THE DESIGN OF SUBOPTIMAL LINEAR REGULATORS

USING REDUCED ORDER AGGREGATED MODELS

by

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CHAPTER I

Objective

The object of this dissertation is to develop and implement a technique for designing approximately optimal (suboptimal) regulators for higher order linear, constant-coefficient dynamic systems. The controls are to be found by determining reduced order models whose states are related to the states of the actual system by a constant linear transformation. These reduced models will then be used to determine control laws for the systems.

Background

In the study of systems it is quite common to approximate very complex processes with simple linear models. These simple models were initially necessary to limit the amount of hand calculation required prior to the advent of the digital computer. Recently reduced models have been studied as a means of reducing the amount of computing time and equipment needed in computer simulation and control of complex processes. In particular, reduced models appear very attractive for design of linear regulators for large-scale dynamic systems. They allow the use of regulators designed and implemented using a model of relatively small order, and help overcome some of the problems involved in computation of optimal controls.

The work of Davison [1], which describes a method for approximating a system of high order with one of lower order, has led to further work by several researchers (see [2]-[11]). The principle of

Davison's technique is to neglect eigenvalues of the original system which are farthest from the origin and retain in the reduced model only those eigenvalues, termed the dominant eigenvalues, whose response lasts longest. This basic idea is also the theme of the follow up work mentioned. While this concept is appealing it is appropriate to ask whether a set of dominant eigenvalues exists and if it does how is it selected from all possible combinations of eigenvalues. The answer to this question has yet to be found. This method does seem to be applicable when the eigenvalues fall into widely separated groups but even then a problem can be encountered if all the eigenvalues of a group are not retained. Further, the dominant eigenvalue method generally requires computation of all the eigenvalues and eigenvectors of a large order system. This task is sometimes difficult for large order systems, exactly the case where use of the technique is most desirable.

Other techniques not explicitly dependent on the idea of dominant eigenvalues have been proposed. Chen [12] has advanced a method based on expanding the transfer function of the system into a continued fraction and ignoring some quotients. Rotherberg [13] develops this method in more detail and offers a very good in-depth treatment of the simplification problem. Although the method appears quite useful it is limited to systems with no positive eigenvalues. Also, the application of this technique to systems which are not single input, single output does not appear straightforward.

Quite recently several authors (see [14]-[16]) have discussed the idea of lower order "aggregated" models and their application to the development of suboptimal regulators. Briefly this work shows that it is possible to retain the dominant eigenvalues of the exact system model in a reduced model and to require that the states of the two models be related by a constant transformation C, the aggregation matrix. This idea and especially the work of Schainker [15] provide the direction for this dissertation.

In addition to these methods, several other researchers have studied the model simplification and suboptimal control problems. References [18]-[21] offer an illustrative cross section of work in the field.

Approach

The approach taken by this dissertation is to examine the linear regulator problem to determine the difficulties involved with designing and implementing the optimal controller. Based on these problems and certain practical considerations a set of requirements that the reduced model must satisfy is presented. Using these requirements as a guide a procedure for determining suitable reduced models will be described and several example problems solved to demonstrate the utility of the method.

CHAPTER II

Introduction

This chapter reviews the linear regulator problem and discusses the difficulties involved with solving for and implementing the optimal controller. The procedure for using a reduced model to generate a suboptimal controller is then discussed. Based on the application for the reduced model and the problems involved with implementing the optimal control, a set of requirements that the reduced model must satisfy is presented.

Review of the Linear Regulator

The linear regulator is one of the most widely studied problems in the field of optimal control. It occurs when the system described by equation (2.1) is to be controlled so that the cost function of equation (2.2) is minimized.

$$\dot{x} = Ax + Bu \qquad x(t_0) = x_0 \qquad 2.1$$

$$z = Hx$$

where

x is the state vector of order n,

u is the control vector of order m,

z is the output vector of order p,

A, B and H are constant matrices of appropriate order .

$$J = \frac{1}{2} z_f^T S' z_f + \frac{1}{2} \int_{t_0}^{t_f} (z^T Q' z + u^T R u) dt \qquad 2.2$$

where

$Q' \geq 0$, $S' \geq 0$, $R > 0$ and z_f is the value of z at t_f.

Notice that this is the form for an output regulator. The problem can be quickly converted to a state regulator by using $z = Hx$ in equation (2.2). This results in $z^T Q' z$ being replaced by $x^T H^T Q' H x$.

This research assumes that the system under study can be described exactly by equation (2.1) and that this model is known. Further, it is assumed that H has been chosen so that the output z reflects the relative importance attached to each state.

In order to present the solution of the regulator problem in the form which was used for computing the controls in Chapter IV, equations (2.1) and (2.2) are given below in their discrete forms.

$$x_{K+1} = Ax_K + Bu_K \qquad 2.1a$$

$$z_K = Hx_K$$

where

A is now the state transition matrix,

B is now the disturbance transition matrix.

$$J = \frac{1}{2} ||x_f||_S^2 + \frac{1}{2} \sum_{0}^{K_f - 1} ||x_K||_Q^2 + ||u_K||_R^2 \qquad 2.2a$$

where

$S = H^T S' H$, $Q = H^T Q' H$ and x_f is the value of x at K_f.

As is well known the solution of the discrete regulator problem is a linear feedback law given by the following difference equations (see [22]-[24]).

$$u_K^* = -(B^T P_K B + R)^{-1} B^T P_K A x_K = -G_K x_K \qquad 2.3$$

$$P_K = Q + A^T P_{K+1}[I - B(B^T P_{K+1} B + R)^{-1} B^T P_{K+1}]A \qquad 2.4$$

$$P_{K_f} = S \ .$$

As can be seen determination of the optimal control u_K^* requires that the matrix difference equation (2.4) be solved backwards from K_f and the solutions stored by some means. This solution is then used along with all the system states to compute the optimal control using equation (2.3). This process is quite costly and time consuming and in general not practical.

The problem can be simplified somewhat if $K_f \to \infty$. In this case P_K is constant for all finite K. The optimal control is then given by

$$u_K^* = -Gx_K \qquad 2.5$$

where

G is a constant mxn matrix.

Even with this simplifying assumption equation (2.4) must be solved backwards in time until the equilibrium solution is obtained, a time consuming task for large order systems, and equation (2.3) used to compute G. In addition all n states of the system must still be determined, if not directly measurable, and fed back. Clearly for large order systems the optimal control is still difficult to realize.

Control Using a Reduced Model

To overcome some of these difficulties let us look at a suboptimal control technique. Suppose that a model of the form given by equation (2.6) can be found so that $\hat{z} \simeq z$ of equation (2.1) for arbitrary inputs u.

$$\dot{y} = \hat{A}y + \hat{B}u \qquad y(t_0) = y_0 \qquad 2.6$$
$$\hat{z} = \hat{H}y$$

where

y is the state vector of order q,

u is the input vector of order m,

$\hat{z}$ is the output vector of order p,

$\hat{A}$, $\hat{B}$ and $\hat{H}$ are constant matrices of appropriate order.

Further suppose that the states of this model are related to the states of the exact model by a constant transformation C, or y = Cx. Now let us find the control $\hat{u}^*$ that minimizes

$$\hat{J} = \frac{1}{2}\,\hat{z}_f^T S' \hat{z}_f + \frac{1}{2}\int_{t_0}^{t_f} (\hat{z}^T Q' \hat{z} + u^T R u)\,dt \qquad 2.7$$

where

S', Q' and R are the same as for equation (2.2).

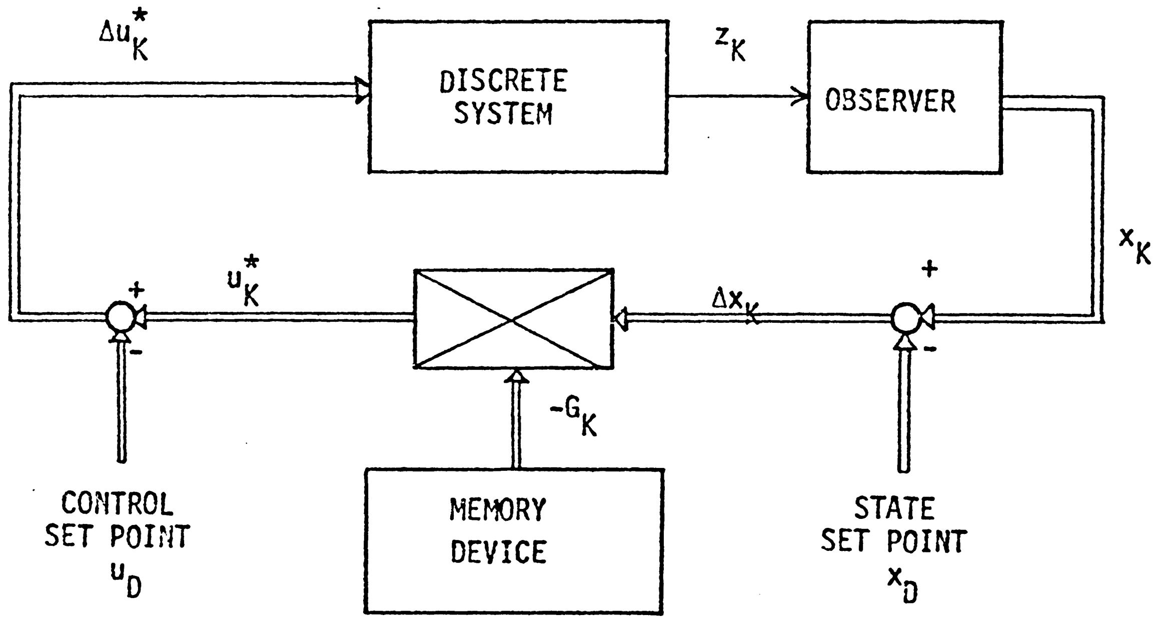
After using $\hat{z} = \hat{H}y$ to convert this to a state regulator problem, equations (2.6) and (2.7) can be converted to discrete form. Using this model equations (2.3) and (2.4) can then be used to find the control that minimizes the discrete counterpart of equation (2.7).

Since $\hat{z} \simeq z$ for arbitrary u, this control should be near optimal or suboptimal for the cost function of equation (2.2). Note that if q is considerably smaller than n, computation and storage of the feedback gains will be much simpler and fewer state variables will be required for feedback. The requirement that $y = Cx$ still allows the system to be driven to an equilibrium point other than zero. Figure 2.1 should help clarify any vagueness concerning implementation of the optimal and suboptimal controls.

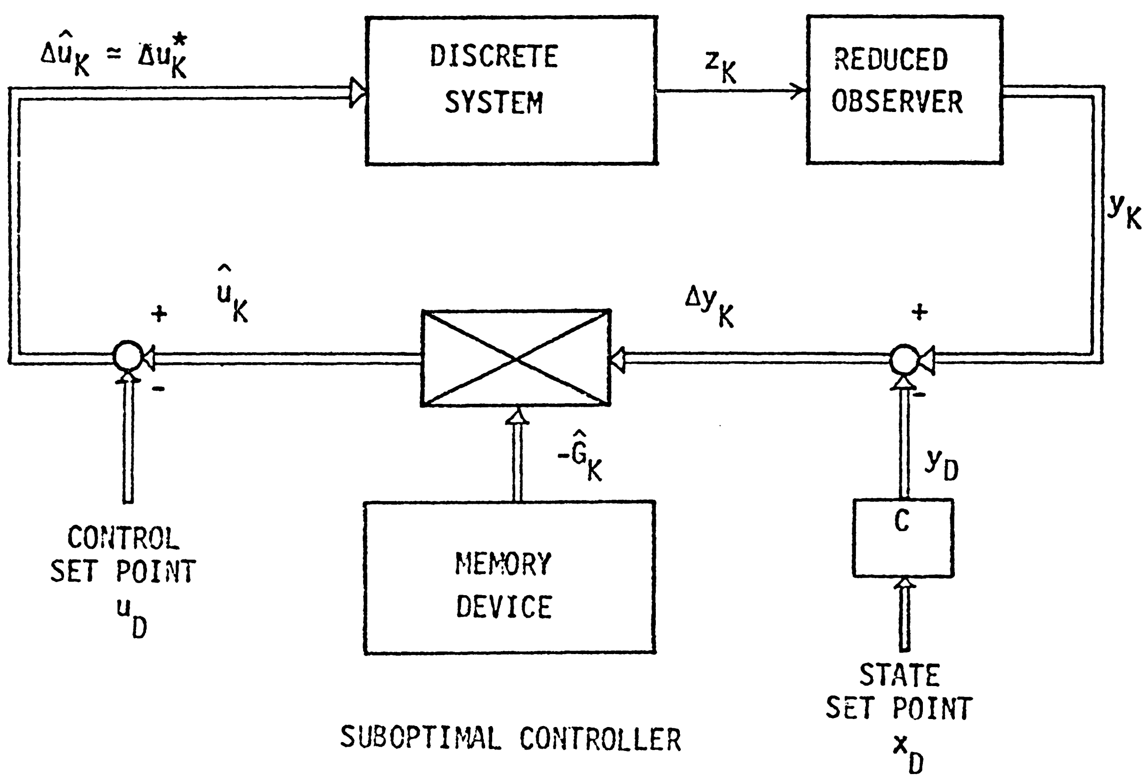
The method of control posed here is certainly feasible since simple models are frequently used to describe very complex systems and control based on these simplified or reduced models is generally valid. Also it must be realized that the system model previously assumed exact is only an approximation and as such neither unique nor sacred. From this viewpoint the further approximation of the system by the model of equation (2.6), which is easier to work with, can be justified if the control laws determined from this model are satisfactory.

Having formulated the idea of controlling a system based on an approximate model of reduced order, it is necessary to determine a practical method for finding this model. The basic method proposed by this research is illustrated in Figure 2.2.

After deciding upon this very general technique certain specific questions arise. Foremost among these is the problem of specifying the reduced model and its free parameters. Certainly the number of free parameters should be kept at a minimum so that the adjustment technique remains fairly simple. Along with this choice a test function,



OPTIMAL CONTROLLER



SUBOPTIMAL CONTROLLER

FIGURE 2.1. IMPLEMENTATION OF OPTIMAL AND SUBOPTIMAL REGULATORS FOR A DISCRETE LINEAR SYSTEM

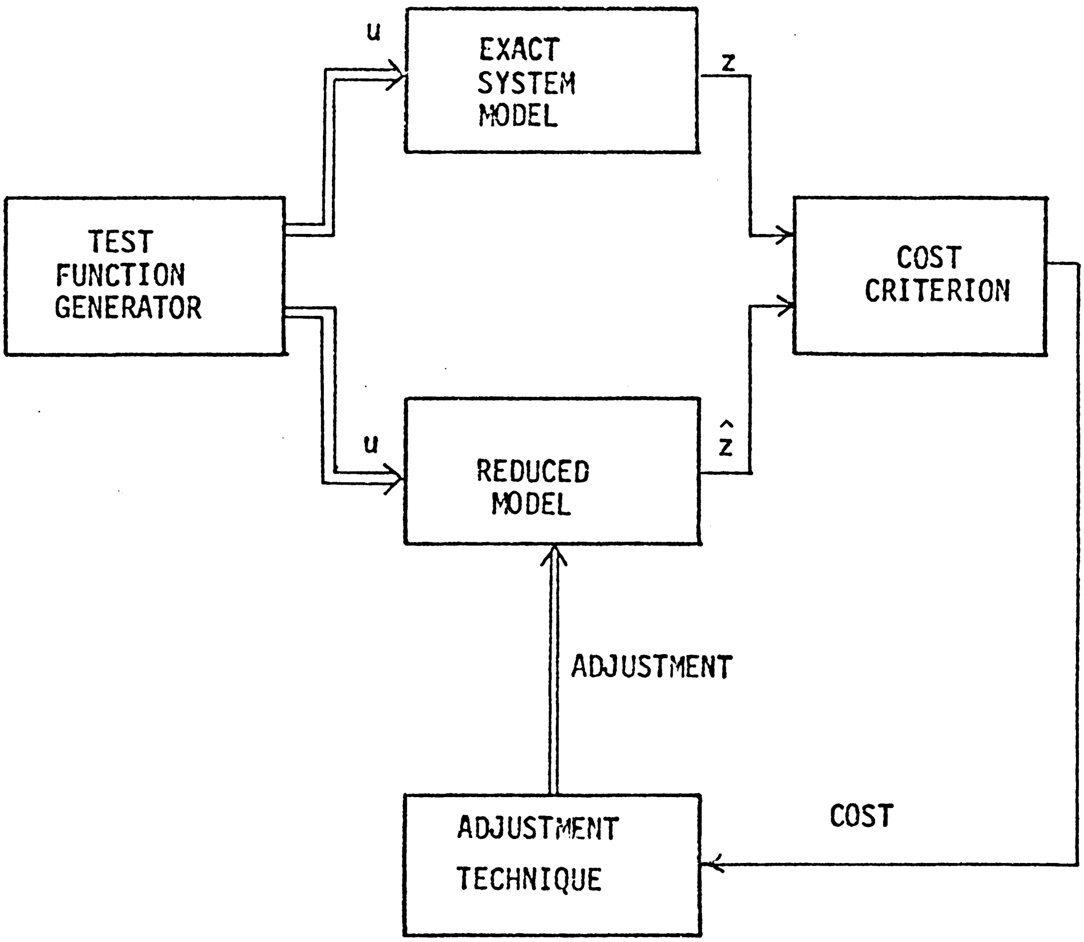


FIGURE 2.2. BLOCK DIAGRAM OF BASIC METHOD FOR DETERMINING THE REDUCED MODEL

cost criterion and adjustment technique must be selected. These selections will be made by the development of four general requirements which the reduced model must satisfy.

Reduced Model Requirements

First it should be noted that the requirements for a reduced model are highly dependent on the application for which the model is desired. Since the application here is the design of a suboptimal regulator the reduced model will be adequate if the resulting closed loop control laws for the non-reduced system are "satisfactory". For this research satisfactory will be defined as meaning that the suboptimal regulator does the following:

(1) stabilizes the system,

(2) drives the system states to the desired final values, and

(3) has an associated cost function (equation (2.2)) which is reasonable when compared with the optimal cost.

There may be cases where the regulator found using a reduced model does not satisfy one or more of the above requirements and the reduced model, no matter how well it performs otherwise, will not be adequate. Further there is no guarantee that a suboptimal regulator exists which will satisfy these requirements. In spite of these considerations, the basic idea of controlling a system based on a properly chosen reduced model still remains sound. To help insure that the reduced model is properly chosen and to avoid some difficult problems involved with finding the reduced model and implementing the resulting control, four general requirements will now be placed on the reduced model. Following each requirement there is a discussion and, where applicable, an explanation of how the requirement can be satisfied.

1) The output of the reduced model must match in some best manner the output of the exact system model when both models are excited by arbitrary inputs.

Since in the general case it will probably be impossible to determine dominant modes, if indeed there are any, and retain them in the reduced model, it is felt that matching input-output relationships is the only feasible method. This requirement also helps insure that the regulator designed using the reduced model stabilizes the exact system. If the classical definition of stability is used then the exact system will be stable if its output is bounded for bounded input. Suppose now that a regulator is designed for the reduced model. The response of the reduced model should then be bounded since the solution of equations (2.3) and (2.4) yield a stable control law for completely controllable and completely observable systems. Since the outputs of the two systems match, if the reduced model has a bounded output the exact system output should also be bounded. As previously stated, however, this cannot be guaranteed.

This requirement necessitates the choice of a cost criterion and a test function. Since the outputs must match, some positive definite function of the error is only logical for the cost function. Further it would be desirable for J of (2.2) and $\hat{J}$ of (2.7) to be close also. For this reason subtract $\hat{J}$ from J.

$$J - \hat{J} = \frac{1}{2}\left(z_f^T S' z_f - \hat{z}_f^T S' \hat{z}_f\right) + \frac{1}{2}\int_{t_o}^{t_f} \left(z^T Q' z - \hat{z}^T Q' \hat{z}\right)dt \qquad 2.7$$

This function is not a suitable criterion since it is not positive definite; however, a similar function was used. For this research the cost function of (2.8) was used. It is simple, forces the outputs to match and helps insure that J and $\hat{J}$ are close.

$$CF = \frac{1}{2}\left(z_f - \hat{z}_f\right)^T S'\left(z_f - \hat{z}_f\right) + \frac{1}{2}\int_{t_o}^{t_f} \left(z - \hat{z}\right)^T Q'\left(z - \hat{z}\right)dt \qquad 2.8$$

The choice of a test function was somewhat more complicated. The use of a step was considered since experience has shown that if a linear system has a good step response it will usually have a transient response that is at least satisfactory for general input. The step was ruled out, however, since it is not applicable to multi-input systems. The test signal decided upon was Gaussian white noise passed through a first order filter. The interested reader is referred to reference [25] which offers a discussion on the use of stochastic inputs for test signals. It should be noted that the filter was used to limit the bandwidth of the test signal to fall within the bandwidth of the systems studied. This was necessary since the adjustment technique used was a gradient descent algorithm. As explained in [25] if the test signal has a bandwidth much wider than the bandwidth of the system, gradient techniques tend to result in adjustment of the approximate system so that $\hat{z}$ approaches the average value of z. Adjustment of the mean value and variance was performed on a trial and error basis from this point to determine a signal that worked well for each system.

2) A relationship between the states of the approximate model and those of the exact model must be maintained.

Certainly this is an obvious requirement since the purpose of the regulator is to drive the system states to prescribed values. This requirement is met by forcing the reduced model to be an approximate aggregated model. For the reduced model proposed here x and y, the states of the exact and approximate models, are assumed related by a constant transformation C, the aggregation matrix or

$$y \simeq Cx \ . \qquad 2.9$$

If C is known then y can be determined using a reduced order Luenberger type observer and the suboptimal control can be implemented quite easily. This idea is discussed again in Chapter III.

3) Implementation of the suboptimal control must require less storage and/or on line computation and require the feedback of fewer states than does the optimal control.

This requirement has been previously mentioned and is closely tied to requirement 2); however, much prior work has centered on approximate solution of the Riccati equation with no reduction in the number of states required for feedback or amount of storage required.

4) The reduced model cannot be based on the assumption that the model of the system is in any special form (i.e. diagonal); neither can there be a requirement to know all the eigenvalues exactly.

To anyone who has attempted the transformation of a large system model to Jordan form, the necessity of this requirement is obvious.

To require that the system be in diagonal form is to require that it be simplified, though not reduced, to begin with. Even the calculation of eigenvalues for systems of 25 or 30 states is by no means trivial, especially when they are widely separated.

Summary

This chapter has reviewed the linear regulator problem and formulated the idea of using a reduced order aggregated model to generate a suboptimal control. A set of criteria which the reduced model must satisfy was also developed. In Chapter III two techniques for determining reduced models that fit within this framework will be developed.

CHAPTER III

Introduction

In this chapter two methods for determining the reduced model and the suboptimal control are presented. Also, implementation of the techniques by means of computer programs is discussed.

Statement of the Reduced Modeling Problem

Chapter II developed the idea of using a reduced order aggregated model to generate a suboptimal control. The criteria that the reduced model must satisfy was stated and a general method to determine the reduced model presented. This chapter is devoted to presenting two specific methods for determining reduced models. For this reason an exact statement of the modeling problem is needed.

Problem statement

Given

$$\dot{x} = Ax + Bu$$

$$z = Hx \qquad 2.1$$

where

x is the state vector of order n,

u is the input vector of order m,

z is the output vector of order p,

A, B and H are constant matrices of appropriate order, determine a reduced model

$$\dot{y} = \hat{A}y + \hat{B}u \qquad 2.6$$

$$\hat{z} = \hat{H}y$$

where

y is the state vector of order q,

u is the input vector of order m,

$\hat{z}$ is the output vector of order p,

$\hat{A}$, $\hat{B}$ and $\hat{H}$ are constant matrices of appropriate order so that

$$1)\quad \hat{z} \simeq z \qquad 3.1a$$

$$2)\quad y \simeq Cx\ . \qquad 3.1b$$

In Chapter II it was stated that (3.1a) would be satisfied by choosing the reduced model so that CF of (3.2) is minimum where

$$CF = \int_{t_0}^{t_f} (z - \hat{z})^T Q'(z - \hat{z})dt \qquad 3.2$$

when both models are excited by filtered white noise. In addition equation (3.1b) must also be satisfied.

The First Reduced Modeling Technique

The first method used for determining a reduced model is based on the technique proposed in references [14] and [15]. It differs from this technique in that the reduced model does not retain the modes of the exact model unless they are unstable. Rather, the modes of the reduced model are chosen so that the error between the outputs of the reduced and exact models is minimized. The technique does not require the exact model to be in any special form, bypassing some difficult problems.

The method can be broken down into steps as follows:

1) Construct a reduced observer for the exact system whose states are related to the states of the exact model by a constant transformation C_1.
2) Determine a measurement matrix for the observer so that the output of the observer is close to the output of the exact model.
3) Combine the observer with an auxiliary linear system to form the reduced model.
4) Select the variables of the auxiliary system so that CF of equation 3.2 is minimized.

A detailed discussion of the technique will now be given.

Construction of the reduced observer

The first step of the technique is the construction of an observer. Equation (3.3) is the general equation describing an observer of the exact system.

$$\dot{y}_1 = A_1 y_1 + B_1 u + Fz \qquad 3.3$$

where

y_1 is the state vector of order ℓ,

u is the input vector of order m,

z is the system output vector of order p,

A_1, B_1 and F are constant matrices of appropriate order.

The desired result is the determination of A_1, B_1 and F so that $y_1 = C_1x$. For this reason consider

$$\dot{y}_1 - C_1\dot{x} = A_1y_1 - C_1Ax + Fz + B_1u - C_1Bu \; . \qquad 3.4$$

Grouping terms and using $z = Hx$ results in

$$\dot{y}_1 - C_1\dot{x} = A_1y_1 - (C_1A - FH)x + (B_1 - C_1B)u \; . \qquad 3.5$$

Now if

$$C_1A - A_1C_1 = FH \qquad 3.6a$$

and

$$B_1 = C_1B \qquad 3.6b$$

then

$$\dot{y}_1 - C_1\dot{x} = A_1(y_1 - C_1x) \; . \qquad 3.7$$

Equations (3.6a) and (3.6b) are simply the requirements for a Luenberger type observer - references [26]-[27].

Solving equation (3.7) yields

$$y_1 - C_1x = e^{A_1t}(y_1(0) - C_1x(0)) \; . \qquad 3.8$$

If all eigenvalues of A_1 have negative real parts the right hand side of (3.8) will decay to zero and $y_1 = C_1x$. Since no restrictions have been placed on A_1 it can be chosen in any special form. This is quite helpful since the solution of equation (3.6a) can be made much simpler by a judicious choice for A_1.

Since there are as yet no restrictions on A_1 or F, let A_1 be in phase variable form and $F = (0 \quad 0 \quad \ldots \quad 0 \; f^T{}_\ell)^T$, where f_ℓ represents the ℓth row of F. This leads to a useful solution for C_1 in equation (3.6a). This solution is given in Appendix A and is repeated here for convenience.

Let c_i represent the ith row of C_1, then $C_1 = (c_1^T c_2^T \ldots c_\ell^T)^T$ and

$$A_1 = \begin{bmatrix} 0 & 1 & & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ a_1 & a_2 & & \cdots & a_\ell \end{bmatrix} . \qquad 3.9$$

The K+1st row of C_1 is given by

$$c_{K+1} = c_1 A^K \qquad K = 1, \ldots \ell - 1 \qquad 3.10$$

where c_1 is the solution of

$$-c_1[a_1 I + a_2 A + \ldots + a_\ell A^{\ell-1} - A^\ell] = f_\ell H . \qquad 3.11$$

Further C_1 will have a unique solution if A and A_1 have no eigenvalues in common and will have full rank if the nxℓ matrix $(c_1^T, A^T c_1^T \ldots (A^T)^{\ell-1} c_1^T)$ has rank ℓ. The rank of C_1 is important since if C_1 has rank less than ℓ some of the states y_1 will not be independent. The relationship between $f_\ell H$ and the rank of C_1 is not apparent, however, and although this is a critical point no hard and fast rules can be given. It is obvious though that if c_1 defines an observable output of the system (2.1), C_1 will have rank ℓ since the nxn matrix $(c_1^T \; A^T c_1^T \; . \; . \; . \; (A^{n-1})^T c_1^T)$ will have rank n (see [22]). From (3.11) c_1 and $f_\ell H$ are related by a nonsingular transformation, therefore considerable liberty is available in choosing c_1 by varying $f_\ell H$. It is felt that the rank of C_1 will not become a problem until the order of the observer becomes large with respect to the system order n, however, and no problem was encountered during this research.

Once the choice for A_1 has been made design of the observer proceeds directly by solving equation (3.6a) for C_1 and equation (3.6b) for B_1, completing the first step.

Construction of the reduced model

At this point an observer of the exact system has been found. This observer meets the required conditions that its states are related to the states of the exact system by a constant linear transformation C_1, or $y_1 = C_1x$. The observer equation is repeated below for convenience.

$$\dot{y}_1 = A_1y_1 + B_1u + Fz \; . \tag{3.3}$$

Notice that this equation requires the exact system output $z = Hx$ as an input. Suppose now that z is replaced by $H_1y_1 + e$. Then

$$\dot{y}_1 = A_1y_1 + B_1u + FH_1y_1 + Fe \tag{3.12}$$

where

$$e = Hx - H_1y_1 = (H - H_1C_1)X \; . \tag{3.13}$$

Suppose now that H_1 can be chosen so that the effect of e on the transient response of (3.12) is small. e can then be approximated by an additional linear system of small order (i.e. 1st or 2nd).

To approximate e define an auxiliary system as follows:

$$\begin{aligned} \dot{y}_2 &= A_2 y_2 + B_2 u \\ z_2 &= H_2 y_2 \end{aligned} \qquad 3.14$$

where

y_2 is the state vector of order r,

u is the input vector of order m,

z_2 is the output vector of order p,

A_2, B_2 and H_2 are constant matrices of appropriate order.

A reduced model can now be found by combining equations (3.12) and (3.14) to form (3.15).

$$\begin{aligned} \dot{y}_1' &= A_1 y_1' + B_1 u + FH_1 y_1' + FH_2 y_2 \\ \dot{y}_2 &= A_2 y_2 + B_2 u \\ z_1' + z_2 &= H_1 y_1' + H_2 y_2 \; . \end{aligned} \qquad 3.15$$

Writing (3.15) in matrix form and grouping terms yields

$$\begin{bmatrix} \dot{y}_1' \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} A_1 + FH_1 & FH_2 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} y_1' \\ y_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$[z_1' + z_2] = H_1 y_1' + H_2 y_2 \; . \qquad 3.16$$

The prime (') is used to denote that y_1' is different from y_1 of (3.12) since $H_2 y_2$ is not identically equal to e.

Examining equation (2.6) it can be seen that (3.16) is in the desired form for the reduced model where

$$\hat{A} = \begin{bmatrix} A_1 + FH_1 & FH_2 \\ 0 & A_2 \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$\hat{z} = [z_1' + z_2] \; . \qquad 3.17$$

Notice now that H_1, H_2, A_2, B_2 and H_2 are as yet unspecified and can be chosen so that $\hat{z} \simeq z$. If these variables can be determined so that $\hat{z}$ is a good approximation for z then

$$y_1' \simeq C_1 x$$

$$H_2 y_2 \simeq (H - H_1 C_1) x \qquad 3.18$$

and

$$\hat{z} \simeq z \; .$$

An important point which may not be apparent is the particular form of $A_1 + FH_1$ in (3.16). Using the phase variable form for A_1 as given in (3.9) and the fact that $F = (0 \quad \ldots \quad f_\ell^T)^T$

$$A_1 + FH_1 = \begin{bmatrix} 0 & 1 & & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ d_1 & d_2 & & \ldots & d_\ell \end{bmatrix} \qquad 3.19$$

where $d_i = a_i + [f_\ell H_1]_i$.

The characteristic polynomial of $A_1 + FH_1$ is

$$\rho_1(\lambda) = \lambda^\ell - (a_\ell + [f_\ell H_1]_\ell)\lambda^{\ell-1} - \ldots - a_1 + [f_\ell H_1]_1 . \qquad 3.20$$

From (3.20) it is obvious that for a given A_1 and f_ℓ the d_i of (3.19) can be placed arbitrarily by specifying H_1. This fact plays an important role in the following development.

Choosing the Free Variables of the Reduced Model

Selection of A_1 and H_1

Before specifying a technique for choosing all the free variables of the reduced model, it is enlightening to study the error term of equation (3.13). First, however, consider the exact system of (2.1).

$$\dot{x} = Ax + Bu$$
$$z = Hx. \qquad 2.1$$

From elementary matrix theory there exists a nonsingular P such that

$P^{-1}AP = J$, where J is the Jordan canonical form of A. Applying this transform to (2.1)

$$P\dot{x} = JPx + PBu \qquad 3.21$$

$$z = Hx \ .$$

Letting $q = Px$ in (3.19)

$$\dot{q} = Jq + PBu$$

$$z = HP^{-1}q \ . \qquad 3.22$$

Notice that each mode of the system now corresponds to an entry in q. Schainker [15] has derived the error terms for the system of equation (3.22). If A has distinct eigenvalues then

$$e = \sum_{K=1}^{n} [HP^{-1}]_K \frac{\rho_1(\lambda_K)}{\rho_2(\lambda_K)} q_K \qquad 3.23$$

where $[HP^{-1}]_K$ represents the K^{th} column in HP^{-1},

λ_K is the K^{th} eigenvalue of A,

q_K represents the state (mode) associated with λ_K,

$\rho_1(\lambda_K)$ is the characteristic polynomial of $A_1 + FH_1$ evaluated for λ_K,

$\rho_2(\lambda_K)$ is the characteristic polynomial of A_1 evaluated for λ_K.

Suppose now that J has a pxp Jordan block associated with some eigenvalue λ_j. Without loss of generality it may be assumed that this block occupies the upper left corner of J and that $\lambda_1 = \lambda_2 = \ldots = \lambda_p = \lambda_j$. For this case Schainker has shown that

$$e = \sum_{K1=1}^{p} [HP^{-1}]_{K1} \frac{1}{(K1-1)!} \left(\frac{d^{K1-1}}{d\lambda^{K1-1}} \frac{\rho_1(\lambda)}{\rho_2(\lambda)} \right) \Bigg|_{\lambda=\lambda_1} q_{K1}$$

$$+ \sum_{K2=p+1}^{n} [HP^{-1}]_{K2} \frac{\rho_1(\lambda_{K2})}{\rho_2(\lambda_{K2})} q_{K2} \qquad 3.24$$

It should be noted that Schainker derived these equations in somewhat different form based on the exact system model in Jordan form. If this is the case the solution for C_1 and manipulation of the defining equation for e are much simpler. The assumption of Jordan form is not necessary however since (3.23) and (3.24) can be derived without it. The manipulation necessary is tedious, however, and since the equations have already been derived, repetitive.

From equations (3.23) and (3.24) several significant conclusions can be drawn.

1) The error terms can be made small by choosing the eigenvalues of A_1 so that the absolute value of $\rho_2(\lambda)$ is large for λ equal to the eigenvalues of A.

2) The error due to any mode can be made zero by choosing H_1 so that the eigenvalue associated with that mode is a root of $\rho_1(\lambda)$. Stated in other words, if q_K is a mode of the exact system associated with λ_K, the error due to that mode will be zero if λ_K is also an eigenvalue of $A_1 + FH_1$.

3) If q_K is an unstable mode (i.e. λ_K does not have a negative real part) the error will increase without bound unless λ_K is also an eigenvalue of $A_1 + FH_1$.

4) The problem of choosing C_1 is bypassed since e is dependent on the characteristic polynomial of A_1.

Considerations similar to the above led Chidambara and Schainker [14] and [15] to propose that A_1 and H_1 be chosen in the following manner:

1) Choose H_1 so that $\rho_1(\lambda) = 0$ for λ equal to the dominant eigenvalues of A,

2) Choose A_1 so that

$$\left|1 - \frac{\rho_1(\lambda)}{\rho_2(\lambda)}\right| < \varepsilon$$

for λ equal to the nondominant eigenvalues of A.

While this procedure can work quite well for some problems, as previously emphasized many systems do not have dominant eigenvalues. An alternate procedure is therefore proposed.

1) Choose the eigenvalues of A_1 so that $\rho_2(\lambda)$ is as large as possible for λ equal to the stable eigenvalues of A.

2) Specify relationships between the entries in H_1 so that $A_1 + FH_1$ retains the unstable eigenvalues of A.

3) Choose H_1 so that

$$CF_1 = \int_{t_o}^{t_f} e^T Q' e dt \qquad 3.25$$

is minimum when the exact model is excited by the test signal.

This criteria is flexible since it allows some of the eigenvalues of $A_1 + FH_1$ to be chosen for the most good. If for some reason it is known that a particular mode is dominant or if it is desired to retain a particular mode in the reduced model, 2) above can be modified so that $\rho_1(\lambda) = 0$ for the eigenvalue associated with that mode.

The above three criteria for choosing H_1 and A_1 are the ones proposed by the first modeling technique. As will be shown later the resulting method still leaves much to be desired. However, there is an advantage over the strictly dominant eigenvalue approach. It is felt that critique of the first method will be more helpful following explanation of the entire technique and further discussion at this point will be omitted.

Selection of A_2, B_2 and H_2

The method used to evaluate the entries in A_2, B_2 and H_2 of the reduced model was to minimize

$$CF_2 = \int_{t_0}^{t_f} (z - z_1' - z_2)^T Q' (z - z_1' - z_2) dt \qquad 3.26$$

using gradient descent. Note that if no assumptions are made this involves considering $r(1 + m + p)$ variables as free where r is the order of the auxiliary system, m is the number of inputs and p the number of outputs. For a single-input single-output system this would involve only three variables, if the auxiliary system is first order, and would be practical. However, for multi-input-output systems the number of free variables can be very large. It was found, however, that good results could be obtained by assuming relationships between the variables. One obvious means to obtain such a relationship is to require that z and $\hat{z}$ have identical steady state values. If this requirement is enforced manipulation of the equations defining the reduced and exact models yields

$$[B_2]_j = \frac{HA^{-1}[B]_j - H_1A_1^{-1}[B_1]_j}{-H_1f_\ell H_2/a_1 + H_2} A_2 \qquad j = 1, \ldots, m \qquad 3.27$$

where

$[\ \ ]_j$ indicates the jth column of the appropriate matrix.

This equation is good for single-output systems with $r = 1$. Similar results can be obtained for multi-output systems and $r > 1$. Notice that the entries in B_2 are now defined in terms of A_2 and H_2, which for this case are scalars. Minimizing CF_2 then requires only the calculation of two gradients and is much simpler. Notice also that without loss of flexibility it can be assumed that $f_\ell H_2$ remains constant. This relationship was used to determine the reduced model of Example 1 in Chapter IV and worked quite well.

This technique would be nice if all systems exhibited a constant steady state response. Unfortunately, many systems exhibit an unbounded response to a constant input and unless some special form is assumed on the A matrix of the exact system, this technique is not feasible. As an alternate procedure CF_2 can be minimized by varying all the entries in A_2, B_2 and H_2. A simplifying assumption in this case is that $f_\ell H_2$ of (3.16) is a constant. This is valid since any change in the contribution of $f_\ell H_2 y_2$ can be affected by simply changing the entries in A_2 and B_2. This method was used in Example 2 of the next chapter.

Once the entries in the model of (3.16) have been determined the matrix C_2 must be determined so that $y_2 \simeq C_2 x$. This was done by assuming $z = \hat{z}$. If this is true then

$$H_2y_2 = (H - H_1C_1)x \tag{3.28a}$$

and C_2 is the solution of

$$H_2C_2 = H - H_1C_1 \; . \tag{3.28b}$$

This completes the development of the first reduced modeling technique. For clarity the steps will be reviewed.

Summary of the method

1) Choose the eigenvalues of A_1 as large in magnitude as possible and form

$$A_1 = \begin{bmatrix} 0 & 1 & & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ a_1 & a_2 & & \cdots & a_\ell \end{bmatrix} . \tag{3.4}$$

2) Select f_ℓ and solve $C_1A - A_1C_1 = FH$ for C_1. Check to see that the rank of C_1 is ℓ.

3) Solve for $B_1 = C_1B$.

4) Specify relationships between the entries in H_1 so that $A_1 + FH_1$ retains the unstable eigenvalues of A. Excite the exact model with the stochastic test function and choose H_1 to minimize

$$CF_1 = \int_{t_o}^{t_f} (Hx - H_1y_1)^T Q'(Hx - H_1y_1)dt$$

to find the best values of H_1 .

5) Form the reduced model of (3.16). Minimize

$$CF_2 = \int_{t_o}^{t_f} (z - z_1' - z_2)^T Q' (z - z_1' - z_2) dt \qquad 3.26$$

to find A_2, B_2 and H_2.

Computer Implementation of the First Reduced Modeling Technique

The technique described in the previous section was implemented on the digital computer and several examples solved. This section discusses the computer program, some of the methods used and problems encountered.

Figure 3.1 is a flow chart for the program used. Notice that solution for a reduced model is by no means trivial and several runs for any particular problem might be necessary to determine a useable model. Some of the possible problem areas will now be discussed.

The test signal

Figure 3.2 shows the model for the input test signal. As mentioned previously a first order filter was used so that

$$W(S) = \frac{a}{s + a} . \qquad 3.29$$

If a is chosen smaller than the lowest corner frequency of the system under study, then the frequency spectrum of the test signal will fall within the passband of the system. If the period h of the random number generator is small with respect to the time constant of the

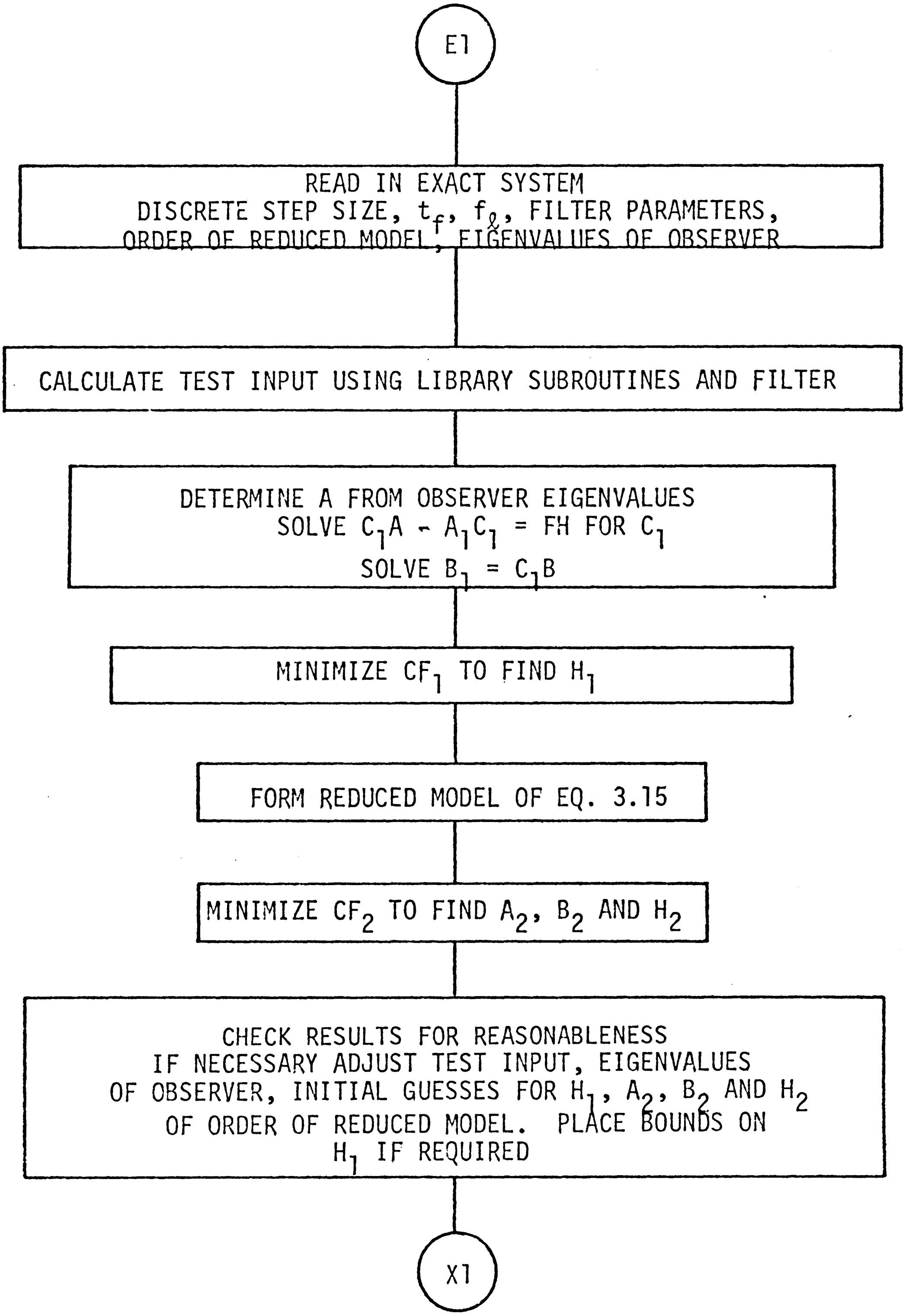


FIGURE 3.1. FLOW CHART FOR COMPUTER PROGRAM USED TO FIND REDUCED MODEL - FIRST MODELING TECHNIQUE

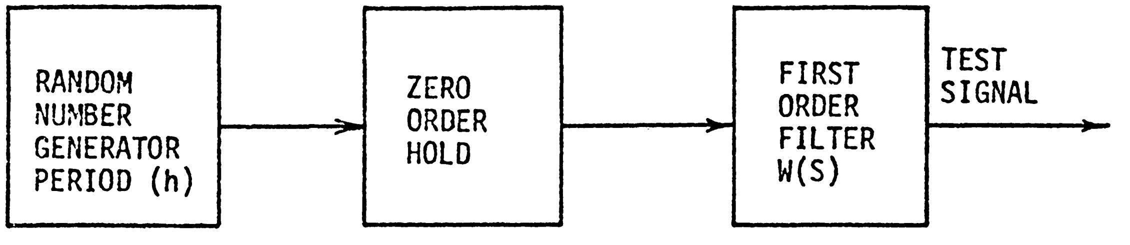


FIGURE 3.2. MODEL FOR TEST SIGNAL GENERATOR

filter, and if the output of the random number generator is Gaussian white noise with variance σ_{rn}^2 then the variance of the test signal is (see [25])

$$\sigma_T^2 = \frac{a}{2}\sigma_{rn}^2 h \; . \qquad 3.30$$

Choosing a as above, σ_{rn}^2 can be adjusted for any given problem to obtain a suitable test signal.

Solution of $C_1A - A_1C_1 = FH$

In Appendix A it is shown that the solution of the equation depends on the solution of

$$c_1P(-a_1I - a_2J - \dots - a_\ell J^{\ell-1} + J^\ell)P^{-1} = FH, \qquad 3.31$$

where J is the Jordan form of A. If A has distinct eigenvalues this reduces to

$$c_1P \begin{bmatrix} (\lambda_1 - \lambda_a)^\ell & & \underline{0}. \\ & (\lambda_2 - \lambda_a)^\ell & \\ \underline{0}. & & \ddots \\ & & & (\lambda_n - \lambda_a)^\ell \end{bmatrix} P^{-1} = f_\ell H, \qquad 3.32$$

where the $\lambda_1 \dots \lambda_n$ are the eigenvalues of A and the eigenvalues of A_1 are all equal to λ_a. Previously it was assumed that λ_a could be chosen much larger than all the eigenvalues of A. If this can be done then the diagonal terms in (3.31) will all be large and not differ greatly in magnitude. Suppose this is not the case and that one or more eigenvalues of A are close to λ_a. In this case the terms associated with these eigenvalues will be small while the other terms are large. If this situation exists the diagonal matrix becomes "almost singular" and the equation can be quite difficult to solve. This problem can arise when there are eigenvalues of the exact system which have large negative real parts. In this case care must be taken to see that the eigenvalues of A_1 are chosen as large as possible and as far away from the eigenvalues of A as possible. This helps assure that a good solution for C_1 is obtained.

Once the eigenvalues of A_1 are chosen, the polynomial $-a_1 I - a_2 A \ldots - a_\ell A^{\ell-1} + A^\ell$ can be evaluated. Solution for c_1 then involves solving an equation of the form $Zc_1^T = Y$. Many different methods for solution of this equation are available in single and double precision as library subroutines. In some cases combinations of techniques might be necessary as well as scaling and partitioning of the coefficient matrix. However, care in choosing the eigenvalues of A_1 greatly simplifies the problem.

The minimization method

Conjugate gradient descent was used to minimize both CF_1 and CF_2 of Figure 3.1. This method was chosen since in general it yields better results in fewer steps than other gradient methods and is fairly simple to implement. The reader is referred to Appendix B for a review of this method.

As with most applications of minimization techniques the user must have some insight into the problem to obtain useable results. Notice that the last block of the flow diagram says check the results for reasonableness and if necessary adjust the initial guess and bounds on H_1. This is necessary since (3.25) does not have a unique minimum and the value of H_1 obtained will depend on the initial guess. Also some values of H_1 may not be useable when substituted into (3.16). In order to make them useable, bounds may need to be placed on the terms of H_1. For instance if it is known that the original system is stable then $A_1 + FH_1$ of (3.16) must have negative eigenvalues to obtain a realistic model. Since A_1 is in phase variable form and $F = (0, \ldots, f_\ell^T)^T$ this can be done by placing bounds on the entries of H_1 so that all roots of the polynomial

$$\rho_1(\lambda) = \lambda^\ell - (a_\ell + [f_\ell H_1]_\ell)\lambda^{\ell-1} - \ldots - (a_2 + [f_\ell H_1]_2)\lambda$$
$$- (a_1 + [f_\ell H_1]_1) \qquad 3.33$$

have negative real parts.

The First Suboptimal Control Law

After obtaining a reduced model using the technique described previously a suboptimal control law can be found. This is done by discretizing the reduced system (3.16), solving equations (2.3) and (2.4) and implementing the control as shown in Figure 2.1. From (2.3) the suboptimal control $\hat{u}_K^*$ will be given by

$$\hat{u}_K^* = -\left[\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}^T \hat{P}_K \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + R\right]^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}^T \hat{P}_K \begin{bmatrix} A_1+FH_1 & fH_2 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} y_1' \\ y_2 \end{bmatrix}$$

Let $t_f \to \infty$ and

$$\hat{u}_K^* = -\begin{pmatrix} \hat{G}_1 & \hat{G}_2 \end{pmatrix} \begin{bmatrix} y_1' \\ y_2 \end{bmatrix} . \qquad 3.34$$

Recall now that $y_1' \simeq C_1x$ and that y_2 was chosen so that $H_2y_2 \simeq e$.

If it is assumed that H_2 and y_2 are scalars then $y_2 = e/H_2$. If this is not the case then $H_2y_2 = e$ must be solved by some appropriate technique, possibly a pseudoinverse, for y_2. Notice in the following argument that e can generally be neglected bypassing the problem. Substituting for y_1' and y_2 in (3.34)

$$\hat{u}_K^* \simeq -\hat{G}_1C_1x - \hat{G}_2e/H_2 . \qquad 3.35$$

Previously it was shown that e could be made arbitrarily small by proper choice of the observer and H_1. With this in mind it was anticipated that the contribution of $\hat{G}_2e/H_2$ would be insignificant when compared to $\hat{G}_1C_1x$. This proved to be a valid assumption. Remember now that an observer whose states are $y_1 = C_1x$ has already been designed. Making use of this observer and neglecting $\hat{G}_2e/H_2$ results in the following control law.

$$\hat{u}_K^* \simeq -\hat{G}_1C_1x . \qquad 3.36$$

This control can now be applied to the unreduced system and should approximately solve the optimal regulator problem.

Critique of the First Method

After developing the first modeling technique and using it to solve the problems presented in Chapter 4 several shortcomings were apparent, notably;

1) The method is not simple or straightforward,
2) The two step procedure used to select the eigenvalues of the reduced model is not satisfactory,
3) The method used to select the eigenvalues of $A_1 + FH_1$ is indirect and difficult to handle since it requires bounds on H_1. This requires that the desired eigenvalues of $A_1 + FH_1$ be known approximately in advance. Further it would be much better if the choice of H_1 was based on its contribution to the reduced model rather than its effect on the observer output error.
4) The auxiliary linear system used to approximate the error e contributes very little to the control law.

Most of these problems could be overcome simply by choosing the entries in H_1, A_2, B_2 and H_2 simultaneously. This could be done by choosing these variables to minimize CF_2 of (3.26). This is unnecessarily complicated, however, and a better, more direct method can be easily formulated.

The Second Reduced Modeling Technique

The second technique is based on the method previously proposed but is much more direct. To develop the method assume that the work up to and including (3.12) is retained (i.e. the design of the reduced observer). Beginning at this point rewrite (3.12)

$$\dot{y}_1 = (A_1 + FH_1)y_1 + B_1u + Fe \ . \qquad 3.37$$

From (3.8) after initial conditions have died out $y_1 = C_1x$. Also recall that

$$e = Hx - H_1y_1 = (H - H_1C_1)x \ . \qquad 3.13$$

In addition (3.23) and (3.24) are valid for e. Consider now the possibility of choosing H_1 so that e has minimal effect on (3.37). This can be done by choosing H_1 so that CF is small when both models are excited by the test input where

$$CF = \int_{t_o}^{t_f} (Hx - H_1y_1')^T Q'(Hx - H_1y_1')dt \qquad 3.38$$

and y_1' is defined by

$$\dot{y}_1' = (A_1 + FH_1)y_1' + B_1u \ . \qquad 3.39$$

From consideration of the error terms given by (3.23) and (3.24) it is obvious that e cannot be chosen to have an insignificant effect on (3.37) unless it is at least bounded. Therefore, the conclusion can be drawn that $A_1 + FH_1$ must retain the unstable eigenvalues of A. This can be done quite handily in the following manner.

From (3.19)

$$A_1 + FH_1 = \begin{bmatrix} 0 & 1 & & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \\ d_1 & d_2 & & \cdots & d_\ell \end{bmatrix} \qquad 3.19$$

where

$$d_i = a_i + [f_\ell H_1]_i \ .$$

If $A_1 + FH_1$ is to retain eigenvalues of A then these eigenvalues must be roots of the polynomial

$$\rho_1(\lambda) = \lambda^\ell - d_\ell\lambda^{\ell-1} - \ldots - d_2\lambda - d_1 \ . \qquad 3.40$$

Suppose now that $A_1 + FH_1$ is to retain j eigenvalues of A, $\lambda_1\ \lambda_2 \ldots \lambda_j$ Then $\rho_1(\lambda)$ must be given by

$$\rho_1(\lambda) = (\lambda^j + b_j\lambda^{j-1} + \ldots + b_2\lambda + b_1)(\lambda^p + d_p'\lambda^{p-1} + \ldots + d_2'\lambda + d_1') \qquad 3.41$$

where

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \ . \ . \ . \ (\lambda - \lambda_j) = \lambda^j + b_j\lambda^{j-1} + \ldots + b_2\lambda + b_1$$

and

$$j + p = \ell \ .$$

Equating (3.40) and (3.41) the following relationships are obtained.

$$-d_\ell = d'_p + b_j$$

$$-d_{\ell-1} = d'_{p-1} + d'_p b_j + b_{j-1}$$

$$-d_{\ell-2} = d'_{p-2} + d'_{p-1} b_j + d'_p b_{j-1} + b_{j-2} \qquad 3.42$$

$$\vdots$$

For the special case of a single output system f_ℓ is a scalar, then from (3.19) and (3.42)

$$h_\ell = -(d'_p + b_j + a_\ell)/f_\ell$$

$$h_{\ell-1} = -(d'_{p-1} + d'_p b_j + b_{j-1} + a_{\ell-1})/f_\ell \qquad 3.43$$

$$\vdots$$

where

$H_1 = (h_1 \;\; h_2 \;\; \ldots \; h_\ell)$ a vector.

The reduced model can now be determined by writing the entries $d_1 \ldots d_\ell$ and $h_1 \ldots h_\ell$ in terms of the $d'_1 \ldots d'_p$ and then substituting these values into equation (3.39). The cost function of (3.38) is then minimized by varying the d'_i.

Suppose now that a two output system is to be modeled. Let

$$H_1 = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1\ell} \\ h_{21} & h_{22} & \cdots & h_{2\ell} \end{bmatrix} \qquad 3.44$$

and

$$f_\ell = (f_{\ell 1} \;\; f_{\ell 2}) \; .$$

Again using (3.19) and (3.42)

$$h_{2\ell} = -\ (d_p' + b_j + a_\ell + f_{\ell 1}h_{1\ell})/f_{\ell 2}$$

$$h_{2\ell-1} = -(d_{p-1}' + d_p'b_j + b_{j-1} + a_{\ell-1} + f_{\ell 1}h_{1\ell-1})/f_{\ell 2}$$

$$\vdots \qquad 3.45$$

Using (3.42) and (3.45) the reduced model of (3.39) can be written in terms of the d_i' and the first row of H_1. The cost function of (3.38) is then minimized by varying the d_i' and the entries in the first row of H_1.

In a similar manner the relations for a system with more outputs can be derived. As is obvious the number of free variables increases by ℓ each time an additional row is added to the output matrix and although the method for determining the reduced model remains the same, the numerical complexity increases rapidly.

Minimizing CF in effect specifies H_1 so that

$$\hat{z} = H_1 y_1' \simeq z \ .$$

To show that $y_1' \simeq C_1 x$ rewrite (3.37) using (3.13).

$$CF = \int_{t_o}^{t_f} (H_1C_1x + e - H_1y_1')^T Q'(H_1C_1x + e - H_1y_1')dt \ . \qquad 3.46$$

From equations (3.23) and (3.24) e can be made quite small by choosing the eigenvalues of A_1 large negative. In practice $e_i < .05[Hx]_i$ is not unreasonable. Certainly the assumption that e is negligible when compared with H_1C_1x is then feasible. Neglecting e in (3.46) note that minimizing CF approximately minimizes

$$CF' = \int_{t_o}^{t_f} (C_1x - y_1')^T H^T Q' H (C_1x - y_1') dt \, . \qquad 3.47$$

Summary of the method

This completes the development of the second reduced modeling technique. For clarity the steps will be reviewed.

1) Choose the eigenvalues of A_1 as large in magnitude as feasible and form the matrix

$$A_1 = \begin{bmatrix} 0 & 1 & & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ a_1 & a_2 & & \cdots & a_\ell \end{bmatrix} . \qquad 3.9$$

2) Select f_ℓ and solve $C_1A - A_1C_1 = FH$ for C_1. Check to see that C_1 has rank ℓ.

3) Solve for $B_1 = C_1B$.

4) Form the reduced model

$$\dot{y}_1' = \begin{bmatrix} 0 & 1 & & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ d_1 & d_2 & & \cdots & d_\ell \end{bmatrix} y_1' + B_1u \qquad 3.48$$

$$z = H_1y_1'$$

where $d_i = a_i + [f_\ell H_1]_i$.

5) Specify relationships between the entries of H_1 so that $A_1 + FH_1$ retains the unstable eigenvalues of A. Excite the exact and reduced models with the test input and minimize

$$CF = \int_{t_0}^{t_f} (H_x - H_1 y_1')^T Q' (H_x - H_1 y_1') dt \qquad 3.49$$

by varying H_1.

Computer Implementation of the Second Reduced Modeling Method

The second method described was implemented on the digital computer and used to solve example 2 of Chapter 4. Figure 3.3 is a flow chart for the program developed. The problems encountered were almost identical to those encountered with the first method. The numerical complexity was greatly reduced, however, due primarily to the fact that only one minimization was required.

The Second Suboptimal Control Law

After obtaining the reduced model a suboptimal control law can be found. As in the previous development this is done by discretizing the reduced model of (3.48), solving equations (2.3) and (2.4) and implementing the control as shown in Figure 2.1. From (2.3) the suboptimal control $\hat{u}_K^*$ will be given by

$$\hat{u}_K^* = -[B_1^T \hat{P}_K B_1 + R]^{-1} B_1^T \hat{P}_K [A_1 + FH_1] y_1' \; . \qquad 3.50$$

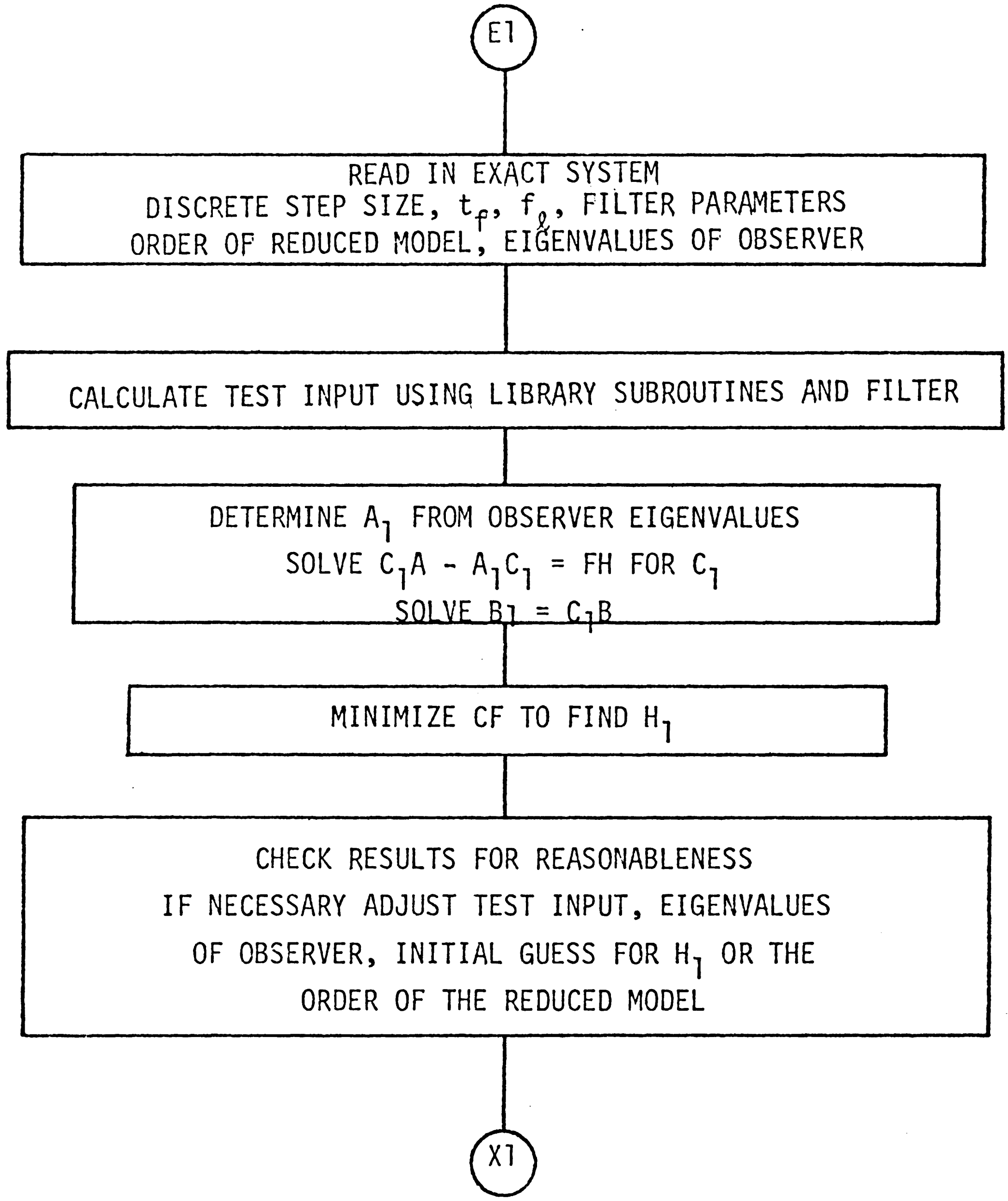


FIGURE 3.3. FLOW CHART FOR COMPUTER PROGRAM USED TO FIND REDUCED MODEL - SECOND MODELING TECHNIQUE

Let $t_f \to \infty$ and

$$\hat{u}_K^* \simeq -\hat{G}_1 y_1' \; . \qquad 3.51$$

As before an observer whose states are $y_1 = C_1 x$ has already been designed. Since $y_1' \simeq y_1$, the suboptimal control is

$$\hat{u}_K^* \simeq - \hat{G}_1 C_1 x \; . \qquad 3.52$$

This control can now be applied to the unreduced system and should approximately solve the optimal regulator problem.

Critique of the Second Method

There are several noteworthy advantages of the second method, namely:

1) The method is as simple as can be reasonably expected,

2) There is only one minimization required and it is straightforward,

3) The free variables of the reduced model are selected so that the model is optimal in accordance with the cost criteria of equation (3.2),

4) All the states contribute to the resulting control law,

5) Specific modes of the exact system may be easily retained in the reduced model.

Summary

This chapter has presented the mathematical foundation for two suboptimal control schemes. They are based on the idea that a suitable reduced model can be found and that control based on this model will provide reasonable results. Such methods can be judged only by applying them to several examples and demonstrating their utility. In Chapter IV it will be demonstrated that these particular techniques are feasible.

CHAPTER IV

Introduction

In this chapter the value of the suboptimal control scheme proposed will be demonstrated by its application to two example problems. Particular problems encountered in solving the examples will be discussed to help clarify points previously mentioned.

Example 1

The first example was taken directly from reference [7]. The system model is

$$\dot{x} = \begin{bmatrix} -1. & & & & & & & \\ & -2. & & & & \underline{0} & & \\ & & -3. & & & & & \\ & & & -4. & & & & \\ & & & & -5. & & & \\ & \underline{0} & & & & -6. & & \\ & & & & & & -7. & \\ & & & & & & & -8. \end{bmatrix} x + \begin{bmatrix} 1. \\ 1. \\ 1. \\ 1. \\ 1. \\ 1. \\ 1. \\ 1. \end{bmatrix} u \qquad 4.1$$

$$z = (-3. \quad 4. \quad -6. \quad 8. \quad 5. \quad -12. \quad 14. \quad 8.)x \ .$$

For this problem two third order reduced models were found using the first modeling technique. The first model was found using a step as the test function. The second model used a filtered white noise source as the test input. In both cases f_ℓ = 163. and the observer eigenvalues were both chosen equal to -20.

The first step is the design of a reduced observer. Using -20. for the observer eigenvalues

$$A_1 = \begin{bmatrix} 0 & 1 \\ -400. & -40. \end{bmatrix} . \qquad 4.2$$

Using the H given in the problem statement and f_ℓ = 163. the equation $C_1A - A_1C_1 = FH$ can be solved for C_1.

$$C_1 = \begin{bmatrix} -1.35 & 2.01 & -3.38 & 5.09 & 3.62 & -9.98 & 13.50 & 9.06 \\ 1.35 & -4.02 & 10.15 & -20.37 & -18.11 & 54.88 & -94.52 & -72.44 \end{bmatrix} \qquad 4.3$$

With the above solution for C_1 and B of the original model $B_1 = C_1B$ can be determined. The observer is then given by

$$\dot{y}_1 = \begin{bmatrix} 0 & 1. \\ -400. & -40. \end{bmatrix} y_1 + \begin{bmatrix} 18.57 \\ -138.09 \end{bmatrix} u \qquad 4.4$$

The next step is to find H_1. This is done by exciting the exact model with a test input and using the model output to drive the observer. If Q' = 1, H_1 is found by minimizing

$$CF_1 = \int_{t_o}^{t_f} (z - H_1y_1)^2 dt . \qquad 4.5$$

Using a step input this procedure yielded

$$H_1 = (2.43 \quad .2) .$$

Using H_1 produced by minimizing CF_1

$$A_1 + FH_1 = \begin{bmatrix} 0. & 1. \\ -4.51 & -7.9 \end{bmatrix} . \qquad 4.6$$

The following reduced model can now be formed.

$$\begin{bmatrix} \dot{y}_1' \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0. & 1. & 0. \\ -4.51 & -7.9 & 163. \\ 0. & 0. & A_2 \end{bmatrix} \begin{bmatrix} y_1' \\ y_2 \end{bmatrix} + \begin{bmatrix} 18.57 \\ -138.09 \\ B_2 \end{bmatrix} u$$

$$\hat{z} = (2.43 \quad .20 \quad H_2) \begin{bmatrix} y_1' \\ y_2 \end{bmatrix} \qquad 4.7$$

Where A_2, B_2 and H_2 are scalars since a first order auxiliary system will be used to approximate the error between observer output and the actual system output.

Using equation (3.27) to force the steady state value of the reduced and exact models to be equal A_2, B_2 and H_2 were found by exciting the models of (4.1) and (4.6) with the test signal, in this case a step, and minimizing

$$CF_2 = \int_{t_o}^{t_f} (z - \hat{z})^2 dt . \qquad 4.8$$

This resulted in the following reduced model.

$$\dot{y} = \begin{bmatrix} 0. & 1. & 0. \\ -4.51 & -7.90 & 163. \\ 0. & 0. & -.75 \end{bmatrix} \underset{\sim}{y} + \begin{bmatrix} 18.57 \\ -138.09 \\ .0003 \end{bmatrix} u \qquad 4.9$$

$$\hat{z} = [2.43 \quad .20 \quad .98]y \ .$$

If $C_2 = (H - H_1C_1/H_2)$ then

$$\begin{bmatrix} C_1 \\ --- \\ C_2 \end{bmatrix} = \left[\begin{array}{cccccccc} -1.35 & 2.01 & -3.38 & 5.09 & 3.62 & -9.98 & 13.50 & 9.06 \\ 1.35 & -4.02 & 10.15 & -20.37 & -18.11 & 59.88 & -94.52 & -72.44 \\ \hline .02 & -.09 & .22 & -.35 & -.23 & .43 & -.15 & .30 \end{array}\right] \quad 4.10$$

A comparison of step responses is shown in Figure 4.1.

Using a step size of .1 and an input weighting constant R = .1, the two systems were discretized and the optimal and suboptimal feedback gains calculated. The optimal feedback gains were

$$G^* = [-2.13 \quad 2.57 \quad -3.49 \quad 4.21 \quad 2.38 \quad -5.38 \quad 5.45 \quad 2.82] \qquad 4.11$$

The feedback gains calculated for the reduced model were

$$\hat{G} = (1.86 \quad .20 \quad 3.57) \ . \qquad 4.12$$

From equation (3.30) the suboptimal control u* is given by $-\hat{G}_1 y = -\hat{G}_1 C_1 x$ where

$$\hat{G}_1 C_1 = [-2.19 \quad 2.69 \quad -3.67 \quad 4.42 \quad 2.47 \quad -5.30 \quad 5.55 \quad 2.93] \ . \qquad 4.13$$

Figure 4.2 shows a typical response of the system for both the optimal and suboptimal regulators. Similar results were obtained for all states.

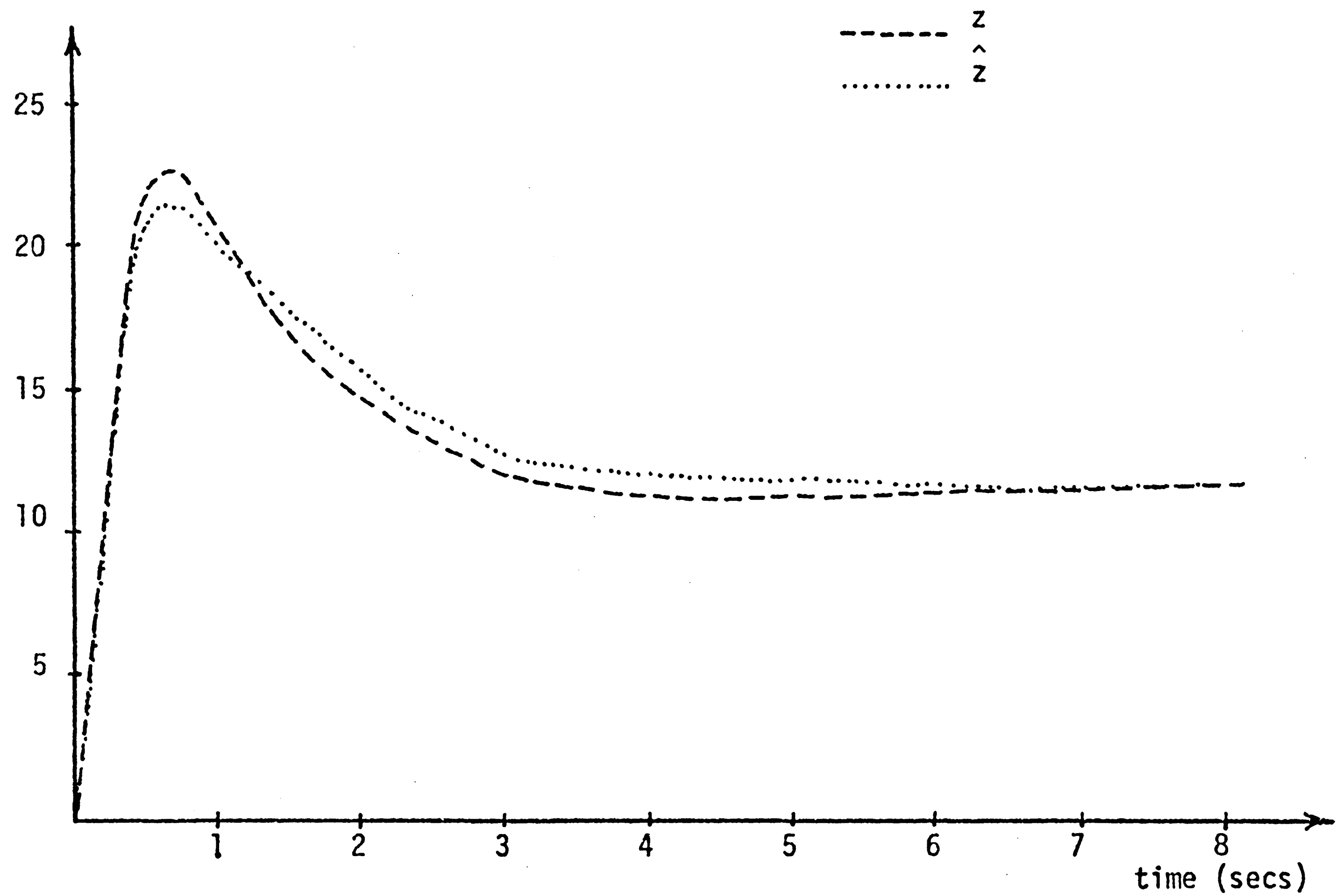


FIGURE 4.1. STEP RESPONSE OF EXACT AND APPROXIMATE MODELS FOR EXAMPLE ONE

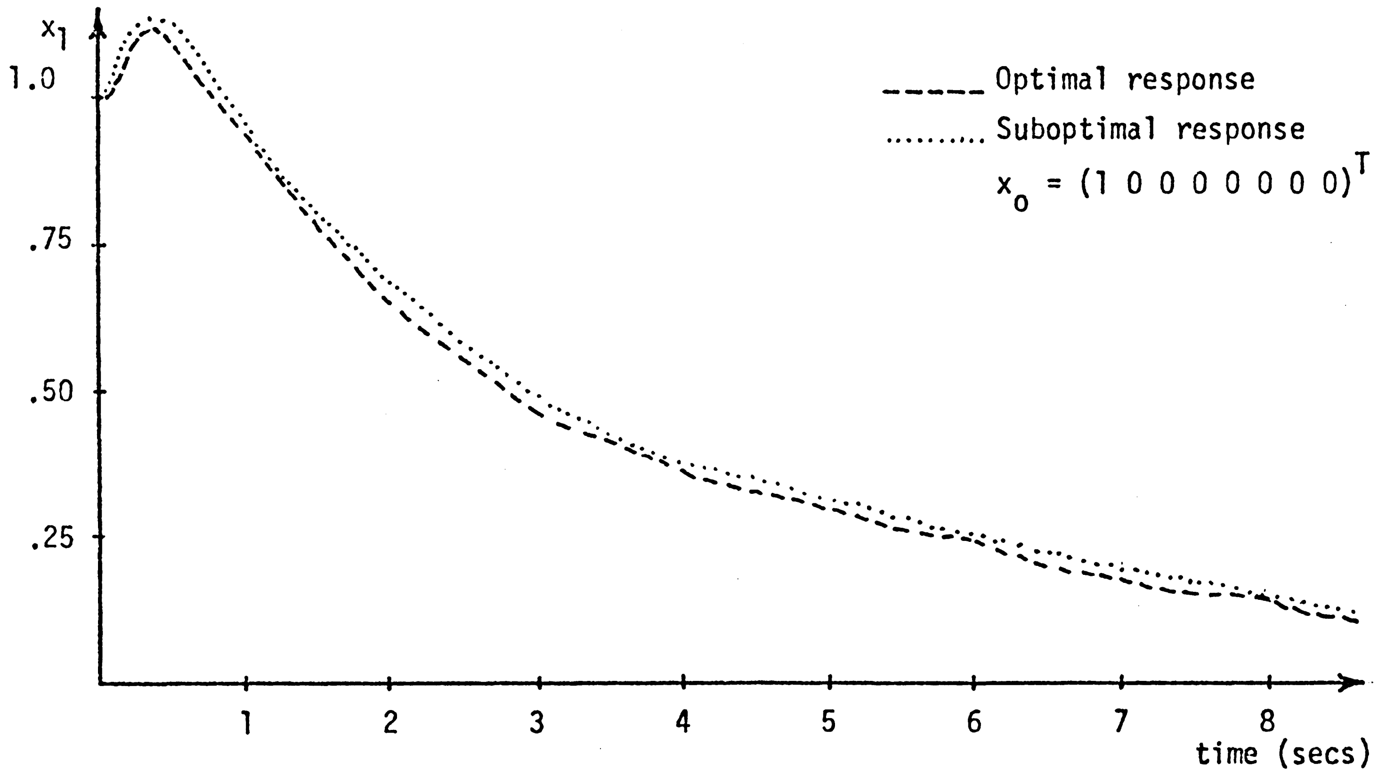


FIGURE 4.2. RESPONSE OF x_1 FOR OPTIMAL AND SUBOPTIMAL CONTROLS FOR EXAMPLE ONE

To obtain an approximate model using a stochastic test function the same procedure is used up to and including (4.4). The exact model is then driven with the test signal and CF_1 of (4.5) minimized by varying H_1. For the test signal used the variance of the random number generator was 1. and the filter was chosen to have a pole at -.05. Using this signal

$$H_1 = (2.40 \quad .19) \ . \tag{4.14}$$

A second reduced model can now be formed.

$$\begin{bmatrix} \dot{y}_1' \\ \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -8.29 & -8.4 & 163. \\ 0 & 0 & A_2 \end{bmatrix} \begin{bmatrix} y_1' \\ \\ y_2 \end{bmatrix} + \begin{bmatrix} 18.57 \\ -138.09 \\ B_2 \end{bmatrix} u$$

$$\hat{z} = (2.40 \quad .19 \quad H_2) \begin{bmatrix} y_1' \\ y_2 \end{bmatrix} \tag{4.15}$$

where

A_2, B_2 and H_2 are scalars since a first order auxiliary system will be found.

Using equation (3.27) to force the steady state value of the two models (4.15) and (4.1) to be equal (4.8) was minimized to find A_2, B_2 and H_2. Using the noisy test signal the reduced model was found to be

$$\dot{y} = \begin{bmatrix} 0. & 1. & 0. \\ -8.29 & -8.40 & 163. \\ 0. & 0. & -.81 \end{bmatrix} y + \begin{bmatrix} 18.57 \\ -138.09 \\ .01 \end{bmatrix} u$$

$$\hat{z} = (2.40 \quad .19 \quad .85)\ y \tag{4.16}$$

where

$$\begin{bmatrix} C_1 \\ \hline C_2 \end{bmatrix} = \left[\begin{array}{cccccccc} -1.35 & 2.01 & -3.38 & 5.09 & 3.62 & -9.98 & 13.50 & 9.06 \\ 1.35 & -4.02 & 10.15 & -20.37 & -18.11 & 59.88 & -94.52 & -72.44 \\ \hline -.101 & -.07 & .19 & -.34 & -.23 & .44 & -.15 & .33 \end{array}\right] \qquad 4.17$$

A comparison of the exact and approximate outputs when both models are excited by the test input is shown in Figures 4.3a and 4.3b. Note here that the model of (4.16) was found by minimizing the cost functions based on the response shown in Figure 4.3a. If $t_f - t_o$ is large, however, (several times longer than the longest time constants of the system to be modeled) it is expected that the approximate model should also follow the exact model output for other inputs from the signal generator. This is verified in Figure 4.3b.

Using a step size of .1 and input weighting constant R = .1 this system was also discretized and the feedback gains calculated. The feedback gains for the reduced model were

$$\hat{G} = (1.77 \quad .1914 \quad 3.41) \qquad 4.18$$

and

$$\hat{G}_1 C_1 = (-2.17 \quad 2.63 \quad -3.57 \quad 4.26 \quad 2.38 \quad -5.12 \quad 5.49 \quad 3.05)\ . \qquad 4.19$$

It can be seen that the feedback gains obtained for this reduced model are very close to those obtained for the first model. In addition, $\hat{G}_1 C_1$ is again almost equal to G*. Figure 4.4 compares typical responses of the system when it is disturbed and the optimal and suboptimal regulators are used to return it to an equilibrium point.

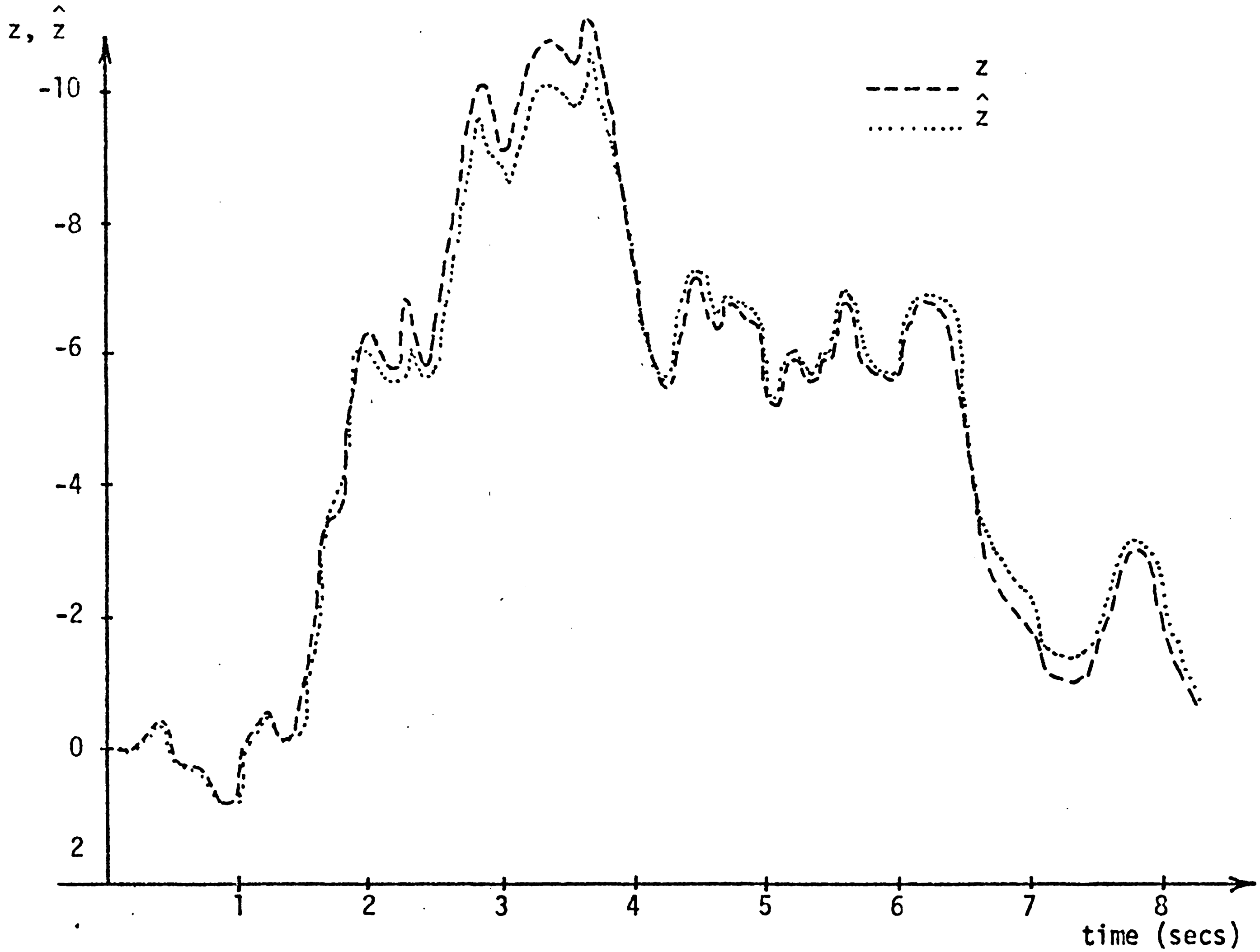


FIGURE 4.3a. RESPONSE OF EXACT AND REDUCED MODELS TO NOISE FOR EXAMPLE ONE

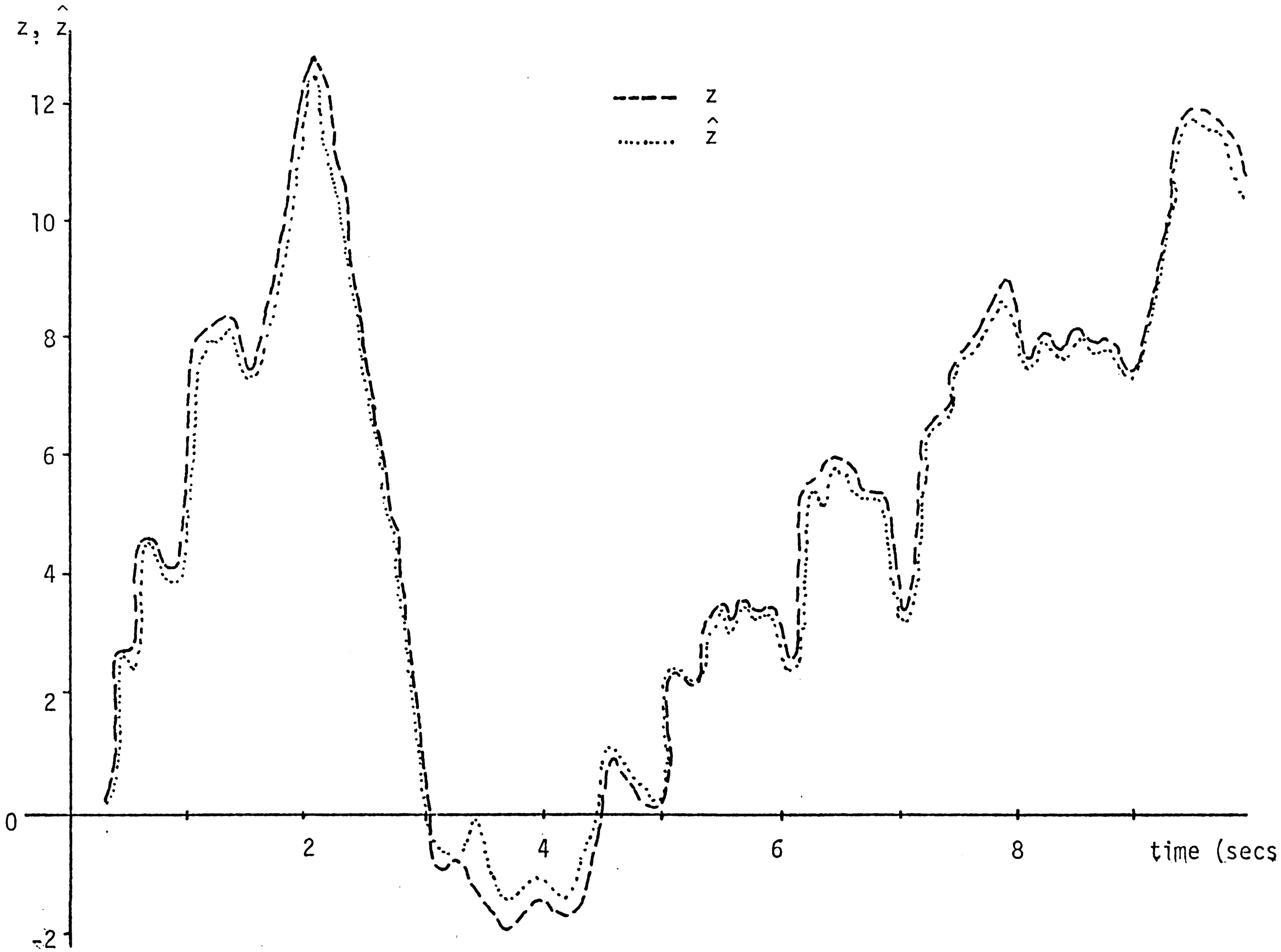


FIGURE 4.3b. RESPONSE OF EXACT AND REDUCED MODELS TO NOISE FOR EXAMPLE ONE

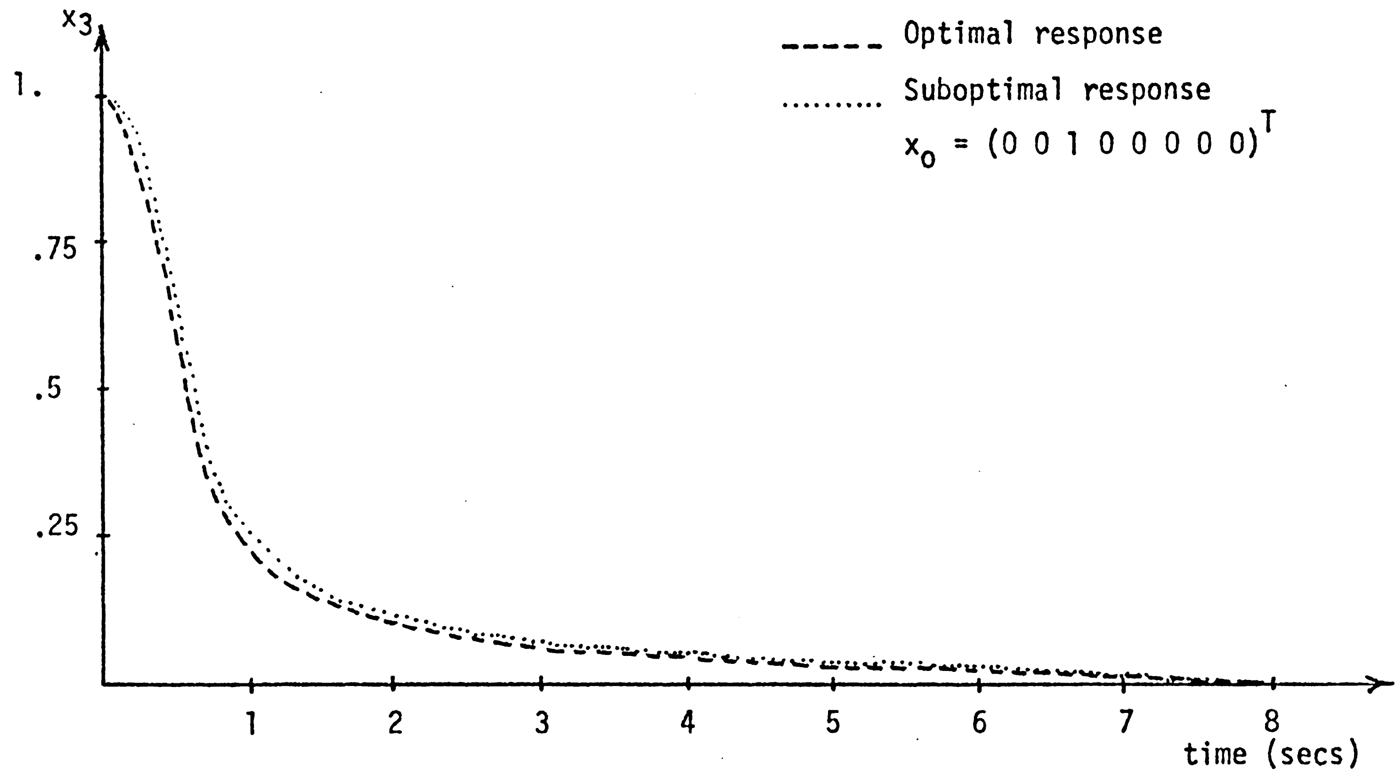


FIGURE 4.4. RESPONSE OF STATE x_3 FOR OPTIMAL AND SUBOPTIMAL CONTROLS FOR EXAMPLE ONE

From observation of the response curves of Figures 4.2 and 4.4 it can be seen that there is very little difference in performance. There is, however, considerable difference in the amount of equipment needed and the time required to compute the feedback gains and multiply them by the appropriate states. Also, there can be a marked difference in the order of the observers required. Figure 4.5 is included to point out the advantages of the suboptimal scheme.

Example 2

The second problem solved was the design of a regulator for a model of the X-14A VTOL aircraft. The linearized dynamic equations for the aircraft, as given in reference [28], are

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -.447 & 0 & .0436 & 0 & -.0133 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -.155 & 0 & -.86 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -.179 & 0 & .653 & 0 & -.20 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 \\ 4.3 & 0 & .076 \\ 0 & 0 & 0 \\ 0 & 2.35 & 0 \\ 0 & 0 & 0 \\ .172 & 0 & 1.14 \end{bmatrix} u \qquad 4.20$$

where x_1, x_3 and x_5 represent displacement in roll, pitch and yaw respectively and x_2, x_4 and x_6 represent the derivatives of these displacements. The scalar output was chosen to weight all states equally giving

$$z = (1. \quad 1. \quad 1. \quad 1. \quad 1. \quad 1.)x \ . \qquad 4.21$$

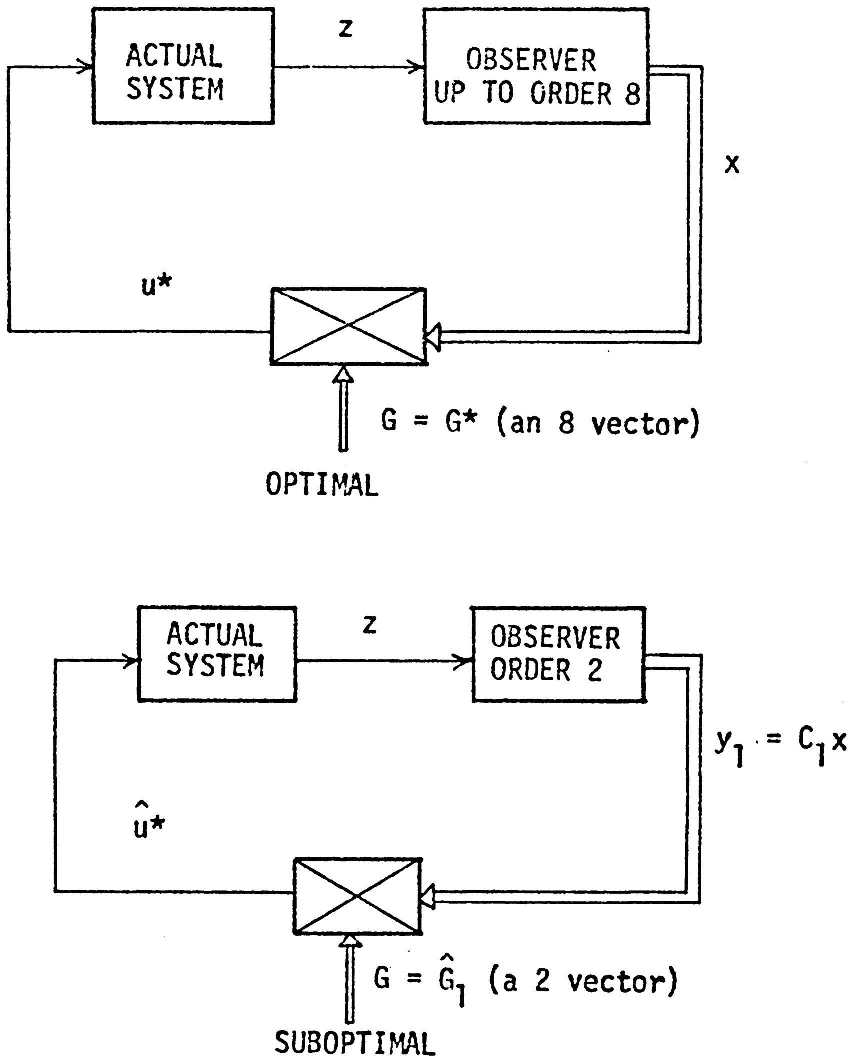


FIGURE 4.5. IMPLEMENTATION OF OPTIMAL AND SUBOPTIMAL CONTROL FOR EXAMPLE 1

From study of the aircraft model it can be seen that for $u = 0$ an equilibrium point will be reached when $x_2 = x_4 = x_6 = 0$. z will then be zero when $x_1 + x_3 + x_5 = 0$. From this reasoning it is expected that the optimal regulator will act as a damper by forcing the derivatives of all states to return to zero should they be disturbed.

Two third order reduced models were found for this system. In both cases $f_\ell = 163.$ and the observer eigenvalues were all equal to -20. A stochastic test function was used with the random number generator variance equal to .5 and the filter eigenvalue at -.05.

Using the first modeling technique, a second order observer is required. Since the observer eigenvalues are -20.,

$$A_1 = \begin{bmatrix} 0 & 1 \\ -400. & -20. \end{bmatrix} . \qquad 4.22$$

Using (4.21) and $f_\ell = 163.$, $C_1A - A_1C_1 = FH$ can be solved for C_1.

$$C_1 = \begin{bmatrix} .41 & .39 & .41 & .34 & .41 & .41 \\ .0 & .16 & .0 & .64 & .0 & .03 \end{bmatrix} . \qquad 4.23$$

The observer is then given by

$$\dot{y}_1 = \begin{bmatrix} 0 & 1 \\ -400. & -40. \end{bmatrix} y_1 + \begin{bmatrix} 1.75 & .81 & .49 \\ .69 & 1.50 & .04 \end{bmatrix} u \qquad 4.24$$

H_1 was found by driving the model of (4.20) with the test input and using z to drive the observer. The values of H_1 were found by minimizing

$$CF_1 = \int_{t_o}^{t_f} (z - H_1 y_1)^2 dt \, . \qquad 4.25$$

This resulted in

$$H_1 = (2.45 \quad -.44) \, . \qquad 4.26$$

The reduced model can now be formed.

$$\begin{bmatrix} \dot{y}_1' \\ \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0. & 1 & 0 \\ 0. & -112.45 & 163 \\ 0 & 0 & A_2 \end{bmatrix} \begin{bmatrix} y_1' \\ \\ y_2 \end{bmatrix} + \begin{bmatrix} 1.75 & .81 & .49 \\ .69 & 1.50 & .04 \\ & \underline{B_2} & \end{bmatrix}$$

$$\hat{z} = (2.45 \quad -.44 \quad H_2) \begin{bmatrix} y_1' \\ y_2 \end{bmatrix} \qquad 4.27$$

where A_2 and H_2 are scalars and B_2 is a vector of order three since a first order auxiliary system will be used to approximate the error between observer output and the actual system output.

Using equations (3.28a-b) A_2, B_2 and H_2 were found by exciting the models of (4.27) and (4.20) with the test input and minimizing

$$CF_2 = \int_{t_o}^{t_f} (z - \hat{z})^2 dt \, . \qquad 4.28$$

This procedure resulted in the following reduced model.

$$\dot{y} = \begin{bmatrix} 0. & 1. & 0. \\ 0. & -112.45 & 163. \\ 0. & 0. & -1.9 \end{bmatrix} y + \begin{bmatrix} 1.75 & .81 & .49 \\ .69 & 1.50 & .04 \\ .35 & .75 & .02 \end{bmatrix} u$$

$$\hat{z} = [2.45 \quad -.44 \quad 1.37] y \qquad 4.29$$

$$\left[\begin{array}{c} C_1 \\ \\ \hline C_2 \end{array}\right] = \left[\begin{array}{cccccc} .41 & .39 & .41 & .34 & .41 & .41 \\ 0. & .16 & 0. & .64 & 0. & .03 \\ \hline 0. & .08 & 0. & .32 & 0. & .01 \end{array}\right] \quad 4.30$$

It should be noted that the zero eigenvalue in the reduced model was forced by specifying that $-400. + 163.*h_{11} = 0$. This was done since for this particular system it is apparent that the response is primarily determined by the three zero eigenvalues of the A matrix. From the previous chapter it could be predicted that this would be necessary. Although several runs were made in an effort to avoid this it became obvious that the integrator was critical. Once this requirement was enforced the above reduced model was obtained. Comparison of the response of the exact model and this reduced model is shown in Figure 4.6.

The two systems were discretized and the feedback gains calculated for R = diag (.1 .1 .1). The optimal feedback gains were

$$G^* = \begin{bmatrix} 1.54 & 1.60 & 1.54 & 1.76 & 1.54 & 1.51 \\ .84 & .88 & .84 & 1.03 & .84 & .81 \\ .42 & .42 & .42 & .46 & .42 & .47 \end{bmatrix} \quad 4.31$$

For the reduced model the feedback gains were

$$\hat{G} = \begin{bmatrix} 3.72 & .03 & 1.34 \\ 2.10 & .02 & .78 \\ .99 & .01 & .36 \end{bmatrix} \quad 4.32$$

and

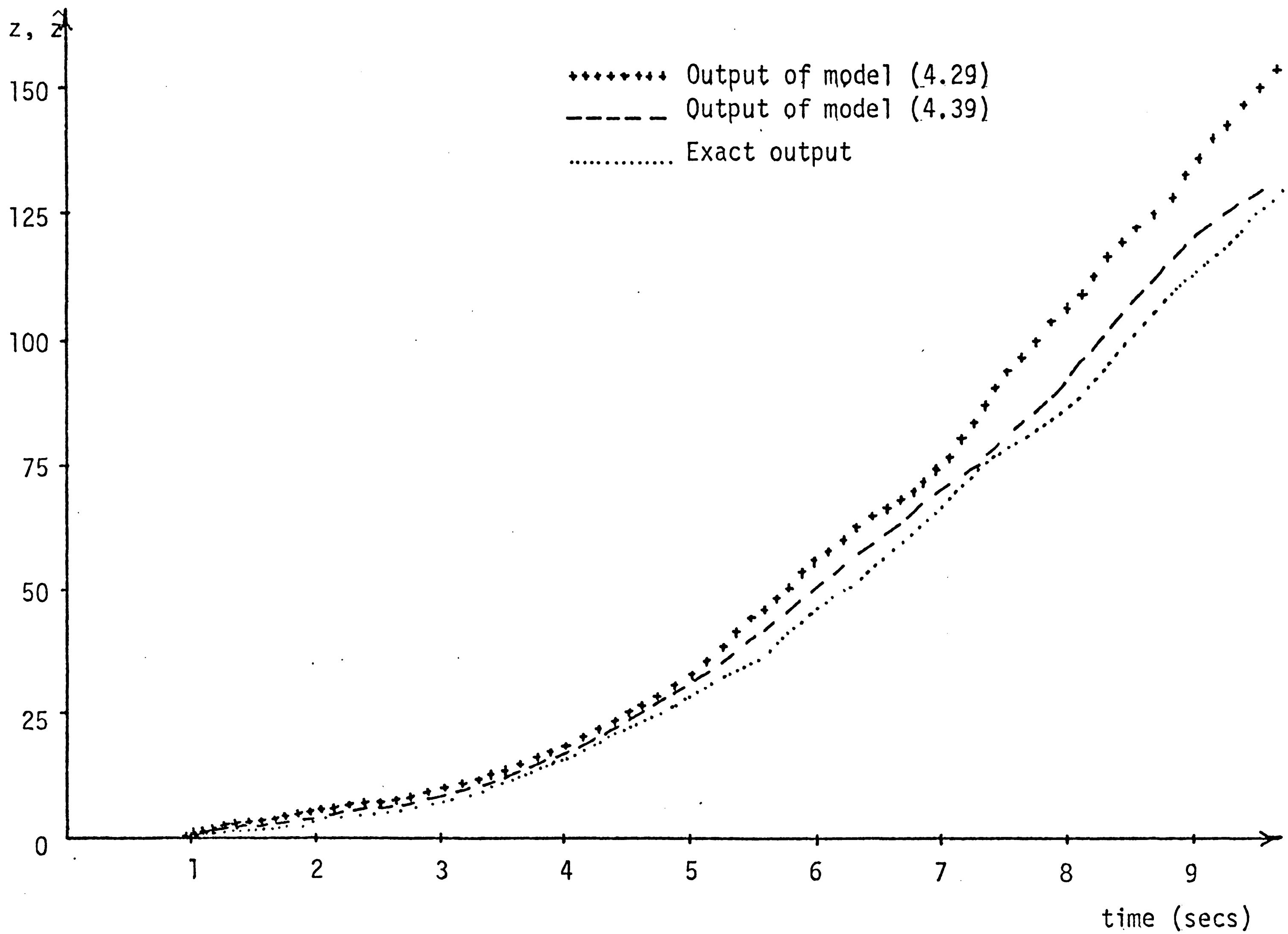


FIGURE 4.6. RESPONSE OF EXACT AND REDUCED MODELS TO TEST INPUT FOR EXAMPLE TWO

$$\hat{G}_1 C_1 = \begin{bmatrix} 1.52 & 1.46 & 1.52 & 1.30 & 1.52 & 1.52 \\ .86 & .83 & .86 & .73 & .86 & .85 \\ .41 & .39 & .41 & .35 & .41 & .41 \end{bmatrix} \qquad 4.33$$

As in Example 1, $G^* \simeq \hat{G}_1 C_1$. Also, the optimal and suboptimal responses were almost identical. Figure 4.7 compares the response of x_2 when the system was disturbed and the optimal gains of (4.31) and suboptimal gains of (4.33) were used to return the states to equilibrium.

The second modeling technique was used to determine another third order model for this system. As in the first technique a reduced observer must be designed. If all observer eigenvalues are equal to -20. for a third order observer

$$A_1 = \begin{bmatrix} 0 & 1. & 0 \\ 0 & 0 & 1. \\ -8000. & -1200. & -60. \end{bmatrix} . \qquad 4.34$$

Using H given by (4.21) and $f_\ell = 163.$ the equation $C_1 A - A_1 C_1 = FH$ can be solved for C_1.

$$C_1 = \begin{bmatrix} .023 & .018 & .022 & .015 & .023 & .020 \\ .0 & .011 & .0 & .034 & .0 & .006 \\ .0 & -.062 & .0 & .0 & .0 & -.030 \end{bmatrix} \qquad 4.35$$

Using this solution for C_1 and B of the original model $B_1 = C_1 B$ can be determined. The reduced observer is then

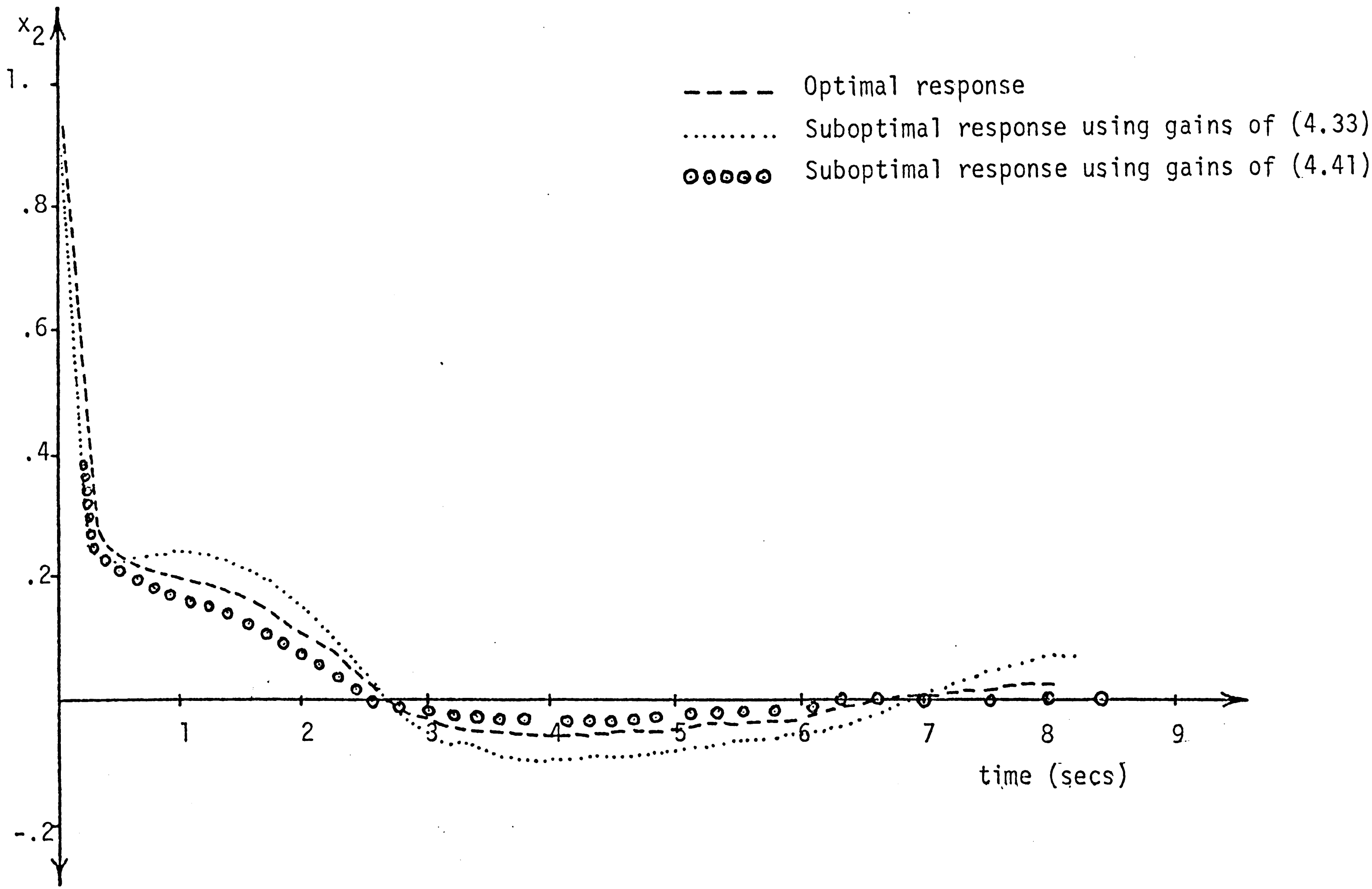


FIGURE 4.7. RESPONSE OF STATE x_2 FOR OPTIMAL AND SUBOPTIMAL CONTROLS FOR EXAMPLE TWO

$$\dot{y}_1 = \begin{bmatrix} 0 & 1. & 0 \\ 0 & 0 & 1. \\ -8000. & -1200. & -60. \end{bmatrix} y_1 + \begin{bmatrix} .082 & .035 & .024 \\ .050 & .080 & .008 \\ -.032 & -.014 & -.035 \end{bmatrix} u \qquad 4.36$$

The next step in this technique is to form the reduced model

$$\dot{y}_1' = \begin{bmatrix} 0 & 1. & 0 \\ 0 & 0 & 1. \\ -8000.+f_\ell h_{11} & -1200+f_\ell h_{12} & -60+f_\ell h_{13} \end{bmatrix} y_1'$$

$$+ \begin{bmatrix} .082 & .035 & .024 \\ .050 & .080 & .008 \\ -.032 & -.014 & -.035 \end{bmatrix} u$$

$$\hat{z} = (h_{11} \quad h_{12} \quad h_{13})\, y_1' \, . \qquad 4.37$$

The values for H_1 were determined by exciting both models with the test input and minimizing

$$CF = \int_{t_o}^{t_f} (z - \hat{z})^2 dt \, . \qquad 4.38$$

This resulted in the reduced model

$$\dot{y}_1' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0. & -2.82 & -.99 \end{bmatrix} y_1' + \begin{bmatrix} .082 & .035 & .024 \\ .050 & .080 & .008 \\ -.032 & -.014 & -.035 \end{bmatrix} u$$

$$\hat{z} = (49.08 \quad 7.35 \quad .36) y_1' \qquad 4.39$$

A comparison of the response of the exact model and this reduced model is shown in Figure 4.6. The response of this model is denoted by $\hat{z}_2$.

This model was discretized and the feedback gain calculated for $R = \text{diag}\ (.1 \quad .1 \quad .1)$. The suboptimal gains were computed to be

$$\hat{G} = \begin{bmatrix} 80.19 & 15.45 & 3.32 \\ 36.00 & 17.11 & 0.09 \\ 20.71 & 4.75 & -1.13 \end{bmatrix} \qquad 4.40$$

$$\hat{G}C_1 = \begin{bmatrix} 1.85 & 1.62 & 1.80 & 1.71 & 1.87 & 1.57 \\ .83 & .85 & .81 & 1.11 & .84 & .81 \\ .48 & .44 & .46 & .47 & .48 & .47 \end{bmatrix} \qquad 4.41$$

Comparison of these gains with the optimal gains of (4.31) shows that they are quite close. Figure 4.7 compares the optimal response of x_2 with the response which resulted from using the gains of (4.41).

The Choice of f_ℓ

In Chapter 3 the choice of f_ℓ was discussed in connection with the rank of C_1. Once $f_\ell H$ has been chosen so that C_1 has full rank the problem of choosing the relative magnitude of f_ℓ arises. A quick glance at (4.37) shows that the magnitude of f_ℓ effects the magnitude of the entries in H_1. Recalling the procedure for finding B_1 of (4.37), the magnitude of f_ℓ also effects the magnitude of the entries in B_1. As a specific example suppose that in Example 2 f_ℓ had been chosen equal to 326. instead of 163. and the reduced model of (4.37) had been determined in the same manner except for this change. The only difference would be that H_1 would be halved and B_1 doubled. The response of the two models would be identical. It follows then that by specifying the relative magnitude of f_ℓ, some control can be exerted over the magnitude of the entries in B_1 and H_1 of the reduced model.

Summary

In this chapter the reduced modeling techniques of Chapter 3 were applied to two example problems and shown to produce adequate models. The suboptimal feedback gains which resulted from using these models to solve the linear regulator problem were near the optimal for all the examples. It should be noted that the second technique was much easier to implement than the first and appears superior, although the results were satisfying in both cases.

CHAPTER V

Summary

This research has presented an approach to the design of linear regulators for high order constant coefficient systems. The method proposed is based on replacing the high order model with a model of smaller dimension and using this approximate model for design of a regulator.

The contribution of this research is threefold. First a set of requirements is stated which helps insure that control based on the reduced order model will prove satisfactory. Secondly, two methods for determining reduced order models which satisfy these requirements are developed. Lastly, the basic concept of using reduced model for designing controllers, and the particular methods proposed by this research are shown to be both useful and practical by the solution of two example problems.

Conclusions and Recommendations for Further Study

During the course of this research several conclusions were reached and many problem areas where future research might be fruitful were encountered. Most of the conclusions and problems were the result of an unsuccessful attempt to apply the modeling techniques of Chapter 3 to a 14th order 4 input state model for a nuclear reactor. This problem was difficult to handle, primarily because of the widely spaced groups of eigenvalues. For example, the eigenvalues of the particular model studied fell into four general groups:

0.20	-0.24	-1.00	-10.83
0.03$\pm$j0.15	-0.33	-2.18	-31.10
0.00	-0.54	-3.00	
	-0.74$\pm$0.68		

For this example neither the dominant eigenvalue approach nor the techniques proposed by this research works particularly well. The former suffers from the vague definition of dominant and the latter methods suffer from the necessity for a relatively high order reduced model which creates problems that will be explained later.

Although the attempt to reduce this model was unsuccessful, the study of the system was extremely enlightening and along with the results of Chapter 3 and the solution of the example problems in Chapter 4 led to the following list of conclusions and recommendations:

1) It appears that no model can adequately describe a system unless it retains all the unstable modes of the system. The model may be satisfactory for some specific purpose, i.e., the design of a regulator, but if it is to accurately describe the input-output relationships of a given system it must retain exactly those unstable modes of the system which appear in the output. This observation prompts the following definition. "The dominant modes of a system are those modes which are unstable."

2) The second modeling technique proposed by this research seems to be quite suitable for determining reduced models of relatively low order. Above approximately fourth order the technique begins to experience

problems due to the method of determining the characteristic polynomial of $A_1 + FH_1$. If reference is made to (3.19) it can be seen that this polynomial is determined by subtracting the entries in $f_\ell H_1$ from the coefficients of the characteristic polynomial of A_1. If the eigenvalues of A_1 are large negative and the order of A_1 substantial then the a_i of (3.19) can differ greatly in magnitude. The problem of subtracting two large numbers to obtain a small one is then encountered. Also, it is obvious that some of the d_i of (3.19) will be very sensitive to small changes in H_1 and that the entries in H_1 will differ by several orders of magnitude.

In connection with 2) there are several interesting areas for study.

a) The development of a method similar to the second modeling technique which avoids these difficulties.

b) The proposal of guidelines for estimating the required order of a reduced model. This would allow accurate prediction of whether the above problems would become significant.

c) The development of a procedure for subdividing a system model into several component models. The procedure of Chapter 3 could then be used to develop several small reduced models.

3) In connection with the attempt to reduce the nuclear reactor model, two very interesting problems indirectly related to model reduction arose. To explain the problems assume that the following system model is given:

$$\dot{x} = Ax + Bu \ .$$

The problems may be concisely stated as follows:

a) Without transforming the A matrix to any special form choose H so that (A, H) is an observable pair. This question may seem trivial, however, when faced with a fairly large A matrix and an infinite choice for H the problem becomes very important. The tests for observability are one way, that is for a particular H one can check for observability; however, if this H fails the test the question of how to change H is not answered. In addition for large order A matrices which are not well scaled (i.e. the entries in A differ widely in magnitude) the tests for observability can be difficult numerically.

A closely related problem arises when some H has been found which satisfies the observability criteria. Surely there are many other matrices which make the system observable. How can other matrices which make the pair (A, H) observable be generated from this one?

b) If the given system is unstable how can K_0 be determined so that $A - BK_0$ has all negative eigenvalues? This should be done without transforming the A matrix.

Obviously what this implies is to determine $u_0 = -K_0 x$ so that the system is stable. There are several interesting constraints which might be put on the problem. First it could be assumed that $K_0^T R K_0$ must be minimized. Second it could be assumed that some of the states x are not available and some columns of K_0 must be zero. Also the question of what combinations of states must be available in order to stabilize the system appears very interesting and unanswered.

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APPENDIX A - SOLUTION OF THE MATRIX EQUATION $C_1A - A_1C_1 = FH$

Theorem. Let A be a nxn square matrix and A_1 be an $\ell x \ell$ square matrix in phase variable form. Also let $F = (0 \;\; 0 \;\; \ldots \;\; 0 \;\; f_\ell^T)^T$ be an ℓxp matrix and H be a pxn matrix. Then the equation

$$C_1A - A_1C_1 = FH \tag{A.1}$$

has a unique solution for C_1 if and only if A and A_1 have no eigenvalues in common. Further if $C_1 = (c_1^T \;\; c_2^T \;\; \ldots \;\; c_\ell^T)^T$ this solution is given by

$$c_1 = -f_\ell H[a_1 + a_2A + \ldots + a_\ell A^{\ell-1} - A^\ell]^{-1}$$

$$c_{K+1} = C_1A^K \qquad K = 1, \ldots, \ell-1$$

where

c_i is the ith row of C_1

and

$$A_1 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & \vdots & & \\ a_1 & a_2 & \cdots & a_\ell \end{bmatrix}.$$

Proof: Using the particular forms of the given matrices and vectors the equation A.1 may be written as

$$\begin{bmatrix} c_1A \\ c_2A \\ \cdot \\ \cdot \\ \cdot \\ c_\ell A \end{bmatrix} - \begin{bmatrix} c_2 \\ c_3 \\ \cdot \\ \cdot \\ \cdot \\ a_1c_1 + a_2c_2 + \dots + a_\ell c_\ell \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \\ \\ \\ \\ f_\ell H \end{bmatrix} \qquad \text{A.2}$$

From A.2 the following relations are readily obtained.

$$\begin{aligned} c_2 &= c_1A \\ c_3 &= c_2A \\ &\vdots \\ c_\ell &= c_{\ell-1}A \end{aligned} \qquad \text{A.3}$$

and

$$[c_\ell A - a_1c_1 - a_2c_2 - \dots - a_\ell c_\ell] = f_\ell H \ . \qquad \text{A.4}$$

By inspection from A.3

$$c_{K+1} = c_1A^K \qquad K = 1, \dots, \ell-1. \qquad \text{A.5}$$

Using A.5, A.4 becomes

$$[c_1A^\ell - a_1c_1 - a_2c_1A - \dots - a_\ell c_1A^{\ell-1}] = f_\ell H \ . \qquad \text{A.6}$$

Manipulation of A.6 yields

$$c_1[-a_1I - a_2A - \dots - a_\ell A^{\ell-1} + A^\ell] = f_\ell H \ . \qquad \text{A.7}$$

From A.7 it can be seen that c_1 will have a unique solution if

$$Q = -a_1 - a_2A - \dots - a_\ell A^{\ell-1} + A^\ell \qquad \text{A.8}$$

has an inverse. Use of A.5 then yields a unique solution for C_1. It remains now to show that Q has an inverse if and only if A and A_1 have no eigenvalues in common.

To show this let $A = PJP^{-1}$ where J is the Jordan canonical form of A and P is a non-singular transformation. Q then becomes

$$Q = PQ'P^{-1}$$

where

$$Q' = -a_1 I - a_2 J \dots - a_\ell J^{\ell-1} + J^\ell . \qquad \text{A.9}$$

Suppose now that all the eigenvalues of A are distinct. If $\lambda_1 \lambda_2 \dots \lambda_n$ are the eigenvalues of A then

$$J = \text{diag } (\lambda_1 \ \lambda_2 \dots \lambda_n) \qquad \text{A.10}$$

and

$$J^K = \text{diag } (\lambda_1^K \ \lambda_2^K \dots \lambda_n^K) \ . \qquad \text{A.11}$$

Q' will then be a diagonal matrix with the KK entry equal to

$$[Q']_{KK} = -a_1 - a_2\lambda_K - \dots - a_\ell \lambda_K^{\ell-1} + \lambda_K^\ell \ . \qquad \text{A.12}$$

Remember now that A_1 was in phase variable form; therefore, the a_i are the negative coefficients of the characteristic polynomial. $[Q']_{KK}$ then is the characteristic polynomial of A_1 evaluated for λ_K, and will be zero only if A_1 has an eigenvalue equal to λ_K. If no eigenvalues of A_1 and A are equal, all terms on the diagonal of Q' will be non-zero and

Q' will be non-singular. Conversely if A_1 and A have common eigenvalues Q' will have zeros on its diagonal and be singular.

Suppose now that A has some Jordan blocks. The terms on the diagonal of J^K will still be equal to the eigenvalues of A raised to the Kth power and J^K will be upper triangular. Q' will then be an upper triangular matrix with terms on its diagonal equal to the characteristic polynomial of A_1 evaluated for the eigenvalues of A. Again Q' will be non-singular if A and A_1 have no common eigenvalues.

Since $Q = PQ'P^{-1}$, Q will be non-singular when Q' is, proving the theorem.

APPENDIX B - THE CONJUGATE GRADIENT ALGORITHM

The algorithm used to minimize the cost functions was designed for minimizing a smooth functional. It differs from the method given in reference [29] only in the details of determining step size. Additional treatments of conjugate gradient descent can be found in references [30] and [31]. The equations of the general algorithm are

$$\begin{aligned} P_0 &= - g_0 \\ x_{i+1} &= x_i + \alpha_i P_i \\ \beta_i &= \frac{(g_{i+1}, g_{i+1})}{(g_i, g_i)} \\ P_{i+1} &= -g_{i+1} + \beta_i P_i \ . \end{aligned} \qquad \text{B.1}$$

In the above algorithm g_i represents the gradient of the cost function with respect to the free parameters x at the ith step. The constant α_i controls the length of step taken at each iteration and is determined so that $CF(x_{i+1})$ is minimized.

For the cost functions used in this research the gradients can be found fairly easily. The basic procedure will be illustrated for CF of (3.38), and needs only slight modification for the other cost functions considered in this research. For illustration purposes assume the system to be modeled is single output and that the relation of (4.42) and (4.43) have been used to write the entries in $A_1 + FH_1$ and H_1 of (4.48) in terms of the d_i' (see (3.41)). Then

$$\frac{\partial CF}{\partial d_i'} = -2 \int_{t_o}^{t_f} (z - \hat{z})(H_1 \frac{\partial y_1'}{\partial d_i'} + \frac{\partial H_1}{\partial d_i'} y_i')dt \ . \qquad B.2$$

The partials of the vector H_1 may be readily evaluated from (3.43). The partials of the states require that the following set of equations be solved (see reference [32]).

$$\frac{d}{dt} \begin{bmatrix} \frac{\partial y_1'}{\partial d_i'} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial y_\ell'}{\partial d_i'} \end{bmatrix} = (A_1 + FH_1) \begin{bmatrix} \frac{\partial y_1'}{\partial d_i'} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial y_\ell'}{\partial d_i'} \end{bmatrix}$$

$$+ \begin{bmatrix} & \underline{0} & \\ \frac{\partial d_1}{\partial d_i'} & \frac{\partial d_2}{\partial d_i'} \cdots & \frac{\partial d_\ell}{\partial d_i'} \end{bmatrix} y_1' \qquad B.3$$

The necessary partial derivatives can now be found by solving B.3 for each i, i = 1, ...p where p is the number of free modes in the reduced system. These partials and the partials of H_1 can now be used in B.2 to find the required gradients.

The method of finding α_i to minimize $CF(x_{i+1})$ was basically a bracketing process and is illustrated in Figure B.1. The procedure determines an initial guess for the value of α_i which minimizes $CF(x_{i+1})$.

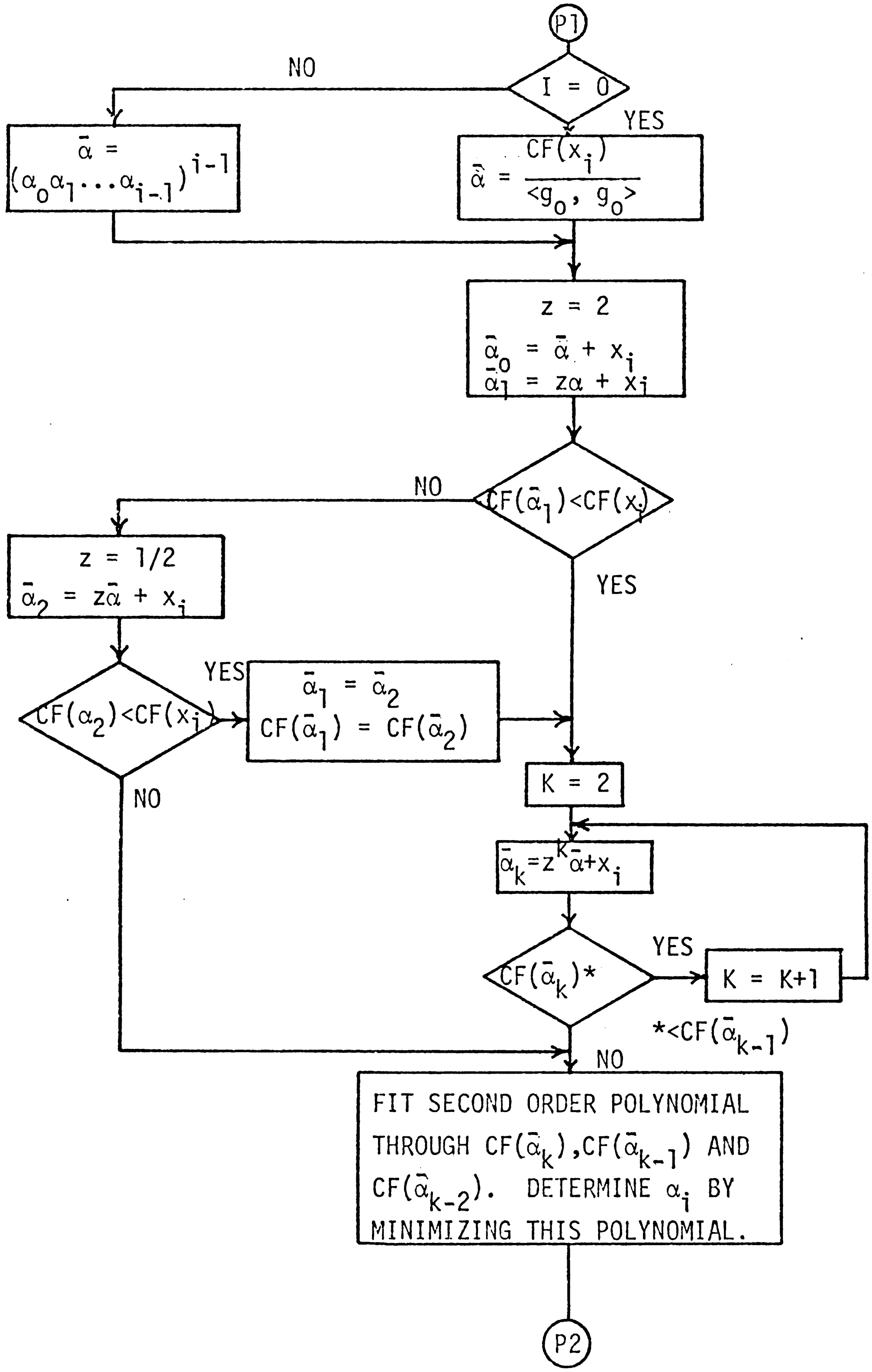

FIGURE B.1. FLOW CHART FOR DETERMINING STEP SIZE

Once this is done the step size is halved or doubled until a minimum is found. At this point a second order polynomial is found which passes through this minimum and the points on either side. The desired value of α_i is then found by minimizing this polynomial.

For a detailed discussion of the properties of the conjugate gradient algorithm the reader is referred to the previous references. The algorithm does possess some very nice qualities, however, which should be mentioned.

1. For a quadratic cost function the sequence converges to a minimum in n steps.
2. The sequence exhibits quadratic convergence but does not require any knowledge of the second derivative.
3. For a smooth function CF the conjugate gradient descent algorithm satisfies

$$CF(x_{i+1}) < CF(x_i),\ g_i \neq 0 .$$

4. The method requires very little more effort to implement than steepest descent.

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THE DESIGN OF SUBOPTIMAL LINEAR REGULATORS

USING REDUCED ORDER AGGREGATED MODELS

by

Luther Lee Joyner, III

(ABSTRACT)

An approach to the design of suboptimal linear regulators is developed. Two techniques are proposed for obtaining a reduced order aggregated model for a constant coefficient dynamic system. This model is then used to determine a suboptimal control law to solve an output regulator problem.

The research is developed by first examining the problems involved when the design and implementation of the optimal regulator is attempted. The idea of using a reduced order model to overcome some of these problems is discussed and a set of criteria that the reduced model must satisfy is presented. Two methods for determining a reduced order model that satisfies the criteria are then developed and used to design controllers for two example systems.

The methods are based on using gradient descent to minimize the error between the exact system output and the output of an observer dependent aggregated model. The use of a stochastic input to serve as the test function for this minimization is proposed and shown to be quite useful. The procedure developed is applicable to multi-input systems and to systems with unstable modes. In addition, there is no requirement that the exact model be in any special form.