ON THE BEHAVIOR OF VISCOELASTIC PLATES IN BENDING

by

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

I

INTRODUCTION

It is well known that the analysis of stress and strain in solid bodies by means of the linear theory of elasticity yields inadequate results when applied to certain engineering materials which exhibit stress-strain relationships that are time-dependent. The increased use of high-polymers and plastics in modern structures, as well as the future requirements of metallic components to operate at higher and higher temperatures, points up the need for a more refined analysis in such applications. The simplest attempt to treat such materials in a more complete manner is to assume a linear relation between the stress, the strain and their derivatives with respect to time. The equations which express these relations are said to define a linear viscoelastic material. Analysis by such linear viscoelastic relations will account for such phenomena as instantaneous elastic response to stress, delayed elastic response and viscous flow occurring after removal of stress. High polymers and many of the plastics are excellent examples of materials which exhibit one or more of the above characteristics.

The general aspects of the various viscoelastic theories and the behavior of viscoelastic bodies under load have received considerable attention in recent years. The books of Freudenthal [11]*,

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*Numbers in square brackets refer to the Bibliography at the end of the paper.

Alfrey [1], Gross [13], Zener [24] and Reiner [20] afford notable examples of the scope and detail of treatment which the subject has received. Freudenthal develops the viscoelastic stress-strain relations from the point of view of thermodynamics. He discusses the application of simple spring and dashpot models to represent viscoelastic behavior and treats in some detail the one-dimensional case. In the book by Alfrey the viscoelastic nature of high polymers is examined and the need for more complex and higher order models to adequately represent such materials is brought out. The mathematical structure of viscoelasticity is developed in a thorough and rigorous manner in the work of Gross. The mathematical analogy between viscoelasticity and electrostatics is explained and discussed. He also considers model representation of viscoelastic media with emphasis on cataloging the various three parameter models. Zener's book contains a somewhat detailed account of the superposition principles as relating to viscoelasticity based on the early work of Boltzmann [8]. The broader field of rheology is the subject of Reiner's book which covers viscoelastic fluids as well as solids. Tensor notation is used throughout and multi-dimensional stress-strain relations are given.

Papers dealing with specific applications of the theory to practical problems are numerous. The books of Freudenthal [11] and Reiner [20] and the article by Pao and Brandt [18] all contain extensive bibliographies of recent work in the field.

The purpose of this investigation is to determine the response of flat thin plates made of a viscoelastic material when subjected to transverse loads. Only the quasi-static case is considered, i.e., inertia forces due to displacement are taken as small and therefore are negligible. The governing differential equation for such plates is developed by three independent methods. The variational method, which is used first, is the most "exact" of the methods and includes the effects of deflection due to shear. The quasi-static equilibrium method has the advantage of yielding, as by-products, expressions for the plate moments. The third approach is based on the correspondence principle and is the quickest and simplest method for arriving at the differential equation. For the case of proportional loading the equation is solved by operational techniques and the solution displayed in terms of the corresponding elastic solution. Under more general loading conditions solutions are obtained by combining a generalized energy approach with the operational techniques.

The mechanical properties of the plate material as expressed by G, the shear modulus, K, the bulk modulus, η, the coefficient of viscosity, are taken as constant for a given plate loading but will be different for different temperature conditions.

II

LINEAR VISCOELASTIC BEHAVIOR

A. General Equations for Viscoelastic Behavior

The simplest description of the inelastic behavior of a structural material under load is afforded by assuming a linear viscoelastic law relating the stress and strain and their respective time derivatives. In the one dimensional case such as a simple tension loading, for example, the relation between the stress σ and strain ϵ is given by the operator equation

$$P\sigma = 2Q\epsilon \tag{1}$$

where P and Q represent linear differential time operators having the form

$$P = \sum_{m=0}^{p} a_m \frac{\partial^m}{\partial t^m} \quad , \quad Q = \sum_{m=0}^{q} b_m \frac{\partial^m}{\partial t^m} \tag{2}$$

A straight forward generalization of Equation (1) to the multi-dimensional case for a homogeneous, isotropic body is given by [16]

$$\left.\begin{aligned} P\, s_{ij} &= 2Q\, e_{ij} \\ M\, \sigma_{ii} &= N \epsilon_{ii} \end{aligned}\right\} \tag{3}$$

where σ_{ij} and ϵ_{ij} are the components of the stress and strain tensors, respectively, the strain tensor considered to be infinitesimal here; and s_{ij} and e_{ij} are the components of the deviators of the stress and strain tensors, respectively, defined by

$$\left.\begin{aligned} s_{ij} &= \sigma_{ij} - \tfrac{1}{3}\,\delta_{ij}\,\sigma_{kk} \\ e_{ij} &= \epsilon_{ij} - \tfrac{1}{3}\,\delta_{ij}\,\epsilon_{kk} \end{aligned}\right\} \qquad (4)$$

where δ_{ij} is the Kronecker delta defined as

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

In the above equations and hereafter throughout this paper small Latin subscripts range over the values x, y and z as indicated, and as usual, a repeated subscript means summation with respect to that index. Thus in Equation 4, for example, when $i = j = x$

$$s_{xx} = \sigma_{xx} - \tfrac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

The operators P and Q in Equation 3 are the same as defined by Equation 2 and M and N are operators of the same form given by

$$M = \sum_{m=0}^{m} c_m \frac{\partial^m}{\partial t^m} \qquad , \qquad N = \sum_{m=0}^{l} d_m \frac{\partial^m}{\partial t^m}$$

The coefficients a_n, b_n, etc. appearing in the operators P, Q, M and N are considered constants throughout this investigation. They are actually the mechanical constants of the material whose behavior is specified by Equation 3. Taking these coefficients as constant somewhat restricts the applicability of Equation 3. In particular, thermal stress problems encountered in dealing with metals at elevated temperatures would involve variable coefficients in all cases where temperature gradients exist since the physical constants of metals are, in general, quite temperature sensitive. Certain restricted problems have been treated taking into account the temperature-dependent nature of the material constants. Freudenthal [12] has discussed thermal stress in flat plates under symmetrical temperature distribution and Hilton [14] considers the case of thermal stresses in thick wall cylinders.

It has been shown experimentally by the application of hydrostatic stress that changes of volume of a body are elastic in nature and since ϵ_{ii} represents the cubical dilatation for small strains the operators M and N in Equation 3 may be taken as constants for almost all materials of practical importance and the equation simplified to read

$$\left.\begin{aligned} P s_{ij} &= 2Q e_{ij} \\ \sigma_{ii} &= 3K\epsilon_{ii} \end{aligned}\right\} \qquad (5)$$

where K is the usual bulk modulus of elasticity theory.

It is of interest to note that Equation 5 incorporates Hooke's Law of the theory of elasticity as a special case. To see this take the operators P and Q as P = a_0 = 1 and Q = b_0 = G, where G is the shear modulus. The resulting equations express Hooke's Law.

$$\left.\begin{aligned} s_{ij} &= 2G e_{ij} \\ \sigma_{ii} &= 3K\epsilon_{ii} \end{aligned}\right\} \qquad (6)$$

Equation 5 may be combined using the relations given in Equation 4 to give

$$P\sigma_{ij} = 2Q\epsilon_{ij} + \delta_{ij}\frac{(3KP - 2Q)}{3}\epsilon_{kk} \qquad (7)$$

which may be re-written as

$$\sigma_{ij} = 2S\epsilon_{ij} + \delta_{ij}R\epsilon_{kk} \qquad (8)$$

with the operators S and R given by

$$S = \frac{Q}{P} \quad , \quad R = \frac{3KP - 2Q}{3P}$$

The elastic case is given by Equation 8 when the operators S and R are taken identically as the LAME constants, μ and λ respectively.

Viscoelastic behavior may also be specified by means of the so-called hereditary integral which may have the form

$$s_{ij} = 2G e_{ij} - 2G \int_0^t \varphi(t-T)\, e_{ij}\, dT \qquad (9)$$

The function φ is called the "memory function". If the memory function is taken as a sum of exponentials the hereditary integral formulation leads to a discrete spectrum of relaxation times as is the same case for the operational formulation of Equation 3. The integral formulation is essentially a superposition integral and was originally suggested by Boltzmann [24]. Volterra [23] and Brull [9] have used such a representation in their investigations.

It has been shown [10] that under quite general conditions the differential operator formulation of viscoelasticity such as given by Equation 3 and the hereditary integral formulation as given by Equation 9 are, in fact, one and the same.

B. Kelvin and Maxwell Representation.

Of the various specific forms that Equation 5 may assume the two that have found most application in engineering design are given by

$$\left.\begin{aligned} s_{ij} &= 2G\,e_{ij} + 2\eta\,\frac{\partial e_{ij}}{\partial t} \\ \sigma_{ii} &= 3K\,\epsilon_{ii} \end{aligned}\right\} \qquad (10)$$

which is said to define a Kelvin (or Voigt) solid and

$$\left.\begin{aligned} \frac{\partial s_{ij}}{\partial t} + \frac{G}{\eta}\,s_{ij} &= 2G\,\frac{\partial e_{ij}}{\partial t} \\ \sigma_{ii} &= 3K\,\epsilon_{ii} \end{aligned}\right\} \qquad (11)$$

which defines the so-called Maxwell solid. In these equations η is the coefficient of viscosity. Such linear viscoelastic materials are often represented by mechanical models which are designed to duplicate the time dependence of the material. Such models give a description only of the phenomenological behavior of the material but tell nothing of the mechanisms which cause this behavior.

Both of the above viscoelastic materials lend themselves especially well to representation by mechanical models made up of combinations of perfectly linear springs and dashpots containing a viscous liquid. The Kelvin model, for example, consists of a spring element coupled in parallel with a dashpot as shown in Figure 1.

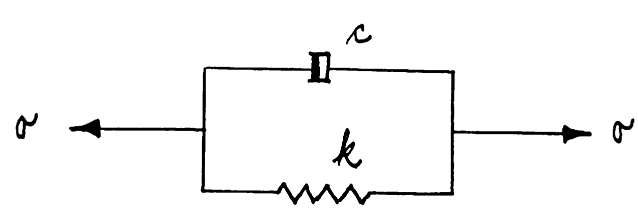

Figure 1. Kelvin Model

If the spring modulus is taken as k and the coefficient of viscosity of the fluid in the dashpot as c the force exerted by the spring when elongated an amount x will be $-kx$ and the force exerted by the dashpot will be $-c\dot{x}$. Therefore, if a constant force σ is applied to the model and maintained, equilibrium of forces demands that

$$\sigma = kx + c\dot{x}$$

which is of the same form as the first expression in Equation 10.

As can be seen from the model representation a Kelvin material behaves like a rigid body at the instant of load application; as time passes the strains asymptotically approach their final elastic values. A material which behaves in such a manner is said to exhibit delayed elasticity.

A model composed of a spring element and a dashpot in series characterizes a Maxwell body. If in Figure 2 which shows such a model

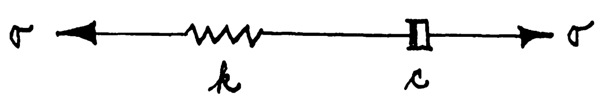

Figure 2. Maxwell Model

a constant force σ is applied and maintained the rate of elongation of the system $\dot{x}$ is equal to the rate of elongation of the spring $\dot{x}_1$ plus the rate of elongation $\dot{x}_2$ due to the dashpot displacement. Since the force σ acts across each element $\dot{x}_1 = \frac{\dot{\sigma}}{k}$ and $\dot{x}_2 = \frac{\sigma}{c}$ so that

$$\dot{x} = \frac{\dot{\sigma}}{k} + \frac{\sigma}{c}$$

which is analogous to the first expression in Equation 11. A Maxwell material therefore exhibits instantaneous elastic strain and viscous flow, as can be seen from its model in Figure 2.

More complex models can be developed by suitable coupling arrangements of Maxwell and Kelvin units together with spring and dashpot elements. The simplest model that could be used to represent a viscoelastic material possessing instantaneous elastic response, delayed elastic response and viscous flow would be a four parameter model consisting of a Kelvin and a Maxwell unit coupled in series. Extensive discussions of models and their place in viscoelastic theory appear in the books by Alfrey [1] and Gross [13]. The question of determining a suitable model from experimental data obtained from the steady state response of the test material to periodic loadings has recently been discussed by Bland and Lee [7].

DERIVATION OF FUNDAMENTAL EQUATION FOR VISCOELASTIC PLATE

A. Variational Method

In the following, the plates considered are assumed to be thin plates of constant thickness h, bounded by a contour of arbitrary shape. Taking the origin of a set of rectangular Cartesian coordinates at some point in the middle plane of the plate, the z axis is directed perpendicular to this plane and the x and y axes are taken in the plane. The deflection of the middle surface is given by w_o and is considered small compared to the thickness h. The plates are subjected to distributed transverse loads which are considered functions of the space coordinates x and y, and of time t.

An approach of considerable generality for deriving the equation governing the flexural deformation of a viscoelastic plate is afforded by the use of a variational principle stemming from the concepts of irreversible thermodynamics [4]. The principle involved is essentially a generalization of the principle of virtual work as it is used in the theory of elasticity. In the case of quasi-static problems in which inertia forces due to deformation are considered small and may be neglected the method is expressed by the equation

$$\delta J = \iint_S F_i \, \delta r_i \, dS \tag{12}$$

where J is the volume integral of the operational invariant I. That is

$$J = \iiint_V I \, dV$$

in which I is defined as

$$I = \frac{1}{2}\left(\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{zz}\epsilon_{zz}\right) + \sigma_{xy}\epsilon_{xy} + \sigma_{yz}\epsilon_{yz} + \sigma_{xz}\epsilon_{xz} \qquad (13)$$

where σ_{ij} and ϵ_{ij} are components of the stress and strain tensors respectively. The right hand side of Equation (12) is the virtual work of the surface forces, these being given by F_i. The virtual displacements are given by $\delta\eta_i$. In words, the principle expressed by Equation (12) states that the variation of the volume integral of a certain operational invariant is equal to the virtual work of the surface forces.

In applying the method to plate problems it is convenient to expand the displacements u, v and w in the x, y and z directions respectively as Taylor series in the coordinate z. Thus the displacement components are given by

$$u = \sum_{i=0}^{\infty} u_i z^i \quad , \quad v = \sum_{i=0}^{\infty} v_i z^i \quad , \quad w = \sum_{i=0}^{\infty} w_i z^i \qquad (14)$$

where the coefficients u_i, v_i and w_i are functions of x and y. The accuracy of the resulting equation for the plate deformation will depend, of course, upon the number of terms taken in the series expressions above. In general, however, the voluminous nature of the calculations involved makes it hardly worthwhile to include terms of higher order than z^3 .

Because of the geometry of flexural deformations of a plate further simplications of the displacement series seems justified for this case. In particular, it seems reasonable to take u and v as odd functions of z, and w as an even function of z. Under this premise the displacement components for the following derivation are taken as

$$\left.\begin{aligned} u &= u_1 z + u_3 z^3 \\ v &= v_1 z + v_3 z^3 \\ W &= W_0 + W_2 z^2 \end{aligned}\right\} \tag{15}$$

From these the components of strain may be calculated according to the usual definitions.

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \qquad \epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} \qquad \epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial W}{\partial y}\right)$$

$$\epsilon_{zz} = \frac{\partial W}{\partial z} \qquad \epsilon_{xz} = \frac{1}{2}\left(\frac{\partial W}{\partial x} + \frac{\partial u}{\partial z}\right)$$

Therefore

$$\left.\begin{aligned} \epsilon_{xx} &= \frac{\partial u_1}{\partial x} z + \frac{\partial u_3}{\partial x} z^3 , & \epsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial y} z + \frac{\partial u_3}{\partial y} z^3 + \frac{\partial v_1}{\partial x} z + \frac{\partial v_3}{\partial x} z^3\right) \\ \epsilon_{yy} &= \frac{\partial v_1}{\partial y} z + \frac{\partial v_3}{\partial y} z^3 , & \epsilon_{yz} &= \frac{1}{2}\left(v_1 + 3v_3 z^2 + \frac{\partial W_0}{\partial y} + \frac{\partial W_2}{\partial y} z^3\right) \\ \epsilon_{zz} &= 2 W_2 z , & \epsilon_{xz} &= \frac{1}{2}\left(u_1 + 3u_3 z^2 + \frac{\partial W_0}{\partial x} + \frac{\partial W_2}{\partial x} z^3\right) \end{aligned}\right. \tag{16}$$

If, now, a further restriction is introduced in the form of requiring the shear strains ϵ_{xz} and ϵ_{yz} to vanish on the top and bottom of the plate the unknowns u_3 and v_3 may be expressed in terms of u_1, v_1, w_o, and w_2 and considerable simplication of the ensuing calculations is achieved. Such an assumption has adequate physical justification in most applications of interest. Therefore, taking

$\epsilon_{xz} = \epsilon_{yz} = 0$ at $z = \pm \frac{h}{2}$ gives

$$
\begin{aligned}
u_3 &= -\frac{4}{3h^2}\left(u_1 + \frac{\partial w_0}{\partial x}\right) - \frac{1}{3}\frac{\partial w_2}{\partial x} \\
v_3 &= -\frac{4}{3h^2}\left(v_1 + \frac{\partial w_0}{\partial y}\right) - \frac{1}{3}\frac{\partial w_2}{\partial y}
\end{aligned}
\tag{17}
$$

from which now

$$
\begin{aligned}
\epsilon_{xx} &= \frac{\partial u_1}{\partial x} z - \left[\frac{4}{3h^2}\left(\frac{\partial u_1}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right) + \frac{1}{3}\frac{\partial^2 w_2}{\partial x^2}\right] z^3 \\
\epsilon_{yy} &= \frac{\partial v_1}{\partial y} z - \left[\frac{4}{3h^2}\left(\frac{\partial v_1}{\partial y} + \frac{\partial^2 w_0}{\partial y^2}\right) + \frac{1}{3}\frac{\partial^2 w_2}{\partial y^2}\right] z^3 \\
\epsilon_{zz} &= 2 w_2 z \\
\epsilon_{xy} &= \frac{1}{2}\left\{\left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right) z - \left[\frac{4}{3h^2}\left(\frac{\partial u_1}{\partial y} + 2\frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial v_1}{\partial x}\right) + \frac{1}{3}\frac{\partial^2 w_2}{\partial x \partial y} - \frac{\partial^2 w_2}{\partial x \partial y}\right] z^3\right\} \\
\epsilon_{yz} &= \frac{1}{2}\left\{v_1 + \frac{\partial w_0}{\partial y} + \left(\frac{\partial w_2}{\partial y} - \frac{4}{h^2}\left(v_1 + \frac{\partial w_0}{\partial y}\right) - \frac{\partial w_2}{\partial y}\right) z^2\right\} \\
\epsilon_{xz} &= \frac{1}{2}\left\{u_1 + \frac{\partial w_0}{\partial x} + \left(\frac{\partial w_2}{\partial x} - \frac{4}{h^2}\left(u_1 + \frac{\partial w_0}{\partial x}\right) - \frac{\partial w_2}{\partial x}\right) z^2\right\}
\end{aligned}
\tag{18}
$$

From the previous section the viscoelastic stress-strain relations are given by Equation (8) as

$$\sigma_{ij} = 2S\epsilon_{ij} + \delta_{ij} R \epsilon_{kk} \tag{8}$$

which when inserted into Equation (13) defining the operational invariant I yields

$$I = \frac{2S+R}{2}\left[\epsilon_{xx}^2 + \epsilon_{yy}^2 + \epsilon_{zz}^2\right] + R\left[\epsilon_{xx}\epsilon_{yy} + \epsilon_{yy}\epsilon_{zz} + \epsilon_{zz}\epsilon_{xx}\right]$$
$$+ 2S\left[\epsilon_{xy}^2 + \epsilon_{yz}^2 + \epsilon_{xz}^2\right]$$

If now the strain components of Equation (18) are substituted in this expression and terms of the order of z^4 and above are neglected the result is

$$I = \frac{2S+R}{2}\left[\left(\frac{\partial u_1}{\partial x}\right)^2 z^2 + \left(\frac{\partial v_1}{\partial y}\right)^2 z^2 + 4 w_2 z^2\right] + R\left[\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial y} z^2\right.$$
$$\left. + 2 w_2 \frac{\partial v_1}{\partial y} z^2 + 2 w_2 \frac{\partial u_1}{\partial x} z^2\right] + \frac{S}{2}\left[\left\{\left(\frac{\partial u_1}{\partial y}\right)^2 + 2\frac{\partial u_1}{\partial y}\frac{\partial v_1}{\partial x} + \left(\frac{\partial v_1}{\partial x}\right)^2\right\} z^2\right.$$
$$+ \left(v_1 + \frac{\partial w_0}{\partial y}\right)^2 + 2\left(v_1 + \frac{\partial w_0}{\partial y}\right)\left\{\frac{\partial w_2}{\partial y} - \frac{4}{h^2}\left(v_1 + \frac{\partial w_0}{\partial y}\right) - \frac{\partial w_2}{\partial y}\right\} z^2$$
$$\left. + \left(u_1 + \frac{\partial w_0}{\partial x}\right)^2 + 2\left(u_1 + \frac{\partial w_0}{\partial x}\right)\left\{\frac{\partial w_2}{\partial x} - \frac{4}{h^2}\left(u_1 + \frac{\partial w_0}{\partial x}\right) - \frac{\partial w_2}{\partial x}\right\} z^2\right]$$

Expanding this and integrating across the thickness of the plate gives J to be

$$J = \iint \Big\langle \frac{h^3}{12}\Big(\frac{2S+R}{2}\Big[\Big(\frac{\partial u_1}{\partial x}\Big)^2 + \Big(\frac{\partial v_1}{\partial y}\Big)^2 + 4 w_2^2 \Big] + R \Big[\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial y}$$

$$+ 2 w_2 \frac{\partial v_1}{\partial y} + 2 w_2 \frac{\partial u_1}{\partial x}\Big]\Big) + \frac{S}{2}\Big\{ \frac{h^3}{12}\Big[\Big(\frac{\partial u_1}{\partial y}\Big)^2 + 2\frac{\partial u_1}{\partial y}\frac{\partial v_1}{\partial x} + \Big(\frac{\partial v_1}{\partial x}\Big)^2$$

$$+ 2 v_1 \frac{\partial w_2}{\partial y} - \frac{8}{h^2} v_1^2 - \frac{8}{h^2} v_1 \frac{\partial w_0}{\partial y} - 2 v_1 \frac{\partial w_2}{\partial y} + 2 \frac{\partial w_0}{\partial y}\frac{\partial w_2}{\partial y}$$

$$- \frac{8}{h^2} v_1 \frac{\partial w_0}{\partial y} - \frac{8}{h^2}\Big(\frac{\partial w_0}{\partial y}\Big)^2 - 2 \frac{\partial w_0}{\partial y}\frac{\partial w_2}{\partial y} + 2 u_1 \frac{\partial w_2}{\partial x} - \frac{8}{h^2} u_1^2$$

$$- \frac{8}{h^2} u_1 \frac{\partial w_0}{\partial x} - 2 u_1 \frac{\partial w_0}{\partial x} + 2 \frac{\partial w_0}{\partial x}\frac{\partial w_2}{\partial x} - \frac{8}{h^2} u_1 \frac{\partial w_0}{\partial x}$$

$$- \frac{8}{h^2}\Big(\frac{\partial w_0}{\partial x}\Big)^2 - 2 \frac{\partial w_0}{\partial x}\frac{\partial w_2}{\partial x}\Big] + h\Big[v_1^2 + 2 v_1 \frac{\partial w_0}{\partial y} + \Big(\frac{\partial w_0}{\partial y}\Big)^2$$

$$+ u_1^2 + 2 u_1 \frac{\partial w_0}{\partial x} + \Big(\frac{\partial w_0}{\partial x}\Big)^2 \Big]\Big\}\Big\rangle \, dx\, dy$$

which reduces at once to

$$J = \iint \Big\langle \frac{h^3}{12}\Big\{ \frac{2S+R}{2}\Big[\Big(\frac{\partial u_1}{\partial x}\Big)^2 + \Big(\frac{\partial v_1}{\partial y}\Big)^2 + 4 w_2^2\Big] + R\Big[\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial y}$$

$$+ 2 w_2 \frac{\partial v_1}{\partial y} + 2 w_2 \frac{\partial u_1}{\partial x}\Big] + \frac{S}{2}\Big[\Big(\frac{\partial u_1}{\partial y}\Big)^2 + 2\frac{\partial u_1}{\partial y}\frac{\partial v_1}{\partial x} + \Big(\frac{\partial v_1}{\partial x}\Big)^2\Big]\Big\}$$

$$+ \frac{h}{3}\Big\{ \frac{S}{2}\Big[v_1^2 + 2 v_1 \frac{\partial w_0}{\partial y} + \Big(\frac{\partial w_0}{\partial y}\Big)^2 + u_1^2 + 2 u_1 \frac{\partial w_0}{\partial x} + \Big(\frac{\partial w_0}{\partial x}\Big)^2\Big]\Big\rangle \, dx\, dy$$

From this the variation of J is found to be

$$\delta J = \iint \Big\langle \frac{h^3}{12}\Big\{\frac{2S+R}{2}\Big[2\frac{\partial u_1}{\partial x}\frac{\partial \delta u_1}{\partial x} + 2\frac{\partial v_1}{\partial y}\frac{\partial \delta v_1}{\partial y} + 8 w_2\, \delta w_2\Big]$$

$$+ R\Big[\frac{\partial u_1}{\partial x}\frac{\partial \delta v_1}{\partial y} + \frac{\partial \delta u_1}{\partial x}\frac{\partial v_1}{\partial y} + 2 w_2 \frac{\partial \delta v_1}{\partial y} + 2\frac{\partial v_1}{\partial y}\delta w_2 + 2 w_2 \frac{\partial \delta u_1}{\partial x}$$

$$+ 2\frac{\partial u_1}{\partial x}\delta w_2\Big] + \frac{S}{2}\Big[2\frac{\partial u_1}{\partial y}\frac{\partial \delta u_1}{\partial y} + 2\frac{\partial u_1}{\partial y}\frac{\partial \delta v_1}{\partial x} + 2\frac{\partial \delta v_1}{\partial x}\frac{\partial v_1}{\partial x} + 2\frac{\partial \delta u_1}{\partial y}\frac{\partial v_1}{\partial x}\Big]\Big\}$$

$$+ \frac{h}{3}\Big\{\frac{S}{2}\Big[2 v_1\, \delta v_1 + 2 v_1 \frac{\partial \delta w_0}{\partial y} + 2\frac{\partial w_0}{\partial y}\delta v_1 + 2\frac{\partial w_0}{\partial y}\frac{\partial \delta w_0}{\partial y} + 2 u_1\, \delta u_1$$

$$+ 2 u_1 \frac{\partial \delta w_0}{\partial x} + 2\frac{\partial w_0}{\partial x}\delta u_1 + 2\frac{\partial w_0}{\partial x}\frac{\partial \delta w_0}{\partial x}\Big]\Big\}\Big\rangle\, dx\, dy$$

If the boundary conditions are such that the boundary forces and moments do no work Equation (12) can now be given explicitly as

$$\iint \Big\langle \Big[\frac{h^3}{12}\Big\{(2S+R)\Big(-\frac{\partial^2 u_1}{\partial x^2}\Big) - R\Big(\frac{\partial^2 v_1}{\partial x \partial y} + 2\frac{\partial w_2}{\partial x}\Big) - S\Big(\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial x \partial y}\Big)\Big\}$$

$$+\frac{h}{3} S\Big(u_1 + \frac{\partial w_0}{\partial x}\Big)\Big]\delta u_1 + \frac{h^3}{12}\Big[\Big\{(2S+R)\Big(-\frac{\partial^2 v_1}{\partial y^2}\Big) - R\Big(\frac{\partial^2 u_1}{\partial x \partial y} + 2\frac{\partial w_2}{\partial y}\Big)$$

$$- S\Big(\frac{\partial^2 u_1}{\partial x \partial y} + \frac{\partial^2 v_1}{\partial x^2}\Big)\Big\} + \frac{h}{3} S\Big(v_1 + \frac{\partial w_0}{\partial y}\Big)\Big]\delta v_1 - \frac{h}{3} S\Big(\frac{\partial^2 w_0}{\partial y^2} + \frac{\partial v_1}{\partial y} + \frac{\partial^2 w_0}{\partial x^2} + \frac{\partial u_1}{\partial x}\Big)\delta w_0$$

$$+\Big[\frac{h^3}{12}(2S+R)(-4w_2) + R\Big(2\frac{\partial v_1}{\partial y} + 2\frac{\partial u_1}{\partial x}\Big)\Big]\delta w_2\Big\rangle\, dx\, dy = \iint f\, \delta w_0\, dx\, dy$$

According to the variational calculus method in use here this equation leads to four partial differential (Euler) equations in u_1, v_1, w_0 and w_2 when coefficients of like variations are equated. The equations so obtained are

$$
\begin{aligned}
&-\frac{h^3}{12}\left\{(2S+R)\frac{\partial^2 u_1}{\partial x^2}+R\left(\frac{\partial^2 v_1}{\partial x\,\partial y}+2\frac{\partial w_2}{\partial x}\right)+S\left(\frac{\partial^2 u_1}{\partial y^2}+\frac{\partial^2 v_1}{\partial x\,\partial y}\right)\right\}+\frac{h}{3}S\left(u_1+\frac{\partial w_0}{\partial x}\right)=0\\
&-\frac{h^3}{12}\left\{(2S+R)\frac{\partial^2 v_1}{\partial y^2}+R\left(\frac{\partial^2 u_1}{\partial x\,\partial y}+2\frac{\partial w_2}{\partial y}\right)+S\left(\frac{\partial^2 u_1}{\partial x\,\partial y}+\frac{\partial^2 v_1}{\partial x^2}\right)\right\}+\frac{h}{3}S\left(v_1+\frac{\partial w_0}{\partial y}\right)=0\\
&-\frac{h}{3}S\left(\frac{\partial^2 w_0}{\partial y^2}+\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial v_1}{\partial y}+\frac{\partial u_1}{\partial x}\right)=f\\
&-\frac{h^3}{12}\left\{4(2S+R)w_2+2R\left(\frac{\partial v_1}{\partial y}+\frac{\partial u_1}{\partial x}\right)\right\}=0
\end{aligned}
\tag{19}
$$

Eliminating u_1, v_1 and w_2 from these equation (see Appendix) gives the fundamental equation for the flexural deformation of the middle surface of the plate, namely

$$\nabla^4 w_0 = \frac{f}{B_1} - \frac{\nabla^2 f}{Sh} \tag{20}$$

where ∇^2 is the Laplacian operator defined as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\nabla^4 w_0 = \nabla^2(\nabla^2 w_0) = \frac{\partial^4 w_0}{\partial x^4} + 2\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + \frac{\partial^4 w_0}{\partial y^4}$$

B_1 is the time operator given by

$$B_1 = \frac{h^3}{3} \frac{S(S+R)}{(2S+R)} \tag{21}$$

It should be noted that the second term on the right hand side of Equation (20) is the shear contribution to the deformation. If only the bending deformation is to be considered the equation can be simplified by dropping the shear deformation term $\frac{\nabla^2 f}{S h}$ to give

$$B_1 (\nabla^4 w_0) = f \tag{22}$$

B. Quasi-Static Equilibrium Method

The simplified form of the fundamental equation involving only the flexural deformation, Equation (22), may be derived by a straight forward application of the concept of instantaneous equilibrium of a small element of volume in much the same manner as is done in elastic plate theory. Beginning with the viscoelastic stress-strain relations as given by Equation (7)

$$P\sigma_{ij} = 2Q\,\epsilon_{ij} + \delta_{ij}\,\frac{3KP-2Q}{3}\,\epsilon_{kk} \tag{7}$$

and making the familiar assumption that the transverse normal stress component σ_{zz} is small compared to σ_{xx} , σ_{yy} and σ_{xy} and hence may be neglected gives ϵ_{zz} to be

$$\epsilon_{zz} = -\frac{3KP-2G}{3KP+4G}\,\epsilon_{\alpha\alpha} \qquad (\alpha = x, y)$$

and leads to the following modified stress-strain relation

$$\sigma_{ij} = \frac{2G}{P}\,\epsilon_{ij} + \delta_{ij}\,\frac{2G(3KP-2G)}{P(3KP+4G)}\,\epsilon_{\alpha\alpha} \qquad (\alpha = x, y) \tag{23}$$

If, now, the further assumption that the deflection of the middle surface of the plate w_o is small compared to the plate thickness "h" is introduced as is customary in classical plate theory the strain components ϵ_{xx} , ϵ_{yy} and ϵ_{xy} are given by

$$\epsilon_{\alpha\beta} = -z\,w_{o,\alpha\beta} \qquad (\alpha, \beta = x, y) \tag{24}$$

where the comma stands for differentiation with respect to the indicated coordinate. From this and Equation (23) σ_{xx} , σ_{yy} and σ_{xy} are expressed in terms of the deflection by the equation

$$\sigma_{\alpha\beta} = -z\left\{\frac{2G}{P}\,w_{o,\alpha\beta} + \delta_{\alpha\beta}\,\frac{2G(3KP-2G)}{P(3KP+4G)}\,w_{o,\gamma\gamma}\right\} \tag{25}$$

These stress components are shown acting on the sides of the elementary volume in Figure 3a

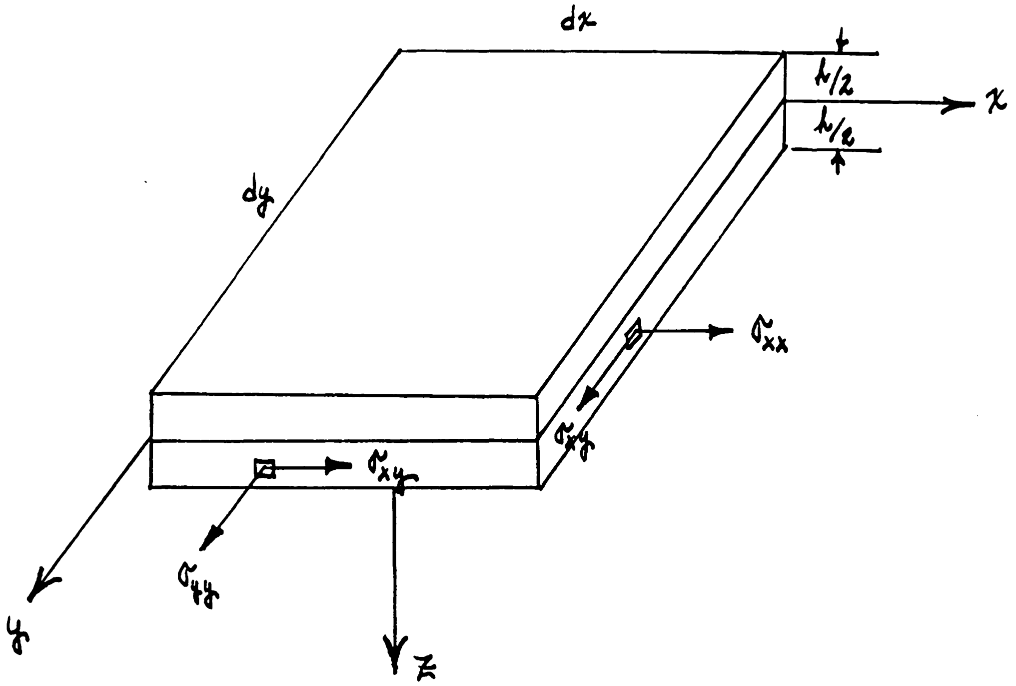

Figure 3. Plate Element Stresses

The plate moments $M_{\alpha\beta}$, defined in terms of the stress components in the usual way by

$$M_{\alpha\beta} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} z\,\sigma_{\alpha\beta}\,dz$$

are given by

$$M_{\alpha\beta} = -\frac{h^3}{12}\left\{\frac{2Q}{P} w_{0,\alpha\beta} + \delta_{\alpha\beta}\frac{2Q(3KP-2Q)}{P(3KP+4Q)} w_{0,\gamma\gamma}\right\} \tag{26}$$

and shown in their positive sense in Figure 4.

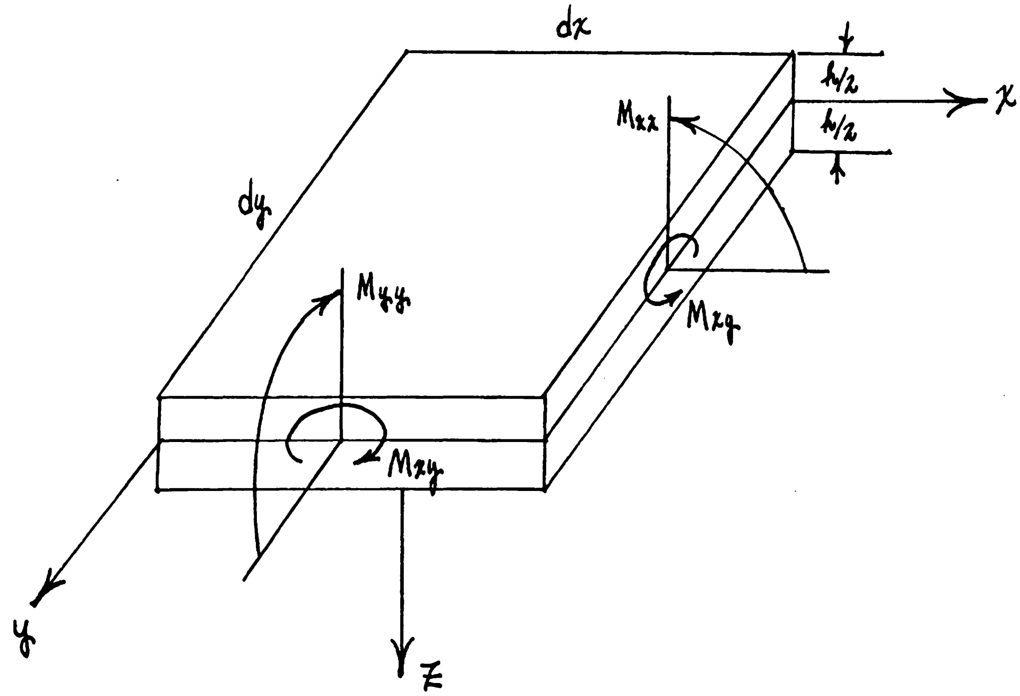

Figure 4. Plate Element Moments

Equilibrium of the element of volume may be expressed by the equation (see 8 , page 87)

$$M_{\alpha\beta,\alpha\beta} = -f$$

where f is the load intensity. Performing the indicated differentiation here gives

$$M_{\alpha\beta,\alpha\beta} = -\frac{h^3}{12}\left\{\frac{2Q}{P} w_{o,\alpha\beta\alpha\beta} + \delta_{\alpha\beta}\frac{2Q(3KP-2Q)}{P(3KP+4Q)} w_{o,\gamma\gamma\alpha\beta}\right\} = -f$$

which reduces immediately to

$$\left\{\frac{h^3}{3}\frac{Q(3KP+Q)}{P(3KP+4Q)}\right\}\nabla^4 w_o = f \qquad (27)$$

This is equivalent to Equation (22) derived by variational methods in the previous section.

C. Principle of Correspondence

The fact that the stress-strain equations of the theory of elasticity are incorporated as a special case of the fundamental stress-strain relations of viscoelasticity has lead a number of investigators to formulate analogies between the two fields with the idea of extending the vast literature of elastic solutions to viscoelastic problems. Alfrey's [2] analogy for the incompressible case has been extended to cover compressible media by Tsien [22]. Biot [5] has given a principle of correspondence based upon the formal analogy of the operational tensor of viscoelastic theory and the elastic moduli of elastic theory. The papers of Lee [16] and Radok [19] contain similar methods and have examples worked out for specific viscoelastic bodies.

All of the above mentioned analogies, in effect, simply replace the elastic constants in a given elasticity equation by differential time operators thus leading to differential equations in time which must be solved for the viscoelastic solution. To show one way this method can be developed [19] consider the basic, quasi-static equations of linear isotropic, elastic or viscoelastic media, namely, the equilibrium equations

$$\sigma_{ij,j} + F_i = 0 \tag{28}$$

the stress-strain relations (repeated here)

$$\left.\begin{aligned} P s_{ij} &= 2 Q e_{ij} \\ M \sigma_{ii} &= N \epsilon_{ii} \end{aligned}\right\} \qquad (3)$$

$$\left.\begin{aligned} s_{ij} &= \sigma_{ij} - \tfrac{1}{3}\delta_{ij}\sigma_{kk} \\ e_{ij} &= \epsilon_{ij} - \tfrac{1}{3}\delta_{ij}\epsilon_{kk} \end{aligned}\right\} \qquad (4)$$

where

$$P = \sum_{n=0}^{p} a_n \frac{\partial^n}{\partial t^n} \qquad Q = \sum_{n=0}^{q} b_n \frac{\partial^n}{\partial t^n}$$

$$M = \sum_{n=0}^{m} c_n \frac{\partial^n}{\partial t^n} \qquad N = \sum_{n=0}^{s} d_n \frac{\partial^n}{\partial t^n}$$

and the strain-displacement equation

$$2\epsilon_{ij} = r_{i,j} + r_{j,i} \qquad (29)$$

r_i being the displacement components. To completely specify an elastic or viscoelastic problem the above equations must be accompanied by boundary conditions given either as prescribed forces T_i on the Boundary L

$$\sigma_{ij}\lambda_j = T_i \qquad \text{ON } L$$

where the λ_j are the direction cosines of the outer normal to L ; or as prescribed displacements g_i on the boundary

$$g_i = r_i \quad \text{on } L_1$$

or some combination of these. In addition, in viscoelastic problems initial conditions have to be satisfied, the number of which depends on the orders of the operators P, Q, M and N.

As has already been shown, when the operators P, Q, M and N are all of zero order Equations (3) and (4) represent the case of an elastic material. For this the constants are

$$\frac{Q}{P} = \frac{b_0}{a_0} = G \quad , \quad \frac{N}{M} = \frac{d_0}{c_0} = 3K$$

Assuming for the purpose of simplicity that the viscoelastic problem has zero initial conditions and that therefore the surface forces and displacements are zero for t < 0, the time dependence of Equations (28) and (3) may be removed by application of the Laplace transform. Then

$$\bar{\sigma}_{ij,j} + \bar{F}_i = 0 \tag{30}$$

$$\left.\begin{aligned} \bar{P}\bar{s}_{ij} &= 2\bar{Q}\bar{e}_{ij} \\ \bar{M}\bar{\sigma}_{ii} &= \bar{N}\bar{\epsilon}_{ii} \end{aligned}\right\} \tag{31}$$

where bars denote transformed quantities. The operators P, Q, M and N become simple polynomials in the Laplace transform vairable. Since Equation (31) also refers to the elastic case when

$$\breve{P} \equiv a_0 \quad , \quad \breve{Q} \equiv b_0 \quad , \quad \breve{M} \equiv c_0 \quad , \quad \breve{N} \equiv d_0 \tag{32}$$

the Laplace transform of the viscoelastic solution may be obtained from that of the elastic solution by replacing the constants of Equation (32) by appropriate polynomials in the transform variable in Equation (31). Inversion of the transform so obtained gives the viscoelastic solution. In view of the above conclusions an alternate approach is also justified. Instead of applying the Laplace transform to the elastic equations the elastic constants a_0, b_0, c_0, d_0 are simply replaced by their corresponding time operators to yield functional equations in the variable time which may now be integrated to give the viscoelastic solution.

To illustrate this procedure consider the classical elastic plate deflection equation (see [8] page 88)

$$\nabla^4 w_0 = \frac{f}{D} \tag{33}$$

where D is the flexural rigidity of theplate given by

$$D = \frac{E h^3}{12(1-\nu^2)}$$

or equivalently by

$$D = \frac{h^3}{3} \frac{G(3K+G)}{(3K+4G)}$$

which in terms of the operator coefficients of order zero is

$$D = \frac{h^3}{3} \frac{\frac{b_0}{a_0}\left(\frac{d_0}{c_0} + \frac{b_0}{a_0}\right)}{\left(\frac{d_0}{c_0} + 4\frac{b_0}{a_0}\right)}$$

Replacing these constants by their parent operators gives the equivalent time operator for the flexural rigidity D.

$$D = \frac{h^3}{3} \frac{\frac{Q}{P}\left(3K + \frac{Q}{P}\right)}{\left(3K + 4\frac{Q}{P}\right)} = \frac{h^3}{3} \frac{Q(3KP+Q)}{P(3KP+4Q)}$$

When this is substituted into Equation (33) the resulting equation

$$\nabla^4 w_0 = \frac{f}{\frac{h^3}{3} \frac{Q(3KP+Q)}{P(3KP+4Q)}}$$

is identical with those obtained previously by the other methods.

IV

SOLUTIONS OF THE FUNDAMENTAL VISCOELASTIC PLATE EQUATION

A. Proportional Loading

For a completely arbitrary load function f, the fundamental viscoelastic plate equation

$$B_1\{\nabla^4 w_o(x,y,t)\} = f(x,y,t) \tag{22}$$

is quite formidable and not easy to solve. However, for the particular case of so-called "proportional loading", i.e. when the load function f (x,y.t) is given as the product of a space function and a time function, as for example,

$$f(x,y,t) = F(x,y)\,\Theta(t) \tag{34}$$

the variables separate and the solution of the resulting differential equations leads to an expression for w_o, also in the form of a space function multiplied by a time function. Therefore, if for this special but very common type of loading

$$w_o(x,y,t) = W(x,y)\,\varphi(t) \tag{35}$$

is assumed as a solution of Equation (22) and is substituted back into that equation the result is

$$B_1\{\varphi\}\,\nabla^4 W = F\,\Theta$$

or dividing through by $(\nabla^4 W)\Theta$

$$\frac{B_1\{\varphi\}}{\theta} = \frac{F}{\nabla^4 W}$$

Since the left hand side of this equation is a function of t only and the right hand side a function of x and y only they must both be equal to the same constant. Taking this constant to be D, the flexural rigidity of the plate, for the purpose of obtaining the deflection in an especially convenient form gives the following two differential equation

$$\nabla^4 W = \frac{F}{D} \tag{36}$$

$$B_1\{\varphi\} = D\,\theta \tag{37}$$

The first of these is the bi-harmonic plate equation of the classical elastic theory. A large number of solutions of this equation for a variety of plate loadings and boundary conditions have been worked out and are readily available [21]. In general, the majority of these solutions may be expressed in the form

$$W = w_e = \frac{q(x,y)}{D} \tag{38}$$

which form will be adopted here also. The term q(x,y) is quite general and may be a finite or infinite series, or other expression.

A convenient method for obtaining solutions of Equation (37) is afforded by the Laplace transformation method. If it is assumed that the plate is at rest in its straight, undeflected position and free of all stress and strain at the instant of application of the load; an assumption that is frequently valid in problems of practical interest, the operator B_1 transforms into a rational function in s, the transform variable. Therefore the transform of Equation (37) may be written

$$\frac{g(s)}{h(s)}\,\bar{\varphi} = D\,\bar{\theta} \tag{39}$$

where barred quantities are the transforms of unbarred quantities and s is the transform parameter, i.e.

$$\bar{\theta}(s) = \mathcal{L}[\theta(t)] \equiv \int_0^\infty \theta(t)\, e^{-st}\, dt$$

Solving Equation (39) for $\bar{\varphi}$ and expanding the quotient $\frac{h(s)}{g(s)}$ of polynomials in s by partial fractions results in

$$\bar{\varphi} = \left[\sum_{i=1}^{M} \frac{A_i}{r_i + s} + A_0 \right] D\,\bar{\theta}$$

where all of the roots r_i can be shown to be real [3]. This equation may be inverted at once by means of the convolution integral to give the general solution of Equation (39) as

$$\varphi = D \int_0^t \left[A_0\,\delta(T) + \sum_{i=1}^{M} A_i\, e^{-r_i T} \right] \theta(t-T)\, dT \tag{40}$$

where $\delta(T)$ is the Dirac delta function and e is the base of the natural logarithm system. From this, taken together with the elastic solution as given by Equation (38) the viscoelastic plate solution becomes

$$w_o(x,y,t) = w_e D \int_0^t \left[A_o \delta(T) + \sum_{i=1}^{n} A_i e^{-r_i T} \right] \theta(t-T)\, dT \quad (41)$$

Further study of this solution is accomplished by evaluation of the integral which depends on the particular form of the loading function f and the expression in the square brackets. The bracketed term comes from the operator B_1 which varies according to the visco-elastic material it represents. In the particularly useful cases of a Kelvin or a Maxwell material its specific character may be determined from Equation (10) and Equation (11), respectively. Thus for a Kelvin plate

$$P = 1 \;, \; \mathcal{G} = G + \eta p$$

where $p = \frac{\partial}{\partial t}$ and η is the coefficient of viscosity; therefore B_1 is expressed as

$$B_1 = \frac{h^3}{3} \frac{(G+\eta p)(3K+G+\eta p)}{(3K+4G+4\eta p)} \quad (42)$$

For a Maxwell plate

$$P = p + \frac{1}{\tau} \;, \; \mathcal{G} = G p$$

where $\tau = \eta/G$ is called the relaxation (sometimes retardation) time, defined as the time for the stress in a tension specimen of the material to relax to 1/e of its original value under a constant strain. Here B_1 is expressed as

$$B_1 = \frac{h^3}{3} \frac{G p \left(\frac{3K}{\tau} + (3K+G) p \right)}{\left(p + \frac{1}{\tau}\right)\left(\frac{3K}{\tau} + (3K+4G) p \right)} \tag{43}$$

Having these expressions for B_1, it is now possible to evaluate the integral for a Kelvin or Maxwell plate for a given time dependence θ of the load function. Take as an example a Kelvin plate of any shape for which the elastic solution is known and loaded by a load applied suddenly at time = 0 and maintained constant thereafter. For this, θ is defined by

$$\begin{aligned} \theta(t) &= 0 \qquad & t < 0 \\ &= 1 \qquad & t > 0 \end{aligned}$$

and Equation (41) becomes

$$w_0 = \frac{3 D w_e}{\eta h^3} \int_0^t \left[e^{-\frac{G}{\eta} T} + 3 e^{-\frac{3K+G}{\eta} T} \right] dT$$

which can be integrated directly to give

$$w_0(x,y,t) = w_e \left\{ 1 - \frac{3K+G}{3K+4G} e^{-\frac{t}{\tau}} - \frac{3G}{3K+4G} e^{-\frac{3K+G}{G} \frac{t}{\tau}} \right\} \tag{44}$$

expressing the viscoelastic solution in terms of the elastic solution of a similarly loaded plate. This form proves especially adaptable for comparing the deflection of the Kelvin (viscoelastic) plate with the

t/τ	W_o/W_e	
	LOAD ON	LOAD OFF
0	0.000	—
1	0.754	—
2	0.910	0.910
3	0.967	0.213
4	0.990	0.078

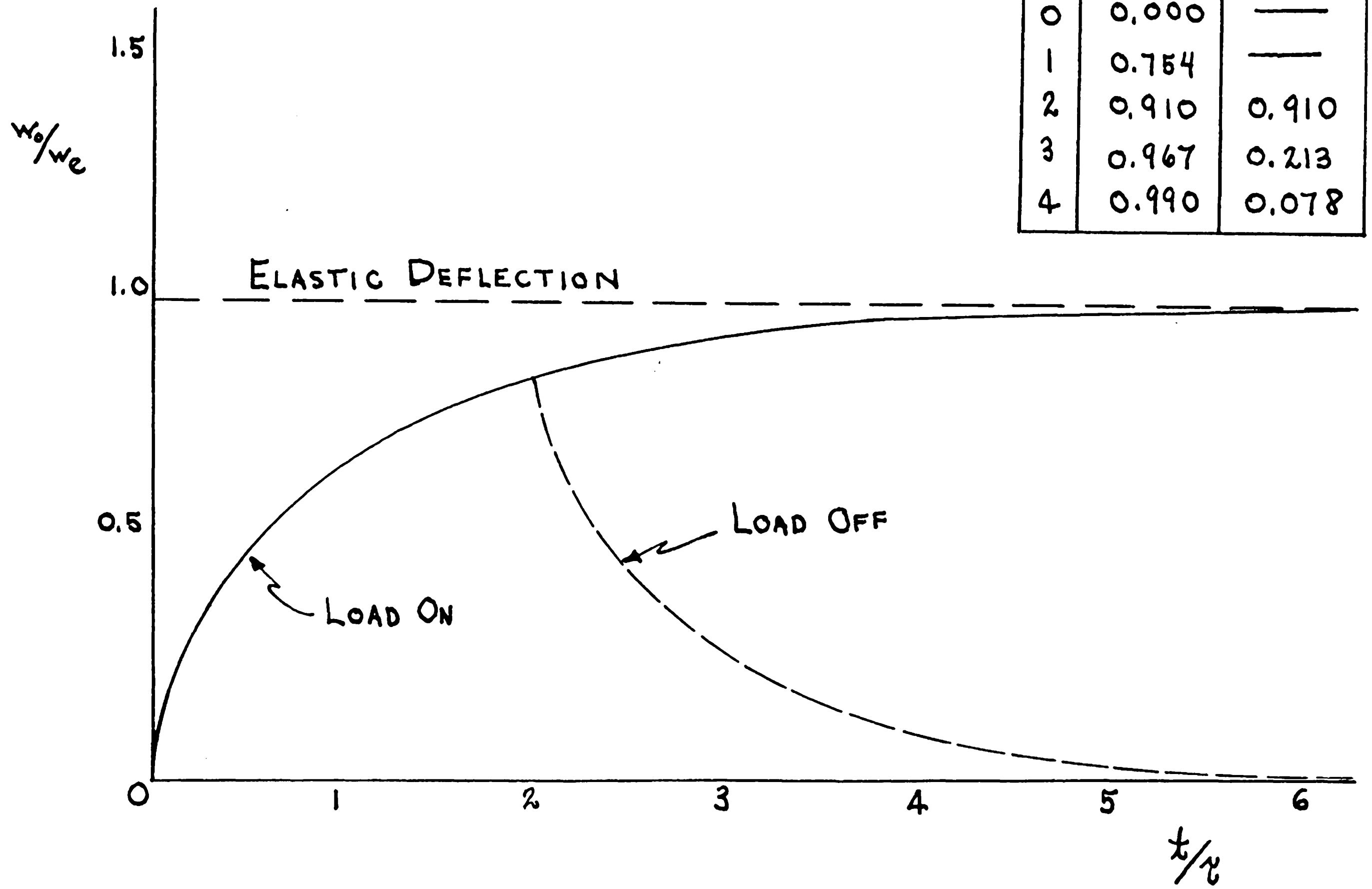

Figure 5. Kelvin Plate Deflection

static deflection of the elastic plate. To illustrate this a plot of the dimensionless deflection W_o/W_e against the dimensionless time t/τ for a plate having a Poisson's ratio of 0.25 is shown in Figure 5. From this it can be observed that the Kelvin plate behaves, (as would be expected from its model representation) as a rigid body at time $t = 0$ and with the deflection approaching asymtotically the elastic deflection as $t \to \infty$.

A rather interesting variation of the problem just discussed is presented when the load is not maintained indefinitely but is suddenly removed at some $t = t_1$. In this case θ is given by

$$\begin{aligned} \theta(t) &= 1 \qquad 0 < t < t_1 \\ &= 0 \qquad t > t_1 \end{aligned}$$

a plot of which is shown in Figure 6.

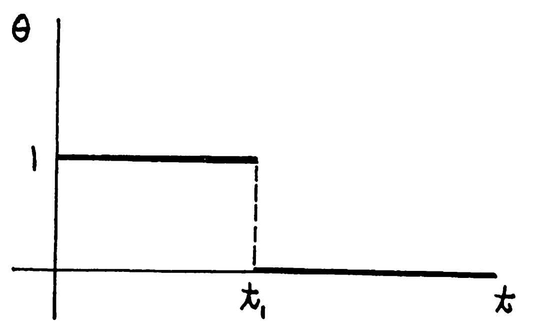

Figure 6. Step Loading Plot

Putting this θ in Equation (41) and evaluating the integral the solution for W_o becomes, as before

$$W_o = W_e \left\{ 1 - \frac{3K+G}{3K+4G} e^{-\frac{t}{\tau}} - \frac{3G}{3K+4G} e^{-\frac{3K+G}{G}\frac{t}{\tau}} \right\}$$

for $0 < t < t_1$

and

$$w_o = w_e \frac{G(3K+G)}{3K+4G}\left\{-\frac{(1-e^{\frac{t_1}{\tau}})e^{\frac{t}{\tau}}}{G} - \frac{3(1-e^{\frac{3K+G}{G}\frac{t_1}{\tau}})e^{-\frac{3K+G}{G}\frac{t}{\tau}}}{3K+G}\right\} \quad (45)$$

for $t > t_1$. The dashed curve in Figure 5 is a plot of this latter equation for the case when $t_1 = 2\tau$ and shows the behavior of the plate after the load is removed. Theoretically, the plate will return to its undeflected position only as $t \to \infty$ but for practical purposes, as can be seen from the curve the residual deflection reaches an insignificant value rather rapidly.

The plate moments $M_{\alpha\beta}$, which are also often an important part of the solution of a plate problem can be readily deduced for the above example by introducing the deflection w_o into Equation (26) and solving the resulting equation by operational methods. Using the known expressions for the operators P and Q in the case of a Kelvin material Equation (26)

$$M_{\alpha\beta} = -\frac{h^3}{12}\left\{\frac{2Q}{P} w_{o,\alpha\beta} + \delta_{\alpha\beta}\frac{2Q}{P}\frac{(3KP-2Q)}{(3KP+4Q)} w_{o,\gamma\gamma}\right\}$$

becomes

$$M_{\alpha\beta} = -\frac{h^3}{6}\left\{(G+\eta p) w_{o,\alpha\beta} + \delta_{\alpha\beta}\frac{(G+\eta p)(3K-2G-2\eta p)}{(3K+4G+4\eta p)} w_{o,\gamma\gamma}\right\}$$

Taking the Laplace transform of this equation and assuming as before zero initial conditions gives

$$\bar{M}_{\alpha\beta} = -\frac{D}{2}\left\{ \frac{\frac{3K+4G}{7} + 4s}{\left(\frac{3K+G}{7} + s\right)s} w_{e,\alpha\beta} + \delta_{\alpha\beta}\, \frac{\frac{3K-2G}{7} - 2s}{\left(\frac{3K+G}{7} + s\right)s} w_{e,\gamma\gamma} \right\}$$

Expanding this by partial fractions results in

$$\bar{M}_{\alpha\beta} = -\frac{D}{2}\left\{ \left(\frac{3K+4G}{(3K+G)s} + \frac{9K}{(3K+G)\left(\frac{3K+G}{7} + s\right)} \right) w_{e,\alpha\beta} \right.$$

$$\left. + \delta_{\alpha\beta}\left(\frac{3K+2G}{(3K+G)s} + \frac{9K}{(3K+G)\left(\frac{3K+G}{7} + s\right)} \right) w_{e,\gamma\gamma} \right\}$$

which may be inverted directly to give the viscoelastic plate moments in terms of the corresponding elastic curvatures

$$M_{\alpha\beta} = -\frac{D}{2}\left\{ \left(\frac{3K+4G}{3K+G} + \frac{9K}{3K+G}\, e^{-\frac{3K+G}{7}t} \right) w_{e,\alpha\beta} \right.$$

$$\left. + \delta_{\alpha\beta}\left(\frac{3K-2G}{3K+G} - \frac{9K}{3K+G}\, e^{-\frac{3K+G}{7}t} \right) w_{e,\gamma\gamma} \right\} \tag{46}$$

Writing these three equations in their individual detail puts them in the most suitable form for comparison with the elastic moments. Therefore from Equation (46)

$$M_{xx} = -\frac{D}{2}\left\{ 2w_{e,xx} + \left(\frac{3K-2G}{3K+G} - \frac{9K}{3K+G}\, e^{-\frac{3K+G}{7}t} \right) w_{e,yy} \right\}$$

$$M_{yy} = -D\left\{ w_{e,yy} + \frac{1}{2}\left(\frac{3K-2G}{3K+G} - \frac{9K}{3K+G}\, e^{-\frac{3K+G}{7}t} \right) w_{e,xx} \right\} \tag{47}$$

$$M_{xy} = -\frac{D}{2}\left\{ \frac{3K+4G}{3K+G} + \frac{9K}{3K+G}\, e^{-\frac{3K+G}{7}t} \right\} w_{e,xy}$$

Consider M_{xx}. As $t \to \infty$

$$M_{xx} \to -D\left\{w_{e,xx} + \frac{3K-2G}{2(3K+G)} w_{e,yy}\right\}$$

or

$$M_{xx} \to -D\left\{w_{e,xx} + \nu\, w_{e,yy}\right\}$$

which is exactly the elastic plate moment M_{xx}. When $t = 0$, M_{xx} has the form

$$M_{xx} = -D\left\{w_{e,xx} - w_{e,yy}\right\}$$

which corresponds to the moment M_{xx} of an elastic plate having a Poisson's ratio $\nu = -1$ which is the case of a rigid body. Therefore, for the plate moments, as well as the deflection, the Kelvin plate behaves initially like a rigid body and approaches the behavior of an elastic plate as t approaches infinity.

As a second example of a viscoelastic plate of practical interest consider the case of a Maxwell plate subjected at time $t = 0$ to a suddenly applied uniformly distributed load which remains constant thereafter. The analysis follows in exactly the same manner as for the Kelvin plate but the solution shows several differences. Equation (41) for w_e now is given by

$$w_0 = w_e \int_0^t \left[\delta(T) + \frac{3K+G}{\tau(3K+4G)} + \frac{3G^2}{\tau(3K+4G)(3K+G)} e^{-\frac{3K}{\tau(3K+G)}T} \right] \theta(t-T)\, dT \qquad (48)$$

which when evaluated for θ equal to the unit step function, as is the case here, becomes

$$w_0 = w_e \left(\frac{3K+G}{3K+4G}\right) \left\{ \frac{K+G}{K} + \frac{t}{\tau} - \frac{G^2}{(3K+G)K} e^{-\frac{3K}{\tau(3K+G)}t} \right\} \qquad (49)$$

If, from this equation w_0/w_e is plotted against t/τ as was done for the Kelvin plate (see Fig. 7) the deflection is observed to reach the elastic deflection immediately and to increase at a constant rate thereafter.

If the load is suddenly removed at time $t_1 = 2\tau$ the elastic deflection is immediately recovered and a residual deflection corresponding to the amount of viscous flow that has taken place remains in the plate. As can be seen from the graph there is a slight "trailing off" of the curve as the plate settles into its final residual deflection. This is not predicted by a simple one-dimensional model of a Maxwell body and probably enters because of the two-dimensional aspects of the problem. The equation for this dashed portion of Figure 7 is determined in the same manner as was Equation (45). It is given by

$$w_0 = w_e \left(\frac{3K+G}{3K+4G}\right) \left\{ \frac{t_1}{\tau} - \frac{G^2\left(1 - e^{\frac{3K}{\tau(3K+G)}t_1}\right)}{K(3K+G)} e^{-\frac{3K}{\tau(3K+G)}t} \right\}$$

which is valid for $t > t_1$.

t/τ	W_o/W_e	
	LOAD ON	LOAD OFF
0	1.000	——
1	1.705	——
2	2.380	1.395
3	3.020	1.360
4	3.730	1.340
5	4.395	1.330

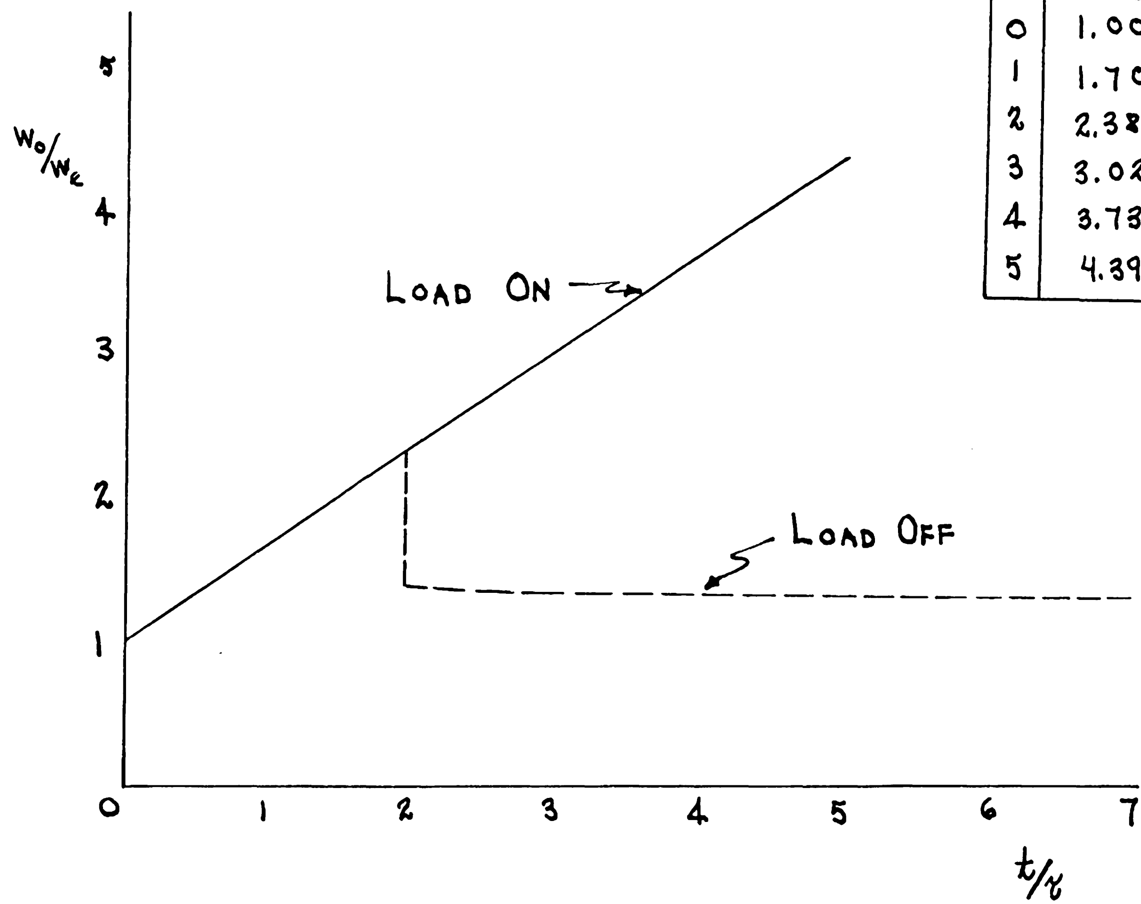

Figure 7. Maxwell Plate Response

The moments in the Maxwell plate may be found in the same way as for the Kelvin plate. Substituting the operators $P = \frac{1}{\tau} + p$ and $Q = Gp$ into Equation (26) and taking the Laplace transform as before gives

$$\bar{M}_{\alpha\beta} = -\frac{D}{2}\left\{ \frac{\frac{3K}{\tau} + (3K+4G)s}{\left(\frac{3K}{\tau} + (3K+G)s\right)s} w_{e,\alpha\beta} + \delta_{\alpha\beta} \frac{\frac{3K}{\tau} + (3K-2G)s}{\left(\frac{3K}{\tau} + (3K+G)s\right)s} w_{e,\gamma\gamma} \right\}$$

Expanding this by partial fractions and taking the inverse gives

$$M_{\alpha\beta} = -\frac{D}{2}\left\{ \left(1 + \frac{3G}{3K+G} e^{-\frac{3K}{\tau(3K+G)}t}\right) w_{e,\alpha\beta} \right. \qquad (51)$$

$$\left. \delta_{\alpha\beta}\left(1 - \frac{3G}{3K+G} e^{-\frac{3K}{\tau(3K+G)}t}\right) w_{e,\gamma\gamma} \right\}$$

From these equations the initial value of M_{xx} is seen to be

$$M_{xx} = -\frac{D}{2}\left\{ 2 w_{e,xx} + \frac{3K-2G}{3K+G} w_{e,yy} \right\}$$

$$= -D\left\{ w_{e,xx} + \nu w_{e,yy} \right\}$$

which is the moment of the equivalent elastic plate. As $t \to \infty$ the plate moment M_{xx} becomes

$$M_{xx} = -D\left\{ w_{e,xx} + \frac{1}{2} w_{e,yy} \right\}$$

which corresponds to an elastic plate moment in which the value of Poisson's ratio is $\frac{1}{2}$. Therefore the Maxwell plate behavior may be characterized as initially elastic with its terminal behavior being that of an incompressible elastic solid.

B. General Loading

In cases where the load on the plate cannot be expressed as simply the product of a time function multiplied by a space function it appears best to represent the load by a series expansion of the form

$$f(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi_{mn} X_m Y_n \tag{52}$$

where the ψ_{mn} are functions of time, X_m and Y_n are functions of x alone and y alone, respectively. Such a series representation permits handling a number of important practical cases of loading including the case of a moving load.

A convenient method for attacking a broad group of such problems is to work directly with the general variational formulation of visco-elastic behavior. This principle, for the case of quasi-static problems, requires that the variation of the operational invariant J be equal to the virtual work of the boundary forces [3]. J is defined as before as the volume integral

$$J = \iiint_V \Big\langle \frac{1}{2} \Big\{ (2S+R)(\epsilon_{xx}^2 + \epsilon_{yy}^2 + \epsilon_{zz}^2) + 2R(\epsilon_{xx}\epsilon_{yy} + \epsilon_{yy}\epsilon_{zz} + \epsilon_{zz}\epsilon_{xx}) \Big\} + 2S(\epsilon_{xy}^2 + \epsilon_{yz}^2 + \epsilon_{xz}^2) \Big\rangle \, dV$$

Under the assumptions usually adopted in studying deflections of thin plates ϵ_{yz} and ϵ_{xz} are taken as zero and σ_{zz} is considered negligible compared with the in-plane stresses. This reduces J to the form

$$J = \iiint_V \frac{1}{2}\left\{(2S+R)(\epsilon_{xx}^2 + \epsilon_{yy}^2) + 2R(\epsilon_{xx}\epsilon_{yy}) - \frac{R}{2S+R}(\epsilon_{xx}+\epsilon_{yy})^2 + 4S\,\epsilon_{xy}^2\right\} dV \tag{53}$$

Now for small strains $\epsilon_{\alpha\beta} = -z\, w_{,\alpha\beta}$ and J may be written in terms of the plate curvatures and twists (after integrating across the thickness) as

$$J = \frac{h^3}{12}\iint_A \left\{\frac{2S(S+R)}{2S+R}(w_{,xx}+w_{,yy})^2 - 2S(w_{,xx}w_{,yy} - w_{,xy}^2)\right\} dA \tag{54}$$

Using this form the variational formulation for plates subjected to bending loads is given by

$$\delta\iint_A \frac{h^3}{12}\left\{\frac{2S(S+R)}{2S+R}(w_{,xx}+w_{,yy})^2 - 2S(w_{,xx}w_{,yy} - w_{,xy}^2)\right\} dA = \iint_A f\,\delta w\, dA \tag{55}$$

where f is the load per unit area of the lateral surface of the plate.

As a simple example to illustrate the method consider the problem of finding the deflection of a simply-supported rectangular plate subjected to a suddenly applied load uniformly distributed over the surface of the plate. Taking the dimensions of the plate as those given in Figure 8,

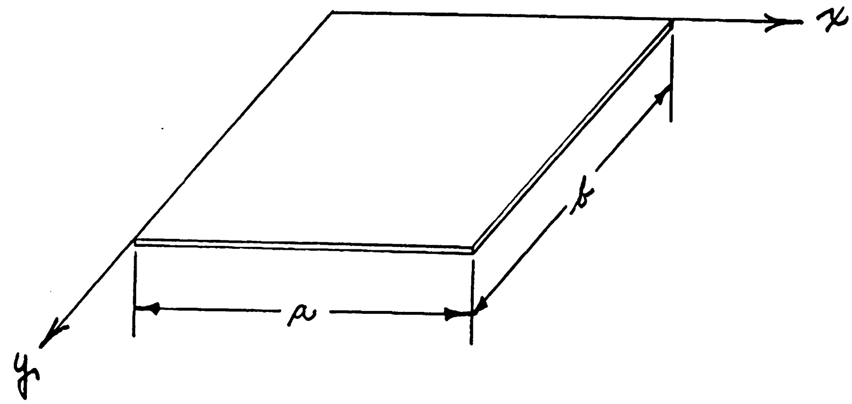

Figure 8. Rectangular Plate

it is convenient to expand the load in a double sine series, namely

$$f(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \psi_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

Noting that for such a loading f is given by

$$f(x,y,t) = f_0\, \theta(t)$$

in which f_0 is a constant and $\theta(t)$ is the unit step function and following the usual procedure employed in obtaining Fourier coefficients the time functions ψ_{mn} may be determined from the expression

$$\psi_{mn} = \frac{4}{ab}\int_0^a\int_0^b f_0\, \theta(t) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy$$

which when integrated yields

$$\varphi_{mn} = \frac{16}{\pi^2 mn} f_0 \, \theta(t) \qquad (m, n = 1, 3, 5, \text{etc.})$$

Now the deflected middle surface of the plate may be taken in the form of the series

$$w(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \varphi_{mn}(t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

in which the sine terms are chosen so that the solution will satisfy the boundary conditions identically. Performing the necessary differentiations on this series for w and putting the results into Equation (54) gives the following value for J after integrating over the area of the plate.

$$J = \frac{h^3}{12} \frac{2s(s+R)}{2s+R} \frac{ab\pi^4}{4} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \varphi_{mn}^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2$$

Taking the variation of this expression and equating it to the virtual work of the load gives

$$\frac{h^3}{3} \frac{s(s+R)}{2s+R} \pi^4 \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \varphi_{mn} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 \delta\varphi_{mn} = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{16}{\pi^2 mn} f_0 \, \theta(t) \, \delta\varphi_{mn}$$

which may be simplified and solved for ψ_{mn} in operational form as

$$\varphi_{mn} = \frac{48 f_0}{h^3 \pi^6 mn \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \left[\frac{2s+R}{s(s+R)} \, \theta(t) \right] \qquad (56)$$

This equation is easily solved for a particular type of visco-elastic plate by use of the Laplace transform method. For a Maxwell plate, for example, the operator $\frac{2S+R}{S(S+R)}$ is given by

$$\frac{2S+R}{S(S+R)} = \frac{\left(p+\frac{1}{\tau}\right)\left(\frac{3K}{\tau}+(3K+4G)p\right)}{Gp\left(\frac{3K}{\tau}+(3K+G)p\right)}$$

and the transformed equation may be simplified with the help of partial fractions and inverted at once to give

$$\varphi_{mn} = \frac{48 f_0}{h^3\pi^6 mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}\left\{\frac{1}{G}\left(\frac{K+G}{G}+\frac{t}{\tau}-\frac{G^2}{(3K+G)K}e^{-\frac{3K}{(3K+G)}\frac{t}{\tau}}\right)\right\}$$

Therefore

$$\begin{aligned} w(x,y,t) = \sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty} & \frac{48 f_0}{G h^3\pi^6 mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}\Big[\frac{K+G}{K}+\frac{t}{\tau} \\ & -\frac{G^2}{(3K+G)K}e^{-\frac{3K}{(3K+G)}\frac{t}{\tau}}\Big]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \end{aligned} \tag{57}$$

This may be written (see [21] page 118)

$$w(x,y,t) = W_e\frac{3K+G}{3K+4G}\left\{\frac{K+G}{K}+\frac{t}{\tau}-\frac{G^2}{(3K+G)K}e^{-\frac{3K}{(3K+G)}\frac{t}{\tau}}\right\}$$

which is, of course, identical with the solution (see Equation (49)) for the same problem as determined by the methods of the previous section.

A problem of considerable more complexity than the simple example above is that of finding the deflection of a plate subjected to a moving load. The method can be used in cases involving such complications, however, the details following essentially the same as above. Consider, for example, a simply supported rectangular plate of the same dimensions as before and imagine a line load parallel to the y axis and of uniform intensity P_o moving with constant velocity v in the x direction. Assuming that the load is on the y axis at time t = 0 its position at any subsequent time is given by x = vt. This is shown in Figure 9.

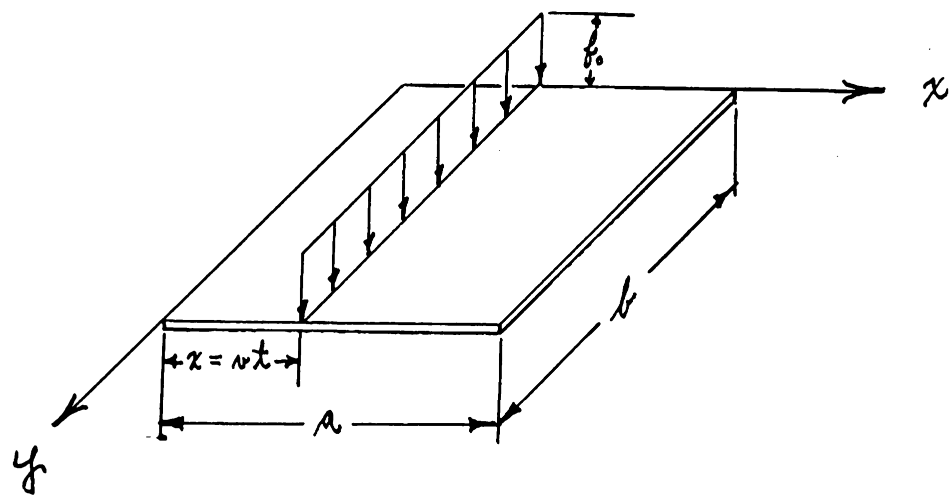

Figure 9. Moving Line Load on a Rectangular Plate

Expanding the load in a double sine series

$$f(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \varphi_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

with ψ_{mn} given by the expression

$$\psi_{mn} = \int_0^b \int_{vt-\varepsilon}^{vt+\varepsilon} \frac{4}{ab} f_0 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

This follows from considering the load as uniform and of intensity f_0 over the small interval of 2ε along the x axis as shown by Figure 10.

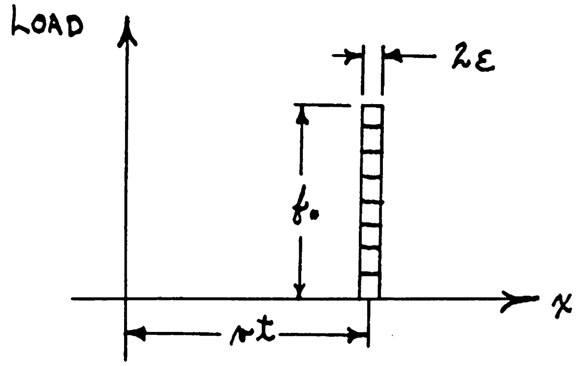

Figure 10. Line Load Representation

Carrying out the indicated integrations and putting in the limits gives

$$\psi_{mn} = \frac{8P_0}{m\pi a} \sin\frac{m\pi v}{a} t \qquad (m, n = 1, 3, 5, \text{etc.})$$

where $P_0 = 2f_0\varepsilon$. Therefore now

$$f(x,y,t) = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{8P_0}{n\pi a} \sin\frac{m\pi v}{a} t \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

For the deflection it is convenient to assume, as before, the series expression

$$w(x,y,t) = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \psi_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

which satisfies the edge conditions for the plate. Using this expression the variation of J is found to be

$$\delta J = \frac{h^3}{3}\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 \frac{s(s+R)}{2s+R}\,\varphi_{mn}\,\delta\varphi_{mn}$$

which, when equated to the virtual work of the load gives

$$\frac{h^3}{3}\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 \frac{s(s+R)}{2s+R}\,\varphi_{mn} = \frac{8P_0}{m\pi a}\sin\frac{m\pi v}{a}t$$

This may be solved for φ_{mn} which in operational form is

$$\varphi_{mn} = \frac{24P_0}{h^3\pi^5 am\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}\left\{\frac{2s+R}{s(s+R)}\sin\frac{m\pi v}{a}t\right\} \tag{59}$$

For the case of a plate made out of a Maxwell type material the operator for such may be substituted in this equation and φ_{mn} determined. Carrying out the details by the usual Laplace transform methods results in

$$\varphi_{mn} = \frac{24P_0}{Gh^3\pi^5 am\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}\left\{\frac{3K+4G}{3K+G}\sin\frac{m\pi v}{a}t\right.$$

$$+\frac{a}{\tau m\pi v}\left(1-\cos\frac{m\pi v}{a}t\right)$$

$$+\frac{3G^2}{\tau(3K+G)^2}\,\frac{1}{A^2+\left(\frac{m\pi v}{a}\right)^2}\left[\frac{3K}{\tau(3K+G)}\sin\frac{m\pi v}{a}t\right.$$

$$- \frac{m\pi v}{a} \cos \frac{m\pi v}{a} t + \frac{m\pi v}{a} e^{-At} \Big] \Big\} \qquad (60)$$

where $A = \dfrac{3K}{\tau(3K+G)}$

Recalling that

$$w(x,y,t) = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \varphi_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

it may be observed from the above expression for φ_{mn} that when $t = 0$, $\varphi_{mn} = 0$ which corresponds to the initially undeflected plate. Also the solution as given is valid only for the period of time while the load is actually on the plate, in other words for the interval $0 \leq t \leq a/v$.

At the instant the load leaves the plate the deflection is given by

$$w(x,y,t) = \frac{24 P_o}{G \lambda^3 \pi^5 a} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{1}{n\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2} \Big[\frac{2a}{\tau m\pi v}$$

$$+ \frac{3G^2}{\tau(3K+G)^2} \frac{1}{A^2 + \left(\frac{m\pi v}{a}\right)^2} \left(\frac{m\pi v}{a} + \frac{m\pi v}{a} e^{-\frac{Aa}{v}} \right) \Big] \Big] \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad (61)$$

Also at this instant the plate velocity is given by

$$\dot{w}(x,y,a/v) = \frac{24 P_0}{G h^3 \pi^5 a} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{1}{n\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2} \left[\frac{3K+4G}{3K+G} \frac{m\pi v}{a} \right.$$

$$\left. + \frac{3G^2}{2(3K+G)^2} \frac{A}{A^2+\left(\frac{m\pi v}{a}\right)^2} \left(-1 + e^{-\frac{Aa}{v}}\right) \frac{m\pi v}{a} \right] \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{61}$$

Using these values as initial conditions for the same plate under no load the subsequent deflection of the plate can be determined by the same methods as used for the loaded plate. The operational equation to be solved in this case is

$$\frac{S(S+R)}{2S+R} \varphi_{mn} = 0 \tag{62}$$

which for the Maxwell operator yields

$$\varphi_{mn} = -\frac{\dot{\Phi}}{A} + \left(\Phi + \frac{\dot{\Phi}}{A}\right) e^{-At} \tag{63}$$

where

$$\Phi \equiv \varphi_{mn}\Big]_{t = a/v}$$

$$\dot{\Phi} \equiv \dot{\varphi}_{mn}\Big]_{t = a/v}$$

Substituting this expression for φ_{mn} back into the deflection expansion results in the solution

$$
\begin{aligned}
w(x,y,t) = {} & \frac{24 P_0}{G h^3 \pi^6 a} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \Bigg\langle \frac{1}{n\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2} \frac{1}{A} \Bigg\{ \frac{3K+4G}{3K+G} \frac{m\pi v}{a} \\
& + \frac{3G^2}{\tau(3K+G)^2} \frac{\frac{m\pi v}{a}}{A^2+\left(\frac{m\pi v}{a}\right)^2} \left(1-e^{-\frac{Aa}{v}}\right) \Bigg\} + \Bigg\{ \frac{h a}{\tau m \pi v} \\
& + \frac{3G^2}{\tau(3K+G)^2} \frac{\frac{m\pi v}{a}}{A^2+\left(\frac{m\pi v}{a}\right)^2} \left(1+e^{-\frac{Aa}{v}}\right) - \frac{1}{A} \Bigg(\frac{3K+4G}{3K+G} \frac{m\pi v}{a} \\
& + \frac{3G^2}{\tau(3K+G)^2} \frac{\frac{m\pi v}{a}}{A^2+\left(\frac{m\pi v}{a}\right)^2} \left(1-e^{-\frac{Aa}{v}}\right) \Bigg) \Bigg\} e^{-At} \Bigg\rangle \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}
\end{aligned}
\tag{64}
$$

As would be expected for a Maxwell plate the deflection does not return to zero as $t \to \infty$ but some residual deflection remains in the plate. This residual deflection is given by

$$
\begin{aligned}
w\Big]_{t\to\infty} = {} & \frac{24 P_0}{G h^3 \pi^6 a} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \Bigg\langle \frac{1}{n\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2} \frac{1}{A} \Bigg\{ \frac{3K+4G}{3K+G} \frac{m\pi v}{a} \\
& + \frac{3G^2}{\tau(3K+G)^2} \frac{\frac{m\pi v}{a}}{A^2+\left(\frac{m\pi v}{a}\right)^2} \left(1-e^{-\frac{Aa}{v}}\right) \Bigg\} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}
\end{aligned}
\tag{65}
$$

V

CORRELATION OF THEORETICAL RESULTS WITH EXPERIMENTAL DATA

As a means of establishing a comparison of the plate deflections predicted by Equation (41) and the actual deflections of plates made of a material known to exhibit viscoelastic characteristics, a series of deflection tests were carried out using plates made of Plexiglas. The test plate was nine inches square and one sixteenth of an inch thick. This was mounted in a heavy steel frame designed specifically for plate deflection experiments. The plate was subjected to a 1.157 pound load over a small area at the center of the plate and readings of the center deflection were obtained by means of a 0.0001 inch increment dial gage attached rigidly to the supporting frame. The holding surfaces of the frame were machined to provide the boundary conditions of a plate clamped on all four edges.

In conducting the deflection experiments it was apparent at the outset that reliable data was to be obtained only if a considerable interval of time elapsed between consecutive test runs. This was to be expected, of course, since, theoretically, the behavior of a visco-elastic test element depends upon the entire past history of the material from which it is made. Accordingly, only one test, usually of about five hours duration was made per day in order to give the plate sufficient time to "relax" to a condition of relatively little residual stress or deflection. Several such tests were run to check uniformity of results. A typical curve taken from the results of these tests is shown in

Figure 11. Also in the same figure two theoretical curves are plotted to form a basis for comparison with the experimental data.

The theoretical curves in Figure 11 are derived directly from Equation (44) for the Kelvin-type plate and Equation (49) for the Maxwell-type plate. Since these equations are in a form that is especially convenient for plotting the dimensionless deflection W/W_e versus a dimensionless time t/τ it was decided to reduce the experimental data to a form suitable for plotting the experimental curve in the same units in order to permit a more realistic comparison. One way to accomplish this is with the help of Poisson's ratio and the relaxation time, τ of the plate material, Plexiglas. Knowing Poisson's ratio permits the shear modulus, G to be expressed directly in terms of the bulk modulus, K by means of the relation

$$2G(1+\nu) = 3K(1-2\nu)$$

and thereby introducing a means of simplifying Equations (44) and (49) by canceling out the constants G and K.

The value of Poisson's ratio for Plexiglas has been determined experimentally by the Rohn and Haas Company of Philadelphia, Pennsylvania, manufacturers of this plastic. It is listed in Table II, page 4 of the Plexiglas Handbook for Aircraft Engineers published by them in 1951. The value given for a temperature of 77°F is 0.35 which is the value used herein. When this value is inserted in the above equation it leads to the relation $K = 3G$ which may in turn be introduced into Equations (44) and (49) and reducing them immediately to a form suitable

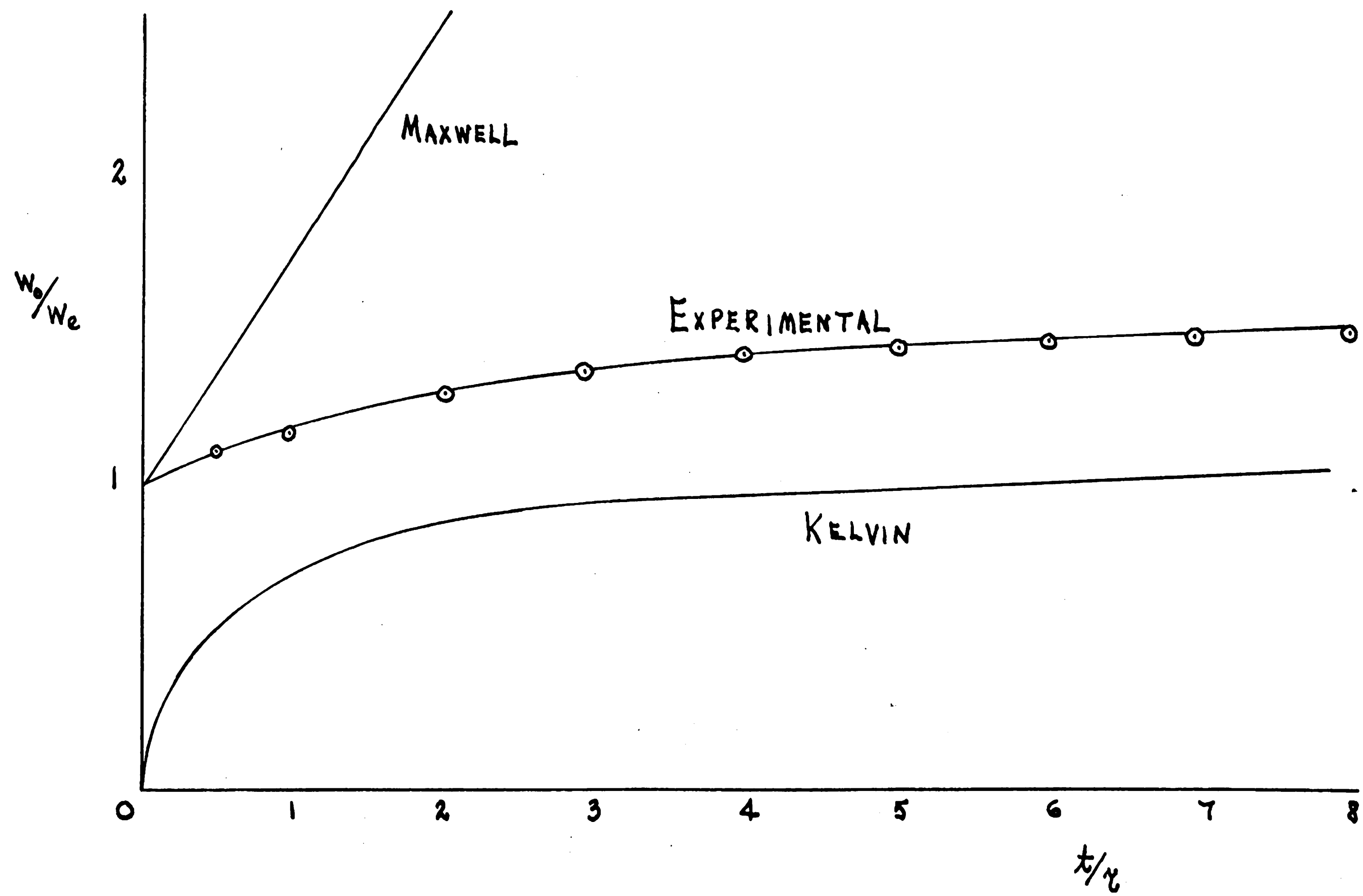

Figure 11. Correlation Curves

for computing the theoretical curves of Figure 11. After carrying out the details Equation (44) becomes

$$W/W_e = \left\{ 1 - \frac{3}{13} e^{-10\frac{t}{\tau}} - \frac{10}{13} e^{-\frac{t}{\tau}} \right\}$$

which is plotted for the Kelvin plate in Figure 11; and Equation (49) becomes

$$W/W_e = \frac{10}{13} \left\{ \frac{4}{3} - \frac{t}{\tau} - \frac{1}{30} e^{-\frac{9}{10}\frac{t}{\tau}} \right\}$$

which is plotted for the Maxwell plate in Figure 11.

The determination of the relaxation time for Plexiglas was based upon a sequence of constant-strain stress-relaxation tests performed upon tensile specimens taken from the same sheet of material from which the plate specimen had been cut. The test pieces were one inch wide and ten inches long. An initial load of 400 pounds was applied rapidly and the stress decay measured at constant strain. The point of intersection of the time axis by the tangent to the stress-time curve at zero time was taken as the relaxation time, τ. Although such a procedure tacitly assumes that the material in question behaves as a Maxwell body, representation of a real material by such a simple model is obviously only an approximation at best. More refined methods of determining viscoelastic constants such as the vibrating reed test [17], [7] or the Fitzgerald transducer apparatus (see United States Patent No. 2,774,239) would no doubt lead to a more authentic representation of the material. Based upon data collected from testing three tensile specimens the relaxation time was determined to be in the neighborhood of 30 minutes, which value was used in plotting the experimental curve of Figure 11. The numerical values for points on all curves of Figure 11 are given in Table I.

t/τ	W_0/W_e		
	KELVIN	MAXWELL	EXPERIMENTAL
0	0.0000	1.0000	1.000
1	0.7170	1.7845	1.181
2	0.8959	2.5598	1.308
3	0.9617	3.3316	1.595
4	0.9859	4.1018	1.450
5	0.9948	4.8715	1.475
6	0.9981	5.6409	1.497
7	0.9993	6.4101	1.519
8	0.9999	7.1794	1.540

Table I. Numerical Values of Correlation Curves

For comparing the experimental with the theoretical curves of Figure 11 it is at once apparent that the viscoelastic nature of Plexiglas is more complicated than either a Kelvin or a Maxwell material and yet seems to incorporate both elements to some degree. The appearance of the experimental curve seems to indicate components of all three of the most common viscoelastic response phenomena, i.e. instantaneous elastic response, delayed elastic response and viscous flow. The first two of these dominate the response curve as would be expected from the "solid" state of the test material. The presence of some flow is indicated, however, by the continued increase in the deflection at higher values of the time.

To adequately represent by a spring and dashpot model a viscoelastic body possessing all three afore-mentioned response patterns would require at least a four parameter model such as the one shown in Figure 12. In this model the free spring represents the instantaneous

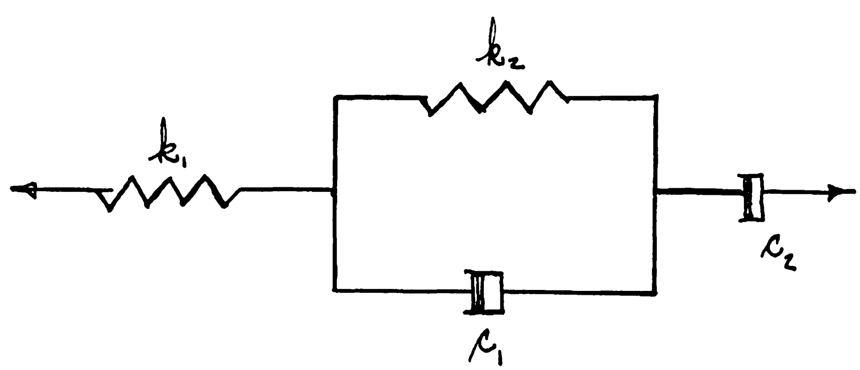

Figure 12. Four Parameter Model

elastic response, the spring and dashpot in parallel represents delayed elasticity and the free dashpot represents viscous flow. The problem of determining the spring and dashpot constants of such a model when it is used to describe a given viscoelastic substance is discussed in considerable detail in the paper by Bland and Lee [7]. The dependence of these constants upon frequency is indicated. In the present investigation, since the deflections are essentially static, such a procedure is hardly necessary and it was for this reason that the relaxation time τ was obtained from the simple static tensile test.

It is interesting to note that the four parameter model of Figure 12 is basically a Kelvin and Maxwell element attached in series. This would seem to indicate that if such a model gives a sufficiently accurate description of Plexiglas, then the two theoretical curves could be blended in a suitable manner to give a curve in good agreement with the experimental curve. Therefore, although the Kelvin and Maxwell models are too much simplified to predict actual plate deflections of a real material, as is evidenced by Figure 11, they are important in that they single out the individual response components and thereby form a basis for combination into more realistic behavior. The question of how to represent the physical material by a model, or what is the same, how to combine the Kelvin and Maxwell response curves is prerequisite to predicting the plate behavior and must be answered by experimental investigation. Thereafter, the theory may be applied in the same manner as for the Kelvin or Maxwell case, although the details will be more complicated and involved.

VI

DYNAMIC RESPONSE OF VISCOELASTIC PLATES

A. Compressible Plate Material

If inertia effects due to deformation are taken into consideration, the fundamental equation for viscoelastic plate behavior must be altered by the addition of an inertia term just as in the elastic case. This additional term is readily found by applying the principle of correspondence to the well known equation for dynamic elastic plate behavior [5]. In this manner Equation (22) is altered and becomes

$$p^2 \rho h w + B_1\{\nabla^4 w\} = f \tag{66}$$

where all previously used symbols retain their identity and ρ is the plate density. Solutions to this equation are best approached through application of the method given in Chapter IV, Part B of this paper. To this end consider the free vibrations of a rectangular plate of dimensions given by Figure 8 and having all edges simply supported. As before assume the middle surface of the plate is given by the double sine series

$$w(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \varphi_{mn}(t) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

Inverting this into Equation (66) and taking "f" to be identically zero, since free vibrations only are considered, results in an ordinary differential equation in the variable, time, that must be solved for ϕ_{mn}. This equation is

$$\left\{ p^2 \rho h + \pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 B_1 \right\} \varphi_{mn} = 0 \tag{67}$$

If the plate material is known to approximate a Maxwell body, the operator B_1 is as given by Equation (43) and Equation (67) takes the form

$$p \left\{ (3K+4G) p^3 + \frac{6K+4G}{\tau} p^2 + \left[\frac{3K}{\tau^2} + \Pi_{mn}(3K+G) \right] p + \frac{3K\Pi_{mn}}{\tau} \right\} \varphi_{mn} = 0 \tag{68}$$

in which

$$\Pi_{mn} = \frac{G h^2 \pi^4}{3\rho} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2$$

For a plate made of a material that may be considered as a Kelvin body, the operator B_1 is given by Equation (42) and Equation (67) becomes

$$\left\{ 4\eta p^3 + \left[(3K+4G) + \overline{\Pi}_{mn} \eta^2 \right] p^2 + \overline{\Pi}_{mn} \eta (3K+2G) p + \overline{\Pi}_{mn} (3K+G) G \right\} \varphi_{mn} = 0 \tag{69}$$

Here $$\overline{\Pi}_{mn} = \frac{\Pi_{mn}}{G}$$

Although, in theory, Equations (68) and (69) may be readily solved by formal application of operational methods this proves extremely difficult, if not impossible, to do with the equations expressed in terms of the physical parameters K, G, τ, η and Π_{mn} as above. To see why this is so consider these equations to be transformed by means of the Laplace transformation. The results are

$$\bar{\varphi}_{mn} = \frac{\Phi_{mn}(s)}{s\left\{(3K+4G)s^3 + \frac{6K+4G}{\tau}s^2 + \left[\frac{3K}{\tau^2} + \Pi_{mn}(3K+G)\right]s + \frac{3K\Pi_{mn}}{\tau}\right\}} \tag{70}$$

from Equation (68) and

$$\bar{\varphi}_{mn} = \frac{\Phi_{mn}(s)}{4\eta s^3 + \left[(3K+4G) + \Pi\eta^2\right]s^2 + \Pi_{mn}\eta(3K+2G)s + \Pi_{mn}(3K+G)G} \tag{71}$$

from Equation (69). In these expression $\Phi_{mn}(s)$ is a polynomial in s, the transform variable, the exact form of which depends on the manner in which the motion of the plate is begun. To invert these equations requires the roots of the cubics appearing in the denominator of each. Explicit expressions for these roots in terms of the parameters given are available in handbooks and algebra texts. It can be shown, moreover, that all the roots of these equations have negative real parts which assures decaying time functions in the solution. Taking the roots of the cubics in question as r_1, r_2, r_3 the solutions of Equation (70) and (71) have the form

$$\varphi_{mn} = \Phi_{mn0} + \sum_{i=1}^{3} \Phi_{mni} e^{-r_i t} \tag{72}$$

and the middle surface of the plate is given by

$$W(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\Phi_{mno} + \sum_{i=1}^{3}\Phi_{mni}\, e^{-r_i t}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad (73)$$

in which the Φ_{mni}'s are constants involving the physical parameters and the initial conditions. It is apparent that for the Kelvin plate, Equation (71), the term Φ_{mno} in this solution is zero.

The importance of Equation (73) rests in the fact that for an actual plate the numerical values of the physical parameters in Equations (70) and (71) may be established and the roots of the resulting numerical cubics determined by direct substitution into standard expressions for these roots. Such calculations would be required for each mode investigated and the amount of work involved in carrying out such a project would be considerable. However, it is quite possible that only a single mode is of interest in a specific problem, in which case the labor would be justified.

Actually a solution somewhat more specific than Equation (73) can be achieved in view of the relation existing between G and K. For the purpose of discussing such a solution consider Equation (70). (Similar calculations could be carried out for Equation (71) also.) Assume Equation (70) is written for a material having a value of Poisson's ratio $\nu = 0.35$ (Plexiglas, for example). In this case K = 3G and $\bar{\varphi}_{mn}$ becomes

$$\bar{\varphi}_{mn} = \frac{\Phi_{mn}}{Gs\left\{13s^3 + \frac{22}{\tau}s^2 + \left(\frac{9}{\tau^2} + 10\Pi_{mn}\right)s + \frac{9\Pi_{mn}}{\tau}\right\}} \qquad (74)$$

In view of the cumbersome nature of the exact solution of the cubic in Equation (74) an approximate solution will be presented by factoring the cubic by trial and error until a neglible remainder is obtained. The factors $(s + \frac{0.9}{\tau})$ and $(13s^2 + \frac{10.3}{\tau} s + 10\,\pi_{mn} - \frac{0.27}{\tau^2})$ are sufficiently refined for the purposes at hand and lead to

$$\bar{\varphi}_{mn} = \frac{\bar{\Phi}_{mn}}{13Gs\left\{\left(s+\frac{0.9}{\tau}\right)\left(s^2+\frac{0.792}{\tau}s+0.77\pi_{mn}-\frac{0.021}{\tau^2}\right)\right\}} \tag{75}$$

Rearranging the quadratic term into a form convenient for inverting gives

$$\bar{\varphi}_{mn} = \frac{\bar{\Phi}_{mn}}{13Gs\left\{\left(s+\frac{0.9}{\tau}\right)\left[\left(s+\frac{0.396}{\tau}\right)^2+\left(0.77\pi_{mn}-\frac{0.178}{\tau^2}\right)\right]\right\}} \tag{76}$$

This may be inverted immediately with the help of a table of function-transform pairs (see formula No. 1,319, page 348 of "Transients in Linear Systems" by M. F. Gardner and J. L. Barnes). The result is

$$\varphi_{mn} = \frac{\Phi_{mn}}{13G}\left\{\frac{1}{\frac{0.9}{\tau}\left(0.77\pi_{mn}-\frac{0.021}{\tau^2}\right)} + \frac{e^{-\frac{0.9}{\tau}t}}{\frac{0.9}{\tau}\left(0.77\pi_{mn}-\frac{0.432}{\tau^2}\right)} \right. \tag{77}$$

$$\left. + \frac{e^{-\frac{0.396}{\tau}t}\sin\left(\sqrt{0.77\pi_{mn}-\frac{0.178}{\tau^2}}\,t-\varphi\right)}{\left[\left(0.77\pi_{mn}-\frac{0.021}{\tau^2}\right)\left(0.77\pi_{mn}-\frac{0.178}{\tau^2}\right)\left(0.77\pi_{mn}-\frac{0.076}{\tau^2}\right)\right]^{\frac{1}{2}}}\right\}$$

where

$$\psi = \tan^{-1}\frac{\sqrt{0.77\pi_{mn} - \frac{0.178}{\tau^2}}}{-\frac{0.396}{\tau}} + \tan^{-1}\frac{\sqrt{0.77\pi_{mn} - \frac{0.178}{\tau^2}}}{\frac{0.504}{\tau}}$$

It is worth noting that the motion is oscillatory as long as $0.77\pi_{mn}$ is larger than $\frac{0.178}{\tau^2}$.

B. Incompressible Plate Material

If the plate material may be assumed to be incompressible, as is frequently done in viscoelastic problems, considerable simplification of Equations (68) and (69) is achieved and general solutions are readily obtained. The condition of incompressibility is expressed by the operators M and N of Equation (3) taking on the values 0 and 1, respectively. This is essentially the same as considering $K \to \infty$. Such an assumption reduces the operational form of the flexural rigidity constant D to

$$B_1 = \frac{h^3}{3}\frac{Q}{P}$$

and Equations (68) and (69) become

$$p\left\{p^2 + \frac{1}{\tau}p + \pi_{mn}\right\}\varphi_{mn} = 0 \tag{78}$$

and

$$\left\{p^2 + \eta\overline{\pi}_{mn}p + G\overline{\pi}_{mn}\right\}\varphi_{mn} = 0 \tag{79}$$

respectively.

Consider first Equation (78) for the Maxwell plate. There is no loss in generality by assuming that when t = 0 the plate has velocity but no displacement or acceleration. In this case Equation (78) may be transformed into

$$\bar{\varphi}_{mn} = \dot{\Phi}_{mn}\left\{\frac{s+\frac{1}{\tau}}{s\left(s^2+\frac{s}{\tau}+\Pi_{mn}\right)}\right\}$$

where $\dot{\Phi}_{mn} = \dot{\phi}_{mn}\Big]_{t=0}$. Rearranging the quadratic in the denominator this may be written in the form

$$\bar{\varphi}_{mn} = \dot{\Phi}_{mn}\left\{\frac{s+\frac{1}{\tau}}{s\left[\left(s+\frac{1}{2\tau}\right)^2+\left(\Pi_{mn}-\frac{1}{4\tau^2}\right)\right]}\right\} \tag{80}$$

If $\Pi_{mn} > \frac{1}{4\tau^2}$, i.e., if the "damping effect" is small enough the response will be oscillatory and this equation may be inverted at once to yield

$$\varphi_{mn} = \dot{\Phi}_{mn}\left\{\frac{1}{\tau\Pi_{mn}} + \frac{e^{-\frac{t}{2\tau}}}{\sqrt{\Pi_{mn}-\frac{1}{4\tau^2}}}\sin\left[\left(\sqrt{\Pi_{mn}-\frac{1}{4\tau^2}}\,t\right)-\psi\right]\right\} \tag{81}$$

where

$$\psi = \tan^{-1}\frac{\sqrt{\Pi_{mn}-\frac{1}{4\tau^2}}}{\frac{1}{2\tau}-\tau\Pi_{mn}}$$

From the form of Equation (81) it may be observed that the response is the sum of a constant term plus a decaying sine term so that the plate will eventually come to rest in some displaced position. This is not surprising in view of the model representation of a Maxwell body.

Some noticeable differences appear in the response of a Kelvin plate as found from solving Equation (79). Here it is convenient to assume that the motion of the plate is initiated by giving it some displacement and releasing it without initial velocity. Transforming Equation (79) under these conditions results in

$$\bar{\varphi}_{mn} = \bar{\Phi}_{mn}\left\{\frac{s + \eta\bar{\pi}_{mn}}{(s^2 + \eta\bar{\pi}_{mn}s + G\bar{\pi}_{mn})}\right\}$$

where Φ_{mn} is $\phi_{mn}\Big]_{t = 0}$

Putting this expression for $\bar{\phi}_{mn}$ into a form suitable for inverting leads to

$$\bar{\varphi}_{mn} = \bar{\Phi}_{mn}\left\{\frac{s + \eta\bar{\pi}_{mn}}{\left(s + \frac{\eta\bar{\pi}_{mn}}{2}\right)^2 + \left[G\bar{\pi}_{mn} + \left(-\frac{\eta^2\bar{\pi}_{mn}^2}{4}\right)\right]}\right\} \qquad (82)$$

From this it is seen that for $4G > \eta^2\bar{\pi}_{mn}$ the motion is oscillatory which clearly indicates that the higher modes will be "damped out" and will not appear. The critically damped mode will be that one for which $4G = \eta^2\bar{\pi}_{mn}$.

Equation (82) may be inverted at once to give

$$\varphi_{mn} = \bar{\Phi}_{mn}\left\{\frac{e^{-\frac{\eta\bar{\pi}_{mn}t}{2}}}{\sqrt{1 - \frac{\eta^2\bar{\pi}_{mn}}{4G}}}\sin\left(\frac{\eta\bar{\pi}_{mn}}{2}\sqrt{\frac{4G}{\eta^2\bar{\pi}_{mn}} - 1}\;t + \psi\right)\right\} \qquad (83)$$

where

$$\psi = \tan^{-1} \frac{\sqrt{G\overline{\pi}_{mn} - \frac{\eta^2 \overline{\pi}_{mn}^2}{4}}}{\frac{\eta \overline{\pi}_{mn}}{2}}$$

The motion here is a decaying sine wave which for the higher modes dies out quite rapidly as mentioned above. As can be seen the plate returns to the undeflected position as $t \to \infty$.

It is worthwhile pointing out that the Kelvin representation of an incompressible material is equivalent to the concept of internal viscous damping, which is frequently used in studying damped vibrations of bars and beams. In general the restriction of imcompressibility need not be invoked in working with beam and bar vibrations since the operational form of E, the modulus of elasticity, is inherently simpler than that of D, the flexural rigidity which enters into plate problems.

Although only free vibrations of plates have been discussed herein, the problem of forced vibrations could be treated by the same methods. The algebra would be more complicated perhaps but the results would follow in the same manner for the usual types of loadings considered in such investigations.

VII

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VIII

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X

APPENDIX

The Euler equations resulting from the variational principle of Equation (12) are

$$-\frac{h^3}{12}\left\{(2S+R)\frac{\partial^2 u_1}{\partial x^2}+R\left(\frac{\partial^2 v_1}{\partial x\,\partial y}+2\frac{\partial w_2}{\partial x}\right)-S\left(\frac{\partial^2 u_1}{\partial y^2}+\frac{\partial^2 v_1}{\partial x\,\partial y}\right)\right\}+\frac{Sh}{3}\left(u_1+\frac{\partial w_0}{\partial x}\right)=0$$

$$-\frac{h^3}{12}\left\{(2S+R)\frac{\partial^2 v_1}{\partial y^2}+R\left(\frac{\partial^2 u_1}{\partial x\,\partial y}+2\frac{\partial w_2}{\partial y}\right)-S\left(\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial x\,\partial y}\right)\right\}+\frac{Sh}{3}\left(v_1+\frac{\partial w_0}{\partial y}\right)=0$$

$$-\frac{Sh}{3}\left(\frac{\partial^2 w_0}{\partial y^2}+\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial v_1}{\partial y}+\frac{\partial u_1}{\partial x}\right)=f$$

$$\frac{h^3}{12}\left\{4(2S+R)w_2+2R\left(\frac{\partial v_1}{\partial y}+\frac{\partial u_1}{\partial x}\right)\right\}=0$$

Differentiating the first of these with respect to x and the second with respect to y and then adding results in

$$-\frac{h^3}{12}\left\{(2S+R)\left[\frac{\partial^3 u_1}{\partial x^3}+\frac{\partial^3 v_1}{\partial y^3}\right]+R\left[\frac{\partial^3 v_1}{\partial x^2\,\partial y}+\frac{\partial^3 u_1}{\partial x\,\partial y^2}\right.\right.$$

$$\left.\left.+2\left(\frac{\partial^2 w_2}{\partial x^2}+\frac{\partial^2 w_2}{\partial y^2}\right)\right]+S\left[\frac{\partial^3 u_1}{\partial x\,\partial y^2}+\frac{\partial^3 v_1}{\partial x^2\,\partial y}+\frac{\partial^3 v_1}{\partial x^2\,\partial y}+\frac{\partial^3 u_1}{\partial x\,\partial y^2}\right]\right\}$$

$$+\frac{hS}{3}\left\{\frac{\partial u_1}{\partial x}+\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial v_1}{\partial y}+\frac{\partial^2 w_0}{\partial y^2}\right\}=0$$

which, with the help of the third Euler equation reduces to

$$-\frac{h^3}{12}\Big\{(2S+R)\Big[\frac{\partial^3 u_1}{\partial x^3}+\frac{\partial^3 v_1}{\partial y^3}\Big]+R\Big[\frac{\partial^3 v_1}{\partial x^2 \partial y}+\frac{\partial^3 u_1}{\partial x \partial y^2}$$

$$+2\Big(\frac{\partial^2 w_2}{\partial x^2}+\frac{\partial^2 w_2}{\partial y^2}\Big)\Big]+2S\Big[\frac{\partial^3 u_1}{\partial x \partial y^2}+\frac{\partial^3 v_1}{\partial x^2 \partial y}\Big]=f \quad \text{(a)}$$

Now from the fourth of the Euler equations

$$2w_2 = -\frac{R}{2S+R}\Big(\frac{\partial v_1}{\partial y}+\frac{\partial u_1}{\partial x}\Big)$$

hence

$$2\Big(\frac{\partial^2 w_2}{\partial x^2}+\frac{\partial^2 w_2}{\partial y^2}\Big) = -\frac{R}{2S+R}\Big(\frac{\partial^3 v_1}{\partial x^2 \partial y}+\frac{\partial^3 u_1}{\partial x^3}+\frac{\partial^3 v_1}{\partial y^3}+\frac{\partial^3 u_1}{\partial x \partial y^2}\Big)$$

and Equation (a) above is further reduced to

$$-\frac{h^3}{12}\Big\{(2S+R)\Big[\frac{\partial^3 u_1}{\partial x^3}+\frac{\partial^3 v_1}{\partial y^3}+\frac{\partial^3 u_1}{\partial x \partial y^2}+\frac{\partial^3 v_1}{\partial x^2 \partial y}\Big]$$

$$-\frac{R}{2S+R}\Big[\frac{\partial^3 v_1}{\partial x^2 \partial y}+\frac{\partial^3 u_1}{\partial x \partial y^2}+\frac{\partial^3 u_1}{\partial x^3}+\frac{\partial^3 v_1}{\partial y^3}\Big]\Big\}=f$$

or

$$-\frac{h^3}{12}\Big\{\Big[(2S+R)-\frac{R}{2S+R}\Big]\Big(\frac{\partial^3 u_1}{\partial x^3}+\frac{\partial^3 v_1}{\partial y^3}+\frac{\partial^3 v_1}{\partial x^2 \partial y}+\frac{\partial^3 u_1}{\partial x \partial y^2}\Big)\Big\}=f$$

This is easily put in terms of the single variable w_0 using relations obtained by differentiating the third Euler equation twice with respect to x and, independently, twice with respect to y. The result is

$$-\frac{h^3}{3}\,\frac{S(S+R)}{2S+R}\left[\frac{\partial^4 w_0}{\partial x^4} + 2\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + \frac{1}{hS}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) + \frac{\partial^4 w_0}{\partial y^4}\right] = -f$$

which may be written

$$\nabla^4 w_0 = \frac{f}{B_1} - \frac{1}{Sh}\nabla^2 f$$

where

$$B_1 = \frac{S(S+R)}{2S+R}$$

Abstract

This investigation is concerned with the flexural response of linear viscoelastic plates of constant thickness. Fundamental equations for both quasi-static and dynamic response of such plates are developed and solved for important cases of each. The term quasi-static is used to indicate that inertia forces due to deformation are neglected. These are included, of course, in the dynamic analysis. Solutions of the quasi-static equation are compared with experimental results obtained by measuring the deflection of a test plate made of Plexiglas.

The basic viscoelastic stress-strain relations used in the derivation of the fundamental plate equations are taken in the form of a differential time operator equation. Use of this equation leads to results that are in a convenient form for reduction to a particular material such as a Kelvin or a Maxwell plate.

Using a generalized virtual work principle based upon irreversible thermodynamic considerations the fundamental plate equation, including shear effects, is established. The procedure involved is that of determining a stationary value of a certain operational invariant by means of the calculus of variations. A simplified form of this equation, omitting the shear effects, is deduced and solutions for various load conditions obtained. An An extended version of this simplified form which includes inertia effects due to deformation is developed by the principle of

correspondence. This is used to study free vibrations of rectangular viscoelastic plates simply supported on all edges.

Solutions of the simplified form of the fundamental equation for the case of so-called proportional loading, i.e. when the load function is the product of a space function multiplied by a time function, are given in terms of the equivalent elastic solution multiplied by a function of time. For more general types of loading the deflection and the load are expanded into suitable infinite series and these series representation inserted directly into the previously mentioned variational expression of the generalized virtual work principle. This leads to a set ordinary differential equations in time the unknowns of which are the coefficients of the deflection expansion. These equations, as were the similar ones arising in the case of proportional loading, are solved by the Laplace transform method of the operational calculus. As an example of such a general loading the case of a moving line load on a rectangular plate is worked out.

As a means of establishing a correlation between the deflection predicted by the analytical solution and actual deflections of inelastic plates a set of static load tests were carried out on a square plate made of Plexiglas. The results are plotted and a comparison of the theoretical and experimental values given.

The problem of determining the dynamic response of viscoelastic plates is treated using the method given above for solving the case of general loading for the quasi-static deflection. Under the assumption of incompressibility of the plate material explicit solutions in terms of the physical parameters involved are presented and discussed. For compressible plate materials methods are developed to give approximate solutions the accuracy of which depends the degree of approximation used in determining the roots of certain cubics appearing in the transformed form of the governing dynamics equation. Conditions for the dynamic solutions to be oscillatory are indicated.