ON MULTIGROUP TRANSPORT THEORY

WITH A DEGENERATE TRANSFER KERNEL

by

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TABLE OF CONTENTS

I. INTRODUCTION

The linearized energy-dependent Boltzmann transport operator has been studied extensively in connection with various boundary value problems.[1-3] In the absence of analytic solutions, except for specially idealized situations, diverse approximate schemes based on numerical techniques have been developed for solving the transport equation. More rigorous mathematical treatments are desirable, even if in simplified cases, to avoid the crude approximations and to obtain standards for determining the accuracy of the numerical solutions. The analysis conducted in this dissertation is intended to be categorized as a treatment of this kind.

The physical idealizations and approximations necessary for the purposes of the present study are those pertinent to the separability of the spatial, angular and energy variables of the angular particle density.[4] A multigroup approximation is employed to describe the energy dependence, and the accuracy can be improved in a trivial manner by increasing the number of energy groups. If the physical properties of the system are not spatially constant, the medium has to be divided into finite regions, in each of which the transport equation (at least approximately) possesses translational invariance.

In cases where the above simplification can be made, one of the widely used approaches is the singular eigenfunction expansion method, whose applications to the monoenergetic theory have been comprehensively discussed in Reference 4.

This method allows, in general, the eigensolutions to be distributions rather than ordinary functions, and it requires the proof of completeness of the eigensolutions. Several generalizations of this approach to multigroup theory have been made, more recently in Refs. 5-7, where the full-range completeness proofs are obtained for the pertinent kernels of isotropic transfer. With respect to semi-infinite medium problems, a complete discussion on a special case in photon transport is given by Siewert and Zweifel.[5] The limitations involved are, however, invalid in neutron transport. For the neutron case, Yoshimura and Katsuragi[6] have considered infinite medium only, whereas Leonard and Ferziger[7] succeeded in proving a half-range completeness theorem except for some detail.[8]

The method of singular eigensolution expansions has also been applied to problems with anisotropic scattering, first by Mika[9] in the one-speed theory. Siewert et al[10] have analyzed a special separable and symmetric two-group kernel in radiative transfer. Shultis[11] has examined a symmetric anisotropic multigroup kernel and reduced the proof of the full-range completeness theorem to the solution of a Fredholm equation.

The invariant imbedding method[4] is particularly suitable for computational purposes in semi-infinite medium problems. In this approach, the emergent distribution is first calculated, while the interior distribution can be obtained in terms of the emergent one. The solution of the non-linear integral equation obtained is not, however, necessarily unique. Pahor and Zweifel[11] have related the appropriate integral equations for one group of neutrons to the formalism of Case,[4] thus proving the uniqueness of the solution. The completeness proofs presented in this thesis can conceivably be used to demonstrate uniqueness in the multigroup cases considered by Pahor and Shultis.[11,13]

Pahor and Shultis also obtained numerical solutions to some standard problems. Recently Clancy[14] has reported some multigroup computational schemes based on invariant imbedding for penetration problems involving anisotropic scattering. On the other hand, Metcalf and Zweifel[15] have carried out numerical calculations in two-group neutron transport by iterating the singular integral equations obtained from the eigensolution expansions.

It has been shown by Leonard and Ferziger[7] that the multigroup approximation is equivalent to a continuous energy scheme, where the energy dependence is expanded in terms of a finite sum of orthogonal functions. The treatments of Bednarz and Mika[16] and Case[17] indicate that maintaining the continuous energy dependence simultaneously with rigorous spatial and angular schemes allows only highly formalistic considerations.

The present work is an extension of earlier papers[18-21] on multigroup transport theory in plane geometry with anisotropic scattering. The latest version[21] considered the N-group problem in the constant cross section limit, the transfer matrix being compact. A degenerate approximation of the kernel was employed. It is emphasized that a compact (square integrable) kernel can be arbitrarily well approximated (in the norm) with a degenerate kernel.[22] The assumption of constant cross section is relaxed and, while the rest of the treatment parallels closely the work in Refs. 18-21, the formalism is generalized to be applicable to neutron transport theory. The terminology of neutron physics is used, although this approach is amenable to all situations where the Boltzmann operator is linearizable. No further studies have been made to determine the class of realistic scattering laws for which this method applies. The limitation to the stationary form of the transport operator is justified because time-dependent problems can be reduced to this form by an integral transform.[4] While analytic solutions to the infinite medium problems are available, the main concern in this thesis has been directed to an investigation of the spectral properties of the operator, to the direct and adjoint eigensolutions and their completeness.

Chapter II reviews the derivation of the multigroup approximation. A general consequence of thermal equilibrium, viz. symmetric transfer, has been discussed. As it will be seen in later sections, the treatment of a symmetric kernel is specially convenient.

The structure of the eigenvalue spectrum and the associated eigensolutions are demonstrated in Chapter III. The degenerate continuum eigensolutions are chosen on a basis of convenience, whereas to obtain an explicit representation of the eigensolutions one prefers a different set. The dispersion matrix is obtained in a block matrix form. The pertinent features of the adjoint spectrum are also exhibited. In fact, it is known[18] that the direct and adjoint spectra are the same and the distributional parts of the corresponding eigensolutions are also identical. Possible degeneracy of the discrete eigenfunctions and embedding of discrete eigenvalues in the continuous spectrum have been overlooked by referring to corresponding treatments of less complicated kernels in Refs. 4, 5 and 9.

The objective in Chapter IV is to derive sufficient conditions for the existence of the completeness properties of the eigensolutions. The proof is reduced to an inhomogeneous Hilbert problem.[23,24] The existence and uniqueness of a proper fundamental system of solutions can be determined from the classical theory of Muskhelisvili[23] and Vekua[24] in the case that the boundary value transformation is performed by a discontinuous mat ix and the conclusions can be mainly drawn from the sign of a parameter ρ. The stringent Hölder or Lipschitz conditions can be partially relaxed in the case where the transformation matrix is continuous.[10,25] The infinite and bounded medium problems are considered separately. The full-range completeness can be obtained under rather general conditions; $\rho = 0$ and the expanded function satisfies the extended Hölder condition. At the present stage of the theory the half-range completeness requires additionally that the dispersion matrix is symmetric (self-adjoint operator) and even, i.e., the elements are even functions, which corresponds closely to reflection symmetry. The case of a nonself-adjoint kernel is also considered and, except for a certain detail, it is directly deducible that either the direct or adjoint eigensolutions form a complete set and the both sets are complete if the discrete spectrum contains fewer than a dertain number of points. The both sets can be argued to be complete under a heuristically justified assumption that the operator relating the direct and adjoint eigensolutions is invertible.[20,26]

As a corollary of the theorems proved it follows that half-range completeness is a consequence of the full-range completeness, and conversely, in case the transfer kernel is symmetric and possesses reflection symmetry.

In Chapter V the standard full-range orthogonality relation is obtained and a brief discussion of the norm integrals is included. The continuous modes require a specific orthogonalization procedure.[10,19] The infinite medium Green's function is solved as an immediate application of the full-range completeness and orthogonality properties.

The diagonalization of the Hilbert problem obtained in Chapter IV leads automatically to analytic solutions. An approach to a general procedure is demonstrated in Appendix A in the constant total cross section limit. In Appendix B Vekua's[24] procedure for determining the parameter ρ is reporduced in general, while in certain cases[19] calculations can be simplified. In Appendix C the non-positivity of the component indices of the Hilbert problem concerned is demonstrated in order rigorously to allow the conclusions of completeness in Section IV. It might be mentioned that some earlier investigations[6,7,11] have received severe polemic[8] because of deficiencies in this respect. The criticism is summarized in Appendix D. Finally, a scalar singular integral equation involving a Fredholm term is obtained in Appendix E. This equation can be indirectly solved using the full-range orthogonality relation.

To summarize, it is the purpose of this work to prove the completeness of the normal modes pertinent to the transport operator with a degenerate transfer kernel and to discuss to which extent analytic solutions are available.

II. MULTIGROUP APPROXIMATION

A concise derivation of the multigroup equations is given in this chapter. While different techniques based on the variational approach[27] or orthogonal expansions[7] are available, the multigroup approximation is obtained in a standard manner by discretizing the energy variable and defining the effective group constants through integration over energy. Neutron regeneration is formally included although the fission modes could be treated separately.[28]

The linear stationary homogeneous Boltzmann equation can be written in plane geometry in the form

$$\mu\frac{\partial}{\partial x}\Psi(x,E,\mu) + \Sigma(E)\Psi(x,E,\mu) = \int_{1}^{1} d\mu' \int_{0}^{\infty} dE'\Sigma(E',E;\mu',\mu)\Psi(x,E',\mu') \quad (2.1)$$

with

$$\Sigma(E',E;\mu',\mu) = \Sigma_s(E',E;\mu',\mu) + \chi(E)\nu(E')\Sigma_f(E'). \quad (2.2)$$

The quantity $\Psi(x,E,\mu)$ is the energy-dependent angular flux, $\Sigma(E)$ is the total cross section, $\Sigma_s(E,E';\mu,\mu')$ denotes the differential scattering cross section involving both elastic and inealstic scattering, $\Sigma_f(E)$ is the fission cross section, $\chi(E)$ is the fission spectrum and $\nu(E)$ denotes the number of neutron produced per fission.

Proceeding in a standard manner[29] the energy variable is split into N regions denoted by

$$\Delta E_n = [E_{n-1}, E_n],$$
$$n = 1,2,\cdots N. \quad (2.3)$$

To make the removal cross sections independent of x and μ, it is assumed that the energy dependence of the angular flux is separable, i.e.,

$$\Psi(x,E,\mu) = F_n(E)\Phi_n(x,\mu), E\varepsilon\Delta E_n. \tag{2.4}$$

In fact, a more general degenerate expansion seems to be possible in the framework of the present formalism as will be proposed later in this section.

Eq. (2.1) is next integrated over ΔE_k, k=1,2,$\cdots$N, consecutively, to obtain

$$\mu\frac{\partial}{\partial x}\Phi_k(x,\mu) + S_k\Phi_k(x,\mu) = \sum_{n=1}^{N}\int_{-1}^{1} d\mu' C_{kn}(\mu,\mu')\Phi_n(x,\mu'), \tag{2.5}$$

$$k = 1,2,\cdots N,$$

where

$$S_k = \frac{\int_{\Delta E_k} F_k(E)\Sigma(E)dE}{\int_{\Delta E_k} F_k(E)dE} \tag{2.6}$$

and

$$C_{kn}(\mu,\mu') = \frac{\int_{\Delta E_k} dE \int_{\Delta E_n} \Sigma(E',E;\mu',\mu)F_n(e')dE'}{\int_{\Delta E_k} F_k(E)dE} \tag{2.7}$$

Distance will be measured in units of the largest mean free path $1/S_{min}$, where $S_{min} \leq S_n$ for all n, the equality holding for some particular n, and Eq. (2.5) is thus divided by S_{min}. Defining

$$s_k = S_k/S_{min} \tag{2.8}$$

and

$$C_{kn}(\mu,\mu') = C_{kn}(\mu,\mu')/S_{min}, \tag{2.9}$$

the system of N equations in Eq. (2.5) can be expressed in the matrix form

$$\mu\frac{\partial}{\partial x}\,\underset{\sim}{\Phi}(x,\mu) + \underset{\sim}{S}\underset{\sim}{\Phi}(x,\mu) = \int_{-1}^{1}\underset{\sim}{C}(\mu,\mu')\underset{\sim}{\Phi}(x,\mu')d\mu', \tag{2.10}$$

where $\underset{\sim}{\Phi}(x,\mu)$ is a N-component vector with component $\Phi_i(x,\mu)$, $\underset{\sim}{S}$ is a diagonal matrix with elements s_i, and $\underset{\sim}{C}(\mu,\mu')$ is the transfer matrix having elements $c_{ij}(\mu,\mu')$.

For convenience, the groups will be reordered by applying a permutation $\underset{\sim}{P}$ defined as[30]

$$\underset{\sim}{P}\underset{\sim}{S}\underset{\sim}{P}^{-1} = \underset{\sim}{\Sigma}, \tag{2.11}$$

where, denoting the elements of the diagonal matrix $\underset{\sim}{\Sigma}$ by σ_i,

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N = 1. \tag{2.12}$$

Eq. (2.10) is now written as

$$\mu\frac{\partial}{\partial x}\,\underset{\sim}{\psi}(x,\mu) + \underset{\sim}{\Sigma}\underset{\sim}{\psi}(x,\mu) = \int_{-1}^{1}\underset{\sim}{K}(\mu,\mu')\underset{\sim}{\psi}(x,\mu')d\mu', \tag{2.13}$$

where

$$\underset{\sim}{\psi}(x,\mu) = \underset{\sim}{P}\underset{\sim}{\Phi}(x,\mu), \tag{2.14}$$

and

$$\underset{\sim}{K}(\mu,\mu') = \underset{\sim}{P}\underset{\sim}{C}(\mu,\mu')\,\underset{\sim}{P}^{-1}. \tag{2.15}$$

A similar multigroup system of equations can be derived by expanding the energy dependence of the angular flux in terms of a finite sum of orthogonal functions.[7,11] This method is particularly applicable in case the eigenfunctions of the scattering operator are known to be a set of orthogonal polynomials, e.g., for the heavy gas or corresponding synthetic models.[31]

The two schemes discussed above can be combined in a manner which makes it possible to relax the assumption on the trial functions in Eq. (2.4). Instead, a degenerate expansion

$$\Psi(x,E,\mu) = \sum_{i=1}^{N_n} F_i^{(n)}(E)\,\phi_i^{(n)}(x,\mu), \qquad E\varepsilon\Delta E_n, \tag{2.16}$$

could be used to generate a submultigroup system inside the selection of original groups. It is not clear, however, whether a similar improvement in accuracy could be achieved more conveniently by reducing the mesh spacings of energy variable, i.e., directly increasing the number of groups.

Symmetric transfer, defined by

$$\underset{\sim}{K}(\mu,\mu') = \underset{\sim}{K}^T(\mu',\mu), \tag{2.17}$$

where superscript T denotes the transpose, would simplify some considerations in the later chapters and it is assumed in the half-range proof. The adjoint operator will be defined in such a manner that Eq. (2.17) implies self-adjointness. Symmetric kernels occur in the thermal neutron problem for a non-multiplying medium where the condition of detailed balance is obeyed,[7,11,13] and also in special astro-physical applications of radiative transport.[5,10] Two-group problems can be symmetrized in general.[13] In fact, symmetric transfer has been assumed in all earlier investigations of anisotropic transport.[10,11]

Some other assumptions are imposed on the transfer kernel $\underset{\sim}{K}(\mu,\mu')$ to make the problem mathematically tractable. $\underset{\sim}{K}(\mu,\mu')$ is assumed to be of a degenerate form,

$$\underset{\sim}{K}(\mu,\mu') = \sum_{i=1}^{M} \underset{\sim}{L}_i(\mu)\underset{\sim}{M}_i(\mu'). \tag{2.18}$$

A compact operator, $\underset{\sim}{K}\varepsilon L^2(-1,1)$, can be arbitrarily well approximated by a kernel of the degenerate form Eq. (2.18).[22] Furthermore, the kernel is assumed to be bounded (although certain parts of the treatment are obviously valid for unbounded kernels.) Certain theorems will also employ a condition pertinent to time reversal invariance[19]

$$\underset{\sim}{K}(\mu,\mu') = \underset{\sim}{K}(-\mu',-\mu). \tag{2.19}$$

III. EIGENSOLUTIONS

In this chapter, the eigenvalue spectrum and the associated eigenfunctions are discussed. The spectral properties of the transport operator are determined by the dispersion matrix which is obtained in the first section from the analysis of the discrete eigenfunctions. The structure of the continuum modes and the adjoint spectrum are demonstrated in subsequent sections.

In the previous chapter the N-group approximation was cast in the form

$$\mu \frac{\partial}{\partial x} \underset{\sim}{\psi}(x,\mu) + \underset{\sim}{\Sigma}\underset{\sim}{\psi}(x,\mu) = \int_{-1}^{1} \underset{\sim}{K}(\mu,\mu')\underset{\sim}{\psi}(x,\mu')d\mu', \tag{3.1}$$

where

$$\underset{\sim}{K}(\mu,\mu') = \sum_{i=1}^{M} \underset{\sim}{L}_i(\mu)\underset{\sim}{M}_i(\mu'). \tag{3.2}$$

The conventional ansatz,[4]

$$\underset{\sim}{\psi}(x,\mu) = e^{-x/\nu}\underset{\sim}{\phi}(\nu,\mu), \tag{3.3}$$

when inserted into Eq. (3.1), yields a general eigenvalue equation

$$(\underset{\sim}{\Sigma} - \mu/\nu\ \underset{\sim}{I})\underset{\sim}{\phi}(\nu,\mu) = \int_{-1}^{1} \underset{\sim}{K}(\mu,\mu')\underset{\sim}{\phi}(\nu,\mu')d\mu'. \tag{3.4}$$

The analysis of the discrete and continuous spectra will be carried out separately. Defining

$$\underset{\sim}{D}(z,\mu) = (\underset{\sim}{\Sigma} - \mu/z\ \underset{\sim}{I})^{-1}, \tag{3.5}$$

it is noticed from Eq. (2.12) that the elements $D_{ii}(z,\mu)$ of the diagonal matrix $\underset{\sim}{D}(z,\mu)$

$$D_{ii}(z,\mu) = \frac{z}{\sigma_i z-\mu} \tag{3.6}$$

remain non-singular whenever $z \notin [-\frac{1}{\sigma_i}, \frac{1}{\sigma_i}]$ but are singular inside this real interval. That is why the interval [-1,1] is distinguished in spectral considerations despite the fact that discrete eigenvalues may lie inside the continuum.

Discrete Eigenfunctions

Assuming $\nu \notin [-1,1]$, Eq. (3.4) is rewritten as

$$\underset{\sim}{\phi}(\nu_k,\mu) = \underset{\sim}{D}(\nu_k,\mu) \sum_{i=1}^{M} \underset{\sim}{L}_i(\mu)\underset{\sim}{n}_i(\nu_k), \tag{3.7}$$

where

$$\underset{\sim}{n}_i(z) = \int_{-1}^{1} \underset{\sim}{M}_i(\mu)\underset{\sim}{\phi}(z,\mu)d\mu. \tag{3.8}$$

In order to determine the discrete eigenvalues one proceeds as in Ref. 18; Eq. (3.7) is multiplied by $\underset{\sim}{M}_j(\mu)$ and integrated over μ. This is done consecutively for $j = 1,2...M$. Defining a dispersion matrix $\underset{\sim}{\Lambda}(z)$ as a block matrix with $[\underset{\sim}{\Lambda}(z)]_{ij}$ as a block element,

$$[\underset{\sim}{\Lambda}(z)]_{ij} = \delta_{ij}\underset{\sim}{I} - \int_{-1}^{1} \underset{\sim}{M}_i(\mu)\underset{\sim}{D}(z,\mu)\underset{\sim}{L}_j(\mu)d\mu, \tag{3.9}$$

the resulting system of equations can be expressed as

$$\sum_{j=1}^{M} [\underset{\sim}{\Lambda}(z)]_{ij}\underset{\sim}{n}_j(z) = \underset{\sim}{0} \qquad \text{for } i = 1,2...M. \tag{3.10}$$

Alternatively, using NM x NM matrices $\underset{\sim}{I}$ and $\underset{\sim}{\Lambda}(z)$ and NM-component vectors $\underset{\sim}{0}$ and $\underset{\sim}{n}(z)$ with

$$\underset{\sim}{n}(z) = \Big((\underset{\sim}{n}_1(z))_1,(\underset{\sim}{n}_1(z))_2,...(\underset{\sim}{n}_k(z))_\ell,...,(\underset{\sim}{n}_M(z))_N\Big)^T, \tag{3.11}$$

Eqs. (3.10) can be cast in a new form of NM linear simultaneous equations,

$$\underset{\sim}{\Lambda}(z)\underset{\sim}{n}(z) = \underset{\sim}{0} \ . \tag{3.12}$$

Denoting the dispersion function by $\Omega(z)$,

$$\Omega(z) = \det \underset{\sim}{\Lambda}(z), \tag{3.13}$$

the discrete eigenvalues ν_k are obtained from

$$\Omega(\nu_k) = 0. \tag{3.14}$$

For simplicity it will be assumed that all these eigenvalues are distinct. In case of multiple roots to Eq. (3.14), the original one-dimensional representation of the translation group in Eq. (3.3) does not suffice but higher order representations are required. This aspect is discussed thoroughly in Refs. 4 and 5. It has also been established[9] that discrete eigenvalues may occur in the continuum, i.e. $\nu_k \epsilon [-1,1]$. These cases will not be considered in this work, either.

In view of Eqs. (3.13,14) the ratio of the components of $\underset{\sim}{n}(\nu_k)$ is uniquely defined by Eq. (3.12). If the cofactor of $(\underset{\sim}{\Lambda}(\nu_k))_{ij}$ is denoted by $(\underset{\sim}{\Lambda}(\nu_k))^{ij}$, one can choose[30]

$$(\underset{\sim}{n}(\nu_k))_j = (\underset{\sim}{\Lambda}(\nu_k))^{ij}, \tag{3.15}$$

where $j = 1,2,\ldots NM$ and $i \leq NM$ is an arbitrary fixed index.

For later convenience, the boundary values $\underset{\sim}{\Lambda}^{\pm}(\nu)$, $\nu \epsilon [-1,1]$, of the dispersion matrix $\underset{\sim}{\Lambda}(z)$ will be introduced. Because of the different explicit forms in each interval $[-\frac{1}{\sigma_i}, \frac{1}{\sigma_i}]$, the expressions will include a unit function $h_k(\nu)$ defined as

$$\begin{aligned} h_k(\nu) &= 1, \ \nu \epsilon [-\frac{1}{\sigma_k}, \frac{1}{\sigma_k}], \\ &= 0 \text{ otherwise.} \end{aligned} \tag{3.16}$$

Consider the ℓmth elements of the ijth block; from Eq. (3.9)

$$([\underset{\sim}{\Lambda}(z)]_{ij})_{\ell m} = (\underset{\sim}{\Lambda}(z))_{(i-1)N+\ell,(j-1)N+m}$$
$$= \delta_{ij}\delta_{\ell m} - \sum_{k=1}^{N} \int_{-1}^{1} \frac{z(\underset{\sim}{M}_i(\mu))_{\ell k}(\underset{\sim}{L}_j(\mu))_{km}}{\sigma_k z - \mu}\, d\mu. \tag{3.17}$$

The boundary values $\underset{\sim}{\Lambda}^{\pm}(\nu)$, defined as

$$\underset{\sim}{\Lambda}^{\pm}(\nu) = \lim_{\varepsilon \to 0} \underset{\sim}{\Lambda}(\nu \pm i\varepsilon), \quad \nu\varepsilon[-1,1], \tag{3.18}$$

are obtained by applying the Plemelj formula to Eq. (3.16);

$$([\underset{\sim}{\Lambda}^{\pm}(\nu)]_{ij})_{\ell m} = \delta_{ij}\delta_{\ell m} - \sum_{k=1}^{N} \left\{ \int_{-1}^{1} \frac{\nu(\underset{\sim}{M}_i(\mu))_{\ell k}(\underset{\sim}{L}_j(\mu))_{km}}{\sigma_k \nu - \mu}\, d\mu \right.$$
$$\left. \mp i\pi\nu(\underset{\sim}{M}_i(\sigma_k\nu))_{\ell k}(\underset{\sim}{L}_j(\sigma_k\nu))_{km} h_k(\nu) \right\}, \tag{3.19}$$

where $h_k(\nu)$ was defined in Eq. (3.16). The integral terms in Eq. (3.19) can be either regular or singular; in the latter case the notation includes the principal value operator.

The notation can be simplified by introducing two transformations mapping an N x N matrix $\underset{\sim}{A}(\nu)$ to $\underset{\sim}{A}^{\sigma}(\nu)$ and $\underset{\sim}{A}_{\sigma}(\nu)$, respectively. The transformations are defined as

$$(\underset{\sim}{A}^{\sigma}(\nu))_{ij} = (\underset{\sim}{A}(\sigma_j\nu))_{ij} h_j(\nu) \tag{3.20}$$

and

$$(\underset{\sim}{A}_{\sigma}(\nu))_{ij} = (\underset{\sim}{A}(\sigma_i\nu))_{ij} h_i(\nu). \tag{3.21}$$

Eq. (3.19) is now rewritten as

$$[\underset{\sim}{\Lambda}^{\pm}(\nu)]_{ij} = \delta_{ij} I - \int_{-1}^{1} \underset{\sim}{M}_i(\mu)\underset{\sim}{D}(\nu,\mu)\underset{\sim}{L}_j(\mu)\, d\mu$$
$$\pm i\pi\nu \underset{\sim}{M}_i^{\sigma}(\nu)\underset{\sim}{L}_{j\sigma}(\nu), \quad \nu\varepsilon[-1,1], \tag{3.22}$$

with a proper interpretation of the integral. The boundary values $\underset{\sim}{\Lambda}^{\pm}(\nu)$ considered here will be employed in later sections where completeness is proved. Furthermore, it is useful to define matrices $\underset{\sim}{\Gamma}_{\pm}(\nu)$ by

$$\underset{\sim}{\Gamma}_{+}(\nu) = \tfrac{1}{2}(\underset{\sim}{\Lambda}^{+}(\nu) + \underset{\sim}{\Lambda}^{-}(\nu)) \tag{3.23}$$

and

$$\underset{\sim}{\Gamma}_{-}(\nu) = \frac{1}{2\pi i\nu}(\underset{\sim}{\Lambda}^{+}(\nu) - \underset{\sim}{\Lambda}^{-}(\nu)). \tag{3.24}$$

Continuum Modes

The general solution to Eq. (3.4) with $\nu\varepsilon[-1,1]$ is considered in this section. Generalized functions are admitted as solutions.[4] Because of the singular form of matrix $\underset{\sim}{D}(\nu,\mu)$ (cf. Eq. (3.6)) it is customary to divide the interval [-1,1] into N subregions labelled (n) and defined as

$$(n) = [-\frac{1}{\sigma_n}, -\frac{1}{\sigma_{n-1}}] \cup [\frac{1}{\sigma_{n-1}}, \frac{1}{\sigma_n}], \tag{3.25}$$

with

$$(1) = [-\frac{1}{\sigma_1}, \frac{1}{\sigma_1}]. \tag{3.26}$$

There is an N-n+1 fold degeneracy in region (n), which prevents an explicit representation of the eigensolutions in the chosen form. While this particular form is preferable in later calculations, another choice of eigensolutions is displayed later in this section. This special combination of linearly independent solutions makes it possible to derive explicit expressions for the functions involved.

Returning to Eq. (3.7), the solutions are sought to be distributions with support [-1,1].[32] In particular, the general solution with the eigenvalue ν lying in region (n) is written as

$$\underset{\sim}{\phi}^{(n)}(\nu,\mu) = (\underset{\sim}{D}(\nu,\mu) + \lambda^{(n)}(\nu)\underset{\sim}{\delta}^{(n)}(\nu,\mu)) \sum_{i=1}^{M} \underset{\sim}{L}_i(\mu)\underset{\sim}{n}_i^{(n)}(\nu), \nu\varepsilon(n), \tag{3.27}$$

where

$$\underset{\sim}{n}_i^{(n)}(\nu) = \int_{-1}^{1} \underset{\sim}{M}_i(\mu)\underset{\sim}{\phi}^{(n)}(\nu,\mu)d\mu \tag{3.28}$$

and the elements of the diagonal matrices $\underset{\sim}{D}(\nu,\mu)$ and $\underset{\sim}{\delta}^{(n)}(\nu,\mu)$ are defined for appropriate test functions[32] $f(\nu)$ as

$$<(\underset{\sim}{D}(\nu,\mu))_{ii}, f(\mu)> = \int_{-1}^{1} \frac{\nu f(\mu)}{\sigma_i\nu-\mu}\, d\mu \tag{3.29}$$

and

$$\begin{aligned} <(\underset{\sim}{\delta}^{(n)}(\nu,\mu))_{ii}, f(\mu)> &= f(\sigma_i\nu),\ i \geq n \\ &= 0 \text{ otherwise, } \nu\varepsilon(n). \end{aligned} \tag{3.30}$$

It is again emphasized that the principle value operator occurs in Eq. (3.29) for $i \geq n$ but it has been omitted by the earlier convention. The conventional form of $(\underset{\sim}{D}(\nu,\mu))_{ii}$ is expressed in Eq. (3.6), whereas $(\underset{\sim}{\delta}^{(n)}(\nu,\mu))_{ii}$ is the usual "delta-function", i.e.

$$(\underset{\sim}{\delta}^{(n)}(\nu,\mu))_{ii} = \delta(\sigma_i\nu - \mu),\ \nu\varepsilon(n). \tag{3.31}$$

To determine $\lambda^{(n)}(\nu)$ Eq. (3.27) is multiplied by $\underset{\sim}{M}_k(\mu)$, k=1,2,...M and integrated over μ. Noticing the relation

$$\int_{-1}^{1} \underset{\sim}{M}_i(\mu)\underset{\sim}{\delta}^{(n)}(\nu,\mu)\underset{\sim}{L}_j(\mu)d\mu = \underset{\sim}{M}_i^{\sigma}(\nu)\underset{\sim}{L}_{j\sigma}(\nu), \tag{3.32}$$

where the σ-operators are defined in Eqs. (3.20,21), one obtains a homogeneous system of simultaneous equations,

$$\left[\underset{\sim}{\Gamma}_+(\nu) - \lambda^{(n)}(\nu)\underset{\sim}{\Gamma}_-(\nu)\right]\underset{\sim}{n}^{(n)}(\nu) = \underset{\sim}{0}, \ \nu\varepsilon(n), \tag{3.33}$$

where Eqs. (3.23,24) have been employed to introduce matrices $\underset{\sim}{\Gamma}_{\pm}(\nu)$ and $\underset{\sim}{n}^{(n)}(\nu)$ is defined in analogy with Eq. (3.11), i.e.,

$$\underset{\sim}{n}^{(n)}(\nu) = \left[(\underset{\sim}{n}_1^{(n)}(\nu))_1, (n_1^{(n)}(\nu))_2 \ldots (\underset{\sim}{n}_M^{(n)}(\nu))_N\right]^T. \tag{3.34}$$

The arbitrary function $\lambda^{(n)}(\nu)$ can now be determined from

$$\det\left[\underset{\sim}{\Gamma}_+(\nu) - \lambda^{(n)}(\nu)\underset{\sim}{\Gamma}_-(\nu)\right] = 0, \qquad \nu\varepsilon(n). \tag{3.35}$$

Because the rank of the matrix $\underset{\sim}{\Gamma}_+(\nu) + \lambda^{(n)}(\nu)\underset{\sim}{\Gamma}_-(\nu)$ is NM, Eq. (3.35) yields, in principle, a characteristic polynomial of NMth order in $\lambda^{(n)}(\nu)$ thus indicating occurance of degeneracy. The actual degree of degeneracy will be established by the following lemma.

Lemma 3.1 The degree of degeneracy of the eigensolutions $\underset{\sim}{\phi}^{(n)}(\nu,\mu)$, $\nu\varepsilon(n)$, is $N - n + 1$.

Proof: One needs to verify that the degree of the characteristic polynomial in $\lambda^{(n)}(\nu)$ is $N - n + 1$ in the nth region. It will be proceeded by decomposing the determinant[20] in Eq. (3.35) according to standard rules of matrix algebra.[30] Immediately, this yields a secular equation

$$\sum_{m=0}^{NM} (-1)^m a_m(\nu)\left[\lambda^{(n)}(\nu)\right]^m = 0, \tag{3.36}$$

with

$$a_m(\nu) = \sum_p \det \underset{\sim}{\Gamma}_{mp}(\nu), \tag{3.37}$$

where matrices $\underset{\sim}{\Gamma}_{mp}(\nu)$ are constructed including m arbitrary distinct rows of $\underset{\sim}{\Gamma}_-(\nu)$ the remaining rows being identical with the corresponding rows of $\underset{\sim}{\Gamma}_+(\nu)$. The sum has been taken over all possible different combinations p. In particular,

$$a_{NM}(\nu) = \det \underset{\sim}{\Gamma}_-(\nu) \tag{3.38}$$

and

$$a_0(\nu) = \det \underset{\sim}{\Gamma}_+(\nu). \tag{3.39}$$

It is required to show that

$$a_m(\nu) \equiv 0, \text{ if } m > N - n + 1. \tag{3.40}$$

Supposing that the p_qth row of $\underset{\sim}{\Gamma}_-(\nu)$ is included in $\underset{\sim}{\Gamma}_{mp}(\nu)$, Eqs. (3.19-24) yield

$$\begin{aligned}\left(\underset{\sim}{\Gamma}_{mp}(\nu)\right)_{p_q,r} &= \left(\underset{\sim}{M}_i^{\sigma}(\nu)L_{\sigma j}(\nu)\right)_{k\ell} \\ &= \sum_{s=1}^{N}\left(\underset{\sim}{M}_i(\sigma_s\nu)\right)_{ks}\left(\underset{\sim}{L}_j(\sigma_s\nu)\right)_{s\ell}h_s(\nu),\end{aligned} \tag{3.41}$$

where

$$p_q = (i-1)N + k, \tag{3.42}$$

and

$$r = (j-1)N + \ell, \tag{3.43}$$

with $i,j \leq M$, $k,\ell \leq N$, $q = 1,2...m$, and $r = 1,2...NM$. In the calculation of det $\underset{\sim}{\Gamma}_{mp}(\nu)$ a decomposition, similar to one used before, is performed on the p_qth row consecutively for all values of q. Defining matrices $\underset{\sim}{\Gamma}_{mpp'}(\nu)$ by

$$\left(\underset{\sim}{\Gamma}_{mpp'}(\nu)\right)_{p_q,r} = \left(\underset{\sim}{M}_i(\sigma_{s_q}\nu)\right)_{ks_q}\left(\underset{\sim}{L}_j(\sigma_{s_q}\nu)\right)_{s_q\ell}h_{s_q}(\nu), \tag{3.44}$$

where Eqs. (3.42,43) are to be applied to relate the indices, the result can be expressed as

$$\det \underset{\sim}{\Gamma}_{mp}(\nu) = \sum_{p'} \det \underset{\sim}{\Gamma}_{mpp'}(\nu), \tag{3.45}$$

where the combinatorial sum is extended over all possible combinations p' obtained when the indices q and s vary simultaneously; $q = 1,2,\ldots m$, $s = 1,2,\ldots N$. Without loss of generality one can choose $s_q = s$, because q denotes the particular row concerned only.

In view of Eqs. (3.37,45), the desired result Eq. (3.40) is equivalent to demonstrating that

$$\det \underset{\sim}{\Gamma}_{mpp'}(\nu) = 0 \qquad \nu\varepsilon(n), \tag{3.46}$$

if $m > N - n + 1$. In Eq. (3.44)

$$h_{s_q}(\nu) = 0 \tag{3.47}$$

if $s_q < n$, in which case the determinant vanishes because all elements of a given row are zero.

Because s_q takes on values $1,2,\ldots N$, there are obviously $N - n + 1$ rows where $h_{s_q} \neq 0$. In addition, in view of Eqs. (3.43,44) it follows that $N - n + 1$ is the maximum number of $\underset{\sim}{\Gamma}$ rows of $\underset{\sim}{\Gamma}_{mpp'}(\nu)$, which are not proportional. Recalling that m is the total number of such rows, it follows that, if $m > N - n + 1$, there are at least two proportional rows included in $\underset{\sim}{\Gamma}_{mpp'}(\nu)$ and Eq. (3.46) results immediately.

Once the degree of degeneracy is established the notation is modified accordingly. The general expression of the eigenfunction $\underset{\sim}{\phi}_j^{(n)}(\nu,\mu)$, $\nu\varepsilon(n)$, where j denotes the degeneracy, can now be written as (cf. Eq. (3.27))

$$\underset{\sim}{\phi}_j^{(n)}(\nu,\mu) = \left[\underset{\sim}{D}(\nu,\mu) + \lambda_j^{(n)}(\nu)\underset{\sim}{\delta}^{(n)}(\nu,\mu)\right]\sum_{i=1}^{M} \underset{\sim}{L}_i(\mu)\underset{\sim}{n}_{ij}^{(n)}(\nu) \tag{3.48}$$

$$\nu\varepsilon(n),\ n = 1,2,3,\ldots N,\ j = 1,2,\ldots N - n + 1,$$

where $\underset{\sim}{D}(\nu,\mu)$, $\underset{\sim}{\delta}^{(n)}(\nu,\mu)$ and $\underset{\sim}{n}_{ij}^{(n)}(\nu)$ are defined in Eqs. (3.28-30) and $\lambda_j^{(n)}(\nu)$ is a root obtained from Eq. (3.35). For the purposes of the rest of this work it is irrelevant to know explicitly the $\lambda_j^{(n)}(\nu)$'s $j = 1,2,\ldots N - n + 1$. However, later in this section the calculation of the corresponding functions will be demonstrated for a particular combination of the eigensolutions.

Again, Eq. (3.33) defines uniquely the ratio of the NM components of the vector $\underset{\sim}{n}_k^{(n)}(\nu)$, $k = 1,2,\ldots N - n + 1$. An explicit expression is obtained in analogy with Eq. (3.15);

$$\left(\underset{\sim}{n}_k^{(n)}(\nu)\right)_j = \left(\underset{\sim}{\Gamma}_+(\nu) - \lambda_k^{(n)}(\nu)\underset{\sim}{\Gamma}_-(\nu)\right)^{ij} \tag{3.49}$$

$j = 1,2,\ldots NM$ and $i \le NM$, the upper index ij denoting a cofactor.

For later reference, Eq. (3.33) is manipulated into a convenient form in terms of the individual blocks. Including the degeneracy, Eq. (3.33) can be written in the form of N - n + 1 equations

$$\sum_{\ell=1}^{M}\left\{[\Gamma_+(\nu)]_{k\ell} - \lambda_j^{(n)}(\nu)[\Gamma_-(\nu)]_{k\ell}\underset{\sim}{n}_{\ell j}^{(n)}(\nu)\right\} = \underset{\sim}{0}\ , \tag{3.50}$$

$$\nu\varepsilon(n),\ k \le M,\ j \le N - n + 1.$$

Simultaneous consideration of all degrees of degeneracy suggests defining a diagonal N - n + 1 x N - n + 1 matrix $\underset{\sim}{\lambda}^{(n)}(\nu)$ by

$$\left(\underset{\sim}{\lambda}^{(n)}(\nu)\right)_{jj} = \lambda_j^{(n)}(\nu), \tag{3.51}$$

and a N - n + 1 x N matrix $\underset{\sim}{N}_\ell^{(n)}(\nu)$;

$$\left(\underset{\sim}{N}_\ell^{(n)}(\nu)\right)_{ij} = \left(\underset{\sim}{n}_{\ell i}^{(n)}(\nu)\right)_j \qquad i \le N - n + 1,\ j \le N. \tag{3.52}$$

Eq. (3.50) now has the form

$$\sum_{\ell=1}^{M} [\underset{\sim}{\Gamma}_+(\nu)]_{k\ell}\underset{\sim}{N}_\ell^{(n)}(\nu) = \sum_{\ell=1}^{M} [\underset{\sim}{\Gamma}_-(\nu)]_{k\ell}\underset{\sim}{N}_\ell^{(n)}(\nu)\underset{\sim}{\lambda}^{(n)}(\nu), \qquad \nu\varepsilon(n),\ k = 1,2,\ldots M. \tag{3.53}$$

which will be employed in the following chapter. Some other pertinent consequences, viz. certain questions of reducibility in a special case are discussed in Appendix A.

As has been mentioned earlier, one prefers a particular combination of the eigensolutions in order to obtain explicit form of $\lambda_j^{(n)}(\nu)$. In terms of Eq. (3.48), this can be accomplished by changing $\underset{\sim}{\delta}^{(n)}(\nu,\mu)$ to $\underset{\sim}{\delta}_j^{(n)}(\nu,\mu)$, defined as[19-21]

$$\left(\underset{\sim}{\delta}_j^{(n)}(\nu,\mu)\right)_{ii} = \left(\underset{\sim}{\delta}^{(n)}(\nu,\mu)\right)_{ii}\delta_{ij}, \tag{3.54}$$

where $\underset{\sim}{\delta}^{(n)}(\nu,\mu)$ was defined in Eq. (3.31); $\lambda_j^{(n)}(\nu)$ is then determined by the same procedure as discussed above. In particular, it can be shown[20] that the $\lambda_i^{(n)}(\nu)$ can be uniquely determined and, in fact, the method of Ref. 20 is applicable to the derivation of the explicit form.

Adjoint Spectrum

The adjoint operator and its eigensolutions corresponding to the direct eigensolutions displayed in Eqs. (3.7,48) will have a prominent role in the subsequent considerations. The completeness of the eigenfunctions and the useful full-range orthogonality are, in general, demonstrated in terms of both direct and adjoint sets. For these purposes certain aspects of the adjoint problem are discussed in this section principally reproducing the results of Ref. 18. It appears unlikely at the present stage of the general theory of linear operators that the main results of this section could be easily derived from the established theorems on linear spaces.

Displaying the multigroup Boltzmann equation (Eq. (2.13)) in an operator form

$$B\underset{\sim}{\psi}(x,\mu) = \underset{\sim}{0}, \tag{3.55}$$

the formal adjoint operator $B^{\dagger}$ is defined by functional equality[22]

$$(\underset{\sim}{\phi}, B\underset{\sim}{\psi}) = (B^{\dagger}\underset{\sim}{\phi}, \underset{\sim}{\psi}), \tag{3.56}$$

where

$$(\underset{\sim}{\phi}, \underset{\sim}{\psi}) = \int_{-\infty}^{\infty} dx \int_{-1}^{1} d\mu \underset{\sim}{\phi}^{T}(x,\mu)\underset{\sim}{\psi}(x,\mu). \tag{3.57}$$

The explicit form Eq. (2.13) is substituted in the left hand side of Eq. (3.56) and the formal inner product is expressed as an integral. The integration involving the streaming term $\mu \frac{\partial \underset{\sim}{\psi}}{\partial x}$ is performed by parts noticing that $\underset{\sim}{\psi}\underset{\sim}{\psi}^{\dagger}$ vanishes at $x = \pm\infty$. The removal term is trivial, and the manipulation in the scattering term consists of relabeling μ and

μ'. Replacing $\underset{\sim}{\phi}(x,\mu)$ by the adjoint angular particle density $\underset{\sim}{\psi}^{\dagger}(x,\mu)$, the adjoint equation can be written as

$$-\mu \frac{\partial}{\partial x} \underset{\sim}{\psi}^{\dagger}(x,\mu) + \underset{\sim}{\Sigma}\underset{\sim}{\psi}^{\dagger}(x,\mu) = \int_{-1}^{1} \underset{\sim}{K}^{T}(\mu',\mu)\underset{\sim}{\psi}^{\dagger}(x,\mu')d\mu'. \tag{3.58}$$

In general $\underset{\sim}{\Sigma}^{T}$ would appear, but the diagonality of $\underset{\sim}{\Sigma}$ is utilized in the expression Eq. (3.58). If general theory were used in the subsequent analysis, it should also be demonstrated that the domain of B is dense.[33] This can be readily proven.[26]

Corresponding to Eq. (3.3) the variables are now separated by

$$\underset{\sim}{\psi}^{\dagger}(x,\mu) = e^{x/\nu}\underset{\sim}{\phi}^{\dagger}(x,\mu), \tag{3.59}$$

and the resulting eigenvalue equation has the form (cf. 3.4)

$$(\underset{\sim}{\Sigma} - \mu/\nu\ \underset{\sim}{I})\underset{\sim}{\phi}^{\dagger}(\nu,\mu) = \int_{-1}^{1} \underset{\sim}{K}^{T}(\mu',\mu)\underset{\sim}{\phi}^{\dagger}(\nu,\mu')d\mu'. \tag{3.60}$$

The rest of the present treatment depends on the following lemma.

Lemma 3.2 Let $\underset{\sim}{\Lambda}^{\dagger}(z)$ denote the dispersion matrix of the adjoint operator defined in Eq. (3.60), then

$$\underset{\sim}{\Lambda}^{\dagger}(z) = \underset{\sim}{\Lambda}^{T}(z), \tag{3.61}$$

and consequently

$$\Omega^{\dagger}(z) = \Omega(z), \tag{3.62}$$

where $\underset{\sim}{\Lambda}(z)$ is defined in Eq. (3.9) and $\Omega(z)$ in Eq. (3.13).

Proof: Comparing Eqs. (3.4) and (3.60) it follows from Eq. (3.2) that an arbitrary matrix block of $\underset{\sim}{\Lambda}^{\dagger}(z)$ has the form

$$[\underset{\sim}{\Lambda}^{\dagger}(z)]_{ij} = \underset{\sim}{I}\delta_{ij} - \int_{-1}^{1} \underset{\sim}{L}_i^T(\mu)\underset{\sim}{D}(z,\mu)\underset{\sim}{M}_j^T(\mu)d\mu. \tag{3.63}$$

Noticing that $\underset{\sim}{D}(z,\mu)$ is a diagonal matrix, see Def. (3.5,6), it can be immediately concluded that

$$[\underset{\sim}{\Lambda}^{\dagger}(z)]_{ij} = [\underset{\sim}{\Lambda}(z)]_{ji}^T, \tag{3.64}$$

which is equivalent to the claim in Eq. (3.61). Furthermore, taking the determinants in Eq. (3.61) one obtains Eq. (3.62).

An important theorem can be now proven utilizing this lemma. The proof was first given in Ref. 18.

<u>Theorem 1</u>. The eigenvalue spectra of the transport operator and its adjoint are identical.

<u>Proof</u>: The discrete eigenvalue ν_i of the direct operator is obtained from Eq. (3.14), i.e.

$$\Omega(\nu_k) = 0. \tag{3.14}$$

From the identity (3.64) it follows immediately that ν_i also belongs to the discrete spectrum of the adjoint operator.

The identity of the continuous spectra follows also from Eq. (3.64); the argument being that the continuous spectrum is defined by the branch cut of $\underset{\sim}{\Lambda}(z)$ which naturally coincides with that of $\underset{\sim}{\Lambda}^T(z)$.

Although the occurance of a discrete eigenvalue imbedded in the continuous spectrum would need some special consideration, it is obvious that this theorem is still valid. As it was mentioned earlier, any further discussion of the consequences in such a case will be omitted in this work, while a special case has been treated in detail in Ref. 9.

The following lemma is confined to the distributional part of the eigensolutions.

Lemma 3.3 Let $\underset{\sim}{\phi}_j^{(n)\dagger}(\nu,\mu)$ be an eigensolution of the adjoint operator corresponding to an eigenvalue in the continuous spectrum, i.e.

$$\underset{\sim}{\phi}_j^{(n)\dagger}(\nu,\mu) = \left[\underset{\sim}{D}(\nu,\mu) + \lambda_j^{(n)\dagger}(\nu)\underset{\sim}{\delta}^{(n)}(\nu,\mu)\right]\sum_{i=1}^{M}\underset{\sim}{M}_i^T(\mu)\underset{\sim}{n}_{ij}^{(n)\dagger}(\nu), \quad (3.65)$$

cf. Eq. (3.48); then

$$\lambda_j^{(n)\dagger}(\nu) = \lambda_j^{(n)}(\nu). \quad (3.66)$$

Proof: By Def. (3.35) $\lambda_j^{(n)\dagger}(\nu)$ is a root of the equation

$$\det\left[\underset{\sim}{\Gamma}_+^T(\nu) - \lambda^{(n)\dagger}(\nu)\underset{\sim}{\Gamma}_-^T(\nu)\right] = 0, \qquad \nu\varepsilon(n), \quad (3.67)$$

where the identity (3.61) is employed. Transposing the matrix in (3.67) one obtains

$$\det\left[\underset{\sim}{\Gamma}_+(\nu) - \lambda^{(n)\dagger}(\nu)\underset{\sim}{\Gamma}_-(\nu)\right] = 0 \quad (3.68)$$

which yields the claim (3.66) when Eq. (3.35) is used.

IV COMPLETENESS OF THE EIGENSOLUTIONS

An inherent necessity in admitting eigensolutions of the forms exhibited in the previous section is to prove the completeness property of this particular set of generalized functions. It is customary to formulate the problem in terms of demonstrating under which conditions an arbitrary function $\underset{\sim}{\psi}(\mu)$ can be represented by a unique eigenfunction expansion. The expansion is cast in a general form of an integral[20]

$$\underset{\sim}{\psi}(\mu) = \int_L \alpha(\nu)\underset{\sim}{\phi}(\nu,\mu)dm(\nu), \tag{4.1}$$

where the region of integration, L, depends on the particular boundary conditions imposed on the solution. The problem is then to determine the conditions on the function $\underset{\sim}{\psi}(\mu)$ and on the operator which permit the existence of a unique function $\alpha(\nu)$ in a region L with appropriate boundary values on $\underset{\sim}{\psi}(\mu)$. Once this problem has been solved the general angular energy and space dependent solution of Eq. (2.13) can be expressed as an eigensolution expansion.

Interpreting the equivalent expression to Eq. (4.1) in more explicit terms, the expansion has the form

$$\underset{\sim}{\psi}(x,\mu) = \sum_{i=1}^{n_L} \alpha_i e^{-x/\nu_i}\, \underset{\sim}{\phi}(\nu_i,\mu)$$

$$+ \sum_{n=1}^{N} \sum_{i=1}^{N-n+1} \int_{(n)_L} \alpha_i^{(n)}(\nu)e^{-x/\nu}\underset{\sim}{\phi}_i^{(n)}(\nu,\mu)d\nu, \tag{4.2}$$

where the different regions and the degeneracy have been taken into account. The discrete and continuous modes were defined in Eqs. (3.7, 48); α_i and $\alpha_i^{(n)}$ are the expansion coefficients, n_L the number of the discrete eigenfunctions involved in any particular case and $(n)_L$ denotes a portion of the region (n) (see Eq. (3.25)) specified by the problem concerned.

Solving the problem amounts then to finding the expansion coefficients. This can be done analytically for the infinite medium problems whereas at the present time, semi-infinite and bounded medium problems necessitate numerical techniques. In that case completeness can be used to argue uniqueness of the numerical solution.

The completeness proof is first reduced to the form of the Hilbert boundary value problem. The transformation across the cut L is represented by a matrix composed of the boundary values of the dispersion matrix.

In case an infinite medium problem is examined, $\mu \varepsilon L = [-1,1]$, the eigensolutions possess the completeness properties under relatively weak restrictions. The partial range problems, where L is a proper subinterval of [-1,1], require special consideration. While the present method of deduction is applicable to an arbitrary partial range, only half-range completeness is discussed in detail. In fact, the half-range case L = [0,1] or [-1,0] is the only application of this kind introduced so far.[4] Furthermore, this particular instance involves *per se* some special simplifications. However, the proof cannot be made constructive in the sense that the expansion coefficients would be solved.

Reduction of the Proof to a Hilbert Problem

The usual method of proof is to attempt an expansion of an arbitrary function $\underset{\sim}{\psi}(\mu)$ satisfying the extended Hölder condition[4] in terms of the continuous modes alone;

$$\underset{\sim}{\psi}(\mu) = \sum_{n=1}^{N} \sum_{j=1}^{N-n+1} \int_{(n)_L} \alpha_j^{(n)}(\nu) \underset{\sim}{\phi}_j^{(n)}(\nu,\mu) d\nu, \qquad \mu \varepsilon L, \tag{4.3}$$

where L is one of the intervals [-1,1], [-1,0] or [0,1] and

$$(n)_L = (n) \cap L, \tag{4.4}$$

where the concept of the subinterval (n) was introduced in Eq. (3.25). Substituting in Eq. (4.3) the explicit expression of $\underset{\sim}{\phi}_i^{(n)}(\nu,\mu)$ from Eq. (3.48), one obtains after some algebra

$$\underset{\sim}{\psi}(\mu) = \sum_{n=1}^{N} \sum_{i=1}^{M} \int_{(n)_L} \{\underset{\sim}{D}(\nu,\mu) \underset{\sim}{L}_i(\mu) \underset{\sim}{N}_i^{(n)}(\nu) \underset{\sim}{\alpha}^{(n)}(\nu)$$

$$+ \underset{\sim}{\delta}^{(n)}(\nu,\mu) \underset{\sim}{L}_i(\mu) \underset{\sim}{N}_i^{(n)}(\nu) \underset{\sim}{\lambda}^{(n)}(\nu) \underset{\sim}{\alpha}^{(n)}(\nu)\} d\nu, \qquad \mu \varepsilon L \tag{4.5}$$

where

$$\underset{\sim}{\alpha}^{(n)}(\nu) = \left(\alpha_1^{(n)}(\nu), \ldots \alpha_{N-n+1}^{(n)}(\nu)\right)^T, \tag{4.6}$$

and the matrices $\underset{\sim}{N}_i^{(n)}(\nu)$ and $\underset{\sim}{\lambda}^{(n)}(\nu)$ are defined in Eqs. (3.51,52).

The δ-function term in Eq. (4.5) would not cause any special difficulty in subsequent calculations. The integral term involving a Cauchy type kernel $\underset{\sim}{D}(\nu,\mu)$ contributes to the mathematical difficulties and differently from Ref. 11, the system of integral equations is transformed to the form of a dominant system.[24] Consequently, the rationale for the following manipulations is to transform Eq. (4.5) into a form where the matrices corresponding to $\underset{\sim}{D}(\nu,\mu)$ and $\underset{\sim}{L}_i(\mu)$ commute. This is achieved by a change of the variable μ to μ/σ_i in the ith equation of the system. The different change in each equation is also urged by notational convenience.[11,34]

In order to elaborate the change of the variable, an arbitrary N-component vector $\underset{\sim}{\xi}(\mu)$ is considered. Defining $\underset{\sim}{\xi}(\mu)$ by

$$\underset{\sim}{\xi}(\mu) = \int_{(n)_L} \left[\underset{\sim}{D}(\nu,\mu)\underset{\sim}{\beta}(\nu,\mu) + \underset{\sim}{\delta}^{(n)}(\nu,\mu)\underset{\sim}{\gamma}(\nu,\mu)\right]d\nu, \qquad \mu\varepsilon L \tag{4.7}$$

where $\underset{\sim}{\beta}(\nu,\mu)$ and $\underset{\sim}{\gamma}(\nu,\mu)$ are certain vectors, one obtains a general form of Eq. (4.5). Recalling Defs. (3.5,31) the ith component obeys the equation

$$\begin{aligned}\left[\underset{\sim}{\xi}(\mu)\right]_i &= \int_{(n)_L} \left\{\frac{\nu}{\sigma_i\nu-\mu}\left[\underset{\sim}{\beta}(\nu,\mu)\right]_i + \delta(\sigma_i\nu - \mu)(\underset{\sim}{\gamma}(\nu,\mu))_i\right\}d\nu \qquad i \geq n \\ &= \int_{(n)_L} \frac{\nu}{\sigma_i\nu-\mu}\left[\underset{\sim}{\beta}(\nu,\mu)\right]_i d\nu \qquad i < n, \qquad \mu\varepsilon L.\end{aligned} \tag{4.8}$$

Performing the integration over the δ-function and letting $\mu\to\mu/\sigma_i$, Eq. (4.8) has the form

$$\begin{aligned}\sigma_i\left[\underset{\sim}{\xi}(\sigma_i\mu)\right]_i h_i(\mu) &= \int_{(n)_L} \frac{\nu}{\nu-\mu}\left[\underset{\sim}{\beta}(\nu,\sigma_i\mu)\right]_i h_i(\mu)d\nu \\ &\quad + \left[\underset{\sim}{\gamma}(\mu,\sigma_i\mu)\right]_i g_n(\mu)h_i(\mu),\end{aligned} \tag{4.9}$$

where

$$\begin{aligned} g_n(\mu) &= 1 \text{ if } \mu\varepsilon(n)_L \\ &= 0 \text{ otherwise}\end{aligned} \tag{4.10}$$

and $h_n(\mu)$ (see Def. (3.16)) is again introduced to retain the arguments of angular dependent functions in the domain [-1,1].

When the change of the variable is performed in Eq. (4.5) and all the stages in Eqs. (4.7-10) have been taken into account, it is convenient to define a N-component vector $\underset{\sim}{\psi}_\sigma(\mu)$ by

$$\left[\underset{\sim}{\psi}_\sigma(\mu)\right]_i = \left[\underset{\sim}{\psi}(\sigma_i\mu)\right]_i h_i(\mu) \tag{4.11}$$

in order to express Eq. (4.5) in the form

$$\begin{aligned}\underset{\sim}{\underset{\sim}{\Sigma}}\underset{\sim}{\psi}_\sigma(\mu) = \sum_{n=1}^{N}\sum_{i=1}^{M}\Bigg\{&\int_{(n)_L}\frac{\nu}{\nu-\mu}\,\underset{\sim}{L}_{\sigma i}(\mu)\underset{\sim}{N}_i^{(n)}(\nu)\underset{\sim}{\alpha}^{(n)}(\nu)d\nu \\ &+ \underset{\sim}{L}_{\sigma i}(\mu)\underset{\sim}{N}_i^{(n)}(\mu)\underset{\sim}{\lambda}^{(n)}(\mu)\alpha^{(n)}(\mu)g_n(\mu)\Bigg\},\end{aligned} \tag{4.12}$$

where $\underset{\sim}{L}_{\sigma i}(\mu)$ was defined in Eq. (3.21). $\underset{\sim}{L}_{\sigma i}(\mu)$ can now be removed from under the integral sign and Eq. (4.12) has the form

$$\begin{aligned}\underset{\sim}{\Sigma}\underset{\sim}{\psi}_\sigma(\mu) = \sum_{i=1}^{M}\underset{\sim}{L}_{\sigma i}(\mu)\Bigg\{&\int_L\frac{\nu}{\nu-\mu}\sum_{n=1}^{N}\underset{\sim}{N}_i^{(n)}(\nu)\underset{\sim}{\alpha}^{(n)}(\nu)g_n(\nu)d\nu \\ &+ \sum_{n=1}^{N}\underset{\sim}{N}_i^{(n)}(\mu)\underset{\sim}{\lambda}^{(n)}(\mu)\underset{\sim}{\alpha}^{(n)}(\mu)g_n(\mu)\Bigg\}.\end{aligned} \tag{4.13}$$

The integral terms appearing in the expression above consist of singular and regular parts. Despite the non-singular contribution, the system of equations Eq. (4.13) can be converted into a Hilbert problem and the solution can be obtained in principle by a familiar technique used in the theory of singular integral equations.

In analogy with Ref. 21 a simplified method is amenable to the conversion procedure. Eq. (4.13) is multiplied by $\underset{\sim}{M}_i^{\sigma}(\mu)$ (Def. (3.20)), $i = 1,2,\ldots M$. Defining a N-component vector $\underset{\sim}{\xi}_i(\mu)$ by

$$\underset{\sim}{\xi}_i(\mu) = \underset{\sim}{M}_i^{\sigma}(\mu)\underset{\sim}{\Sigma}\underset{\sim}{\psi}_{\sigma}(\mu), \tag{4.14}$$

the resulting system of equations has the form

$$\begin{aligned}\underset{\sim}{\xi}_i(\mu) = \sum_{j=1}^{M}[\underset{\sim}{\Gamma}_-(\mu)]_{ij}\Bigg\{\int_L \frac{\nu}{\nu-\mu}\sum_{n=1}^{N}\underset{\sim}{N}_j^{(n)}(\nu)\underset{\sim}{\alpha}^{(n)}(\nu)g_n(\nu)d\nu \\ + \sum_{n=1}^{N}\underset{\sim}{N}_j^{(n)}(\mu)\underset{\sim}{\lambda}^{(n)}(\mu)\underset{\sim}{\alpha}^{(n)}(\mu)g_n(\mu)\Bigg\},\end{aligned} \tag{4.15}$$

where $\underset{\sim}{\Gamma}_-(\mu)$ was defined in Eq. (3.24). Alternatively, noticing Eq. (3.53), Eq. (4.15) can be written as

$$\begin{aligned}\underset{\sim}{\xi}_i(\mu) = \sum_{j=1}^{M}\Big\{[\underset{\sim}{\Gamma}_-(\mu)]_{ij}\int_L \frac{\nu}{\nu-\mu}\underset{\sim}{\rho}_j(\nu)d\nu \\ + [\underset{\sim}{\Gamma}_+(\mu)]_{ij}\underset{\sim}{\rho}_j(\mu)\Big\},\end{aligned} \tag{4.16}$$

where

$$\underset{\sim}{\rho}_j(\nu)g_n(\nu) = \underset{\sim}{N}_j^{(n)}(\nu)\underset{\sim}{\alpha}^{(n)}(\nu). \tag{4.17}$$

The transformation (4.17) indicates the N-component vector $\underset{\sim}{\rho}_j(\nu)$ has only N-n+1 independent components when $\nu\varepsilon(n)$. In a sense the transformation is unique if $\underset{\sim}{N}_j^{(n)}(\nu)$ possesses a square non-singular submatrix of rank N-n+1.

In order to remove ambiguity on this point, Eq. (4.13) is analyzed in more detail. Letting $\mu\varepsilon(n)_L$ in Eq. (4.13), it follows from Defs. (3.21), (4.11) that $\underset{\sim}{\psi}_{\sigma}(\mu)$ has at most N-n+1 non-zero components and

$\underset{\sim}{L}_{\sigma i}(\mu)$ N-n+1 non-zero rows. In view of Eq. (4.17) only the components $\left(\rho_i(\nu)\right)_k$, $k \geq n$, appear in the term involving the integral. The second term of the right hand side of Eq. (4.13) formally contains all the components of $\underset{\sim}{\rho}_i(\nu)$ but the first n components can be eliminated because of the linear dependence. Including the n-1 first components of $\underset{\sim}{\rho}_j(\nu)$ in $\tilde{\underset{\sim}{\rho}}_j(\nu)$ and the rest N-n+1 components in $\bar{\underset{\sim}{\rho}}_j(\nu)$, i.e.,

$$\left(\rho_j(\nu)\right)_k = \begin{cases} \left(\tilde{\rho}_j(\nu)\right)_k & k < n\,, \\ \left(\bar{\rho}_j(\nu)\right)_{k-n+1} & k \geq n, \end{cases} \tag{4.18}$$

and similarly deleting the first n-1 rows of the matrix $\underset{\sim}{N}_j^{(n)}(\nu)$, i.e.,

$$\left(N_j^{(n)}(\nu)\right)_{k\ell} = \begin{cases} \left(\tilde{\underset{\sim}{N}}_j^{(n)}(\nu)\right)_{k\ell} & k < n\,, \\ \left(\bar{\underset{\sim}{N}}_j^{(n)}(\nu)\right)_{k-n+1,\ell} & k \geq n\,, \end{cases} \tag{4.19}$$

Equation (4.17) has the equivalent form

$$\begin{aligned} \tilde{\underset{\sim}{\rho}}_j(\nu) &= \tilde{\underset{\sim}{N}}_j^{(n)}(\nu)\underset{\sim}{\alpha}(\nu)\,, \\ \bar{\underset{\sim}{\rho}}_j(\nu) &= \bar{\underset{\sim}{N}}^{(n)}(\nu)\underset{\sim}{\alpha}(\nu). \end{aligned} \tag{4.20}$$

Therefore, the transformation is unique if the matrices $\bar{\underset{\sim}{N}}_j^{(n)}(\nu)$, $n \leq N$, $j \leq M$, are non-singular and any possible solution of the system of N equations in Eq. (4.13) yields only the last N-n+1 components of $\underset{\sim}{\rho}_i(\nu)$ in a linearly independent manner. Furthermore, there are only N-n+1 linearly independent equations in Eq. (4.16).

A system of NM simultaneous equations is obtained when Eq. (4.16) is rewritten to include all values of $i \leq M$. For this purpose NM-component vectors $\underset{\sim}{\xi}(\mu)$ and $\underset{\sim}{\rho}(\mu)$ are constructed from the auxiliary vectors $\underset{\sim}{\xi}_i(\mu)$ and $\underset{\sim}{\rho}_i(\mu)$, respectively, by defining in general

$$\underset{\sim}{\alpha}(\mu) = \left[\left(\underset{\sim}{\alpha}_1(\mu)\right)_1, \left(\underset{\sim}{\alpha}_1(\mu)\right)_2, \cdots \left(\underset{\sim}{\alpha}_M(\mu)\right)_N\right]^T . \tag{4.21}$$

With these definitions Eq. (4.16) can be written

$$\underset{\sim}{\xi}(\mu) = \underset{\sim}{\Gamma}_-(\mu)\int_L \frac{\nu}{\nu-\mu}\, \underset{\sim}{\rho}(\nu)d\nu + \underset{\sim}{\Gamma}_+(\mu)\underset{\sim}{\rho}(\mu), \tag{4.22}$$

$$\mu\varepsilon L.$$

Finally, introducing the boundary values $\underset{\sim}{\phi}^{\pm}(\mu)$ of a sectionally holomorphic vector $\underset{\sim}{\phi}(z)$,[23] where

$$\underset{\sim}{\phi}(z) = \frac{1}{2\pi i}\int_L \frac{\nu}{\nu-z}\, \underset{\sim}{\rho}(\nu)d\nu, \tag{4.23}$$

one obtains the customary factorization of Eq. (4.22);

$$\mu\underset{\sim}{\xi}(\mu) = \underset{\sim}{\Lambda}^+(\mu)\underset{\sim}{\phi}^+(\mu) - \underset{\sim}{\Lambda}^-(\mu)\underset{\sim}{\phi}^-(\mu) , \tag{4.24}$$

where Eqs. (3.23,34) are employed when substituting $\underset{\sim}{\Lambda}^{\pm}(\mu)$ for $\underset{\sim}{\Gamma}_{\pm}(\mu)$. In terms of classical mathematical literature Eq. (4.24) is cast into a form of the Hilbert problem

$$\underset{\sim}{\phi}^+(\mu) = \underset{\sim}{g}(\mu)\underset{\sim}{\phi}^-(\mu) + \underset{\sim}{f}(\mu), \tag{4.25}$$

$$\mu\varepsilon L.$$

where the transformation matrix $\underset{\sim}{g}(\mu)$ is

$$\underset{\sim}{g}(\mu) = \left(\underset{\sim}{\Lambda}^+(\mu)\right)^{-1}\underset{\sim}{\Lambda}^-(\mu) , \tag{4.26}$$

and the inhomogeneous term $\underset{\sim}{f}(\mu)$ has the form

$$\underset{\sim}{f}(\mu) = \mu\left(\underset{\sim}{\Lambda}^+(\mu)\right)^{-1}\underset{\sim}{\xi}(\mu). \tag{4.27}$$

It has been implicitly assumed above that $\underset{\sim}{\Lambda}^{\pm}(\mu)$ is a nonsingular matrix.

The question of completeness was studied by demonstrating under which conditions an arbitrary function $\underset{\sim}{\psi}(\mu)$ can be uniquely expanded in terms of the continuous modes. The N-component $\underset{\sim}{\psi}(\mu)$ is now included in the NM-component $\underset{\sim}{f}(\mu)$ and the original expansion coefficients $\alpha_i^{(n)}(\nu)$ are transformed to the components of $\underset{\sim}{\rho}(\nu)$, which can be obtained from the boundary values of $\underset{\sim}{\phi}(z)$ by

$$\nu\underset{\sim}{\rho}(\nu) = \underset{\sim}{\phi}^{+}(\nu) - \underset{\sim}{\phi}^{-}(\nu), \ \nu\epsilon L. \tag{4.28}$$

By the previous discussion it is observed that $\underset{\sim}{\rho}(\nu)$ contains only N-n+1 independent components whenever $\nu\epsilon(n)$. The follows from Def (4.21) and the fact that Eq. (4.20) implies

$$\bar{\underset{\sim}{\rho}}_i(\nu) = \underset{\sim}{\bar{N}}_i^{(n)}(\nu)\left(\underset{\sim}{\bar{N}}_j^{(n)}(\nu)\right)^{-1}\bar{\underset{\sim}{\rho}}_j(\nu); \tag{4.29}$$

$$\nu\epsilon(n).$$

In finding $\underset{\sim}{\phi}(z)$ the intrinsic nature of the problem requires certain conditions of continuity on both the matrix $\underset{\sim}{g}(\mu)$ and the admissible function $\underset{\sim}{f}(\mu)$. Further restrictions arise because of the special analyticity requirements on $\underset{\sim}{\phi}(z)$, viz $\underset{\sim}{\phi}(z)$ is analytic in the entire complex plane except the cut L and $\underset{\sim}{\phi}(z) \sim 1/z$ as $|z|\to\infty$. These restrictions facilitate the determination of the discrete expansion coefficients or, in the half-range case, whether there exists a proper number of linearly independent discrete eigenfunctions. All these questions will be examined in a manner developed recently[19,20] for special cases of this problem. Before any further restrictions some general remarks will be made on the present status of the theory of Hilbert-problem

In order for the conventional theory to applicable, the inhomogeneous term $\underset{\sim}{f}(\mu)$ in Eq. (4.25) has to belong to class H_ε, i.e., it is required that $\underset{\sim}{f}(\mu)$ satisfies the Hölder condition everywhere on L except possibly at a finite number of points μ_i where

$$\lim_{\mu\to\mu_i} |\mu-\mu_i|^\varepsilon \underset{\sim}{f}(\mu_i) = \underset{\sim}{0}. \tag{4.30}$$

The class of admissible functions has been extended[4] to cover appropriate generalized functions, e.g., δ-functions. Therefore Green's functions can be constructed, as will be illustrated in Chapter V.

The essential problem is to establish the existence of a non-singular fundamental matrix $\underset{\sim}{X}(z)$ whose boundary conditions obey the equation

$$\underset{\sim}{g}(\mu) = \underset{\sim}{X}^+(\mu)\left(\underset{\sim}{X}^-(\mu)\right)^{-1}, \tag{4.31}$$

$$\mu\varepsilon L.$$

If Eq. (4.31) were soluble, Eq. (4.25) could be solved immediately,

$$\underset{\sim}{\phi}(z) = \frac{1}{2\pi i} X(z)\left\{\int \left(\underset{\sim}{X}^+(\mu)\right)^{-1} \underset{\sim}{f}(\mu)\frac{\mu}{\mu-z} + \underset{\sim}{p}(z)\right\}, \tag{4.32}$$

where $\underset{\sim}{p}(z)$ is an arbitrary vector whose components are polynomials. Regardless of the solubility of Eq. (4.31) sufficient conditions can be derived which imply that a matrix $\underset{\sim}{X}(z)$ exists. The original theory of Muskhelisvili and Vekua assumes that the elements of the transformation matrix $\underset{\sim}{g}(\mu)$ obey the Lipschitz condition everywhere on L with the exception of a finite number of points where they may have discontinuities of the first kind. This condition is necessitated since the theory of Fredholm equations has been utilized in deducing the existence of the canonical matrix. The conditions are again far too severe to permit practical application of the theory. However, in case the Hilbert problem has continuous coefficients, i.e., $\underset{\sim}{g}(\mu)$ is continuous and obtains the same limit values at the endpoints of L, Mandzhavidze and Khvedelidze[25] have shown that the Lipschitz condition can be relaxed and mere continuity suffices to ensure the existence of the $\underset{\sim}{X}$-matrix. The method does not concern the classical theory but argues the proof using a certain sequence of matrices. It is at least conceivable that the stringent H-condition could be relaxed even in the cse of discontinuities, but no such work seems to be available. The cumbersome endpoint condition arises since the endpoints of L must be connected by a smooth curve to obtain a closed contour C, on which the boundary value problem is restated. The matrix $\underset{\sim}{g}(\mu)$ is continued to have a constant value on the appended portion of C. This constant matrix is chosen to be the identity matrix. In the full-range case the problem can be solved on the contour [-1,1].

The question of existence and uniqueness of a solution is finally reduced to considerations on the total index k of the problem. While the total index appears only in a fictitious manner, it is decomposed into two integer-valued numbers ℓ and m as

$$k = \ell - m, \tag{4.33}$$

where ℓ and m are related to the component indices or to the associate problem where the transformation is performed by $\left(g(\mu)^T\right)^{-1}$. All the details have been discussed in Ref. 20 and are reproduced in Appendix C.

The numbers ℓ and m are of essential importance in final conclusions, viz Eq. (4.25) will possess a solution $\underset{\sim}{\phi}(z)$ with previously specified analytic properties if the vector $\underset{\sim}{f}(\mu)$ satisfies m conditions of the form

$$\int_L \underset{\sim}{w}_i^T \underset{\sim}{f}(\mu)d\mu = 0 \,, \qquad i = 1,2,\cdots m, \tag{4.34}$$

where the vectors $\underset{\sim}{w}_i(\mu)$ are certain linearly independent quantities related to the $\underset{\sim}{X}$-matrix. The required degrees of freedom will be furnished by the subsidiary discrete expansion coefficients. In fact, the relations in Eq. (4.34) could be employed to determine the expansion coefficients if the $\underset{\sim}{X}$-matrix were known and thus the analytic forms of $\underset{\sim}{w}_i(\mu)$'s were available. The explicit expression are not required for the purposes of merely demonstrating that a unique set of expansion coefficients exists. Analytic solution can be obtained only in the full-range, and these aspects are discussed in the following section. The general solution will contain ℓ arbitrary constants appearing in a linear manner. These are the polynomial coefficients in Eq. (4.32). Therefore it is required to prove that $\ell = 0$, otherwise the problem defies a unique solution. This particular proof turns out to be the ordeal that actually eliminates a comprehensive consideration of an arbitrary partial-range case using the present formalism. The proof for the full- and half-range cases is given in Appendix C and involves only weak restrictions.

The index k is given by

$$k = \frac{1}{2\pi} \Delta_c \arg\left((z-z_o)^{-\rho}\det \underset{\sim}{g}(z)\right), \tag{4.35}$$

where

$$\begin{aligned} \underset{\sim}{g}(z) &= \underset{\sim}{g}(\mu) \text{ if } z = \mu\varepsilon L \\ &= \underset{\sim}{I} \text{ otherwise.} \end{aligned} \tag{4.36}$$

Δ_c arg denotes the change of argument of the operand as the contour C is traversed in the positive direction, and z_o is a point inside C. The number ρ is determined by the discontinuities of $g(\mu)$. The procedure is given in detail by Vekua and only the main points are repeated in Appendix B. ρ enters the index equation when the discontinuous problem is reduced to one with continuous coefficients. A priori, $\rho = 0$ if the Hilbert problem has continuous coefficients. Although all precedent kernels entail this property, it cannot apparently be asserted in general.

Referring to Eq. (4.35) and simultaneously observing the requirement $\ell = 0$ in Eq. (4.33), the number m can be calculated from

$$m = n + \rho \tag{4.37}$$

where n is given by

$$n = -\Delta_L \arg \det \underset{\sim}{g}(\mu). \tag{4.38}$$

Def. (4.36) was employed in obtaining Eq. (4.38). For later convenience it is necessary to consider the Hilbert problem pertaining to the proof of completeness of the adjoint set. Pursuant to Lemma 3.2 the transpose of the direct dispersion matrix occurs throughout the adjoint problem. Consequently the transformation matrix of the adjoint completeness proof denoted by $\underset{\sim}{h}(\mu)$ obeys the equation

$$\underset{\sim}{h}(\mu) = \left(\underset{\sim}{\Lambda}^{T+}(\mu)\right)^{-1}\underset{\sim}{\Lambda}^{T-}(\mu), \tag{4.39}$$

which is the counterpart of Eq. (4.27).

Since

$$\det \underset{\sim}{g}(\mu) = \det \underset{\sim}{h}(\mu), \tag{4.40}$$

the number n calculated from Eq. (4.38) remains unchanged if the direct expansion is substituted by the adjoint one. In case of an endpoint discontinuity the number ρ appearing in Eq. (4.37) remains also the same, because it depends only on the eigenvalues of $\underset{\sim}{g}(\mu)$ and $\underset{\sim}{h}(\mu)$, respectively. While there is no elementary matrix operation relating $\underset{\sim}{g}(\mu)$ and $\underset{\sim}{h}(\mu)$ it can be shown[35] that these matrices have the same eigenvalues. In fact, letting $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be arbitrary non-singular matrices, then the matrices $\underset{\sim}{A}\underset{\sim}{B}$ and $\underset{\sim}{A}^T\underset{\sim}{B}^T$ have the same eigenvalues (with identical multiplicity.) In particular this applies to $\underset{\sim}{g}(\mu)$ and $\underset{\sim}{h}(\mu)$.

With this requisite the completeness of the eigensolutions will be considered on full- and half-ranges. For subsequent reference the overall assumptions are summarized, viz

1. $\underset{\sim}{\Lambda}^{\pm}(\nu)$ is nonsingular, i.e.,

$$\Omega^{\pm}(\nu) \equiv \det \underset{\sim}{\Lambda}^{\pm}(\nu) \neq 0, \ \nu\varepsilon L, \tag{4.41}$$

2. $\det \underset{\sim}{\bar{N}}_j^{(n)}(\nu) \neq 0$

$$\nu\varepsilon(n), \ n \leq N, \ j \leq M, \tag{4.42}$$

and

3. $\underset{\sim}{f}(\nu)$ satisfies the extended Holder condition on $\nu\varepsilon L$.

Reportedly[24] condition 1 can be violated while the theory of

Muskhelisvili and Vekua is still valid.

Assuming the conditions 1-3 above the conclusions will be drawn in terms of the number ρ appearing in the index relation Eq. (4.37), and the restrictions imposed are to be regarded sufficient rather than necessary in character.

Full-Range Completeness

For the sake of coherence the general formalism is first applied to derive the sufficient conditions. Subsequently in this section alternative conclusions are based on a preferable deviant procedure whose analogue is customarily employed in this occasion.[4]

The associated discrete modes are formally introduced by the following Lemma which is virtually established in earlier studies.[4]

Lemma 4.1. Letting L correspond to the full-range, i.e., L = [-1.1], in Eq. (4.38) then n is equal to the number of discrete eigenfunctions.

Proof. Considering Eq. (4.38) in this special case

$$n = -\frac{1}{2\pi}\Delta_{-1,1} \arg \det \underset{\sim}{g}(\mu), \qquad (4.43)$$

and observing Defs. (3.13) and (4.26) the defining equation has the form

$$n = -\frac{1}{2\pi}\Delta_{-1,1} \arg (\Omega^{-}(\mu)/\Omega^{+}(\mu)). \qquad (4.44)$$

The number of the discrete eigenfunctions is the number of the zeros of the dispersion function $\Omega(z)$.

In order to employ the argument principle, the change of arg $\Omega(z)$ has to be calculated as z varies along the contour enclosing the plane.

As it has been noticed previously, $\Omega(z)$ is an analytic function in the entire complex plane cut from [-1,1] and, furthermore, $\Omega(z)$ assumes a constant value at infinity. Therefore a non-zero contribution may only be obtained from the path surrounding the cut [-1,1].

Rewriting Eq. (4.36) as

$$n = \frac{1}{2\pi} \Delta_{-1,1} \arg \Omega^{+}(\mu) + \frac{1}{2\pi} \Delta_{1,-1} \arg \Omega^{-}(\mu), \tag{4.45}$$

the desired proof is obtained instantaneously.

In view of Eq. (4.37), where m represents the number of conditions to be satisfied (ℓ=0 by the proof in Appendix C), and according to the preceeding lemma n is the number of undetermined coefficients available, it is required $\rho = 0$ for the eigensolutions to be complete. In case $\rho > 0$ the set is incomplete and it is overcomplete if $\rho > 0$.

Since the formalism has not been exhaustively applied to deduce the conditions of completeness in a given instance, it is appropriate to recapitulate the results established on the eigensolutions in Eq. (3.48).

Theorem 2. Assuming that the conditions 1-3, Eqs. (4.41,42) are valid, then the eigensolution are complete on the full range $\mu \varepsilon L = [-1,1]$; that is, a unique expansion of the form Eq. (4.1) exists if the elements of $\underset{\sim}{g}(\mu)$ are continuous functions and

$$\underset{\sim}{g}(\pm 1) = \underset{\sim}{I},$$

or if $\underset{\sim}{g}(\mu)$ is piecewise Lipschitz-continuous and the number calculated from the discontinuities of $\underset{\sim}{g}(\mu)$ is zero.

An alternative approach is based on Eq. (4.24). Observing that the branch cuts of $\underset{\sim}{\Lambda}(z)$ and $\underset{\sim}{\Lambda}(z)$ coincide onto [-1,1], one can define

$$\underset{\sim}{\eta}(z) = \underset{\sim}{\Lambda}(z)\underset{\sim}{\phi}(z), \tag{4.47}$$

and the equation has the form

$$\underset{\sim}{\eta}^{+}(\mu) - \underset{\sim}{\eta}^{-}(\mu) = \mu\underset{\sim}{\xi}(\mu), \tag{4.48}$$

yielding a solution

$$\underset{\sim}{\eta}(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{\mu\underset{\sim}{\xi}(\mu)}{\mu - z}\, d\mu. \tag{4.49}$$

The polynomial part corresponding to $\underset{\sim}{p}(z)$ in Eq. (4.32) can be argued to vanish by a trival application of the method displayed in Appendix C noticing that the transformation matrix has been reduced to the identity matrix.

The transformation in Eq. (4.47) is not invertable if there exists at least one discrete eigenvalue in the complex plane, because $\underset{\sim}{\Lambda}(\nu_i)$, $i \le n$, is singular. Defining

$$\underset{\sim}{\Lambda}^{-1}(z) = \left(\Omega(z)\right)^{-1}\underset{\sim}{\Lambda}^{c}(z) \tag{4.50}$$

it is required that $\underset{\sim}{\Lambda}^{c}(z)\eta(z)$ also vanishes at ν_i, $i \le n$. The discrete expansion coefficients could be determined from these conditions.[4,11]

Since the present formalism is unnecessarily involved for this purpose the calculation is omitted here. However, in the following chapter the results are obtained conveniently from the orthogonality relations. This section is concluded by summarizing these particular results in a theorem of a preferable form to Theorem 2.

Theorem 3. If the conditions 1 and 2 (Eqs. (4.41,42)) are full-filled, the set of eigensolutions is complete on the full-range in the sense that an arbitrary N-component vector $\underset{\sim}{\psi}(\mu)$, $\underset{\sim}{M}_i(\mu)\underset{\sim}{\psi}(\mu)$ $i \leq M$ satisfying the extended Hölder condition, can be uniquely expanded in terms of the eigensolutions.

Half-Range Completeness

The completeness property of the set of the eigensolutions must be extablished also on the half-range, $\mu \varepsilon L = [-1,0]$ or $[0,1]$, in order to permit the use of the expansion Eq. (4.2) in connection with semi-infinite or bounded medium transport problems. Physically meaningful situations always include boundaries.

While the mathematical formalism displayed in Section 4.1 has no such implications, the discussion will be restricted to situations where physical arguments require completeness on both intervals [-1,0] and [0,1] simultaneously. In this case it can be shown[19] that if $\rho = 0$ for the full-range then the corresponding numbers $\rho_{\pm}$ equal to zero on the half-ranges provided that the kernel is bounded at the origin, i.e.,

$$\lim_{\mu \to 0} \mu \underset{\sim}{M}_i(\mu)\underset{\sim}{L}_j(\mu) = \underset{\sim}{0}, \tag{4.51}$$

for all $i,j \leq M$. The proof is briefly reviewed in Appendix B. In particular, Eq. (4.51) implies that if the full-range Hilbert problem has continuous coefficients $(\underset{\sim}{g}(\pm 1) = \underset{\sim}{I})$ then the half-range problems have continuous coefficients and conversely.

As the most stringent limitation it will be assumed that the scattering operator is self-adjoint, i.e., $\underset{\sim}{\Lambda}(z)$ is a symmetric matrix. However, an attempt to generalize the proof to cover non-symmetric transfer has been made and the lacking fragment of the proof is discussed in Appendix C.

In addition to symmetric transfer, it is required that $\underset{\sim}{\Lambda}(z)$ be an even function of z,

$$\underset{\sim}{\Lambda}(z) = \underset{\sim}{\Lambda}(-z). \qquad (4.53)$$

The Eq. (4.53) is to be regarded a weak condition. In fact, it is valid for isotropic scattering and, in the anisotropic case, if the matrices $\underset{\sim}{L}_i(\mu)$ and $\underset{\sim}{M}_j(\mu)$ are simultaneously either even or odd matrices. This is almost equivalent to the kernel $\underset{\sim}{K}(\mu,\mu')$ possessing reflection symmetry. Both these conditions are required to prove that the partial indices of the fundamental matrix $\underset{\sim}{X}(z)$ are non-positive. Referring to Eq. (4.32), this means that the polynomial vanishes in the general solution and, simultaneously, there exists a correct number of discrete modes to fullfill the conditions Eq. (4.34).

All earlier investigations considered kernels possessing the property (4.53). As it has been demonstrated in Ref. 19, the calculation of can be simplified for such a kernel. As a further consequence the discrete eigenvalues occur in pairs, i.e., if ν_k is a root of Eq. (3.14) then so is $-\nu_k$.

In accordance with the formalism developed in Section (4.1) the equations corresponding to Eq. (4.37) are written as

$$m_\pm = n_\pm + \ell_\pm + \rho_\pm \qquad (4.54)$$

where the ± labelling refers to the half-ranges $L_- = [-1,0]$ and $L_+ = [0,1]$.

The members $n_\pm$ are defined in analogy with the general definition Eq. (4.38) i.e.,

$$n_\pm = -\Delta_{L_\pm} \arg \det g(\mu). \tag{4.55}$$

Since the operator Δarg is additive and

$$L_+ \cup L_- = [-1,1] , \tag{4.56}$$

it follows from Eq. (4.43) that

$$n = n_+ + n_- \tag{4.57}$$

and observing Eq. (4.53) one obtains

$$n = 2n_\pm . \tag{4.58}$$

As it is shown in Appendix B a similar relationship exists for the numbers $\rho_\pm$ and ρ. That is

$$\rho_+ + \rho_- = \rho \tag{4.59}$$

in general, and applying the symmetry condition Eq. (4.53)

$$\rho = 2\rho_\pm . \tag{4.60}$$

Pursuant to the discussion in Sec. 4.1 it has to be shown that the numbers $\ell_\pm$ vanish in Eq. (4.54). This is proven in Appendix C.

The Eqs. (4.54) valid on $L_\pm$ are added to obtain

$$m_+ + m_- = n + \rho . \tag{4.61}$$

where Eqs. (4.57,59) and the result $\ell_\pm = 0$ are observed. Recalling the meaning of the quantities in Eq. (4.61), viz $m_+ + m_-$ is the number of conditions of the form Eq. (4.34) which have to be fullfilled in order to permit a unique expansion on both half-intervals simultaneously; and n is the number of linearly independent free constants available; the eigensolutions are complete on the half range if $\rho = 0$. The ρ-dependence of the conclusion is identical with that in the full-range case.

The fragments of the proof of this section are now gathered together. Referring to Sec. 4.1 the conditions 1-3 (page 40) were assumed to be valid. In addition, in the half-range proof the following sufficient conditions have been imposed:

4. The kernel is self-adjoint and $\underset{\sim}{\Lambda}(z) = \underset{\sim}{\Lambda}(-z)$.

Theorem 4. Assuming the conditions 1-4 are fullfilled, then the eigensolutions are complete on the half-range, if $\rho = 0$.

Corollary. If condition 4 is valid and the transfer kernel is bounded in a neighborhood of the origin, then half-range completeness is a consequence of full-range completeness, and conversely.

The corollary is obtained by comparing the conclusions in Theorems 2 and 4 and the discussion of ρ in Appendix B. To reiterate, self-adjointness in condition 4 can be rigorously relaxed as soon as the non-positivity of the partial indices in established.

V. FULL-RANGE ORTHOGONALITY AND APPLICATIONS

The results derived in the previous chapters are not directly applicable to solving any practical boundary value problems but permit the use of numerical techniques based on the normal modes. This is in part due to the intrinsic complexity of the problem but also because the formalism developed in terms of matrices of rank NM, while appropriate in the consideration of completeness, is unnecessarily complicated for solving for the expansion coefficients. This is emphatically obvious for problems involving infinite medium boundary conditions. In this case the convenient orthogonality properties of the eigensolutions are easily established.

In the general case the derivation of orthogonality relations, assuming such relations exist, consists of determining the weight function matrix $\underset{\sim}{W}(\mu)$ possessing the property

$$\int_L \left(\underset{\sim}{\phi}^{\dagger}(\nu',\mu)\right)^T \underset{\sim}{W}(\mu)\underset{\sim}{\phi}(\nu,\mu)d\mu = \delta(\nu-\nu') \,. \tag{5.1}$$

In fact the degeneracy of the continuous spectrum necessitates an orthogonalization procedure to be applied on the continuous eigensolutions. The weight function has been determined in a special multigroup case[5] whereas no general proof has been given so far. Fortunately, the full-range completeness relation can be obtained in a straight-forward manner directly from the eigenvalue equation.[4] To illustrate the utility of the formalism, the infinite medium Green's function is determined in a subsequent section. Finally, some remarks are made on the asymptotic behavior of the solution of the Milne problem.

Orthogonality and Normalization

Theorem 5. The eigenfunctions $\underset{\sim}{\phi}(\nu,\mu)$ and $\underset{\sim}{\phi}^{\dagger}(\nu,\mu)$ are orthogonal on the full range with respect to weight function μ. As proof, consider the eigenvalue equation Eq. (3.4)

$$(\Sigma - \mu/\nu\ I)\underset{\sim}{\phi}(\nu,\mu) = \int_{-1}^{1} \underset{\sim}{K}(\mu,\mu')\underset{\sim}{\phi}(\nu,\mu')d\mu' \tag{5.2}$$

and the transpose of the adjoint equation Eq. (3.60) associated with eigenvalue ν',

$$\left(\underset{\sim}{\phi}^{\dagger}(\nu',\mu)\right)^{T}(\underset{\sim}{\Sigma}-\mu/\nu' I)= \int_{-1}^{1} \left(\underset{\sim}{\phi}^{\dagger}(\nu',\mu')\right)^{T}\underset{\sim}{K}(\mu',\mu)d\mu' . \tag{5.3}$$

Eq. (5.2) is pre-multiplied by $\left(\phi^{\dagger}(\nu',\mu)\right)^{T}$ and Eq. (5.3) post-multiplied by $\underset{\sim}{\phi}(\nu,\mu)$. The resulting equations are integrated over μ from -1 to 1 and subtracted to yield the desired result,

$$\left(\frac{1}{\nu} - \frac{1}{\nu'}\right)\int_{1}^{1} \mu\left(\underset{\sim}{\phi}^{\dagger}(\nu',\mu)\right)^{T}\underset{\sim}{\phi}(\nu,\mu)d\mu = 0. \tag{5.4}$$

The practical value of Eq. (5.4) is established by Theorem 1 stating that the direct and adjoint spectra are identical and therefore eigenvalues $\nu=\nu'$ exist.

The normalization integral for a discrete eigenfunction $\underset{\sim}{\phi}(\nu',\mu)$ (Eq. 3.7), denoted by $N(\nu_i)$, where

$$N(\nu_i) = \int_{-1}^{1} \mu\left(\underset{\sim}{\phi}^{\dagger}(\nu_i,\mu)\right)^{T}\underset{\sim}{\phi}(\nu_i,\mu)d\mu, \tag{5.5}$$

is evaluated from

$$N(\nu_i) = \sum_{k=1}^{M}\sum_{\ell=1}^{M} \left(\underset{\sim}{n}_k^{\dagger}(\nu_i)\right)^T \int_{-1}^{1} \mu \underset{\sim}{M}_k(\mu)\left(\underset{\sim}{D}(\nu_i,\mu)\right)^2 \underset{\sim}{L}_\ell(\mu)d\mu \underset{\sim}{n}_\ell(\nu_i). \tag{5.6}$$

The adjoint norm vectors $\underset{\sim}{n}_k^{\dagger}(\nu_i)$ correspond to the vectors $\underset{\sim}{n}_\ell(\nu_i)$ of the direct kernel.

Noticing Def. (3.9) the preceeding equation is cast in the form

$$N(\nu_i) = \nu_i^2\left(\underset{\sim}{n}^{\dagger}(\nu_i)\right)^T\left\{\frac{d}{dz}\underset{\sim}{\Lambda}(z)\right\}_{z=\nu_i} \underset{\sim}{n}(\nu_i), \tag{5.7}$$

where $\underset{\sim}{n}(\nu_i)$ was defined in Eq. (3.11) and again $\underset{\sim}{n}^{\dagger}(\nu_i)$ is the corresponding vector for the adjoint problem. Analogous expressions have been exhibited in one-speed theory[4] and in earlier multigroup studies.[36]

The use of the orthogonality relations to determine the continuous expansion coefficients involves normalization integrals $N_{ij}^{(n)}(\nu)$ of the form

$$N_{ij}^{(n)}(\nu)\alpha(\nu) = \int_{-1}^{1} d\mu\mu\left(\underset{\sim}{\phi}_i^{(n)\dagger}(\nu,\mu)\right)^T \int_{(n)} \alpha(\nu')\underset{\sim}{\phi}_j^{(n)}(\nu',\mu)d\nu', \tag{5.7'}$$

where $\alpha(\nu)$ is an expansion coefficient defined on $\nu\varepsilon(n)$. Using the explicit forms of the eigensolutions given in Eqs. (3.48,65) the above expression has the form

$$N_{ij}^{(n)}(\nu)\alpha(\nu) = \left(\underset{\sim}{n}_i^{\dagger}(\nu)\right)^T \int_{-1}^{1} d\mu\mu \int_{(n)} d\nu'\alpha(\nu')\underset{\sim}{\Gamma}_{ij}^{(n)}(\nu,\nu',\mu)\underset{\sim}{n}_j(\nu'), \tag{5.8}$$

where the NM component vectors are obtained from Def (3.34) and an arbitrary N×N matrix block of the matrix $\underset{\sim}{\Gamma}_{ij}^{(n)}(\nu,\nu',\mu)$ is given by

$$[\underset{\sim}{\Gamma}_{ij}^{(n)}(\nu,\nu',\mu)]_{k\ell} =$$

$$\underset{\sim}{M}_k(\mu)\left(\underset{\sim}{D}(\nu,\mu)+\lambda_i^{(n)}(\nu)\underset{\sim}{\delta}^{(n)}(\nu,\mu)\right)\left(\underset{\sim}{D}(\nu',\mu)+\lambda_j^{(n)}(\nu')\underset{\sim}{\delta}^{(n)}(\nu',\mu)\right)\underset{\sim}{L}_\ell(\mu). \tag{5.9}$$

More detailed expressions of $N_{ij}^{(n)}(\nu)$ are not needed for the purposes of the rest of this work. From the point of view of performing the integrations in Eq. (5.8) it is observed, however, that Eqs. (3.22, 32) yield

$$[\underset{\sim}{\Lambda}^{\pm}(\nu)]_{k\ell} = I\delta_{k\ell} - \int_{-1}^{1}\underset{\sim}{M}_k(\mu)\left(\underset{\sim}{D}(\nu,\mu)\mp i\pi\nu\underset{\sim}{\delta}^{(n)}(\nu,\mu)\right)\underset{\sim}{L}_\ell(\mu)d\mu, \tag{5.10}$$

$$\nu\varepsilon(n),$$

which relates the matrices appearing in Eq. (5.9) to the boundary values of the dispersion matrix.

As has been indicated in Eq. (5.7), the orthogonality relation Eq. (5.4) does not provide a vanishing scalar product of the eigen-solutions associated with the same eigenvalue but different functions $\lambda_i^{(n)}(\nu)$, where $i \le N-n+1$. Besides using a generalized Gram-Schmidt process[10,19] or some other feasible procedure,[11] an alternative scheme involves solution of a linear system. In order to explain the calculation, consider the expansion of a given admissible function $\underset{\sim}{\psi}(\mu)$ (c.f., Eq. 4.1)

$$\underset{\sim}{\psi}(\mu) = \sum_{i=1}^{n}\alpha_i(\nu)\underset{\sim}{\phi}(\nu_i,\mu) + \sum_{n=1}^{N}\sum_{i=1}^{N-n+1}\int_{(n)}\alpha_i^{(n)}(\nu)\underset{\sim}{\phi}_i^{(n)}(\nu,\mu)d\nu, \tag{5.11}$$

where the discrete coefficient α_i is determined applying Eqs. (5.5,7). The coefficient $\alpha_i^{(n)}(\nu)$ is determined by multiplying Eq. (5.11) by

$\mu(\underset{\sim}{\phi}_i^{(n)\dagger}(\nu,\mu))^T$ and integrating over μ from -1 to 1. Proceeding in this manner for all values of $i \le N-n+1$ consecutively, one obtains a system of N-n+1 linear equations;

$$\underset{\sim}{\beta}^{(n)}(\nu) = \underset{\sim}{N}^{(n)}(\nu)\underset{\sim}{\alpha}^{(n)}(\nu), \tag{5.12}$$

with

$$\left(\underset{\sim}{\beta}^{(n)}(\nu)\right)_i = \int_{-1}^{1} \mu\left(\underset{\sim}{\phi}_i^{(n)\dagger}(\nu,\mu)\right)^T \underset{\sim}{\psi}(\mu)d\mu\ , \tag{5.13}$$

and

$$\left(\underset{\sim}{N}^{(n)}(\nu)\right)_{ij} = N_{ij}^{(n)}(\nu)\ , \tag{5.14}$$

where $N_{ij}^{(n)}(\nu)$ was defined in Eq. (5.7) and $\underset{\sim}{\alpha}^{(n)}(\nu)$ has the components $\alpha_i^{(n)}(\nu)$ (c.f., Eq. (4.6)). Inversion of Eq. (5.12) yields the coefficient provided that $\underset{\sim}{N}^{(n)}(\nu)$ is non-singular.

The Infinite-Medium Green's Function

The full-range completeness and orthogonality properties of the eigenfunctions can be applied immediately to solve the infinite-medium Green's function. Consider a planar source at the origin emitting q_i neutrons collimated at μ_i in the ith group. The particle density, conventionally called $\underset{\sim}{g}(x,\mu)$, satisfies the Boltzmann equation

$$\underset{\sim}{B}\underset{\sim}{g}(x,\mu) = \underset{\sim}{q}(\mu,\mu_o)\delta(x) \tag{5.15}$$

where

$$\left(\underset{\sim}{q}(\mu,\mu_o)\right)_i = q_i\delta(\mu-\mu_i). \tag{5.16}$$

In order to specify $\underset{\sim}{g}(x,\mu)$ uniquely some other appropriate conditions are required. Supposing the system is subcritical the Green's function vanishes at infinity, i.e.,

$$\lim_{x\to\infty} \underset{\sim}{g}(x,\mu) = \underset{\sim}{0}. \tag{5.17}$$

While no further studies have been made it can be conjectured on the basis of earlier results[4] that subcriticality implies the discrete eigenvalues to be real. In any event, the condition (5.17) and reality of the spectrum are assumed in the subsequent discussion. Letting p eigenvalues in the discrete spectrum be positive, the relevant expansions of the Green's function are

$$\begin{aligned}\underset{\sim}{g}(x,\mu) = &\sum_{i=1}^{p} \alpha_i e^{-x/\nu_i}\underset{\sim}{\phi}(\nu_i,\mu)\\ &+ \sum_{n=1}^{N}\sum_{i=1}^{N-n+1}\int_{(n)_{L_+}} \alpha_i^{(n)}(\nu)e^{-x/\nu}\underset{\sim}{\phi}_i^{(n)}(\nu,\mu)d\nu, \qquad x > 0,\end{aligned} \tag{5.18}$$

$$\begin{aligned}\underset{\sim}{g}(x,\mu) = &\sum_{i=1}^{n-p} \alpha_i e^{-x/\nu_i}\underset{\sim}{\phi}(\nu_i,\mu)\\ &- \sum_{n=1}^{N}\sum_{i=1}^{N-n+1}\int_{(n)_{L_-}} \alpha_i^{(n)}(\nu)e^{-x/\nu}\underset{\sim}{\phi}_i^{(n)}(\nu,\mu)d\nu, \qquad x < 0,\end{aligned} \tag{5.19}$$

where Eq. (5.17) is now manifestly fullfilled. Applying the jump condition,[4]

$$\mu\left(\underset{\sim}{g}(0_+,\mu) - \underset{\sim}{g}(0_-,\mu)\right) = \underset{\sim}{q}(\mu,\mu_o), \tag{5.20}$$

one obtains immediately Eq. (5.11) with

$$\mu\underset{\sim}{\psi}(\mu) = \underset{\sim}{q}(\mu,\mu_o). \tag{5.21}$$

Employing Eqs. (5.4,5) the discrete coefficients α_i are obtained from

$$\alpha_i = \frac{1}{N(\nu_i)} \int_{1}^{1} \left(\underset{\sim}{\phi}^{\dagger}(\nu_i,\mu)\right)^T \underset{\sim}{q}(\mu,\mu_o)d\mu, \tag{5.22}$$

and pursuant to the discussion leading to Eq. (5.12) the continuous expansion coefficient are determined by

$$\underset{\sim}{\alpha}^{(n)}(\nu) = \left(\underset{\sim}{N}^{(n)}(\nu)\right)^{-1}\underset{\sim}{\beta}_g^{(n)}(\nu), \tag{5.23}$$

where

$$\left(\underset{\sim}{\beta}_g^{(n)}(\nu)\right)_i = \int_{-1}^{1} \left(\underset{\sim}{\phi}_i^{\dagger(n)}(\nu,\mu)\right)^T \underset{\sim}{q}(\mu,\mu_o)d\mu. \tag{5.24}$$

Once the Green's functions are known one could derive pertinent integral equations for any boundary value problem involving semi-infinite or bounded media.[17]

Application to the Milne Problem

The infinite medium Green's function can be applied to certain problems involving finite media introducing fictitious sources.[25] As an example the Milne problem is considered in the following.

The problem consists of determining the angular flux in a half-space with the non-reentrant boundary and with a source at infinity. Choosing a properly normalized source the problem is to find a solution $\underset{\sim}{\phi}(x,\mu)$ of the transport equation with the boundary conditions

$$\lim_{x\to\infty} \underset{\sim}{\phi}(x,\mu) = e^{-x/\nu}\underset{\sim}{\psi}(\nu,\mu), \quad \nu<0 \ , \tag{5.25}$$

and

$$\underset{\sim}{\Phi}(0,\mu) = \underset{\sim}{0} \quad , \mu>0, \tag{5.26}$$

where ν is some real eigenvalue belonging either to the point or continuous spectrum.

For the purposes of the subsequent discussion it will be assumed that the emergent distribution $\underset{\sim}{\Phi}(0,\mu)$, $\mu<0$, is known, and it is regarded as a negative source at x=0. The total angular flux is then given by

$$\underset{\sim}{\Phi}(x,\mu) = e^{-x/\nu}\underset{\sim}{\psi}(\nu,\mu) + \int_{-1}^{0} \mu_o \underset{\sim}{G}(0,\mu_o;x,\mu)\underset{\sim}{\Phi}(0,\mu_o)d\mu_o, \tag{5.27}$$

where $\underset{\sim}{G}(x_o,\mu_o;x,\mu)$ is a matrix whose ith column is the Green's function $\underset{\sim}{g}_i(x,\mu)$ derived above corresponding to a source q with $q_k = \delta(\mu-\mu_o)\delta_{ki}$.

In order to consider the asymptotic contribution of the source at the surface, it is assumed that there exists a dominant real positive discrete eigenvalue ν_o. This holds at least in known cases. The term containing the Green's function will then have an asymptotically dominant behavior as $e^{-x/\nu}$ and the asymptotic expression $\underset{\sim}{\Phi}^{as}(x,\mu)$ can be written as

$$\underset{\sim}{\Phi}^{as}(x,\mu) = e^{-x/\nu}\underset{\sim}{\psi}(\nu,\mu) + \alpha_o e^{-x/\nu_o}\underset{\sim}{\psi}(\nu_o,\mu), \tag{5.28}$$

where it is easy to verify that α_o is the full-range expansion coefficient of the emergent distribution $\underset{\sim}{\Phi}(0,\mu)$, i.e.,

$$N(\nu_o)\alpha_o = \int_{-1}^{0} \mu_o \left(\underset{\sim}{\psi}^{\dagger}(\nu_o,\mu)\right)^T \underset{\sim}{\Phi}\,(0,\mu)d\mu. \tag{5.29}$$

Introducing the corresponding asymptotic density $\underset{\sim}{\rho}^{as}(x)$ with

$$\underset{\sim}{\varrho}^{as}(x) = \int_{-1}^{1} \underset{\sim}{\Phi}^{as}(x,\mu)d\mu, \tag{5.30}$$

the extrapolation distance x_o is defined by

$$\underset{\sim}{\varrho}^{as}(-x_o) = \underset{\sim}{0}, \tag{5.31}$$

and in this particular case it is deduced from

$$\underset{\sim}{\varrho}^{as}(-x_o) = e^{x_o/\nu}\underset{\sim}{n}(\nu) + \alpha_o e^{x_o/\nu_o}\underset{\sim}{n}(\nu_o) = \underset{\sim}{0}, \tag{5.32}$$

that there exists in general a different extrapolation distance for each group. However, if $\underset{\sim}{\Lambda}(z) = \underset{\sim}{\Lambda}(-z)$ holds and if the source at infinity is normalized to correspond to the eigenvalue $\nu=-\nu_o$, then $\underset{\sim}{n}(-\nu_o) = \underset{\sim}{n}(\nu_o)$ and a single extrapolation distance is readily determined to be

$$x_o = -\frac{\nu_o}{2}\log|\alpha_o|. \tag{5.33}$$

In case the emergent distribution is calculated using a technique based on the half-range completeness proof given in the present study, then the condition $\underset{\sim}{\Lambda}(z) = \underset{\sim}{\Lambda}(-z)$ is concomitant. The general discussion in this chapter is not restricted by this condition.

VI. CONCLUSION

The multigroup transport equation has been studied in plane geometry assuming the angular dependence of the transfer kernel could be represented in a degenerate form. The analysis was conducted employing the method of singular eigenfunctions.

The dispersion matrix, whose determinant defines the eigenvalue spectrum, was found in a block matrix form. The associated eigenfunctions can be derived explicitly for a certain combination of the set of eigensolutions chosen for the analysis. Since the full-range orthogonality properties of the eigenfunctions involve the adjoint eigenfunctions, the adjoint operator was considered in some detail. In particular, it was demonstrated that the direct and adjoint eigenvalue spectra are the same and that the distributional part of the singular eigensolutions is identical for the direct and adjoint Boltzmann operators studied.

An indispensable part of this work was directed to establishing sufficient conditions under which the angular eigenfunctions form a complete set in the sense that an arbitrary vector satisfying the extended Hölder condition can be uniquely expanded in terms of these normal modes. Attempting an expansion in terms of the continuous modes alone leads to a system of integral equations which is found to be equivalent to a matrix Hilbert problem on a given range. The Hilbert problem possesses a solution with appropriate analyticity requirements if a certain number of conditions is imposed on the inhomogeneous term.

The main complication arises from the fact that the general solution includes a polynomial contribution which has to be proven to vanish in order for the set to be complete.

In the full-range case it is shown that no arbitrary polynomial can occur in the solution and that the necessary analytic properties are achieved when the discrete expansion coefficients are introduced. In fact, the matrix Hilbert problem is reducible to a diagonal form and therefore the theory of the scalar Hilbert problem will suffice in the treatment. Thus, it was proven that the eigenfunctions are complete on the full-range, i.e., an arbitrary vector $\underset{\sim}{\psi}(\mu)$ is expandable, if any of the following conditions are fulfilled.

1) If the vectors $\mu \underset{\sim}{M}_1(\mu)\underset{\sim}{\psi}(\mu)$ satisfy the extended Hölder condition.

2) If a certain vector $\underset{\sim}{f}$ constructed from $\mu \underset{\sim}{M}_1(\mu)\underset{\sim}{\psi}(\mu)$ and from the boundary values of the dispersion matrix (see Eq. 4.27) satisfies the extended Hölder condition, and

a) the Hilbert problem has continuous coefficients, or

b) if the Hilbert problem has piecewise Lipschitz continuous coefficients, and an index ρ (which must be calculated from the discontinuities for each particular kernel) vanishes. While the first case above can be used to assert completeness in all known problems, only the second alternative is readily transferable to the half-range problems.

In the half-range completeness proof the main obstruction is caused by the difficulty in demonstrating that the solution cannot include a polynomial. For this reason further restrictions have to be introduced on admissible kernels. The half-range completeness holds if condition 2 above is satisfied and, if in addition, the scattering operator is self-adjoint (a symmetric dispersion matrix) and the elements of the dispersion matrix are even functions. These conditions pertain to the thermal equilibrium of the system and to reflection symmetry of the scattering kernel, respectively.

An approach to establish the half-range completeness for nonself-adjoint kernels is also proposed but, unfortunately, the proof is inconclusive because of some detail. Once the nonpositivity of the partial indices has been established for an arbitrary case, completeness would follow immediately. To date a rigorous proof is available for a two-group nonself-adjoint problem with isotropic scattering in which case the transfer kernel can be symmetrized.

The condition on the index ρ is found to be equivalent in the full- and half-ranges provided that the elements of the transfer matrix are bounded in a neighborhood of the origin.

As an application of the full-range completeness and orthogonality properties, the infinite medium Green's function is derived in a closed form. At the present time no analytic solution appears to be available for a general half-range problem. The half-range completeness proof can evidently be used to show the existence and uniqueness of a solution in numerous cases where numerical schemes have been developed, e.g., in invariant embedding or direct iteration.

The question whether an analytic solution exists in the half-range case is tantamount to exploring whether an appropriate fundamental matrix can be constructed. Besides the general considerations, it would be interesting to study the diagonalization of the transformation by a properly discontinuous matrix. A diagonal Hilbert problem would immediately be amenable to analytic solution. A convenient similarity transformation has been found for separable kernels. This would require, however, the continuation of the transforming matrix into the entire complex plane, since the similarity transformation concerned is defined only on the branch cut. In any event, the eigenvalues of the transformation matrix can be readily obtained for other purposes.

In this connection it should be noted that certain representations in terms of matrices with rational elements have been used in the development of the basic mathematical theory. Even in this case the diagonalized treatment would be more feasible not to mention possible numerical approximation in this direction.

Appendix A

Characterization of the Hilbert Problems

Since the solvability of various boundary value problems occurring in transport theory is determined by the attributes of the transformation matrix of the appropriate Hilbert problem, different possibilities are considered in the following. A new approach to these problems is also proposed.

Consider the homogeneous Hilbert problem (cf., Eq. 4.25)

$$\underset{\sim}{\phi}^{+}(\mu) = \underset{\sim}{g}(\mu)\underset{\sim}{\phi}^{-}(\mu), \quad \mu\varepsilon L, \tag{A-1}$$

where L is a given subinterval of [-1,1]. The problems are divided into two cases.[19]

I. $\underset{\sim}{g}(\mu)$ is a diagonal matrix or diagonalizable. In this connection, by $\underset{\sim}{g}(\mu)$ being diagonalizable it is meant that there exists a non-singular matrix $\underset{\sim}{U}(z)$, discontinuous on L in general, such that the matrix $\underset{\sim}{U}^{+}(\mu)\underset{\sim}{g}(\mu)\left(\underset{\sim}{U}^{-}(\mu)\right)^{-1}$ is diagonal.

II. $\underset{\sim}{g}(\mu)$ is not diagonalizable.

In case I, the problems can be solved analytically on an open arc L.[23] In case II, one has to close the contour and no general method is reported to obtain the fundamental matrix in a closed form. However, general analysis can be conducted if either[24,25]

1) $\underset{\sim}{g}(\mu)$ is a continuous matrix whose value at both endpoints of L is the same, or

2) $\underset{\sim}{g}(\mu)$ is a piecewise Lipschitz continuous matrix.

Besides these general considerations it is interesting to examine the diagonalization of $\underset{\sim}{g}(\mu)$ by an ordinary similarity transformation in the particular case of transport theory. For simplicity, consider Eq. (3.53) for a separable kernel in the constant total cross section limit. The equation has the form

$$i\pi\mu\left(\underset{\sim}{\Lambda}^{+}(\mu)+\underset{\sim}{\Lambda}^{-}(\mu)\right)\underset{\sim}{N}(\mu) = \left(\underset{\sim}{\Lambda}^{+}(\mu)-\underset{\sim}{\Lambda}^{-}(\mu)\right)\underset{\sim}{N}(\mu)\underset{\sim}{\lambda}(\mu), \tag{A-2}$$

which is equivalent to

$$\underset{\sim}{N}^{-1}(\mu)\underset{\sim}{g}(\mu)\underset{\sim}{N}(\mu) = \left(\underset{\sim}{\lambda}(\mu)+i\pi\mu\underset{\sim}{I}\right)^{-1}\left(\underset{\sim}{\lambda}(\mu)-i\pi\mu I\right). \tag{A-3}$$

Since $\underset{\sim}{\lambda}(\mu)$ is a diagonal matrix by definition, it is noticed that the normalization matrix $\underset{\sim}{N}(\mu)$ diagonalizes the transformation. Furthermore, the diagonal elements, denoted by $\gamma_i(\mu)$, have the form

$$\gamma_i(\mu) = \left(\lambda_i(\mu)-i\pi\mu\right)/\left(\lambda_i(\mu)+i\pi\mu\right). \tag{A-4}$$

The problem is reduced to constructing a non-singular, analytic matrix $\underset{\sim}{N}(z)$ having the value $\underset{\sim}{N}(\mu)$ on the cut. Even if such a matrix does not exist, it is noticed that in the standard treatments of this kind (c.f., Ref. 25) the occurance of poles is circumvented by introducing appropriate rational matrices. Because of these unresolved questions only a brief sketch is given on the procedure. Defining an auxiliary diagonal matrix $\underset{\sim}{X}_o(z)$ conventionally by[23]

$$\left(X_o(z)\right)_{ii} = \exp\left(\frac{1}{2\pi i}\int_L \log\gamma_i(\mu)\frac{d\mu}{\mu-z}\right), \tag{A-5}$$

the appropriate fundamental matrix $\underset{\sim}{X}(z)$ is obtained from

$$\underset{\sim}{X}(z) = \underset{\sim}{R}(z)\underset{\sim}{X}_o(z), \tag{A-6}$$

where $\underset{\sim}{R}(z)$ is a diagonal rational matrix explicitly determined by the behavior of $\underset{\sim}{X}_o(z)$. Incidentally, the partial indices are also explicitly determined by $\underset{\sim}{X}_o(z)$. Finally, the solution of the inhomogeneous equation pertinent to Eq. (A-1) is obtained in the form

$$\underset{\sim}{\phi}(z) = \frac{1}{2\pi i}\,\underset{\sim}{N}(z)\underset{\sim}{X}(z)\int_N \left(\underset{\sim}{N}(\mu)\underset{\sim}{X}^+(\mu)\right)^{-1}\underset{\sim}{f}(\mu)\frac{d\mu}{\mu - z} + \underset{\sim}{N}(z)\underset{\sim}{X}(z)\underset{\sim}{p}(z), \tag{A-7}$$

where $\underset{\sim}{f}(\mu)$ is the inhomogeneous term and $\underset{\sim}{p}(z)$ is a vector of polynomials, whose occurance is determined by the known partial indices.

Appendix B

Calculation of the Index ρ

In the completeness proofs, the possibility for a discontinuous transformation in the Hilbert problem was allowed. In the general theory of Vekua[24] the problem is reduced to another transformation, which has continuous coefficients, and a parameter ρ enters the index equation. While this case appears to be rather uninteresting because of the stringency of the Lipschitz condition required, it is plausible that the condition can be relaxed to piecewise continuity, which would be a reasonable limitation. Furthermore, the possibility of discontinuities on $\underset{\sim}{g}(\mu)$ has not been ruled out, and especially the points $\mu=1/\sigma_i$ should be checked. For these reasons, the calculation of ρ is briefly quoted[24] and the relation between the full- and half-range cases is considered in this Appendix.

Letting ν_k be an arbitrary point of discontinuity of $\underset{\sim}{g}(\mu)$, the eigenvalues of the matrix $\underset{\sim}{\gamma}(\nu)$, defined as

$$\underset{\sim}{\gamma}(\nu_k) = \lim_{\varepsilon\to 0} \underset{\sim}{g}^{-1}(\nu_k+\varepsilon)\underset{\sim}{g}(\nu_k-\varepsilon), \tag{B-1}$$

determine the number ρ in the following way. Since $\underset{\sim}{g}(\mu)$ is nonsingular, the eigenvalues λ_i^k can be represented as

$$\lambda_i^k = \exp(2\pi i\rho_i^k)$$
$$-1 < \mathrm{Re}\rho_i^k < 1. \tag{B-2}$$

In fact, the numbers ρ_i^k are chosen either with $-1 < \mathrm{Re}\rho_i^k \leq 0$ or with $0 \leq \mathrm{Re}\rho_i^k < 1$ based on some other considerations.[24] Recalling that the rank of the matrix $\underset{\sim}{\nu}$ is NM, ρ is obtained from

$$\rho = \sum_{i=1}^{NM} \sum_{k=1}^{K} \rho_i^k, \tag{B-3}$$

where K is the number of discontinuities. There are two main simplifications which may have some practical importance. The symmetry $\underset{\sim}{\Lambda}(z) = \underset{\sim}{\Lambda}(-z)$ implies that $\gamma(\nu_k)$ and $\underset{\sim}{\gamma}(-\nu_k)$ have the same eigenvalues since $\underset{\sim}{\gamma}(-\nu_k)$ reduces to

$$\underset{\sim}{\gamma}(-\nu_k) = \lim_{\varepsilon\to o} g(\nu_k-\varepsilon)\underset{\sim}{g}^{-1}(\nu_k+\varepsilon). \tag{B-4}$$

Therefore the contribution to ρ can be calculated from the half-range. Secondly, the calculation of the endpoint discontinuities is simplified by the fact that $\underset{\sim}{g}=\underset{\sim}{I}$ on the appended contour. The eigenvalues can be immediately obtained from the similarity transformation (Eqs. A-3,A-4). In particula , since the adjoint transformation matrix has the same diagonal form, the number ρ is the same for the adjoint problem. It is noticed that this result is independent of the restriction to a separable kernel made in Eq. (A-3).

Finally, it is convenient to notice that from Eqs.(3.22 and 4.27) it follows $g(o)=\underset{\sim}{I}$, provided that the elements of the matrix $\underset{\sim}{M}_i(\mu)\underset{\sim}{L}_j(\mu)$ are bounded in a neighborhood of the origin. This implies that

$$\rho_+ + \rho_- = \rho, \tag{B-5}$$

where $\rho_\pm$ are calculated from the half-ranges and ρ from the full-range.

Appendix C

Non-positivity of the Component Indices

It has been noted in Chapter IV that in order to obtain a solution of the Hilbert problem (Eq. 4.25) one has to apply a certain number of independent conditions on the inhomogeneous term, and that a general solution will include a polynomial part depending on the behavior at infinity of the $\underset{\sim}{X}$-matrix. In particular, it was shown that the discrete expansion coefficients will furnish the proper number of degrees of freedom to satisfy these conditions on the full- and half-range cases provided that the component indices of the $\underset{\sim}{X}$-matrix are non-positive. Simultaneously the polynomial would vanish. In fact, it would be adequate to prove that the non-zero component indices have the same algebraic sign, since the sum of the indices (the total index) is known to be non-positive.

In the full-range case the Hilbert problem can be treated in an alternative way. Nevertheless, the appropriate inspection of the number of the discrete modes needed would eventually require as much calculation as the brief proof presented below. Because more severe conditions must be imposed in the half-range than in the full-range, these cases are discussed separately.

Full-range

The fundamental matrix $\underset{\sim}{X}(z)$ satisfies the equation

$$\underset{\sim}{X}^{+}(\mu) = \underset{\sim}{g}(\mu)\underset{\sim}{X}^{-}(\mu) \qquad \mu\varepsilon[-1,1], \tag{C-1}$$

with

$$\underset{\sim}{g} = (\underset{\sim}{\Lambda}^{+})^{-1}\underset{\sim}{\Lambda}^{-}, \tag{C-2}$$

The matrix $\underset{\sim}{\Lambda}^{-1}$ would formally satisfy Eq. (C-1), but since it does not exist in the entire complex plane, it is more convenient to consider the associated problem[24] to Eq. (C-1). The corresponding fundamental matrix $\underset{\sim}{Y}(z)$ obeys

$$\underset{\sim}{Y}^{+}(\mu) = \left(\underset{\sim}{g}^{T}(\mu)\right)^{-1}\underset{\sim}{Y}^{-}(\mu). \tag{C-3}$$

It is readily observed that $\underset{\sim}{\Lambda}^{T}(z)$ satisfies Eq. (C-3), and being analytic outside the branch cut [-1,1], it can be expressed as[24]

$$\underset{\sim}{\Lambda}^{T}(z) = \underset{\sim}{Y}(z)\underset{\sim}{P}(z), \tag{C-4}$$

where $\underset{\sim}{P}(z)$ is a matrix of polynomials. If z approaches the origin, where $\underset{\sim}{\Lambda}(0)=\underset{\sim}{I}$, Eq. (C-4) has the form

$$\underset{\sim}{Y}(0)\underset{\sim}{P}(0) = \underset{\sim}{I}, \tag{C-5}$$

where $\underset{\sim}{Y}(0)$ is non-singular[24] and therefore at least one element of a given row of $\underset{\sim}{P}(z)$ has a term of order zero.

Letting κ_i, $i \leq NM$, denote the component indices of $\underset{\sim}{X}(z)$, the $Y(z)$ has the indices $-\kappa_i$.[24] Since $Y(z)$ must be of normal form at infinity,[24,25] it follows that

$$\lim_{z\to\infty} \underset{\sim}{Y}(z) \begin{pmatrix} z^{-\kappa_1} & & & & 0 \\ & z^{-\kappa_2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & z^{-\kappa_{NM}} \end{pmatrix} = \underset{\sim}{I} \tag{C-6}$$

Finally, letting z approach infinity in Eq. (C-4) and noting that $\underset{\sim}{\Lambda}(z)$ then approaches a constant value, it follows immediately by Eq. (C-6) and by the fact that some element of each row of $\underset{\sim}{P}(z)$ is non-zero that all the component indices κ_i are non-positive.

Half-range

In regard to the half-range no definitive index proof has been found under the relatively weak conditions applied to the rest of the analysis. In fact, while the ensuing discussion will be conducted in rather general terms, the conclusive proof will be limited to the same class of scattering operators as is considered in Ref. 10, i.e., to symmetric transfer.

The proof is commenced by introducing the relevant fundamental matrix of the half-range associate problem denoted by $\underset{\sim}{Y}(z)$. The boundary values of $\underset{\sim}{Y}$ satisfy Eq. (C-3) on the half-range, where as $\underset{\sim}{Y}$ is continuous on the opposite section of the branch cut. Furthermore, let $\underset{\sim}{Z}(z)$ be the fundamental matrix of the Hilbert problem related to the adjoint kernel $\underset{\sim}{K}^T(\mu',\mu)$. On the half-range $\underset{\sim}{Z}(z)$ obeys the equation

$$\underset{\sim}{Z}^+(\mu) = \underset{\sim}{h}(\mu)\underset{\sim}{Z}^-(\mu), \tag{C-7}$$

where

$$\underset{\sim}{h} = (\underset{\sim}{\Lambda}^{T+})^{-1}\underset{\sim}{\Lambda}^{T-}. \tag{C-8}$$

Consider a subsidiary function $\underset{\sim}{V}(z)$ defined as

$$\underset{\sim}{V}(z) = \underset{\sim}{\Lambda}^T(z)\underset{\sim}{Z}(-z). \tag{C-9}$$

Assuming that $\underset{\sim}{\Lambda}(z) = \underset{\sim}{\Lambda}(-z)$ it is readily verified that $\underset{\sim}{V}(z)$ satisfies

Eq. (C-3) on the half-range, and is also analytic in the rest of the plane. Consequently, $\underset{\sim}{V}(z)$ can be represented as

$$\underset{\sim}{V}(z) = \underset{\sim}{Y}(z)\underset{\sim}{P}(z) \tag{C-10}$$

Let ξ_i, $i \leq NM$, now denote the component indices of $\underset{\sim}{Z}(z)$. In view of Eqs. (C-9) and (C-10) it is seen immediately that assuming $\kappa_i > 0$ and $\xi_i > 0$ simultaneously leads to a contradiction in the limit as $z \to \infty$ provided that

$$\lim_{z \to \infty} \left[\underset{\sim}{P}(z)\right]_{ii} \neq 0. \tag{C-11}$$

In particular, if Eq. (C-11) were valid for i=1 it could be concluded that the partial indices of either the $\underset{\sim}{X}$-matrix or the $\underset{\sim}{Z}$-matrix are non-positive, i.e., either the direct or adjoint set is complete. This follows from the fact that the indices can be ordered. It would also follow immediately[21] that the both sets are complete if the total number of discrete eigenfunctions is less than 2NM or if the operator relating the direct and adjoint eigenfunctions is invertible. This could be verified in each case separately considering the spectral representation of the operator which can be constructed.[21]

Unfortunately, any attempt to establish Eq. (C-11) in general appears to lead to an impasse. Letting z appraoch the origin in Eqs. (C-9) and (C-10) leads to

$$\underset{\sim}{P}(0) = \underset{\sim}{X}^T(0)\underset{\sim}{Z}(0). \tag{C-12}$$

For Eq. (C-11) to hold it is sufficient to have $\underset{\sim}{X}(0) = \underset{\sim}{Z}(0)$, which of course is trivially true for a self-adjoint problem where $\underset{\sim}{X}(z) \equiv \underset{\sim}{Z}(z)$.

Appendix D

Comments on Some Previous Investigations

One has to exercise certain care in applying the results of the established mathematical theory to the proofs of completeness. Some inaccuracies persistent in the literature were pointed out throughout this study. These are summarized in this Appendix along the lines of a forthcoming paper.[37]

First, it was observed in Chapter IV that the proof of completeness can be reduced to a Hilbert problem rather than to the solution of a system of Fredholm equations.[11] This is achieved by transforming the appropriate system of singular integral equations into a dominant form[24] Eq. (4.13). Hence the occuring Fredholm term is eliminated.

The rest of the remarks are confined to the half-range proof. In the full-range case the corresponding difficulties can be easily circumvented.

The completeness proofs subject to criticism[7,11,36,38] are based on Muskhelisvili's[23] or Vekua's[24] analyses of the matrix Hilbert problem. Expressing the Hilbert problem as (c.f. Eq. 4.25)

$$\underset{\sim}{\phi}^{+}(\mu) = \underset{\sim}{g}(\mu)\underset{\sim}{\phi}^{-}(\mu) + \underset{\sim}{f}(\mu), \quad \mu\varepsilon L, \tag{D-1}$$

their formalism requires the transformation matrix $g(\mu)$ to be at least Hölder continuous on L. This condition is too stringent for the works referred above. Fortunately, the equivalent theory of

Mandzhavidze and Khvedelidze[25] is applicable if $\underset{\sim}{g}(\mu)$ is a continuous matrix.

Most importantly, several papers are either incomplete[36] or assumptious[7,11,38] in regard to establishing the algebraic sign of the partial indices of the $\underset{\sim}{X}$-matrix. The importance of these considerations is thereby dismissed.

To clarify this point, consider the two-group case as an example. To solve Eq. (D-1), it is necessary to obtain a solution $\underset{\sim}{X}(z)$ of the homogeneous equation (i.e., to perform the Wiener-Hopf factorization of the $\underset{\sim}{g}$-matrix.) This factorization is indeterminate to within a vector of entire functions, call it $\underset{\sim}{p}(z)$. Then a solution to Eq. (D-1), if it exists, can be written in

$$\underset{\sim}{\phi}(z) = \frac{1}{2\pi i}\,\underset{\sim}{X}(z)\left\{\int\left(\underset{\sim}{X}^{+}(\mu)\right)^{-1}\underset{\sim}{f}(\mu)\frac{d\mu}{\mu-z} + \underset{\sim}{p}(z)\right\}. \tag{D-2}$$

His known, from the original equation (4.33), that each component of $\underset{\sim}{\phi}(z)$ must vanish at least like $1/z$ as $|z|\to\infty$. Furthermore, at infinity $\underset{\sim}{X}(z)$ behaves as

$$\underset{\sim}{X}(z) \sim \begin{pmatrix} az^{-\kappa_1}+\cdots & bz^{-\kappa_2}+\cdots \\ cz^{-\kappa_1}+\cdots & dz^{-\kappa_2}+\cdots \end{pmatrix}, \quad |z| \to \infty, \tag{D-3}$$

where κ_1 and κ_2 are the partial (or component) indices and $\kappa_1 + \kappa_2 = n$, where $2n$ is the number of discrete roots of the dispersion function. Suppose, for example, that $\kappa_1 = +1$. Then the first column of the $\underset{\sim}{X}$-matrix will vanish at infinity as z^{-1}, and the first component of the solution is acceptable. But now $\kappa_2 = -n-1$, so that the second column of $\underset{\sim}{X}(z)$ will diverge at infinity as z^{n+1}. This requires n+1 conditions a the integral in Eq. (2) to insure that the produce vanishes at 1/z, but, in half-range, there are only n conditions available, corresponding to half the discrete eigenvalues; hence, no solution exists. It is clear that this type of behavior holds if either κ_1 or κ_2 is positive. Thus, a solution to the Hilbert problem exists iff $\kappa_i \leq 0 \forall i$. Although the above analysis is given for the two group case, the conclusion is valid for any number of groups. In cases where the number of conditions is not limited a priori, the positivity of the partial indices implies that th? polynomial contribution $\underset{\sim}{p}(z)$ is nonzero.

No counter example is given above and, in fact, the eigensolutions in these particular cases may be complete. However, it might be mentioned that employing the continuous energy approach Nicolaenko[28] constructed a counter-example where the normal modes are incomplete. This may be expected to occur when the scattering operator is noncompact. In fact, Nicolaenko's example was based on the existence of a residual spectrum which possibility has been ruled out by establishing the identiy of the direct and adjoint spectra in Theorem 1.

Appendix E

Elimination Procedure for a System of Singular Integral Equations

Attempting to solve the system of integral equations in Eq. (4.12) in component form leads, in general, to a scalar singular integral equation involving a non-degenerate Fredholm term.[15,19] While there exists no general method for solving such an equation, an analytic solution can be obtained in the full-range case from the orthogonality relations. For simplicity, Eq. (4.12) is considered in two-group case ($\underset{\sim}{\Sigma}=\underset{\sim}{I}$) with a separable kernel ($\underset{\sim}{K}=\underset{\sim}{L}\underset{\sim}{M}$). The subscripts indicating different regions and the degenerate sum of the kernel are omitted.

In view of Eq. (4.12);

$$\underset{\sim}{\psi}(\mu) = \underset{\sim}{L}(\mu)\int_L \frac{\nu}{\nu-\mu}\underset{\sim}{N}(\nu)\underset{\sim}{\alpha}(\nu)d\nu + \underset{\sim}{L}(\mu)\underset{\sim}{N}(\mu)\underset{\sim}{\lambda}(\mu)\underset{\sim}{\alpha}(\mu), \quad \mu\varepsilon L\,, \tag{E-1}$$

it would be more convenient to consider the expansion of a function $\left(\underset{\sim}{L}(\mu)\right)^{-1}\underset{\sim}{\psi}(\mu)$. However, $\underset{\sim}{L}$ may fail to have an inverse at some point on [-1,1], e.g., as in Ref. 10, and hence it is appropriate to consider Eq. (E-1) as it has been expressed above. Defining (slightly differently than in Chapter IV)

$$\nu\underset{\sim}{N}(\nu)\underset{\sim}{\alpha}(\nu) = \underset{\sim}{\rho}(\nu), \tag{E-2}$$

and

$$\nu\underset{\sim}{A}(\nu) = \underset{\sim}{L}(\nu)\underset{\sim}{N}(\nu)\underset{\sim}{\lambda}(\nu)\underset{\sim}{N}^{-1}(\nu), \tag{E-3}$$

Eq. (E-1) has the form

$$\psi_1(\mu) = L_{11}(\mu)P\int_L \frac{\rho_1(\nu)}{\nu-\mu}d\nu + L_{12}(\mu)P\int_L \frac{\rho_2(\nu)}{\nu-\mu}d\nu + A_{11}(\mu)\rho_1(\mu) + A_{12}(\mu)\rho_2(\mu), \tag{E-4a}$$

$$\psi_2(\mu) = L_{21}(\mu)P\int_L \frac{\rho_1(\nu)}{\nu-\mu}d\nu + L_{22}(\mu)P\int_L \frac{\rho_2(\nu)}{\nu-\mu}d\nu + A_{21}(\mu)\rho_1(\mu) + A_{22}(\mu)\rho_2(\mu). \tag{E-4b}$$

$\rho_2(\mu)$ is eliminated from Eq. (E-4b) as a function of $\rho_1(\mu)$ and substituted in Eq. (E-4a). It will appear that the resulting singular integral equation for $\rho_1(\mu)$ will, in general, involve a non-degenerate Fredholm term. However, one can solve for the vector $\underset{\sim}{\alpha}(\nu)$ from Eq. (5.23) by employing the full-range orthogonality of the eigenfunctions. The vector $\underset{\sim}{\rho}(\nu)$ can then be solved from Eq. (E-2).

In the following the singular integral equation with a Fredholm term is constructed. The procedure is commenced by defining

$$\psi_2'(\mu) = \psi_2(\mu) - L_{21}(\mu)P\int_L \frac{\rho_1(\nu)}{\nu-\mu}d\nu - A_{21}(\mu)\rho_1(\mu) \tag{E-5}$$

and

$$N_2(z) = \frac{1}{2\pi i}\int_L \frac{\rho_2(\nu)}{\nu-z}d\nu. \tag{E-6}$$

Introducing the boundary functions $N_2^{\pm}(\mu)$;

$$N_2^{\pm}(\mu) = \frac{1}{2\pi i}P\int_L \frac{\rho_2(\nu)}{\nu-\mu}\, d\nu \pm \frac{1}{2}\rho_2(\mu), \tag{E-7}$$

Eq. E-4b becomes

$$\psi_2'(\mu) = i\pi L_{22}(\mu)\left(N_2^+(\mu) + N_2^-(\mu)\right) + A_{22}(\mu)\left(N_2^+(\mu) - N_2^-(\mu)\right) \quad \text{(E-8)}$$

Defining

$$\theta(\nu) = \arg\left(A_{22}(\nu) + i\pi L_{22}(\nu)\right) \quad \text{(E-9)}$$

the appropriate X-function is

$$X(z) = (\ell_1 - z)^{-\frac{\theta(\ell_1)}{\pi}} (\ell_2 - z)^{-\frac{(\ell_2)}{\pi}} \exp\left(\frac{1}{\pi}\int_L \frac{\theta(\nu)}{\nu - z} d\nu\right), \quad \text{(E-10)}$$

where ℓ_1 and ℓ_2 are the endpoints of the cut L. Eq. (E-6) now becomes

$$\gamma(\mu)\psi_2'(\mu) = X^+(\mu)N_2^+(\mu) - X^-(\mu)N_2^-(\mu), \quad \text{(E-11)}$$

where

$$\gamma(\mu) = \frac{X^+(\mu) - X^-(\mu)}{2\pi i L_{22}(\mu)} . \quad \text{(E-12)}$$

The solution of Eq. (E-11) can now be written[23] as

$$N\ (z) = \frac{1}{2\pi i X(z)} \int_L \frac{\gamma(\mu)\psi_2'(\mu)}{\mu - z} d\mu. \quad \text{(E-13)}$$

$N_2(\mu)$ is supposed to vanish as $1/z$ at infinity. To make this possible, n_1 discrete eigenfunctions have to be introduced with

$$n' - 1 \leq n_1 < n', \quad \text{(E-14)}$$

where

$$n' = \frac{\theta(\ell_1)}{\pi} + \frac{\theta(\ell_2)}{\pi} . \quad \text{(E-15)}$$

A new function N(z) is defined by letting

$$\psi_2'(\mu) \to \psi_2'(\mu) - \sum_{i=1}^{n_1} a_{ji}\phi(\nu_{ji},\mu) \tag{E-16}$$

in Eq. (E-13). By an abbreviation

$$\psi_2''(\mu) = \sum_{i=1}^{n_1} a_{ji}\phi(\nu_{ji},\mu), \tag{E-17}$$

the definition of N(z) is

$$N\left(z,\psi_2'(\mu)\right) = N_2\left(z,\psi_2'(\mu)-\psi_2''(\mu)\right). \tag{E-18}$$

The functions N(z) and $N_2(z)$ will have the same discontinuity across the cut L, and from Eq. (E-7) one has

$$\rho_2(\mu) = N^+(\mu) - N^-(\mu), \; \mu\varepsilon L. \tag{E-19}$$

The substitution of Eq. (E-19), after using Eqs. (E-13) and (E-18), into Eq. (E-4a) leads to a rather lengthy calculation. The expression can be simplified by changing the order of integrations, using the Poincare-Bertrand formula[23] when appropriate.

Defining

$$\beta(\nu) = \frac{1}{2\pi i}\left(\frac{1}{X^+(\nu)} - \frac{1}{X^-(\nu)}\right) \tag{E-20a}$$

$$\varepsilon(\nu) = \frac{1}{2}\left(\frac{1}{X^+(\nu)} + \frac{1}{X^-(\nu)}\right), \tag{E-20b}$$

$$a(\nu) = \pi^2\beta(\nu)\gamma(\nu)L_{21}(\nu) - \varepsilon(\nu)\gamma(\nu)A_{21}(\nu), \tag{E-20c}$$

$$b(\nu,\mu) = -\beta(\nu)\gamma(\nu)A_{21}(\mu) - \varepsilon(\nu)\gamma(\nu)L\quad(\nu), \tag{E-20d}$$

$$k(\nu,\mu) = P\int_L \frac{\gamma(\eta)L_{21}(\eta)}{(\eta-\nu)(\mu-\eta)}\,d\eta, \tag{E-20e}$$

$$x(\mu) = A_{11}(\mu) + A_{12}(\mu)a(\mu) - L_{12}(\mu)b(\mu,\mu) \tag{E-20f}$$

$$y(\mu,\nu) = L_{11}(\mu) + L_{12}(\mu)a(\nu) + A_{12}(\mu)b(\mu,\nu), \tag{E-20g}$$

$$z(\mu,\nu) = L_{12}(\mu)P\int_L \left[\frac{b(\eta,\nu)}{\nu-\eta} - \beta(\eta)K(\nu,\eta)\right]\frac{d\eta}{\eta-\mu} - A_{12}(\mu)\beta(\mu)k(\nu,\mu), \tag{E-20h}$$

and

$$\psi_1'(\mu) = \psi_1(\mu) - A_{12}(\mu)\left[\beta(\mu)P\int_L \frac{\gamma(\nu)\left[\psi_2(\nu)-\psi_2''(\nu)\right]}{\nu-\mu}d\nu + \varepsilon(\mu)\gamma(\mu)\left(\psi_2(\mu)-\psi_2''(\mu)\right)\right], \tag{E-20i}$$

one finally obtains the integral equation

$$\psi_1'(\mu) = x(\mu)\rho_1(\mu) + P\int_L \frac{y(\mu,\nu)\rho_1(\nu)}{\nu-\mu}d\nu + \int_L z(\mu,\nu)\rho_1(\nu)d\nu, \tag{E-21}$$

involving both a singular term and, in a general case, a non-degenerate Fredholm term.

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ON MULTIGROUP TRANSPORT THEORY

WITH A DEGENERATE TRANSFER KERNEL

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Abstract

The multigroup transport operator is studied in plane geometry assuming that the transfer kernel can be represented in a degenerate form. The eigenvalue spectrum is analyzed constructing the pertinent dispersion matrix in a block matrix form. The associated eigensolutions are obtained in terms of generalized functions. The adjoint operator is also considered for the purpose of demonstrating the full-range orthogonality relations. In particular, it is proven that the direct and adjoint eigenvalue spectra are identical. The full-range completeness of the eigensolutions is established under rather general conditions. For the half-range completeness to hold it is additionally required that the scattering kernel is self-adjoint and possesses reflection symmetry, i.e., the dispersion matrix is symmetric and even. Finally, the infinite medium Green's function is derived employing the orthogonality relations, and the extrapolation distance for the Milne problem is calculated in terms of the emergent distribution.