NONLINEAR DEFLECTIONS OF A CIRCULAR PLATE

WITH VARYING THICKNESS

by

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Thesis submitted to the Graduate Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY

in

Engineering Mechanics

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August, 1972

Blacksburg, Virginia
ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation and thanks to his advisor, Dr. G. W. Swift for his guidance, suggestions and encouragement in the development of this dissertation. He also expresses his gratitude to the members of his graduate committee, Professors K. L. Reifsnider, R. P. McNitt, D. T. Mook and J. Kaiser.
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NOMENCLATURE

dA \quad \text{element of surface area}

ds \quad \text{element of arc length}

b_{\alpha \beta} \quad \text{curvature tensor}

f^\alpha \quad \text{surface components of } q^i

f \quad \text{normal component of } q^i

g_{\alpha \beta} \quad \text{metric tensor}

g \quad \text{determinate of } g_{\alpha \beta}

h \quad \text{undeformed plate thickness}

M^i \quad \text{moment vector per unit undeformed length of middle surface}

N^i \quad \text{unit normal to surface}

n^\alpha x^i \big|_\alpha \quad \text{unit surface vector normal to boundary}

p \quad \text{pressure loading per unit deformed area of surface}

Q^i \quad \text{force vector per unit undeformed length of middle surface}

Q \quad \text{component of } Q^i \text{ normal to surface}

q^i \quad \text{loading force per unit undeformed area}

r \quad \text{radius of undeformed plate}

r^\alpha \quad \text{components of } Q^i \text{ in the surface}

U^i \quad \text{displacement of middle surface}

u^\alpha \quad \text{surface components of displacement}

w \quad \text{normal component of displacement}

x^i \quad \text{position vector}

x^i \big|_\alpha \quad \text{surface base vectors}
\( z \) \hspace{1cm} \text{point in shell away from middle surface}

\( \xi^\alpha \) \hspace{1cm} \text{material point in middle surface of shell}

A bar over a symbol is used to denote a quantity that refers to the deformed plate.

Slashes represent covariant differentiation based on the geometry of the deformed shell.

Commas represent ordinary differentiation.
I. INTRODUCTION AND REVIEW OF THE LITERATURE

Inflated structures, which are used, for example, in our satellite and space programs, quite often involve large displacements and strains. Therefore, it is apparent that a general nonlinear theory for large displacements and strains is required. For a practical application of this nonlinear theory, a suitable set of constitutive relations relating stress and strain must also be chosen.

The exact nonlinear equilibrium equations for thin plates and shells in deformed coordinates have been derived by several authors. Among them are Green and Zerna (1954), Naghdi (1963), Koiter (1966), Sanders (1963), and Leonard (1961). The deformed coordinates or geometry of a body is generally not prescribed or known, and therefore an indirect method must be used for solving problems involving large deformations.

A more direct solution could be obtained by applying the equilibrium equations written in undeformed coordinates. Budiansky (1968), following the work of Sanders (1963), Leonard (1961) and Koiter (1966), used a variational approach to derive exact tensor equations of equilibrium for nonlinear membrane shell theory. He also outlined a procedure for obtaining the non-linear membrane and bending equations for plates and shells in the undeformed coordinate system. Junkin (1970), following the procedure suggested by Budiansky (1968), wrote out explicitly the "first approximation" nonlinear, membrane and bending equations for an arbitrary plate in tensor notation. The term
"first approximation" is used in the sense of Love's (1944) first approximation as interpreted by Koiter (1960); that is, the contribution to the strain energy of deformation of stresses acting in the direction normal to the reference surface can be neglected compared to the contributions of bending and stretching. These tensor equations are then converted into physical equations for a rotationally symmetric plate with arbitrary symmetrical loadings using a method suggested by Frederick (1956).

Mooney (1940) was one of the first to set down in the literature a constitutive relation for elastic materials which undergo large deformations. Other constitutive relations have been proposed by several authors. Among them are Rivlin (1948), Treloar (1958), Rivlin and Saunders (1951), Biderman (1958), Hart-Smith (1966) and Alexander (1968).

The Mooney theory gives a fair approximation for moderate deformations under uniaxial extension but gives very poor correlation with biaxial tension experiments. Rivlin (1948) has shown that the constitutive relation for an elastic, incompressible material can be derived from a strain-energy density function. Rivlin and Saunders (1951) proposed a form for the strain energy density function that has been used by various investigators in specifying the elastic response of high polymeric materials. Treloar (1958) suggested another form for the strain energy density function and tried to verify it with his previous uniaxial and biaxial experiments on rubber. His theory did not yield good results with large strains. A good constitutive
relation for uniaxial tension and uniaxial compression was suggested by Biderman (1958). However, poor results were obtained for biaxial tests on rubber. By elaborating upon the theory of Rivlin and Saunders, Hart-Smith was able to modify the strain energy density function such that good results between theory and experiments were obtained for biaxial tests for moderate deformations and good results for uniaxial tests in the large deformation range. The shortcomings of the Hart-Smith theory were that it did not give good results for biaxial tests in the large deformation range and for uniaxial tests in the moderate deformation range. Alexander (1968) set up a combined form of the Hart-Smith theory and a modified version of the Rivlin-Saunders (1951) theory to accurately represent the response of an elastic, incompressible material well into the large deformation range. His experiments on neoprene, which he showed was nearly incompressible, give excellent correlation between theory and experimental data for both uniaxial and equi-biaxial tension tests throughout a large range of deformations.

The specific problem of a rotationally symmetric plate with a uniform pressure will be studied. A review of the literature shows that no one has applied the Alexander, Hart-Smith or the Rivlin-Saunders constitutive relations along with the exact tensor equilibrium equations to an axisymmetric plate made of a rubber-like material. In this paper we will use the Alexander constitutive relations along with the exact tensor "first approximation" equilibrium equations for a plate to study the large deflections of a rotationally symmetric
thin plate with (1) uniform thickness and (2) varying thickness in the radial direction and uniform thickness in the \( \theta \) direction. We will also apply the Hart-Smith and the Rivlin-Saunders constitutive relations and compare the results with Alexander's Theory.

The equilibrium equations used in this study were developed for a plate thin enough so that the state of stress is approximately "plane" in the sense that the contribution to the strain energy of deformation of stress acting in the direction normal to the reference surface can be neglected. Koiter (1960) pointed out that this implies that \((h/L)^2 \ll 1\) where \(L\) is the shortest wave length of deformation of the deformed reference surface and \(h\) is the thickness.
II. DEVELOPMENT OF THE EQUATIONS

This chapter is concerned with selected preliminaries in tensor analysis as applied in a brief outline of the development of specific relationships concerning nonlinear shell theory. These relationships are used in the development of the exact "first approximation" tensor equilibrium equations for the nonlinear membrane and bending problem for an arbitrary thin flat plate. This development closely follows the works of Budiansky (1968) and Junkin (1970).

A. Tensor Analysis Applied to Nonlinear Shell Theory

Consider an arbitrary shell whose undeformed middle surface is prescribed by the equations

\[ x^i = x^i (\xi^1, \xi^2) \]  

(2.1)

where \( x^1, x^2, x^3 \) are rectangular Cartesian coordinates and \( \xi^1 \) and \( \xi^2 \) are general coordinates in the middle surface. The displacement of the middle surface, \( U^i \), is defined by

\[ U^i = \bar{x}^i - x^i \]  

(2.2)

where barred quantities refer to the deformed state of the middle surface. If we define surface tensor components of displacement by \( u^\alpha \) and normal displacements by \( w \), then

\[ U^i = u^\alpha x^i|_\alpha + w N^i \]  

(2.3)

where \( x^i|_\alpha \) are the surface base vectors and \( N^i \) is the unit normal to surface.
In the equations Latin indexes will denote three-dimensional Cartesian components and Greek superscripts and subscripts will denote contravariant and covariant surface tensors respectively. Vertical slashes will denote covariant differentiation based on the geometry of the undeformed shell.

We know from tensor analysis (see Sokolnikoff (1966)) that the metric, curvature and alternating tensors can be defined by

\[ e_{\alpha \beta} = x^i |_{\alpha} x^1 |_{\beta}, \]  
\[ b_{\alpha \beta} = b_{\beta \alpha} = N^i |_{\alpha} x^1 |_{\beta}, \]  
\[ \epsilon_{12} = - \epsilon_{21} = \sqrt{g} \]  
and

\[ \epsilon_{11} = \epsilon_{22} = 0. \]  

The following two equations relate quantities in the deformed and undeformed shell as

\[ \frac{dA}{dA} = \sqrt{\frac{g}{g}} \]  
and

\[ \bar{n}_\alpha \, d\bar{s} = \sqrt{\frac{g}{g}} n_\alpha \, ds. \]  

The Gauss and Weingarten relations are given in the form

\[ x^i |_{\alpha \beta} = - b_{\alpha \beta} N^i \]  
and

\[ N^i |_{\alpha} = b^{\gamma}_{\alpha} x^i |_{\gamma}. \]
The definition of curvature, equation (2.5), following Sanders (1963), is taken opposite in sign to that given in most books on tensors. The present definition gives a positive curvature to a sphere when the surface normal points outward.

From equations (2.2), (2.10), and (2.11) we can show that

$$\bar{x}^i = x^i + d^\gamma x^i - \phi \ N^i$$  \hspace{1cm} (2.12)

where

$$d_{\gamma\alpha} = u_{\gamma\alpha} + b_{\gamma\alpha}$$  \hspace{1cm} (2.13)

and

$$\phi = - w_{\alpha} + b^\gamma u_{\gamma}$$  \hspace{1cm} (2.14)

Leonard (1961) has shown that the unit normal to the deformed middle surface is

$$\bar{N}^i = \sqrt{g/g} \ [(\theta^\gamma + R^\gamma) x^i + (1 + d^{\omega}_{\omega} + H) N^i]$$  \hspace{1cm} (2.15)

where

$$R^\gamma = \phi_{\gamma} d_{\omega} - \phi d_{\omega\gamma}$$  \hspace{1cm} (2.16)

and

$$H = 1/2 \ (d_{\omega\rho} - d_{\omega\rho} d_{\omega\rho}) .$$  \hspace{1cm} (2.17)

The membrane strain tensor $E_{\alpha\beta}$ is defined by

$$E_{\alpha\beta} = 1/2 \ (\bar{g}_{\alpha\beta} - g_{\alpha\beta})$$  \hspace{1cm} (2.18)

and can be written as

$$E_{\alpha\beta} = e_{\alpha\beta} + 1/2 \ (d_{\alpha} d_{\gamma\beta} + \phi_{\alpha} \phi_{\beta})$$  \hspace{1cm} (2.19)
where the linear part of $E_{\alpha\beta}$ is

$$e_{\alpha\beta} = \frac{1}{2} (u_{\alpha|\beta} + u_{\beta|\alpha}) + b_{\alpha\beta} w . \tag{2.20}$$

Consider two types of surface loading, a pressure, $p$, per unit deformed area acting in the direction of $N^i$ and a loading of intensity $q^i$ prescribed as a force per unit undeformed area written as

$$q^i = f^\alpha x^i|\alpha + f N^i . \tag{2.21}$$

Thus the total force acting on the undeformed area is

$$p \sqrt{g/g} N^i + f^\alpha x^i|\alpha + f N^i . \tag{2.22}$$

In general on any edge there may exist a force vector $Q^i$ per unit undeformed length of middle surface and a moment vector $M^i$ per unit undeformed length. The force on any element of deformed arc length $ds$ having unit normal

$$n^i = \frac{n^\alpha}{n_\alpha} x^i|\alpha \tag{2.23}$$

is given by

$$(N^\alpha\beta \frac{n^\alpha}{n_\alpha} x^i|\beta + Q N^i) ds = Q^i ds \tag{2.24}$$

where $N^\alpha\beta$ is the membrane stress resultant and $Q$ is the transverse shear. The force vector $Q^i$ can also be written as

$$Q^i = T^\alpha x^i|\alpha + Q N^i \tag{2.25}$$

where $T^\alpha$ are the components in the surface and $Q$ is perpendicular to the surface. The moment vector $M^i$ is given by
\[
\frac{-g^{\alpha \omega}}{c_{\beta \alpha}} M^\gamma_\beta \frac{1}{n_\gamma} x_\omega ds = M^i_1 ds \tag{2.26}
\]

where \(M^{\alpha \beta}\) is the unsymmetrical stress-couple tensor and \(g^{-1}_{\alpha \omega}\) is the inverse of \(g_{\alpha \omega}\) and constitutes the only exception in this paper to the convention that indices are raised or lowered by means of the metric tensor of the undeformed shell. In the absence of couple stresses one restriction on \(M^i_1\) is that

\[
M^i_1 N^{-1} = 0 . \tag{2.27}
\]

B. Exact Tensor Equations for Arbitrary Plates

The exact tensor equilibrium equations for a plate can be developed from the variational equation of equilibrium for an arbitrary shell. From the variational approach the boundary conditions can also be developed. Only a brief outline of this development will be given.

The variational equation of equilibrium as presented by Budiansky (1968) for an arbitrary shell is

\[
\int_A (\tilde{n}^{\alpha \beta} \delta E_{\alpha \beta} + \tilde{M}^{\alpha \beta} \delta K_{\alpha \beta}) dA = \int_A p N^1 \delta U^1 dA
\]

\[
+ \int_A q^1 \delta U^1 dA + \int_C q^1 \delta U^1 ds \tag{2.28}
\]

where \(A\) and \(C\) are the initial area and boundary of the shell middle surface, \(\tilde{A}\) is the deformed area and

\[
\tilde{n}^{\alpha \beta} = \tilde{n}^{\alpha \beta} + 1/2 \left( b^\alpha_\gamma M^\gamma_\beta + b^\beta_\gamma M^\alpha_\gamma \right)
\]

\[
+ g^{\alpha \beta} b^i_\omega M^i_\omega, \tag{2.29}
\]
\[
\ddot K_{\alpha\beta} = \ddot K_{\alpha\beta} - 1/2 (b_\gamma^\gamma e_{\gamma\beta} + b_\gamma^\gamma e_{\gamma\alpha}) - b_{\alpha\beta} E_\gamma^\gamma,
\]
(2.30)

\[
\bar M^{\alpha\beta} = 1/2 (M^{\alpha\beta} + M^\beta\alpha),
\]

\[
\tilde n^{\alpha\beta} = \sqrt{\frac{g}{\bar g}} [n^{\alpha\beta} - b_{\gamma\rho} (g^{\rho\beta} M^{\gamma\alpha} + g^{\alpha\beta} M^{\gamma\rho})],
\]
(2.31)

\[
\ddot K_{\alpha\beta} = \sqrt{\frac{g}{\bar g}} \tilde n^{\alpha\beta} - b_{\alpha\beta},
\]
(2.32)

and

\[
\bar b_{\alpha\beta} = 2 \sqrt{\frac{g}{\bar g}} [(1 + e_\omega^\omega + H) (b_{\alpha\beta} + \phi_{\alpha|\beta} + b_{\gamma\alpha}^\gamma d_{\gamma\alpha}) - (\phi^\gamma + R^\gamma) (d_{\gamma\alpha|\beta} - b_{\alpha\beta} \phi_{\gamma})].
\]
(2.33)

For a plate

\[
b_{\alpha\beta} = 0;
\]
(2.34)

Therefore, when the variational equation (2.29) is applied to a plate, the equation is simplified.

We can then write the above quantities as

\[
\ddot n^{\alpha\beta} = n^{\alpha\beta} = \sqrt{\frac{g}{\bar g}} [n^{\alpha\beta} - b_{\gamma\rho} (g^{\rho\beta} M^{\gamma\alpha} + g^{\alpha\beta} M^{\gamma\rho})],
\]
(2.35)

\[
\ddot K_{\alpha\beta} = \ddot K_{\alpha\beta} = \sqrt{\frac{g}{\bar g}} \tilde n^{\alpha\beta},
\]
(2.36)

\[
\bar M^{\alpha\beta} = 1/2 (M^{\alpha\beta} + M^\beta\alpha)
\]
(2.37)

and

\[
\bar b_{\alpha\beta} = \sqrt{\frac{g}{\bar g}} [(1 + e_\omega^\omega + H) (\phi_{\alpha|\beta}) - (\phi^\gamma + R^\gamma) (d_{\gamma\alpha|\beta})].
\]
(2.38)
With the use of variational calculus the following equilibrium equations and boundary conditions are obtained. The equilibrium equations are

\[
\left( g_{\kappa\alpha} + d_{\kappa\alpha} \right) \tilde{n}^{\alpha\beta} - \left[ \bar{M}^{\alpha\beta} \left( \left[ \delta^\rho_\kappa + (g^\omega_\lambda \delta^\rho_\kappa - g^\rho_\omega \delta^\lambda_\kappa) u_\omega | \lambda \right] w | \alpha \beta \right.ight.
\]

\[
+ \left[ g^{\rho \nu} \omega | \kappa - g^{\lambda \nu} \phi^\rho_\kappa \omega | \lambda \right] u_\nu | \alpha \beta \left\} | \rho \right. + \left[ \bar{M}^{\alpha\beta} (\phi_\kappa + R_\kappa) \right] | \beta \alpha
\]

\[
+ p(\phi_\kappa + R_\kappa) + f_\kappa = 0
\]

(2.39)

and

\[
\left( \tilde{n}^{\alpha\beta} w | \alpha \right) | \beta - \left[ \bar{M}^{\alpha\beta} \left( g^{\lambda \nu} + (g^{\phi \nu} \lambda^\rho_\kappa - g^\rho \phi^\lambda_\kappa) n_\omega | \phi \right) u_\nu | \alpha \beta \right| \lambda
\]

\[
+ \left[ \bar{M}^{\alpha\beta} (1 + e_\omega + H) \right] | \beta \alpha + p(1 + e_\omega + H) + f = 0,
\]

(2.40)

and the boundary conditions prescribe

\[
n_\rho \left\{ \tilde{n}^{\phi \rho} + \tilde{n}^{\alpha \rho} \phi^\lambda_\alpha u_\lambda | \alpha - \bar{M}^{\alpha\beta} \left( \left[ \phi^\rho_\kappa + (g^\omega_\lambda \phi^\rho_\kappa - g^\rho \phi^\lambda_\kappa) u_\omega | \lambda \right] w | \alpha \beta \right. \right.
\]

\[
+ \left[ g^{\rho \nu} \omega | \omega - g^{\lambda \nu} \phi^\rho_\lambda \omega | \lambda \right] u_\nu | \alpha \beta \left\} + \left[ \bar{M}^{\alpha\beta} \left( \phi^\phi + R^\phi \right) \right] | \beta \}
\]

\[
= T^\phi
\]

(2.41)

or the displacement \( u_\phi \), and

\[
n_\lambda \left\{ \tilde{n}^{\phi \lambda} w | \phi - \bar{M}^{\alpha\beta} \left( g^{\phi \lambda} - g^\rho \phi^\lambda_\omega \right) u_\omega | \phi \right\} u_\nu | \alpha \rho
\]

\[
+ \left[ \bar{M}^{\lambda\beta} (1 + e_\omega + H) \right] | \beta \} = Q
\]

(2.42)

or the displacement \( w \).
C. Exact Tensor and Physical Equations for Symmetric Circular Plates

For a rotationally symmetric circular plate

\[ \xi^1 = r, \]  
\[ \xi^2 = \theta. \]  

Therefore

\[ x = x^1 = r \cos \theta, \]  
\[ y = x^2 = r \sin \theta. \]  

and

\[ z = x^3 = 0. \]  

The metric is given by

\[ g_{\alpha\beta} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \]  
\[ \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}. \]  

and

\[ g = |g_{\alpha\beta}| = r^2. \]  

Symmetry requires that

\[ \frac{\partial}{\partial \theta} ( ) = 0, \]  
\[ u_2 = 0. \]
and

\[ \frac{\mathbf{M}^{12}}{\mathbf{n}^{12}} = \frac{\mathbf{M}^{21}}{\mathbf{n}^{21}} = \frac{\mathbf{n}^{12}}{\mathbf{n}^{21}} = \frac{\mathbf{n}^{21}}{\mathbf{n}^{12}} = 0. \]  

(2.52)

By substituting the above into the equilibrium equations (2.39) and (2.40) the tensor equilibrium equations for a rotationally symmetric circular plate are found to be

\[
\begin{align*}
\mathbf{n}^{11} & (1 + u_{1,1}) + \mathbf{n}^{11} [u_{1,11} + 1/r (1 + u_{1,1})] \\
- \mathbf{n}^{22} & (1 + \frac{u_{1}}{r}) r - \mathbf{M}_{11}^{11} [(1 + \frac{u_{1}}{r}) w_{,1}] \\
- \mathbf{M}_{11}^{11} & (3(1 + \frac{u_{1}}{r}) w_{,11} + \frac{2}{r} (1 + u_{1,1}) w_{,1}] \\
- \mathbf{M}_{11}^{11} & (2(1 + \frac{u_{1}}{r}) w_{,111} + \frac{2}{r} (1 + u_{1,1}) w_{,11}} \\
+ \frac{2}{r} & u_{1,11} w_{,1} + \mathbf{M}_{11}^{22} [2(1 + \frac{u_{1}}{r}) w_{,1}] \\
- \mathbf{M}_{11}^{11} & (1 + \frac{u_{1}}{r}) w_{,1} + f_{1} = 0
\end{align*}
\]  

(2.53)

in the r direction, and

\[
\begin{align*}
\dot{\mathbf{n}}^{11} & w_{,1} + \dot{\mathbf{n}}^{11} (w_{,11} + \frac{w_{,1}}{r}) + \mathbf{M}_{11}^{11} [(1 + \frac{u_{1}}{r}) \\
(1 + u_{1,1})] + \mathbf{M}_{11}^{11} [3(1 + \frac{u_{1}}{r}) u_{1,11} + \frac{2}{r} (1 + u_{1,1})^2] \\
+ \mathbf{M}_{11}^{11} & (2(1 + \frac{u_{1}}{r}) u_{1,11} + \frac{4}{r} (1 + u_{1,1}) u_{1,11}] \\
- \mathbf{M}_{11}^{22} & [r(1 + \frac{u_{1}}{r})^2] - \mathbf{M}_{11}^{22} [w(1 + \frac{u_{1}}{r}) (1 + u_{1,1})] \\
+ \mathbf{p} & [(1 + \frac{u_{1}}{r}) (1 + u_{1,1})] + f = 0
\end{align*}
\]  

(2.54)
in the z direction.

The equilibrium equation in the θ direction is identically satisfied. The boundary conditions and displacements in tensor notation are found from equations (2.41) and (2.42).

The boundary conditions in tensor notation prescribe

\[ T^1 = \tilde{\mathbf{n}}^{11} (1 + u_{1,1}) - \tilde{\mathbf{M}}^{11} \frac{u_{1}}{r} \left( (1 + \frac{u_{1}}{r}) w_{,1} \right) \]

\[ - \tilde{\mathbf{M}}^{11} \left[ 2(1 + \frac{u_{1}}{r}) w_{,11} - \frac{1}{r} \left( (1 + u_{1,1}) w_{,1} \right) \right] \quad (2.55) \]

or the displacement \( u_1 \), and

\[ Q = \tilde{\mathbf{n}}^{11} w_{,1} + \tilde{\mathbf{M}}^{11} \left[ (1 + \frac{u_{1}}{r}) \left( (1 + u_{1,1}) \right) \right] \]

\[ + \tilde{\mathbf{M}}^{11} \left[ 2(1 + \frac{u_{1}}{r}) u_{1,11} + \frac{1}{r} \left( (1 + u_{1,1})^2 \right) \right] \]

\[ - \tilde{\mathbf{M}}^{22} \left[ r(1 + \frac{u_{1}}{r})^2 \right] \quad (2.56) \]

or the displacement \( w \).

Commas in the above equations denote ordinary differentiation.

To convert these equations into physical equations the relations given by Frederick (1956) were used. They are

\[ u_r = u_1 \]  \quad (2.57)

\[ w = w \]  \quad (2.58)

\[ \tilde{n}_r = \tilde{n}^{11} \]  \quad (2.59)

\[ \tilde{n}_\theta = r^2 \tilde{n}^{22} \]  \quad (2.60)
\[ \bar{M} = \bar{M}^{-11} \]  
(2.61)

and

\[ \bar{M}_{\theta} = r^{2-22}. \]  
(2.62)

By using the above relations the equilibrium equations in physical
variables are written as

\[
\begin{align*}
n_r' (1 + u_r') + n_r \left[ u_r'' + \frac{1}{r} (1 + u_r') \right] \\
- n \left[ \frac{1}{r} (1 + \frac{u_r}{r}) \right] - \bar{M}_r'' \left[ (1 + \frac{u_r}{r}) w' \right] \\
- \bar{M}_r' \left[ 3(1 + \frac{u_r}{r}) w'' + \frac{2}{r} (1 + u_r') w' \right] \\
- \bar{M}_r \left[ 2(1 + \frac{u_r}{r}) w''' + \frac{2}{r} (1 + u_r') w'' \right] \\
+ \frac{2}{r} u_r w' + \bar{M}_\theta \left[ \frac{2}{r^2} (1 + \frac{u_r}{r}) w' \right] \\
- p[(1 + \frac{u_r}{r}) w'] + f_r = 0
\end{align*}
\]  
(2.63)

in the \( r \) direction, and

\[
\begin{align*}
n_r' w' + n_r' (w'' + \frac{w'}{r}) + \bar{M}_r'' \left[ (1 + \frac{u_r}{r}) (1 + u_r') \right] \\
+ \bar{M}_r' \left[ 3(1 + \frac{u_r}{r}) u_r'' + \frac{2}{r} (1 + u_r') \right]^2 \right] \\
+ \bar{M}_r \left[ 2(1 + \frac{u_r}{r}) u_r''' + \frac{4}{r} (1 + u_r') u_r'' \right] \\
- \bar{M}_r' \left[ \frac{1}{r} (1 + \frac{u_r}{r}) \right]^2 \right] - \bar{N} \left[ \frac{2}{r^2} (1 + \frac{u_r}{r}) (u_r' - \frac{u_r}{r}) \right] \\
p[(1 + \frac{u_r}{r}) (1 + u_r')] + \epsilon = 0
\end{align*}
\]  
(2.64)

in the \( z \) direction.
The boundary conditions in physical variables prescribe

\[ T_r = \tilde{n}_r (1 + u_r) - \overline{M}_r [(1 + \frac{u_r}{r}) w'] \]

\[ - \overline{M}_r [2(1 + \frac{u_r}{r}) w'' + \frac{1}{r} (1 + u_r') w'] \]  

or the displacement \( u_r \), and

\[ Q = \tilde{n}_r w' + \overline{M}_r [(1 + \frac{u_r}{r}) (1 + u_r')] \]

\[ + \overline{M}_r [2(1 + \frac{u_r}{r}) u_r'' + \frac{1}{r} (1 + u_r')^2] \]

\[ - \overline{M}_0 \left[ \frac{1}{r} (1 + \frac{u_r}{r})^2 \right] \]  

or the displacement \( w \).

Primes in the above equations represent ordinary differentiation

with respect to \( r \).
III. CONSTITUTIVE EQUATIONS

The equilibrium equations and boundary conditions given in the previous chapter are exact. However, we need a relationship between stress and moment resultants and the strain in addition to the equilibrium equations and boundary conditions.

Since the purpose of this paper is to study the large deflections of a circular thin plate made from a rubber-like material, we want to choose a set of constitutive equations that will most accurately describe the response of this material under large deformations. Of the several proposed forms of the constitutive relations mentioned in Chapter I, three seem most suited for our problem. They are the Rivlin-Saunders theory, the Hart-Smith theory and the Alexander theory. In order to apply these theories to our problem we need to understand the parameters involved. Each of the three theories will be discussed separately and in the necessary detail later in this chapter.

It has been shown by Rivlin (1948) that for an initially isotropic, elastic, incompressible material a constitutive relation can be derived from a strain energy density function, $W$, as

$$\sigma_i = 2 \left[ \lambda_1^2 \frac{\partial W}{\partial I_1} - \frac{1}{\lambda_1^2} \frac{\partial W}{\partial I_2} \right] + p \quad (i=1,2,3) \quad (3.1)$$

where $\sigma_i$ are the principal true stresses, $\lambda_1$ is the principal extension in the $i$ direction, $p$ is an arbitrary hydrostatic pressure which, due to incompressibility, does not affect the deformation, and $I_1$ and $I_2$ are the first two invariants of the extensions defined as
and, for an incompressible material,

\[ I_3 = \lambda_1 \lambda_2 \lambda_3 = 1. \]  

(3.4)

The stress deviator is the only part of the stress that causes deformation for an incompressible material. The stress deviator is defined as

\[ s_{ij} = \sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk}. \]  

(3.5)

Therefore, for an incompressible material, equation (3.1) can be written as

\[ s_i = \frac{2}{3} \left\{ (2\lambda_1^2 - \lambda_1^2 - \lambda_{1+1}^2) \frac{\partial W}{\partial I_1} + (\lambda_{1-1}^2 \lambda_1 + \lambda_{1+1}^2 \lambda_1) \frac{\partial W}{\partial I_2} \right\} \]  

(3.6)

with the indices cyclic in the form 1, 2, 3, 1, 2, 3, etc. and i not summed.

A. Rivlin and Saunders Constitutive Relations

Rivlin and Saunders (1951) performed a series of experiments using different types of vulcanized rubbers and found, assuming incompressibility, that \( \partial W / \partial I_1 \) is substantially a constant and that \( \partial W / \partial I_2 \) is independent of \( I_1 \) but varies with \( I_2 \). Therefore a strain energy density function was proposed in the form
\[ W = c_1(I_1-3) + f(I_2-3) \]  

(3.7)

where \( c_1 \) and the function \( f(I_2-3) \) are to be determined by experiments.

From equation (3.6) the constitutive relations become

\[ s_i = \frac{2}{3} \left\{ (2\lambda_i^2 - \lambda_i^2 - \lambda_i^2) c_1 ight\} \]
\[ + (\lambda_i^2 \lambda_i^{1-1} + \lambda_i^2 \lambda_i^{1+1} - 2\lambda_i^{1-1} \lambda_i^{1+1}) \frac{\partial f}{\partial I_2} \]  

(3.8)

with \( i \) not summed.

B. Hart-Smith Constitutive Relations

Using the experiments of Treloar (1944), Hart-Smith found that up to a value of \( I_1 = 12 \), \( \partial W/\partial I_1 \) is essentially a constant and that for values of \( I_1 > 12 \), \( \partial W/\partial I_1 \) increases with increasing \( I_1 \). It was assumed therefore that the partial derivative of the strain energy density with respect to the strain invariant \( I_1 \) could be expressed as

\[ \frac{\partial W}{\partial I_1} = Ge \ k_1(I_1-3)^2 \]  

(3.9)

where \( G \) and \( k_1 \) are constants. He also assumed the form for \( \partial f(I_2-3)/\partial I_2 \) as proposed by Gent and Thomas (1958), given as

\[ \frac{\partial W}{\partial I_2} = \frac{Gk_2}{I_2} \]  

(3.10)

where \( G \) and \( k_2 \) are constants. These two partial derivatives are known as the exponential-hyperbolic elastic parameters. Using these parameters, the strain energy density function becomes

\[ W = G \ k_1(I_1-3)^2 \]
\[ + k_2 \ln \left( \frac{I_2}{3} \right) \]

(3.11)
which gives the constitutive relations,

\[
s_i = \frac{2}{3} G \left\{ (2\lambda_{i}^{2} - \lambda_{i-1}^{2} - \lambda_{i+1}^{2}) e^{k_{1}(I_{1}-3)^{2}} + \left( \lambda_{i}^{2} \lambda_{i-1}^{2} + \lambda_{i}^{2} \lambda_{i+1}^{2} - 2\lambda_{i-1}^{2} \lambda_{i+1}^{2} \right) \frac{k_{2}}{I_{2}} \right\}
\]  

(3.12)

For equi-biaxial tension experiments, this theory gives good results up to \( \lambda_{b} = 2.6 \), and for very large values of \( \lambda_{b} \) the theory seems to follow the experimental results. However, the intermediate values of \( \lambda_{b} \) fall below the experimental data. For uniaxial tension the results are somewhat better than for biaxial tension throughout all the ranges of \( \lambda_{u} \).

C. Alexander Constitutive Relations

H. Alexander (1968) extended the Rivlin-Saunders theory by showing that \( f(I_{2} - 3) \) can be expressed as a simple function that has the same basic form for many natural and synthetic rubbers.

For uniaxial stretching the constitutive equation (3.6) becomes

\[
s_{u} = 2(\lambda_{u}^{2} - \frac{1}{\lambda_{u}}) \left( C_{1} + \frac{1}{\lambda_{u}} \frac{\partial f}{\partial I_{2}} \right)
\]  

(3.13)

which can be rearranged as

\[
\left( \frac{\partial f}{\partial I_{2}} \right)_{u} = \frac{s_{u}}{2(\lambda_{u} - \frac{1}{\lambda_{u}^{2}})} - C_{1} \lambda_{u}
\]  

(3.14)
Also, the second invariant can be expressed as

\[ I_2 = 2\lambda_u + \frac{1}{\lambda_u^2}, \tag{3.15} \]

For equi-biaxial tension, equation (3.6) becomes

\[ S_b = 2\left(\lambda_b^2 - \frac{1}{\lambda_b^4}\right) C_1 + \lambda_b^2 \frac{\partial f}{\partial I_2}, \tag{3.16} \]

which can be rearranged as

\[ \frac{\partial f}{\partial I_2} = \frac{S_b}{2\left(\lambda_b^4 - \frac{1}{\lambda_b^2}\right)} - \frac{C_1}{\lambda_b^2}, \tag{3.17} \]

and the second invariant can be expressed as

\[ I_2 = \lambda_b^4 + \frac{2}{\lambda_b^2}. \tag{3.18} \]

The curve of \( (\partial f/\partial I_2)_b \) vs. \( I_2 \) should be identical with the curve of \( (\partial f/\partial I_2)_u \) vs. \( I_2 \) for some value of \( C_1 \) if equation (3.7) is a valid expression for \( W \). These curves were evaluated for a large range of values of \( C_1 \). The assumption of incompressibility expressed in the form of equation (3.4) was verified by conducting a uniaxial test on a strip of neoprene (see Fig. 1). Then using his experimental data from uniaxial and biaxial tension tests over a wide range of deformations, \( C_1 \) was evaluated and then a transposed hyperbola was fitted to the resulting curve (see Fig. 2), yielding

\[ \frac{\partial f}{\partial I_2} = \frac{C_2}{(I_2 - 3) + \gamma} + C_3, \tag{3.19} \]
with

\[ c_1 = 17.00 \text{ psi,} \quad (3.20) \]

\[ c_2 = 19.85 \text{ psi,} \quad (3.21) \]

\[ c_3 = 1.0 \text{ psi} \quad (3.22) \]

and

\[ \gamma = 0.735. \quad (3.23) \]

By integrating equation (3.19) and substituting into equation (3.7), we have a strain-energy density of the form

\[ W = c_1(I_1-3) + c_2 \ln \left( \frac{I_2-3}{\gamma} \right) + c_3(I_2-3). \quad (3.24) \]

For small strain, equation (3.6) can be shown to reduce to Hooke's Law with

\[ \frac{\partial W}{\partial I_1} \bigg|_{\varepsilon=0} + \frac{\partial W}{\partial I_2} \bigg|_{\varepsilon=0} = \frac{E}{6} \quad (3.25) \]

where \( E \), the elastic modulus, should be thought of as the slope of the uniaxial stress vs. strain curve at zero strain,

\[ E = \frac{\partial \sigma}{\partial \varepsilon} \bigg|_{\varepsilon=0} \quad (3.26) \]

Therefore, using equations (3.24) and (3.25), we have

\[ c_1 + \frac{c_2}{\gamma} + c_3 = \frac{E}{6}. \quad (3.27) \]
To non-dimensionalize the constants $C_1$, $C_2$ and $C_3$ a shear modulus is defined as

$$\mu = \frac{E}{3}$$ \hspace{1cm} (3.28)

and the constants are then written in non-dimensional form,

$$\overline{C}_i = \frac{2C_i}{\mu} \hspace{1cm} i = 1, 2, 3.$$ \hspace{1cm} (3.29)

The strain energy density function can then be expressed as

$$W = \frac{\mu}{2} \left\{ \overline{C}_1 (I_1 - 3) + \overline{C}_2 \ln \left[ \frac{(I_2 - 3) + \gamma}{\gamma} \right] + \overline{C}_3 (I_2 - 3) \right\}$$ \hspace{1cm} (3.30)

and the constitutive relation becomes

$$s_1 = \frac{\mu}{3} \left\{ (2\lambda_1^2 - \lambda_{1-1}^2 - \lambda_{1+1}^2) \overline{C}_1 
+ (\lambda_1^2 \lambda_{1-1} + \lambda_1^2 \lambda_{1+1}^2 - 2\lambda_{1-1} \lambda_{1+1}) \left[ \frac{\overline{C}_2}{(I_2 - 3) + \gamma + \overline{C}_3} \right] \right\}.$$ \hspace{1cm} (3.31)

Equation (3.31) is called the extended Rivlin-Saunders theory. Correlation between experiments and theory was found to be very good for $\lambda_b < 3.5$ and $\lambda_u < 6$.

Noting that the Hart-Smith theory gives good results for very large extensions, Alexander proposed a combined Hart-Smith and extended Rivlin-Saunders theory. A strain energy density function giving the $\partial W/\partial I_1$ of the Hart-Smith theory and the $\partial W/\partial I_2$ of the extended Rivlin-Saunders theory was proposed in the form
The constitutive relation then becomes,

\[ W = \frac{\mu}{2} \left\{ \bar{C}_1 \int e^{k(I_1-3)^2} dI_1 + \bar{C}_2 \ln \left[ \frac{(I_2-3) + \gamma}{\gamma} \right] \right\} + \bar{C}_3 (I_2-3). \] (3.32)

The constitutive relation then becomes,

\[ s_i = \frac{\mu}{3} \left\{ (2\lambda_1^2 - \lambda_{i-1}^2 - \lambda_{i+1}^2) \bar{C}_1 e^{k(I_1-3)^2} + \frac{\bar{C}_2}{(I_2-3) + \gamma + \bar{C}_3} \right\}. \] (3.33)

Equation (3.33) is called the Alexander constitutive relation and with a value of

\[ k = 0.00015 \] (3.34)

correlation between theory and experimental data was excellent throughout all ranges of deformation (see Fig. 3).

Note that equations (3.32) and (3.33) reduce to the extended Rivlin-Saunders theory for \( k = 0 \) and to those of the Hart-Smith theory for \( \gamma = 3 \) and \( C_3 = 0 \).

D. Stress-Resultants and Stress-Couples for a Thin Plate

The true stress resultants and stress-couple resultants for a shell can be written in the following forms:

\[ N_r = \int_{-h/2}^{+h/2} \sigma_1 \left( 1 + \frac{z}{R_2} \right) dz, \] (3.35)
\[ N_\theta = \int_{-\tilde{h}/2}^{+\tilde{h}/2} \sigma_2 \left( 1 + \frac{z}{R_2} \right) dz \quad (3.36) \]

\[ \bar{M}_r = \int_{-\tilde{h}/2}^{+\tilde{h}/2} z\sigma_1 \left( 1 + \frac{z}{R_1} \right) dz \quad (3.37) \]

and

\[ \bar{M}_\theta = \int_{-\tilde{h}/2}^{+\tilde{h}/2} z\sigma_2 \left( 1 + \frac{z}{R_1} \right) dz \quad (3.38) \]

where \( R_1 \) and \( R_2 \) are principal radii of curvature and \( \tilde{h} \) is the thickness of the deformed shell. For a thin plate the ratios \( z/R_1 \) and \( z/R_2 \) are considered small with respect to unity. Therefore the stress resultants and stress-couple resultants can be simplified to the form

\[ N_r = \int_{-\tilde{h}/2}^{+\tilde{h}/2} \sigma_1 \, dz \quad (3.39) \]

\[ N_\theta = \int_{-\tilde{h}/2}^{+\tilde{h}/2} \sigma_2 \, dz \quad (3.40) \]

\[ \bar{M}_r = \int_{-\tilde{h}/2}^{+\tilde{h}/2} z\sigma_1 \, dz \quad (3.41) \]

and

\[ \bar{M}_\theta = \int_{-\tilde{h}/2}^{+\tilde{h}/2} z\sigma_2 \, dz \quad (3.42) \]

By expanding equation (3.5) we can solve for the principal true stresses in terms of the deviator stresses for an incompressible
material. They are
\[ \sigma_1 = 2s_1 + s_2 \]  
(3.43)
and
\[ \sigma_1 = s_1 + 2s_2 . \]  
(3.44)

For an incompressible material, the relation between the thickness of the deformed and the undeformed plate can be shown by using equation (2.8) to be
\[ \tilde{h} = \sqrt{\frac{g}{\tilde{g}}} h . \]  
(3.45)

Assuming a uniform stress distribution across the thickness of a thin plate due to the stretching, the equations for the true stress resultants can be integrated. They are
\[ N_r = h \sqrt{\frac{g}{\tilde{g}}} (2s_1 + s_2) \]  
(3.46)
and
\[ N_\theta = h \sqrt{\frac{g}{\tilde{g}}} (s_1 + 2s_2) . \]  
(3.47)

In order to write the stress resultants and the stress-couple resultants in terms of displacements, the metric tensor of the deformed plate must be found. The metric tensor of the deformed plate is written as
\[- g_{\alpha\beta} = \frac{1}{x} |x^{-1} | \beta . \]  
(3.48)
Using equations (2.2), (2.3), (2.5), (2.10) and (2.11), we can show that

\[ x^i_\alpha = x^i_\alpha + g^{ik} u_k^j x^j_\alpha + w^i_\alpha N^i. \]  

(3.49)

By substituting equations (2.45), (2.46) and (2.47) for an axisymmetric plate into equation (3.49) and using equation (3.48), we can obtain the metric tensor in the deformed plate in terms of physical components of displacement,

\[
\hat{g}_{\alpha \beta} = \begin{pmatrix}
(1 + u^1_r)^2 + (w^1_r)^2 & 0 \\
0 & r^2(1 + u^1_r) 
\end{pmatrix},
\]

(3.50)

and

\[
\hat{g}_{\alpha \beta} = \begin{pmatrix}
1 \\
0 
\end{pmatrix} = \begin{pmatrix}
(1 + u^1_r)^2 + (w^1_r)^2 \\
0 
\end{pmatrix}
\]

(3.51)

and

\[
\hat{g} = \sqrt{\hat{g}_{\alpha \beta}} = r^2(1 + u^1_r)^2 \left[ (1 + u^1_r)^2 + (w^1_r)^2 \right].
\]

(3.52)

The principal extension in the i direction is defined as

\[
\lambda_i = \frac{\text{final length}}{\text{initial length}} = \frac{(\ell_i^f)}{(\ell_i^0)}. \]

(3.53)
i not summed. If the physical strains, $\varepsilon_1$, are defined as

$$\varepsilon_1 = \frac{(l_i)_{f1} - (l_o)_{i1}}{(l_o)_{i1}}$$

(3.54)

i not summed, we can write

$$\lambda_1 = \varepsilon_1 + 1$$

(3.55)

or

$$\lambda_1^2 = \varepsilon_1^2 + 2\varepsilon_1 + 1.$$  

(3.56)

The relations between the physical strains and the strain tensor, $E_{\alpha\beta}$, shown in the Appendix are

$$\varepsilon_1 = \sqrt{1 + 2E_{11}} - 1$$

(3.57)

and

$$\varepsilon_2 = \sqrt{1 + 2\frac{E_{22}}{r^2}} - 1.$$  

(3.58)

Therefore, from equation (3.56), (3.57) and (3.58) we can relate the principal extensions to the strain tensor. These relations are presented as

$$\lambda_1^2 = 1 + 2E_{11}$$

(3.59)

and

$$\lambda_1^2 = 1 + 2\frac{E_{22}}{r^2}.$$  

(3.60)
The strain-displacement relations for the stress resultants are found by substituting equations (2.13), (2.14), (2.20), and (2.34) into equation (2.19). They are

\[ E_{11} = \frac{1}{2} (2 + u'_r) u'_r + \frac{1}{2} (w')^2 \]  

(3.61)

and

\[ E_{22} = \frac{r}{2} (2 + \frac{u_r}{r}) u_r. \]  

(3.62)

Expanding equation (2.38) for a rotationally symmetric thin flat plate, the curvatures of the deformed plate become

\[ \bar{b}_{11} = \sqrt{\frac{E}{g}} \left[ - (1 + u'_r) (1 + \frac{u_r}{r}) w'' + u''_r (1 + \frac{u_r}{r}) w' \right] \]  

(3.63)

and

\[ \bar{b}_{22} = - \sqrt{\frac{E}{g}} \left[ r (1 + \frac{u_r}{r}) w' \right]. \]  

(3.64)

The modified stress resultants used in the equilibrium equations and the boundary conditions can be found by expanding equation (2.35) and substituting in equations (3.4-6), (3.47), (3.51), (3.63) and (3.64). The modified stress resultants are given as

\[ \bar{n}_r = h (2s_1 + s_2) + \frac{2 u_r [(1 + u'_r) w'' - u''_r w'}{[(1 + u'_r)^2 + (w')^2]} M_r \]

\[ + \frac{1}{r} \frac{u}{[(1 + u'_r)^2 + (w')^2]} M_\theta \]  

(3.65)

and
\[\tilde{n}_\theta = h(s_1 + 2s_2) + \frac{2(1+\frac{u_r}{r})[(1+u'_r)w''-u''_r w']}{(1+\frac{u_r}{r})} M_r \]

\[+ \frac{2}{r} [(1+\frac{u_r}{r}) w'] M_\theta \cdot \] (3.66)

Koiter (1960) has shown that the last two terms in equations (3.65) and (3.66) are terms of the same order as those that are neglected in a first approximation theory. He shows that if

\[(h/L)^2 \ll 1 \] (3.67)

then the last two terms in equations (3.65) and (3.66) are small compared to the other terms. The L in equation (3.66) is the smallest wave length of deformation of the deformed reference surface. L is defined by Koter as

\[\frac{|d\varepsilon_1|}{|ds|}, \frac{|d\varepsilon_2|}{|ds|} = 0(\varepsilon/L) \] (3.68)

\[\frac{|dK_1|}{|ds|}, \frac{|dK_2|}{|ds|} = 0(K/L) \] (3.69)

where \(\varepsilon_1\) and \(\varepsilon_2\) are the principal strains, \(K_1\) and \(K_2\) are the principal curvatures, \(\varepsilon\) and \(K\) are the maximum principal strain and curvature respectively, and \(ds\) is any element on the middle surface.

Since \(E_{\alpha\beta}\) and \(\tilde{K}_{\alpha\beta}\) are suitable measures of the deformed shell, relate the strain tensor to the modified curvature tensor for the stress-couple resultants by
Expanding equation (2.36) and using equation (3.63) and (3.64), we find

\[ \tilde{K}_{11} = (1 + \frac{u_r^2}{r}) \left[-(1 + u_r') w'' + u_r'' w'\right] \]  

(3.72)

and

\[ \tilde{K}_{22} = -r(1 + \frac{u_r^2}{r}) w'. \]  

(3.73)

The strain-displacement relations for the stress-couple resultants then become

\[ E_{11} = z(1 + \frac{u_r}{r}) \left[-(1 + u_r') w'' + u_r'' w'\right] \]  

(3.74)

and

\[ E_{22} = z[-r(1 + \frac{u_r}{r}) w']. \]  

(3.75)

By using equations (3.43) and (3.44) the stress-couple resultants can be written in terms of the stress deviators as

\[ \bar{M}_r = \int_{-\bar{h}/2}^{+\bar{h}/2} z[2s_1 + s_2] \, dz \]  

(3.76)

and

\[ \bar{M}_\theta = \int_{-\bar{h}/2}^{+\bar{h}/2} z[s_1 + 2s_2] \, dz. \]  

(3.77)
The Alexander constitutive relations are to be used in the stress resultants and the stress-couple resultants. Expanding equation (3.33), the stress deviators in terms of the principal extensions are

\[
s_1 = \frac{\mu}{3} \left\{ \left[ 2\lambda_1^2 - \lambda_2^2 - \frac{1}{\lambda_1^2\lambda_2^2} \right] \bar{c}_1 + \frac{k(I_1-3)^2}{c_1} \right\} + \left[ \lambda_1^2\lambda_2^2 + \frac{1}{\lambda_2^2} - \frac{2}{\lambda_1^2} \right] \left[ \frac{c_2}{(I_2-3) + \gamma + c_3} \right] (3.78)
\]

and

\[
s_2 = \frac{\mu}{3} \left\{ \left[ 2\lambda_2^2 - \lambda_1^2 - \frac{1}{\lambda_1^2\lambda_2^2} \right] \bar{c}_1 + \frac{k(I_1-3)^2}{c_1} \right\} + \left[ \lambda_1^2\lambda_2^2 + \frac{1}{\lambda_1^2} - \frac{2}{\lambda_2^2} \right] \left[ \frac{c_2}{(I_2-3) + \gamma + c_3} \right] (3.79)
\]

with the first two invariants of the extensions, written in terms of the principal extensions, found by using equations (3.2), (3.3) and (3.4),

\[
I_1 = \lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2\lambda_2^2} (3.80)
\]

and

\[
I_2 = \lambda_1^2\lambda_2^2 + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} (3.81)
\]

The principal extensions in terms of the displacements for the stress resultants are found by substituting equations (3.61) and (3.62) into equations (3.59) and (3.60), which give
\[ \lambda_1^2 = (1 + u'_r)^2 + (w')^2 \]  
(3.82)

and

\[ \lambda_2^2 = \frac{u_r}{r} (2 + \frac{u_r}{r}) . \]  
(3.83)

The principal extensions in terms of the displacements for the stress-couple resultants are obtained by substituting equations (3.74) and (3.75) into equations (3.59) and (3.60),

\[ \lambda_1^2 = 1 + 2z (1 + \frac{u_r}{r}) [-(1 + u'_r)w'' + u''_r w'] \]  
(3.84)

and

\[ \lambda_2^2 = 1 + 2z [-(1 + \frac{u_r}{r})^2 w'] . \]  
(3.85)

From equations (2.49a) and (3.52) the ratio \( \sqrt{g/g} \) can be written in terms of the displacements as

\[ \sqrt{\frac{g}{g}} = \frac{1}{\{(1 + \frac{u_r}{r})^2 [((1+u'_r)^2 + (w')^2)]\}^{1/2}} . \]  
(3.86)

Therefore, the thickness of the deformed shell can be related to the thickness of the undeformed plate and the displacements as

\[ \bar{h} = \frac{h}{\{(1 + \frac{u_r}{r})^2 [((1+u'_r)^2 + (w')^2)]\}^{1/2}} . \]  
(3.87)
IV. ANALYSIS OF A CIRCULAR PLATE PROBLEM

In this chapter we will study the deflections of a circular plate made from a rubber-like material with (1) uniform thickness, and (2) varying thickness in the radial direction and uniform thickness in the \( \theta \) direction. The plate is deformed due to a uniform pressure, \( p \), acting normal to the deformed middle surface. The outer edge of the plate is considered to be clamped such that it undergoes no rotation or displacement. The rest of the plate is free to move. To solve this problem we will use the exact first approximation tensor equilibrium equations for a symmetric circular plate and the constitutive relation given by Alexander (1968).

A. Plate With Uniform Thickness

The equilibrium equations used in this problem are obtained from equation (2.63) and (2.64) with \( f = f_r = 0 \) and are written in physical components as

\[
\begin{align*}
&\tilde{n}_r (1+u'_r) + \tilde{n}_r [u''_r + \frac{1}{r} (1+u'_r)] - \tilde{n}_\theta \left[ \frac{1}{r} (1 + \frac{u_r}{r}) \right] \\
&- \tilde{M}_r'' [(1 + \frac{u_r}{r})w'] - \tilde{M}_r' [3(1 + \frac{u_r}{r}) w'' + \frac{2}{r} (1 + u'_r)w'] \\
&- \tilde{M}_r [2(1 + \frac{u_r}{r}) w''' + \frac{2}{r} (1+u'_r) w'' + \frac{2}{r} u''_r w'] \\
&+ \tilde{M}_\theta \left[ \frac{2}{r^2} (1 + \frac{u_r}{r}) w' \right] - p[(1 + \frac{u_r}{r}) w'] = 0 \\
\end{align*}
\]

(4.1)

in the \( r \) direction, and
\[ \hat{n}'_r (w') + \hat{n}'_r (w'' + \frac{w'}{r}) + \hat{M}'_r [(1 + \frac{u_r}{r}) (1+u'_r)] \]
\[ + \hat{M}'_r [3(1 + \frac{u_r}{r}) u''_r + \frac{2}{r} (1 + u'_r)^2] \]
\[ + \hat{M}_r [2(1 + \frac{u_r}{r}) u''_r + \frac{4}{r} (1 + u'_r) u''_r] \]
\[ - \hat{M}_0 [\frac{1}{r} (1 + \frac{u_r}{r})^2] - \hat{M}_0 [\frac{2}{r^2} (1 + \frac{u_r}{r}) (u'_r - \frac{u_r}{r})] \]
\[ + p [\frac{u_r}{r} (1+u'_r)] = 0 \tag{4.2} \]

in the z direction.

The constitutive relations as given in equations (3.78) and (3.79) are to be used in the modified stress resultants and the stress-couple resultants. Since the deviator stresses, invariants of the extensions and the principal extensions are functions of the displacements and the radius, r, in the stress resultants, show this relationship by writing

\[ s_1 = s_1(r), \tag{4.3} \]
\[ s_2 = s_2(r), \tag{4.4} \]
\[ I_1 = I_1(r), \tag{4.5} \]
\[ I_2 = I_2(r), \tag{4.6} \]
\[ \lambda_1 = \lambda_1(r) \tag{4.7} \]

and

\[ \lambda_2 = \lambda_2(r). \tag{4.8} \]

The modified stress resultants for a thin plate are found from equations (3.65) and (3.66). Using the notation above, the
modified stress resultants become

\[
\tilde{\mathbf{n}}_r = h \left[ 2s_1(r) + s_2(r) \right]
\]  

(4.9)

and

\[
\tilde{\mathbf{n}} = h \left[ s_1(r) + 2s_2(r) \right]
\]  

(4.10)

where \(s_1(r)\) and \(s_2(r)\) are given by equations (3.78) and (3.79) and written as

\[
s_1(r) = \frac{\mu}{3} \left\{ [2\lambda_1^2(r) - \lambda_2(r)^2 - \frac{1}{\lambda_1^2(r) \lambda_2^2(r)}] 
\right. 
\]

\[
- \frac{k(I_1(r) - 3)^2}{C_1 e} + \left[ \frac{\lambda_1^2(r)}{\lambda_2^2(r)} + \frac{1}{\lambda_2^2(r)} - \frac{2}{\lambda_1^2(r)} \right] 
\]

\[
\left[ \frac{C_2}{(I_2(r) - 3) + \gamma + C_3} \right] \} \]  

(4.11)

and

\[
s_2(r) = \frac{\mu}{3} \left\{ [2\lambda_2^2(r) - \lambda_1^2(r) - \frac{1}{\lambda_1^2(r) \lambda_2^2(r)}] 
\right. 
\]

\[
- \frac{k(I_1(r) - 3)^3}{C_1 e} + \left[ \frac{\lambda_1^2(r)}{\lambda_2^2(r)} + \frac{1}{\lambda_1^2(r)} - \frac{2}{\lambda_2^2(r)} \right] 
\]

\[
\left[ \frac{C_2}{(I_2(r) - 3) + \gamma + C_3} \right] \} \]  

(4.12)

with \(I_1(r)\) and \(I_2(r)\) given by equations (3.80) and (3.81) as
The principal extensions for the modified stress resultants, written in terms of the displacements, are given by equations (3.82) and (3.83) and presented as

\[
\lambda_1^2(r) = (1 + \frac{u'_r}{r})^2 + (w')^2
\] (4.15)

and

\[
\lambda_2^2(r) = \frac{u_r}{r} (2 + \frac{u_r}{r}).
\] (4.16)

In the stress-couple resultants the deviator stresses, invariants of the extension and the principal extensions are functions of \(z\) as well as \(r\) and the displacements. These functional relationships follow. The \(r\)-dependence has been omitted from these equations for the sake of brevity.

\[
s_1 = s_1(z),
\] (4.17)

\[
s_2 = s_2(z),
\] (4.18)

\[
I_1 = I_1(z),
\] (4.19)

\[
I_2 = I_2(z),
\] (4.20)

\[
\lambda_1 = \lambda_1(z)
\] (4.21)
The stress-couple resultants are given by equations (3.76) and (3.77) and are rewritten in the notational form

\[
\overline{M}_r = \int_{-h/2}^{+h/2} z [2s_1(z) + s_2(z)] \, dz
\]

(4.23)

and

\[
\overline{M}_\theta = \int_{-h/2}^{+h/2} z[s_1(z) + 2s_2(z)] \, dz
\]

(4.24)

where \( s_1(z) \) and \( s_2(z) \) have the same form as equations (3.78) and (3.79), and are rewritten to show the functional notation,

\[
s_1(z) = \frac{\mu}{3} \left\{ [2\lambda_1^2(z) - \lambda_2^2(z) - \frac{1}{\lambda_1^2(z) + \lambda_2^2(z)}] \overline{c}_1 e^{k(I_1(z)-3)\lambda_1^2(z) + \lambda_2^2(z)} \right. \\
+ \left. [\lambda_1^2(z) + \lambda_2^2(z) + \frac{1}{\lambda_1^2(z) + \lambda_2^2(z)} - \frac{2}{\lambda_1^2(z) + \lambda_2^2(z)}] \right\}
\]

\[
\frac{\overline{c}_2}{(I_2(z)-3) + \gamma + \overline{c}_3}
\]

(4.25)

and

\[
s_2(z) = \frac{\mu}{3} \left\{ [2\lambda_2^2(z) - \lambda_1^2(z) - \frac{1}{\lambda_1^2(z) + \lambda_2^2(z)}] \overline{c}_1 e^{k(I_1(z)-3)\lambda_1^2(z) + \lambda_2^2(z)} \right. \\
+ \left. [\lambda_1^2(z) + \lambda_2^2(z) + \frac{1}{\lambda_1^2(z) + \lambda_2^2(z)} - \frac{2}{\lambda_1^2(z) + \lambda_2^2(z)}] \right\}
\]

\[
\frac{\overline{c}_2}{(I_2(z)-3) + \gamma + \overline{c}_3} \}
\]

(4.26)
with \( I_1(z) \) and \( I_2(z) \) given by equations (3.80) and (3.81)

\[
I_1(z) = \lambda_1^2(z) + \lambda_2^2(z) + \frac{1}{\lambda_1^2(z) \lambda_2^2(z)}
\]

and

\[
I_2(z) = \lambda_1^2(z) \lambda_2^2(z) + \frac{1}{\lambda_1^2(z)} + \frac{1}{\lambda_2^2(z)}
\]

The principal extensions written in terms of the displacements for the stress-couple resultants are given by equations (3.84) and (3.85) in functional notation as

\[
\lambda_1^2(z) = 1 + 2z(1 + \frac{u_r}{r}) \left[-(1+u_r') w'' + u_r'' w'\right]
\]

and

\[
\lambda_2^2(z) = 1 + 2z[-(1 + \frac{u_r}{r})^2 w']
\]

It is noted that at the center of the circular plate

\[
g_{11} = g_{22}
\]

and the metric for the undeformed circular plate at the center has the forms

\[
g_{\alpha\beta} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
\( g^{\alpha\beta} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) \quad (4.33)

and

\[ g = \left| g_{\alpha\beta} \right| = 1. \] \quad (4.34)

When a uniform pressure, \( p \), is applied normal to the deformed surface of the circular plate the deflections will be symmetrical and at the center we see that

\[ \lambda_1 = \lambda_2 \] \quad (4.35)

and

\[ E_{11} = E_{22}. \] \quad (4.36)

From the equation

\[ E_{\alpha\beta} = \frac{1}{2} (g_{\alpha\beta} - g_{\beta\alpha}), \] \quad (2.18)

we have

\[ \bar{g}_{11} = \bar{g}_{22}. \] \quad (4.37)

Therefore, the metric of the deformed circular plate at the center can be written in the forms
\[ g_{\alpha\beta} = \begin{pmatrix}
(1+u_r')^2 + (w')^2 & 0 \\
0 & (1+u_r')^2 + (w')^2
\end{pmatrix}, \quad (4.38) \]

\[ g_{\alpha\beta} = \begin{pmatrix}
\frac{1}{(1+u_r')^2 + (w')^2} & 0 \\
0 & \frac{1}{(1+u_r')^2 + (w')^2}
\end{pmatrix} \quad (4.39) \]

and

\[ \bar{g} = \left| g_{\alpha\beta} \right| = \left[ (1+u_r')^2 + (w')^2 \right]^2. \quad (4.40) \]

Since \( g_{11} = g_{22} = 1 \), the Christoffel Symbols of the First and Second kind are zero,

\[ [ij,k] = \frac{1}{2} \left[ \frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k} \right] = 0, \quad (4.41) \]

\[ \{ k \} = g^{km} [i, j, m] = 0 \quad (4.42) \]

and the covariant derivative becomes the partial derivative.

It is noted that the \( u_r \) deflections are of the same magnitude but of opposite signs on opposite sides of center for the same value of \( r \).

From the five-point finite difference method used to solve the two fourth order differential equilibrium equations, we can write
where \( i \) is a node point, \( i = 1 \) at the center and increases with positive \( r \), and \( \Delta r \) is the distance between node points, held constant along \( r \).

Since, at the center

\[
u_{r_1} = -u_{r_{i+1}}
\]

we can see from equation (4.43) by setting \( u_{r_i} = 0 \) for \( i = 1 \), that

\[
u_{r_1}'' = 0
\]

at the center, since \( \lambda_1 = \lambda_2 \) we can write

\[
\lambda = \lambda_1 = \lambda_2.
\]

Since, \( \sigma_1 = \sigma_2 \) at the center we find from equations (3.43) and (3.44) that

\[
s = s_1 = s_2.
\]

Therefore, from equations (4.9) and (4.17) we can write the modified stress-resultant as

\[
\hat{n} = \hat{n}_r = \hat{n}_\theta = h [3 s(r)]
\]

and from equations (4.46) and (4.11) the deviator stress is found to be
\[ s(r) = \frac{\mu}{3} \left\{ \left[ \lambda^2(r) - \frac{1}{\lambda^4(r)} \right] + \frac{c_1}{c_e} \right\} \left( k(I_1(r)-3)^2 \right) \]

\[ + \left[ \lambda^4(r) - \frac{1}{\lambda^2(r)} \right] \left[ \frac{c_2}{(I_2(r)-3)^2 + c_3} \right] \]  

(4.49)

with \( I_1(r) \) and \( I_2(r) \) found from equations (4.13) and (4.14) as

\[ I_1(r) = 2\lambda(r)^2 + \frac{1}{\lambda(r)^4} \]  

(4.50)

and

\[ I_2(r) = \lambda(r)^4 + \frac{2}{\lambda(r)^2} \]  

(4.51)

From equations (2.19), (4.33) and (3.59) we find

\[ \lambda^2(r) = (1+u_r')^2 + (w')^2 \]  

(4.52)

From equations (4.23) and (4.24) we note that the stress-couple resultants can be written as

\[ \bar{M} = \bar{M}_r = \bar{M}_\theta = \int_{-h/2}^{+h/2} z \left[ 3s(z) \right] dz \]  

(4.53)

where \( s(z) \) can be found by using equations (4.46) and (4.25),

\[ s(z) = \frac{\mu}{3} \left\{ \left[ \lambda^2(z) - \frac{1}{\lambda(z)^4} \right] + \frac{c_1}{c_e} \right\} \left( k(I_1(z)-3)^2 \right) \]

\[ + \left[ \lambda(z)^4 - \frac{1}{\lambda(z)^2} \right] \left[ \frac{c_2}{(I_2(z)-3)^2 + c_3} \right] \]  

(4.54)
with $I_1(z)$ and $I_2(z)$ from equation (4.27), (4.28) and (4.46) given as

$$I_1(z) = 2\lambda(z)^2 + \frac{1}{\lambda(z)^4} \quad (4.55)$$

and

$$I_2(z) = \lambda(z)^4 + \frac{2}{\lambda^2(z)} \quad (4.56)$$

again omitting $r$ for brevity. From equations (2.38) and (4.33) the curvature becomes

$$\bar{b}_{11} = \sqrt{\frac{g}{g}} [w' u'' - (1 + u'_r) w''] \quad (4.57)$$

and from equations (2.36) and (3.70) we find the strain displacement relations for the stress-couple resultant to be

$$E_{11} = z [w'u'' - (1 + u'_r)w'']. \quad (4.58)$$

Using equation (3.59), the principal extension at the center of the plate for the stress-couple resultant is

$$\lambda^2(z) = 1 + 2z[w'u'' - (1 + u'_r)w'']. \quad (4.59)$$

The ratio $\sqrt{\frac{g}{g}}$ for the center of the plate is found by using equations (4.34) and (4.40),

$$\sqrt{\frac{g}{g}} = \frac{1}{(1 + u'_r)^2 + (w')^2} \quad (4.60)$$

Therefore, at the center the deformed thickness in terms of the undeformed thickness and the displacements is given by equations (3.95) and (4.60) as
\[ h = \frac{h}{(1 + u'_r)^2 + (w')^2} \]  \hspace{1cm} (4.61)

The equilibrium equations and boundary condition equations at the center of the circular plate are found by expanding equations (2.39), (2.40), (2.41), (2.42) and using equations (4.33). The equilibrium equation in physical variables is

\[ \bar{n}_r (w'') + \bar{M}' (1 + u'_r) + \bar{M}_r (2 u'_r'') + p (1 + u'_r) = 0 \]  \hspace{1cm} (4.62)

in the \( z \) direction, and the boundary conditions in physical variables prescribe

\[ \bar{n}_r (1 + u'_r) - \bar{M}_r (2w'') = T_r \]  \hspace{1cm} (4.63)

or the deflection, \( u_r \), and

\[ Q = 0 \]  \hspace{1cm} (4.64)

or the deflection, \( w \).

The equilibrium equation in the surface is satisfied identically. Since \( Q = 0 \) at the center, this implies that \( w'''' = 0 \), which also can be seen by using the finite difference equation for \( w'''' \).

At the outer edge of the clamped circular plate it is noted that on the middle surface

\[ \lambda_2 = 1 \]  \hspace{1cm} (4.65)

and from equation (3.60)

\[ \lambda_2^2 = 1 = 1 + 2 \frac{E_{22}}{r^2} \]  \hspace{1cm} (4.66)
Therefore,
\[ E_{22} = 0 \]  \hspace{1cm} (4.67)

and from equation (2.18)
\[ E_{22} = \frac{1}{2} (\bar{g}_{22} - g_{22}) \]  \hspace{1cm} (4.68)
we obtain
\[ \bar{g}_{22} = g_{22} \]  \hspace{1cm} (4.69)

Thus, the metric for the undeformed and deformed plate at its outer edge can be written in the forms

\[ g_{\alpha\beta} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix}, \]  \hspace{1cm} (4.70)

\[ g = |g_{\alpha\beta}| = a^2 \]  \hspace{1cm} (4.71)

and

\[ g_{\alpha\beta} \rightarrow \begin{pmatrix} (1 + u_r')^2 + (w')^2 & 0 \\ 0 & a^2 \end{pmatrix} \]  \hspace{1cm} (4.72)

\[ \bar{g} = |\bar{g}_{\alpha\beta}| = a^2 [(1 + u_r')^2 + (w')^2], \]  \hspace{1cm} (4.73)

where \( a \) is the outer radius of the plate. From equation (3.45) we get
\[ \frac{h}{\bar{h}} = 1 + u_r'. \]  \hspace{1cm} (4.74)
If we prescribe for the clamped circular thin plate that

\[ \frac{h}{\tilde{h}} = 1 \]  

(4.75)
along its outer edge, this implies from equation (4.74) that

\[ u_r' < < 1 . \]  

(4.76)

The boundary conditions for a clamped circular thin plate with a uniform pressure applied normal to the deformed middle surface are prescribed by

\[ u_r = 0 \text{ at } r = a \text{ and } r = 0 \]  

(4.77)

\[ w = 0 \text{ at } r = a \]  

(4.78)

\[ u_r' = 0 \text{ at } r = a \]  

(4.79)

\[ w' = 0 \text{ at } r = a \text{ and } r = 0 \]  

(4.80)

\[ u_r'' = 0 \text{ at } r = 0 \]  

(4.81)

\[ w''' = 0 \text{ at } r = 0 . \]  

(4.82)

To solve the two fourth order nonlinear differential equations let \( f \) represent the \( r \) equilibrium equation and \( g \) represent the \( z \) equilibrium equation,

\[ f = \tilde{n}_r' \left( 1 + u_r' \right) + \tilde{n}_r \left[ u_r'' + \frac{1}{r} \left( 1 + u_r' \right) \right] \]

\[ - \tilde{n}_0 \left[ \frac{1}{r} \left( 1 + \frac{u_r}{r} \right) \right] - \bar{M}_r'' \left[ \left( 1 + \frac{u_r}{r} \right) w' \right] \]
\[- \bar{M}_r' [3(1 + \frac{u_r}{r}) w'' + \frac{2}{r} (1 + u_r') w']\]

\[- \bar{M}_r [(1 + \frac{u_r}{r}) w'' + \frac{2}{r} (1 + u_r') w'' + \frac{2}{r} u_r'' w']\]

\[+ \bar{M}_\theta [\frac{2}{r^2} (1 + \frac{u_r}{r}) w'] - p[(1 + \frac{u_r}{r}) w'] \quad (4.83)\]

and

\[g = \tilde{n}_r' (w') + \tilde{n}_r (w'' + \frac{w'}{r}) + \bar{M}_r'' [(1 + \frac{u_r}{r}) (1 + u_r')]\]

\[+ \bar{M}_r' [3(1 + \frac{u_r}{r}) u_r'' + \frac{2}{r} (1 + u_r')^2]\]

\[+ \bar{M}_r [2(1 + \frac{u_r}{r}) u_r'' + \frac{4}{r} (1 + u_r') u_r'']\]

\[= \bar{M}_r' [(1 + \frac{u_r}{r})^2] - \bar{M}_r' [\frac{2}{r^2} (1 + \frac{u_r}{r}) (u_r' - \frac{u_r}{r})]\]

\[+ p [(1 + \frac{u_r}{r}) (1 + u_r')] \quad (4.84)\]

Let the (n) superscript represent the iteration number and the \(i\) subscript represent the node point. Applying Newton's method of quasilinérisation to equations (4.83) and (4.84), we write

\[f_i^{(n)} + (u_{ri}^{(n+1)} - u_{ri}^{(n)}) \frac{\partial f_i^{(n)}}{\partial u_r^{ri}} + (u_i^{(n+1)} - u_i^{(n)}) \frac{\partial f_i^{(n)}}{\partial u_r^{ri}}\]

\[\frac{\partial f_i^{(n)}}{\partial u_r^{ri}} + (u''_{ri}^{(n+1)} - u''_{ri}^{(n)}) \frac{\partial f_i^{(n)}}{\partial u_r^{ri}}\]

\[+ (u_{ri}^{(n+1)} - u_{ri}^{(n)}) \frac{\partial f_i^{(n)}}{\partial u_r^{ri}} + (u'''_{ri}^{(n+1)} - u'''_{ri}^{(n)}) \frac{\partial f_i^{(n)}}{\partial u_r^{ri}}\]
Using a five point finite difference formula we can write the derivatives of the displacements \( u_r \) and \( w \) in the form

\[
\begin{align*}
\frac{u_r}{\Delta r} &= \frac{u_{r+1} - u_{r-1}}{2\Delta r} + O(\Delta r^2), \\
\frac{u_{r+1}}{\Delta^2} &= \frac{u_{r+1} - 2u_r + u_{r-1}}{\Delta^2} + O(\Delta r^2),
\end{align*}
\]
\[ u''_i = \frac{u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}}{2\Delta r^3} + 0(\Delta r^2), \quad (4.89) \]

\[ u'''_i = \frac{u_{i+2} - 4u_{i+1} + 6u_{i} - 4u_{i-1} + u_{i-2}}{\Delta r^4} + 0(\Delta r^2), \quad (4.90) \]

\[ w'_i = \frac{w_{i+1} - w_{i-1}}{2\Delta r} + 0(\Delta r^2), \quad (4.91) \]

\[ w''_i = \frac{w_{i+1} - 2w_{i} + w_{i-1}}{\Delta r^2} + 0(\Delta r^2), \quad (4.92) \]

\[ w'''_i = \frac{w_{i+2} - 2w_{i+1} + 2w_{i-1} - w_{i-2}}{2\Delta r^3} + 0(\Delta r^2) \quad (4.93) \]

and

\[ w''''_i = \frac{w_{i+2} - 4w_{i+1} + 6w_{i} - 4w_{i-1} + w_{i-2}}{\Delta r^4} + 0(\Delta r^2). \quad (4.94) \]

By substituting equations (4.87) through (4.94) into equations (4.85) and (4.86) and rearranging, we get

\[ u_r^{(n+1)} = \frac{1}{2\Delta r^3} \left[ \frac{\partial f_1^{(n)}}{\partial u^{'''}} \right] + \frac{1}{4} \left[ \frac{\partial f_1^{(n)}}{\partial u^{''''}} \right] \]

\[ u_r^{(n+1)} = \frac{1}{2\Delta r} \left[ \frac{\partial f_1^{(n)}}{\partial u^{''}} \right] + \frac{1}{2} \left[ \frac{\partial f_1^{(n)}}{\partial u^{'''}} \right] + \frac{1}{3} \left[ \frac{\partial f_1^{(n)}}{\partial u^{''''}} \right] \]
\[
- \frac{4}{\Delta r^2} \left[ \frac{\partial f_i}{\partial u_{i+1}} \right] + u_{r_1}^{(n+1)} \left[ \frac{\partial f_i}{\partial u_r} \right] - \frac{2}{\Delta r^2} \left[ \frac{\partial f_i}{\partial u_r} \right]
\]

\[
+ \frac{6}{\Delta r^4} \left[ \frac{\partial f_i}{\partial u_{i+1}} \right] + u_{r_{i+1}}^{(n+1)} \left[ \frac{1}{2\Delta r} \frac{\partial f_i}{\partial u_r} \right]
\]

\[
+ \frac{1}{\Delta r^2} \left[ \frac{\partial f_i}{\partial u_r} \right] - \frac{1}{\Delta r^3} \left[ \frac{\partial f_i}{\partial u_r} \right] - \frac{4}{\Delta r^4} \left[ \frac{\partial f_i}{\partial u_r} \right]
\]

\[
+ u_{r_{i+2}}^{(n+1)} \left[ \frac{1}{2\Delta r^3} \frac{\partial f_i}{\partial u_r} \right] + \frac{1}{\Delta r^4} \left[ \frac{\partial f_i}{\partial u_r} \right]
\]

\[
+ w_{i+2}^{(n+1)} \left[ - \frac{1}{2\Delta r^3} \frac{\partial f_i}{\partial w_{i+2}} \right] + \frac{1}{\Delta r^4} \left[ \frac{\partial f_i}{\partial w_{i+2}} \right]
\]

\[
+ w_{i-1}^{(n+1)} \left[ - \frac{1}{2\Delta r} \frac{\partial f_i}{\partial w_i} \right] + \frac{1}{\Delta r^2} \left[ \frac{\partial f_i}{\partial w_i} \right] + \frac{1}{\Delta r^3} \left[ \frac{\partial f_i}{\partial w_i} \right]
\]

\[
- \frac{4}{\Delta r^4} \left[ \frac{\partial f_i}{\partial w_{i+1}} \right] + w_{i}^{(n+1)} \left[ - \frac{2}{\Delta r^2} \frac{\partial f_i}{\partial w_i} \right] + \frac{6}{\Delta r^4} \left[ \frac{\partial f_i}{\partial w_i} \right]
\]

\[
+ w_{i+1}^{(n+1)} \left[ \frac{1}{2\Delta r} \frac{\partial f_i}{\partial w_i} \right] + \frac{\partial f_i}{\partial w_i} - \frac{1}{\Delta r^3} \left[ \frac{\partial f_i}{\partial w_i} \right]
\]
\[-\frac{4}{\Delta r^4} \frac{\partial f_i(n)}{\partial w_{i+2}} (n+1) \geq \frac{1}{2\Delta r^3} \frac{\partial f_i(n)}{\partial w_i} + \frac{1}{\Delta r^4} \frac{\partial f_i(n)}{\partial w_{i+2}} \]

\[= u_{r_1-2}(n) \geq \frac{1}{2\Delta r^3} \frac{\partial f_i(n)}{\partial u_i} + \frac{1}{\Delta r^4} \frac{\partial f_i(n)}{\partial u_{i+2}} \]

\[+ u_{r_1-1}(n) \geq \frac{1}{2\Delta r^3} \frac{\partial f_i(n)}{\partial u_i} + \frac{1}{\Delta r^2} \frac{\partial f_i(n)}{\partial u_{i+2}} - \frac{1}{\Delta r^3} \frac{\partial f_i(n)}{\partial u_{i+1}} \]

\[-\frac{4}{\Delta r^4} \frac{\partial f_i(n)}{\partial u_{i+2}} + u_{r_1}(n) \geq \frac{1}{2\Delta r^3} \frac{\partial f_i(n)}{\partial u_i} - \frac{2}{\Delta r^2} \frac{\partial f_i(n)}{\partial u_{i+2}} \]

\[+ \frac{6}{\Delta r^4} \frac{\partial f_i(n)}{\partial u_{i+2}} + u_{r_1+1}(n) \geq \frac{1}{2\Delta r^3} \frac{\partial f_i(n)}{\partial u_i} + \frac{1}{\Delta r^2} \frac{\partial f_i(n)}{\partial u_{i+2}} \]

\[-\frac{1}{\Delta r^3} \frac{\partial f_i(n)}{\partial u_i} - \frac{4}{\Delta r^4} \frac{\partial f_i(n)}{\partial u_{i+2}} \geq u_{r_1+2}(n) \geq \frac{1}{2\Delta r^3} \frac{\partial f_i(n)}{\partial u_i} \]

\[+ \frac{1}{\Delta r^4} \frac{\partial f_i(n)}{\partial u_{i+2}} + w_{i-2}(n) \geq \frac{1}{2\Delta r^3} \frac{\partial f_i(n)}{\partial w_i} \]

\[+ \frac{1}{\Delta r^4} \frac{\partial f_i(n)}{\partial w_{i+2}} + w_{i-1}(n) \geq \frac{1}{2\Delta r^3} \frac{\partial f_i(n)}{\partial w_i} + \frac{1}{\Delta r^2} \frac{\partial f_i(n)}{\partial w_{i+2}} \]
\[
\begin{align*}
+ \frac{1}{\Delta r^3} \frac{\partial f_1(n)}{\partial w'''} & - \frac{4}{\Delta r} \frac{\partial f_1(n)}{\partial w'''} + w_1(n) \left[ - \frac{2}{\Delta r^2} \frac{\partial f_1(n)}{\partial w''} \right] \\
+ 6 \frac{\partial f_1(n)}{\partial r} \frac{\partial f_1(n)}{\partial w''''} & + w_{i+1}(n) \left[ \frac{1}{2\Delta r} \frac{\partial f_1(n)}{\partial w'} + \frac{1}{\Delta r} \frac{\partial f_1(n)}{\partial w''} \right] \\
- \frac{1}{\Delta r^3} \frac{\partial f_1(n)}{\partial w'''} & - \frac{4}{\Delta r} \frac{\partial f_1(n)}{\partial w'''} + w_{i+2}(n) \left[ \frac{1}{2\Delta r^3} \frac{\partial f_1(n)}{\partial w''''} \right] \\
+ \frac{1}{\Delta r^4} \frac{\partial f_1(n)}{\partial w''''} & - f_1(n) \quad (4.95)
\end{align*}
\]

and

\[
\begin{align*}
& u_{r_{i-2}}^{(n+1)} \left[ - \frac{1}{2\Delta r^3} \frac{\partial g_1(n)}{\partial u'''} + \frac{1}{\Delta r^4} \frac{\partial g_1(n)}{\partial u''''} \right] \\
& + u_{r_{i-1}}^{(n+1)} \left[ - \frac{1}{2\Delta r} \frac{\partial g_1(n)}{\partial u'} + \frac{1}{\Delta r^2} \frac{\partial g_1(n)}{\partial u''} + \frac{1}{\Delta r^3} \frac{\partial g_1(n)}{\partial u'''} \right] \\
& - \frac{4}{\Delta r^4} \frac{\partial g_1(n)}{\partial u''''} \right] + u_{r_1}^{(n+1)} \left[ \frac{1}{\Delta r^2} \frac{\partial g_1(n)}{\partial u''} - \frac{2}{\Delta r^3} \frac{\partial g_1(n)}{\partial u'''} \right] \\
& + 6 \frac{\partial g_1(n)}{\partial r} \frac{\partial g_1(n)}{\partial u''''} \right] + u_{r_{i+1}}^{(n+1)} \left[ \frac{1}{2\Delta r} \frac{\partial g_1(n)}{\partial u'} + \frac{1}{\Delta r^2} \frac{\partial g_1(n)}{\partial u''} \right]
\end{align*}
\]
\[-\frac{1}{\Delta r^3} \frac{\partial g_i(n)}{\partial u_r'''} - \frac{4}{\Delta r^4} \frac{\partial g_i(n)}{\partial u_r''} + u_{r_{i+2}}^{(n+1)} \left[ \frac{1}{2\Delta r^3} \frac{\partial g_i(n)}{\partial u_r'''} \right] \]

\[+ \frac{1}{\Delta r^4} \frac{\partial g_i(n)}{\partial u_r'''} + w_{i-2}^{(n+1)} \left[ -\frac{1}{2\Delta r^3} \frac{\partial g_i(n)}{\partial w'''} + \frac{1}{\Delta r^2} \frac{\partial g_i(n)}{\partial w''} + \frac{1}{\Delta r^3} \frac{\partial g_i(n)}{\partial w'''} \right] \]

\[+ w_{i-1}^{(n+1)} \left[ -\frac{1}{2\Delta r^3} \frac{\partial g_i(n)}{\partial w'''} + \frac{1}{\Delta r^2} \frac{\partial g_i(n)}{\partial w''} + \frac{1}{\Delta r^3} \frac{\partial g_i(n)}{\partial w'''} \right] \]

\[\frac{4}{\Delta r^4} \frac{\partial g_i(n)}{\partial u_r'''} + w_{i}^{(n+1)} \left[ -\frac{2}{\Delta r^2} \frac{\partial g_i(n)}{\partial w''} + \frac{6}{4} \frac{\partial g_i(n)}{\partial w'''} \right] \]

\[+ w_{i+1}^{(n+1)} \left[ \frac{1}{2\Delta r^3} \frac{\partial g_i(n)}{\partial w'''} + \frac{1}{\Delta r^2} \frac{\partial g_i(n)}{\partial w''} - \frac{1}{\Delta r^3} \frac{\partial g_i(n)}{\partial w'''} \right] \]

\[\frac{4}{\Delta r^4} \frac{\partial g_i(n)}{\partial u_r'''} + w_{i+2}^{(n+1)} \left[ \frac{1}{2\Delta r^3} \frac{\partial g_i(n)}{\partial w'''} + \frac{1}{4} \frac{\partial g_i(n)}{\partial w'''} \right] \]

\[= u_{r_{i-2}}^{(n)} \left[ -\frac{1}{2\Delta r^3} \frac{\partial g_i(n)}{\partial u_r'''} + \frac{1}{\Delta r^4} \frac{\partial g_i(n)}{\partial u_r'''} \right] + u_{r_{i-1}}^{(n)} \]

\[\left[ -\frac{1}{2\Delta r} \frac{\partial g_i(n)}{\partial u_r'} + \frac{1}{\Delta r^2} \frac{\partial g_i(n)}{\partial u_r''} + \frac{1}{\Delta r^3} \frac{\partial g_i(n)}{\partial u_r'''} - \frac{4}{\Delta r^4} \frac{\partial g_i(n)}{\partial u_r'''} \right] \]
\[ + u_{r_1}^{(n)} \left( \frac{\partial g_1^{(n)}}{\partial u_r} - \frac{2}{\Delta r} \frac{\partial g_1^{(n)}}{\partial u_r''} + \frac{6}{\Delta r^2} \frac{\partial g_1^{(n)}}{\partial u_r'''} \right) \]

\[ + u_{r_{i+1}}^{(n)} \left( \frac{1}{2\Delta r} \frac{\partial g_1^{(n)}}{\partial u_r'} + \frac{1}{\Delta r^2} \frac{\partial g_1^{(n)}}{\partial u_r''} - \frac{1}{\Delta r^3} \frac{\partial g_1^{(n)}}{\partial u_r'''} \right) \]

\[ - \frac{4}{\Delta r^4} \frac{\partial g_1^{(n)}}{\partial u_r^{(n)}} \] + \[ u_{r_{i+2}}^{(n)} \left( \frac{1}{2\Delta r^3} \frac{\partial g_1^{(n)}}{\partial u_r'''} + \frac{1}{\Delta r^4} \frac{\partial g_1^{(n)}}{\partial u_r^{(n)}} \right) \]

\[ + w_{i-2}^{(n)} \left[ - \frac{1}{2\Delta r^3} \frac{\partial g_1^{(n)}}{\partial w_r'''} + \frac{1}{\Delta r^4} \frac{\partial g_1^{(n)}}{\partial w_r^{(n)}} \right] \]

\[ + w_{i-1}^{(n)} \left[ - \frac{1}{2\Delta r} \frac{\partial g_1^{(n)}}{\partial w_r'} + \frac{1}{\Delta r^2} \frac{\partial g_1^{(n)}}{\partial w_r''} + \frac{1}{\Delta r^3} \frac{\partial g_1^{(n)}}{\partial w_r'''} \right] \]

\[ - \frac{4}{\Delta r^4} \frac{\partial g_1^{(n)}}{\partial w_r^{(n)}} \] + \[ w_1^{(n)} \left[ - \frac{2}{\Delta r^2} \frac{\partial g_1^{(n)}}{\partial w_r''} + \frac{6}{\Delta r^4} \frac{\partial g_1^{(n)}}{\partial w_r^{(n)}} \right] \]

\[ + w_{i+1}^{(n)} \left[ \frac{1}{2\Delta r} \frac{\partial g_1^{(n)}}{\partial w_r'} + \frac{1}{\Delta r^2} \frac{\partial g_1^{(n)}}{\partial w_r''} - \frac{1}{\Delta r^3} \frac{\partial g_1^{(n)}}{\partial w_r'''} \right] \]

\[ - \frac{4}{\Delta r^4} \frac{\partial g_1^{(n)}}{\partial w_r^{(n)}} \] + \[ w_{i+2}^{(n)} \left[ \frac{1}{2\Delta r^3} \frac{\partial g_1^{(n)}}{\partial w_r'''} - \frac{1}{\Delta r^4} \frac{\partial g_1^{(n)}}{\partial w_r^{(n)}} \right] \]

\[ - g_i^{(n)}. \] (4.96)
The quasilinearized equilibrium equations (4.95) and (4.96) are programmed for a digital computer. Initial guessed values are given for the displacements \( u_r \) and \( w \) along the radius of the plate. Note that the right sides of equations (4.95) and (4.96) are known as well as the coefficients of \( u_r^{(n+1)} \) and \( w^{(n+1)} \) on the left side.

The coefficients of \( u_r \) and \( w \) are put into a band array matrix to allow for enough storage space in the computer. The problem is allowed to iterate until it converges to a specified difference between the \((n+1)\)th and the \(n\)th iteration.

A separate subprogram was written for the center of the plate \((i = 1)\). By letting \( f \) and \( g \) represent the \( r \) equilibrium equation and the \( z \) equilibrium equation respectively, we write

\[
f = 0 \tag{4.97}
\]

and

\[
g = \tilde{n} (w'') + \tilde{M}'' (1 + u_r') \\
+ \overline{M_r} (2u_r'''') + p (1 + u_r') . \tag{4.98}
\]

Using equations (4.87) through (4.94) and the boundary conditions at the center of the plate along with the symmetry of deformation around the center, the finite difference equations at the center \((i = 1)\) become

\[
u_r'_{i} = \frac{u_r^{i+1} - u_r^i}{\Delta r} , \tag{4.99}
\]
The quasilinearized equilibrium equations are

\[ u_{ri}'' = 0, \quad (4.100) \]

\[ u_{ri}''' = \frac{u_{ri+2} - 2u_{i+1}}{\Delta r^3}, \quad (4.101) \]

\[ u_{ri}'''' = 0, \quad (4.102) \]

\[ w' = 0, \quad (4.103) \]

\[ w'' = \frac{2w_{i+1} - 2w_i}{\Delta r^2}, \quad (4.104) \]

\[ w'''' = 0 \quad (4.105) \]

and

\[ w''''' = \frac{2w_{i+2} - 8w_{i+1} + 6w_i}{\Delta r^4}. \quad (4.106) \]

The quasilinearized equilibrium equations are

\[ f_i^{(n)} = 0 \quad (4.107) \]

and

\[ g_i^{(n)} + (u'_{ri}^{(n+1)} - u'_{ri}^{(n)}) \frac{\Delta g_i^{(n)}}{\Delta u'_{ri}} + (u'''_{ri}^{(n+1)} - u'''_{ri}^{(n)}) \frac{\Delta g_i^{(n)}}{\Delta w'''} + (w'''_{i}^{(n+1)} - w'''_{i}^{(n)}) \frac{\Delta g_i^{(n)}}{\Delta w''''} = 0. \quad (4.108) \]
By substituting equations (4.99) through (4.107) in equation (4.108), and rearranging, we write

\[
\begin{align*}
    \frac{u_{r_i+1}}{r_i} (n+1) &= \left[ \frac{1}{\Delta r} \frac{\partial g_{1_i}}{\partial u_{r_i}} - \frac{2}{\Delta r^3} \frac{\partial^3 g_{1_i}}{\partial u_{r_i}^3} \right] u_{r_i+2} (n+1) + \left[ \frac{1}{\Delta r^3} \frac{\partial^3 g_{1_i}}{\partial u_{r_i}^3} \right] \\
    + w_i (n+1) &= \left[ -\frac{2}{\Delta r^2} \frac{\partial g_{1_i}}{\partial u_{r_i}''} + \frac{6}{\Delta r^4} \frac{\partial^4 g_{1_i}}{\partial u_{r_i}'''} \right] \\
    + w_{i+1} (n+1) &= \left[ \frac{2}{\Delta r^2} \frac{\partial^2 g_{1_i}}{\partial u_{r_i}'^2} - \frac{8}{\Delta r^4} \frac{\partial^4 g_{1_i}}{\partial u_{r_i}'''} \right] \\
    + w_{i+2} (n+1) &= \left[ \frac{2}{\Delta r^4} \frac{\partial^4 g_{1_i}}{\partial u_{r_i}'''} \right] u_{r_i+1} (n) + \left[ \frac{1}{\Delta r} \frac{\partial g_{1_i}}{\partial u_{r_i}'} \right] \\
    - \frac{2}{\Delta r^3} \frac{\partial^3 g_{1_i}}{\partial u_{r_i}^3} \right] + u_{r_i+2} (n) + \left[ \frac{1}{\Delta r^3} \frac{\partial^3 g_{1_i}}{\partial u_{r_i}^3} \right] \\
    + w_i (n) &= \left[ -\frac{2}{\Delta r^2} \frac{\partial g_{1_i}}{\partial u_{r_i}''} + \frac{6}{\Delta r^4} \frac{\partial^4 g_{1_i}}{\partial u_{r_i}'''} \right] + w_{i+1} (n) + \left[ \frac{2}{\Delta r^2} \frac{\partial^2 g_{1_i}}{\partial u_{r_i}'^2} \right] \\
    - \frac{8}{\Delta r^4} \frac{\partial^4 g_{1_i}}{\partial u_{r_i}'''} \right] + w_{i+2} (n) + \left[ \frac{2}{\Delta r^4} \frac{\partial^4 g_{1_i}}{\partial u_{r_i}'''} \right] - g_i (n). \quad (4.109)
\end{align*}
\]
The coefficients of the displacements $u_r$ and $w$ are evaluated in a separate subprogram and are included in the band array matrix.

To satisfy the boundary conditions at the center of the plate for $i = 2$ the finite difference equations become

\[ u''_r = \frac{u_{r+1} - 2u_r + u_{r-1}}{\Delta r^2}, \]  

(4.110)  

\[ u'''_r = \frac{u_{r+2} - 2u_{r+1} + u_r}{\Delta r^3}, \]  

(4.111)  

\[ u''''_r = \frac{u_{r+2} - 4u_{r+1} + 5u_r - u_r}{\Delta r^4}, \]  

(4.112)  

\[ w'_i = \frac{w_{i+1} - w_{i-1}}{2\Delta r}, \]  

(4.113)  

\[ w''_i = \frac{w_{i+1} - 2w_i + w_{i-3}}{\Delta r^2}, \]  

(4.114)  

\[ w'''_i = \frac{w_{i+2} - 2w_{i+1} - w_i + 2w_{i-1}}{2\Delta r^3}, \]  

(4.115)  

and

\[ w''''_i = \frac{w_{i+2} - 4w_{i+1} + 7w_i - 4w_{i-1}}{\Delta r^4}. \]  

(4.116)
For \( i = 3 \) the finite difference equations are rewritten to satisfy the boundary conditions at the center and are presented as

\[
\frac{u_{r_{i+1}} - u_{r_{i-1}}}{2\Delta r},
\]

\[
\frac{u''_{r_{i+1}} - 2u_{r_i} + u_{r_{i-1}}}{\Delta r^2},
\]

\[
\frac{u''''_{r_{i+1}} - 2u_{r_i} + 2u_{r_{i-1}}}{2\Delta r^3},
\]

\[
\frac{u''''''_{r_{i+1}} - 4u_{r_i} + 6u_{r_{i-1}} - 4u_{r_{i-2}}}{\Delta r^4},
\]

\[
w' = \frac{w_{i+1} - w_{i-1}}{2\Delta^2},
\]

\[
w'' = \frac{w_{i+1} - 2w_i + w_{i-1}}{\Delta r^2},
\]

\[
w''' = \frac{w_{i+2} - 2w_{i+1} + 2w_{i-1} - w_{i-2}}{2\Delta r^3},
\]

and

\[
w'''' = \frac{w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2}}{\Delta r^4}.
\]
Values for \( u_r \) and \( w \) are found for each node point up to \( \Delta r \) distance from the outer edge of the plate. The node point one \( \Delta r \) distance from the outer edge is called \( i = MNP \). To satisfy the boundary conditions on the outer edge, the finite difference equations for \( i = MNP - 1 \) are written as

\[
\frac{u'_r}{\Delta r} = \frac{u_{r+1} - u_{r-1}}{\Delta r}, \quad (4.126)
\]

\[
\frac{u''_r}{\Delta r^2} = \frac{u_{r+1} - 2u_r + u_{r-1}}{\Delta r^2}, \quad (4.127)
\]

\[
\frac{u'''_r}{2\Delta r^3} = \frac{2u_{r+1} + 2u_{r-1} - u_{r-2}}{2\Delta r^3}, \quad (4.128)
\]

\[
\frac{u''''_r}{\Delta r^4} = \frac{4u_{r+1} + 6u_r - 4u_{r-1} + u_{r-2}}{\Delta r^4}, \quad (4.129)
\]

\[
\frac{w'_i}{2\Delta r} = \frac{w_{i+1} - w_{i-1}}{2\Delta r}, \quad (4.130)
\]

\[
\frac{w''_i}{\Delta r^2} = \frac{w_{i+1} - 2w_i + w_{i-1}}{\Delta r^2}, \quad (4.131)
\]

\[
\frac{w''''_i}{2\Delta r^3} = \frac{2w_{i+1} + 2w_{i-1} - w_{i-2}}{2\Delta r^3}, \quad (4.132)
\]
\[ w'''' = \frac{-4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2}}{\Delta r^4} . \] (4.133)

For \( i = \text{NMP} \), the finite difference equations become

\[ u_i'' = \frac{-u_{r_{i-1}}}{2\Delta r} , \] (4.134)

\[ u_i''' = \frac{-2u_r + u_{r_{i-1}}}{\Delta r^2} , \] (4.135)

\[ u_i'''' = \frac{u_r + 2u_{r_{i-1}} - u_{r_{i-2}}}{2\Delta r^3} , \] (4.136)

\[ u_i''''' = \frac{7u_r - 4u_{r_{i-1}} + u_{r_{i-2}}}{\Delta r^4} . \] (4.137)

\[ w_i' = \frac{w_{i-1}}{2\Delta r} , \] (4.138)

\[ w_i'' = -2w_i + w_{i-1} , \] (4.139)

\[ w_i''' = \frac{w_i + 2w_{i-1} - w_{i-2}}{2\Delta r^3} . \] (4.140)
and

\[ w'''_i = \frac{7w_i - 4w_{i-1} + w_{i-2}}{\Delta r^4} \]. \quad (4.141)

The principal extensions and their derivatives were programmed in terms of the displacements and their derivatives. The stress deviators and their derivatives and the invariants of the extensions and their derivatives were programmed in terms of the principal extensions and their derivatives. The stress resultants and the stress-couple resultants and their derivatives were programmed in terms of the stress deviators and their derivatives. The deformed thickness was programmed in terms of the displacements and their derivatives.

The material constants for the incompressible rubber-like material used in this paper are

\[ E = 270 \text{ psi}, \quad (4.142) \]
\[ v = 0.5 \quad (4.143) \]

and

\[ \mu = 90 \text{ psi} \]

The radius, \( a \), of the thin plate was chosen to be

\[ a = 10 \text{ inches}, \quad (4.145) \]

and the maximum number of node points, MNP, along the radius

\[ MNP = 100. \quad (4.146) \]

The number of node points, \( N \), through the thickness of the plate is set at

\[ n = 20. \quad (4.147) \]
Simpson's rule was used in a separate subprogram to evaluate the integrals in equations (4.23) and (4.24) for the stress-couple resultants.

The value of the uniform thickness, \( h \), of the thin plate was chosen to be

\[
h = 0.5 \text{ inches}
\]

(4.148)

to compare the deflections caused by applying different values of the uniform pressure, \( p \).

B. Plate with Varying Thickness

The previous equations can also be used to study plates with varying thickness. A thin circular plate with varying thickness in the radial direction and uniform thickness in the \( \theta \) direction was considered. A linear variation in the thickness from 0.5 inches at the center to 0.25 inches on the outer edge was set up in the form

\[
h(i) = h(1 - \frac{1}{2} \frac{r}{a})
\]

(4.149)

where (i) represents the node point along the radius \( r \). Also, a 10.0 inch radius plate 0.5 inches thick at the outer edge with a 2.0 inch thick portion at the center along the first 2.0 inches of radius was examined. A linear transition in the thickness from 2.0 inches to 0.5 inches between \( r = 2.0 \) inches and \( r = 3.0 \) inches connected the thick and thin portions of the plate. We are aware that for the center inclusion the first approximation assumption used in the equilibrium equations may be violated, however for very large deflections the
thickness near the center decreases to a value less than 0.5 inches. We will proceed to solve this problem keeping in mind our first approximation assumption. The results for these cases will be discussed in Chapter VI.
A thin circular plate made from a rubber-like material which undergoes very large deflections will be affected more by membrane stretching than by bending. Therefore, it is important to analyze the membrane solution for very large deflections.

For the membrane problem the equilibrium equations (2.63) and (2.64) in physical components reduce to

\[
\tilde{n}'_r (1 + u'_r) + \tilde{n}_r [u''_r + \frac{1}{r} (1 + u'_r)] - \tilde{n}_\theta [\frac{1}{r} (1 + \frac{u_r}{r})]
- p (1 + \frac{u_r}{r}) w' = 0
\]

(5.1)
in the r direction, and

\[
\tilde{n}'_r w' + \tilde{n}_r (w'' + \frac{w'}{r}) + p (1 + \frac{u_r}{r}) (1 + u'_r) = 0
\]

(5.2)
in the z direction. The boundary conditions (2.65) and (2.66) in physical components reduced for the membrane problem prescribe

\[
\tilde{n}_r (1 + u'_r) = T_r
\]

(5.3)
or the displacement \( u_r \), and

\[
\tilde{n}_r w' = Q
\]

(5.4)
or the displacement \( w \).

The modified stress resultants, the stress deviators, the invariants of the extensions and the principal extensions are the same
as for the thin plate and are given in equations (4.9) through (4.16) respectively. Therefore, we see that the equilibrium equations (5.1) and (5.2) and the boundary conditions (5.3) and (5.4) for the membrane solution are second order.

The boundary conditions for the circular membrane solution with a uniform pressure applied normal to the deformed middle surface are prescribed by

\[ u_r = 0 \quad \text{at } r = a \quad \text{and } r = 0 \]  
\[ w = 0 \quad \text{at } r = a \]  
\[ w' = 0 \quad \text{at } r = 0 \]  

At the center of the circular membrane the equilibrium equations reduce to

\[ \tilde{n}_r w'' + p (1 + u_r') = 0 \]  

in the z direction and the equilibrium equation in the r direction is satisfied identically. The boundary conditions at the center reduce to the forms: prescribe

\[ \tilde{n}_r (1 + u_r') = T_r \]  

or the displacement \( u_r \), and

\[ Q = 0 \]  

or the displacement \( w \).
To solve the two second order nonlinear differential equations for the circular membrane let \( f \) represent the \( r \) equilibrium equation and \( g \) represent the \( z \) equilibrium equation,

\[
f = \tilde{n}_r \cdot (1 + u'_r) + \tilde{n}_r \left[ u''_r + \frac{1}{r} (1 + u'_r) \right] - \tilde{n}_0 \left[ \frac{1}{r} (1 + \frac{u'_r}{r}) \right] - p (1 + \frac{u'_r}{r}) w' \tag{5.11}
\]

and

\[
g = \tilde{n}_r w' + \tilde{n}_r (w'' + \frac{w'}{r}) + p (1 + \frac{u'_r}{r}) (1 + u'_r) \tag{5.12}
\]

Let the \((n)\) superscript represent the iteration number and the \( i \) subscript represent the node point. By applying Newton's method of quasilinearization to equations (5.11) and (5.12), we obtain

\[
f_i^{(n)} + \left[ u_{r_i}^{(n+1)} - u_{r_i}^{(n)} \right] \frac{\partial f_i^{(n)}}{\partial u_r} + \left[ u_{r_i}^{(n+1)} - u_{r_i}^{(n)} \right] \frac{\partial f_i^{(n)}}{\partial u_{r_i}'}
\]

\[
+ \left[ u_{r_i}''^{(n+1)} - u_{r_i}''^{(n)} \right] \frac{\partial f_i^{(n)}}{\partial u_{r_i}''} + \left[ w_i^{(n+1)} - w_i^{(n)} \right] \frac{\partial f_i^{(n)}}{\partial w_i''} = 0 \tag{5.13}
\]

and

\[
g_i^{(n)} + \left[ u_{r_i}^{(n+1)} - u_{r_i}^{(n)} \right] \frac{\partial g_i^{(n)}}{\partial u_r} + \left[ u_{r_i}^{(n+1)} - u_{r_i}^{(n)} \right] \frac{\partial g_i^{(n)}}{\partial u_{r_i}'}
\]
Using a three point finite difference formula we write the derivatives of the displacements $u_r$ and $w$ in the form

$$u_r' = \frac{u_{r+1} - u_{r-1}}{2\Delta r} + O(\Delta r^2) \quad (5.15)$$

$$u_r'' = \frac{u_{r+1} - 2u_r + u_{r-1}}{\Delta r^2} + O(\Delta r^2) \quad (5.16)$$

$$w_i' = \frac{w_{i+1} - w_{i-1}}{2\Delta r} + O(\Delta r^2) \quad (5.17)$$

and

$$w_i'' = \frac{w_{i+1} - 2w_i + w_{i-1}}{\Delta r^2} + O(\Delta r^2) \quad (5.18)$$

By substituting equations (5.15) through (5.18) into equations (5.13) and (5.14) and rearranging, we get

$$u_r^{(n+1)} = \frac{1}{2\Delta r} \frac{\partial f_i}{\partial u_r'} + \frac{1}{\Delta r^2} \frac{\partial f_i}{\partial u_r''} + u_r^{(n)} \frac{\partial f_i}{\partial u_r'} \quad (5.19)$$
\[ - \frac{2}{\Delta r^2} \frac{\partial f_i(n)}{\partial u_{r+1}^{n+1}} + u_{r+1}^{n+1} \left[ \frac{1}{2\Delta r} \frac{\partial f_i(n)}{\partial u_r} + \frac{1}{\Delta r^2} \frac{\partial f_i(n)}{\partial u_r} \right] \]

\[ + w_{i-1}^{n+1} \left[ - \frac{1}{2\Delta r} \frac{\partial f_i(n)}{\partial w_{r+1}} + \frac{1}{\Delta r^2} \frac{\partial f_i(n)}{\partial w_{r+1}} \right] + w_{i-1}^{n+1} \]

\[ - \frac{2}{\Delta r^2} \frac{\partial f_i(n)}{\partial w_{r+1}} + w_{i+1}^{n+1} \left[ \frac{1}{2\Delta r} \frac{\partial f_i(n)}{\partial w_{r}} + \frac{1}{\Delta r^2} \frac{\partial f_i(n)}{\partial w_{r}} \right] \]

\[ = u_{r+1}^{n+1} \left[ - \frac{1}{2\Delta r} \frac{\partial f_i(n)}{\partial u_r} + \frac{1}{\Delta r^2} \frac{\partial f_i(n)}{\partial u_r} \right] + u_{r+1}^{n+1} \left[ \frac{1}{2\Delta r} \frac{\partial f_i(n)}{\partial u_r} \right] \]

\[ + \frac{1}{\Delta r^2} \frac{\partial f_i(n)}{\partial u_r} + w_{i-1}^{n+1} \left[ - \frac{1}{2\Delta r} \frac{\partial f_i(n)}{\partial w_{r}} + \frac{1}{\Delta r^2} \frac{\partial f_i(n)}{\partial w_{r}} \right] \]

\[ + w_{i+1}^{n+1} \left[ - \frac{2}{\Delta r^2} \frac{\partial f_i(n)}{\partial w_{r}} \right] + \frac{1}{2\Delta r} \frac{\partial f_i(n)}{\partial w_{r}} + \frac{1}{\Delta r^2} \frac{\partial f_i(n)}{\partial w_{r}} \]

\[ - f_i(n) \]  

(5.19)
and

\[ u_{r_{i-1}}^{(n+1)} \left[ - \frac{1}{2\Delta r} \frac{\partial g_i}{\partial u_{r_i}} + \frac{1}{\Delta r^2} \frac{\partial g_i}{\partial u_{r_{i-1}}} \right] + u_{r_i}^{(n+1)} \left[ \frac{1}{2\Delta r} \frac{\partial g_i}{\partial u_{r_i}} + \frac{1}{\Delta r^2} \frac{\partial g_i}{\partial u_{r_{i+1}}} \right] \]

\[ - \frac{2}{\Delta r^2} \frac{\partial g_i}{\partial u_{r_{i-1}}} \] + \[ w_{i-1}^{(n+1)} \left[ - \frac{1}{2\Delta r} \frac{\partial g_i}{\partial u_{r_i}} + \frac{1}{\Delta r^2} \frac{\partial g_i}{\partial u_{r_{i-1}}} \right] + w_i^{(n+1)} \left[ \frac{1}{2\Delta r} \frac{\partial g_i}{\partial u_{r_i}} + \frac{1}{\Delta r^2} \frac{\partial g_i}{\partial u_{r_{i+1}}} \right] \]

\[ = u_{r_{i-1}}^{(n)} \left[ - \frac{1}{2\Delta r} \frac{\partial g_i}{\partial u_{r_i}} + \frac{1}{\Delta r^2} \frac{\partial g_i}{\partial u_{r_{i-1}}} \right] + u_i^{(n)} \left[ \frac{1}{2\Delta r} \frac{\partial g_i}{\partial u_{r_i}} + \frac{1}{\Delta r^2} \frac{\partial g_i}{\partial u_{r_{i+1}}} \right] \]

\[ - \frac{2}{\Delta r^2} \frac{\partial g_i}{\partial u_{r_{i-1}}} \] + \[ w_{i-1}^{(n)} \left[ - \frac{1}{2\Delta r} \frac{\partial g_i}{\partial u_{r_i}} + \frac{1}{\Delta r^2} \frac{\partial g_i}{\partial u_{r_{i-1}}} \right] + w_i^{(n)} \left[ \frac{1}{2\Delta r} \frac{\partial g_i}{\partial u_{r_i}} + \frac{1}{\Delta r^2} \frac{\partial g_i}{\partial u_{r_{i+1}}} \right] \]

\[ + w_{i+1}^{(n)} \left[ \frac{1}{2\Delta r} \frac{\partial g_i}{\partial u_{r_i}} + \frac{1}{\Delta r^2} \frac{\partial g_i}{\partial u_{r_{i+1}}} \right] - g_i^{(n)}. \quad (5.20) \]
The quasilinearized equilibrium equations (5.19) and (5.20) for the circular membrane are programmed for a digital computer. It should be noted that the solution to the equilibrium equations in this chapter follows the method used in chapter 4. With initial guessed values given for the displacements $u_r$ and $w$ along each node point, the right sides of equations (5.19) and (5.20) are known as well as the coefficients of $u_r^{(n+1)}$ and $w^{(n+1)}$. The coefficients of $u_r$ and $w$ are stacked in a band array matrix and the problem is allowed to iterate until it converges.

As in chapter IV, a separate subprogram was written for the center of the plate ($i = 1$). The equilibrium equations as given in (5.11) and (5.12) reduce to

$$f = 0 \quad (5.21)$$

and

$$g = \hat{n}_r w'' + p(1 + u_r') \quad (5.22)$$

for the center. The boundary conditions at the center of the plate and the symmetry of deformation around the center require that the finite difference equations have the form

$$u_r' = \frac{u_r^{i+1} - u_r^i}{\Delta r} \quad , \quad (5.23)$$

$$u_r'' = 0 \quad , \quad (5.24)$$

$$w_j' = 0 \quad , \quad (5.25)$$
and
\[ w''_i = \frac{2w_{i+1} - 2w_i}{\Delta r^2} \]  \hfill (5.26)

Equations (5.21) and (5.22) are quasilinearized and by using equations (5.23) through (5.26) and rearranging we have the equilibrium equation for the center given as

\[ u_{i+1}^{(n+1)} \left[ \frac{1}{\Delta r} \frac{\partial g_i}{\partial u_r} \right] + w_{i+1}^{(n+1)} \left[ \frac{2}{\Delta r^2} \frac{\partial g_i}{\partial w''} \right] 
+ w_i^{(n+1)} \left[ -\frac{2}{\Delta r^2} \frac{\partial g_i}{\partial w''} \right] = u_i^{(n)} \left[ \frac{1}{\Delta r} \frac{\partial g_i}{\partial u_r} \right] 
+ w_i^{(n)} \left[ \frac{2}{\Delta r^2} \frac{\partial g_i}{\partial w''} \right] + w_i^{(n)} \left[ -\frac{2}{\Delta r^2} \frac{\partial g_i}{\partial w''} \right] - g_i^{(n)} \]  \hfill (5.27)

The coefficients of the displacements \( u_r \) and \( w \) are evaluated in a separate subprogram and are included in the band array matrix for the main program.

The boundary conditions at the center require that the finite difference equations for \( i = 2 \) to have the form
The boundary conditions along the outer edge of the membrane require that the difference equations at \( i = MNP \) have the form

\[
\begin{align*}
    u'_r \bigg|_i &= \frac{u_{r,i+1}}{2\Delta r}, \quad (5.28) \\
    u''_r \bigg|_i &= \frac{u_{r,i+1} - 2u_{r,i}}{\Delta r^2}, \quad (5.29) \\
    w'_i &= \frac{w_{i+2} - w_{i-1}}{2\Delta r}, \quad (5.30)
\end{align*}
\]

and

\[
\begin{align*}
    w''_i &= \frac{w_{i+1} - 2w_i + w_{i-1}}{\Delta r^2}. \quad (5.31)
\end{align*}
\]

The principal extensions, the stress deviators, the invariants of the extensions, the stress resultants and their derivatives were programmed in chapter IV and are also used in the solution of the
membrane problem in this chapter. The same material, dimensions and numbers of node points are used in the membrane problem as was used in the thin plate problem.
VI. DISCUSSION OF RESULTS AND CONCLUSIONS

The exact tensor equilibrium equations for circular plates were used with the Alexander constitutive equations for a rubber-like material to determine the deflections, stress resultants and change in thickness for a thin plate with constant thickness and a plate with varying thickness.

The deflections of a circular plate with varying thickness in the r direction, constant thickness in the θ direction and clamped along its outer edge are shown in Figure 3. The 10.0 inch radius plate is 0.5 inches thick at the outer edge and has a 2.0 inch thick portion at the center along the first 2.0 inches of radius. A linear transition in thickness from 2.0 inches to 0.5 inches between r = 2.0 inches and r = 3.0 inches connects the thick and thin portions of the plate. A uniform pressure of 0.2 psi was applied normal to the middle surface and allowed to increase in steps of 0.2 psi until a pressure of 3.2 psi was reached. The problem was allowed to converge for each value of pressure before an increase in pressure was made. The total computer time was approximately 90 minutes to convergence at 3.2 psi. The effect of bending can be seen near the clamped edge in Figure 3. A deflection profile comparison with a uniformly thick 0.5 inch plate with bending included and with no bending is shown in Figure 12. The stress resultants shown in Figures 4 and 5 have approximately the same value near the center of the plate, but the circumferential stress resultant is slightly smaller than the meridional stress resultant.
near the outer edge. From Figures 6 and 7 we see that the meridional and circumferential stress-couples have a maximum positive value along the outer edge of the inclusion. The thickness variation as seen in Figure 11 shows that the inclusion is slightly thinner at the center than along its outer edge. The change in thickness at the center of the plate for 3.2 psi is approximately 16.5 percent and the maximum change in thickness for the 0.5 inch thick portion of the plate is 8 percent and occurs near the point where the linearly varying portion joins the 0.5 inch thick portion. The plate seems to inflate when the deflection at the center reaches approximately one-half the radius, and membrane action begins to take a dominate role. The membrane solution was used to find the deflection for pressures greater than 3.2 psi. These deflections are shown in Figure 8 for pressures up to 90.0 psi. A uniform pressure of 2.0 psi was applied initially to the middle surface and allowed to increase in steps of 2.0 psi after each convergence. The total computer time was approximately 10 minutes to convergence at 90.0 psi. A comparison with the deflection profiles of a plate with a uniform thickness as given in Figure 13 shows that the vertical deflection at the center is less, but the maximum horizontal displacement is approximately the same. The "flattening" effect at the top is apparently caused by the thick inclusion at the center. If it is desirable to have a relatively large horizontal displacement for a given pressure and less vertical displacement, the plate should be made thicker in the center portion. The meridional and circumferential stress resultants are shown in
Figures 9 and 10, and the thickness changes are shown in Figure 11.

The deflections of another circular plate with a linearly varying thickness from 0.5 inches at the center to 0.25 inches at the outer edge along the radius and constant thickness in the \( \theta \) direction and clamped along the outer edge are shown in Figure 14. The pressure was applied in the same way as the previous problem and allowed to converge at 3.0 psi. The stress resultants shown in Figures 15 and 16 have approximately the same value near the center, but the meridional stress resultant is slightly larger than the circumferential stress resultant near the outer edge. The same observation is true for the meridional stress-couples and the circumferential stress-couples as shown in Figures 17 and 18, respectively. The membrane solution was used to find large deflections above 3.0 psi, since membrane action takes the dominate role. Again, the pressure was applied in the same manner as the membrane portion of the previous problem and allowed to converge at 64.0 psi. The deflection at the center of the plate is approximately 3 percent less than the deflection of the uniformly thick plate at the center for 64.0 psi, and the maximum horizontal deflection is approximately 15.6 percent greater for the linearly varying thickness plate than for the uniformly thick plate at 64.0 psi. For pressures over 8.0 psi the meridional stress resultants shown in Figure 20, reaches a minimum value at approximately 3.0 inches from the outer edge. The circumferential stress resultants shown in Figure 21 are maximum near the center and decrease toward the outer edge.
A thin plate with a uniform thickness of 0.12 inches was analyzed with the bending equations and the membrane equations for a pressure of 1.0 psi. The deflections as seen in Figure 22 are almost identical for both solutions and the effect of bending is seen to be very small near the outer edge.

Different constitutive relations were used to determine the deflections of a uniformly thin plate for pressures up to 64.0 psi. As seen from Figure 13, the Rivlin-Saunders and the Alexander constitutive relations gave approximately the same results up to a pressure of 30.0 psi. As the pressure was increased, the Rivlin-Saunders relations gave a slightly larger deflection than the Alexander relations. The Hart-Smith constitutive relations gave larger deflections throughout the given range of pressures. Experimental results by Alexander (1968), on neoprene under biaxial stresses show that the Hart-Smith theory predicts a higher deflection for a given stress than the real deflection.

The deflections of a circular plate with a radius of 10.0 inches and 0.667 inches thick were found by using the tensor membrane equilibrium equations with the Alexander constitutive relations. The results were compared in Figure 23 with those given in Oden (1972), for a similar plate simply supported on the outer edge. The results given in Oden (1972), were found by using the finite element approach with the Mooney constitutive theory. Good agreement was found between the two different solutions. For a pressure of 42.0 psi the difference in deflections at the center was only 3.8 percent, and the difference
in maximum horizontal deflections was approximately 8 percent. Several investigators, including Rivlin and Saunders (1951), Treloar (1958), and Alexander (1968), have shown that the Mooney theory gives poor correlation with biaxial tension experiments. The Mooney theory predicts less deflection for a given value of stress than found from experiments. This fact could possibly be used to explain the slight difference between the solution given in Oden (1972) and the solution found in this paper.
Figure 1. Verification of the Incompressibility Assumption for Neoprene.
Figure 2. Derivative of the Strain Energy Density with respect to the Second Invariant vs. the Second Invariant for Neoprene.

C1 = 17.00 psi

○ - Equi-biaxial Experiments

□ - Uniaxial Experiments

NOTE: Scale Change
a = 10 in.
E = 270 psi
v = 1/2
h is variable

Figure 3. Deflection Profile vs. Radius.
\[ a = 10 \text{ in.} \]
\[ E = 270 \text{ psi} \]
\[ v = \frac{1}{2} \]

h is variable

Figure 4. Meridional Stress Resultant vs. Radius.
Figure 5. Circumferential Stress Resultant vs. Radius.

- \( p = 3.2 \) psi
- \( p = 2.6 \) psi
- \( p = 2.0 \) psi

**Parameters:**

- \( a = 10 \) in.
- \( E = 270 \) psi
- \( \nu = 1/2 \)
- \( h \) is variable
Figure 6. Meridional Stress Couple vs. Radius.

- $p = 2.6$ psi
- $p = 3.2$ psi
- $p = 2.0$ psi

$a = 10$ in.
$E = 270$ psi
$\nu = 1/2$
$h$ is variable
Figure 7. Circumferential Stress Couple vs. Radius.

\[ \begin{align*}
a &= 10 \text{ in.} \\
E &= 270 \text{ psi} \\
\nu &= 1/2 \\
h &= \text{variable}
\end{align*} \]
\[
\begin{align*}
a &= 10 \text{ in.} \\
E &= 270 \text{ psi} \\
\nu &= 1/2 \\
h \text{ is variable}
\end{align*}
\]

Figure 8. Deflection Profile vs. Radius
Figure 9. Meridional Stress Resultant vs. Radius.

- $a = 10\text{ in.}$
- $E = 270\text{ psi}$
- $\nu = 1/2$
- $h$ is variable
\[ a = 10.0 \text{ in.} \]

\[ E = 270 \text{ psi} \]

\[ \nu = 1/2 \]

\[ h \text{ is variable} \]

Figure 10. Circumferential Stress Resultant vs. Radius.
Figure 11. Thickness Variation vs. Radius.

Figure 11. Thickness Variation vs. Radius.

\( a = 10 \text{ in.} \)

\( E = 270 \text{ psi} \)

\( v = 1/2 \)
Figure 12. Deflection Profile vs. Radius.

\[ a = 10 \text{ in.} \]
\[ E = 270 \text{ psi} \]
\[ v = 1/2 \]
\[ p = 3.2 \text{ psi} \]

- --- h is constant
- --- h is variable
- --- No Bending

h = 0.5 in.
a = 10 in.
E = 270 psi
ν = 1/2
h = 0.5 in.

--- Alexander
--- Rivlin-Saunders
--- Hart-Smith

Figure 13. Deflection Profile vs. Radius.
a = 10 in.

E = 270 in.

ν = 1/2

h is variable

Figure 14. Deflection Profile vs. Radius.
Figure 15. Meridional Stress Resultant vs. Radius.

- $a = 10$ in.
- $E = 270$ psi
- $\nu = \frac{1}{2}$
- $h$ is variable
Figure 16. Circumferential Stress Resultant vs. Radius.

- p = 3.0 psi
- p = 2.0 psi

a = 10 in.
E = 270 psi
v = 1/2
h is variable
Figure 17. Meridional Stress Couple vs. Radius.

\[
a = 10 \text{ in.} \\
E = 270 \text{ psi} \\
\nu = 1/2 \\
h \text{ is variable}
\]
Figure 18. Circumferential Stress Couple vs. Radius.
\[ a = 10 \text{ in.} \]
\[ E = 270 \text{ psi} \]
\[ \nu = 1/2 \]
\[ h \text{ is variable} \]

Figure 19. Deflection Profile vs. Radius.
Figure 20. Meridional Stress Resultant vs. Radius.
Figure 21. Circumferential Stress Resultant vs. Radius.

-\( p = 64.0 \text{ psi} \)
-\( p = 32.0 \text{ psi} \)
-\( p = 8.0 \text{ psi} \)

\[
a = 10 \text{ in.}
\]
\[
E = 270 \text{ psi}
\]
\[
v = 1/2
\]
\[
h \text{ is variable}
\]
a = 10 in.
E = 270 psi
v = 1/2
h = 0.12 in.
p = 1.0 psi

Figure 22. Deflection Profile vs. Radius.
Figure 23. Deflection Profile vs. Radius.
VII. BIBLIOGRAPHY


Relations Between the Physical Strains and the Strain Tensor in Polar Coordinates

In polar coordinates let $ds_0$ be the initial length of a line element in the $\theta$ direction and $ds$ be the final length. Let $E_{ij\theta}$ be the Lagrangian nonlinear strain tensor. From Green and Zerna (1954) we know that

$$ (ds)^2 - (ds_0)^2 = 2 E_{ij\theta} da^i da^j \quad (A.1) $$

where $da^i$ are the tensorial coordinates. The left side of equation (A.1) can be written in the form

$$ (ds)^2 - (ds_0)^2 = \left(\frac{ds - ds_0}{ds_0}\right) \left(\frac{ds - ds_0 + 2ds_0}{ds_0}\right) (ds_0)^2 . \quad (A.2) $$

If we let the physical strain, $\varepsilon_2$, in the $\theta$ direction be defined as

$$ \varepsilon_2 = \frac{ds - ds_0}{ds_0} , \quad (A.3) $$

the left side of equation (A.1) can be written as

$$ (ds)^2 - (ds_0)^2 = (\varepsilon_2^2 + 2\varepsilon_2) (ds_0)^2 . \quad (A.4) $$

The right side of equation (A.1) can be expressed in the $\theta$ direction as

$$ 2E_{22} da^2 da^2 = 2E_{22} \frac{da^2}{ds_0} \frac{da^2}{ds_0} (ds_0)^2 . \quad (A.5) $$

Also, noting that in the $\theta$ direction
and

\[ ds_0 = rd\theta \]

(A.6)

and

\[ da^2 = d\theta, \]

(A.7)

we can express the right side of equation (A.1) in the form

\[ 2E_{22} da^2 da^2 = 2 \frac{E_{22}}{r^2} (ds_0)^2. \]

(A.8)

By substituting equations (A.4) and (A.8) into equation (A.1), we get

\[ \varepsilon_2^2 + 2\varepsilon_2 = 2 \frac{E_{22}}{r^2} \]

(A.9)

or

\[ \varepsilon_2 = \sqrt{1 + 2 \frac{E_{22}}{r^2}} - 1. \]

(A.10)

By using a similar procedure, we find that

\[ \varepsilon_1 = \sqrt{1 + 2E_{11}} - 1. \]

(A.11)

Equations (A.10) and (A.11) represent the relations between the physical strains and the Lagrangian nonlinear strain tensor.
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NONLINEAR DEFLECTIONS OF A CIRCULAR PLATE

WITH VARYING THICKNESS

by

Leighton A. Caldwell

(ABSTRACT)

A theoretical analysis of large deflections and large strains in a circular plate with varying thickness and a circular membrane is considered.

The exact tensor first approximation equilibrium equations, converted into physical equations for a rotationally symmetric thin plate are used with the Alexander constitutive relations for a rubber-like material to analyze the deflections, stress resultants and change in the thickness for a plate clamped along the outer edge and deflected by a uniform pressure applied normal to the deformed surface. The equations are quasilinearized and solved numerically with the aid of a digital computer.

The thickness is allowed to vary in the radial direction but is held constant in the circumferential direction. Several variations in thickness were considered. The solutions found by using the Alexander constitutive relations were compared with the solutions using the Rivlin and Saunders constitutive relations and the Hart-Smith constitutive relations. Numerical results from the solution of a plate with uniform thickness were compared with those for a similar plate given by J. T. Oden.