AN ANALYTICAL APPROXIMATION RELATING INTERNAL DAMPING AND STRAIN AMPLITUDE

by

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LIST OF SYMBOLS

Symbol	Name	Unit
m	mass	$\frac{\text{lb.-sec.}^2}{\text{ft.}}$
$\ddot{x}$	acceleration: $d^2x/d\tau^2$	ft./sec.^{2}
$\dot{x}$	velocity: $dx/d\tau$	ft./sec.
τ	time	sec.
c	damping coefficient	$\frac{\text{lb.-sec.}}{\text{ft.}}$
k	spring constant	lb./ft.
x	displacement	ft.
δ	logarithmic decrement	- -
T	period $(2\pi)/\omega$	sec.
ω	circular frequency	1/sec.
W	vibrational energy $\frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2$	lb.-ft.
ΔW	energy loss per cycle of vibration (damping capacity)	lb.-ft.
ψ	specific damping	- -
Ar	amplification ratio	- -
b'	damping coefficient	(lb.-sec.)/ft.
B'	damping coefficient	(lb.-sec.)/ft.
u	unit strain	ft./ft.
$g(u)$	arbitrary strain function	- -
b	dimensionless damping coefficient	- -

Symbol	Name	Unit
B	dimensionless damping coefficient	- -
t	dimensionless time	- -
x_o	characteristic length	ft.
a_o	initial strain amplitude	ft./ft.
k_n	dimensionless constant	- -
L	dimensionless constant	- -
Γ	total phase angle	rad.
f_n	dimensionless constant	- -
α	modified damping constant	$(ft.^2/lb.)^n$
n	exponent of the strain	- -
s	stress amplitude	$lb./ft.^2$
G	modulus of elasticity	$lb./ft.^2$

I

INTRODUCTION

Internal Friction in Solids

Definition: Internal friction, or damping capacity, has been defined as that property of a solid material which results in energy absorption when the material is stressed cyclically.[22*] It is well known that a solid material subjected to free oscillations will vibrate with amplitudes of decreasing magnitude until the motion ceases. This is true in the complete absence of external damping. That property of the material responsible for the condition described above is the internal friction or damping capacity.

Importance: Internal friction has become of increasing importance with the demand for higher speeds in production and transportation equipment. High operating speeds have intensified the dynamic effects on machine members. The operation of machine parts at or near resonant frequencies is difficult to avoid in many cases. Internal friction limits the deflection of such parts and, in general, provides a factor of safety in that it limits the maximum stress to which components of the equipment may be subjected.

In other cases the presence of internal friction may be detrimental. Kimball[13] has shown that internal friction is responsible for the phenomenon of "whirl" in shafts.

A knowledge of the laws governing internal friction is, then, important. Such knowledge would enable the engineer to evaluate the

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*Superscripts refer to the Bibliography.

effect of dynamic loads on machines, buildings and transport equipment to a higher degree of precision. In time such a knowledge should lead to a more efficient utilization of structural resources and greater safety in design.

Equation of Motion: The basic equation of motion of a one degree of freedom system subjected to free damped vibration is:

$$m\ddot{x} + c\dot{x} + kx = 0. \qquad \text{- - - - -(1)}$$

Since this equation is well known, discussion will be limited to the damping term ($c\dot{x}$).

Physically the damping term represents the effect of a force continuously acting in opposition to the motion. Thus, energy is extracted from the system at the expense of the vibrational energy. In any case, the damping force eventually depletes the store of vibrational energy and the motion ceases unless the system is disturbed by outside forces.

For small vibrating masses, low velocities, or high viscosity fluids the damping force is nearly proportional to the velocity. The damping coefficient, c, is then constant. This is the so-called "viscous damping" which has given good results in many cases. For example, the damping due to lubricated surfaces and shock absorbers nearly satisfies the viscous law.

High velocities, large vibrating masses, or low viscosity fluids are represented by a damping force proportional to the square of the velocity. Air resistance often fits this case. The Reynolds number determines the nature of the resistance. A low Reynolds number indicates viscous damping.

If the form of the damping term for internal friction is considered, it is found that no relation is proposed since, unfortunately, the physical processes underlying internal friction are not yet definitely established.

Never the less, equation (1), with the damping term taken as proportional to the velocity, is retained as the defining equation. It is generally recognized that the damping is not correctly represented by this form. However, it is argued that for small deflections and low damping this form is justified as an approximation.[5,10,11] Further, it should be noted that this form of the equation is the basis for certain definitions through which the problem can be attacked experimentally.

Concept of Internal Friction: Among the first to show interest in the subject of internal friction was Lord Kelvin[24] who presented a report on "The Elasticity and Viscosity of Metals" to the Royal Society in 1865. He termed the phenomenon "internal viscosity", an unfortunate appellation that tended to obscure the issue for many years. Sporadic research in this field was conducted from time to time but no great progress was recorded until the second decade of this century. In 1912 and 1914 two papers were published in England[9,23] which ushered in what might be called the modern era.

After this time, interest in this field increased and a large number of publications appeared. Perhaps one of the most productive schools was the Wöhler Institute, Braunschweig, where Professor O. Foppl[4] and his associates produced a large amount of data on the damping capacity of various structural materials. Meanwhile, in the United States such men as Kimball,[11,12,13,14] Lazan,[16] Robertson,[22] Zener,[26,27] and many others were contributing to the sum of knowledge in this field.

As mentioned above, Lord Kelvin[24] considered internal friction as a "viscous" phenomenon. His experiments on wires in torsion did not support this supposition however; it being observed that the damping in certain ranges was too great to properly account for by the "viscous" damping theory. In fact, Kelvin observed that the damping was a function of the amplitude but minimized the importance of this effect.

Quimby,[20] in 1925, investigated the effect of "internal viscosity" on the propagation of sound waves in a continuous medium. He considered that the damping forces were proportional to the first power of the velocity, but distinguished between damping due to dilation and shear. All damping was considered due to the shearing forces, a concept supported by later investigators.[22] Accurate results were claimed for the theory for materials with low damping and for small amplitudes. However, for large displacements the theory did not hold.

Hopkinson and Williams,[9] as well as Rowett,[23] were among the first to recognize the dependence of the damping on stress (or strain) amplitude. The first two authors presented internal friction as a hysteresis effect. The loss of energy was considered to be due to the noncoincidence of the upward and downward portions of the stress-strain curves. The stress-strain diagram for a complete cycle of loading thus formed a loop, the area of which is an indication of the damping capacity of the material since the area is the energy loss per unit volume of material per cycle. This concept forms the basis of the static or hysteresis test.[4,7,9,23]

Föppl[4] considered the strain in a solid material to be composed of an elastic and plastic part. The total potential energy in the system then was considered to be the sum of the elastic energy and the plastic energy. The energy associated with the plastic strain (plastic energy) was considered to be irrecoverable and thus lost in the form of heat. Föppl's concept strengthened the dependency of internal damping on stress amplitude.

Rowett[23] first demonstrated that damping of a solid material was practically independent of frequency. He found that the speed of performance of a complete cycle of stresses did not alter the area under the hysteresis loop. Kimball's[13] experiments, performed in 1927, supported Rowett's conclusions. These conclusions were generally true only for metals.

Zenner[26,27] assumed that the variation of stress in a solid body produced a temperature gradient. Thermal flow, essentially an irreversible process, then leads to an increase in the enthalopy of the system. The energy loss is at the expense of the vibrational energy. The non-uniform stress distribution might be caused by the type of loading, by the inhomogeneity of the material, or by the residual stresses. Only the first case permits a rigorous mathematical treatment. Zenner's theory indicates a marked dependence of the damping on the frequency.

The concept of the basic nature of internal friction thus changed from the "viscous" phenomenon of Lord Kelvin to the stress dependent function accepted today. It is to be noted that the various methods of expressing this idea, viz: the hysteresis effect of Hopkinson and Williams, the plastic strain effect due to Föppl, are basically the same. The work of many investigators, not mentioned here, support this concept.

Other important variables, of which temperature is perhaps the most important, also influence the damping.

<u>Methods of Expression</u>: Internal friction has been expressed and measured in the form of various quantities. These expressions are based on the equation:

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad \text{- - - - -(1)}$$

Generally the quantity used depends upon the particular experimental technique chosen. Only those expressions which have the most general application will be reviewed here.

The logarithmic decrement, (δ), is defined as the logarithm of the ratio of the amplitude of vibration at time (τ) to the amplitude at time ($T + \tau$).

$$\delta = \ln \frac{x(\tau)}{x(T+\tau)} \qquad \text{- - - - -(2)}$$

This expression is generally applied to the case of free vibrations since it can be completely determined from an experimental time-displacement curve. If the decrease in amplitude per cycle (ΔX) is small, δ can be expressed approximately as:

$$\delta = \frac{\Delta X}{X} \qquad \text{- - - - -(2 a)}$$

The specific damping, (Ψ), is defined as the ratio of the energy loss per cycle of vibration, (ΔW), to the total vibrational energy of the system (w)

$$\Psi = \frac{\Delta W}{W} \qquad \text{- - - - -(3)}$$

The term ΔW is defined as the damping capacity and is often used alone since it is the quantity which must be determined experimentally to evaluate (3). It is generally used when the experiment involves forced vibrations in which case ΔW is the work per cycle required to maintain the vibration at constant amplitude.

The logarithmic decrement may be related to the specific damping by the equation:

$$\Psi = 1 - e^{-2\delta}, \qquad \text{- - - - -(4)}$$

For δ small, equation (4) may be expressed as:

$$\Psi = 2\delta \ , \qquad \delta < 0.01 \qquad \text{- - - - -(4 a)}$$

The error in equation (4a) is less than one per cent for δ less than 0.01, as occurs in many metals[16] such as high carbon steel. For higher values of δ the expression

$$\psi = 2(\delta - \delta^2), \qquad \delta < 0.12 \qquad \text{-----(4 b)}$$

gives results of sufficient accuracy. The error in equation (4b) is less than one per cent for δ less than 0.12 as occurs in most structural materials.[16]

Lazan[16] defines the amplification ratio (Ar) as:

$$Ar = \frac{2\pi}{\psi} \qquad \text{-----(5)}$$

This relation is used by Lazan to present experimental data. Applying equations (4), (4a) and (4b) to (5), we obtain:

$$Ar = \frac{2\pi}{1 - e^{-2\delta}}, \qquad \delta > 0 \qquad \text{-----(5 a)}$$

$$Ar = \frac{\pi}{\delta}, \qquad \delta < 0.01 \qquad \text{-----(5 b)}$$

$$Ar = \frac{\pi}{\delta - \delta^2}, \qquad \delta < 0.12 \qquad \text{-----(5 c)}$$

<u>General Laws Governing Internal Friction</u>: Out of the mass of experimental data available, several empirical laws emerge. Gemant[5] has summarized the results, in terms of the logarithmic decrement, as follows:

"1. In general, the variation of the logarithmic decrement with frequency is small.

2. The decrement usually increases with increasing temperature, slowly at first, and then rapidly.

3. The decrement, usually constant at small amplitudes, increases considerably at higher amplitudes.

4. The higher the elastic modulus of a material, the lower its decrement."

In addition, the damping is known to be a function of the grain size,[15] previous stress history,[5,14,16,21] type of stress[20,22] and, for the case of free vibrations, the initial stress.[2] Many deviations from the general laws outlined above may be cited (see Gemant).[5]

Of the above, the most important is the variation of the damping with the stress (or strain) amplitude. Lazan[16] has shown that for some metals, for example Duralumin, the effect of temperature changes up to as much as one hundred degrees above room temperature is small. The heat generated by the internal friction would have little effect on the damping in a case such as this.

The actual stress level in the material also depends on the dynamic modulus of elasticity. Unfortunately, the dynamic modulus changes with the stress level as well as with temperature. Still, for many metals of low damping capacity, the effect of stress and temperature on the dynamic modulus is only a few per cent, the dynamic modulus differing only slightly from the static modulus.[16]

There are, then, cases in which the damping is primarily a function of the stress amplitude. The discussion may be limited to the effect of stress amplitude on internal damping.

Effect of Stress Amplitude: The following discussion is concerned with materials which have a relatively small damping capacity, of which metals are the most important. Research in the field of plastics, plywood, glass, and other materials has been conducted but will not be discussed here.

There is wide disagreement among the various investigators concerning the effect of the stress amplitude. Hopkinson and Williams[9] found that the damping capacity (ΔW) was approximately proportional to the fourth power of the strain amplitude. Rowett[23] suggested that the relation was cubic. Other investigators[3,22] have also noted a third power variation.

Kimball and Lovell[13] found the damping capacity to vary as the square of the stress amplitude, from which the logarithmic decrement was found to be constant (i.e., the rate of decay was exponential). Okleston,[19] in attempting to verify the results obtained by Kimball and Lovell, found the decrement to vary with the stress amplitude. It is to be noted that both investigated low stress ranges and used different experiments.

If the cubic relation previously noted for the damping capacity is expressed in terms of the specific damping, a linear variation of specific damping with stress is indicated. This assumes, among other things, that the dynamic modulus of elasticity is constant. The results of investigators who have presented data in the form of graphs of δ or Ψ versus the strain (or stress) do not in general show a linear relation.[2,4,7,8,16]

Experimental results do, however, definitely indicate that the damping is a function of the stress amplitude. Although experimental data is not in agreement on the power of this variation, insofar as the damping capacity is concerned, values observed have ranged from two through four. The value of three seems to be the one most often found for this relation.

Improvement of the Basic Equation: The object of the many experimental investigations conducted in this field is, of course, to define the exact relations controlling the internal damping. Experiment, however, seems capable of yielding only the trend of these relations. The ideal solution will be achieved only when the mathematical relations governing the motion of systems influenced by internal damping have been written and solved.

Since the physical processes underlying internal friction have not been developed, approximate methods are necessary. Thus, equation (1) is retained as an approximation.

A step in the proper direction, then, would consist of improving the degree of approximation represented by equation (1).

Object: The object of this paper is to improve the degree of approximation represented by the damping term in equation (1). An improvement in the damping term is suggested, the solution of the improved equation presented and the results are compared with the experimental results obtained by previous investigators.

II

REVIEW OF THE LITERATURE

Listed below is a brief summary of the more important books and articles read in connection with this paper:

Contractor, G. P. and Thompson, F. C.

"The Damping Capacity of Steel and Its Measurement"

Journal of The Iron and Steel Institute 138, 157-218 (1940)

The Foppl-Pertz apparatus for the measurement of "damping capacity" was found to yield unduly high values, mainly as the result of frictional losses between the styles and the recording medium. These were corrected by employing an optical recording device. Particular attention has been paid to the effect on damping capacity of factors such as the initial stress, bending, size and shape of the speciman, etc. A detailed investigation of temperature effects is given.

It was found that the damping capacity was a material property independent of the shape of the speciman. This supported Föppl who reached the same conclusion.

Data are presented in terms of the variation of the specific damping with stress amplitude and temperature.

Föppl, O.

"The Practical Importance of The Damping Capacity of Metals, Especially Steels"

Journal of The Iron and Steel Institute 2, 134, 393-455 (1936)

It is concluded that damping does not predict fatigue failure but the damping capacity of most materials tends to increase under repeated reversal of stress. The damping capacity is a material property independent of the size or shape of the speciman.

As a possible application, the use of material with high damping characteristics is suggested for machine parts operated at or near critical frequencies. The maximum stresses might thus be limited so that a safer design is achieved even when a sacrifice of strength is required to obtain the desired damping characteristics.

Results of experiments conducted at the Wöehler Institute, Braunschweig, Germany, are presented in terms of the logarithmic decrement.

Gemant, Andrew

Frictional Phenomena

New York, Chemical Publishing Company, Inc. (1950)

The general nature of friction, both internal and external, for fluids, gases, plastics, and solids is discussed. The equations of viscous flow are developed for gas and liquids, and a physical as well as a mathematical discussion is included.

In the case of plastic and solid friction, the effects of internal friction are investigated through the concept of the logarithmic decrement. Empirical and analytical formulae are developed for the internal damping effect and the results of theoretical calculations compared with experiment by use of charts and graphs. It is found in many cases that the theory inadequately explains the observed behavior of materials. A discussion of experimental techniques is included.

For plastic type deformations the author states that the logarithmic decrement might be represented as a constant plus the hyperbolic cosine of the stress (or strain) amplitude.

Von Hedekampf, G. S.

"Damping Capacity of Materials"

Proc. A. S. T. M., 31, II, 157 (1931)

The damping capacity of materials is defined and shown to be identical with the "mechanical hysteresis effect or internal friction of solids." Different methods of determining the damping capacity are described. Although it is generally true that higher tensile or hardness values are connected with a low damping capacity, many experiments have shown that the damping capacity may be quite different for steels of identical chemical and tensile properties. It is noted that cast iron has a surprisingly high damping capacity.

Results of tests conducted at the Wöhler Institute, Braunschweig, are presented in the form of graphs of the specific damping capacity plotted against the maximum torsional fiber stress.

Lazan, B. J.

"Some Mechanical Properties of Plastics and Metals Under Sustained Vibrations"

Trans. of the A. S. M. E. 65, 87-104 (1943)

The development of the hypocyclic oscillator, the technique of its use and the results obtained are presented. Experimental data on the damping capacity and dynamic moluli of elasticity are presented for mild steel, Duralumin, and several plastics. The dynamic modulus is shown to be related to the damping capacity of materials.

The dynamic modulus of metals was found to decrease by only a few per cent as the alternating stress increased. The dynamic moduli of the metals tested were within a few per cent of the static moduli. For plastics the dynamic modulus was found to decrease by as much as forty per cent.

The data were presented in terms of the amplification ratio plotted as a function of the stress amplitude.

Minorsky, N.

Introduction to Non-Linear Mechanics

Ann Arbor, Michigan, J. W. Edwards (1947)

Minorsky surveys the field of non-linear mechanics summarizing in a systematic order the results of such men as Poincare', Van der Pol, Kryloff and Bogoliuboff, and others.

The fundamental concept of the phase plane and interpretation of results is discussed. It is shown that the introduction of the displacement velocity plane (phase plane) permits a graphical procedure which yields a qualitative description of the motion. Criteria are developed by which the linear solutions, applicable to small displacements, determine the initial stability or instability of the system under investigation.

In the second part of the book analytical methods are developed for the solution of non-linear equations of certain types. These are generally restricted to the so-called quasi-linear systems, or systems which possess a small degree of non-linearity. At the other extreme, solutions are developed for equations with large non-linear elements. Here the so-called "relaxation oscillation" is introduced. In the range for which the equations are neither quasi-linear nor highly non-linear, no satisfactory analytical methods exist and recourse to the phase plane with graphical methods is unavoidable.

The principal analytical methods discussed are the perturbation theory, method of Van der Pol, equivalent linearization, Theory of the First Approximation of Kryloff and Bogoliuboff, and relaxation oscillations. Numerous examples from the fields of electronics and mechanics are used, the former more frequently, since the nature of the non-linearity of many mechanics problems is as yet indefinite.

Robertson, J. M. and Yorgiadis, A. J.

"Internal Friction in Engineering Materials"

Transactions of the American Institute of Mechanical Engineers

68 (1946)

Experimental data on internal friction was collected for a number of materials including plastics, plywood, and metals. The method of investigation utilized forced vibrations. The equipment was developed by Lazan and a discription is available in reference 16. Tests were conducted with the speciman subjected to pure torsion and also in pure tension and compression.

The authors concluded that the damping capacity of a solid material was in general proportional to the cube of the stress amplitude, independent of frequency, and due only to the distortion of the material.

Data were presented in the form of graphs of the damping capacity (ΔW) plotted against the maximum stress amplitude.

Thompson, W. (Lord Kelvin)

"On the Elasticity and Viscosity of Metals"

Proceedings of the Royal Society 14, 289-297 (1865)

The result of investigations conducted at the University of Glasgow on "Viscosity" and "Moduli of Elasticity" were presented. Experiments were conducted using flat disks attached to round wires oscillating in torsion.

The loss of energy in a given range was found to increase with velocity, but the difference between the losses at high and low speeds was much less than that predicted by the "viscous damping" law. The "viscosity" was found to increase with increased loading when measured immediately after loading, but then decreased with time. The damping over a given range was found to vary with the initial amplitude.

III

ABSTRACT

An improvement in the degree of approximation represented by the basic equation of motion of a vibratory system influenced by internal damping is advanced and an approximate solution given. The improvement lies in assuming that the damping coefficient in the basic equation is proportional to the n^{th} power of the absolute value of the strain amplitude.

The solution is expressed in terms of the logarithmic decrement and specific damping capacity to facilitate comparison with experimental data. The restrictions imposed by the approximate solution limit comparison to materials which posses a small degree of damping.

The constants in the solution are determined for three cases and a fair fit to experimental curves is obtained.

IV

THE MODIFIED EQUATION OF MOTION OF A VIBRATING SYSTEM CONSIDERING INTERNAL DAMPING AND ITS SOLUTION

The Dissipative Forces: Since the damping in a solid material, as represented by the energy loss per cycle of vibration, is a function of the stress (or strain) amplitude, it would seem logical to conclude that the dissipative force responsible for the energy loss is also a function of the strain amplitude.

In the equation

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad \text{-----(1)}$$

the second term ($c\dot{x}$) then becomes $g(u)\,\dot{x}$, where u is the strain and is related to the displacement. The velocity term is retained since in this manner a non-conservative force may be introduced. It is now proposed that g(u) be taken as

$$g(u) = b' + B'|u|^n \qquad \text{-----(6)}$$

where the damping coefficient, g(u), is considered to vary as the n^{th} power of the strain amplitude. The absolute value of the strain amplitude is used so that the sign of the damping term is automatically given by the sign of the velocity.

Equation of Motion: Substituting equation (6) in equation (1) the equation of motion becomes

$$m\ddot{x} + (b' + B'|u|^n)\,\dot{x} + kx = 0 \qquad \text{-----(7)}$$

in which the assumption that the spring force is directly proportional to the displacement is retained.

Solution: If the assumption is now made that the second term in equation (7) is small compared to the first and third terms, equation (7) becomes quasi-linear and the theory for the solution of such equations may be used. It is to be noted that this assumption is the one usually made since the damping due to internal friction is usually small.[5,10,11,13]

The theory to be used in obtaining a solution of equation (7) is the "Theory of the First Approximation of Kryloff and Bogoliuboff." A development of the method is given by Minorsky[17] and also by Andronow and Chaikin.[1]

Since equation (7) is non-linear, no method of obtaining the exact solution exists. However, the assumption of small damping permits the use of the approximate method mentioned above.

The solution is obtained in the form

$$x = a \sin \Gamma \qquad \text{-----(8)}$$

where a and Γ are defined as

$$\frac{da}{dt} = -\frac{1}{m}\,\frac{1}{2\pi\omega}\int_0^{2\pi} f(a\sin\phi,\, a\omega\cos\phi)\cos\phi\, d\phi$$

$$\text{-----(8 a)}$$

$$\frac{d\Gamma}{dt} = \omega + \frac{1}{m}\,\frac{1}{2\pi a\omega}\int_0^{2\pi} f(a\sin\phi,\, a\omega\cos\phi)\, d\phi$$

in which

$$f(a \sin\phi, a\omega\cos\phi) = f(x, \mathring{x}) \qquad \text{-----(8 b)}$$

If we now express equation (7) in the dimensionless form

$$\mathring{\mathring{u}} + (b + B|u|^n)\mathring{u} + u = 0 \qquad \text{-----(7 a)}$$

where $\quad u = x/x_o, \qquad t = \omega\tau,$

$$\omega = \sqrt{K/m}, \qquad B = B'/m\omega,$$

$$\mathring{u} = du/dt, \qquad b = b'/m\omega,$$

and $\quad \mathring{\mathring{u}} = d^2u/dt^2$

equation (8b) becomes

$$(b + B|a\sin\phi|^n)\, a\cos\phi = (b + B|u|^n)\mathring{u}$$

and equation (8a)

$$\frac{da}{dt} = -\frac{a}{2\pi}\int_0^{2\pi}(b + Ba^n|\sin\phi|^n)\cos^2\phi\, d\phi$$

$$\frac{d\Gamma}{dt} = 1 + \frac{1}{2\pi}\int_0^{2\pi}(b + B\,a^n|\sin\phi|^n)\cos\phi\sin\phi\, d\phi \qquad \text{-----(8 c)}$$

in which $\omega = 1$.

From (8c) the solutions for a and Γ are

$$a = \frac{a_o e^{-bt/2}}{[1 + K_n a_o^n(1 - e^{-(bnt)/2})]^{\frac{1}{n}}} \qquad \text{-----(8 d)}$$

and $\quad \Gamma = t + \Gamma_o$

where a_0 is the initial strain in this case and Γ_0 the initial phase angle. K_n is defined as

$$K_n = \frac{BL}{b\pi} \qquad \text{-----(8 c)}$$

where

$$L = \int_0^{2\pi} |\sin\phi|^n \cos^2\phi \, d\phi$$

For even values of n

$$L = -2\pi \prod_{m=0}^{n/2} \left(\frac{n-1-2m}{n+2-2m}\right), \quad n \geq 0$$

For odd values of n

$$L = -\frac{4}{3} \prod_{m=0}^{(n-3)/2} \left(\frac{n-1-2m}{n+2-2m}\right), \quad n > 3$$

For n = 1, L = 4/3.

As was to be expected, no frequency correction was found since the spring force was linear.

Substituting equation (8d) in equation (8), the solution becomes:

$$u = \frac{a_0 e^{-\frac{b}{2}t}}{\left[1 + K_n a_0^n (1 - e^{-\frac{bn}{2}t})\right]^{\frac{1}{n}}} \sin(t + \Gamma_0) \qquad \text{-----(9)}$$

For n = 0, equation (9) reduces to the solution of the linear case, equation (1).

If b = 0, we have the case where $g(u) = B' \, |u|^n$. Equation (9) then becomes

$$u = \frac{a_o}{[1 + f_n a_o^n t]^{\frac{1}{n}}} \sin(t + \Gamma_o) \qquad \text{- - - - -(10)}$$

where

$$f_n = \frac{nBL}{2\pi}$$

The restrictions placed on the strain in this solution are:

1. Hooke's Law must be satisfied.

2. The strain may be expressed as $u = x/x_o$.

Logarithmic Decrement: Using the definition, equation (2), of the logarithmic decrement we obtain from equation (9):

$$\delta = \frac{b}{2} + \frac{1}{n} \ln\left[\frac{1 + K_n a_o^n(1 - e^{-\frac{bn}{2}(t + 2\pi)})}{1 + K_n a_o^n(1 - e^{-\frac{bn}{2}t})}\right] \qquad \text{- - - - -(11)}$$

Since, for most practical cases, the points of tangency of the time-displacement curve, equation (9), and the envelope of this curve differ from the actual maxima of equation (9) by a negligible amount,[18] the envelope of equation (9) may be used to solve for the time in terms of the maximum strain amplitude. The points of tangency occur for $\sin(t + \Gamma_o) = 1$. The envelope of equation (9) is:

$$u = \frac{a_o e^{-\frac{b}{2}t}}{[1 + K_n a_o^n(1 - e^{-\frac{bn}{2}t})]^{\frac{1}{n}}} \qquad \text{- - - - -(9 a)}$$

Solving equation (9a) we find:

$$-\frac{bn}{2}t = \ln\left\{\left(\frac{u_{max}}{a_0}\right)\left[\frac{1+K_n a_0^n}{1+K_n u^n}\right]\right\} \qquad \text{-----(9 b)}$$

Substituting equation (9b) in equation (11),

$$\delta = \frac{b}{2} + \frac{1}{n}\ln\left[1 + K_n u^n\left(1 - e^{-\pi b n}\right)\right] \qquad \text{-----(12)}$$

where it is understood that $u = (u)_{max}$.

Specific Damping: Using equation (12) and the definition of specific damping one obtains:

$$\psi = 1 - e^{-b}\left[1 + K_n u^n\left(1 - e^{-\pi b n}\right)\right]^{-\frac{2}{n}} \qquad \text{-----(13)}$$

If δ is assumed to be small, ψ can be expressed as:

$$\psi = b + \frac{2}{n}\ln\left[1 + K_n u^n\left(1 - e^{-\pi b n}\right)\right] \qquad \text{-----(13 a)}$$

From equation (12) the expression for δ may be obtained for the case where $g(u) = B'\,|u|^{n}$ by allowing b to approach zero. This yields

$$\delta = \frac{1}{n}\ln\left[1 + 2\pi f_n u^n\right] \qquad \text{-----(14)}$$

from which ψ may be expressed, using relation (4) as:

$$\psi = 1 - \left[1 - 2\pi f_n u^n\right]^{-\frac{2}{n}} \qquad \text{-----(14 a)}$$

V

DISCUSSION OF RESULTS

Theoretical Results: Equation (9) is the time-displacement relation of a vibrating body influenced by an internal damping constant proportional to the n^{th} power of the strain amplitude. The motion is sinusoidal (see Fig. 1) as expected, since the solution was based on the assumption of small damping.

The solution to equation (9) is a modification of the solution of the linear equation (1). The effect of strain is represented by the term

$$\left[1 + K n' a_o^n \left(1 - e^{-\frac{h\Omega}{2} t} \right) \right]^{\frac{1}{n}}$$

in the denominator. This term increases with time and the motion thus decays at a faster rate than predicted by the linear theory.

More interesting, and most surprising, is the expression defining

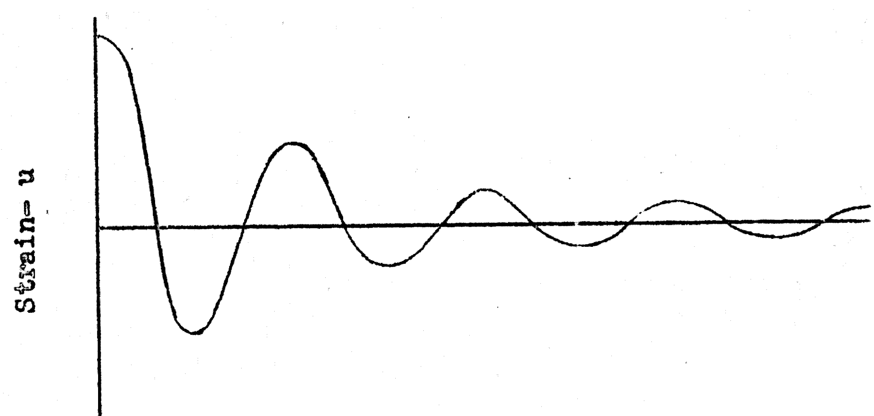

Fig. 1 - General form of the strain-time curve

the logarithmic decrement, equation (12). For small values of the strain, the curve appears to be a power function (see Fig. 2). In fact, if the ln $(1 + Q' u^n)$ is expanded in a power series and terms of the second order or higher are neglected, the decrement then becomes

$$\delta = \frac{b}{2} + \frac{1}{n} Q' u^n \qquad \text{- - - - -(12 a)}$$

where $$Q' = 1 + Kn(1 - e^{-\pi b n})$$

For large values of the strain, the quantity $Q'u^n$ is large compared to one and the decrement varies as the logarithm of the strain.

In Fig. 3 is shown a plot of the specific damping versus strain amplitude. The specific damping is seen to increase slowly at first and then more rapidly as the strain increases. For large values of the strain the specific damping approaches the value $\psi = 1$ asymptotically.

For the case of no initial damping ($b = 0$), the expressions for the decrement and the specific damping vary in the same manner as the functions discussed above, except that the curves are shifted to the origin. Since experimental evidence[2] indicates that damping exists even for zero strain (or stress), it is expected that equations (12) and (13) will show closer agreement with experiment than equations (14) and (14a).

<u>Comparison With Experiment</u>: Since a large portion of experimental data are presented in terms of the logarithmic decrement or the specific damping, it was necessary to express equation (9) in those forms if the theoretical results are to be compared with experiment.

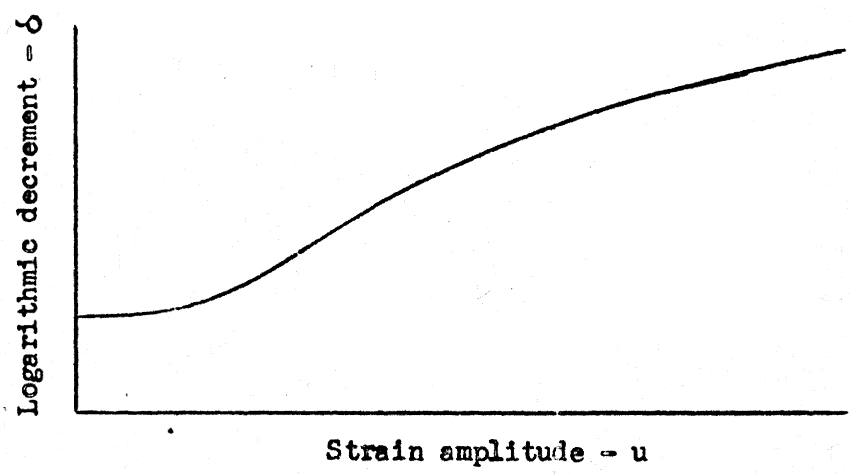

Fig. 2 - General form of the decrement-strain curve

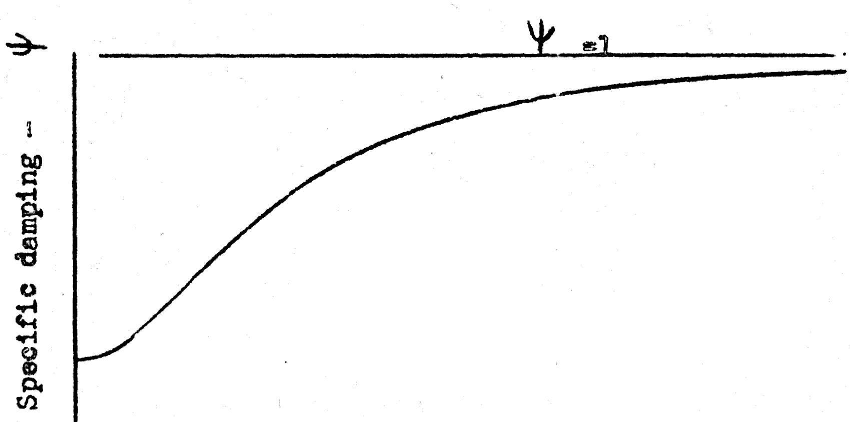

Fig. 3 - General form of the specific damping-strain curve

Comparison of theoretical results with experimental results was accomplished by two steps. First, the literature was scanned to see if any curves obtained by the various experimenters were similar to the plots of equations (12) and (13). An attempt was then made to duplicate the curves that appeared most promising by solving for the constants n, b, and Kn in equation (12).

In Figs. 4, 5, 6 and 7 are reproduced experimental curves found in two reports. In Figs. 4 and 5 the curves demonstrate the variation of the decrement with stress amplitude in the manner predicted by equation (12), (see Fig. 2). The decrement increases slowly at first with increasing amplitude, an inflection in the curve is present and the decrement then increases at a decreasing rate.

Figures 6 and 7 are not as strikingly similar to the curve of Fig. 2 as the first mentioned, but a close examination reveals several similarities. The lower portions show an increasing slope with increasing stress. The slope then becomes practically constant, and in two cases a point of inflection is shown. However, the curves do not show a definite decrease in slope at higher stresses such as seen in Figs. 4 and 5.

In Fig. 8 is shown a plot of the specific damping versus stress for a 0.20% carbon steel tested at various temperatures. It is noted that the point of inflection, as shown in Fig. 3, is not present. The decrease in slope for higher stress levels predicted by equation (13) is apparent, though it occurs at a much lower value of ψ than predicted by the theory.

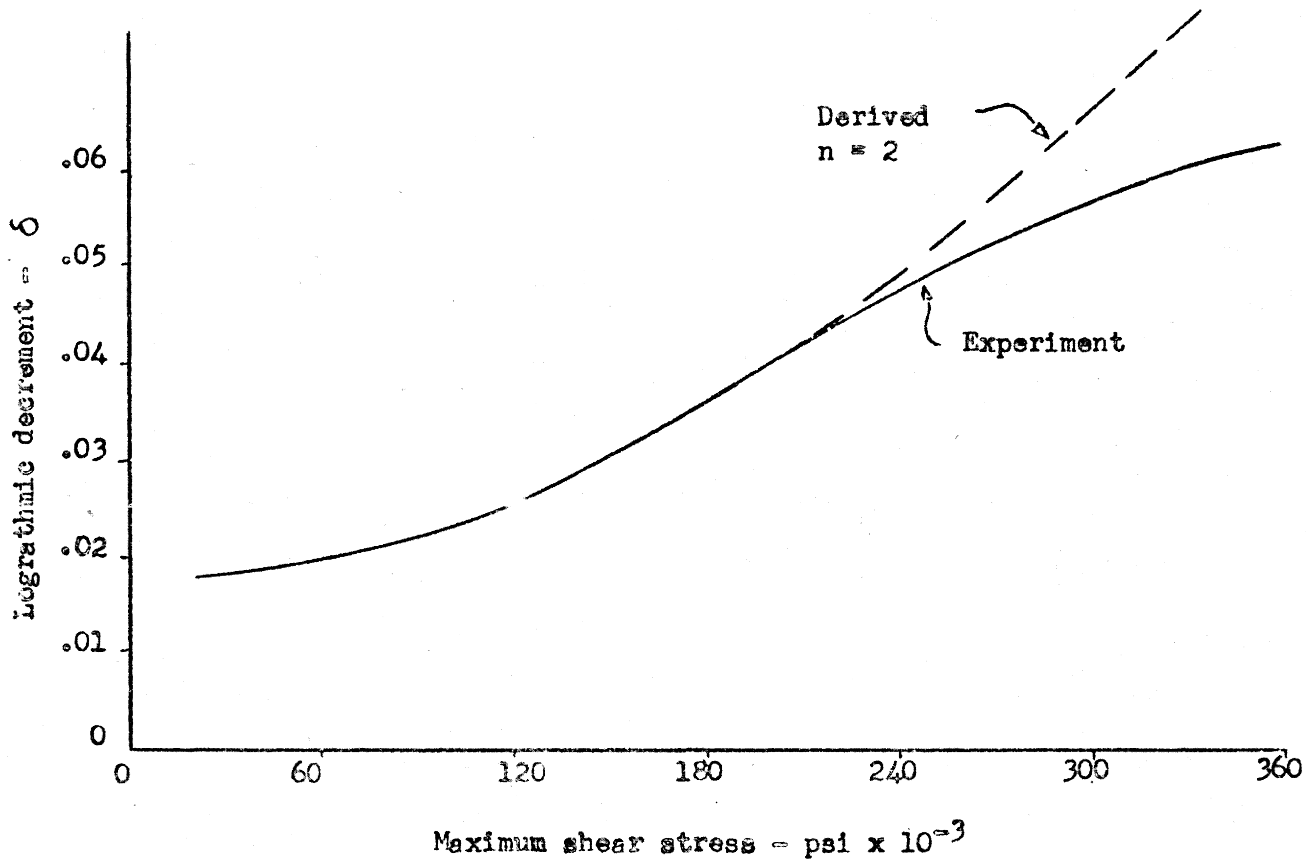

Fig. 4 - Comparison of derived and experimental curves. 17 ST Duraluminum - Reference 16 - Fig. 6.

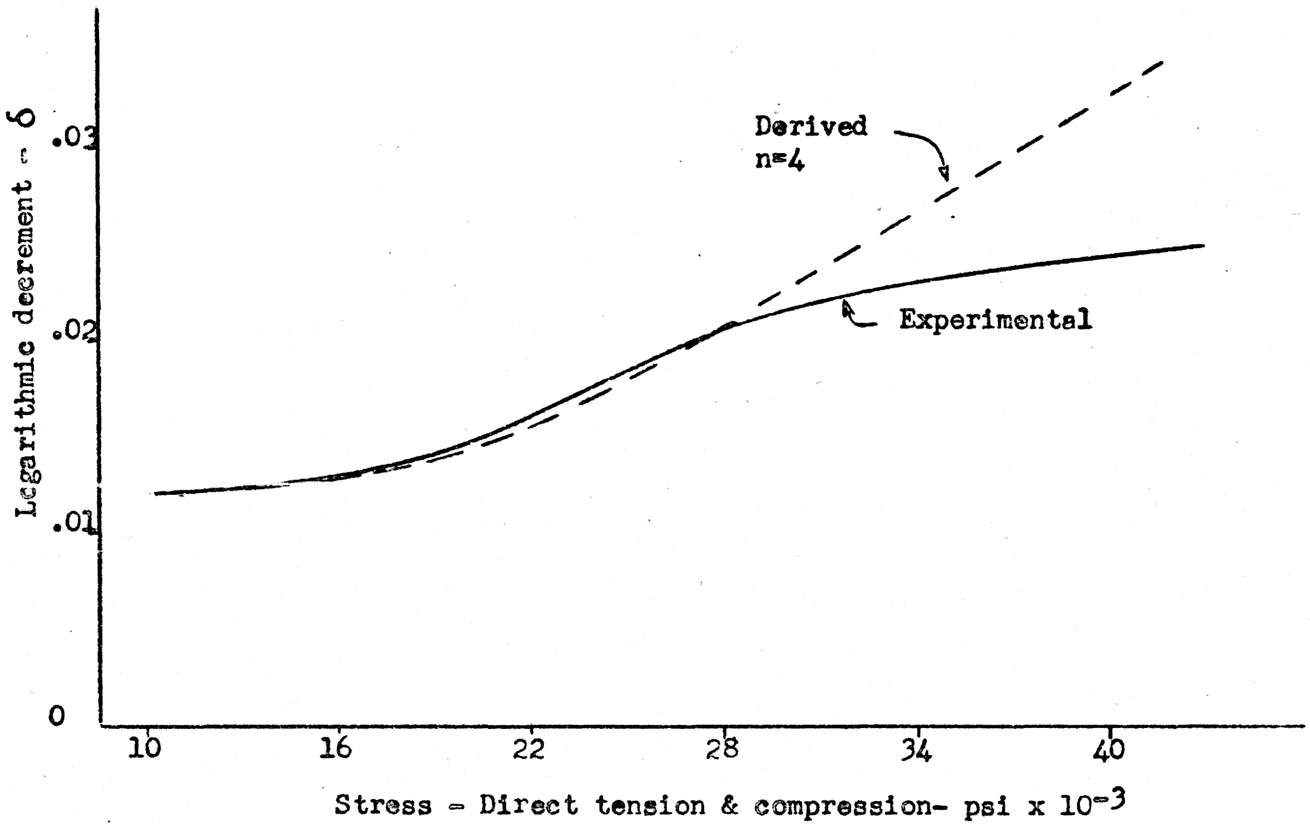

Fig. 5 - Comparison of experimental and derived curve. 17-ST Duraluminum-Reference 16, Fig. 7.

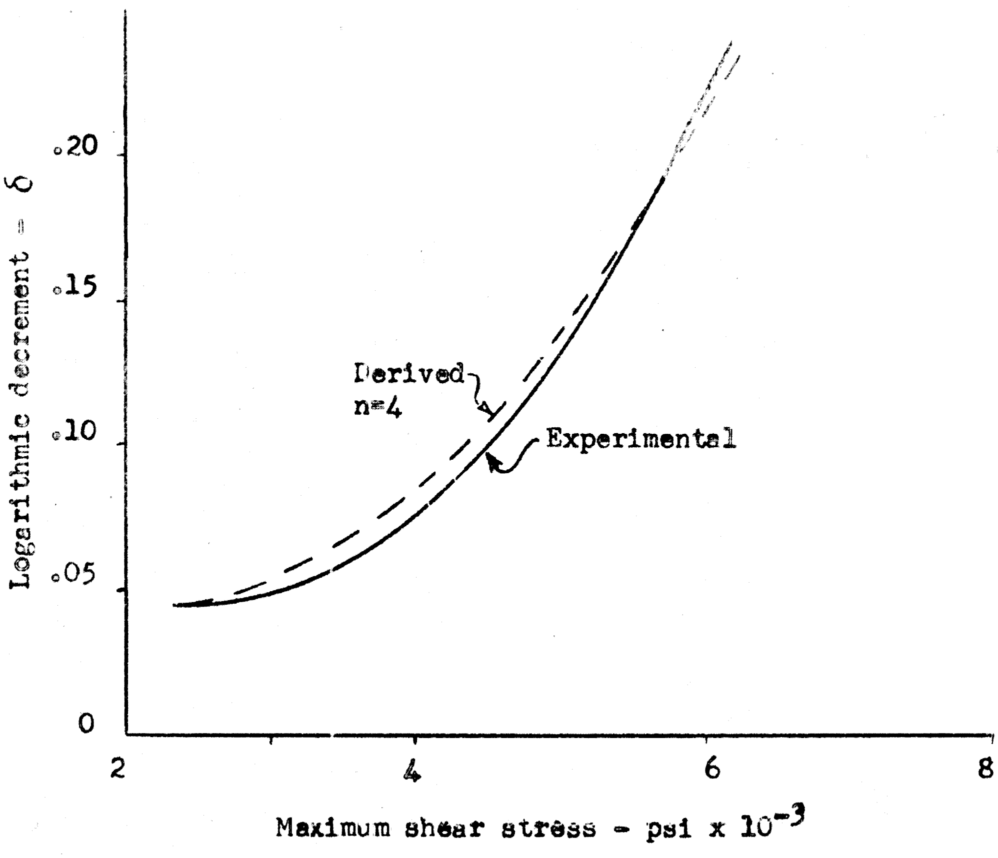

Fig. 6 - Comparison of experimental and derived curve.
Magnesium Alloy - Reference 15, Fig. 7.

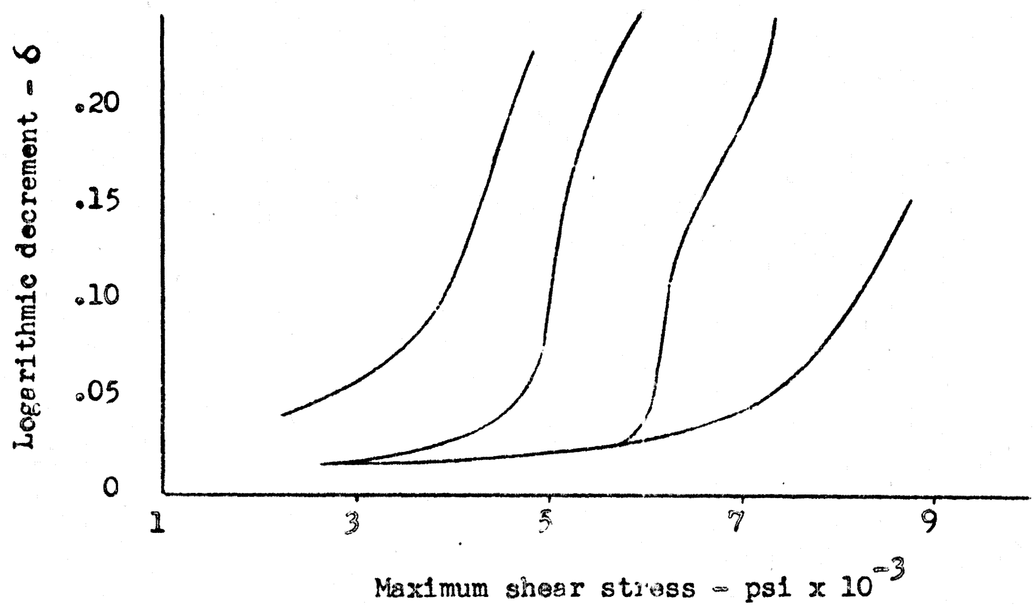

Fig. 7 - Damping curves for Magnesium alloy. Reference 15 - Fig. 6.

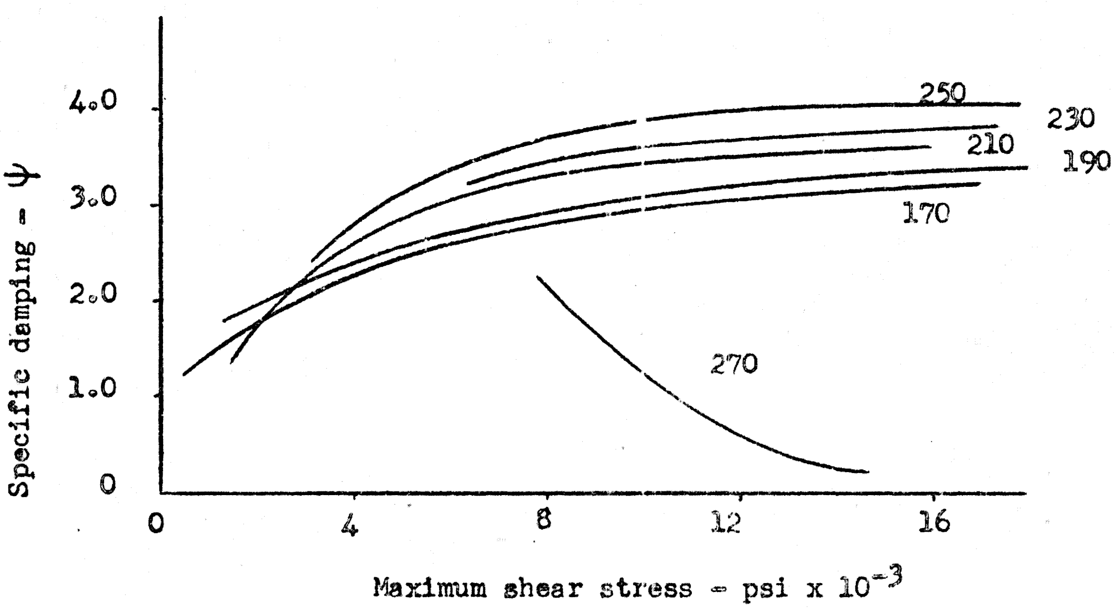

Fig. 8 - Damping curves for 0.2% carbon steel, hot rolled - Reference 2 - Fig. 19. Numbers refer to temperature at time of test in degrees centigrade.

A companion set of curves in the same reference[2] for temperatures from 20 to 110 degrees centigrade shows the same general variation of ψ with stress. The curve for 270 degrees has been included only to indicate the capricious nature of the damping when the temperature is included as a parameter.

Although many experimental curves may be found which do not have the general appearance of the curves developed from equations (12) and (13), a number do show the tendency toward constant slope over a portion of the curve followed by a slight decrease in slope. In some the decrease does not occur, while in others the point of inflection occurs and is followed by another increase in slope. For example, see Fig. 5 and 6, reference 7.

In attempting to fit the experimental curves by use of equation (12), it was found convenient to set the quantity

$$Kn\left(1-e^{-\pi b n}\right) = Q'$$

Since the data are generally presented in terms of the stress amplitude, it is convenient to express the strain in terms of the stress. If we assume Hooke's Law to be valid, equation (12) becomes

$$\delta = \frac{b}{2} + \frac{1}{n}\ln\left[1 + Q'\frac{S^n}{G^n}\right] \qquad \text{-----(12 a)}$$

where G is the modulus of elasticity.

The constant b is readily evaluated by setting the stress equal to zero if experimental data is available for very low stresses. This is usually not the case, however, since the damping is very difficult to measure for low stresses. For this reason it is necessary to solve for Q' and then evaluate b .

By choosing corresponding values of the decrement and stress from the experimental curves, such as S_1, δ_1 and S_2, δ_2, b may be eliminated between the two equations. If the resulting equation is then solved for Q' the formula

$$Q' = \left(\frac{e^{n(\delta_1 - \delta_2)} - 1}{S_2^n \, e^{n(\delta_1 - \delta_2)} - S_1^n} \right) G^n$$

$$Q' = Q \, G^n \qquad \text{- - - - -(12 b)}$$

is obtained. A value of n must be assumed and tested in each case, to determine the value which gives the best results.

Substituting equation (12b) in (12a):

$$\delta = \frac{b}{2} + \frac{1}{n} \ln\left[1 + Q S^n\right] \qquad \text{- - - - -(12 c)}$$

The equation is now expressed in terms of the stress amplitude. Since Q has units of inches squared per pound, all to the n^{th} power, equation (12c) is still in dimensionless form.

In choosing experimental examples for comparison, the assumptions involved in the problem must be considered. Since the basic equation implies a lumped mass system with one degree of freedom, care must be exercised to pick only those examples which approximate this condition. The examples discussed in this section were chosen with the above in mind.

A comparison of the derived equations with experimental curves is shown in Figs. 4, 5, and 6. It was found that the value n = 2 best satisfied the curve in Fig. 2. The derived curve fits the experimental curve very closely up to a stress level of approximately 210 p. s. i. Above that value the damping predicted by the derived curve is higher than shown by experiment. The point of inflection in the derived curve occurs at a much higher stress than in the experimental curve.

For the curve shown in Fig. 5, the value of n which best satisfied the curve was found to be four. It should be noted that the two figures are for different types of stresses.

Fig. 4 was obtained from a torsion test, while Fig. 5 was obtained from a direct stress test. The derived curve follows the same pattern as in Fig. 4.

In Fig. 6 the experimental curve does not show a point of inflection. Here the value n = 4 again gave the best fit. Although the derived curve varies slightly from the experimental curve, the difference is not large over the range of stress shown.

VI

CONCLUSIONS

The assumption that the damping coefficient is proportional to the n^{th} power of the strain amplitude is an improvement in the approximation represented by the linear equation of motion.

In the lower range of the strain (or stress) amplitude the damping (as interpreted through the logarithmic decrement) is accurately represented by the assumption stated above.

For higher values of the strain (or stress) the damping is not represented so accurately by the assumption. Therefore, no conclusion regarding the over-all correctness of the assumption can be made at this time.

VII

SUMMARY

The degree of the approximation represented by the basic equation of motion of a vibratory system influenced by internal damping was improved by the assumption that the damping coefficient is proportional to the n^{th} power of the strain (or stress) amplitude.

The solution of the improved equation was presented by use of approximate methods. The results were expressed in terms of the logarithmic decrement and specific damping to facilitate comparison with experimental data.

Three theoretical curves were fitted to experimental curves of the logarithmic decrement. It was found that the theoretical curves accurately described the damping (that is, they fit the experimental curves) for low values of the strain. The curves were found to diverge for higher values of the strain.

VIII

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