NEAR AGGREGATION: A TIME AND FREQUENCY DOMAIN ANALYSIS
USING STATE TRAJECTORIES AND TRANSFER FUNCTION RESIDUES

by

Justin Eric Babendreier

Thesis submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of
MASTER OF SCIENCE
in
Electrical Engineering

APPROVED:

Douglas K. Lindner, Chairman

William Luse

S.P. Bingulac

August, 1987

Blacksburg, Virginia

NEAR AGGREGATION: A TIME AND FREQUENCY DOMAIN ANALYSIS
USING STATE TRAJECTORIES AND TRANSFER FUNCTION RESIDUES

by

Justin Eric Babendreier

Committee Chairman: Douglas K. Lindner
Electrical Engineering

(ABSTRACT)

In this thesis we investigate concepts associated with aggregation. The basic idea of aggregation is that there exists a reduced-order model such that, for an appropriate initial condition, the trajectories of the reduced-order model are linear combinations of the trajectories of the full-order model. We study systems which do not aggregate exactly, but which "nearly aggregate". It is shown that for "nearly aggregable" systems there exists a reduced-order model such that, for an appropriate initial condition, the trajectories of the reduced-order model are near a linear combination of the trajectories of the full-order model.

Under certain conditions it has also been shown that near-aggregation is equivalent to near-unobservability (roughly, an invariant subspace close to the null space of C). Here we establish a relationship between near-unobservability and modal measures of observability as suggested by Selective Modal Analysis. With this result we then obtain an upper bound on the norm of the transfer function residue using near-unobservability measures. The Generalized Hessenberg Representation (GHR) and Dual GHR are examined throughout this analysis. It is finally shown that for SISO systems, the residue norm may be expressed in terms of certain parameters of the Dual GHR.

ACKNOWLEDGEMENTS

The completion of this work would not have been possible without the goodwill and constant encouragement from my friends and family. I would like to especially thank Dr. William Luse and Dr. S.P. Bingulac for their lasting patience in my frequent solicitations of their advice. Also, special thanks to Dr. A. M. Hamdan whose discussion and insight made a good deal of this work possible. Finally, I extend my sincerest appreciation and gratitude to my advisor, Dr. Douglas K. Lindner. Without doubt, I am forever indebted to his wisdom and guidance.

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

1.0 Introduction

In this thesis, we generalize previous results and interpretations on a topic in Control Theory known as near-aggregation. We do this by establishing the behavior of nearly-aggregable systems in terms of two characterizations of system dynamics.

The first characterization is obtained through the trajectories of state space representations. The second is given by the residues of a partial fraction expansion on the transfer function matrix. With these generalizations of near-aggregation, we obtain new results which quantitatively relate near-aggregation to already established concepts of system analysis.

At the root of near-aggregation is the more basic concept of aggregation introduced by Aoki [1]. This original concept has proven to be useful in model reduction schemes. Fundamentally, the idea is that the trajectories of the reduced-order system are a linear combination of the trajectories of the full-order system. This relationship is expressed by the aggregation functional.

The Generalized Hessenberg Representation (GHR) was introduced by Tse et al. [4] to extend Aoki's aggregation concept. The idea was to attach the aggregation functional to the full-order model as an output equation, and to then study the structure induced by this output equation. This structure was represented by the GHR in that the A matrix has a block lower Hessenberg form. Then the system aggregates if and only if one of the super diagonal blocks is zero.

In this thesis, we review a generalization of aggregation and the GHR by examining systems in which none of the super diagonal blocks in the GHR are zero but one of them is small. Loosely speaking, this concept of aggregation is called near-aggregation. The generalization introduced here complicates the reduction methodology in two ways. First, the reduced-order model may not be controllable even if the full-order model is controllable. Secondly, the reduction methodology is dependent on the scaling in the system.

Both of these issues can be addressed by using the Dual Generalized Hessenberg Representation (Dual GHR) [8]. It is a canonic representation of a single input single output system represented in state space. The Dual GHR is unique in the forum of model reduction in that it provides a set of reduced-order models which are controllable and observable, and such that the scaling in the system is optimized. In terms of the ideas presented, both the GHR and Dual GHR provide natural settings to study system aggregational behaviour.

In order to develop our trajectory and residue analysis, we will review other concepts associated with near-aggregation and the Hessenberg forms. In Lindner and Perkins [6], near-aggregation was related to near-unobservability, near-uncontrollability, and almost pole zero cancellation. We will in turn ultimately relate these four ideas to state trajectories and transfer function residues. As will be shown, for a nearly-aggregable system, the trajectories of the full-order model are near a linear combination of the trajectories of the reduced-order model.

We will also see that under certain conditions, the magnitude of a residue is small if the system is nearly-unobservable and nearly-uncontrollable. Under certain assumptions, this in turn may be related to almost pole zero cancellations. To obtain this description of the residue in terms of the state space, we must interpose with another notion of unobservability and uncontrollability. This notion is based on modal techniques, and will allow us to compare residues with the concepts of near-aggregation above.

Chapter 2 outlines the basic results of aggregation and the state space developments of the GHR and Dual GHR. Chapter 3 introduces the extension of these results to near-aggregation. Chapter 4 contains the trajectory analysis, and Chapter 5 discusses the extension of aggregation to transfer function residues. Finally, Chapter 6 gives the conclusions, and possible ideas for further investigation.

2.0 Aggregation and State Space Representations

The underlying concept which ties together the fundamental ideas of this thesis, is that of "aggregation". The application of aggregation may be found throughout the literature of Linear Systems Theory, notedly in the application of reduced-order modelling [1]-[10]. We will first layout the basics of aggregation, and later extend this concept using Generalized Hessenburg Representations of the state space.

2.1 The Model and Aggregation

To first lend a more intuitive sense, we can formulate the idea of aggregation as a clustering together of information into a denser mass which still constitutes or describes the entire picture. In terms of the control system then, we may characterize this idea by the following: given a linear time-invariant system in state space representation with a large dimensional state vector, we wish to produce a reduced-order model which approximates the output of the given system.

Though the concept had earlier applications in the field of economics, Aoki [1] was the first to apply aggregation to the above control problem. In his formulation, aggregation as a model reduction technique seeks a reduced-order model which approximates the output variables. Essentially it exploits the information structure of the system by determining only that part of the system which contributes to the output. As we shall see, aggregation is inherently involved in many of the structures of the state space.

As first order of business, we introduce the model set. Unless otherwise noted in this thesis, we shall generally consider multiple input multiple output, continuous, linear, time-invariant systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.1.1a}$$

$$y(t) = Cx(t) \tag{2.1.1b}$$

where $x \in R^n$, $u \in R^m$, $y \in R^l$, and A, B, C are appropriately dimensioned constant matrices. We denote the state space by $\mathscr{X} = R^n$, and further assume, without loss of generality, that B and C are of full rank.

Next we present a summary review of aggregation and basic results, most of which can be found in Aoki [1] and Medanic et al. [2]. Assume for the moment that C in (2.1.1b) has not yet been specified. Now suppose we define a new state vector by

$$z = \begin{bmatrix} z_a \\ z_r \end{bmatrix} \tag{2.1.2}$$

where the dimension of z_a, $d(z_a) < n$. Here we use the subscript "a" to denote the aggregate state, and "r" to denote the residual states. Furthermore, suppose

$$z_a = Cx \ . \tag{2.1.3}$$

With such a relationship, we shall refer to C as the aggregation matrix.

In the sense of Aoki, we would like to construct a reduced-order model for the output variables of the form

$$\begin{aligned} \dot{z}_a &= F_{11} z_a + G_1 u \\ y &= z_a \ . \end{aligned} \tag{2.1.4}$$

Equation (2.1.4) in mind, we can now characterize aggregation in terms of the system of (2.1.1).

<u>Definition 2.1.1</u> [1] The system (2.1.1) is said to be <u>completely aggregable with respect to C</u> if there exists an lxl matrix F_{11} such that $CA = F_{11}C$. □

Thus, such an F_{11} exists iff an aggregate model of the form (2.1.4) exists.

On the other hand, as might be implied by the decomposition of z in (2.1.2), we can view complete aggregation as an outcome of a basis transformation T on (2.1.1). Let

$$x = Tz \tag{2.1.5}$$

where

$$T = \left[\bar{R}[C^T] \quad \bar{N}[C] \right] , \tag{2.1.6}$$

and $\bar{R}$ and $\bar{N}$ denote basis representations of the range and null space, respectively. Now apply (2.1.5) to (2.1.1) to obtain

$$\begin{aligned} \begin{bmatrix} \dot{z}_a \\ \dot{z}_r \end{bmatrix} &= \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} z_a \\ z_r \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u \\ y &= [\ H_1 \quad 0\ ] \begin{bmatrix} z_a \\ z_r \end{bmatrix} . \end{aligned} \tag{2.1.7}$$

Note first that T may be chosen such that $H_1 = I$. Now then, with regard to equations (2.1.2) and (2.1.4), the above representation can be thought of as an interconnection between to subsystems S_1 and S_2. The aggregate subsystem S_1 is given by

$$S_1: \begin{cases} \dot{z}_a = F_{11}z_a + F_{12}z_r + G_1u \\ y = z_a , \end{cases} \tag{2.1.8}$$

and the residual subsystem by

$$S_2: \left\{ \dot{z}_r = F_{21}z_a + F_{22}z_r + G_2u \right. . \tag{2.1.9}$$

In terms of the transformation viewpoint then, the system (2.1.1) is completely aggregable iff $F_{12} = 0$. If so, the composite system reduces to a tandem configuration where the aggregate drives the residual system, but where residual feedback into the aggregate is not present. See Figure 1 on the next page for a diagram of this interaction. As we see from the block diagram, the residual subsystem is unobservable through the aggregated variables. In other words, via the output information structure, the aggregation matrix has been used to identify and separate out the unobservable part of the system.

From these points we may also conclude that in the case $F_{12} = 0$, the transfer function of the system exhibits some pole zero cancellations. This and the above results, along with a few more insights on aggregation, are summarized by the following theorems. Note the cases in which a converse argument does not exist to imply complete aggregability.

<u>Theorem 2.1.1</u> [2] The following statements are equivalent:

1. The system (2.1.1) is completely aggregable with respect to C.
2. With regard to (2.1.6) $F_{12} = 0$.
3. The $\mathscr{N}[C]$ is A-invariant.
4. The $\mathscr{R}[C^T]$ is A^T-invariant. □

<u>Theorem 2.1.2</u> If the system (2.1.1) is completely aggregable with respect to C then:

1. The pair (A,C) is unobservable.
2. The transfer function of the system (2.1.1) is reducible. □

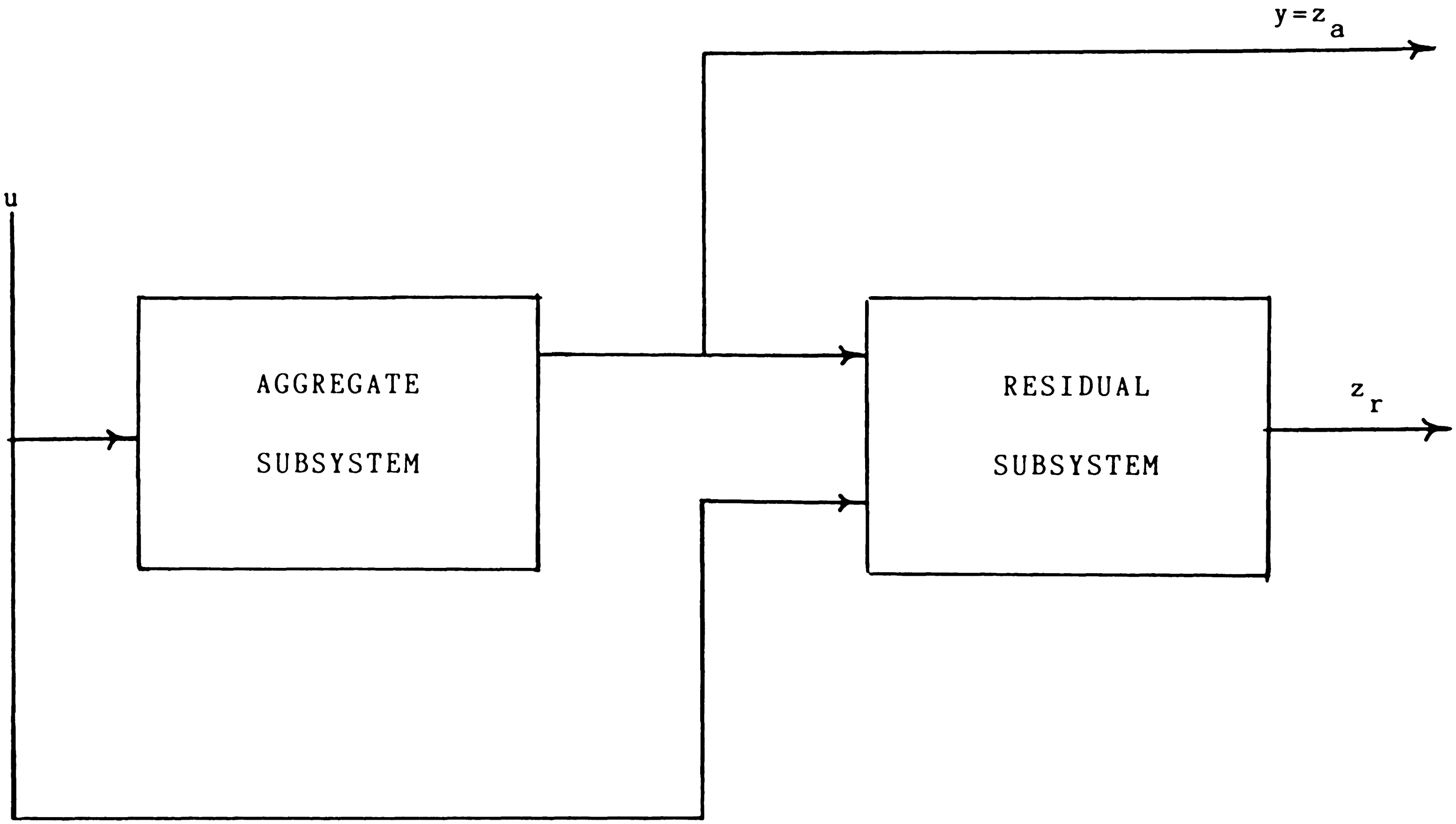

FIGURE 1: [2] Tandem Configuration of Aggregate and Residual Subsystems Under Complete Aggregation.

It is apparent from the above theorems that the requirements, and hence the applicability of technique of the completely aggregable system are somewhat restrictive. Some immediate questions which arise in this discussion are the following:

1. Given the system (2.1.1a) when does there exist an aggregation matrix C of (2.1.1b) such that the system is completely aggregable?
2. Given the existence of such a C, can we derive its precise form from (2.1.1a)?
3. If we can answer questions 1 and 2, what can be done when the required outputs of such a C lie outside the set of physically measurable outputs?

The development of a straightforward procedure for determining the aggregation matrix C was given by Hickin [3]. Thus, questions 1 and 2 were answered. The answer to the third question was introduced by Tse et al. [4,5], and formulated itself in terms of a generalization of complete aggregation known as chained aggregation. Significant and illuminating connections between aggregation, model reduction, system observability, and other facets of the state space were significantly advanced by this concept of chained aggregation.

2.2 The Generalized Hessenburg Representation

Let us consider again the system of (2.1.1)

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \ . \end{aligned} \tag{2.2.1}$$

but now let us assume that the output has been specified completely by an lxl matrix C of full rank. In the case that the subsystem is completely aggregable with respect to this C, we may proceed as in the last section to obtain

$$\begin{aligned} \begin{bmatrix} \dot{z}_a \\ \dot{z}_r \end{bmatrix} &= \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} z_a \\ z_r \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u \\ y &= [\ H_1 \quad 0 \ ] \begin{bmatrix} z_a \\ z_r \end{bmatrix} \end{aligned} \tag{2.2.2}$$

where $F_{12} = 0$, and the dynamics of the output are completely described by the reduced-order triple (F_{11}, G_1, H_1).

The more interesting and physically realistic case occurs though when $F_{12} \neq 0$. This less trivial situation will present itself when the pair (A,C) is unobservable, and thus, the system does not perfectly aggregate in the sense of Aoki. In order to extend the applicability of aggregation as a model reduction technique to such systems, an extension of aggregation called chained aggregation was introduced by Tse, et al. [5]. Chained aggregation consists of a finite sequence of aggregation steps which reduce (2.2.1) to a form called the Generalized Hessenburg Representation (GHR).

This process of chained aggregation is as follows [6]:

1. Apply Aoki's aggregation transformation T_1 (2.1.5) to obtain (2.2.2). If $F_{12} = 0$ or $d(z_r) = 0$ then the algorithm terminates and the system is in GHR.
2. If neither of these conditions hold, apply a second aggregation transformation T_2 to the residual subsystem S_2. However this time, we replace C with the residual output matrix F_{12}.
3. This in turn will give us an aggregate and residual subsystem similar to (2.2.2), but of smaller dimension. We may now test this decomposition for complete aggregability of the original residual with respect to the aggregation matrix F_{12}.

We continue in this fashion until one of the residual outputs $F_{i,i+1} = 0$, or until the column rank of $F_{i,i+1}$ matches the dimension of its associated residual subsystem, thus implying $d(z_r^{i+1}) = 0$.

After i steps of chained aggregation, the system (2.2.1) is transformed into the GHR form given by:

$$\dot{z}^i = \begin{bmatrix} F_{11} & F_{12} & 0 & \cdots\cdots & 0 \\ F_{21} & F_{22} & F_{23} & & \vdots \\ \vdots & & & \ddots & 0 \\ & & & & \ddots \\ F_{i,1} & \cdots\cdots & & \cdots\cdots & F_{i,i+1} \\ A_{i+1,1} & \cdots\cdots & & \cdots\cdots & A_{i+1,i+1} \end{bmatrix} z^i + \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_i \\ B_{i+1} \end{bmatrix} u \tag{2.2.3}$$

$$y = [\ H_1 \quad 0 \quad 0 \cdots\cdots 0\ ]\ z^i .$$

In terms of the aggregate and residual characterization, we might represent (2.2.3) more concisely by

$$\begin{bmatrix} \dot{z}_a^i \\ \dot{z}_r^i \end{bmatrix} = \begin{bmatrix} F^i & E^i \\ K^i & A^i \end{bmatrix} \begin{bmatrix} z_a^i \\ z_r^i \end{bmatrix} + \begin{bmatrix} G^i \\ B^i \end{bmatrix} u$$

$$y = [\ H^i \quad 0\ ] \begin{bmatrix} z_a^i \\ z_r^i \end{bmatrix} . \tag{2.2.4}$$

If we denote the dimensions of the square F_{ii} blocks by

$$d(F_{ii}) = r_i , \tag{2.2.5}$$

then by the construction of the algorithm, it is easy to see that $r_i \geq r_{i+1}$, and $\Sigma r_i = n$. The representation (2.2.3) is called the Generalized Hessenburg Representation since the F blocks generalize the scaler concept of an n^{th} order Hessenburg matrix, characterized by $f_{ij} = 0$ for $i = 1,2,\ldots,n-2$ and $j = i+2,\ldots,n$. Also note that the transformations are chosen such that $\mathcal{N}[H_1] = \mathcal{N}[F_{j,j+1}] = 0$ for $j = 1,\ldots,i-1$.

As we see, the GHR explicitly displays an internal structure which was originally induced by the output matrix C. Thus the internal structure is said to be induced by the information structure of the system. Now if $F_{i,i+1} = 0$ in (2.2.3), then an obvious reduced order model induced by the information structure is

$$\begin{aligned} \dot{z}_a^i &= F^i z_a + G^i u \\ y &= H^i z_a \end{aligned} \tag{2.2.6}$$

This motivates the following definition.

<u>Definition 2.2.1</u> [7] The system (2.2.1) is said to <u>exactly aggregate</u> if $F_{i,i+1} = 0$ after i steps of aggregation. □

If i = 1 in (2.2.6) then this aggregation process is the same as introduced by Aoki. Thus, the GHR extends the notion of complete aggregation.

A property of the GHR is that this system representation is not defined by a unique basis. For instance, $\mathcal{N}[C]$ can be represented by any number of bases. In this sense then, the GHR cannot be considered to be a canonic form. However, it does represent a family of system representations which have some very appealing structural characteristics. A few of the more important ones are summarized below.

<u>Lemma 2.2.1</u> [2] Any linear system of the form (2.2.1) can be transformed into the Generalized Hessenburg Representation (2.2.3). □

Lemma 2.2.2 [2] The internal structure of the GHR characterized by the indices $r_1, \ldots r_k$ is unique. □

Assume that an inner product has been specified along with (2.2.1). In order to preserve this inner product under state space transformations, we require that these transformations be orthogonal. Under such conditions, we have the following:

Lemma 2.2.3 [7] Suppose the GHR (2.2.3) is constructed from (2.2.1) using orthogonal transformations. Then the norm $\|F_{j,j+1}\|$ is unique. □

Finally, with the extension to chained aggregation we are able to strengthen the result of Theorem 2.1.2.

Lemma 2.2.4 The system (2.1.1) exactly aggregates iff the pair (A,C) is unobservable. □

Note however, that a reducible transfer function does not necessarily imply exact aggregation and/or unobservability. In short, the system may just be uncontrollable. We close this introduction on chained aggregation and the GHR with a few remarks.

Remark 2.2.1 To this point we have only considered the output information structure. To characterize the input structure, we may simply apply the GHR and Definition 2.2.1 to the dual system (A^T, B^T).□

Remark 2.2.2 Note that in Aoki's original concept of aggregation, the reduced-order model was restricted to be the same dimension as the number of outputs. The GHR removes this restriction on the reduced-order model to induce a larger set of possible choices. The order of these reduced models is then imposed by the r_i in general, and not just on $r_i = 1$. □

<u>Remark 2.2.3</u> A computer algorithm written in the L-A-S software package [11] is given for the reduction of a system into GHR form (see Appendix). It is noted that this subroutine employs orthogonal matrices to perform this task. □

2.3 The Dual Generalized Hessenburg Representation

In the last section it was shown that complete aggregation could be generalized by the GHR to fully develop an internal structure induced by the output structure. As per Remark 2.2.1 we can dualize this statement in terms of the input information structure. A question which arises then, is can we simultaneously expose the internal structure in terms of both the inputs and outputs? This question was resolved for single input single output systems. The resulting representation is called the Dual Generalized Hessenburg Representation (Dual GHR) and is now discussed.

Consider the SISO model (2.1.1) where $l = m = 1$. There exists a state transformation

$$x = Tz \qquad (2.3.1)$$

such that the triple (A,B,C) takes on the following canonic form:

$$(F,G,H) = \tag{2.3.2}$$

$$\left(\begin{bmatrix} F_1 & H_2 & 0 & \cdots & 0 \\ G_2 & F_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & H_r \\ 0 & \cdots & 0 & G_r & F_r \end{bmatrix}, \begin{bmatrix} G_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} H_1 & 0 & \cdots & 0 \end{bmatrix} \right)$$

where

$$F_k = \begin{bmatrix} a_{k1} & 1 & & \\ \vdots & & \ddots & 1 \\ a_{k\sigma_k} & & & \end{bmatrix}$$

and

$$G_k = \begin{bmatrix} & \cdots & 0 \\ & \iddots & \vdots \\ \gamma_k & & \end{bmatrix}, \quad H_k = \begin{bmatrix} & \cdots & 0 \\ & \iddots & \vdots \\ \varepsilon_k \gamma_k & & \end{bmatrix}. \tag{2.3.3}$$

The state matrix has a block tridiagonal structure with the dimension of the F_k block as $\sigma_k \times \sigma_k$. The diagonal blocks are in a standard phase canonical form. The off diagonal blocks G_k and H_k have only one non-zero element which always appears in the lower left hand corner. Furthermore, $\gamma_k > 0$ and $\varepsilon_k = \pm 1$. Therefore, the input and output matrices have only one non-zero element with the output matrix's non-zero element appearing in the (1,1) position.

The transformation matrix T in (2.3.1) can be constructed algorithmically from (2.1.1). This algorithm is called extended chained aggregation and is derived from chained aggregation. A discussion of this algorithm is given in [8]-[10].

The generic form of the Dual GHR occurs when $\sigma_k = 1$ for all k; that is, F is a tridiagonal matrix and $H_k = G_k^T$ modulo a sign. In this case the system (2.3.2) will be called regular. Note that if (2.3.2) is regular then considering (F,H) we see that (2.3.2) is a GHR. Furthermore, we see that (F^T, G^T) is also a GHR. By an easy rearrangement of variables, $(\bar{F}^T, \bar{G}^T)$ will also be a GHR. Herein lies the motivation of the notation Dual, where in the above sense the Dual GHR is an extension of the GHR introduced by Tse, et al.

To gain a more intuitive feeling for the Dual GHR we give the following example.

<u>Example 2.3.1</u> Consider this system representation already in Dual GHR form:

$$\dot{x} = \left[\begin{array}{ccc:c} -21.59 & 1 & 0 & 0 \\ -77.21 & 0 & 1 & 0 \\ -25.06 & 0 & 0 & -.419 \\ \hdashline .419 & 0 & 0 & -0.07 \end{array}\right] z + \left[\begin{array}{c} 0 \\ 0 \\ 1 \\ \hdashline 0 \end{array}\right] u \tag{2.3.4}$$

$$y = \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \end{array}\right].$$

For this example, $k = 2$, $\sigma_1 = 3$, and $\sigma_2 = 1$. □

Suppose we divide the state vector of (2.3.2) into an aggregate and residual part as before. Then a similar decomposition is induced in the Dual GHR system as follows:

$$\begin{bmatrix} \dot{z}_a^i \\ \dot{z}_r^i \end{bmatrix} = \begin{bmatrix} F^i & H^i \\ G^i & E^i \end{bmatrix} \begin{bmatrix} z_a^i \\ z_r^i \end{bmatrix} + \begin{bmatrix} B^i \\ 0 \end{bmatrix} u$$
$$y = [\; C^i \quad 0 \;] \begin{bmatrix} z_a^i \\ z_r^i \end{bmatrix} . \tag{2.3.5}$$

Here

$$F^i = \begin{bmatrix} F_1 & H_2 & 0 \cdots 0 \\ G_2 & F_2 & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots \\ \vdots & & \ddots & H_i \\ 0 \cdots 0 & & G_i & F_i \end{bmatrix} , \tag{2.3.6}$$

and the other subblocks in (2.3.5) are defined accordingly. Thus, these decompositions do not split the diagonal blocks F_k. For example:

$$F^2 = \begin{bmatrix} F_1 & H_2 \\ G_2 & F_2 \end{bmatrix} , \; G^2 = \begin{bmatrix} G_3 & \vdots \\ & \ddots \bigcirc \end{bmatrix} , \; H^2 = \begin{bmatrix} & \ddots \bigcirc \\ H_3 & \vdots \end{bmatrix} . \tag{2.3.7}$$

Also define the indices

$$\rho_i = \sum_{j=1}^{i} \sigma_j , \quad \rho_0 = 0 . \tag{2.3.8}$$

Then the decomposition of (2.3.6) has diagonal blocks F^i and E^i which are dimensioned by $(\rho_i \times \rho_i)$ and $(n - \rho_i) \times (n - \rho_i)$ respectively.

Since the Dual GHR is quite similar to the GHR in form, we would hope to be able to generalize the results of Lemma's 2.2.1 through Lemma's 2.2.4. Indeed we can, and so make the following notes on these generalizations.

Lemma 2.3.1 [9] Any linear time-invariant SISO system of the form (2.1.1) can be transformed into the Dual GHR (2.3.2). □

Lemma 2.3.2 [9] The Dual GHR is a canonic form. Specifically the Dual GHR characterized by the indices $\sigma_1,\ldots,\sigma_k$ is unique. □

Lemma 2.3.3 The system (2.2.1) exactly aggregates iff the pair (A,C) is unobservable. □

Remark 2.3.1 Lemma 2.3.1 through Lemma 2.3.3 follow directly from the algorithm given in [8]. Note also that we may infer a similar relationship between reducible transfer functions and exact aggregation. □

Remark 2.3.2 Furthermore, the Dual GHR has one additional property which the GHR cannot claim. In the Dual GHR basis, the reduced-order model defined by the triple (F^i, B^i, C^i) is controllable and observable. In general, reduced-order models obtained by the GHR may be uncontrollable, even if the full-order model is completely controllable. □

A unique feature of the Dual GHR is its ability to easily display the pole zero structure of the system. From (2.3.6) we define

$$
\begin{aligned}
q_i(s) &= \det[\, sI - F^i \,] \ , \quad q_0(s) = 1 \ , \\
r_i(s) &= \det[\, sI - E^i \,] \ , \quad r_{r+1}(s) = 1 \ ,
\end{aligned}
\tag{2.3.9}
$$

and

$$
\bar{q}_i(s) = \det \left\{ sI - \begin{bmatrix} F_1 & H_2 & 0 & \cdots & 0 \\ G_2 & F_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & H_i \\ 0 & \cdots & 0 & G_i & F_i \end{bmatrix} \right\} i = 2,\ldots,r \ ,
$$

$$
\bar{q}_1(s) = 1 \ . \tag{2.3.10}
$$

Let the transfer function of (2.3.5) be

$$
G(s) = \frac{\Gamma z(s)}{p(s)} \ . \tag{2.3.11}
$$

That is, z(s) is monic.

<u>Theorem 2.3.1</u> [9]

$$
\begin{aligned}
p(s) &= q_i(s) r_i(s) - \varepsilon_{i+1}\gamma_{i+1}^2 q_{i-1}(s) r_{i+1}(s) \ , \\
z(s) &= \bar{q}_i(s) r_i(s) - \varepsilon_{i+1}\gamma_{i+1}^2 \bar{q}_{i-1}(s) r_{i+1}(s) \ , \\
\Gamma &= \varepsilon_1 \gamma_1^2 \ .
\end{aligned}
\tag{2.3.12}
$$

□

Proof: The formula for p(s) can be verified by Laplace expansion along the ρ_ith row. To see z(s) note that in the calculation of adj(sI-A), we need only look at the cofactor associated with the ρ_ith row and first column. The left upper block of the cofactor is an identity matrix, and therefore the zero polynominal obtained is given by

$$z(s) = r_1(s) = \det[\, sI - E^1 \,] . \tag{2.3.13}$$

Now the form of z(s) follows from the form of p(s). The gain Γ may be verified by direct computation. □

By iterative application of Theorem 2.3.1, the transfer function of the system can be obtained by inspection from the Dual GHR.

Example 2.3.2 Consider a regular Dual GHR

$$\dot{x} = \begin{bmatrix} a_1 & \gamma_2 & 0 \\ \gamma_2 & a_2 & \gamma_3 \\ 0 & \gamma_3 & a_3 \end{bmatrix} x + \begin{bmatrix} \gamma_1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [\ \gamma_1 \quad 0 \quad 0\ ]\, x . \tag{2.3.14}$$

By repeated application of Theorem 2.3.1 we have

$$\begin{aligned} p(s) &= (s-a_1)r_1(s) - \gamma_2^2 r_2(s) \\ &= (s-a_1)[(s-a_2)(s-a_3) - \gamma_3^2] - \gamma_2^2(s-a_3) , \end{aligned} \tag{2.3.15}$$

and

$$z(s) = (s-a_2)(s-a_3) - \gamma_3^2 . \tag{2.3.16}$$

Now the transfer function is

$$G(s) = \frac{\gamma_1^2 z(s)}{p(s)} . \tag{2.3.17}$$

Starting with (2.3.14) and working backwards through (2.3.17) we obtain

$$\frac{\gamma_1^2 z(s)}{p(s)} = \frac{\gamma_1^2}{(s-a_1) - \gamma_2^2 \dfrac{(s-a_3)}{z(s)}} \tag{2.3.18}$$

$$= \frac{\gamma_1^2}{(s-a_1) - \dfrac{\gamma_2^2}{(s-a_2) - \dfrac{\gamma_3^2}{(s-a_3)}}} . \tag{2.3.19}$$

□

Kalman [12] originally developed the form (2.3.2) by applying a Euclidean algorithm to the transfer function. This Euclidean algorithm is easily recovered as shown by the above example. Also note the close relationship with a particular continued fraction (2.3.19). The similarity between the Dual GHR and another continued fraction expansion proposed by Mitra and Sherwood [13] may be found in DeBrunner [14].

<u>Remark 2.3.3</u> Theorem 2.3.1 implys that the Dual GHR explicitly displays the invariant zeros [16] of the system. In general, the GHR may also be exploited to obtain the invariant zeros of a MIMO system [15].
□

Remark 2.3.4 A computer algorithm written in LAS is also given for the reduction of a SISO system into the Dual GHR (see the Appendix).□

2.4 Geometry

In this section we introduce some geometrical quantities which play a fundamental role in interpreting the structures associated with aggregation.

Definition 2.4.1 The i^{th}-unobservable subspace $\mathscr{L}_i$ is defined by

$$\mathscr{L}_i = \bigcap_{j=0}^{i-1} \mathscr{N}[CA^j] ,$$

$$\mathscr{L}_0 = \mathscr{X}.$$
□

These unobservable subspaces satisfy the following properties, and are further discussed in [15].

Lemma 2.4.1 [6]

1. $\mathscr{L}_i$ is a subspace.
2. $\mathscr{L}_i \supset \mathscr{L}_{i+1}$
3. There exists an $m \leq n$ such that $\mathscr{L}_i \supset \mathscr{L}_{i+1}$ for $i < m$ and $\mathscr{L}_m = \mathscr{L}_i$ for all $i \geq m$.
4. $\mathscr{L}_i$ are invariant to a change of basis in the state and output spaces.
5. $\mathscr{L}_m$ is the standard unobservable subspace. □

These unobservable subspaces are intimately related to the GHR. After i steps of aggregation, let the system be represented as in (2.2.3). If we denote e_i as the i^{th} natural basis vector with respect to (2.2.3) then we have:

Theorem 2.4.1 [15]

1. $\mathscr{L}_j = \sum_{k=\rho_{i+1}}^{n} sp[e_k] \quad ; \quad 1 \le j \le i.$
2. $\mathscr{L}_{i+1} = \mathscr{O}$ if $F_{i,i+1} \neq 0$.
3. $\mathscr{L}_{i+1} = \mathscr{L}_i$ if $F_{i,i+1} = 0$. □

With Theorem 2.4.1 we may provide a few geometric interpretations of chained aggregation. First note that at the i^{th} step of aggregation we are actually selecting a transformation T_i which explicitly identifies the subspace $\mathscr{L}_i$. Secondly we see that the GHR as a system representation easily identifies the unobservable subspaces. Thus, the GHR lends itself nicely to the geometrical viewpoint.

With Theorem 2.4.1 we may now state the following.

Lemma 2.4.2 [7] The system (2.1.1) is exactly aggregable iff $\mathscr{L}_i$ coincides with an A-invariant subspace for some i. □

Remark 2.4.1 For large systems direct evaluation of $\mathscr{L}_i$ via Definition 2.4.1 is numerically unfeasible for general basis representations. This is due to the successive multiplication of A with itself. Using orthogonal transformations we may use a GHR for the system to obtain the $\mathscr{L}_i$ by inspection. With Lemma 2.4.1;4 we may then transform back to the original basis to obtain the desired $\mathscr{L}_i$ with respect to (2.1.1). Due to the numerical properties of orthogonal matrices, this algorithm makes the GHR especially attractive. □

We may also define the following structure which is also useful.

Definition 2.4.2 The i^{th} reachable subspace $\mathscr{R}_i$ is defined by

$$\mathscr{R}_i = \sum_{j=0}^{i-1} A^j \mathscr{B} , \quad \text{where } \mathscr{B} = \mathrm{Im}[B] ,$$

$$\mathscr{R}_0 = \mathscr{O} .$$

□

In this characterization of the input we also obtain an analogous set of properties similar to Lemma 2.4.1, though generally opposing in nature. For instance, as we view the $\mathscr{L}_i$ as a decreasing sequence of subspaces, we in turn view the $\mathscr{R}_i$ as an increasing sequence of subspaces.

Theorem 2.4.2 Consider the Dual GHR in (2.3.2). Then

$$\mathscr{L}_i = \sum_{j=i+1}^{n} \mathrm{sp}[e_j] ,$$

$$\mathscr{R}_i = \mathscr{R}_{m+k} = \mathscr{R}_{m} + \sigma_{m-1} - 2 \quad + \quad \sum_{j=\rho_m - k}^{m} \mathrm{sp}[e_j] ,$$

where $m = 1,\ldots,r$, and $k = 0,\ldots,\sigma_m - 1$. □

The subspaces $\mathscr{L}_i$ characterize the output structure of the system. By duality, we could use the GHR to identify the subspaces $\mathscr{R}_i$, which characterize the input structure. Theorem 2.4.2 says that the Dual GHR simultaneously identifies both sets of subspaces in a simple way. In other words, the Dual GHR is identifying the input-output interaction in terms of the internal state. It is this input-output interpretation which makes the Dual GHR unique among all other canonical forms.

<u>Example 2.4.1</u> Consider the following single input single output system already in Dual GHR form:

$$(F,G,H) = \left\{ \left[\begin{array}{c:ccc:cc} 1 & -1 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right], \left[\begin{array}{c} 1 \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 1 \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]^T \right\}.$$

From Theorem (2.4.2) we have

$$\begin{array}{ll} \mathscr{L}_1 = \mathrm{sp}[e_2,\ldots,e_6] & \mathscr{R}_1 = \mathrm{sp}[e_1] \\ \mathscr{L}_2 = \mathrm{sp}[e_3,\ldots,e_6] & \mathscr{R}_2 = \mathrm{sp}[e_1,e_4] \\ \mathscr{L}_3 = \mathrm{sp}[e_4,e_5,e_6] & \mathscr{R}_3 = \mathrm{sp}[e_1,e_3,e_4] \\ \mathscr{L}_4 = \mathrm{sp}[e_5,e_6] & \mathscr{R}_4 = \mathrm{sp}[e_1,\ldots,e_4] \\ \mathscr{L}_5 = \mathrm{sp}[e_6] & \mathscr{R}_5 = \mathrm{sp}[e_1,\ldots,e_4,e_6] \\ \mathscr{L}_6 = \mathscr{O} & \mathscr{R}_6 = \mathscr{X}. \end{array}$$

Note the typical circling effect exhibited in the $\mathscr{R}_i$ spaces. Also we have $\sigma_1 = 1$, $\sigma_2 = 3$, and $\sigma_3 = 2$. □

2.4.1 Subspace Decompositions

The Dual GHR naturally identified the subspaces $\mathscr{L}_i$ and $\mathscr{R}_i$ by imposing a specific basis for the system in which $\mathscr{L}_{\rho_i}$ is always orthogonal to $\mathscr{R}_{\rho_i}$. With such a relationship, the idea that these subspaces somehow naturally decompose the state space comes to mind. We now proceed to characterize this decomposition.

<u>Definition 2.4.3</u> [8] Let $\mathscr{X} = R^n$ and suppose an inner product is defined on R^n. Let $\mathscr{X}_i$, $i = 1,\ldots,r$ form a set of subspaces of $\mathscr{X}$ such that

$$1.\ \mathscr{X}_1 \oplus \ldots \oplus \mathscr{X}_r = R^n ,$$
$$2.\ \mathscr{X}_i \perp \mathscr{X}_j , \quad i \neq j .$$

We say that $\mathscr{X}_i$, $i = 1,\ldots,r$ is a <u>decomposition of</u> R^n or that the $\mathscr{X}_i$ <u>decompose</u> the state space R^n. □

The interaction of the subspaces $\mathscr{L}_i$ and $\mathscr{R}_i$ induce a decomposition of the state space in the following way. Define the subspaces $\mathscr{K}_i$ by

$$\begin{gathered} \mathscr{R}_{\rho_{i-1}} + \mathscr{K}_i + \mathscr{L}_{\rho_i} = R^n \\ \mathscr{R}_{\rho_{i-1}} \perp \mathscr{K}_i \perp \mathscr{L}_{\rho_i} , \quad i = 1,\ldots,r . \end{gathered} \tag{2.4.1}$$

<u>Lemma 2.4.4</u> [8] Suppose $\mathscr{K}_i$ are defined as in (2.4.1). Then with respect to the Dual GHR basis we have:

$$\mathscr{K}_i = \sum_{k=\rho_{i-1}+1}^{\rho_i} sp[e_k] \quad ; \quad i = 1,\ldots,r \; . \qquad \square$$

Example 2.4.4 (Revisited) Here we obtain

$$\mathscr{K}_1 = sp[e_1]$$
$$\mathscr{K}_2 = sp[e_2,\ldots,e_4]$$
$$\mathscr{K}_3 = sp[e_5,e_6] \; . \qquad \square$$

Example 2.4.5 In a regular system $\mathscr{K}_i = sp[e_i]$, $i = 1,\ldots,n$. $\square$

The above development gives a detailed geometrical description of the basis of the Dual GHR. Equation (2.4.1) implies that the decomposition induced by the interaction of the subspaces $\mathscr{L}_i$ and $\mathscr{R}_i$ does not completely specify the basis. Instead there remains some freedom to choose the internal scaling of the subspaces $\mathscr{K}_i$. In the Dual GHR basis we use this freedom in two ways.

The first and less significant choice is used to impose a phase canonical form on the F_{ii} blocks. One advantage to this is that it readily displays the pole polynomial structure of the aggregated blocks. The second choice is used to simultaneously scale the off-diagonal elements such that $\|G_i\| = \|H_i\|$. The idea here is to simultaneously balance the input and output. This will be shown in later chapters to be an optimum, and hence a rather important quality of the Dual GHR when it is applied to systems which do not exactly aggregate.

Remark 2.4.1 As per the above comments, Lindner's algorithm completely fixes the internal scaling in the subspaces $\mathscr{K}_i$. This along with the structures associated with the $\mathscr{L}_i$ and $\mathscr{R}_i$ completely specifies the basis of the Dual GHR. Hence, the Dual GHR is indeed a canonical representation. $\square$

3.0 Concepts of Nearness in the State Space

In the last chapter we discussed the concept of aggregation in the state space. We also discussed how this concept related to other ideas; specifically the ideas of unobservability, uncontrollability, and pole zero cancellations. We now move to further extend the aggregation concept by examining systems in which none of the super diagonal blocks in the GHR are zero but one is small. This extension is referred to as near-aggregation, and similarly evokes concepts of near-unobservability, near-uncontrollability, and almost pole zero cancellations.

3.1 Near Aggregation

Consider the following two state space systems:

$$S_1: \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} F & H \\ G & E \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u \\ y_1 = [\, C \quad 0 \,] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} , \end{cases} \tag{3.1.1}$$

and

$$S_2: \begin{cases} \dot{z} = Fz + Bu \\ y_2 = Cz \, . \end{cases} \tag{3.1.2}$$

Note that S_2 is derived from S_1. In review of the ideas presented before:

<u>Definition 3.1.1</u> [9] The system S_2 is said to be an <u>output aggregation</u> of S_1 if H=0. □

<u>Definition 3.1.2</u> [9] The system S_2 is said to be an <u>input aggregation</u> of S_1 if G=0. □

Next we characterize the system (3.1.1) when $\|H\|$ and $\|G\|$ are non-zero but relatively small.

<u>Definition 3.1.3</u> [9] The system S_1 is said to be μ_0-<u>nearly output aggregable</u> if $\|H\| \le \mu_0$. □

Definition 3.1.4 [9] The system S_1 is said to be μ_0-nearly input aggregable if $\|G\| \leq \mu_0$. □

Definition 3.1.5 [9] The system S_1 is said to be μ_0 - nearly input-output aggregable if $\|G\| \leq \mu_0$ and $\|H\| \leq \mu_0$. □

We are interested in systems for which μ_0 is very small. In short we call such systems nearly-aggregable. Implicit in this generalization of aggregation is the assumption that in addition to the usual algebraic structure of (3.1.1), the state space $\mathscr{X}$ has an attached inner product. Of important note is that an arbitrary change in basis will, in general, change this inner product.

Example 3.1.1 [9] Consider the second order system

$$\dot{x} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x .$$

If $\bar{x}_2 = tx_2$, then

$$\dot{\bar{x}} = \begin{bmatrix} a_1 & a_2 t^{-1} \\ ta_3 & a_4 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \bar{x} .$$

By choosing t very large or very small we can make this system nearly-output aggregable or nearly-input aggregable, but not both. □

Thus, in order to apply the above measure consistently, we must assume an inner product a priori. In turn then, all transformations must somehow preserve this information structure. As per Remark 2.4.1 and Lemma 2.2.3, the Dual GHR and orthogonally transformed GHR induce this desired uniqueness. Therefore, the inner product associated with the GHR and Dual GHR basis provides a natural setting to investigate near-aggregation. This inner product leads to a 2-norm on the underlying vector space given by

$$\|x\| = (x^T x)^{1/2} \tag{3.1.3}$$

In this paper, unless otherwise specified, we will assume all norms are the 2-norm in (3.1.3) or the corresponding induced operator norm.

We may now confidentially associate the μ_0 measure with the GHR and Dual GHR basis:

Lemma 3.1.1 Suppose that (3.1.1) is a GHR. Then the system S_1 is μ_0-output aggregable for $\|F_{i,i+1}\| \leq \mu_0$. □

Lemma 3.1.2 Suppose that (3.1.1) is a Dual GHR. Then the system S_1 is μ_0-input-output aggregable for $\gamma_i \leq \mu_0$. □

Example 3.1.2 Consider again the model in Example 2.3.1. This system is 0.419 - I/O aggregable. □

Remark 3.1.1 By choosing $\|G\| = \|H\| = \gamma_i$ in the Dual GHR basis, we are simultaneously minimizing the strength of these coupling terms. Thus, Lemma 3.1.2 directly implies that the Dual GHR basis gives the minimum μ_0 such that a system is μ_0 - I/O aggregable. In this sense then, the Dual GHR optimizes the scaling in the system. This optimization will in turn lead to unique reduced-order models. □

Remark 3.1.2 When none of the super-diagonal blocks of the GHR are small, we may still successfully apply model reduction techniques based on this form. The process involves adjusting certain parameters of the GHR in order to minimize the neglected terms [2]. □

3.2 Near Unobservability and Near Uncontrollability

We have already seen that a system is exactly aggregable iff it is unobservable iff there exists an A-invariant subspace in $\mathscr{N}[C]$. If (3.1.1) is μ_0-I/O aggregable is the system nearly-unobservable? We investigate this question next. Intuitively we say the system (3.1.1) is nearly-unobservable if there exists an A-invariant subspace $\mathscr{V}$ close to $\mathscr{L}_1$. To make this statement precise we next introduce a distance measure between subspaces.

Definition 3.2.1 [19] Let $\mathscr{U}$ and $\mathscr{V}$ be subspaces of C^n. The gap between $\mathscr{U}$ and $\mathscr{V}$ is the number

$$\tau(\mathscr{U},\mathscr{V}) = \max \left\{ \sup_{\substack{\|u\|=1,\, v\varepsilon\mathscr{V} \\ u\varepsilon\mathscr{U}}} \inf \|v-u\| \,,\; \sup_{\substack{\|v\|=1,\, u\varepsilon\mathscr{U} \\ v\varepsilon\mathscr{V}}} \inf \|v-u\| \right\} .$$

□

Remark 3.2.1 In the original model description (2.1.1) we assumed the state space was given by $\mathscr{X} = R^n$. In the event the system contains complex poles then the corresponding eigenvectors belong to C^n. We will later address this issue, and for now present the foregoing results which are based on the inner product $x^H y$ for vectors $x,y \in C^n$.

A consequence of Definition 3.3.1 is that $\tau(\mathcal{U}, \mathcal{V}) = 1$ if $d(\mathcal{U}) \neq (\mathcal{V})$. If $d(\mathcal{U}) = d(\mathcal{V})$, we can give a geometrical interpretation of the gap in terms of canonical angles. Let the conjugate transpose be denoted by superscript H.

<u>Definition 3.2.2</u> Let $\mathcal{U}$ and $\mathcal{V}$ be subspaces of C^n with orthonomal bases U and V respectively. Let σ_i be the singular values of $U^H V$. Then the <u>canonical angles</u> between $\mathcal{U}$ and $\mathcal{V}$ are the numbers

$$\theta_i = \cos^{-1}\sigma_i \ . \qquad \square$$

Also we define

$$\Phi(\mathcal{U},\mathcal{V}) = \max\{\theta_i\} \ . \tag{3.2.1}$$

The arguments of $\Phi(.,.)$ may be subspaces themselves, or matrices which consist of basis vectors of the respective subspace.

The gap function is related to canonical angles as follows:

<u>Lemma 3.2.1</u> [20] $\qquad \tau(\mathcal{U},\mathcal{V}) = |\sin\Phi(\mathcal{U},\mathcal{V})| \ . \qquad \square$

Hence, if all the canonical angles between two subspaces are small, they are close in the gap topology.

With the concept of distance between subspaces well defined we may now introduce a quantitative measure of observability. Let $0 \leq \varepsilon_0 < 1$ be given.

Definition 3.2.3 Let $\mathscr{V}$ be an A-invariant subspace of (3.1.1). Then $\mathscr{V}$ is said to be ε_0-unobservable if there exists some j such that

$$(\mathscr{V}, \mathscr{L}_j) \leq \varepsilon_0 \, . \qquad \square$$

By duality we get a controllability measure:

Definition 3.2.4 Let $\mathscr{V}$ be an A-invariant subspace of (3.1.1). Then $\mathscr{V}$ is said to be ε_0-uncontrollable if there exists some j such that

$$(\mathscr{V}, \mathscr{L}_j) \leq \varepsilon_0 \, . \qquad \square$$

Remark 3.2.2 Clearly we are interested in those subspaces which are ε_0-unobservable or ε_0-uncontrollable for small ε_0. In short we call such subspaces nearly-unobservable or nearly-uncontrollable, respectively. Furthermore, we shall in general refer to this set of measures as ε_0-measures. $\square$

Next we give another method [6] to compute the gap between unobservable subspaces $\mathscr{L}_i$ and A-invariant subspaces, similarly applicable to $\mathscr{R}_i$. Although the method given by Stewart via canonical angles can be easily implemented on computer for any pair $(\mathscr{U}, \mathscr{V})$, this method gives a more intuitive feeling to near-unobservability.

Suppose that the natural orthonormal basis of R^n (C^n) yields a basis for $\mathscr{L}_i^{\perp}$ and $\mathscr{L}_i$, respectively. In matrix form

$$L = \begin{bmatrix} I_{\rho_i} & 0 \\ 0 & I_{n-\rho_i} \end{bmatrix} = [\; L_c \quad L_i \;] \tag{3.2.2}$$

where the first r columns span $\mathscr{L}_i^{\perp}$ and the last n-r columns span $\mathscr{L}_i$. Let a second n - ρ_i dimensional subspace $\mathscr{V}$ and its complement be spanned by the orthogonal basis

$$\bar{P} = \begin{bmatrix} I_r & P \\ -P^T & I_{n-\rho_i} \end{bmatrix} \begin{bmatrix} (I+PP^T)^{-1/2} & 0 \\ 0 & (I+P^TP)^{-1/2} \end{bmatrix}$$

$$= [\ V_c V\ ]\ . \tag{3.2.3}$$

To compute the canonical angles between $\mathscr{L}_i$ and $\mathscr{V}$, note that

$$L_i^T V = (I+P^TP)^{-1/2}\ . \tag{3.2.4}$$

Let P have singular values σ_i. Then the canonical angles between $\mathscr{L}_i$ and $\mathscr{V}$ are given by

$$\theta_i = \cos^{-1}(1+\sigma_i^2)^{-1/2}\ . \tag{3.2.5}$$

It follows that

$$\sigma_i = \tan\theta_i\ . \tag{3.2.6}$$

Thus,

$$\tau(\mathscr{V},\mathscr{L}_i) = |\sin\Phi(\mathscr{V},\mathscr{L}_i)\ | \le \|P\| = |\tan\Phi(\mathscr{V},\mathscr{L}_i)\ |\ . \tag{3.2.7}$$

Remark 3.2.3 Note here that if $\mathscr{V}$ is an ε_0-unobservable subspace then we may express $\|P\|$ in (3.2.3) by

$$\|P\| = \frac{\varepsilon_0}{\sqrt{1-\varepsilon_0^2}}\ . \tag{3.2.8}$$

□

To test a system for near-unobservability then, we may employ the GHR or Dual GHR to help identify an invariant subspace near $\mathscr{L}_i$. To check for this geometry, we look for a small super-diagonal element, and then search for $(n - \rho_i)$ dimensional invariant subspaces such that

$$\mathscr{V} = \text{sp}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \text{sp}\begin{bmatrix} P \\ I \end{bmatrix}, \tag{3.2.9}$$

where

$$P = V_1 V_2^{-1}. \tag{3.2.10}$$

If such a subspace exists, then (3.2.7) holds from which the gap $\tau(\mathscr{V}, \mathscr{L}_i)$ can be computed.

<u>Example 3.2.1</u> Consider again the model of Example (2.3.1). Though we probably wouldn't consider this system nearly-aggregable, we may still calculate an observability measure. Looking over all one dimensional eigenvectors, we find that the eigenvector corresponding to the pole at -.079 is computed as

$$V^T = [\ -.022 \quad -.472 \quad -1.657 \quad 1\ ]\ . \tag{3.2.11}$$

Using (3.2.7) we get

$$(\mathscr{V}, \mathscr{L}_3) = .865\ . \tag{3.2.12}$$

Now suppose in general, the sytem is nearly-aggregable. Does this imply that the system is also nearly-unobservable? Roughly speaking, nearly-aggregable systems are nearly-unobservable if there exists an appropriately dimensioned invariant subspace. We will now derive conditions to determine when these invariant subspaces of a given dimension exist.

Define the operator

$$T(P) = FP - PE \tag{3.2.13}$$

which is linear in P.

Definition 3.2.5 [18] The separation of F and E denoted by δ, is defined as

$$\delta = \begin{cases} \|T^{-1}\|^{-1} & 0 \notin \lambda(T) \\ 0 & 0 \in \lambda(T) \end{cases}$$

where $\lambda(T)$ denotes the eigenvalues of T. □

It is well known that the eigenvalues of T, $\lambda(T)$ are non-zero iff F and E have no common eigenvalues. Thus the separation measure gives an indication of the separation between the candidate reduced-order model and the residual system. If $\delta \neq 0$ then we say that the system is separable.

Theorem 3.2.1 [18] Consider (3.1.1) and suppose that $\delta > 0$. Let $\gamma = \|H\|$ and $\eta = \|G\|$. If

$$\frac{\gamma\eta}{\delta^2} < \frac{1}{4} \tag{3.2.14}$$

then there exist matrices P_1 and P_2 such that

$$\begin{aligned} \|P_1\| &\le \frac{2\gamma}{\delta} \\ \|P_2\| &\le \frac{2\eta}{\delta} \end{aligned} \tag{3.2.15}$$

where the columns of

$$\begin{bmatrix} P_1 \\ I \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I \\ P_2 \end{bmatrix} \tag{3.2.16}$$

span the invariant subspaces $\mathscr{V}_1$ and $\mathscr{V}_2$, respectively. □

Remark 3.2.4 Theorem 3.2.1 says that if (3.1.1) is a GHR which satisfies (3.2.14) then $\mathscr{V}_1$ is ε_0-unobservable for $\varepsilon_0 = 2\gamma/\delta$. If (3.1.1) is also a Dual GHR then under these conditions $\eta = \gamma$ and $\mathscr{V}_2$ is ε_0-uncontrollable. Note how the Dual GHR balances the distance between $\mathscr{L}_i$ and $\mathscr{V}_1$, and $\mathscr{R}_i$ and $\mathscr{V}_2$. □

With Theorem 3.2.1 we can now formally relate near-aggregability with near-unobservability. Consider (3.1.1) and parametrize the system in μ such that

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} F & \mu H \\ G & E \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u \,. \tag{3.2.17}$$

Corollary 3.2.1 [7] Suppose that (3.1.1) is separable. Then there exists a $\bar{\mu}$ such that for all $0 < \mu \leq \bar{\mu}$, there exists an $(n - \rho_i)$ dimensional invariant subspace. $\square$

Lemma 3.2.2 [7] Let ε_0 be given and suppose that (3.1.1) is separable. Then there exists a μ_0 such that for all $0 < \mu \leq \mu_0$, the system is ε_0-unobservable. $\square$

We can establish a converse to Lemma 3.2.2 as follows. From (3.1.1) define a parameterized system by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} I & -\varepsilon P \\ 0 & I \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ \Lambda_2 & \Lambda_3 \end{bmatrix} \begin{bmatrix} I & \varepsilon P \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

$$y = [\, C \quad 0 \,] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} . \tag{3.2.18}$$

In (3.2.18) we assume that an appropriately dimensioned invariant subspace exists. Note that at $\varepsilon = 1$, (3.2.18) and (3.1.1) are identical. Also note that $\tau(\mathscr{L}_i, \mathscr{V}) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Lemma 3.2.3 [7] Let μ_0 be given. Consider the system defined by (3.2.18). There exists an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the system (3.1.1) is μ_0-aggregable. $\square$

Roughly speaking then, if there exists an appropriately dimensioned invariant subspace, near-aggregability and near-unobservability are equivalent concepts.

3.3 Almost Pole Zero Cancellations

In the previous section we related near-aggregation to near-unobservability. From Lemma 2.2.4 we remember that if the system is exactly aggregable (i.e., if it is unobservable) then the transfer function matrix exhibits a pole zero cancellation. What then can be said of the transfer function in the case of a nearly-aggregable and/or nearly-unobservable system?

Consider again S_1 and S_2 in (3.1.1-2). If S_1 aggregates exactly, then the poles of S_2 are contained among the poles of S_1. In this case, the transfer function of S_1 exhibits an exact pole zero cancellation. By carrying out this cancellation, we obtain the transfer function of S_2.

If S_1 is only μ_0-aggregable, then it is clear that the poles of the reduced-order model are not contained among the poles of the full-order model. It seems intuitively obvious, however, that if μ_0 is small enough, then the poles of S_2 should closely approximate some of the poles of S_1. In the case of the Dual GHR, these ideas are directly exposed.

Suppose that in (2.3.6) we parameterize again in μ by letting $\gamma_{i+1} = \gamma_{i+1}(\mu)$ where $\gamma_{i+1}(\mu) \longrightarrow 0$ as $\mu \longrightarrow 0$. From Theorem 2.3.1 we can obtain the transfer function of the full system S_1,

$$G(s;\mu) = \frac{\Gamma z(s;\mu)}{p(s;\mu)} \quad . \tag{3.3.1}$$

Theorem 3.3.1 [10] Suppose (3.1.1) is a Dual GHR parameterized on μ as above. Then

$$G(s;\mu) \longrightarrow G_a(s) = \frac{\Gamma \bar{q}_i(s)}{q_i(s)} \quad \text{as} \quad \mu \longrightarrow 0 \ .$$ □

Theorem 3.3.1 follows directly from Theorem 2.3.1. Using root locus arguments, it is easily seen that for small γ_{i+1}, the roots of p(s) associated with $r_i(s)$ will lay close to the roots of z(s) associated with $r_i(s)$. Thus deleting the states of the system which are associated with small γ_{i+1} is equivalent to removing almost pole zero cancellations from the transfer function. Note in this process that the poles of the reduced-order model are modified slightly.

To this point we have only given an intuitive sense to the concept of almost pole zero cancellation. With the help of the following theorem we can validate this idea more concretely. Let

$$A = M \Lambda M^{-1} , \tag{3.3.2}$$

where Λ is the Jordon matrix of A. Further, denote the condition number of A by

$$\mathcal{K}(A) = \|M\| \|M^{-1}\| \tag{3.3.3}$$

Theorem 3.3.2 [7] Suppose (3.1.1) is a Dual GHR and that the poles of E are simple. If (3.1.1) is μ_0 - I/O aggregable and ε_0-unobservable then

$$\min_j |p_j - z_i| \le \mathcal{K}(E) \frac{\mu_0 \varepsilon_0}{\sqrt{1 - \varepsilon_0^2}} .$$ □

The above result is derived by applying a method of eigenvalue perturbation given by Bauer and Fike [19]. Note that this theorem may be more generally applied to the GHR. However, the application is restricted since the system must also be input aggregable.

<u>Remark 3.3.1</u> Roughly speaking then, a system which is nearly-aggregable and nearly-unobservable exhibits an almost pole zero cancellation. □

In general, Theorem 3.3.2 allows us to describe the pole zero interaction based on the parameters μ_0 and ε_0 of a given system representation. It would be benficial if we could similarly extend a feeling for the magnitudes of μ_0 and ε_0 based on pole zero information. As the next example indicates this is not generally true even for the SISO Dual GHR.

<u>Example 3.3.1</u> Consider once again the Dual GHR model of Example 2.3.1. Using Theorem 3.2.1 we can write by inspection

$$\begin{aligned} p(s) &= (s^3 + 21.59s^2 + 77.21s + 25.06)(s + .07) - (.419)^2 \\ z(s) &= (s + .07) \ . \end{aligned} \tag{3.3.4}$$

Root locus arguments will show an almost pole-zero cancallation. This fact can be verified from the transfer function of the full system

$$\frac{Y(s)}{U(s)} = P(s) = \frac{(s + .07)}{(s + .0791)(s + .350)(s + 4.05)(s + 17.12)} \tag{3.3.5}$$

which shows an almost pole zero cancellation. By example 3.2.1 though we would not call the associated one-dimensional eigenspace nearly-unobservable. $\square$

In order to further expose the complicated nature of pole zero interactions, we conclude this discussion with an interesting anomaly which can present itself in MIMO systems.

Example 3.3.2 [10] The transfer function matrix of a MIMO system is given by

$$G(s) = \begin{bmatrix} \dfrac{1}{(s+1)^2} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{1}{(s+1)(s+2)} & \dfrac{(s+3)}{(s+2)^2} \end{bmatrix} \tag{3.3.6}$$

This matrix has Smith-McMillan form

$$M(s) = \begin{bmatrix} \dfrac{1}{(s+1)^2(s+2)^2} & 0 \\ 0 & s+2 \end{bmatrix} \tag{3.3.7}$$

which shows a pole and zero at -2 but does not exhibit a pole-zero cancellation. $\square$

3.4 Interpretation of Complex Eigensystems

In view of Remark 3.2.1 we now discuss the interpretation of complex vector spaces. Consider the real valued matrix A with eigenvalues λ_i, $i = 1,\ldots,n$. It may be that some λ_i appear as complex conjugate pairs in the solution of

$$Ap_i = \lambda_i p_i \; . \tag{3.4.1}$$

In order to accommodate complex solutions of (3.4.1), we must accordingly view the associated eigensystem as a vector space defined over the field of complex numbers.

In the analysis of near-unobservability we must interactively compare the state space with the eigenspace of the system. If our system has complex eigenvalues, we may still apply our ε_0-measure analysis by considering real-valued vectors as vectors in C^n with identically zero imaginary parts. Such a set up is well defined by the inner product

$$x^H y \tag{3.4.2}$$

for vectors $x, y \in C^n$.

Via canonical angles then, Definition 3.2.2 gives a geometric interpretation of the gap topology between for instance, a real vector spanning an unobservable subspace and a complex eigenvector. In Section 3.2, we characterized near-unobservability by inferring the existence of an A-invariant subspace which lies close to $\mathscr{N}[C]$. Suppose a complex eigenvector y spans $\mathscr{V}$ where $\mathscr{V}$ is close in gap topology to some $\mathscr{L}_i$. Does the complex $\mathscr{V}$ lie close to $\mathscr{L}_i$?

Let x span $\mathscr{L}_i$ and y span $\mathscr{V}$. Then in general, we may write

$$\begin{aligned} x &= [c_j] \\ y &= [a_j] + i[b_j] = y_r + iy_c \; . \end{aligned} \tag{3.4.3}$$

From (3.4.2) then

$$x^H y = \left\{ \left(\sum_{i=1}^{n} c_i a_i \right)^2 + \left(\sum_{i=1}^{n} c_i b_i \right)^2 \right\}^{1/2}, \qquad (3.4.4)$$

and thus

$$x^H y = \left\{ (x^T y_r)^2 + (x^T y_c)^2 \right\}^{1/2} . \qquad (3.4.5)$$

Note first that if $y_c = 0$ in (3.4.3), then (3.4.5) reduces to the inner product of R^n. As we also see, (3.4.5) indicates how the real and imaginary portions of the eigenvector influence the inner product.

We may use the above description then, to extend the notion of near-unobservability to complex vector spaces. With this extension, the idea of $\mathcal{V}$ near $\mathcal{L}_i$ or $\mathcal{V}$ in $\mathcal{L}_i$ gains understanding in the idea that a composite projection in $\mathcal{V}$ lies close to $\mathcal{L}_i$ if $d(\mathcal{L}_i) = 1$. If $d(\mathcal{L}_i) > 1$ we make the same analogy, but reference the projection of $\mathcal{V}$ onto $\mathcal{L}_i$, instead of $\mathcal{L}_i$ itself. In either case, this composite projection is defined according to (3.4.5). In this way then, we may interpret spatial relationships between vector subspaces defined over the fields of real and complex numbers respectively.

4.0 A Trajectory Analysis of Near Aggregation

As can be seen from the material presented thus far, advantageous viewpoints on the dynamic behavior of a system model come forth in the application of aggregation. These points of view became even more insightful with the extension to chained aggregation, and near-aggregation.

In the review of aggregation, we were also witness to the logical developments which espoused each of the generalizations above to the concepts of system observability, system controllability, and pole zero interaction. From the control design perspective, these system characteristics often provide successful starting points in the design methodology. Another successful starting point in time-domain control is found in the analysis of state trajectories.

In studying the way state variables change over time, the control engineer may gain considerable knowledge in how the plant actually operates. This is especially true when modes and/or physical parameters of the system can be directly associated with certain states of the model. With this in mind, and the concept of aggregation so appealing, we would like to relate the two ideas, and proceed now to do so.

4.1 State Trajectories and Exact Aggregation

From the viewpoint of observability we saw that exact-aggregation separated apart the observable and unobservable parts of the system by inducing a reduced-order model which was completely observable. In that this occurs with no relevant loss of information, a bit of introspection would indicate that the states of the system have been rearranged so as to expose that set which essentially duplicates the behavior of other states.

This may be readily seen analytically. Consider again our general form after aggregation, yet let the input remain less restricted.

$$S_1: \left\{ \begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} F & H \\ G & E \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B \\ K \end{bmatrix} u \\ y_1 &= [\,C \quad 0\,] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} , \end{aligned} \right. \tag{4.1.1}$$

and

$$S_2: \left\{ \begin{aligned} \dot{z} &= Fz + Bu \\ y_2 &= Cz \,. \end{aligned} \right. \tag{4.1.2}$$

We may characterize aggregation in terms of the state trajectories of the systems S_1 and S_2.

Theorem 4.1.1

1. If S_2 is an output aggregation of S_1 then for all inputs $u(t)$ and all initial conditions $x(0) = x_0$, there exists an initial condition $z(0) = x_{10}$ such that

$$y_1(t) = y_2(t).$$

2. If S_2 is an input aggregation of S_1 then for all inputs $u(t)$ and zero initial conditions,

$$y_1(t) = y_2(t). \qquad \square$$

Thus we may view aggregation in different light. For systems which exactly aggregate, there exists a reduced-order model such that, for an appropriate initial condition, the trajectories of the reduced-order model are linear combinations of the trajectories of the full-order model. This relationship is expressed by the aggregation functional.

4.2 State Trajectories and Near Aggregation

Given a candidate aggregation functional, we now wish to study those systems which do not aggregate, but which are nearly-aggregable. It will be shown that for these systems, there exists a reduced-order model such that, for an appropriate initial condition, the trajectories of the reduced order model are near a linear combination of the trajectories of the full-order model.

Consider again the systems S_1 and S_2 in (4.1.1-2). It follows that

$$\|y_1(t) - y_2(t)\| \leq \|C\|\|x_1(t) - z(t)\| . \qquad (4.2.1)$$

We will derive a bound on the states so that the bound on the output follows from (4.2.1). In this derivation we will first derive bounds assuming the existence of a certain transformation of the state. Then we will relate this to near-aggregation.

4.2.1 Output Error Bounds

We assume that the inputs to each system are identical and that $x_1(0) = z(0)$. To derive an expression for $x_1(t)$ we assume that there exists a transformation of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \qquad (4.2.2)$$

such that S_1 becomes

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} F-PG & 0 \\ G & E+PG \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} B-PK \\ K \end{bmatrix} u \qquad (4.2.3)$$

$$= \begin{bmatrix} \bar{F} & 0 \\ G & \bar{E} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{B} \\ K \end{bmatrix} u . \qquad (4.2.4)$$

Corresponding to the system representation in (4.1.1), decompose the state space by

$$\mathscr{X} = \mathscr{X}_1 \oplus \mathscr{X}_2 \ . \tag{4.2.5}$$

Then the existence of P in (4.2.2) is equivalent to assuming the existence of an $(n - \rho_i)$ dimensional invariant subspace $\mathscr{V}$ such that

$$\mathscr{V} \cap \mathscr{X}_2 = \mathscr{O}. \tag{4.2.6}$$

To calculate such a P see (3.2.9-10).

<u>Remark 4.2.1</u> Note by Theorem 3.2.1 that if a system is separable and μ_0-aggregable for some μ_0 small enough, then this existence is guaranteed. However. it is not a necessary condition for existence. □

Direct evaluation of (4.2.2) yields

$$\begin{aligned} \bar{x}_{10} &= x_{10} - P\bar{x}_{20} = x_{10} - Px_{20} \\ x_1 &= \bar{x}_1 + P\bar{x}_2 \ . \end{aligned} \tag{4.2.7}$$

Next using (4.2.4) we may employ the variation of constants formula to obtain both $\bar{x}_1$ and $\bar{x}_2$:

$$\begin{aligned} \bar{x}_1 &= e^{\bar{F}t}\bar{x}_{10} + \int_0^t e^{\bar{F}(t-\tau)}\bar{B}u(\tau)d\tau \\ \bar{x}_2 &= e^{\bar{E}t}\bar{x}_{20} + \int_0^t e^{\bar{E}(t-\tau)}[G\bar{x}_1(\tau) + Ku(\tau)]d\tau \ . \end{aligned} \tag{4.2.8}$$

Note than in obtaining $\bar{x}_2$ we interpret both $\bar{x}_1(t)$ and $u(t)$ as inputs into the residual subsystem. Substituting (4.2.8) back into (4.2.7) we obtain the following expression for $x_1(t)$:

$$
\begin{aligned}
x_1(t) = e^{\bar{F}t}(x_{10} - Px_{20}) + Pe^{\bar{E}t}x_{20} + \int_0^t e^{\bar{F}(t-\tau)}\bar{B}u(\tau)d\tau \\
+ P\int_0^t e^{\bar{E}(t-\tau)}Ge^{\bar{F}\tau}(x_{10} - Px_{20})d\tau \\
+ P\int_0^t e^{\bar{E}(t-\tau)}G\left(\int_0^\tau e^{\bar{F}(\tau-s)}\bar{B}u(s)ds\right)d\tau \\
+ P\int_0^t e^{\bar{E}(t-\tau)}Ku(\tau)d\tau \ .
\end{aligned} \tag{4.2.9}
$$

Using the variation of constants formula again we easily obtain an explicit expression for $z(t)$:

$$
z(t) = e^{Ft}z(0) + \int_0^t e^{F(t-\tau)}Bu(\tau)d\tau \ . \tag{4.2.10}
$$

Combining (4.2.9) and (4.2.10) we finally have

$$
\begin{aligned}
z(t) - x_1(t) = (e^{Ft} - e^{\bar{F}t})x_{10} + (e^{\bar{F}t}P - Pe^{\bar{E}t})x_{20} \\
- P\int_0^t e^{\bar{E}(t-\tau)}Ge^{\bar{F}\tau}(x_{10} - Px_{20})d\tau \\
- \int_0^t [e^{\bar{F}(t-\tau)} - e^{F(t-\tau)}]Bu(\tau)d\tau \\
- P\int_0^t e^{\bar{E}(t-\tau)}G\left(\int_0^\tau e^{\bar{F}(\tau-s)}\bar{B}u(s)ds\right)d\tau \\
- P\int_0^t e^{\bar{E}(t-\tau)}Ku(\tau)d\tau + \int_0^t e^{\bar{F}(t-\tau)}PKu(\tau)d\tau \ .
\end{aligned}
$$

In the above, the last term has been separated out in order to facilitate the use of the next result. This result will allow us to expose the quantity $\|P\|$ in every term.

<u>Lemma 4.2.1</u> Let F and $\bar{F}$ be as defined in (4.2.3-4). Then

$$
e^{Ft} - e^{\bar{F}t} = \int_0^t e^{F(t-\tau)}PGe^{\bar{F}\tau}d\tau \ . \qquad \square
$$

<u>Proof</u> Consider the differential equation

$$\dot{\bar{x}}(t) = \bar{F}\bar{x}(t) = F\bar{x}(t) - PG\bar{x}(t)\ , \ t \geq 0\ , \ \bar{x}(0) = \bar{x}_0\ . \qquad (4.2.12)$$

From the variation of constants formula we have

$$\bar{x}(t) = e^{Ft}\bar{x}_0 - \int_0^t e^{F(t-\tau)}PG\bar{x}(\tau)d\tau\ . \qquad (4.2.13)$$

Since $\bar{x}(t) = e^{\bar{F}t}\bar{x}_0$ from (3.11) it follows that

$$e^{\bar{F}t}\bar{x}_0 = e^{Ft}\bar{x}_0 - \int_0^t e^{F(t-\tau)}PGe^{\bar{F}\tau}\bar{x}_0 d\tau\ . \qquad (4.2.14)$$

Because (4.2.14) holds for all $\bar{x}_0$ then the lemma follows. □

With Lemma 4.2.1 we may now involve the entire dynamics of (4.2.11) in terms of the matrix P. We next invoke a matrix measure which in turn will allow us to bound the free response of our linear system.

Let M be the modal matrix of A, and Λ the Jordon form. Then

$$e^{At} = Me^{\Lambda t}M^{-1}\ . \qquad (4.2.15)$$

Therefore, assuming the 2-norm we get

$$\begin{aligned} \|e^{At}\| &\leq \|M\|\|M^{-1}\|\|e^{\Lambda t}\| \\ &\leq \|M\|\|M^{-1}\|\, e^{\bar{\lambda}t} \end{aligned} \qquad (4.2.16)$$

where

$$\bar{\lambda} = \max \left\{ Re(\lambda(A)) \right\}\ . \qquad (4.2.17)$$

Remark 4.2.2 Note the appearance of the condition number in our bound on the transition matrix. Also we see that this bound is stable for bounded-input bounded-output systems. □

In applying the bound of (4.2.16) to F, $\bar{F}$, and $\bar{E}$ we assume the following notation. Let

$$\begin{aligned} \|e^{Ft}\| &\le me^{\lambda t} \\ \|e^{\bar{F}t}\| &\le m_a e^{\lambda_a t} \qquad (4.2.18) \\ \|e^{\bar{E}t}\| &\le m_r e^{\lambda_r t} \ . \end{aligned}$$

With the bounds in (4.2.18) we may now obtain an upper bound on the error

$$\|z(t) - x_1(t)\| . \qquad (4.2.19)$$

Applying the triangle inequality to the norm of (4.2.11) we can calculate term by term using (4.2.18), Cauchy's inequality, and integral calculus. For example, the first term is given by:

$$\|(e^{Ft} - e^{\bar{F}t})x_{10}\| = \|\int_0^t e^{F(t-\tau)}PGe^{\bar{F}\tau}d\tau \ x_{10}\| \qquad (4.2.20)$$

$$\le \|P\|\|G\|\|x_{10}\|mm_a e^{\lambda t} \times \int_0^t e^{-\lambda\tau} e^{\lambda_a \tau} d\tau \qquad (4.2.21)$$

$$= \|P\|\|G\|\|x_{10}\|mm_a \times \frac{e^{\lambda t} - e^{\lambda_a t}}{\lambda - \lambda_a} \ . \qquad (4.2.22)$$

Similarly, we may calculate the other terms, and then sum these results to obtain the trajectory error bound. We divide the results in x(0) and u(t) for simplicity of presentation:

<u>Theorem 4.2.1</u> Consider S_1 (4.1.1) and the reduced-order model S_2 (4.1.2). Assume there exists a matrix P which satisfies (4.2.2) and (4.2.3). Then if $u(t) = 0$ and $x_1(0) = z(0)$ we have

$$\|z(t) - x_1(t)\| \le \|P\|\|G\|\|x_{10}\| \, m_a\left[m\left(\frac{e^{\lambda t} - e^{\lambda_a t}}{\lambda - \lambda_a}\right) + m_r\left(\frac{e^{\lambda_a t} - e^{\lambda_r t}}{\lambda_a - \lambda_r}\right)\right]$$

$$+ \|P\|\|x_{20}\|\left[m_a e^{\lambda_a t} + m_r e^{\lambda_r t} + \|P\|\|G\| \, m_a m_r \frac{e^{\lambda_a t} - e^{\lambda_r t}}{\lambda_a - \lambda_r}\right]. \qquad \square$$

<u>Theorem 4.2.2</u> Consider S_1 (4.1.1) and the reduced-order model S_2 (4.1.2). Assume there exists a matrix P which satisfies (4.2.2-3), and also that the initial conditions in S_1 and S_2 are zero. Suppose the control satisfies

$$\bar{u} = \max_{[0,t]} \|u(t)\| .$$

Then

$$\|z(t) - x_1(t)\| \le \|P\|\|G\|\bar{u}\cdot m_a$$

$$\times\left[\|\bar{B}\|\frac{m_r}{\lambda_a}\left(\frac{e^{\lambda_a t} - e^{\lambda_r t}}{\lambda_a - \lambda_r} - \frac{e^{\lambda_r t} - 1}{\lambda_r}\right) + \|B\|\frac{m}{\lambda - \lambda_a}\left(\frac{e^{\lambda t} - 1}{\lambda} - \frac{e^{\lambda_a t} - 1}{\lambda_a}\right)\right]$$

$$+ \|P\|\|K\|\bar{u}\cdot\left(m_a\frac{e^{\lambda_a t} - 1}{\lambda_a} + m_r\frac{e^{\lambda_r t} - 1}{\lambda_r}\right) . \qquad \square$$

<u>Remark 4.2.3</u> Note that the bounds above are pointwise in t. □

<u>Remark 4.2.4</u> The theorems above were derived independently of any assumption on stability. Of course, if F, $\bar{F}$, or $\bar{E}$ are unstable then the bounds become meaningless with increasing time. □

Remark 4.2.5 The bounds given in Theorem 4.2.1 and Theorem 4.2.2 are extremely poor. This is quite evident from actual calculations. More will be said about this in the Example section. However, this bound for the difference between the trajectories of the full and reduced-order models does provide important theoretical interest. □

We review the results of Theorem 3.2.1.

Theorem 4.2.3 Suppose that (4.1.1) is separable. If

$$\frac{\|H\|\|G\|}{\delta^2} < \frac{1}{4}$$

then there exists a matrix P which satisfies (4.2.2-4) such that

$$\|P\| \le \frac{2\|H\|}{\delta} \quad .$$

□

Theorem 4.2.4 Let S_1 (4.1.1) be μ_0-aggregable and satisfy Theorem 4.2.3. Then there exists a transformation (4.2.2) such that (4.2.4) holds and

$$\|P\| \le \frac{2\mu_0}{\delta} \quad .$$

□

Theorem 4.2.4 gives us a bound on $\|P\|$ in terms of μ_0. With this we may now extend near-aggregation to the trajectories of S_1 and S_2.

Theorem 4.2.5 Suppose S_1 (4.1.1) satisfies Theorem 4.2.1 and Theorem 4.2.4. Then

$$\|z(t) - x_1(t)\| \leq \frac{2\mu_0}{\delta}\left[\|G\|\|x_{10}\|m_a\left(m\left(\frac{e^{\lambda t} - e^{\lambda_a t}}{\lambda - \lambda_a}\right) + m_r\left(\frac{e^{\lambda_a t} - e^{\lambda_r t}}{\lambda_a - \lambda_r}\right)\right)\right.$$

$$\left. + \|x_{20}\| \; m_a e^{\lambda_a t} + m_r e^{\lambda_r t} + \|P\|\|G\|m_a m_r \frac{e^{\lambda_a t} - e^{\lambda_r t}}{\lambda_a - \lambda_r}\right]. \qquad \square$$

Remark 4.2.6 As per Remark 3.2.3 we may also characterize the trajectory error norm in terms of near-unobservability. Here we can replace $\|P\|$ with

$$\frac{\varepsilon_0}{\sqrt{1 - \varepsilon_0^2}}$$

to obtain the analogous result. Similarly, we can derive a result for the input u(t). $\square$

Theorem 4.2.5 and Remark 4.2.6 generalize Theorem 4.1.1. This is, if S_1 is nearly-aggregable and/or nearly-unobservable and satisfies Theorem 4.2.4 then a linear combination of its trajectories are close to a linear combination of the trajectories of the reduced-order model. Thus, we have extended a trajectory analysis to near-aggregation and near-observability.

4.2.2 Function Space Bounds: L^2 and L^∞

Since the bounds derived above are pointwise in t, then we can derive function space bounds on the trajectories. We shall now obtain expressions for two such bounds induced by the L^2 and L^∞ norms.

Consider

$$\|z(t) - x_1(t)\|_{L^2}^2 = \int_0^\infty \|z(t) - x_1(t)\|^2 dt \quad . \tag{4.2.23}$$

Denote the bound obtained in Theorem 4.2.1 by ψ such that

$$\|z - x_1\| \leq \psi(t) \quad . \tag{4.2.24}$$

We note then that

$$\|z - x_1\|_{L^2}^2 \leq \int_0^\infty \psi^2(t) dt \quad . \tag{4.2.25}$$

The expression on the right can easily be obtained by squaring out, and then directly evaluating the integral. After doing so, a collecting of terms and simplification give the next result.

<u>Theorem 4.2.6</u> Consider (4.2.2-4). Assume that F, $\bar{F}$, and $\bar{E}$ are BIBO stable. Suppose further that $x_1(0) = z(0)$ and $u(t) = 0$. Then

$$\|z - x_1\|_{L^2}^2 \leq -\left[\frac{\alpha}{2\lambda} + \frac{\alpha_a}{2\lambda_a} + \frac{\alpha_r}{2\lambda_r} + \frac{2\alpha\alpha_a}{\lambda+\lambda_a} + \frac{2\alpha\alpha_r}{\lambda+\lambda_r} + \frac{2\alpha_a\alpha_r}{\lambda_a+\lambda_r}\right]$$

where

$$\alpha = \|P\|\|G\|\|x_{10}\|\frac{mm_a}{\lambda-\lambda_a} \quad ,$$

$$\alpha_a = \|P\|\|G\|\frac{m_a m_r}{\lambda_a-\lambda_r}\left(\|x_{10}\| + \|P\|\|x_{20}\|\right) + \|P\|\|x_{20}\|m_a - \alpha \; ,$$

$$\alpha_r = \|P\|\|x_{20}\|\left(m_a + m_r\right) - \alpha - \alpha_a \quad . \qquad \square$$

Remark 4.2.7 Here then we can use this bound to obtain a maximum error estimate on the difference in trajectories. Note we could also obtain such a bound in terms of u(t). □

Next we consider another function space bound which is associated with the L^{∞} norm:

$$\|z(t) - x_1(t)\|_{L^{\infty}} = \sup_t \|z(t) - x_1(t)\| . \qquad (4.2.26)$$

In order to evaluate (4.2.26) we employ the bound ψ again. Thus,

$$\|z - x_1\|_{L^{\infty}} \leq \sup_t \psi . \qquad (4.2.27)$$

Examining ψ in Theorem 4.2.1 we note that its derivative may be reduced to the following form:

$$\frac{d}{dt} = ae^{\lambda t} + be^{\lambda_a t} + ce^{\lambda_r t} = 0 . \qquad (4.2.28)$$

In general there does not exist an analytical solution to (4.2.28). However, we may still obtain an upper bound on (4.2.26). In ψ we see there exist basically two positive semidefinite terms which can be maximized separately, and then added to obtain a bound on (4.2.27). We can represent both terms by the following single expression:

$$f(t) = a_1e^{\lambda_1 t} - a_2e^{\lambda_2 t} . \qquad (4.2.29)$$

Using derivative calculus, f(t) may be analyzed directly for a maximum:

$$\frac{d}{dt}[f(t)] = a_1\lambda_1e^{\lambda_1 t} - a_2\lambda_2e^{\lambda_2 t} = 0 . \qquad (4.2.30)$$

The solution to (4.2.30) is given by

$$t = \infty , \tag{4.2.31}$$

and

$$t = \frac{\log_e\left(\dfrac{a_1\lambda_1}{a_2\lambda_2}\right)}{\lambda_2 - \lambda_1} \quad . \tag{4.2.32}$$

Furthermore we deduce that (4.2.32) is the global maximum.

Adding together the key two terms evaluated at time t in (4.2.32) we obtain the following result.

<u>Theorem 4.2.7</u> (Consider 4.2.2-4). Assume that F, $\bar{F}$, and $\bar{E}$ are BIBO stable. Suppose further that $x_1(0) = z(0)$ and $u(t) = 0$. Then

$$\|z - x_1\|_{L^\infty} \leq \|P\|\|G\|\|x_{10}\|\cdot m_a$$

$$\times \left[\frac{m}{\lambda-\lambda_a}\left(\left(\frac{\lambda_a}{\lambda}\right)^{\frac{\lambda}{\lambda-\lambda_a}} - \left(\frac{\lambda_a}{\lambda}\right)^{\frac{\lambda_a}{\lambda-\lambda_a}}\right) + \frac{m_r}{\lambda_a-\lambda_r}\left(\left(\frac{\lambda_r}{\lambda_a}\right)^{\frac{\lambda_a}{\lambda_a- r}} - \left(\frac{\lambda_r}{\lambda_a}\right)^{\frac{\lambda_r}{\lambda_a-\lambda_r}}\right)\right]$$

$$+ \|P\|\|x_{20}\|\left[(a+m_a)\left(\frac{a-m_r\lambda_r}{a+m_a\lambda_a}\right)^{\frac{\lambda_a}{\lambda_a-\lambda_r}} - (a-m_r)\left(\frac{a-m_r\lambda_r}{a+m_a\lambda_a}\right)^{\frac{\lambda_r}{\lambda_a-\lambda_r}}\right]$$

where

$$a = \frac{\|P\|\|G\|m_a m_r}{\lambda_a - \lambda_r} \quad .$$

□

As in the L^2 discussion, similar comments found in Remark 4.2.7 apply to the analysis of the L^∞ discussion.

4.3 Examples

In this section we present two systems of the form (4.1.1) for which we calculate the following:

1. Actual values of $\|x_1(t) - z(t)\|$ for initial conditions, and unit step inputs.
2. Initial condition bounds given in Theorem 4.2.1
3. Unit step input bounds given in Theorem 4.2.2.
4. Continuous time simulations of certain state trajectories for both the full and reduced-order models.

These calculations summarize the basic points and results of this section. All of the algorithms used were simple and straightforward, and are therefore not presented.

4.3.1 Example 1

Consider again the system given in Example 3.3.1. Partition the state vector x

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4.3.1}$$

such that $d(x_1) = 3$. The reduced-order model is then given by

$$z = \begin{bmatrix} -21.59 & 1 & 0 \\ -77.21 & 0 & 1 \\ -25.06 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [\ 1 \quad 0 \quad 0\ ]\, z\ . \tag{4.3.2}$$

From Example 3.3.1 we have an almost pole zero cancellation with the pair (.070, .079). Therefore we will examine the invariant subspace associated with the pole of this pair. From Example 3.2.1, the P of (4.2.2-3) is given by

$$P = \begin{bmatrix} -.022 \\ -.472 \\ -1.657 \end{bmatrix} . \tag{4.3.3}$$

The eigenvalues of F, $\bar{F}$, $\bar{E}$ are then found to be

$$\begin{aligned} \lambda(F) &= \{\ -17.181,\ -4.0486,\ -.36035\ \} \\ \lambda(\bar{F}) &= \{\ -17.181,\ -4.0495,\ -.35029\ \} \\ \lambda(\bar{E}) &= \{\ -.079191\ \}\ . \end{aligned} \tag{4.3.4}$$

Summarizing the necessary parameter calculations then

$$\begin{aligned} \lambda &= -.36035 & m &= 14.110 \\ \lambda_a &= -.35029 & m_a &= 14.019 \\ \lambda_r &= -.079191 & m_r &= 1.000\ , \end{aligned} \tag{4.3.5}$$

and

$$
\begin{array}{llll}
\|P\| = 1.723 & \|B\| = 1.000 & \\
\|G\| = .419 & \|\bar{B}\| = 1.000 & (4.3.6) \\
\|K\| = 0 & \bar{u} = 1.000 \ . &
\end{array}
$$

Calculations of error norms and state trajectories induced by an initial condition follow in Figure 2 through Figure 5.

<u>Remark 4.3.1</u> From Example 3.2.1 we have shown that $\mathcal{V}$ would not be considered nearly-unobservable. Note however that the trajectories of the full and reduced order models are very close to each other. This effect is therefore more attributed to the eigenvalue separation between F and E. A discussion of the error norm bounds may be found in Section 4.3.3. □

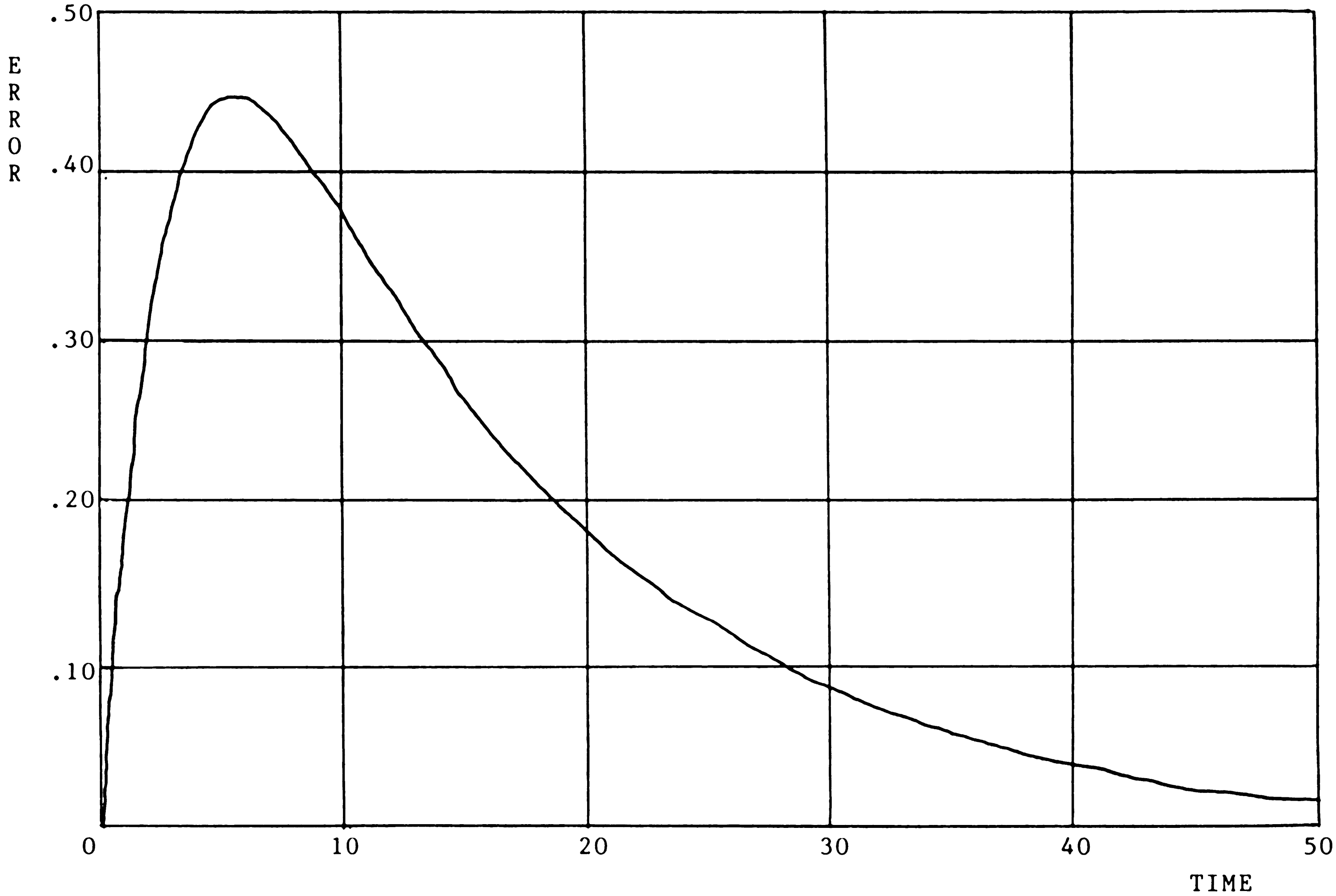

FIGURE 2: Example 1 — Actual $\|x_1 - z\|$ for $x_o = [1\ 0\ 0 \,\vdots\, .5]^T$ and $u(t)=0$.

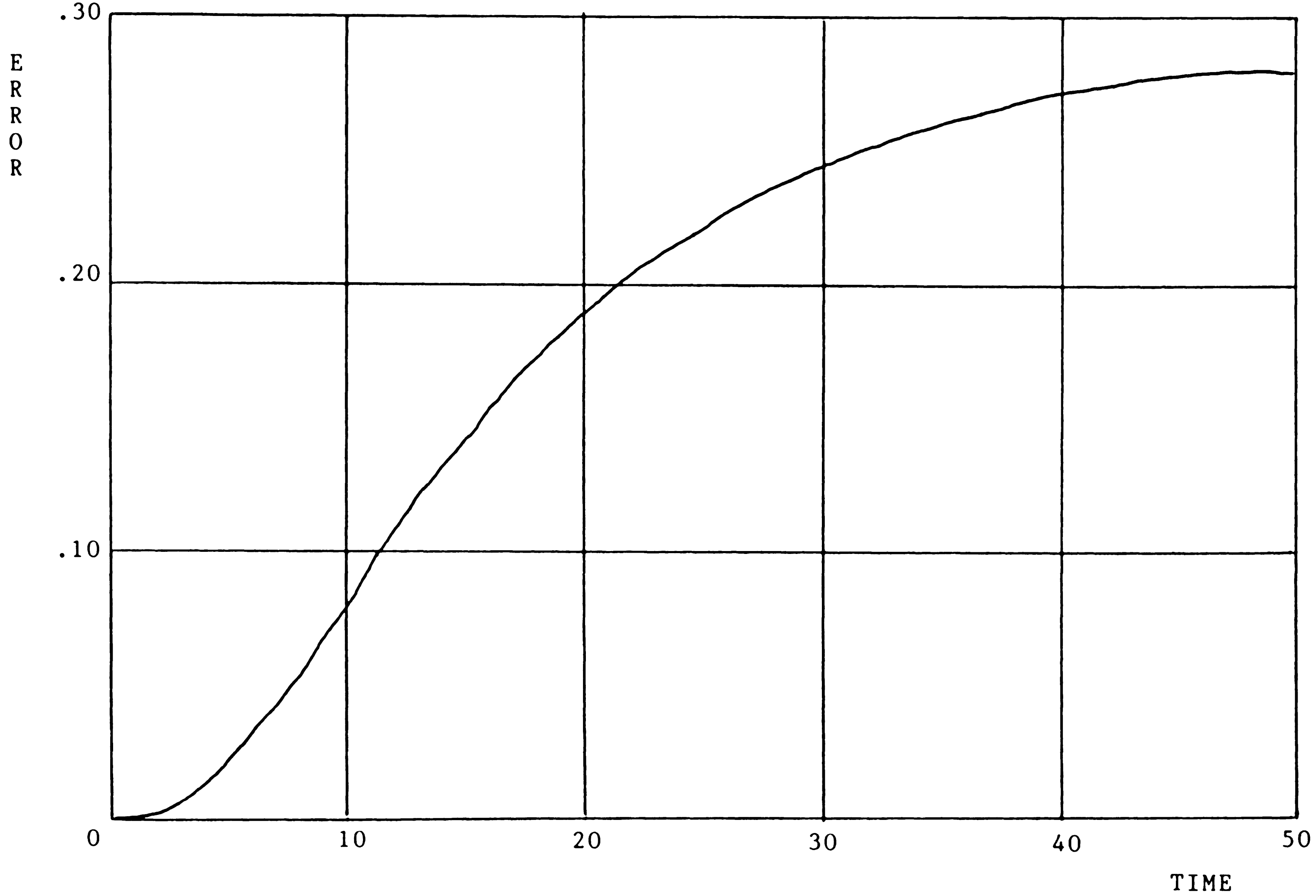

FIGURE 3: Example 1 — Actual $\|x_1 - z\|$ for $x_o = 0$ and unit step input.

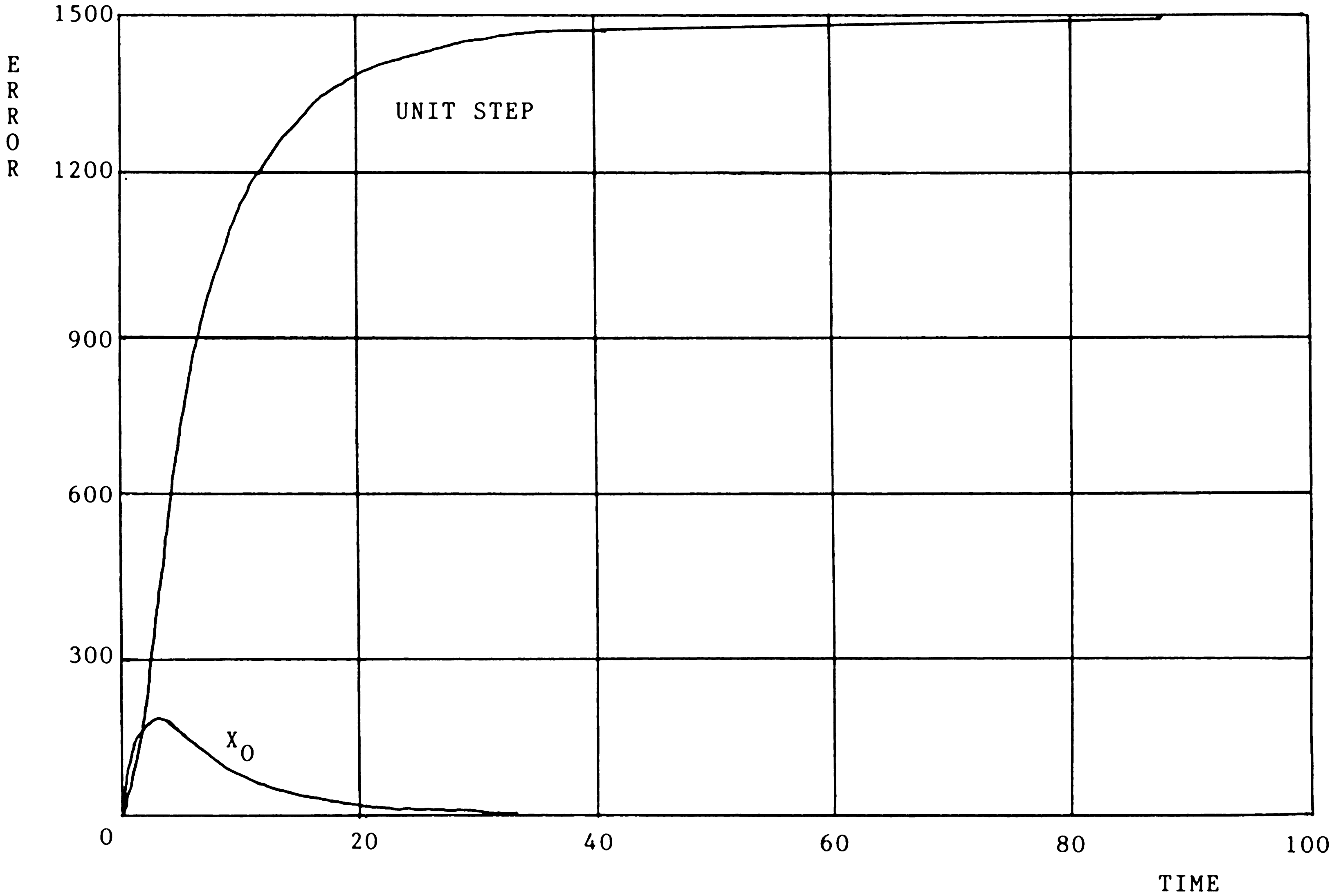

FIGURE 4: Example 1 — Bound of $\|x_1 - z\|$ for $x_o = [1\ 0\ 0 \vdots .5]^T$ and unit step, respectively.

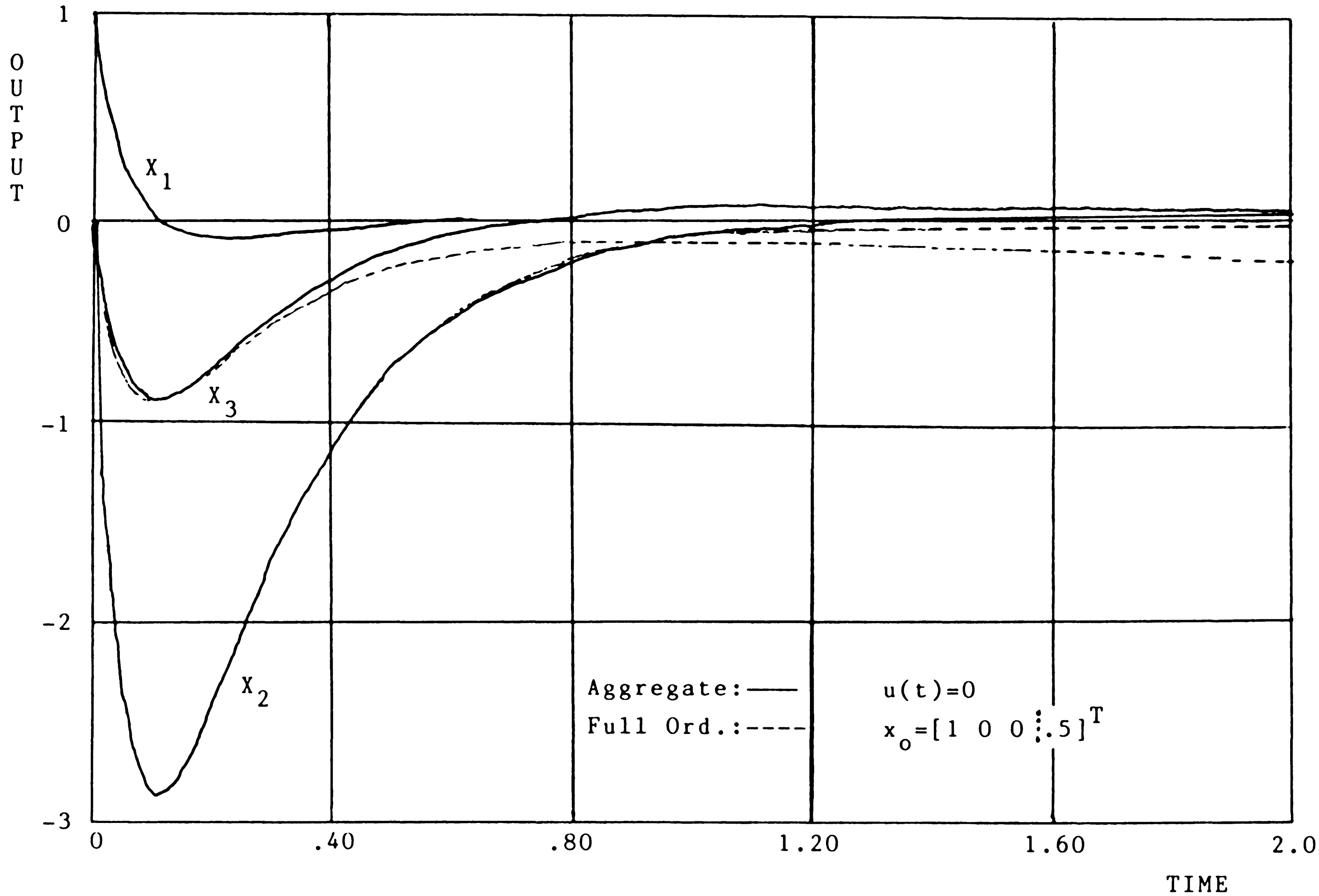

FIGURE 5: Example 1 — State trajectories of aggregate and full order models.

4.3.2 Example 2

Next we consider the linear model of a Hoop Column Antenna. The system matrices A, B, and C are given in Dual GHR form by

$$
A = \begin{bmatrix}
-0.013 & -6.760 & 0 & 0 & 0 & 0 & 0 \\
6.760 & -0.001 & -1.599 & 0 & 0 & 0 & 0 \\
0 & 1.599 & -0.023 & -6.148 & 0 & 0 & 0 \\
0 & 0 & 6.148 & 0.010 & -1.477 & 0 & 0 \\
0 & 0 & 0 & 1.477 & -0.001 & -0.605 & 0 \\
0 & 0 & 0 & 0 & 0.605 & -0.019 & -1.249 \\
0 & 0 & 0 & 0 & 0 & 1.249 & 0.009
\end{bmatrix}
$$

$$
C = B^T = [\ 0.010 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\ ] \tag{4.3.7}
$$

Noting the small superdiagonal element .605, we partition the state vector x again as in (4.3.1) where $d(x_1) = 5$. Here we examine the subspace associated with the lower right hand block. P is given by

$$
P = \begin{bmatrix}
0.000 & -0.029 \\
0.005 & 0.000 \\
-0.001 & -0.115 \\
0.025 & 0.000 \\
-0.003 & -0.455
\end{bmatrix} \tag{4.3.8}
$$

The eigenvalues of F, $\bar{F}$, and $\bar{E}$ are

$$
\lambda(F) = \left\{ \begin{matrix}
-.0072884 & j7.4015 \\
-.0055950 & j5.7837 \\
-.0022333 &
\end{matrix} \right\}
$$

$$\lambda(\bar{F}) = \begin{Bmatrix} -.0072874 & j7.4018 \\ -.0059260 & j5.7852 \\ -.00019877 & \end{Bmatrix} \qquad (4.3.9)$$

$$\lambda(\bar{E}) = \{ -.0060207 \quad j1.3797 \} .$$

In this case we have

$$\begin{aligned} \lambda &= -.0022333 & m &= 1.167 \\ \lambda_a &= -.00019877 & m_a &= 1.167 \qquad (4.3.10) \\ \lambda_r &= -.0060207 & m_r &= 1.105 , \end{aligned}$$

and

$$\begin{aligned} \|P\| &= .470 & \|B\| &= .0100 \\ \|G\| &= .605 & \|\bar{B}\| &= .0100 \qquad (4.3.11) \\ \|K\| &= 0 & \bar{u} &= 1.000 . \end{aligned}$$

Calculations of error norms and state trajectories induced by both initial conditions, and unit step inputs are given in Figure 6 through Figure 12.

<u>Remark 4.3.2</u> State trajectories for x_1, x_2 and x_4 are not displayed since the difference between the full and reduced models is not distinguishable by graph. Also note that in this case, the small diagonal element .605 might have been used to predict the closeness of the full and reduced-order trajectories.

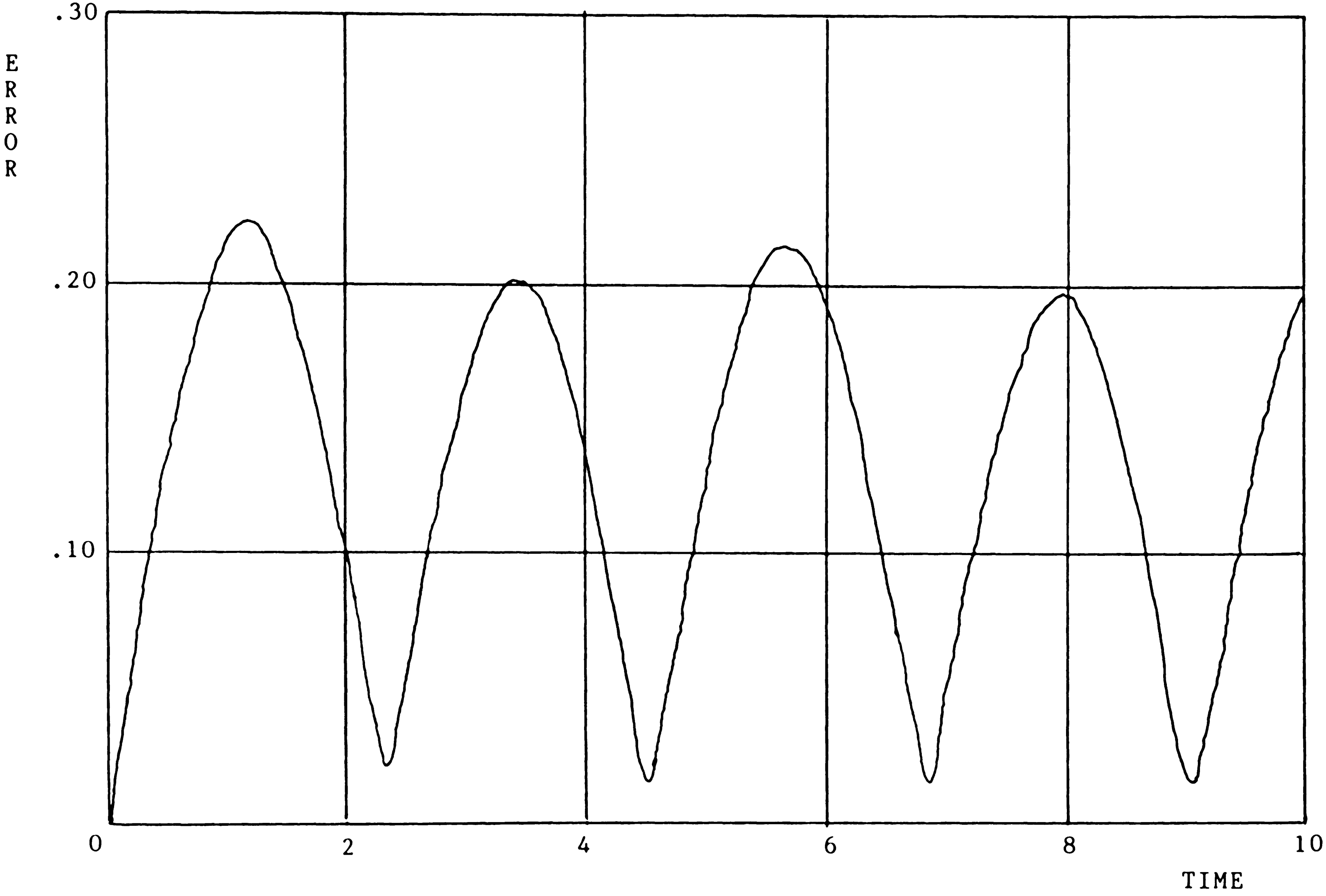

FIGURE 6: Example 2 — Actual $\|x_1 - z\|$ for $x_o = [1\ 0\ 0\ 0\ 0 \vdots .5\ 0]^T$ and $u(t) = 0$.

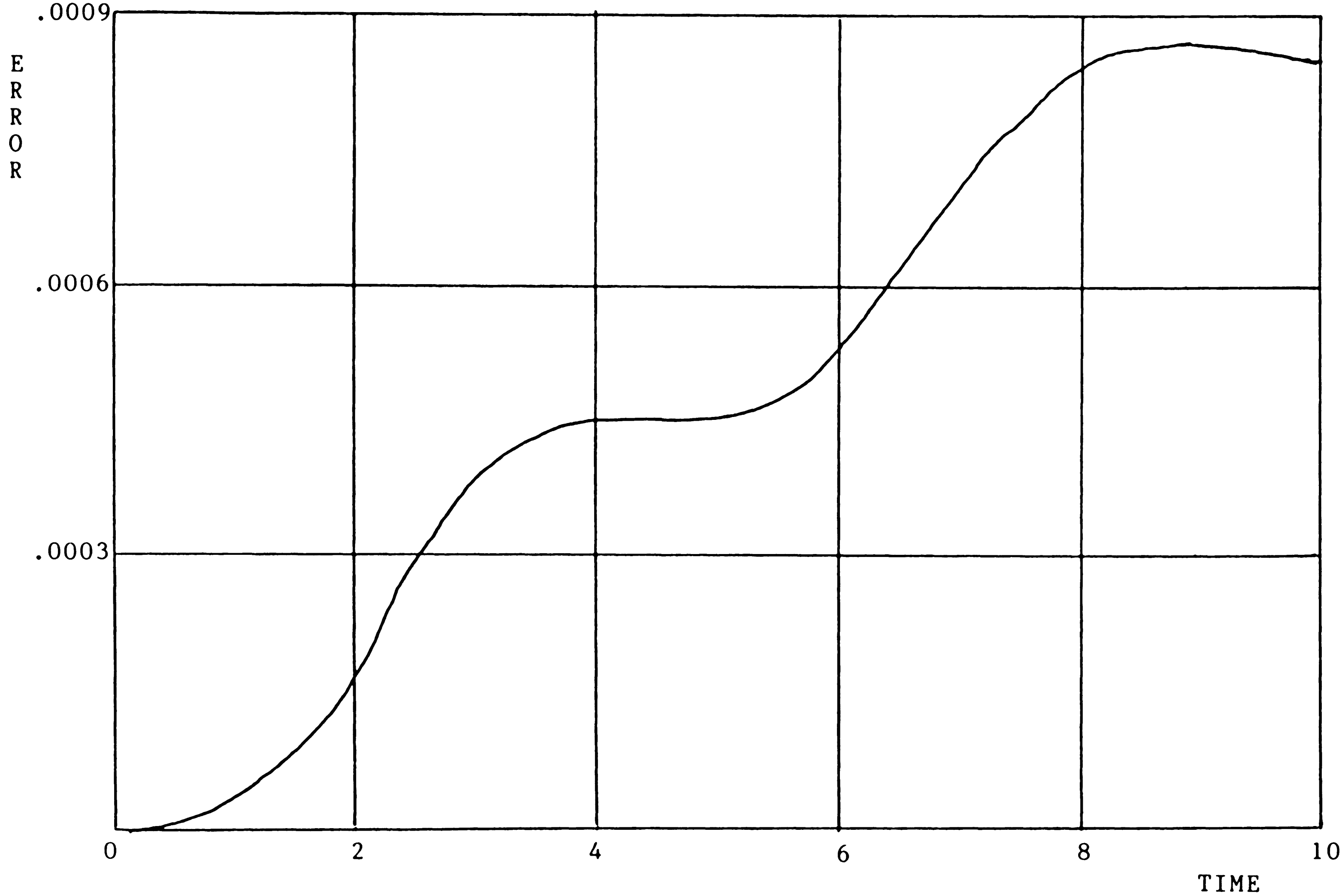

FIGURE 7: Example 2 — Actual $\|x_1 - z\|$ for $x_o = 0$ and unit step input.

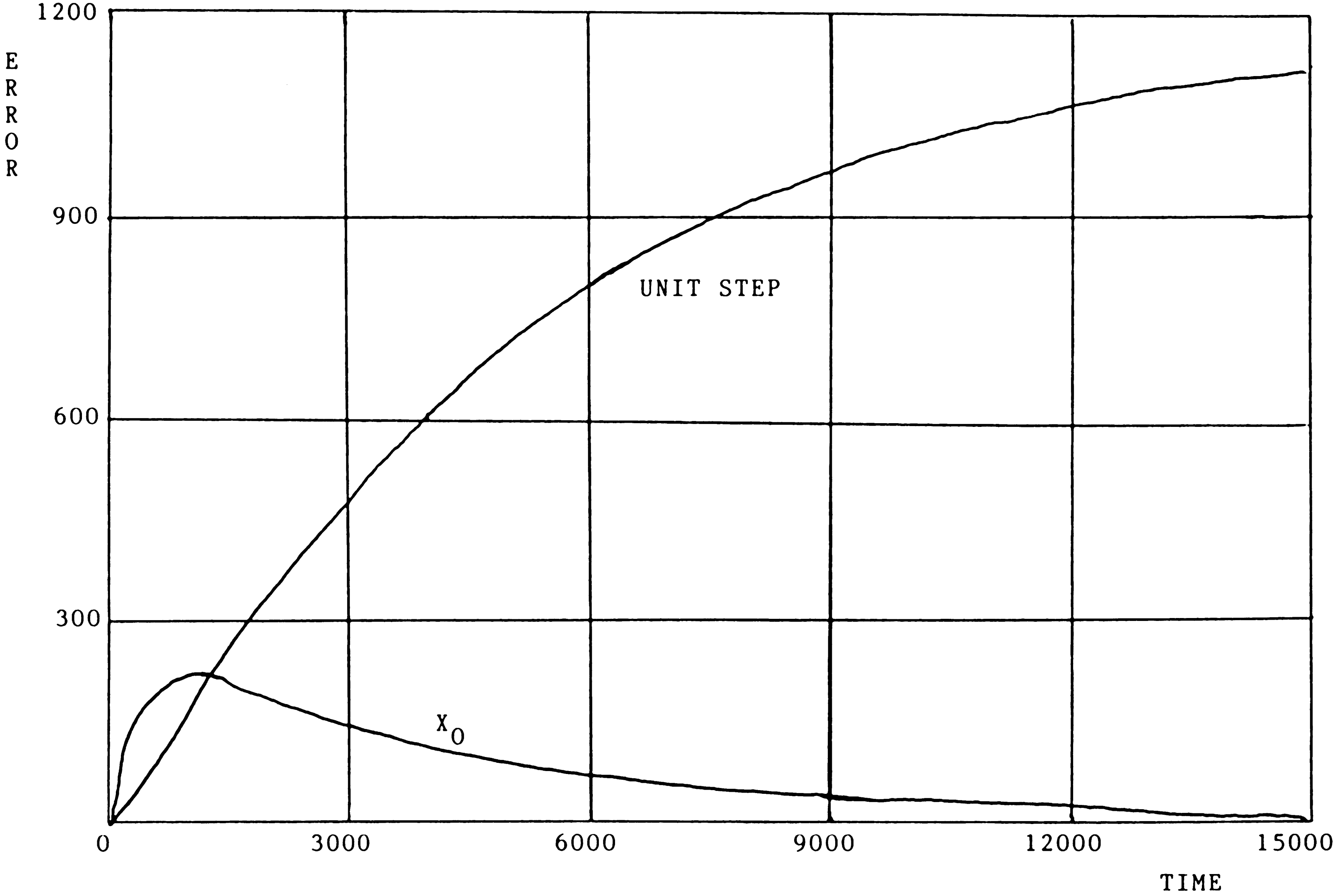

FIGURE 8: Example 2 — Bound of $\|x_1-z\|$ for $x_o = [1\ 0\ 0\ 0\ 0 \vdots .5\ 0]^T$ and unit step.

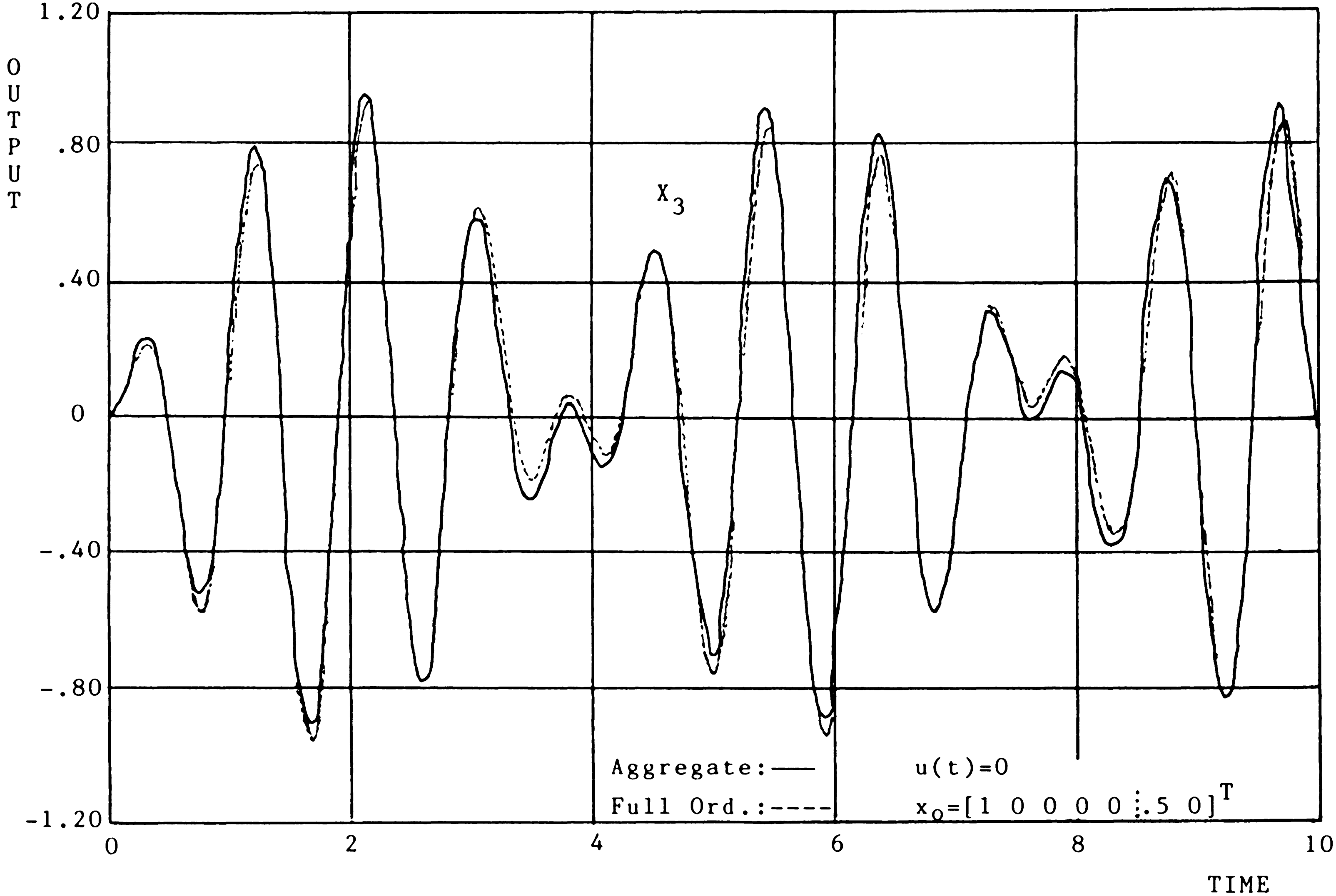

FIGURE 9: Example 2 — State trajectories of aggregate and full order models.

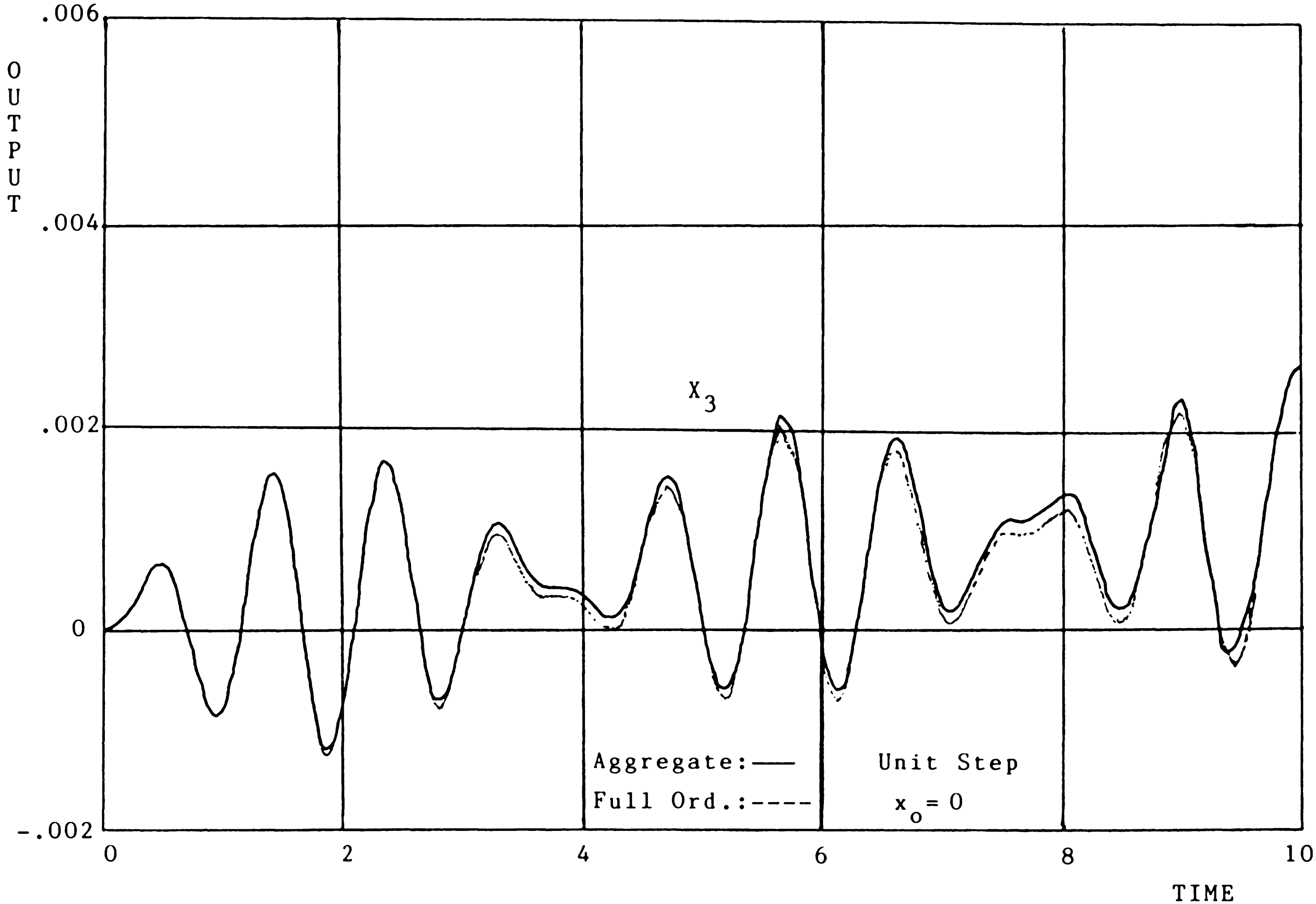

FIGURE 10: Example 2 — State trajectories of aggregate and full order models.

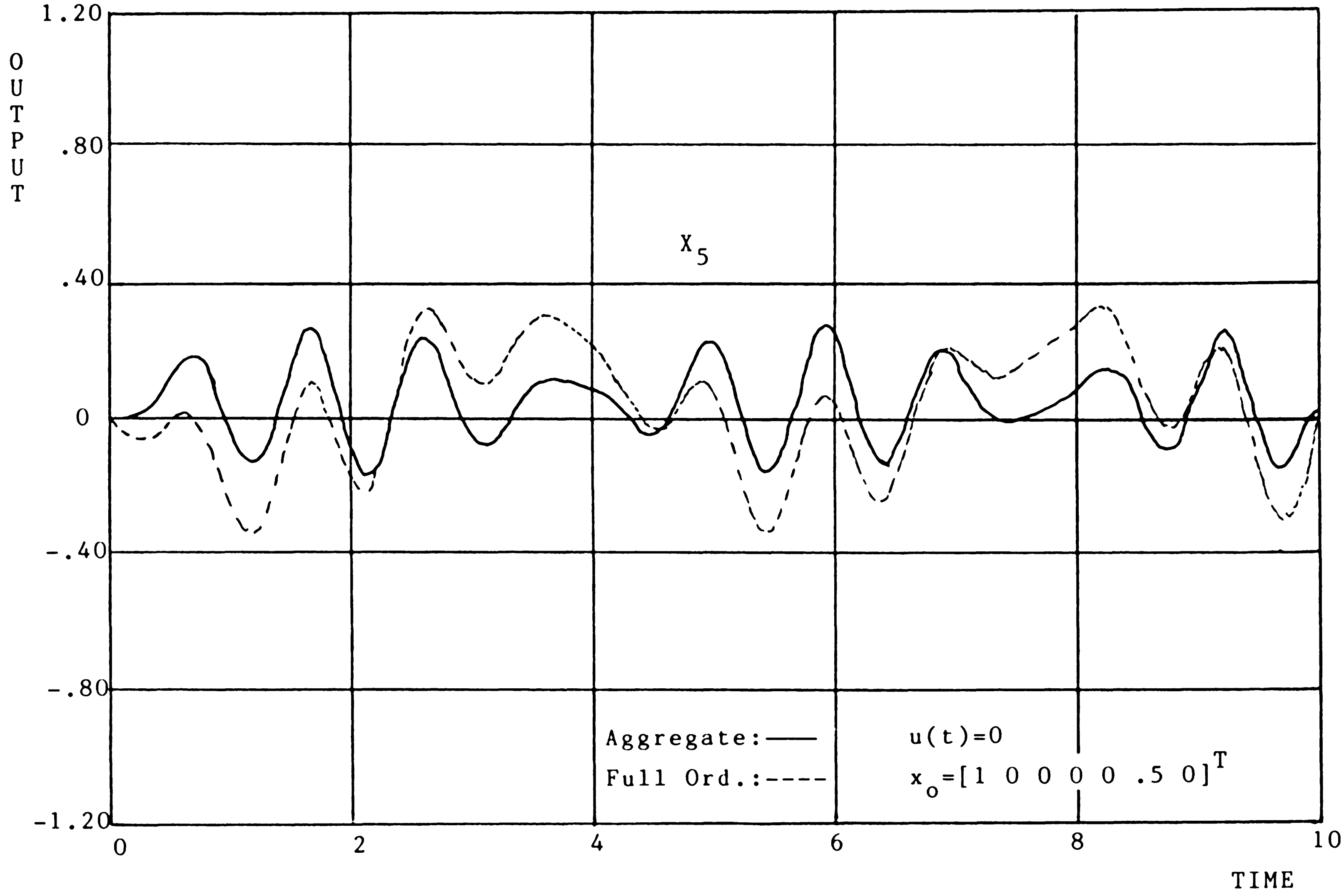

FIGURE 11: Example 2 — State trajectories of aggregate and full order models.

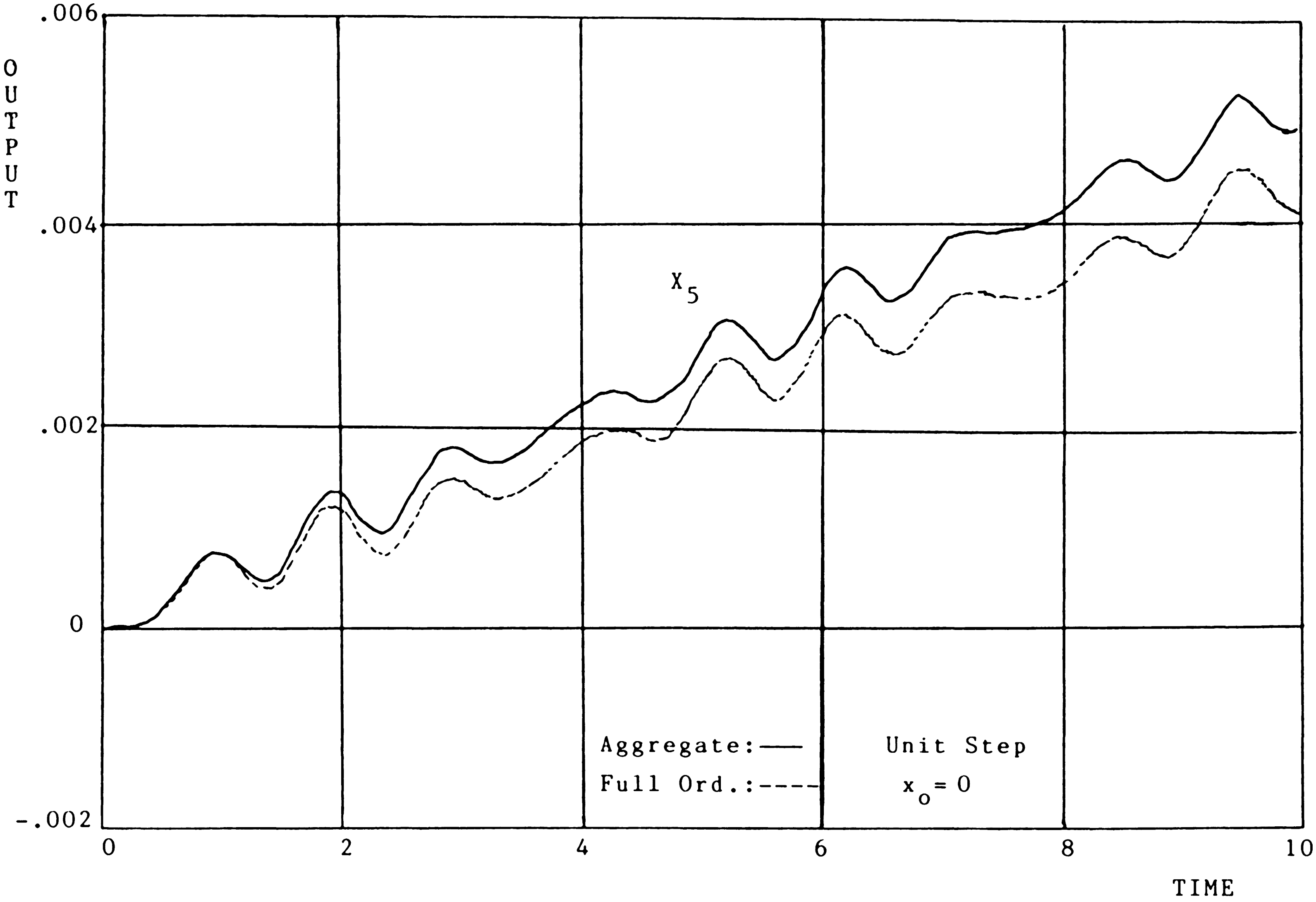

FIGURE 12: Example 2 — State trajectories of aggregate and full order models.

4.3.3 Discussion on $\|e^{At}\|$

From Figure 4 and Figure 8 we see that the bounds given for the error norm (4.2.19) are rather crude. A few things which account for this are:

1. The use of the triangle inequality.
2. The bound on e^{At} given in (4.2.16).

To bound an equation of the type (4.2.11) we have little choice but to apply the first of these. However, there are other matrix measures available to us.

Since we are using the two 2-norm, an obvious choice might be to use the matrix measure function naturally defined by this induced-norm. Let A be square, then this measure is given by

$$\mu(A) = \lambda_{max} \left(\frac{A^T + A}{2} \right) . \qquad (4.3.12)$$

With this, we may bound the transition matrix:

<u>Lemma 4.3.1</u> [22] $\quad \|e^{At}\| \leq e^{\lambda_{max} t}$. $\square$

In retrospect of (4.2.11) we would like to apply the above bound on the subsystem matrices F, $\bar{F}$, and $\bar{E}$. As it is though, the measure $\mu(A)$ was positive for most examples considered. This, as one can easily see from the above lemma, gives a bound which does not converge as $t \longrightarrow \infty$. From (4.3.12) we notice that the measure is focused on the symmetric portion of the system matrix. For a general representation, nothing guarantees that this portion of the system will be stable even if the full system is stable. Partitioning the state matrix further complicates this issue.

As the reader will recall from the discussion on the Dual GHR, we had some remaining freedom after aggregation to adjust the scaling in the subspaces $\mathscr{K}_i$. We used part of this freedom to impose a canonical form on the subsystem blocks F_{ii}. That choice was motivated by reasons without regard to the measure in (4.3.12). A question then, is could it be possible to approach the F_{ii} block form such that it would be optimized with respect to the measure given above?

In short, the available transformations which preserve the Dual GHR structure are not powerful enough to accomplish this goal. While it may be that some of the available transformations give better results in some cases, nothing appeared to be generically superior.

In conclusion then, we cannot use the natural measure of the 2-norm to obtain a finite bound in the general case. The bound in (4.2.16) however, does converge for BIBO stable systems. Though this bound is not very tight, it does provide the theoretical connection which allows us to extend the concept of near-aggregation to full and reduced-order trajectory behaviors.

5.0 Residues and Almost Pole Zero Cancellations.

In Section 3.3 we gave conditions under which a nearly aggregable system exhibits an almost pole zero cancellation. In this formulation, Theorem 3.3.2 characterized the minimum distance between a pole and zero in terms of μ_0-aggregation and ε_0-unobservability. We will now examine another viewpoint of almost pole zero cancellation which can also be related to the associated concepts of aggregation. This development is focused on the residues of a transfer function matrix expansion.

5.1 Residues and the State Space

Consider again the multiple input multiple output system of (2.1.1).

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \;. \end{aligned} \tag{5.1.1}$$

Let the transfer function of (5.1.1) be given by

$$y(s) = G(s)u(s) \ . \tag{5.1.2}$$

If the poles of G(s) are distinct, then G(s) may be expanded via partial fractions to obtain

$$G(s) = \sum_{i=1}^{n} \frac{R_i}{(s - \lambda_i)} \quad . \tag{5.1.3}$$

The 1xm matrix R_i is called the residue matrix of G(s) at pole $s = \lambda_i$, and is generally complex valued and of rank one. The relationship between (5.1.1) and the residues in (5.1.3) is given by the following well known formula. Let the eigenvalues of A, λ_i $i=1,\ldots n$, be distinct. Also define the sets $\{p_i\}$ and $\{q_i\}$ as the associated sets of right and left eigenvectors respectively, such that $\{p_i\}$ is the reciprocal basis of $\{q_i\}$.

<u>Lemma 5.1.1</u> $\quad R_i = Cp_iq_i^TB$ □

<u>Proof:</u> Let $\Lambda = PAQ$ be the Jordon form of A where

$$P^{-1} = [\ p_1 \mid p_2 \mid \cdots \mid p_n\ ] = \begin{bmatrix} q_1^T \\ \hline q_2^T \\ \hline \vdots \\ \hline q_n^T \end{bmatrix} = Q \ . \tag{5.1.4}$$

Note then we may write

$$G(s) = CP(sI - \Lambda)^{-1}QB \ . \tag{5.1.5}$$

Using the adjoint to evaluate, then

$$G(s) = CP \frac{\mathrm{adj}(sI - \Lambda)}{\det(sI - \Lambda)} QB \ . \tag{5.1.6}$$

Expanding to (5.1.3), then

$$R_i = (s - \lambda_i)CP \frac{\mathrm{adj}(sI - \Lambda)}{\det(sI - \Lambda)} QB \Bigg|_{s=\lambda_i} \ . \tag{5.1.7}$$

Simplifying we get

$$R_i = CP \left(\frac{\mathrm{diag}[0 \cdots \prod_{\substack{j=1 \\ i\neq j}}^{n} (\lambda_i - \lambda_j) \cdots 0]}{\prod_{\substack{j=1 \\ i\neq j}}^{n} (\lambda_i - \lambda_j)} \right) QB \tag{5.1.8}$$

where the nonzero diagonal element is α_{ii}. From (5.1.8) the result is easily established. □

<u>Remark 5.1.1</u> Note that p and q in Lemma 5.1.1 satisfy

$$p_i^T q_i = \delta_{ij} = \left\{ \begin{array}{cc} 1 & i=j \\ 0 & i\neq j \end{array} \right] \ . \tag{5.1.9}$$

This property is sometimes referred to as biorthogonality. Further note that only one of the sets $\{p_i\}$ and $\{q_i\}$ can be taken as normalized. For a more detailed conversation of the residue and other properties, see [24]. □

From (5.1.3) we see that if $R_i = 0$, then G(s) exhibits an exact pole zero cancellation. Therefore, if $\|R_i\|$ is small then G(s) should have an almost pole zero cancellation. In short then, the residues R_i provide a direct means for detecting almost pole zero cancellations. Noting the terms Cp_i and q_i^TB, Lemma 5.1.1 immediately suggests a direct connection between observability, controllability, and the residue. Thus, we might use these first two structures to infer pole zero interaction directly from the state space.

In the rest of this chapter, we pursue this idea. To do this we will develop a relationship between two different measures of controllability and observability, and in turn, show how each set of measures influences the residuals of the transfer function matrix. The first measure of observability is based on the cosine of the angle between each eigenvector and c_i^T, where c_i is a row vector of the output matrix [21]. This measure is closely related to Selective Modal Analysis [22]. The second measure of observability is the one already introduced in Definition 3.2.4.

Recall that this second measure is based on the sine of the angle between an invariant subspace and the null space of C. We will show that these two measures of observability are related by complementary angles. In our analysis, we must also include the concept of controllability. This set of measures follows immediately from the dual of the observability measures, and similarly we can extend a complimentary relationship within this set.

We will eventually show that these controllability and observability measures can be used to bound the magnitude of the residues of the poles of the transfer function. It follows then, that as a mode becomes more unobservable and/or more uncontrollable, the residue associated with that mode tends to zero.

To see that we must simultaneously consider both observability and controllability, consider the following property of the residue.

<u>Lemma 5.1.2</u> R_i (and therefore $\|R_i\|$) is invariant to a change of basis in the state. □

<u>Proof:</u> Consider the state basis transformation

$$x = T\bar{x} . \tag{5.1.10}$$

With respect to x we have

$$R_i = Cp_iq_i^TB . \tag{5.1.11}$$

Also we may write with respect to $\bar{x}$

$$\bar{R}_i = \bar{C}\bar{p}_i\bar{q}_i^T\bar{B} . \tag{5.1.12}$$

Using (5.1.10) directly in the left and right eigenvector equations

$$\begin{aligned} Ap_i &= \lambda_i p_i \\ A^T q_i &= \lambda_i q_i \end{aligned} \tag{5.1.13}$$

we easily obtain that

$$R_i = \bar{R}_i \tag{5.1.14}$$

<u>Remark 5.1.1</u> As Example 3.1.1 shows, the internal scaling of the states can effect the measures of observability and controllability. However, as the above lemma implies, use of the residue as a measure by itself is independent of the scaling issue. This is consistent in that the residue is a frequency domain concept. □

As per the above remark though, if we are to gain the sought after state space interpretation, then clearly we must simultaneously address observability, controllability, and the underlying scaling issue. For SISO systems the Dual GHR basis provides the most natural setting for such an endeavor.

From Section 3.2 we note that if one of the superdiagonal blocks of the Dual GHR is nearly zero, then under certain conditions, the system is nearly-unobservable/uncontrollable. In such a case, the system exhibits an almost pole zero cancellation. Here we will also establish that such a system has a small residue.

In the SISO case it is then clearly evident that a small residue is equivalent to an almost pole zero cancellation. In the MIMO model this is not generally true. Here it is possible to have an almost pole zero cancellation while the associated residue is quite large. (See #4 of the Example Section). Also, a small residue may exist without finite invariant zeros (see #3 of the Example Section). For MIMO systems though, we can show that a small residue implies that the mode associated with that residue is nearly-unobservable/ uncontrollable.

We have then that for both SISO and MIMO systems, we can establish a quantitative relationship between measures of the state space and $\|R_i\|$. This is useful since $\|R_i\|$ itself can be used to describe system dynamics. For instance, in [23] it is suggested that the magnitude of the residues, normalized by the magnitude of the corresponding pole, can be used as a criteria for model reduction. Here, $\|R_i\|$ is viewed as a gain in the exponential input output behavior description. A relatively small residue indicates a zero output characteristic at the associated frequency. Hence, it does not dominate the dynamics of the control variable, and in this respect may be discarded from the model.

In summary, we now propose to extend the residue measure $\|R_i\|$ to the associated concepts of near-aggregation developed earlier in this thesis. With this development we bring new interpretation and application of near-aggregation and the Hessenburg forms.

5.2 Modal Measures

In comparing the ideas of near-aggregation developed earlier we first introduce another set of measures of observability and controllability referred to as modal observability and modal controllability. These measures natural manifest themselves in the discussion of the residue matrices of G(s). Let

$$c_j = j^{th} \text{ row of } C \quad , \quad j = 1,\ldots,l$$
$$b_k = k^{th} \text{ column of } B \quad , \quad k = 1,\ldots,m \; . \tag{5.2.1}$$

We will reference (5.2.1) as the j^{th} output and the k^{th} input, respectively.

<u>Definition 5.2.1</u> For each right eigenvector p_i , we say that mode is ε_{oj}-<u>modal unobservable</u> if

$$\cos\Phi(c_j^T,p_i) \leq \varepsilon_{oj} \; . \qquad \square$$

Similarly,

<u>Definition 5.2.2</u> For each left eigenvector q_i, we say that mode is ε_{ok}-<u>modal uncontrollable</u> if

$$\cos\Phi(q_i,b_k) \leq \varepsilon_{ok} \; . \qquad \square$$

<u>Remark 5.2.1</u> From Definition 5.2.1 it is easily seen that

$$\cos\Phi(c_j^T,p_i) = \frac{|\; c_j^T p_i \;|}{\|c_j^T\|\cdot\|p_i\|} \; . \tag{5.2.2}$$
$\square$

Geometrically, we can interpret ε_{oj}-unobservability as the cosine of the angle between the vectors c_j^T and p_i. If this cosine is nearly zero, then c_j^T and p_i are nearly orthogonal. In this case, p_i is close to the null space of c_j, $\mathcal{N}[c_j]$. Hence we expect little contribution of the i^{th} mode in the j^{th} output.

Remark 5.2.2 Definition 5.2.1 and Definition 5.2.2 also hold for complex eigenvectors. It is easily seen that if $q_i = q_j^*$, then

$$\cos\Phi(q_i, b_k) = \cos\Phi(q_j^*, b_k) \ . \tag{5.2.3}$$

In general then, we also have that

$$\|R_i\| = \|R_i^*\| . \tag{5.2.4}$$

□

Now the relationship of ε_0-modal measures to the residue is easily established. Define the diagonal matrices

$$D_o = \mathrm{diag}[\ \|c_1^T\|, \ldots, \|c_l^T\|\ ]$$

and (5.2.5)

$$D_c = \mathrm{diag}[\ \|b_1\|, \ldots, \|b_m\|\ ] \ .$$

Also let

$$\bar{C}\bar{p}_i = \begin{bmatrix} \cos\Phi(c_1^T, p_i) \\ \vdots \\ \cos\Phi(c_l^T, p_i) \end{bmatrix} , \text{ and } \ \bar{q}_i^T\bar{B} = \begin{bmatrix} \cos\Phi(q_i, b_1) \\ \vdots \\ \cos\Phi(q_i, b_m) \end{bmatrix}^T . \tag{5.2.6}$$

If we denote the absolute value of a matrix by

$$|\ A\ | = \Big[\ |\ a_{ij}\ |\ \Big] \tag{5.2.7}$$

then we have the following :

<u>Lemma 5.2.1</u> $\quad | R_i | = \|p_i\| \|q_i\| D_o \cdot \bar{C}\bar{p}_i \; \bar{q}_i^T \bar{B} \cdot D_c \quad .$ □

<u>Proof:</u>

$$R_i = \begin{bmatrix} c_1 \\ \vdots \\ \vdots \\ c_l \end{bmatrix} \cdot p_i q_i^T [\, b_1 \; \cdots\cdots \; b_m \,] \; . \tag{5.2.8}$$

Thus,

$$|R_i| = \mathrm{diag}[\|c_1^T\| \| p_i\|, \ldots, \|c_l^T\| \|p_i\|] \cdot \begin{bmatrix} \cos\phi(c_1^T, p_i) \\ \vdots \\ \cos\phi(c_l^T, p_i) \end{bmatrix} \tag{5.2.9}$$

$$\times [\cos\phi(q_i, b_1) \; \ldots\ldots \; \cos\phi(q_i, b_m)] \cdot \mathrm{diag}[\|b_1\| \|q_i\|, \ldots, \|b_m\| \|q_i\|]$$

from which we obtain the result. □

<u>Remark 5.2.3</u> A similar result is derived by Hamdan [21]. In this derivation, a specific scaling in the input and output spaces is assumed. □

With the above lemma then, we can connect the state space modal descriptions of observability and controllability to the magnitude of the residue. This concept of modal measures is very popular, and in its own right has found much attention and interpretation in the literature. For more discussion on this, see [23]. For now though, our intention is to employ ε_0-modal measures simply as an avenue through which we may relate the ε_0-measures of near-unobservability and near-uncontrollability to the residue. We will do this next, but first a few closing remarks.

<u>Remark 5.2.4</u> In Lemma 5.2.1 suppose that $\|p_i\|$, $\|q_i\|$ and the diagonal elements of D_0 and D_c are all approximately one, but that $\|R_i\| << 1$. It follows that $\bar{C}\bar{p}_i \cdot \bar{q}_i^T\bar{B}$ must be small. From Definition 5.2.1 and Definition 5.2.2 we conclude that the mode associated with R_i must be nearly modal-unobservable and/or uncontrollable. Also by Lemma 5.1.2 we conclude that the degree of controllability or observability can be changed by a scaling of the basis, but in this scaling, the degrees will always be traded off against each other. □

5.3 Comparing ϵ_0-Modal Measures to ϵ_0-Measures.

In this section we compare the ε_0-modal measures introduced in Section 5.2 to the ε_0-measures introduced in Section 3.2. In review of the ε_0-measures we recall that $\mathscr{L}_j \subseteq \mathscr{L}_1 = \mathscr{N}[C]$ for all j. Therefore, if a system is ε_0-unobservable, then there exists an A-invariant subspace close to $\mathscr{N}[C]$. We also note that $\mathscr{R}_j \supseteq \mathscr{R}_1 = \mathrm{Im}[B]$ for all j, and therefore an ε_0-uncontrollable system implies that Im[B] lies close to some A-invariant subspace.

In view then of ε_0-modal measures, we note that right and left eigenvectors form A-invariant and A^T-invariant subspaces respectively. To mark another difference, note that the modal measures pertain to a specific input or output while the ε_0-measures encompass the entire input and output, respectively. Still though, these two sets of measures are quite similar in that the two angles describing these sets are in a sense, complimentary.

We may exploit the above noted differences and similarities to arrive at the following relationship between ε_0-measures and ε_0-modal measures.

Theorem 5.3.1 Let $\mathscr{V}_{oi}$ and $\mathscr{V}_{ci}$ be subspaces of C^n such that $p_i \varepsilon \mathscr{V}_{oi}$, and $q_i \perp \mathscr{V}_{ci}$. Then

$$\cos\Phi(c_j^T, p_i) \leq \tau(\mathscr{V}_{oi}, \mathscr{L}_h), \text{ and } \cos\Phi(q_i, b_k) \leq \tau(\mathscr{V}_{ci}, \mathscr{L}_h)$$

for every $h = 1,\ldots n$, $i = 1,\ldots n$, $j = 1,\ldots,l$, and $k = 1,\ldots,m$. $\square$

We will present two discussions of the above theorem. The first discussion provides a special case example. It is based solely on geometric arguments, and is presented here to endow the reader with a more intuitive grasp on how ε_0-modal measures relate to ε_0-measures. The second is a proof which is presented for purposes of rigorous verification of the claimed result.

5.3.1 Geometric Interpretation

Here we present a geometrical analysis of Theorem 5.3.1 for a SISO 3rd order system. We consider the observability case where $\mathscr{V}_{oi}$ is spanned by p_i and $\mathscr{L}_{n-1} = \mathscr{L}_2$ is the unobservable space chosen to measure from. Note that $d(\mathscr{L}_2) = 1$ if the system is completely observable. To set the ground for this discussion we introduce the concept of orthogonal projectors in Linear Algebra.

<u>Definition 5.3.1</u> Let $\mathcal{W}$ be a subspace of $\mathcal{X}$ and $x \varepsilon \mathcal{X}$. We denote $P_{\mathcal{W}}(x)$ as the <u>orthogonal projection</u> of x onto $\mathcal{W}$. □

Furthermore,

<u>Definition 5.3.2</u> Let θ be the angle between some $w \varepsilon \mathcal{W}$ and some $x \varepsilon \mathcal{X}$. Define the <u>projection angle</u> of x, θ_p by

$$\cos\theta_p = \frac{|\, w^T P_{\mathcal{W}}(x) \,|}{\|w\| \|P_{\mathcal{W}}(x)\|} .$$

□

<u>Lemma 5.3.1</u> $\theta_p \leq \theta$. □

<u>Proof:</u>

$$\cos\theta = \frac{|\, w^T x \,|}{\|w\| \|x\|} \tag{5.3.1}$$

$$= \frac{|\, w^T [P_{\mathcal{W}}(x) + x - P_{\mathcal{W}}(x)] \,|}{\|w\| \|x\|} \tag{5.3.2}$$

$$= \frac{|\, w^T P_{\mathcal{W}}(x) \,|}{\|w\| \|x\|} + \frac{|\, w^T [x - P_{\mathcal{W}}(x)] \,|}{\|w\| \|x\|} . \tag{5.3.3}$$

Since w and $x - P_{\mathcal{W}}(x)$ are orthogonal then the second term vanishes. Noting that $\|P_{\mathcal{W}}(x)\| \leq \|x\|$ we obtain the necessary result:

$$\cos\theta \leq \frac{| w^T \cdot P_\omega(x) |}{\|w\| \cdot \|P_\omega(x)\|} = \cos\theta_p \ . \tag{5.3.4}$$

□

In our chosen situation, we are trying to show that

$$\cos\phi(C^T, p_i) \leq | \sin\phi(p_i, \mathscr{L}_2) | \ . \tag{5.3.5}$$

Figure 13, on the next page, gives us a geometrical picture of how $C^T, \mathscr{L}_2$, and p_i are related. In terms of applying Lemma 5.3.2 we form the projection space $\mathscr{W} = sp[e_1, e_3]$. From the lemma then, it follows that

$$\theta_p \leq \theta \quad \text{and} \quad \phi_p \leq \phi \ . \tag{5.3.6}$$

Also though, $\mathscr{W}$ defines the complimentary plane, and thus

$$\phi_p = 90^o - \theta_p . \tag{5.3.7}$$

Using (5.3.6) then

$$\phi \geq 90^o - \theta \ . \tag{5.3.8}$$

A little bit of trigonometry then readily gives the result of (5.3.5).

5.3.2 Analytic Development

We now present the following proof of the results given in Theorem 5.3.1. Assume that $\mathscr{V}_{oi}$ and $\mathscr{V}_{ci}$ are as required by Theorem 5.3.1. Similar to Definition 2.4.1, define the i^{th} uncontrollable subspace by

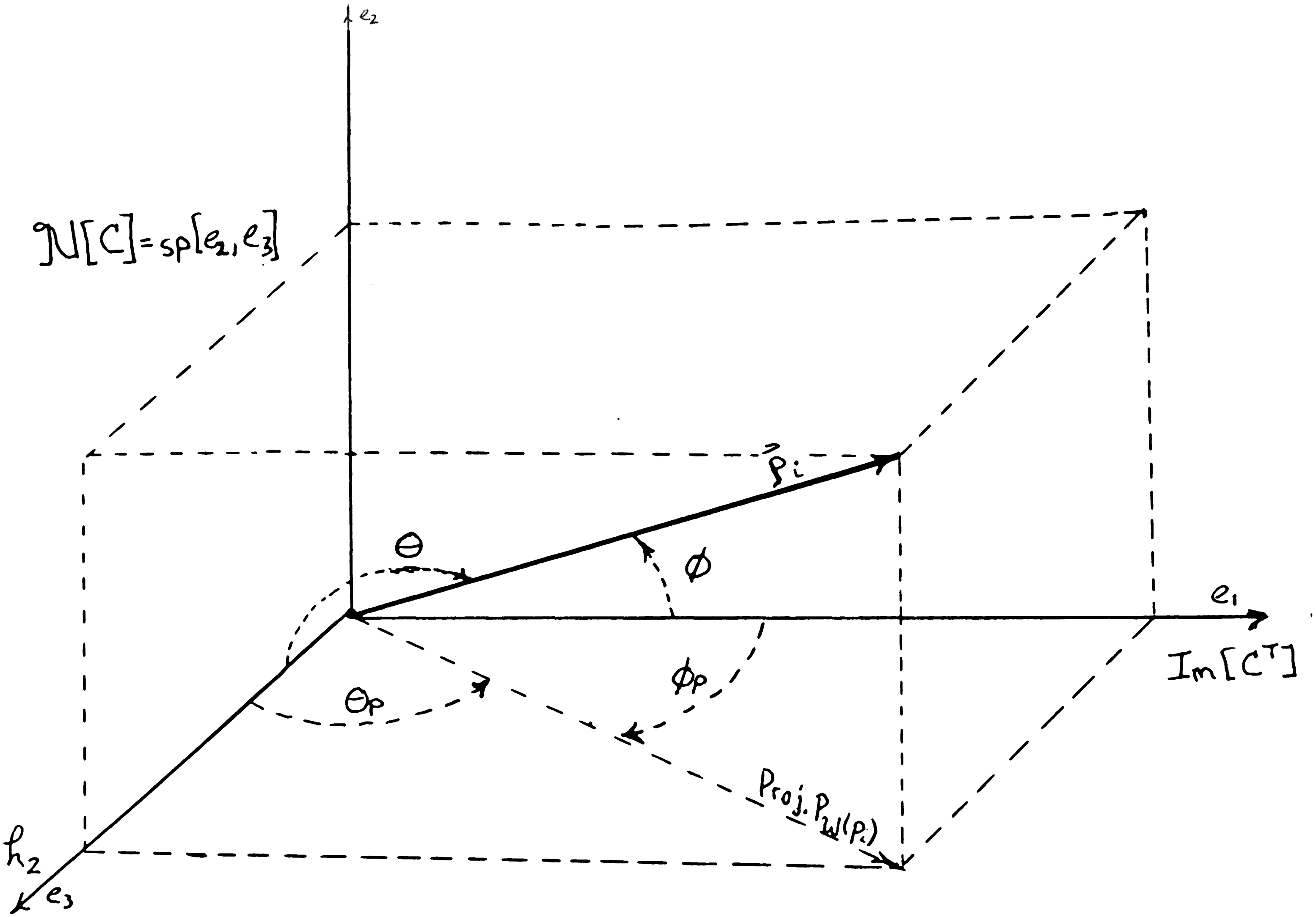

FIGURE 13: Geometrical Picture Comparing Modal-Observability with Near-Unobservability.

$$\mathcal{M}_h = \bigcap_{j=0}^{i-1} \mathcal{N}[(A^jB)^T] \ . \tag{5.3.9}$$

Denote $\mathcal{V}_{ci}^{\perp}$ as the orthogomal compliment of $\mathcal{V}_{ci}$, and let $P_{Mh}(qi)$ be the orthogonal projection of q_i onto $\mathcal{M}_h$. Then we have from Definition (3.2.1)

$$\tau(\mathcal{V}_{ci}^{\perp}\mathcal{M}_h) \geq \inf_{m \varepsilon \mathcal{M}_h} \|q_i - m\| \tag{5.3.10}$$

where q_i has unit length. From the Projection Theorem

$$\inf_{m \varepsilon \mathcal{M}_h} \|q_i - m\| = \|q_i - P_{Mh}(q_i)\| = |\ \sin\Phi(q_i, P_{Mh}\{q_i\})\ | \ . \tag{5.3.11}$$

Now since $\mathcal{M}_h \perp \mathrm{Im}[b_k]$ for every $h = 1,\ldots,n$ and $k = 1,\ldots,m$, then

$$\Phi(q_i, P_{Mh}\{q_i\}) = 90^o - \Phi(q_i, b_k) \ . \tag{5.3.12}$$

Therefore,

$$\tau(\mathcal{V}_{ci}^{\perp}\mathcal{M}_h) \geq |\ \sin(90^o - \Phi\{q_i, b_k\})\ | = |\ \cos\Phi(q_i, b_k)\ | \ . \tag{5.3.13}$$

Also, since $\mathcal{M}_h$ is the orthogonal compliment of $\mathcal{R}_h$ then [25]

$$(\mathcal{V}_{ci}^{\perp}\mathcal{M}_h) = (\mathcal{V}_{ci}\mathcal{R}_h) \tag{5.3.14}$$

from which the result for the controllable case directly follows. If we next note that $\mathcal{L}_h \perp \mathrm{Im}[c_j{}^T]$ for every $j = 1,\ldots,l$, then the observable case follows with a similar discussion using $\mathcal{L}_h$ and $\mathcal{V}_{oi}$.

With the proof in hand, we end this measure comparison with one additional comment:

Remark 5.3.1 Theorem 5.3.1 shows that if an A-invariant subspace $\mathcal{V}_{oi}$ is ε_o-unobservable, then all modes associated with the eigenvectors in $\mathcal{V}_{oi}$ will be ε_o-modal unobservable for all j. Since ε_o-unobservability accounts for several modes and all outputs simultaneously, we may think of it as a generalization of ε_o-modal unobservability. □

5.4 Measure Bounds of the Residue

Theorem 5.3.1 now allows us to describe the residue magnitude directly from state space in terms of near-unobservability and near-uncontrollability. Define the numbers

$$\bar{C}_l = \max \left\{ \|c_j^T\| \;:\; j = 1,\ldots,l \right\} \tag{5.4.1}$$

and

$$\bar{B}_m = \max \left\{ \|b_k\| \;:\; k = 1,\ldots,m \right\}. \tag{5.4.2}$$

Theorem 5.4.1 Let $\mathcal{V}_{oi}$ and $\mathcal{V}_{ci}$ be subspaces of C^n such that $p_i \in \mathcal{V}_{oi}$ and $q_i \perp \mathcal{V}_{ci}$. Then for every h_1, $h_2 = 1,\ldots n$

$$\|R_i\| \leq \sqrt{l \cdot m} \cdot \bar{C}_l \bar{B}_m \|p_i\| \|q_i\| \cdot \tau(\mathcal{V}_{oi}, \mathcal{L}_{h_1}) \tau(\mathcal{V}_{ci}, \mathcal{R}_{h_2}) \;. \qquad \square$$

Proof: Using Cauchy's inequality, Lemma 5.2.1, Theorem 5.3.1, and Remark 5.2.3 the result is established. □

With Theorem 5.4.1 we can then establish an explicit relationship between the residues R_i and the parameters μ_o and ε_o associated with near-aggregation, and near-unobservability/uncontrollability, respectively.

<u>Corollary 5.4.1</u> $\|R_i\| \le \sqrt{1\,m}\cdot\bar{C}_1\bar{B}_m\|p_i\|\|q_i\|\cdot\varepsilon_o^2$

where

$$\varepsilon_o = \max\left\{ \min \varepsilon_{oOBS} \;,\; \min \varepsilon_{oCON} \right\} . \qquad \square$$

<u>Corollary 5.4.2</u> Let (5.1.1) be μ_o-I/O aggregable, and assume Theorem 3.2.1 holds. Then

$$\|R_i\| \le \sqrt{1\,m}\cdot\bar{C}_1\bar{B}_m\|p_i\|\|q_i\|\cdot\frac{4\mu_o^2}{\delta^2} . \qquad \square$$

<u>Remark 5.4.1</u> From the above discussion, we may obtain a best bound on the modal measure of observability by finding that subspace $\mathcal{V}_{oi}$ which contains the mode, and is also closest in gap topology to some $\mathcal{L}_h$ subspace. Note that in a given system, there are many such possible choices ($\mathcal{V}_{oi}, \mathcal{L}_h$). Hence, the task of choosing the best one can be quite involved. Therefore, a general basis representation is quite unsuited to the idea of applying Theorem 5.3.1 to bound the modal measure. However, the Dual GHR can be easily adapted to such a process. More will be said about this below. $\square$

<u>Remark 5.4.2</u> In Theorem 5.3.1, the subspaces $\mathcal{V}_{oi}$ and $\mathcal{V}_{ci}$ are not necessarily related in anyway except through p_i and q_i. Thus, for a general system we must endure the process described above twice, in order to include controllability. $\square$

Lemma 5.2.1 and Theorem 5.4.1 show that the magnitude of the residual is proportional to the degree of controllability and

observability in either choice of measures. Because the degree of observability of a particular mode can be changed, by a change of basis in state space (i.e. changing the inner product), an observability measure or controllability measure alone is not enough to estimate the magnitude of the residual. Roughly speaking, a change of basis will trade off the magnitudes of the controllability and observability measures.

The basic overtones of the above discussion evoke us to examine the Dual GHR. Recall that the Dual GHR readily identified the subspaces $\mathscr{L}_i$ and $\mathscr{R}_i$. Also, the Dual GHR's super diagonal-structure, along with eigenvalue separation consideration, gives immediate ideas as to which subspaces $\mathscr{V}_{oi}$ and $\mathscr{V}_{ci}$ we should consider. In essence then, the Dual GHR allows us to make an explicit connection between almost pole zero cancellation, ε_0-measures of observability and controllability, and residues. We can formalize this assertion by the following.

<u>Theorem 5.4.2</u> Consider the modes of the Dual GHR decomposition associated with $\mathscr{V}_1 = sp[P_1 \;\; I]^T$. Suppose Theorem 3.2.1 holds. If

$$\det[\, I_{n-\rho_i} - P_2 P_1 \,] \neq 0$$

then the corresponding residues R_j associated with $\mathscr{V}_1$ satisfy

$$\|R_j\| \le 4\gamma_1 \|p_j\| \|q_j\| \frac{\gamma_{i+1}^2}{\delta^2} \quad . \qquad \square$$

Proof: Let

$$\mathscr{L}_{\rho_i} = \text{sp}\begin{bmatrix} 0 \\ I_{n-\rho_i} \end{bmatrix} \quad \text{and} \quad \mathscr{R}_{\rho_i} = \text{sp}\begin{bmatrix} I_{\rho_i} \\ 0 \end{bmatrix} \quad . \tag{5.4.2}$$

We have then that

$$\tau(\mathscr{L}_{\rho_i}, \mathscr{V}_1) \leq \|P_1\| \quad \text{and} \quad \tau(\mathscr{R}_{\rho_i}, \mathscr{V}_2) < \|P_2\| \, . \tag{5.4.3}$$

If $p_j \in \mathscr{V}_1$, then $q_j \perp p_k$ for every $j \neq k$. Also if

$$\det[\, I_{n-\rho_i} - P_2 P_1 \,] \neq 0 \;, \tag{5.4.4}$$

then $\mathscr{V}_1 \oplus \mathscr{V}_2 = \mathscr{X}$. Therefore $p_j \in \mathscr{V}_2$, and since $\mathscr{V}_2$ is A-invariant then

$$\mathscr{V}_2 = \text{sp}[\; p_k : k = 1,\ldots,n \;,\; j \neq k \;] \; . \tag{5.4.5}$$

From this we may conclude that

$$q_j \perp \mathscr{V}_2 \; . \tag{5.4.6}$$

Using Theorem 5.4.1 and (5.4.3) the desired result is obtained. □

Now compare Theorem 5.4.2 with Theorem 3.3.1 which describes the interaction of $G(s;\mu)$ and $G_a(s)$. We see that as $\gamma_{i+1}(\mu) \longrightarrow 0$, the system exhibits $(n - \rho_i)$ almost pole zero cancellations, and that the residues corresponding to these modes also go to zero. Thus, model reduction via the Dual GHR is equivalent under certain considerations to deleting modes associated with small residues.

5.5 Examples

To give an overview of the results devived in this chapter we present four examples. The first two are in Dual GHR forms. The second two are MIMO systems. For each type we calculate the following:

1. ϵ_0-modal measures for each mode.
2. $\|R_i\|$.
3. Minimum ε_0-measures for each mode.
4. $\|R_i\|$ bound given in Theorem 5.4.1.

The first two are computed using the LAS program Residue, the second two are given by the LAS program Residue Continued. Both may be found in the Appendix, and were written for general systems of the form (2.1.1). Note that in determining the minimum ε_0-measures, a linear search is made examining all possible combinations of ($\mathcal{V}_{0i}$,$\mathcal{L}_{h1}$) and ($\mathcal{V}_{ci}$,$\mathcal{R}_{h2}$). Also note that we choose the $\mathcal{L}$ or $\mathcal{R}$ subspaces above to match the dimension of the $\mathcal{V}_{0i}$ or $\mathcal{V}_{ci}$ in question.

5.5.1 Example 1

We first examine the 4x4 model considered in Examples 2.3.1, 3.2.1, 3.3.1 and Example 1 of the last chapter. The generated data may be found in Table 1 on the next page.

TABLE 1: Calculations for Example 1.

i	MODE	ε_o-MODAL MEASURES OBS.	ε_o-MODAL MEASURES CON.	ε_o-MEASURES OBS.*	ε_o-MEASURES CON.*	$\lVert p_i\rVert\cdot\lVert q_i\rVert$	$\lVert R_i\rVert$	$\lVert \bar{R}_i\rVert$
1	-4.0497	0.054	0.059	0.058 (1,3,4)	0.247 (2,3)	6.502	.020633	.09315
2	-17.181	0.211	0.003	0.247 (2,3,4)	0.058 (1,3)	6.359	.004527	.09200
3	-.35019	0.014	0.544	0.058 (1,3,4)	0.555 (4)	2.225	.016606	.07162
4	-.07920	0.011	0.022	0.058 (1,3,4)	0.022 (1,2,3)	2.067	.000500	.00264

* $\longrightarrow$ Represents tightest bound.
Ordered tuple gives V_{oi} and V_{ci} subspaces, respectively.
$\lVert \bar{R}_i\rVert$ denotes the bound on the residue.

Zeros: s=-.070

Note that this system is not regular. Also, by Example 3.2.1 we would not call $\mathcal{V}_{01}$ = sp[p_4] ε_0-unobservable, however $\mathcal{V}_{02}$ = sp[p_1,p_3,p_4] would be considered ε_0-unobservable. In view of the actual $\|R_i\|$, we might delete the fourth mode from this model. From (4.3.4) it is obvious that this is somewhat equivalent to deleting the state x_4.

Remark 5.5.1 This Dual GHR example then presents an interesting phenomena. On one hand, the fourth mode is not ε_0-unobservable, yet on the other, it resides in a space which is ε_0-unobservable, and consequently its residue is small. Explanations for this are unclear at the moment but the issue is under investigation. The reasons for this appear to lie in the irregularity of the system. We may conclude though, that a small residue does not necessarily imply the assocaited 1-dimensional invariant subspace is ε_0-unobservable, even in a Dual GHR. Finally, note that $\mathcal{V}_{ci}$ = sp[q_1,q_2,q_3] is .022-uncontrollable.

5.5.2 Example 2

Next we apply the residues analysis to the Hoop Column Antenna also examined in the last chapter. Here though, we scale the non-zero input and output elements (i.e., γ_1) to unity. The results are given in Table 2 on the next page.

From the pole and zero locations, there are three almost pole zero cancellations, the 5th, 6th, and 7th modes. These almost cancellations are predicted separately by the ε_0-modal measures and collectively by the ε_0-measures.

TABLE 2: Calculations for Example 2.

i	MODE	$\mathcal{E}_o$-MODAL MEASURES OBS.	CON.	$\mathcal{E}_o$-MEASURES OBS.*	CON.*	$\lVert p_i \rVert \cdot \lVert q_i \rVert$	$\lVert R_i \rVert$	$\lVert \bar{R}_i \rVert$
1,2	-.00729 ±j7.40	0.538	0.538	0.799 (1,4-7)	0.799 (2,3)	1.000	.28926	.63840
3,4	-.00559 ±j5.79	0.457	0.457	0.649 (3-7)	0.649 (1,2)	1.000	.20916	.42120
5	-.000198	0.050	0.050	0.242 (5,6,7)	0.242 (1-4)	1.000	.00249	.05856
6,7	-.00602 ±j1.38	0.018	0.018	0.242 (5,6,7)	0.242 (1-4)	1.000	.00033	.05856

* ⟶ Represents tightest bound.
Ordered tuple gives V_{oi} and V_{ci} subspaces, respectively for +j mode.
$\lVert \bar{R}_i \rVert$ denotes the bound on the residue.

Zeros: s= -.00628 ±j6.51
s= -.000209 ±j.326
s= -.00602 ±j1.39

5.5.3 Example 3

The following system description models the flexible dynamics of a rocket, [2].

$$A = \begin{bmatrix} -.2105 & -.1056 & -.0007 & 0 & -.0706 & 0 \\ 1.0 & -.0354 & -.0001 & 0 & -.0004 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -605.1 & -4.92 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3906 & -12.5 \end{bmatrix}$$

$$B = [\ -7.211 \quad -.0523 \quad 0 \quad 794.7 \quad 0 \quad -448.5\ ]^T$$

$$C = \begin{bmatrix} 1.0 & 0 & .0003 & 0 & -.0077 & 0 \\ 0 & 1.0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The modal measures, ε_0-measures, residues and bounds on the residues are summarized in Table 3 on the following page.

In this MIMO model we see that the residue analysis yields the 3^{rd} through 6^{th} modes as having relatively small residues. Note however, in this case there do not exist any finite invariant zeros. Thus, the small residue in a MIMO system does not always equate to the concept of almost pole zero cancellation. This example does show, though, that an ε_0-unobservable/uncontrolable subspace does indicate the presence of small residues.

TABLE 3: Calculations for Example 3.

i	MODE	$\mathcal{E}_o$-MODAL MEASURES OBS.	CON.	$\mathcal{E}_o$-MEASURES OBS.*	CON.*	$\lVert p_i \rVert \cdot \lVert q_i \rVert$	$\lVert R_i \rVert$	$\lVert \overline{R}_i \rVert$
1,2	-.12295 $\pm j.312$	0.309(c_1) 0.951(c_2)	.00753	1.000 (1,2)	0.091 (3-6)	1.769	121.57	189583.5
3,4	-2.4600 $\pm j24.4$	$.137 \times 10^{-4}$ $.197 \times 10^{-6}$	.03537	0.008 (3-6)	0.298 (4,5)	12.382	.005489	34763.88
5,6	-6.2500 $\pm j62.1$	$.127 \times 10^{-3}$ $.297 \times 10^{-6}$	.00786	0.008 (3-6)	0.298 (3,6)	31.415	.028551	88201.20

* ⟶ Represents tightest bound.
Ordered tuple gives V_{oi} and V_{ci} subspaces, respectively for +j mode.
$\lVert \overline{R}_i \rVert$ denotes the bound on the residue.

Zeros: No Finite Zeros

5.5.4 Example 4

A 23 state linear MIMO model of an F-100 Jet Engine may be found in [25]. This system has a pole at -.648 and a zero at -.649. The associated residue, however, is one of the largest comparitively. Note that in this case, the pole is very close to the $j\omega$-axis with respect to the rest of the system. In a dominance argument, this pole would usually be kept just based on this position information.

6.0 Conclusions and Further Study

6.1 Conclusions

In this thesis we have presented an indepth review summarizing basic results and interpretations previously reported on aggregation, notedly in the extension of aggregation to near-aggregation, near-unobservability, and almost pole zero cancellations. In this review, the role of the GHR and Dual GHR state space representations have been emphasized.

In our analysis, we have extended the above concepts in terms of the state trajectory behaviors of the full and reduced-order models. Here we have shown that if a system is nearly-aggregable, then the trajectories of the reduced-order model are near linear combinations of the trajectories of the full model. This generalizes the well known concept of exact aggregation in which the trajectories of the reduced-order model are precisely given by linear combinations of the full system.

We have also examined the relationship between transfer function residues and two measures of observability and controllability, denoted by modal measures and near measures, respectively. Using a geometrical analysis, we have shown that near-unobservability /uncontrollability generalizes the concept of modal unobservability/ uncontrollability. In particular we have demonstrated that for both SISO and MIMO systems, a nearly-unobservable/uncontrollable system implies that some of the system residues will be small. In addition, for SISO systems we have generalized this relationship to the concept of almost pole zero cancellations.

6.2 Further Study

The research presented in this thesis suggests many points of interest which might be further investigated. The most obvious of these would be in the generalization of the Dual GHR to MIMO systems. As the results for the SISO Dual GHR indicate, the potential for such a canonical MIMO representation is exceptional.

Another line of investigation to follow would be in relating the two measures of observability and controllability discussed here, to other measures present in the control literature.

Finally, even without the MIMO Dual GHR, the examples of the last chapter invite further study into the interaction of the pole and zero structures of the system. The subject of pole zero interaction is still very open, and it would appear that the concepts of aggregation discussed in this paper could be further explored in this issue. Particularly, in view of the relationship between transfer function residues and the internal state space description.

BIBLIOGRAPHY

[1] Aoki, M., "Control of Large-Scale Dynamic Systems by Aggregation", IEEE Transactions on Automatic Control, Vol. AC-13, 1968, pp. 246-253.

[2] Medanic, J.V., W.R. Perkins, and E. Tse, "Reduced Order Modeling by the Restricted QL Algorithm", Decision and Control Report T-57, Coordinated Science Laboratory, University of Illinois, Urbana, IL, 1978.

[3] Elrazaz, Z., "A Review of Some Model Reduction Techniques", Can.Elec. Eng. J., Vol. 6, 1981, pp. 34-40.

[4] Tse, E.C.Y., J. Medanic, and W.R. Perkins, "Generalized Hessenberg Transformations for Reduced Order Modeling of Large Scale Systems", Int. J. Control, Vol 27, 1978, pp. 493-512.

[5] Tse, E.C.Y., J.V. Medanic, and W.R. Perkins, "Chained Aggregation of Linear Time Invariant Systems," JACC, 1977.

[6] Lindner, D.K. and W.R. Perkins, "Generalized Hessenberg Representations and Model Reduction", in Advances in Large Scale Systems, J.B. Cruz, Jr., editor, 1984, pp. 27-59.

[7] Lindner, D.K. and W.R. Perkins, "Generalized Hessenberg Representations and Near Aggregations," 1987, to appear in Automatica.

[8] Lindner, D.K., "The Dual GHR", Proc. of 22nd Annual Allerton Conf. on Communication, Control, and Computing, Monticello, IL, 1984, pp. 745-752.

[9] Lindner, D.K. and J. Babendreier, "The Dual GHR", Internal Technical Report, Dept. Elec. Eng., Virginia Polytechnic Institute and State University, Blacksburg, VA, 1986.

[10] Lindner, D.K., "Near Aggregation, the Dual GHR, and Pole Zero Cancellations", 1987, submitted for publication.

[11] Bingulac, S.P. and M.A. Farias, "L-A-S (Linear Algebra and System) Language and Its Use in Control Education and Research", Computing, Springer-Verlag, Vol. 23, 1979, pp. 1-23.

[12] Kalman, R., "On Partial Realizations, Transfer Functions and Canonical Forms", Acta Polytech. Scand., MA31, 1979, p. 9.

[13] S.K. Mitra and R.J. Sherwood, "Canonic Realizations of Digital Filters Using the Continued Fraction Expansion", IEEE Trans. Audio Electroacoust., v AU-20, 1972, pp. 185-194.

[14] DeBrummer, V.E., "Sensitivity Analysis of Digital Filter Structures", Masters Thesis, Virginia Polytechnic Institute and State University, Blacksburg, VA, May, 1986.

[15] Lindner, D.K., W.R. Perkins, and J. Medanic, "Chained Aggregation and Three-Control Component Design: A Geometric Analysis", Int. J. Control, Vol. 35, 1982, pp. 621-636.

[16] McFarlane, A.G.J.and N. Karcanias, "Poles and Zeros of Linear Multivariable Systems: A Survey of the Algebraic, Geometric and Complex-Variable Theory", Int. J. Control, Vol. 24, 1976, pp. 33-74.

[17] Kato, T., Perturbation Theory for Linear Operators, Springer, NY, 1966.

[18] Stewart, G.W., "Error and Perturbation Bounds for Subspaces Associated with Certain Eigenvalue Problems", SIAM Review, Vol. 15, 1973, pp. 727-764.

[19] Bauer, F.L. and C.T. Fike, "Norms and Exclusion Theorems", Numer Math., Vol. 2, 1960, pp. 137-144.

[20] Vidyasagar, M., Nonlinear System Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1978.

[21] Hamdan, A.M.A. and Nayfeh, A., "Modal Observability and Controllability Measures for First and Second Order Linear Systems and Model Reduction", to be presented at Sixth VPI&SU Symposium on Dynamics and Control of Large Structures, Blacksburg, Va, June, 1987.

[22] Verghese, G., I Perez-Arriage, and F. Schweppe, "Selective Modal Analysis with Applications to Electric Power Systems, Part II: The Dynamic Stability Problem", IEEE Trans. on PAS. Vol. PAS-101, 1982, p. 3126.

[23] Athans, M., P. Kapasouris, E. Kappos, and H. Spang III, "Linear-Quadratic Gaussian with Loop-Transfer Recovery Methodology for the F-100 Engine", J. of Guidance, Control and Dynamics, Vol. 9, 1986, pp. 45-52.

[24] Kailath, T., Linear Systems, Prentice-Hall, Englewood Cliffs, NH, 1980.

[25] Gohberg, I., P. Lancaster, and L. Rodman, Invariant Subspaces of Matrices with Applications, Wiley, New York, 1986.

APPENDIX

```
  _Program_Subroutine_GHR
  ‾
  ‾This_Program_Calculates_the_GHR_Representation
  _of_a_General_MIMO_System.
  _
  Ā,B,C(GHR,SUB)=AGHR,BGHR,CGHR,GHRT,BLOC,LN
  (NLI)=
  A,B,C(MCP)=AO,BO,CO
  O(DVC)=RHO
  O,O(DZM)=LN
  CO(MCP)=CI
  AO(CDI)=N
  N(MCP)=NI
  N,N(DIM)=I
  I(MCP)=GHRT
  CI(T)(RSP)=D1
  D1(CDI)=RI
  RI(MCP)=BLOC

a:C̄I(NSP)=D2
  D1,D2(CTI)(ORD)=DI
  N,NI(-)(INC)=N*
  I,DI,N*,N*(RMP)=TI

  ḠHRT,TI(*)=GHRT
  TI(T),AO,TI(*)(*)=AO
  RHO,RI(+)=RHON
  NI,RI(-)=NI
  AO,RHO(INC),RHON(INC),RI,NI(EXM)=CI
  RHON(MCP)=RHO

  Ō,O(DZM)=D1
  CI(T)(RSP)=D1
  D1(CDI)=RI
  RI(IFJ)=b,b,c
b:N,RHO(-)=RI
  CI(NSP)=D2
  RHO,D2(CDI)(DZM),D2(RTI)=LN
c:BLOC,RI(CTI)=BLOC
  NI,RI(IFJ)=d,d,a

d:ĀO(MCP)=AGHR
  GHRT(T),BO(*)=BGHR
  CO,GHRT(*)=CGHR
  (LIS)=
```

```
__PROGARM_DUAL_GHR__

¯(TXT,T,THIS"PROGRAM"TRANSFORMS"A"STATE)=
(TXT,T,REPRESENTED"SISO"SYSTEM"INTO"ITS)=
(TXT,T,DUAL"GHR"FORM.)=
(TXT,T,")=

0̄(DVC)=SIG
SIG(MCP)=RHO
0(DVC)=U0
1(DVC)=U1
2(DVC)=U2
0,1(DZM)=SIGM
_
_
__Input_System_and_Check_Validity__

¯(RDF)=A,B,C
A(CDI)=NA
A(RDI)=RA
NA,RA(IFJ)=e,a,e
a:B(CDI)=NB
B(RDI)=RB
RB,NA(IFJ)=e,b,e
b:U1,NB(IFJ)=e,c,e
c:C(CDI)=N
C(RDI)=R
N,NA(IFJ)=e,d,e
d:U1,R(IFJ)=e,f,e
e:A,B,C(OUT,T)=
(TXT,T,A"B"C"IS"NOT"A"VALID"SISO"SYSTEM)=
(TXT,T,TYPE""'JUM,17'""AND"TRY"AGAIN.)=
(STO)=
f:N,U1(IFJ)=F,E,E
F:(TXT,T,TRIVIAL"QUESTIONS"NEED"NO"ANSWER)=
(STO)=
E:A(MCP)=AORG
B(MCP)=BORG
C(MCP)=CORG
N(MCP)=NORG
_
_
__This_section_performs_i-steps_of_aggregation__
__on_subsystem_(A,B,C)_untill_bi_is_non-zero.__

¯(TXT,T,PLEASE"ENTER"A"ZERO"COMPARISON)=
(DSC)=Eo
Eo,Eo(*)(SQR)=Eo

G:N̄,SIG(-)=NSIG
B(T),B(*)(SQR)=BDOT
```

```
  BDOT,Eo(IFJ)=P,P,I
 P:N(MCP)=SIG
  UO(MCP)=NSIG
  (JMP)=y
 I:C,C(T)(*)(SQR)=CDOT
  CDOT,Eo(IFJ)=P,P,J

J:N̄,U1(IFJ)=H,L,H
H:SIG(IFJ)=i,h,i
h:C(MCP)=CSIG
  (JMP)=j
i:A,SIG,SIG,U1(+),1,NSIG(EXM)=CSIG
j:CSIG(CDI)=FLAG
  FLAG,U1(IFJ)=L,L,K
K:CSIG,Eo(NSP)=NSPC
  CSIG(T),Eo(RSP)=RSPC
  RSPC,NSPC(CTI)=T
  SIG(IFJ)=k,l,k
k:SIG,SIG(DIM)=I
  SIG,NSIG(DZM)=Z
  I,Z(T)(RTI),Z,T(RTI)(CTI)=T
l:A,B,C,T(STR)=A,B,C

L̄:SIG,U1(+)=SIG
  N,SIG(-)=NSIG
  N,U1(IFJ)=M,w,M
M:NSIG(IFJ)=o,o,O
O:B,SIG,1,1,1(EXM)=BSIG
  BSIG,BSIG(*)(SQR)=ABS
  ABS,Eo(IFJ)=J,J,m

m:B̄,SIG(CTR)=BA,BR
  SIG,SIG(DIM)=S1
  NSIG,NSIG(DIM)=S4
  SIG,NSIG(DZM)=S2
  NSIG,SIG,U1(-)(DZM),BR,BSIG(-1)(*)(CTI)=S3
  S1,S2(CTI),S3,S4(CTI)(RTI)=S
  A,B,C,S(STR)=A,B,C
  _
  _
  __In_this_section_we_transform_the_Residual__
  __into_an_input-output-state_triple,leaving__
  __the_Aggregate_unchanged.

N:Ō(DVC)=K
  SIG,U2(IFJ)=w,n,n

n:Ā,SIG,U1(+),SIG,K(-),NSIG,1(EXM)=ARK
  A,SIG,U1(-),K(-),SIG,K(-),1,1(EXM)=CSK
  NSIG,U1,K(+)(DZM)=ZR
  NSIG,SIG,U2(-),K(-)(DZM)=ZL
```

```
  ZL,ARK,CSK(-1)(*),ZR(CTI)=V3
  S1,S2(CTI),V3,S4(CTI)(RTI)=VK
  A,B,C,VK(STR)=A,B,C
  K,U1(+)=K
  SIG,K(-),U2(IFJ)=o,n,n

  _
  _
  __Here_we_transform_the_Aggregate_diagonal_block__
  __into_phase_canonical_form.__

o:Ā,1,1,SIG,SIG(EXM)=AS
  B,1,1,SIG,1(EXM)=BS
  C,1,1,1,SIG(EXM)=CS

  ĀS(MCP)=A1
  CS(MCP)=OB
  SIG,U1(-)=K

p:ŌB,CS,A1(*)(RTI)=OB
  K,U1(-)=K
  A1,AS(*)=A1
  K(IFJ)=q,q,p

q:ŌB(RKC)=DUM1,DUM2,RANK
  RANK,1,1,1,1(EXM)=RANK
  RANK,SIG(IFJ)=w,S,S
S:SIG,U1(-)=K
  OB(-1)=OBIN
  OBIN,K(CTC)=DUM,T
  T(MCP)=TO
  AS(MCP)=A1
r:TO,A1,T(*)(CTI)=TO
  K,U1(-)=K
  A1,AS(*)=A1
  K(IFJ)=s,s,r

s:S̄IG,SIG(DIM)=I
  I,1(CTC)=I1
  I1(MCP)=IP
  SIG,U1(-)=K

u:Ī1(SHD)=I2
  I2,IP(CTI)=IP
  I2(MCP)=I1
  K,U1(-)=K
  K(IFJ)=v,v,u

v:T̄O,IP(*)=M
  N,N(DZM)=MZ
  NSIG,NSIG(DIM)=MI
  MZ,M(RMP)=M
```

```
  M,MI,SIG,U1(+),SIG,U1(+)(RMP)=M
  A,B,C,M(STR)=A,B,C
  _
  _
  __Finally_we_balance_the_input_and_output_elements__
  __via_scaling.__

w:B,SIG,1,1,1(EXM)=Bi
  C,1,1,1,1(EXM)=C1
  Bi,Bi(*)(SQR)=ABi
  Bi,ABi(-1)(*)=X1
  Bi,C1(-1)(*),Bi,C1(-1)(*)(*)(SQR)(SQR)=X2
  X1,X2(*)=ALPA
  N,N(DIM)=I
  U1(MCP)=K

x:I,ALPA,K,K(RMP)=N1
  A,B,C,N1(STR)=A,B,C
  K,U1(+)=K
  SIG,K(IFJ)=y,x,x
  _
  _
  __In_this_section_we_update_the_DGHR_form__
  __and_repeat_the_above_on_the_residual_system__

y:RHO,SIG(+)=RHO
  SIGM,SIG(RTI)=SIGM
  A,1,1,SIG,SIG(EXM)=ATEM
  RHO,SIG(IFJ)=A,z,A

z:C(MCP)=CGHR
  B(MCP)=BGHR
  N,N(DZM)=AGHR
  AGHR,ATEM,1,1(RMP)=AGHR
  (JMP)=B

A:RHO,SIG(-),U1(+)=L
  AGHR,ATEM,L,L(RMP)=AGHR
  AGHR,B,L,L,SIGO(-)(RMP)=AGHR
  AGHR,C,L,U1(-),L(RMP)=AGHR

B:NSIG(IFJ)=D,D,C
C:A,SIG,U1(+),SIG,U1(+),NSIG,NSIG(EXM)=ANEX
  A,SIG,U1(+),1,NSIG,1(EXM)=B
  A,SIG,SIG,U1(+),1,NSIG(EXM)=C
  ANEX(MCP)=A
  SIG(MCP)=SIGO
  NSIG(MCP)=N
  O(DVC)=SIG
  (JMP)=G
  _
```

```
D:AORG,BORG,CORG(OUT,T,ORIGINAL"FORM)=
  AORG,BORG,CORG(OUT,L,E,ORIGINAL"FORM)=
  AGHR,BGHR,CGHR(OUT,T,DUAL"GHR"FORM)=
  AGHR,BGHR,CGHR(OUT,L,E,DUAL"GHR"FORM)=
  SIGM(OUT,T,SIGMA"BLOCK"BREAKDOWN)=
  SIGM(OUT,L,SIGMA"BLOCK"BREAKDOWN)=
  AGHR,BGHR,CGHR(COT)=
  AGHR,BGHR,CGHR(MTF)=F1,G1
  G1(T)=G1
  G1(ELZ)=G1
W:G1(CHE)=EGV1
  F1(CHE)=EGV2
  EGV1(OUT,T,SYSTEM"ZEROS"ARE)=
  EGV1(OUT,L,E,SYSTEM"ZEROS"ARE)=
  EGV2(OUT,T,SYSTEM"POLES"ARE)=
  EGV2(OUT,L,E,SYSTEM"POLES"ARE)=
  AGHR,BGHR,CGHR(WDF)=
  AGHR(JFR)=MA,AJ
  MA,AJ(OUT,L,E,MODAL"MATRIX"&"JORDEN"FORM)=
```

```
  _PROGRAM_RESIDUE
  _
  _This_Program_Calculates_Modal_Measures_of
  _Observability_and_Controllability._It_Also
  _Calculates_the_Norm_of_the_Residue_Matrix.
  _
  (RDF)=A,B,C
  A(RDI)=N
  -1(DVC)=MN1
  1(DVC)=K

  B(CDI)=M
  C(RDI)=L
  1,0(DZM)=BBAR
  BBAR(MCP)=CBAR

  A(EGV)=POLE
  A(JFR)=MODE
  N,1(DZM)=PZER
  N,0(DZM)=PR
  PR(MCP)=PC
X:POLE,K,2,1,1(EXM)=TES
  TES(ABS)(IFJ)=W,W,V
V:MODE,1(CTC)=PIR,MODE
  MODE,1(CTC)=PIC,MODE
  PR,PIR,PIR(CTI)=PR
  PC,PIC,PIC,MN1(S*)(CTI)=PC
  K,MN1(-),MN1(-)=K
  (JMP)=U
```

```
W:MODE,1(CTC)=PIR,MODE
  PR,PIR(CTI)=PR
  PC,PZER(CTI)=PC
  K,MN1(-)=K
U:K,N(IFJ)=X,X,T
T̄:1(DVC)=K
  A,B,C(OUT,L)=
  (TXT,L,*********************************)=
  (TXT,L,*********************************)=
  (TXT,L,SYSTEM"POLES"ARE)=
  POLE(OUT,L,E)=
  (TXT,L,*********************************)=
  (TXT,L,*********************************)=
  (TXT,L,TRANSMISION"ZEROS"ARE)=
  L̄,M(-)=DIM*
  DIM*(IFJ)=A,B,A
A:(TXT,L,UNAVAILABLE"--->NON-SQUARE"SYSTEM)=
  (JMP)=m
B:A,B(CTI),MN1(S*)=X1
  C,L,L(DZM)(CTI)=X2
  X1,X2(RTI)=X1
  N,L(+),N,L(+)(DZM),N,N(DIM),1,1(RMP)=X2
  X1,X2(TZS)=ZERO
  ZERO(OUT,L,E)=
  (TXT,L,*********************************)=
  (TXT,L,*********************************)=
m:Ī(DVC)=K
  PR,PC(CTI)=P
  P,1(ITC,SUB)=Q
  Q,2(ITC,SUB)=Q
  Q,N(CTC)=QR,QC
  P̄R(OUT,L,REAL"PART"RIGHT"EIGENVECTORS)=
  PC(OUT,L,IMAG"PART"RIGHT"EIGENVECTORS)=
  (TXT,L,*********************************)=
  (TXT,L,*********************************)=
  QR(OUT,L,REAL"PART"LEFT"EIGENVECTORS)=
  QC(OUT,L,IMAG"PART"LEFT"EIGENVECTORS)=
a:B̄,1,K,N,1(EXM)=BM
  BBAR,BM(GHR)(CTI)=BBAR
  K,MN1(-)=K
  K,M(IFJ)=a,a,b
b:BBAR(DDM)=DC
  DC(GHR),1,1,1,1(EXM)=BMAX
  1(DVC)=K
c:C̄,K,1,1,N(EXM)(T)=CLT
```

```
  CBAR,CLT(GHR)(CTI)=CBAR
  K,MN1(-)=K
  K,L(IFJ)=c,c,d
d:CBAR(DDM)=DO
  DO(GHR),1,1,1,1(EXM)=CMAX
  M,L(*)(SQR),BMAX,CMAX(*)(*)=CONS

  Ō(DVC)=K2
  N(MCP)=K
  K2,K2(CTI)=INL
  INL,INL,INL,INL(MCP)=L1,L2,L3,L4
  INL,INL,INL,INL(MCP)=R1,R2,R3,R4
  N,N(DIM)=I
  1,0(DZM)=LDIM
  1,0(DZM)=RDIM

  Ā,B,C(GHR,SUB)=AGHR,BGHR,CGHR,GHRT,BLOC,DUM
  (TXT,L,*********************************)=
  (TXT,L,*********************************)=
  BLOC(OUT,L)=
  AGHR,BGHR,CGHR(OUT,L)=
e:BLOC,1(CTC)=BLO*,BLOC
  K,BLO*(-)=K
  K2,BLO*(+)=K2
  K,LDIM(CTI)=LDIM
  I,1,K2(INC),N,K(EXM)=L*
  GHRT,L*(*)=L*
  L*,N,K(DZM)(CTI)=L*
  L4,L3,L2,L1,L*(MCP)=L5,L4,L3,L2,L1
  BLOC(CDI)(DEC)(IFJ)=f,f,e

f:(̄TXT,L,*********************************)=
  (TXT,L,*********************************)=
  LDIM(CDI),5(IFJ)=E,E,D
D:LDIM,5(CTC)=LDIM
E:LDIM,L1,L2,L3,L4,L5(OUT,L)=
  N(MCP)=K
  O(DVC)=K2

  Ā(T),C(T),B(T)(MCP)=AR,BR,CR
  AR,BR,CR(GHR,SUB)=AGHR,BGHR,CGHR,GHRT,BLOC,DUM
  BLOC,BLOC(CDI)(DEC)(CTC)=BLOC
g:BLOC,BLOC(CDI)(DEC)(CTC)=BLOC,BLO*
  K,BLO*(-)=K
  K2,BLO*(+)=K2
  K2,RDIM(CTI)=RDIM
  I,1,1,N,K2(EXM)=R*
  GHRT,R*(*)=R*
  R*,N,K2(DZM)(CTI)=R*
  R4,R3,R2,R1,R*(MCP)=R5,R4,R3,R2,R1
  BLOC(CDI)(IFJ)=h,h,g
```

```
h:(TXT,L,*******************************)=
  (TXT,L,*******************************)=
  RDIM(CDI),5(IFJ)=G,G,F
F:RDIM,5(CTC)=RDIM
G:RDIM,R1,R2,R3,R4,R5(OUT,L)=
  (TXT,L,*******************************)=
  (TXT,L,*******************************)=

  I(DVC)=K
  2,0(DZM)=DAT1
  L,MN1(-),0(DZM)=OBSV
  M,MN1(-),0(DZM)=CONT
  0,1(DZM)=NMAR

q:PR,1,K,N,1(EXM)=PIR
  PC,1,K,N,1(EXM)=PIC
  QR,1,K,N,1(EXM)=QIR
  QC,1,K,N,1(EXM)=QIC
  POLE,K,1,1,2(EXM)=MODE
  PIR(T),PIR(*),PIC(T),PIC(*)(+)(SQR)=PINM
  QIR(T),QIR(*),QIC(T),QIC(*)(+)(SQR)=QINM
  PINM,QINM(*)=PQNM
  NMAR,PQNM(RTI)=NMAR
  1(DVC)=K2

  I,0(DZM)=CQBM
  0,1(DZM)=CPCL
r:BBAR,1,K2,1,1(EXM)=BMNM
  B,1,K2,N,1(EXM)=BM
  QIR(T),BM(*)=X1
  QIC(T),BM(*)=X2
  X1,X1(*),X2,X2(*)(+)(SQR)=X3
  CQBM,X3,QINM,BMNM(*)(-1)(*)(CTI)=CQBM
  K2,MN1(-)=K2
  K2,M(IFJ)=r,r,s

s:I(DVC)=K2

t:CBAR,1,K2,1,1(EXM)=CLNM
  C,K2,1,1,N(EXM)=CL
  CL,PIR(*)=X1
  CL,PIC(*)=X2
  X1,X1(*),X2,X2(*)(+)(SQR)=X3
  CPCL,X3,PINM,CLNM(*)(-1)(*)(RTI)=CPCL
  K2,MN1(-)=K2
  K2,L(IFJ)=t,t,u

u:DO,CPCL,CQBM,DC(*)(*)(*),PQNM(S*)=RI
  RI(GHR)=RINM
  RINM,1,1,1,1(EXM)=RINM
```

```
      MODE,1(CTC)=MR,MC
      K,RINM(RTI)=DATI
      DAT1,DATI(CTI)=DAT1
      K,CPCL(RTI)=OBSI
      OBSV,OBSI(CTI)=OBSV
      K,CQBM(T)(RTI)=CONI
      CONT,CONI(CTI)=CONT
      K,MN1(-)=K
      K,N(IFJ)=q,q,v

    v:(TXT,L,*********************************)=
      DO,DC,BMAX,CMAX,NMAR(OUT,L)=
      CONS(OUT,L,E)=
      (TXT,L,*********************************)=
      (TXT,L,RESIDUE"NORMS"BY"COLUMN)=
      DAT1(OUT,L,E)=
      (TXT,L,*********************************)=
      (TXT,L,MODAL"MEASURES"BY"COLUMN)=
      OBSV,CONT(OUT,L)=
      (TXT,L,*********************************)=
      PR,PC,LDIM,RDIM,[RES](WBF)=
      L1,L2,L3,L4,L5,[LS](WBF)=
      R1,R2,R3,R4,R5,[RS](WBF)=
```

```
      _PROGRAM_RESIDUE_CONTINUED
      _
      _IN_THIS_PART_BOUNDS_ON_RI_ARE_OBTAINED
      _USING_EO_MEASURES
      _
      (RBF)=PR,PC,LDIM,RDIM,[RES]
      (RBF)=S*,S2,S3,S4,S5,[LS]
      2(DVC)=TWO
      100(DVC)=HUND
      PR(CDI)=N
      1(DVC)=FLAG
      -1(DVC)=MN1
      LDIM(MCP)=SDIM
      (TXT,L,*********************************)=
      (TXT,L,*********************************)=
      (TXT,L,UNOBSERVABILITY"MEASURES)=

    a:SDIM,1(CTC)=SD*,SDIM
      SD*(IFJ)=m,m,b
    b:SD*(INC),0(DZM)=GAPS
      1,0(DZM)=PULL
      1(DVC)=PUL*
    c:PULL,PUL*(CTI)=PULL
      PUL*(INC)=PUL*
      PUL*,SD*(IFJ)=c,c,d
```

```
d:1(DVC)=K
  1,0(DZM)=PORG
p:K,SD*(IFJ)=q,r,r
q:PORG,PULL(CTI)=PORG
  K(INC)=K
  (JMP)=p
r:1(DVC)=K
ē:N,0(DZM)=PTR
  N,0(DZM)=PTC
  PULL(MCP)=PUL
f:PUL,1(CTC)=PUL*,PUL
  PR,1,PUL*,N,1(EXM)=PRTM
  PC,1,PUL*,N,1(EXM)=PCTM
  PTR,PRTM(CTI)=PTR
  PTC,PCTM(CTI)=PTC
  PUL(CDI)(IFJ)=g,g,f
g:P̄TC(GHR)=TES
  TES,1,1,1,1(EXM)=TES
  TES(IFJ)=h,h,i
h:PTR(ORD)=QPTR
  QPTR,PTR(CDI)(CTC)=PTR
  PTR,PTC(CTI)=PT
  (JMP)=z
i:P̄TR(CDI)=M
  N,N(+)=2N
  M,M(+)=2M
  M,M(DIM)=I
  I,I(MCP)=T12,T22
  P̄TR,PTC,MN1(S*),PTC,PTR(M41,SUB)=PAUG
  PAUG(ORD)=QQ,RR
  RR,2M(CTR)=RR
  RR,M,M(M14,SUB)=R11,R21,R12,R22
  R11(-1)=R11I
  R12,R11I(*)=R12I
  I,R12I,R12I(*)(+)(-1)=X1
  R22,R11I,R12I,X1(*)(*)(*)=X22
  R22,R11I,X1(*)(*)=X12
  T̄22,X22(*),T12,X12(*)(+),MN1(S*)=T21
  T21,R12I(*),T22,R22,R11I(*)(*)(+)=T11
  T11,T21,T12,T22(M41,SUB)=T1
  2N,2N(DIM),T1,1,1(RMP)=T1
  QQ,T1(-1)(*)=QQM
  QQM(T),QQM(*)=T2
  T2(-1)=DUM,DET
  .1(DMA),DET(S*)=EPS
  T2,EPS(SQM)=T2
  QQM,T2(-1)(*)=QBAR
```

```
  QBAR,2M(CTC)=QBAR
  QBAR,N,M(M14,SUB)=Q11,Q21,Q12,Q22

  Q11,Q12(CTI)=PT

z:S*(MCP)=V
  PT,2(ITC,SUB)=UT
  UT(CDI),TWO(-1)(*)=KL
  UT,KL(CTC)=UT1,UT2
  UT2,MN1(S*)=UT2
  UT1,UT2(CTI)=UH
  UH,V(MLC,SUB)=UHV
  UHV,UHV(CDI),TWO(-1)(*)(CTC)=UHVR,UHVC
  UHVC,MN1(S*),UHVR(CTI)=UHVC
  UHV,UHVC(RTI)=UHV
  UHV(SVD)=W
  W(DDM)(GHR)=KL
  KL,1,KL(CDI),1,1(EXM)=MINV
  1(DVC),MINV,MINV(*)(-)(ABS)(SQR)=GAP

  PULL(T),GAP(RTI)=GAP
  GAPS,GAP(CTI)=GAPS

  PULL,SD*(DEC)(CTC)=P1,P2
  P2(INC)=P2
  P1,P2(CTI)=PULL
  GAPS(CDI),HUND(IFJ)=x,y,y
y:GAPS(OUT,L)=
  SD*(INC),0(DZM)=GAPS
x:P2,N(IFJ)=e,e,j

j:SD*(DEC)(IFJ)=1,1,k
k:PORG,SD*,SD*(DEC)(*),K,SD*(*)(-)(CTC)=P1,P2
  P2,SD*(CTC)=P3
  P3,SD*,K(INC)(-)(CTC)=P4,P5
  P5(INC)=P5
  P4,P5(CTI)=P3
  P5,K(CTC)=DUM,TES
  TES,N(IFJ)=s,s,w

s:P3(MCP)=PULL
  1(DVC)=KK
t:KK,K(IFJ)=u,v,v
u:PULL,P3(CTI)=P3
  KK(INC)=KK
  (JMP)=t
v:P1,P3(CTI)=PORG
  1(DVC)=K
  (JMP)=e

w:K(INC)=K
  P1(CDI)(IFJ)=1,1,k
```

```
l:(̄TXT,L,**********************************)=
  GAPS(OUT,L)=
  S5,S4,S3,S2(MCP)=S4,S3,S2,S*
  SDIM(CDI)(IFJ)=m,m,a

m:F̄LAG(IFJ)=o,o,n
n:(RBF)=S*,S2,S3,S4,S5,[RS]
  RDIM(MCP)=SDIM
  FLAG(DEC)=FLAG
  (TXT,L,**********************************)=
  (TXT,L,**********************************)=
  (TXT,L,UNCONTROLLABILTY"MEASURES)=
  (JMP)=a
o:END
```

The vita has been removed from
the scanned document