2$ for $i = 1, 2, \ldots, n$. Let $-v\nu + r\delta$ be a positive affine root with $\nu \in \Phi_{\text{fin}}$. Then $y = s_{-v\nu+r\delta}x$ is a cocover of x if and only if one of the following holds:*

(1) $r = 0$ *and* $\ell(v) = \ell(vs_\nu) + 1$.
(2) $r = 1$ *and* $\ell(v) = \ell(vs_\nu) + 1 - \langle\nu, 2\rho\rangle$.
(3) $r = \langle\lambda, \nu\rangle$ *and* $\ell(w^{-1}vs_\nu) = \ell(w^{-1}v) + 1$.
(4) $r = \langle\lambda, \nu\rangle - 1$ *and* $\ell(w^{-1}vs_\nu) = \ell(w^{-1}v) + 1 - \langle\nu, 2\rho\rangle$.

Proof. Because λ is dominant and regular, we may use Proposition 2.4.9 to say $\ell(x) = \langle\lambda, 2\rho\rangle - \ell(w^{-1}v) + \ell(v)$. We can use a similar formula for $\ell(y)$ since $\lambda - \nu$ will also be dominant and regular.

Case One: If $r = 0$, then $y = Y^{vs_\nu\lambda} s_{v\nu} w$. So by Proposition 2.4.9,

$$\ell(y) = \langle \lambda, 2\rho \rangle - \ell((s_{v\nu} w)^{-1} v s_\nu) + \ell(v s_\nu) = \langle \lambda, 2\rho \rangle - \ell(w^{-1} v) + \ell(v s_\nu).$$

Then $\ell(x) - \ell(y) = \ell(v) - \ell(v s_\nu)$, and y is a cocover of x if and only if $\ell(v) = \ell(v s_\nu) + 1$.

Case Two: If $r = 1$, then $y = Y^{vs_\nu(\lambda - \nu)} s_{v\nu} w$. So by Proposition 2.4.9,

$$\ell(y) = \langle \lambda, 2\rho \rangle - \langle \nu, 2\rho \rangle - \ell(w^{-1} v) + \ell(v s_\nu).$$

Then $\ell(x) - \ell(y) = \ell(v) - \ell(v s_\nu) + \langle \nu, 2\rho \rangle$, and y is a cocover of x if and only if $\ell(v) = \ell(v s_\nu) + 1 - \langle \nu, 2\rho \rangle$.

Case Three: If $r = \langle \lambda, \nu \rangle$, then $y = Y^{v\lambda} s_{v\nu} w$. So by Proposition 2.4.9,

$$\ell(y) = \langle \lambda, 2\rho \rangle - \ell((s_{v\nu} w)^{-1} v) + \ell(v) = \langle \lambda, 2\rho \rangle - \ell(w^{-1} v s_\nu) + \ell(v).$$

Then $\ell(x) - \ell(y) = \ell(w^{-1} v s_\nu) - \ell(w^{-1} v)$, and y is a cocover of x if and only if $\ell(w^{-1} v s_\nu) = \ell(w^{-1} v) + 1$.

Case Four: If $r = \langle \lambda, \nu \rangle - 1$, then $y = Y^{v(\lambda - \nu)} s_{v\nu} w$. So by Proposition 2.4.9,

$$\ell(y) = \langle \lambda, 2\rho \rangle - \langle \nu, 2\rho \rangle - \ell(w^{-1} v s_\nu) + \ell(v).$$

Then $\ell(x) - \ell(y) = \ell(w^{-1} v s_\nu) - \ell(w^{-1} v) + \langle \nu, 2\rho \rangle$, and y is a cocover of x if and only if $\ell(w^{-1} v s_\nu) = \ell(w^{-1} v) + 1 - \langle \nu, 2\rho \rangle$. □

Appendix A: The Level Zero Elements

We have seen why using $X = Q + \mathbb{Z}\delta + \mathbb{Z}\Lambda_0$ instead of $\mathcal{T}$ when creating W will not work with our length function, but it is not immediately clear why we need higher levels (elements $x \in X$ with $\text{lev}(x) > 0$). When first examining the double affine case, we attempted to use Q_{fin}, a subset of the level zero elements. In doing so, we were able to define a double affine Weyl group that was generated by reflections (and when considering type A we were able to define double affine permutations), but we encountered problems with the Bruhat intervals. In this section, we will explain our first attempt and the problems we encountered.

1. Double Affine Weyl Group

Definition A.1.1. Let $Q_X = Q_Y = Q_{\text{fin}}$. We define the **double affine Weyl group** W^{XY} by

$$W^{XY} = (Q_X \times Q_Y) \rtimes W_{\text{fin}}.$$

An element of W^{XY} is written in the form $X^\mu Y^\lambda w$ where $\mu \in Q_X, \lambda \in Q_Y, w \in W_{\text{fin}}$.

Definition A.1.2. The set of **double affine roots** is defined as

$$\Phi = \{\nu + r\delta^Y + j\delta^X : \nu \in \Phi_{\text{fin}},\ r \in \mathbb{Z},\ j \in \mathbb{Z}\},$$

where δ_X corresponds to the δ used in $Q_X \rtimes W_{\text{fin}}$ and δ_Y corresponds to the δ used in $Q_Y \rtimes W_{\text{fin}}$.

We take the pairing $\langle \cdot, \cdot \rangle : Q_X \times Q_Y \to \mathbb{Z}$ to be the usual inner product on V and extend this to $\langle \cdot, \cdot \rangle : (Q_X \oplus \mathbb{Z}\delta_X) \times (Q_Y \oplus \mathbb{Z}\delta_Y) \to \mathbb{Z}$ by

$$\langle Q_X, \delta_Y \rangle = \langle \delta_X, Q_Y \rangle = \langle \delta_X, \delta_Y \rangle = 0.$$

Definition A.1.3. The **reflection** associated to $\alpha = \nu + r\delta^Y + j\delta^X$ is given by

$$s_{\nu+r\delta^Y+j\delta^X} = X^{-j\nu}Y^{-r\nu}s_\nu.$$

Definition A.1.4. We say a double affine root $\alpha = \nu + r\delta^Y + j\delta^X$ is **positive** if $\nu + r\delta^Y$ is a positive affine root and $j \geq 0$ or if $\nu + r\delta^Y$ is a negative affine root and $j > 0$. We denote the set of all such roots by Φ^+. We say α is **negative** if $\nu + r\delta^Y$ is a negative affine root and $j \leq 0$ or if $\nu + r\delta^Y$ is a positive affine root and $j < 0$. We denote the set of all such roots by Φ^-.

Remark A.1.5. By using this definition of positive and negative roots, we break the symmetry between X and Y because the coefficient for δ^X has more control on whether the root is positive or negative.

Let $\alpha = \nu + r\delta^Y + j\delta^X$ be a general element of Φ. Then W^{XY} acts on Φ by

$$\begin{aligned} X^\mu Y^\lambda w(\alpha) &= w(\nu) + (r - \langle w(\nu), \lambda\rangle)\delta^Y + (j - \langle w(\nu) + (r - \langle w(\nu), \lambda\rangle)\delta^Y, \mu\rangle)\delta^X \\ &= w(\nu) + (r - \langle w(\nu), \lambda\rangle)\delta^Y + (j - \langle w(\nu), \mu\rangle)\delta^X. \end{aligned}$$

Definition A.1.6. Let $\alpha \in \Phi$ and $\ddot{w} \in W^{XY}$. Then α is said to be an **inversion** of $\ddot{w}$ if $\alpha > 0$ and $\ddot{w}(\alpha) < 0$. We write $\mathrm{Inv}(\ddot{w})$ to denote the set of inversions of $\ddot{w}$.

Definition A.1.7. Define $s_0^X = X^\theta s_\theta = s_{-\theta+\delta^X}$ and $s_0^Y = Y^\theta s_\theta = s_{-\theta+\delta^Y}$. Then $s_0^X, s_0^Y \in W^{XY}$, and any $X^\mu Y^\lambda w \in W^{XY}$ can be written as a finite product of the reflections $s_0^X, s_0^Y, s_1, \ldots, s_n$. For this reason, we will call these reflections the **"simple reflections"**.

2. Permutations of the Plane

For the rest of this chapter, we specialize to the case of $W_{\text{fin}} = S_n$ acting on $Q_X = Q_Y = Q_{\text{fin}}(A_{n-1})$, the root lattice of type A_{n-1}. For simplicity, we will refer to Q_{fin} as Q, and we will refer to Q_X and Q_Y as Q; however, we will continue to use δ_X for the δ corresponding to Q_X and δ_Y for the δ corresponding to Q_Y to distinguish between the two.

So $S_n^{XY} := W^{XY} = (Q^X \times Q^Y) \rtimes S_n$ and an element $\ddot{w}$ of S_n^{XY} is of the form $X^\mu Y^\lambda w$ where $w \in S_n$ and $\mu, \lambda \in Q$.

When considering the affine case $\widetilde{S}_n$, we will use the notation $t(\lambda)$ to represent the translation by $\lambda \in Q$. For example, $s_0 \in \widetilde{S}_n$ will be written as $t(\theta)s_\theta$.

Proposition A.2.1. *The double affine permutations of S_n^{XY} can be viewed as permutations of the plane $\mathbb{Z}^2$.*

Proof. Every element of S_n^{XY} has an X-weight, a Y-weight, and a direction and we can associate it with the triple $(\mathrm{wt}_X, \mathrm{wt}_Y, w)$ where w is the direction. We can map S_n^{XY} to $\widetilde{S}_n \times \widetilde{S}_n$ by sending $(\mathrm{wt}_X, \mathrm{wt}_Y, w)$ to (w_X, w_Y) where $w_X = t(\mathrm{wt}_X)w, w_Y = t(\mathrm{wt}_Y)w$. Note that while the weights of w_X and w_Y may be different, the direction is the same for both w_X and w_Y.

This mapping allows us to view S_n^{XY} as a subgroup of $\widetilde{S}_n \times \widetilde{S}_n$, which can be realized as a subgroup of $S_{\mathbb{Z}} \times S_{\mathbb{Z}}$ and thus of $S_{\mathbb{Z}^2}$. $\square$

3. Muthiah's Length Function

In the double affine case, we can no longer use the order of the inversion set to define the length of an element because in some cases we have infinite inversion sets.

Example A.3.1. Consider $X^\theta \in S_3^{XY}$ and $\alpha = \alpha_1 + r\delta$ where $r \geq 0$. Then $\alpha > 0$ and $X^\theta(\alpha) = \alpha_1 + r\delta - \pi < 0$. So $\{\alpha_1 + r\delta : r \geq 0\} \subset \mathrm{Inv}(X^\theta)$.

Instead, we follow the work done by Muthiah [M, Section 4.3] to define a length function for W^{XY}.

Definition A.3.2. For $\ddot{w} = X^\mu Y^\lambda w \in W^{XY}$, we define the **length** of $\ddot{w}$ to be

$$\ell(\ddot{w}) = \ell_{\mathrm{big}}(\ddot{w}) + \ell_{\mathrm{small}}(\ddot{w})$$

where

$$\ell_{\mathrm{big}}(X^\mu Y^\lambda w) = \langle \mu_{+_{\mathrm{fin}}}, 2\rho_{\mathrm{aff}} \rangle,$$

with $\mu_{+_{\mathrm{fin}}}$ being the finite dominant weight associated to $\mu \in Q$, and

$$\ell_{\mathrm{small}}(X^\mu Y^\lambda w) = \left| \left\{ \widetilde{\alpha} = \nu + r\delta \in \mathrm{Inv}((Y^\lambda w)^{-1}) : \langle \mu, \widetilde{\alpha} \rangle \leq 0 \right\} \right|$$

$$
\begin{aligned}
&- \left|\left\{\widetilde{\alpha} = \nu + r\delta \in \mathrm{Inv}((Y^{\lambda}w)^{-1}) : \langle \mu, \widetilde{\alpha}\rangle > 0\right\}\right| \\
=&\left|\left\{\widetilde{\alpha} = \nu + r\delta \in \mathrm{Inv}((Y^{\lambda}w)^{-1}) : \langle \mu, \nu\rangle \le 0\right\}\right| \\
&- \left|\left\{\widetilde{\alpha} = \nu + r\delta \in \mathrm{Inv}((Y^{\lambda}w)^{-1}) : \langle \mu, \nu\rangle > 0\right\}\right|.
\end{aligned}
$$

Proposition A.3.3. *The big length of $\ddot{w} = X^{\mu}Y^{\lambda}w \in W^{XY}$ is equal to the number of roots of the form $\nu + j\delta^X$ that are positive and made negative under X^{μ}:*

$$\ell_{\mathrm{big}}(X^{\mu}Y^{\lambda}w) = \ell_{\mathrm{aff}}(t(\mu)).$$

Proof. By our definition of W^{XY}, $\mu \in Q$ (and so has no δ part), so $\langle \mu_{+_{\mathrm{fin}}}, \rho_{\mathrm{fin}}\rangle = \langle \mu_{+_{\mathrm{fin}}}, \rho_{\mathrm{aff}}\rangle$, and we have $\ell_{\mathrm{big}}(X^{\mu}Y^{\lambda}w) = \langle \mu_{+_{\mathrm{fin}}}, 2\rho_{\mathrm{aff}}\rangle = \langle \mu_{+_{\mathrm{fin}}}, 2\rho_{\mathrm{fin}}\rangle = \ell_{\mathrm{aff}}(t(\mu))$. □

Example A.3.4. We calculate the length of the generators for S_n^{XY}:

- Let $\ddot{w} = s_i$ for $i = 1, 2, \ldots, n$. Then $X^{\mu} = X^0$, so $\ell_{\mathrm{big}}(s_i) = 0$, and $\langle \mu, \widetilde{\alpha}\rangle = 0$ for all $\widetilde{\alpha} \in \Phi_{\mathrm{aff}}$. To calculate $\ell_{\mathrm{small}}(s_i)$, we only need to count the inversions of s_i. Therefore $\ell(s_i) = 1$.
- Let $\ddot{w} = s_0^Y$. Again we have $\mu = 0$, so $\ell_{\mathrm{big}}(s_0^Y) = 0$, and $\langle \mu, \widetilde{\alpha}\rangle = 0$ for all $\widetilde{\alpha} \in \Phi_{\mathrm{aff}}$. To calculate $\ell(s_0^Y)$, we count the inversions of $(s_0^Y)^{-1} = s_0^Y$, and again we have length 1.
- Let $\ddot{w} = s_0^X$. In this case $\mu = \theta$, so $\ell_{\mathrm{big}}(s_0^X) = \ell_{\mathrm{aff}}(t(\theta)) = 2n - 2$. This comes from counting the inversions of $t(\theta) \in W_{\mathrm{aff}}$.

  To calculate the small length, we examine the inversions of $s_{\theta}^{-1} = s_{\theta}$. The inversions of s_{θ} are $\epsilon_1 - \epsilon_j$ where $1 < j \le n$ and $\epsilon_i - \epsilon_n$ where $1 < i < n$. Since $\langle \theta, \epsilon_1 - \epsilon_j\rangle > 0$ for $1 < j \le n$ and $\langle \theta, \epsilon_i - \epsilon_n\rangle > 0$ for $1 < i < n$, $\ell_{\mathrm{small}}(s_0^X) = -(2n - 3)$. Therefore, $\ell(s_0^X) = 1$.

4. Elements with Length 0

We've checked that the length function matches what we believe the length of a generating element should be, but what about an element $\ddot{w}$ that isn't a generating element?

Example A.4.1. Consider S_2^{XY}. We've shown in Example A.3.4 that $\ell(s_0^X) = 1$ and $\ell(s_0^Y) = 1$. Now consider $w = s_0^X s_0^Y = X^{\theta}s_{\theta}Y^{\theta}s_{\theta} = X^{\theta}Y^{-\theta}$. This is generated by one copy of s_0^X and one copy of s_0^Y, so we would like it to have length 2, but when checking our length definition, we see $\ell(s_0^X s_0^Y) = 0$.

First note that $\ell_{\text{big}}(X^\theta Y^{-\theta}) = \ell_{\text{big}}(X^\theta) = \ell_{\text{aff}}(t(\theta)) = 2n - 2 = 2$.

To determine the small length, we need to look at the inversions of Y^θ.

$$\begin{aligned} Y^\theta(\theta + r\delta^Y) &= \theta + (r - \langle \theta, \theta \rangle)\delta^Y \\ &= \theta + (r-2)\delta^Y \end{aligned}$$

So the only inversions with finite part θ would need $0 \leq r < 2$.

$$\begin{aligned} Y^\theta(-\theta + r\delta^Y) &= -\theta + (r - \langle \theta, -\theta \rangle)\delta^Y \\ &= -\theta + (r+2)\delta^Y \end{aligned}$$

So the only inversions with finite part $-\theta$ would need $0 < r \leq -2$, which is impossible. So $\text{Inv}(Y^\theta) = \{\theta + \delta^Y, \theta\}$.

Since $\langle \theta, \theta + k\delta^Y \rangle = 2 > 0$ for k any integer, $\ell_{\text{small}}(X^\theta Y^\theta) = -2$.

Therefore $\ell(s_0^X s_0^Y) = \ell_{\text{big}}(X^\theta Y^{-\theta}) + \ell_{\text{small}}(X^\theta Y^{-\theta}) = 2 - 2 = 0$.

This example shows that the length function cannot be said to count the number of "simple reflections" (the reflections $s_0^X, s_0^Y, s_1, \ldots s_n$) required in a "reduced word".

5. Infinite Intervals

We have seen that some elements of W^{XY} have infinite inversion sets, but by definining our length function in a different fashion, we are able to obtain finite lengths for every element of W^{XY}. However, we run into problems when looking at intervals with respect to the Bruhat order.

Definition A.5.1. Let $\alpha \in \Phi$ be a positive root and $x, y \in W^{XY}$. We say $x \to xs_\alpha$ if $x(\alpha) > 0$. The **Bruhat order** is the partial order generated by these relations. So $x \leq y$ with respect to the Bruhat order if there is some chain

$$x \to xs_{\alpha_1} \to xs_{\alpha_1}s_{\alpha_2} \to \cdots \to xs_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_k} = y.$$

EXAMPLE A.5.2. Let $n = 2$ and consider S_n^{XY}. The interval $[X^{-\theta}Y^{2\theta}, X^{3\theta}Y^{\theta}]$ is infinite because it contains every element of the form $X^{-2\theta}s_\theta Y^{b\theta}$ where b is an integer.

First we'll show $X^{-\theta}Y^{2\theta} \leq X^{-2\theta}s_\theta Y^{b\theta}$. Let $x = X^{-\theta}Y^{2\theta}$ and $z = X^{-2\theta}s_\theta Y^{b\theta}$. Then

$$\begin{aligned} z = X^{-2\theta}s_\theta Y^{b\theta} &= X^{-\theta}s_{\theta+\delta^X}Y^{b\theta} \\ &= X^{-\theta}Y^{2\theta}s_{\theta+(2+b)\delta^Y+\delta^X} \\ &= xs_\alpha \end{aligned}$$

where $\alpha = \theta + (2+b)\delta^Y + \delta^X$, and

$$\begin{aligned} x(\alpha) &= X^{-\theta}Y^{2\theta}(\theta + (2+b)\delta^Y + \delta^X) \\ &= \theta + (2 + b - \langle 2\theta, \theta\rangle)\delta^Y + (1 - \langle \theta, -\theta\rangle)\delta^X \\ &= \theta + (b-2)\delta^Y + 3\delta^X, \end{aligned}$$

which is always positive since $3 > 0$. So $x \leq z$ because $z = xs_\alpha$ where $\alpha > 0$ and $x(\alpha) > 0$.

Next we'll show $X^{-2\theta}s_\theta Y^{b\theta} \leq X^{3\theta}Y^{\theta}$. Let $z = X^{-2\theta}s_\theta Y^{b\theta}$ and $y = X^{3\theta}Y^{\theta}$. Then

$$\begin{aligned} z = X^{-2\theta}s_\theta Y^{b\theta} &= X^{3\theta}s_{\theta+5\delta^X}Y^{b\theta} \\ &= X^{3\theta}Y^{\theta}s_{\theta+(b+1)\delta^Y+5\delta^X} \\ &= ys_\beta \end{aligned}$$

where $\beta = \theta + (b+1)\delta^Y + 5\delta^X$, and

$$\begin{aligned} y(\beta) &= X^{3\theta}Y^{\theta}(\theta + (b+1)\delta^Y + 5\delta^X) \\ &= \theta + (b + 1 - \langle \theta, \theta\rangle)\delta^Y + (5 - \langle 3\theta, \theta\rangle)\delta^X \\ &= \theta + (b-1)\delta^Y - 1\delta^X, \end{aligned}$$

which is always negative since $-1 < 0$. So $z \leq y$ since $z = ys_\beta$ and $y(\beta) < 0$.

So $\{X^{-2\theta}s_\theta Y^{b\theta} : b \in \mathbb{Z}\} \subset [X^{-\theta}Y^{2\theta}, X^{3\theta}Y^{\theta}]$, and we see that our definition of W^{XY} allows infinite Bruhat intervals.

Bibliography

[B] N. Bourbaki Lie Groups and Lie Algebras. Chapters 4-6. Elements of Mathematics (Berlin) Springer-Verlag, 2002. Translated from the 1968 French original by Andrew Pressley.

[BB] Anders Björner and Francesco Brenti Combinatorics of Coxeter Groups. Graduate Texts in Mathematics, 231. Springer Science+Business Media, Inc., 2005.

[BFP] Brenti, Fomin, Postnikov. Mixed Bruhat Operators and Yang-Baxter Equations for Weyl Groups. International Mathematics Research Notices. 1999, No. 8. arXiv:math/9805079

[BKP] A. Braverman, D. Kazhdan, and M. Patnaik. Iwahori-Hecke algebras for p-adic loop groups. Invent. Math. 204 (2016), no. 2, 347-442.

[D] M. J. Dyer, Hecke algebras and shellings of Bruhat intervals. Compositio Mathematic, tome 89, no. 1 (1993), p. 91-115.

[H] J. E. Humphreys. Reflection groups and Coxeter groups. Cambridge Studies in Advanced Mathematics, 29. Cambridge University Press, Cambridge, 1990.

[K] Victor G. Kac. Infinite dimensional Lie algebras. Third edition. Cambridge University Press, Cambridge, 1990.

[LS] T. Lam and M. Shimozono. Quantum cohomology of G/P and homology of affine Grassmannian. Acta Math, 204(1):49-90, 2010

[M] D. Muthiah. On Iwahori-Hecke algebras for p-adic loop groups: Double coset basis and Bruhat order. American Journal of Mathematics 140 (2018), no. 1, 221-224. arXiv:1502.00525

[Mi] E. Milićević. Maximal Newton points and the quantum Bruhat graph. arXiv:1606.07478

[MM] Augustin-Liviu Mare and Leonardo C. Mihalcea. An Affine Quantum Cohomology Ring For Flag Manifolds and the Periodic Toda Lattice. Proceedings of the London Mathematical Society (3) 116 (2018), no. 1, 135-181. arXiv:1409.3587

[MO] D. Muthiah and D. Orr. On the double-affine Bruhat order: the $\epsilon = 1$ conjecture and classification of covers in ADE type. arXiv:1609.03653