



## On the diagonal subalgebra of an Ext algebra

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## ABSTRACT

Let  $R$  be a Koszul algebra over a field  $\mathbb{k}$  and  $M$  be a linear  $R$ -module. We study a graded subalgebra  $\Delta_M$  of the Ext-algebra  $\text{Ext}_R^*(M, M)$  called the diagonal subalgebra and its properties. Applications to the Hochschild cohomology ring of  $R$  and to periodicity of linear modules are given. Viewing  $R$  as a linear module over its enveloping algebra, we also show that  $\Delta_R$  is isomorphic to the graded center of the Koszul dual of  $R$ .

When  $R$  is selfinjective and not necessarily graded, we study connections between periodic modules  $M$ , complexity of  $M$  and existence of non-nilpotent elements of positive degree in the Ext-algebra of  $M$ . Characterizations of periodic algebras are given.

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## 1. Introduction

Let  $\mathbb{k}$  be a field and let  $R = R_0 \oplus R_1 \oplus \cdots$  be a Koszul  $\mathbb{k}$ -algebra with  $R_0 = \mathbb{k} \times \cdots \times \mathbb{k}$  and with graded radical  $\mathfrak{r} = \bigoplus_{i>0} R_i$ . Let  $M$  be a linear  $R$ -module and let  $\text{Ext}_R^*(M, M)$  be its Ext (or Yoneda) algebra. It is well-known that  $\text{Ext}_R^*(M, M)$  has a bigrading induced from the homological grading and from the internal grading of  $M$ . One of the main goals of this paper is to study a particular subalgebra  $\Delta_M$ , of  $\text{Ext}_R^*(M, M)$  called the *diagonal subalgebra*. Namely, this subalgebra will be generated by all the elements of  $\text{Ext}_R^*(M, M)$  of bidegree  $(i, -i)$  for  $i \geq 0$ . We study the graded algebra decomposition

$$\text{Ext}_R^*(M, M) = \Delta_M \oplus \mathcal{N}_M$$

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of the Ext-algebra given in [Proposition 2.2](#), so viewing  $R$  as a linear module over its enveloping algebra, we obtain

$$\mathrm{HH}^*(R) = \Delta_R \oplus \mathcal{N}_R,$$

where  $\mathrm{HH}^*(R)$  denotes the Hochschild cohomology ring of  $R$ . We show that

- (1) If  $M$  is a linear  $R$ -module of finite graded width, then the two-sided ideal  $\mathcal{N}_M$  consists of nilpotent elements.
- (2) If  $Z_{\mathrm{gr}}(\mathcal{E}_R)$  denotes the graded center of the Koszul dual  $\mathcal{E}_R$  of  $R$ , that is, of  $\mathrm{Ext}_R^*(R/\mathfrak{r}, R/\mathfrak{r})$ , then

$$\Delta_R \cong Z_{\mathrm{gr}}(\mathcal{E}_R).$$

- (3) If  $\mathcal{E}_R$  is a finitely generated  $Z_{\mathrm{gr}}(\mathcal{E}_R)$ -module and there exists a non-nilpotent element in  $(\mathcal{E}_R)_{\geq 1}$ , then there exists a non-nilpotent element of positive degree in the Hochschild cohomology ring of  $R$ .
- (4) We use properties of  $\Delta_M$  to obtain a characterization of the linear  $R$ -modules  $M$  that are eventually periodic over a Koszul algebra.

In addition, we prove in the last section the following when  $R$  is a selfinjective (but not necessarily graded) algebra:

- (1) The algebra  $R$  is a periodic algebra if and only if all the simple  $R$ -modules have complexity one and there exists a non-nilpotent element of positive degree in the Hochschild cohomology ring of  $R$ .
- (2) If  $R$  is an algebra over an algebraically closed field and satisfies the **Fg** finiteness condition (see the paragraph after [Proposition 4.5](#) for the definition), then the property that all the simple modules are periodic is equivalent to the algebra being periodic when viewed as a module over its enveloping algebra.

Throughout this paper we only consider graded  $R$ -modules that have graded projective resolutions such that each projective module occurring in the resolution is finitely generated. It is well-known that minimal graded resolutions exist in our situation. We start by recalling some of the basic notation that will be used in this paper. Let  $M$  and  $N$  be two graded  $R$ -modules. By a homomorphism of degree  $i$  we mean a module homomorphism from  $M$  to  $N$  taking  $M_k$  into  $N_{i+k}$  for each integer  $k$ . By abuse of language we will use the term “graded homomorphism” for degree zero homomorphisms. Let  $M$  be a finitely generated graded  $R$ -module and let  $m \in \mathbb{Z}$ . Then  $M(m)$  will denote the graded shift of the module  $M$ , that is, the graded module whose  $i$ -th graded piece is  $M(m)_i = M_{i+m}$ . Similarly, if  $(P_M^\bullet, d_M^\bullet)$  is a graded projective resolution of  $M$ , then its graded shift  $(P_M^\bullet(m), d_M^\bullet(m))$  is defined in the obvious way.

Denote by  $\Gamma$ , the Ext-algebra of  $M$ , that is,

$$\Gamma = \mathrm{Ext}_R^*(M, M) = \bigoplus_{n \geq 0} \mathrm{Ext}_R^n(M, M).$$

It is well-known that  $\Gamma$  is an associative  $\mathbb{k}$ -algebra with the multiplication given by the Yoneda product. For the convenience of the reader we recall one way of looking at this product. Let

$$P_M^\bullet \cdots \rightarrow P_M^{n+1} \xrightarrow{d_M^{n+1}} P_M^n \xrightarrow{d_M^n} \cdots \rightarrow P_M^1 \xrightarrow{d_M^1} P_M^0 \xrightarrow{d_M^0} M \rightarrow 0$$

be a minimal graded projective resolution of  $M$ . Let  $\eta$  be an element of  $\mathrm{Ext}_R^n(M, M)$ . We may represent it as a homomorphism  $\eta: P_M^n \rightarrow M$  having the property that the composition  $\eta d_M^{n+1} = 0$ . Now let  $\xi: P_M^m \rightarrow M$

represent an element of  $\text{Ext}_R^m(M, M)$ . For each  $i = 0, \dots, m$ , we have liftings  $l^i: P_M^{n+i} \rightarrow P_M^i$  of  $\eta$  and we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & P_M^{n+m} & \xrightarrow{d_M^{m+n}} & \dots & \longrightarrow & P_M^{n+1} & \xrightarrow{d_M^{n+1}} & P_M^n & & \\
 & & \downarrow l^m & & & & \downarrow l^1 & & \downarrow l^0 & \searrow \eta & \\
 \dots & \longrightarrow & P_M^m & \xrightarrow{d_M^m} & \dots & \longrightarrow & P_M^1 & \xrightarrow{d_M^1} & P_M^0 & \xrightarrow{d_M^0} & M \\
 & & \downarrow \xi & & & & & & & & \\
 & & M & & & & & & & & 
 \end{array}$$

It is well-known that the composition  $\xi l^m$  represents the element  $\xi \eta$  of  $\text{Ext}_R^{m+n}(M, M)$  and that this multiplication does not depend on the choice of the liftings  $l^i$ . Note also that if  $\eta$  is a homomorphism of degree  $j$ , then we may assume without loss of generality that each lifting  $l^i$  has degree  $j$  as well.

The Ext-algebra of  $M$  is a bigraded  $\mathbb{k}$ -algebra in the following way: For each integer  $i$ , let  $\text{Hom}_R(M, M)_i$  denote the space of all graded homomorphisms of degree  $i$  from  $M$  to  $M$ . Then  $\text{End}_R(M, M)$  is a graded  $R_0$ -module by putting  $\text{End}_R(M, M) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_R(M, M)_i$ . Passing to derived functors, for each  $n \geq 0$ ,  $\text{Ext}_R^n(M, M)$  has a grading induced from the grading on  $\text{End}_R(M, M)$ . More precisely, let

$$P_M^\bullet \quad \dots \rightarrow P_M^{n+1} \xrightarrow{d_M^{n+1}} P_M^n \xrightarrow{d_M^n} \dots \rightarrow P_M^1 \xrightarrow{d_M^1} P_M^0 \xrightarrow{d_M^0} M \rightarrow 0$$

be a minimal graded projective resolution of  $M$ . If  $\eta: P_M^n \rightarrow M$  represents an element of  $\text{Ext}_R^n(M, M)$ , then  $\eta$  is a sum of graded homomorphisms of various degrees. Denoting by  $\text{Ext}_R^n(M, M)_i$  the elements of  $\text{Ext}_R^n(M, M)$  represented by elements of  $\text{Hom}_R(P_M^n, M)_i$  we obtain the bigrading

$$\text{Ext}_R^*(M, M) = \bigoplus_{n \geq 0} \bigoplus_{i \in \mathbb{Z}} \text{Ext}_R^n(M, M)_i.$$

The reader may easily check that the Yoneda product is compatible with this bigrading.

Finally, we remark that most of the results in this paper concern Koszul algebras and linear modules, and we recall that a graded  $R$ -module generated in degree zero is called a *linear module* if it has a graded projective resolution such that, for each  $n$ , the  $n$ -th term of the resolution is finitely generated and generated in degree  $n$ . So in the case when  $M$  is a linear module, this implies that for each nonnegative integer  $n$ , we have  $\text{Ext}_R^n(M, M) = \bigoplus_{i \geq -n} \text{Ext}_R^n(M, M)_i$ .

### 2. The diagonal subalgebra of an Ext-algebra

Throughout this section  $R = R_0 \oplus R_1 \oplus \dots$  is a Koszul  $\mathbb{k}$ -algebra, where  $R_0 = \mathbb{k} \times \dots \times \mathbb{k}$ , and  $M$  is a linear  $R$ -module. We define the  $n$ -th graded part  $\Delta_M^n$  of the diagonal subalgebra  $\Delta_M$  of  $\text{Ext}_R^*(M, M)$  as

$$\Delta_M^n = \text{Ext}_R^n(M, M)_{-n}.$$

The whole diagonal subalgebra  $\Delta_M$  of  $\text{Ext}_R^*(M, M)$  is then defined as

$$\Delta_M = \bigoplus_{n \geq 0} \Delta_M^n$$

and we set

$$\mathcal{N}_M = \bigoplus_{i > -n} \bigoplus_{n \geq 0} \text{Ext}_R^n(M, M)_i.$$

Note that the Yoneda product induces a two-sided  $\Delta_M$ -module structure on  $\mathcal{N}_M$ . It is also clear that  $\mathcal{N}_M$  is an ideal of  $\text{Ext}_R^*(M, M)$ . We start with a very simple example. A less obvious one is presented later in this section.

**Example 2.1.** Let  $R$  be a Koszul algebra and let  $M$  be a semisimple linear module. It is easy to show in this case that  $\Delta_M = \text{Ext}_R^*(M, M)$ .

We have the following immediate consequence of the definition:

**Proposition 2.2.** *Let  $R$  be a Koszul algebra and let  $M$  be a linear  $R$ -module. Then we have a decomposition  $\text{Ext}_R^*(M, M) = \Delta_M \oplus \mathcal{N}_M$  and  $\mathcal{N}_M$  is an ideal of  $\text{Ext}_R^*(M, M)$ .  $\square$*

The following general results concern modules of finite graded width. Recall that a graded module  $M$  has *finite graded width*  $d$ , if, for some integers  $a \leq b$ ,  $M_a \neq 0$ ,  $M_b \neq 0$ ,  $M_i = 0$  for  $i < a$  and  $i > b$ , and  $d = b - a + 1$ .

**Lemma 2.3.** *Let  $R$  be a Koszul algebra, let  $M$  be a linear  $R$ -module of finite graded width  $d$ , and let  $P_M^n$  be the  $n$ -th term in a minimal graded projective resolution of  $M$ . If  $\eta: P_M^n \rightarrow M$  represents an element of  $\text{Ext}_R^n(M, M)_s$  and  $s > -n + d$ , then  $\eta = 0$  in  $\text{Ext}_R^n(M, M)$ .*

**Proof.** Note that  $M$  lives in degrees  $0, \dots, d - 1$ . Since the top of  $P_M^n$  lies in degree at least  $n$ , every homomorphism  $P_M^n \rightarrow M$  of degree greater than  $-n + d$  is zero so the result follows.  $\square$

**Proposition 2.4.** *Let  $R$  be a Koszul algebra and  $M$  a linear  $R$ -module of finite graded width. Then every element of  $\mathcal{N}_M$  is nilpotent.*

**Proof.** Assume that the graded width of  $M$  is  $d$ , and let  $x \in \mathcal{N}_M$ . Then  $x$  can be written as a finite sum  $\sum_{k \geq 0} \sum_{j > -k} x_{k,j}$  where  $x_{k,j} \in \text{Ext}_R^k(M, M)_j$ . By abuse of notation, we view  $x_{k,j}$  as a homomorphism from  $P_M^k$  to  $M$  of degree  $j$  greater than  $-k$  such that  $x_{k,j} d_M^{k+1} = 0$ , where  $(P_M^\bullet, d_M^\bullet)$  is a minimal graded projective resolution of  $M$ . Consider  $x^{d+1} = (\sum_{k \geq 0} \sum_{j > -k} x_{k,j})^{d+1}$ . Each term  $y$  in  $x^{d+1}$  is a product of  $d + 1$   $x_{k,j}$ 's, so, as an element in some  $\text{Ext}_R^n(M, M)$ ,  $y$  is represented by a graded map of degree greater or equal to  $-n + d + 1$ . By Lemma 2.3, each  $y = 0$  and hence  $x^{d+1} = 0$ .  $\square$

We have the following slightly more explicit way of describing the subalgebra  $\Delta_M$  and the ideal  $\mathcal{N}_M$  for a linear module  $M$ . First, let  $\mathfrak{r} = R_1 \oplus R_2 \oplus \dots$  denote the graded radical of  $R$ , and let  $(P_M^\bullet, d_M^\bullet)$  be a linear resolution of  $M$ . Let  $\eta: P_M^n \rightarrow M$  be a  $1 \times k$  matrix with entries in  $M$  representing an element of  $\text{Ext}_R^n(M, M)$ , where  $k$  is the number of indecomposable summands of  $P_M^n$ . Then  $\eta$  represents an element of  $\Delta_M^n$  if and only if  $\eta = [m_1, \dots, m_k]$  where each  $m_i \in M_0$ , or equivalently,  $\eta$  is a degree  $-n$  map from  $P_M^n$  to  $M$ . Similarly,  $\eta: P_M^n \rightarrow M$  represents an element of  $\mathcal{N}_M$  if and only if  $\text{Im } \eta \subseteq M\mathfrak{r}$  (that is, each  $m_i$  is in the radical of  $M$ ).

In view of the preceding results we introduce the following definitions. Let  $R$  be a Koszul algebra and let  $M$  be a linear  $R$ -module. We say that a nonzero homogeneous element  $\eta \in \text{Ext}_R^n(M, M)$  is *strongly radical*, if it is in  $\mathcal{N}_M$ . If, in addition,  $R$  is finite dimensional, then we call  $\eta$  *strongly nilpotent*. Note that it is possible to have a nilpotent homogeneous element in  $\text{Ext}_R^*(M, M)$  that is not strongly nilpotent. For instance if we look at the polynomial ring in  $n$  variables over a field, and we let  $M$  be the unique graded simple module, then  $\text{Ext}_R^*(M, M)$  is finite dimensional and in this case  $\text{Ext}_R^*(M, M) = \Delta_M$ , so every element of  $\Delta_M$  of positive (homological) degree is nilpotent.

We have the following illustration of [Proposition 2.2](#). Viewing a Koszul algebra  $R$  as a module over its enveloping algebra  $R^e = R^{\text{op}} \otimes_{\mathbb{k}} R$ , the module  $R$  is linear (see [\[6\]](#) for instance) and we get a decomposition of the Hochschild cohomology ring as

$$\text{HH}^*(R) = \Delta_R \oplus \mathcal{N}_R.$$

Let  $\mathcal{E}_R$  denote the Ext-algebra of  $R$ , that is,  $\mathcal{E}_R = \text{Ext}_R^*(R/\mathfrak{r}, R/\mathfrak{r})$ . Recall also that the *graded center* of  $\mathcal{E}_R$  is the graded subring  $Z_{\text{gr}}(\mathcal{E}_R)$  generated by all the homogeneous elements  $u$  such that  $uv = (-1)^{|u||v|}vu$  for every homogeneous element  $v$  of  $\mathcal{E}_R$ , where  $|x|$  denotes the degree of the homogeneous element  $x$ . There is a homomorphism  $T$  of graded algebras

$$T: \text{HH}^*(R) \rightarrow \mathcal{E}_R$$

given by  $T(\eta) = R/\mathfrak{r} \otimes_R \eta$ . Since  $R$  is Koszul, the image of this homomorphism is the graded center  $Z_{\text{gr}}(\mathcal{E}_R)$  (see [\[1\]](#)). Summarizing, we have the following characterization of the graded center of a Koszul algebra:

**Theorem 2.5.** *Let  $S$  be a Koszul algebra and let  $R$  be its Koszul dual, so  $S = \mathcal{E}_R$ . Let  $\text{HH}^*(R) = \Delta_R \oplus \mathcal{N}_R$ . Then the homomorphism  $T$  induces an isomorphism of graded  $\mathbb{k}$ -algebras  $\Delta_R \cong Z_{\text{gr}}(S)$ .*

**Proof.** Let  $(P^\bullet, d^\bullet)$  be a linear resolution of  $R$  over its enveloping algebra  $R^e$  and let  $\varepsilon \in \mathcal{N}_R$  be homogeneous of degree  $n$ . Now,  $\varepsilon$  can be represented by a matrix  $[z_1, \dots, z_k]: P^n \rightarrow R$  where  $k$  denotes the number of indecomposable summands of  $P^n$  and where each entry  $z_i \in \mathfrak{r}$ , so it is clear that  $T(\varepsilon) = 0$ . Therefore we have an induced homomorphism of graded algebras  $T: \Delta_R \rightarrow Z_{\text{gr}}(\mathcal{E}_R)$ . By [\[1\]](#) it is clear that this restriction of  $T$  to  $\Delta_R$  is surjective.

Now let  $Q$  be an indecomposable projective  $R^e$ -summand of  $P^n$ . We may write  $Q = (e_v^o \otimes_{\mathbb{k}} e_w)R^e$  where  $e_v^o$  and  $e_w$  are primitive idempotents in  $R^{\text{op}}$  and  $R$  respectively. If we have a nonzero map  $\eta: Q \rightarrow R(-n)$  such that  $\eta(e_v^o \otimes_{\mathbb{k}} e_w) = r_0 \in R_0$ , then clearly  $R/\mathfrak{r} \otimes_R \eta \neq 0$ . This clearly implies that the restriction of  $T$  to  $\Delta_R$  is one-to-one.  $\square$

It has been conjectured in [\[9\]](#) that for a finite dimensional algebra  $R$ , the Hochschild cohomology modulo the ideal generated by the nilpotent elements is finitely generated as an algebra over the ground field. This conjecture has been disproved by Xu [\[10\]](#), see also [\[8\]](#). However, there are many instances where this quotient is finitely generated, to which we add the following result:

**Proposition 2.6.** *Let  $R$  be a finite dimensional Koszul  $\mathbb{k}$ -algebra with Koszul dual  $\mathcal{E}_R$  and let  $\tilde{\mathcal{N}}$  be the ideal of  $\text{HH}^*(R)$  generated by the homogeneous nilpotent elements. Assume that the graded center of  $\mathcal{E}_R$  is a finitely generated  $\mathbb{k}$ -algebra. Then  $\text{HH}^*(R)/\tilde{\mathcal{N}}$  is also a finitely generated  $\mathbb{k}$ -algebra.*

**Proof.** Let  $\Delta_R$  be the diagonal subalgebra of the Hochschild cohomology ring. By [Theorem 2.5](#),  $\Delta_R$  is isomorphic as a graded  $\mathbb{k}$ -algebra to the graded center of  $\mathcal{E}_R$ , so  $\Delta_R$  is also finitely generated. By [Proposition 2.2](#),  $\text{HH}^*(R)/\tilde{\mathcal{N}}$  is a quotient of  $\Delta_R$ , so the result follows.  $\square$

We also have the following immediate consequence:

**Theorem 2.7.** *Let  $R$  be a finite dimensional Koszul  $\mathbb{k}$ -algebra having the property that its Ext-algebra is commutative and let  $\tilde{\mathcal{N}}$  be the ideal of  $\text{HH}^*(R)$  generated by the homogeneous nilpotent elements. Then  $\text{HH}^*(R)/\tilde{\mathcal{N}}$  is a finitely generated  $\mathbb{k}$ -algebra.*

**Proof.** Assume that  $S = \mathcal{E}_R$  is commutative. If  $\text{char } \mathbb{k} = 2$ , then  $Z_{\text{gr}}(S) = S$  so the graded center is finitely generated. If the characteristic of  $\mathbb{k}$  is different from 2, then every element  $x \in Z_{\text{gr}}(S)$  of odd degree is

nilpotent with nilpotency index 2. So the graded center of  $S$  decomposes as  $Z_{\text{gr}}(S) = S_{\text{ev}} \oplus I$  where  $S_{\text{ev}}$  denotes the even degree part of  $S$  and  $I$  is a two sided ideal of the graded center generated by nilpotent elements of odd degree. This means that we have an isomorphism of graded algebras between  $\Delta_R$  modulo the ideal generated by its nilpotent elements and  $S_{\text{ev}}$  modulo its nilpotent elements. Since  $S$  is a Koszul algebra,  $S_{\text{ev}}$  is finitely generated and the result follows.  $\square$

The following example shows that we can have nonzero elements in  $\Delta_M^n$  that are nilpotent.

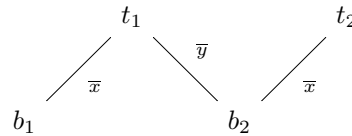
**Example 2.8.** We show that there are periodic linear modules  $M$  with constant Betti number 2 such that there is a nonzero nilpotent element  $\eta \in \Delta_M^n$ . We let

$$R = \mathbb{k}\langle x, y \rangle / \langle xy - qyx, x^2, y^2 \rangle$$

where  $0 \neq q \in \mathbb{k}$ . Let  $\bar{x}$  and  $\bar{y}$  denote the residue classes in  $R$  of  $x$  and  $y$  respectively. Assume  $q \neq 0$ . Then  $R$  is a special biserial, selfinjective Koszul algebra. We now let

$$M = \text{coker} \left( R^2 \xrightarrow{\begin{pmatrix} -\bar{y} & 0 \\ \bar{x} & q\bar{y} \end{pmatrix}} R^2 \right).$$

We note that  $M$  is four-dimensional with basis  $t_1, t_2, b_1, b_2$  such that  $t_1\bar{x} = b_1, t_1\bar{y} = b_2, t_2\bar{x} = b_2,$  and  $t_2\bar{y} = 0$ . We see that  $M$  is a string module with shape



Let  $p: R^2 \rightarrow M$  be the map sending  $(1, 0)^t$  to  $t_1$  and  $(0, 1)^t$  to  $t_2$ . Viewing  $M$  as a graded module generated in degree 0, a minimal graded projective resolution of  $M$  is given by

$$\dots \longrightarrow R^2(-2) \xrightarrow{\begin{pmatrix} -\bar{y} & 0 \\ \bar{x} & q\bar{y} \end{pmatrix}} R^2(-1) \xrightarrow{\begin{pmatrix} -\bar{y} & 0 \\ \bar{x} & q\bar{y} \end{pmatrix}} R^2 \xrightarrow{p} M \longrightarrow 0.$$

Therefore, for each  $n, \Omega^n M = M(-n)$ , so  $M$  is a periodic linear module with constant Betti numbers equal to 2.

We now define a nonzero nilpotent element  $\eta \in \Delta_M^n$ . Let  $N$  be the submodule of  $M$  generated by  $t_2$ . We note that  $M/N \cong N$ . Let  $\eta \in \Delta_M^1$  be given by composition of the following maps:

$$R^2(-1) \xrightarrow{p^{(-1)}} M(-1) \xrightarrow{\pi} (M/N)(-1) \xrightarrow{\cong} N(-1) \hookrightarrow M(-1),$$

where  $\pi$  is the canonical surjection. The map  $\eta: R^2(-1) \rightarrow M(-1)$  lifts as follows:

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & R^2(-2) & \xrightarrow{\begin{pmatrix} -\bar{y} & 0 \\ \bar{x} & q\bar{y} \end{pmatrix}} & R^2(-1) & & \\
 & & \downarrow l^1 & & \downarrow l^0 & \searrow \eta & \\
 \dots & \longrightarrow & R^2(-2) & \xrightarrow{\begin{pmatrix} -\bar{y} & 0 \\ \bar{x} & q\bar{y} \end{pmatrix}} & R^2(-1) & \xrightarrow{p^{(-1)}} & M(-1) \longrightarrow 0
 \end{array}$$

where  $l^i$  is given by the matrix  $\begin{pmatrix} 0 & 0 \\ (-1/q)^i & 0 \end{pmatrix}$ . It is clear that  $\eta$  is nonzero and that  $\eta^2 = 0$ .

We end this section with the following observation:

**Remark 2.9.** Let  $R$  be a Koszul algebra,  $M$  a linear  $R$ -module, and let  $\eta: P_M^n \rightarrow M$  be a map representing an element of  $\text{Ext}_R^n(M, M)$  for some  $n \geq 1$  and assume also that the image of  $\eta$  is not entirely contained in the radical of  $M$ . This can happen, for instance, if  $\eta$  is a surjective homomorphism. We claim that in this case  $\eta$  cannot represent the zero element in cohomology. To see this, assume by contradiction that  $\eta$  factors through  $P_M^{n-1}$ , so that there is a map  $\xi: P_M^{n-1} \rightarrow M$  such that  $\eta = \xi \circ d$ . Since the resolution of  $M$  is minimal,  $d$  has image in the radical of  $P_M^{n-1}$ , and this implies that  $\eta$  has image in the radical of  $M$ , contradicting our assumption.

### 3. Liftings

In this section we present some technical results about consecutive liftings associated with elements of  $\Delta_M$  and of  $\mathcal{N}_M$ . Let  $\eta: P_M^n \rightarrow M$  represent an element of  $\text{Ext}_R^n(M, M)$ . Consider the following diagram

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & P_M^{n+m} & \xrightarrow{d_M^{m+n}} & \dots & \longrightarrow & P_M^{n+1} & \xrightarrow{d_M^{n+1}} & P_M^n & & \\
 & & \downarrow l^m & & & & \downarrow l^1 & & \downarrow l^0 & \searrow \eta & \\
 \dots & \longrightarrow & P_M^m & \xrightarrow{d_M^m} & \dots & \longrightarrow & P_M^1 & \xrightarrow{d_M^1} & P_M^0 & \xrightarrow{d_M^0} & M
 \end{array}$$

where we view each lifting  $l^i$  as a matrix with entries in  $R$ .

**Lemma 3.1.** *Let  $R$  be a Koszul algebra, let  $M$  be a linear  $R$ -module, and let  $(P_M^\bullet, d_M^\bullet)$  be a linear resolution of  $M$ . Let  $\eta: P_M^n \rightarrow M$  be a  $1 \times k$  matrix with entries in  $M$  representing an element of  $\text{Ext}_R^n(M, M)$ , where  $k$  is the number of indecomposable summands of  $P_M^n$ .*

- (1) *Suppose that  $\text{Im } \eta \subseteq M\mathfrak{r}$ . Then the lifting  $l^0$  can be chosen in such a way so that its entries are all in  $\mathfrak{r}$ .*
- (2) *Suppose that  $\eta = [m'_1, \dots, m'_k]$  where each  $m'_i \in M_0$ . Then the lifting  $l^0$  can be chosen in such a way so that its entries are all in  $R_0$ .*

**Proof.** We prove only (1), as the other part is similar. Since the image of  $\eta$  is in the radical of  $M$ ,  $\eta$  is a sum of graded homomorphisms of degrees greater than  $-n$ . Without loss of generality we may assume that  $\eta$  is a graded homomorphism of degree  $s > -n$ . As  $d_M^0$  is of degree zero, the lifting  $l^0$  is also of degree  $s$  and can be chosen with entries in  $\mathfrak{r}$ .  $\square$

It turns out that if one of the liftings has its entries in  $\mathfrak{r}$ , or entirely in  $R_0$ , then so do all its successors:

**Lemma 3.2.** *Keeping the above notation, let  $R$  be a Koszul algebra and let  $M$  be a linear  $R$ -module. Let  $\eta: P_M^n \rightarrow M$  be a  $1 \times k$  matrix with entries in  $M$  representing an element of  $\text{Ext}_R^n(M, M)$ , where  $k$  is the number of indecomposable summands of  $P_M^n$ .*

- (1) *Assume that for some  $i$ , the lifting  $l^i$  has all its entries in  $\mathfrak{r}$ . Then we may choose the subsequent liftings  $l^{i+1}, l^{i+2}, \dots$  in such a way that all their entries are also in  $\mathfrak{r}$ .*
- (2) *Assume that for some  $i$ , the lifting  $l^i$  has all its entries in  $R_0$ . Then we may choose the subsequent liftings  $l^{i+1}, l^{i+2}, \dots$  in such a way that all their entries are also in  $R_0$ .*  
*In addition, if the map  $l^i$  has all its entries in  $R_0$ , then there exists in fact a unique lifting  $l^{i+1}$  having all its entries in  $R_0$ .*

**Proof.** (1) Consider the following commutative diagram:

$$\begin{array}{ccc}
 P_M^{n+i+1} & \xrightarrow{d_M^{n+i+1}} & P_M^{n+i} \\
 \downarrow l^{i+1} & & \downarrow l^i \\
 P_M^{i+1} & \xrightarrow{d_M^n} & P_M^i
 \end{array}$$

Since the entries of  $l^i d_M^{n+i+1}$  are in  $\mathfrak{r}^2$  and those of the differentials are in  $R_1$  it follows that we may choose  $l^{i+1}$  as a matrix with entries in  $\mathfrak{r}$ .

(2) The proof of the first part of (2) is similar to the arguments for (1), so we leave this to the reader. For the second part observe the following. Given two liftings of  $l^i$ , the difference of any two such liftings would have to factor through the radical of  $P_M^{n+i}$ . The claim follows from this.  $\square$

The following corollary discusses two situations in which all the liftings of a nonzero element of  $\Delta_M^n$  must be nonzero matrices with entries in  $R_0$ :

**Corollary 3.3.** *Let  $R$  be a Koszul algebra and let  $M$  be a linear  $R$ -module. Let  $\eta: P_M^n \rightarrow M$  be a map representing a nonzero homogeneous element of degree  $n$  in  $\Delta_M$  for some  $n \geq 1$ . Assume that either  $\eta$  is non-nilpotent, or that  $R$  is selfinjective. Then all the successive liftings  $l^i$  of  $\eta$  are nonzero and can be chosen to be matrices whose nonzero entries are all in  $R_0$ .*

**Proof.** Since  $\eta$  is nonzero it is clear that the first lifting  $l^0 \neq 0$  and that we may choose all the other liftings with entries in  $R_0$ . Assume first that  $\eta$  is non-nilpotent. If a lifting  $l^i$  was zero, then all the successive liftings would also be zero, implying that  $\eta$  must be nilpotent. Let us consider now the situation when  $R$  is selfinjective. If one of these liftings was zero, then the previous one would have to factor through its injective envelope hence its entries would not be in  $R_0$ .  $\square$

We continue with two observations. They hold in the more general context in which  $R$  is a not necessarily Koszul graded  $\mathbb{k}$ -algebra. Assume that  $g: P \rightarrow Q$  is a degree zero homomorphism between two projective  $R$ -modules generated in the same degree. Let  $m$  and  $n$  be the number of indecomposable summands of  $P$  and of  $Q$  respectively. Then  $g$  can be represented as an  $n \times m$  matrix with entries in  $R_0$ . We have the following:

**Lemma 3.4.** *Let  $\mathbb{k}$  be a field, let  $R = R_0 \oplus R_1 \oplus \dots$  be a graded algebra with  $R_0 = \mathbb{k} \times \dots \times \mathbb{k}$ , and let  $P$  and  $Q$  be two finitely generated projective  $R$ -modules generated in the same degree. Let  $g: P \rightarrow Q$  be a degree zero homomorphism. Then we have the following.*

- (1)  $\text{Ker } g$  is a direct summand of  $P$ .
- (2)  $\text{Im } g$  is a direct summand of  $Q$ .
- (3) If  $P = \text{Ker } g \oplus P'$  and  $Q = \text{Im } g \oplus Q'$ , then  $g|_{P'}: P' \rightarrow \text{Im } g$  is an isomorphism and  $Q'$  is isomorphic to the cokernel of  $g$ .

**Proof.** It is enough to prove that the cokernel  $M$  of  $g$  is a direct summand of  $Q$ . Assume not. Without loss of generality we may assume that both  $P$  and  $Q$  are generated in degree zero. So  $M$  is also generated in degree zero. Let  $P^1 \rightarrow P^0 \rightarrow M \rightarrow 0$  be a minimal projective presentation of  $M$ . It is clear that  $P^0$  is generated in degree zero and  $P^1$  is generated in degree(s) one and higher. We have the following commutative diagram with exact rows of graded modules and degree zero homomorphisms:



$$\begin{array}{ccccccc}
 P & \xrightarrow{g} & Q & \xrightarrow{p} & M & \longrightarrow & 0 \\
 \downarrow h^1 & & \downarrow h^0 & & \parallel & & \\
 P^1 & \xrightarrow{d^1} & P^0 & \xrightarrow{d^0} & M & \longrightarrow & 0
 \end{array}$$

where  $p$  is the canonical surjection. Since the degree zero component of  $P^1$  vanishes and  $P$  is generated in degree zero, the map  $h^1$  is zero. So  $h^0$  factors through  $M$ . This implies that  $d^0$  splits so  $M$  is projective. The result follows.  $\square$

We mention without proof the following dual result.

**Lemma 3.5.** *Let  $\mathbb{k}$  be a field, let  $R = R_0 \oplus R_1 \oplus \dots$  be a graded algebra with  $R_0 = \mathbb{k} \times \dots \times \mathbb{k}$ , and let  $E$  and  $I$  be two finitely generated injective  $R$ -modules cogenerated in the same degree. Let  $g: E \rightarrow I$  be a degree zero homomorphism. Then we have the following.*

- (1)  $\text{Ker } g$  is a direct summand of  $E$ .
- (2)  $\text{Im } g$  is a direct summand of  $I$ .
- (3) If  $E = \text{Ker } g \oplus E'$  and  $I = \text{Im } g \oplus I'$ , then  $g|_{E'}: E' \rightarrow \text{Im } g$  is an isomorphism and  $I'$  is isomorphic to the cokernel of  $g$ .  $\square$

We have the following application of Lemma 3.4:

**Proposition 3.6.** *Let  $R$  be a graded algebra with  $R_0 = \mathbb{k} \times \dots \times \mathbb{k}$ , and let  $f: L \rightarrow M$  be a degree zero homomorphism between two non-projective modules  $L$  and  $M$  generated in the same degree. Denote by  $P_L$  and  $P_M$  the projective cover of  $L$  and  $M$  respectively. If  $f$  is a monomorphism, then the morphisms  $\Omega f: \Omega L \rightarrow \Omega M$  and  $l: P_L \rightarrow P_M$  induced by  $f$  are also monomorphisms of degree zero.*

**Proof.** We may assume without loss of generality that both  $L$  and  $M$  are generated in degree zero. We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Omega L & \longrightarrow & P_L & \xrightarrow{p} & L \longrightarrow 0 \\
 & & \downarrow \Omega f & & \downarrow l & & \downarrow f \\
 0 & \longrightarrow & \Omega M & \longrightarrow & P_M & \longrightarrow & M \longrightarrow 0
 \end{array}$$

The Snake Lemma implies that the kernels of  $l$  and  $\Omega f$  are isomorphic, and the latter is clearly generated in positive degrees, since  $\Omega L$  is. At the same time,  $\text{Ker } l$  is a direct summand of  $P_L$  by Lemma 3.4, so it is generated in degree zero. This is only possible if  $\text{Ker } l$  is actually zero, so  $\text{Ker } \Omega f$  equals zero.  $\square$

As an immediate consequence we obtain:

**Proposition 3.7.** *Let  $R$  be a Koszul algebra and let  $M$  be an indecomposable linear  $R$ -module. Let  $n \geq 1$  and let  $0 \neq \eta: P_M^n \rightarrow M$  represent an element of  $\Delta_M^n$  and, for each  $i \geq 0$ , let  $l^i: P_M^{n+i} \rightarrow P_M^i$  be liftings of  $\eta$  with entries in  $R_0$ . Assume that  $l^j$  is a monomorphism for some  $j \geq 0$ . Then all the subsequent liftings  $l^{j+k}$  with  $k \geq 0$  are monomorphisms.*

**Proof.** Assume that the lifting  $l^j$  is a monomorphism. Then, according to Proposition 3.6, this implies that the induced morphism  $f^{j+1}: \Omega^{n+j+1} M \rightarrow \Omega^{j+1} M$  is also a monomorphism so we have the following commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Omega^{n+j+2}M & \longrightarrow & P_M^{n+j+1} & \longrightarrow & \Omega^{n+j+1}M \longrightarrow 0 \\
 & & \downarrow f^{j+2} & & \downarrow l^{j+1} & & \downarrow f^{j+1} \\
 0 & \longrightarrow & \Omega^{j+2}M & \longrightarrow & P_M^{j+1} & \longrightarrow & \Omega^{j+1}M \longrightarrow 0
 \end{array}$$

where the lifting  $l^{j+1}$  is a matrix with entries in  $R_0$  by Lemma 3.2. Clearly  $l^{j+1} \neq 0$  since  $f^{j+1}$  is a monomorphism. By the previous result,  $l^{j+2}$  is also a monomorphism and so is  $f^{j+2}$ . The result follows now by induction.  $\square$

It turns out that one can use Lemma 3.4 to show that in some interesting cases we have a behavior similar to the one in Proposition 3.7 when one of the liftings is an epimorphism:

**Proposition 3.8.** *Let  $R$  be a Koszul algebra and let  $M$  be an indecomposable linear  $R$ -module with the property that none of its syzygies has a nonzero projective summand. Assume that for some  $n \geq 1$ ,  $\eta: P_M^n \rightarrow M$  represents a nonzero element of  $\Delta_M^n$ . For each  $i \geq 0$ , let  $l^i: P_M^{n+i} \rightarrow P_M^i$  be liftings of  $\eta$  with entries in  $R_0$ . Assume that  $l^j$  is an epimorphism for some  $j \geq 0$ . Then all the previous liftings  $l^k$  with  $0 \leq k \leq j$  are epimorphisms.*

**Proof.** Assume that the lifting  $l^{j-1}$  is not onto. Then, by Lemma 3.4, its cokernel  $Q$  is projective and generated in degree  $j - 1$ . We have the following commutative diagram:

$$\begin{array}{ccccccc}
 P_M^{j+n} & \longrightarrow & P_M^{j+n-1} & \longrightarrow & \Omega^{j+n-1}M & \longrightarrow & 0 \\
 \downarrow l^j & & \downarrow l^{j-1} & & \downarrow f^{j-1} & & \\
 P_M^j & \xrightarrow{d_M^j} & P_M^{j-1} & \longrightarrow & \Omega^{j-1}M & \longrightarrow & 0 \\
 & & \downarrow \pi & & & & \\
 & & Q & & & & 
 \end{array}$$

Since  $l^j$  is onto, we have that  $\pi d_M^j = 0$ , so  $\pi$  must factor through the syzygy  $\Omega^{j-1}M$ . This induces an epimorphism  $\Omega^{j-1}M \rightarrow Q$  and we get a contradiction to our assumption.  $\square$

**Remark 3.9.** Notice that the condition on the syzygies of the indecomposable module  $M$  is automatically satisfied if the algebra  $R$  is also selfinjective and  $M$  is non-projective.

**4. Modules with bounded Betti numbers**

In this section we consider two set-ups, our algebra  $R$  is either a selfinjective algebra (not necessarily graded) or  $R$  is a Koszul algebra as considered so far in the paper. We start by studying selfinjective algebras, then switch to Koszul algebras and in the final result investigate selfinjective Koszul algebras.

The topic of this section is periodic modules, and more generally, modules with bounded Betti numbers, that is, having complexity one. We start with the following:

**Proposition 4.1.** *Let  $R$  be a selfinjective algebra and let  $M$  be an indecomposable module. The following statements are equivalent.*

- (a) *The module  $M$  is periodic.*
- (b) *The module  $M$  has complexity one and there exists a non-nilpotent element in  $\text{Ext}_R^*(M, M)$  of positive degree.*

**Proof.** Assume that the module  $M$  is periodic of period  $n$  for some  $n \geq 1$ , so  $M \cong \Omega^n M$ . Then  $M$  has complexity one and the extension

$$0 \rightarrow M \rightarrow P_M^{n-1} \rightarrow \dots \rightarrow P_M^1 \rightarrow P_M^0 \rightarrow M \rightarrow 0$$

induced from the minimal projective resolution of  $M$  gives rise to a non-nilpotent element in  $\text{Ext}_R^*(M, M)$  of positive degree. For the reverse implication, assume that the module  $M$  has complexity one and that there is a non-nilpotent element  $\eta$  of positive degree  $n$  in  $\text{Ext}_R^*(M, M)$ . Since  $\text{Ext}_R^k(M, M)$  can be identified with  $\underline{\text{Hom}}_R(\Omega^k M, M)$  for each  $k \geq 1$ , we may consider  $\eta$  as being a homomorphism that we also denote by  $\eta: \Omega^n M \rightarrow M$ . Then a power  $\eta^t$  of  $\eta$  is represented by the composition

$$f_t = \eta \Omega^n(\eta) \cdots \Omega^{(t-1)n}(\eta),$$

where  $\Omega^{jn}(\eta)$  are liftings of  $\eta$ . Since  $\eta^t$  is nonzero for all  $t \geq 1$ , the homomorphism  $f_t$  is nonzero for all  $t \geq 1$ . Since all the modules  $\Omega^{jn} M$  are indecomposable, and their lengths are bounded, we infer by the Harada–Sai Lemma that one of the homomorphisms  $\Omega^{jn}(\eta)$  is an isomorphism. That is,  $\Omega^{(j+1)n} M$  is isomorphic to  $\Omega^{jn} M$  for some  $j$  with  $1 \leq j \leq t - 1$ . Since  $R$  is selfinjective and  $M$  is a non-projective indecomposable module, we have that  $\Omega^{-i} \Omega^i(M) \simeq M$  for all  $i \geq 1$ . It follows that  $M$  is periodic of period dividing  $n$ .  $\square$

Since tensoring a minimal projective  $R^e$ -resolution of  $R$  with  $R/\mathfrak{r}$  yields a minimal projective  $R$ -resolution of  $R/\mathfrak{r}$ , it follows that, viewed as an  $R^e$ -module,  $R$  has complexity one if and only if all the simple  $R$ -modules have complexity one. The following is a direct consequence of our discussion.

**Corollary 4.2.** *Let  $R$  be a selfinjective algebra. Then the following statements are equivalent.*

- (a)  *$R$  is a periodic algebra.*
- (b) *As an  $R^e$ -module,  $R$  has complexity one, and there exists a non-nilpotent element of positive degree in the Hochschild cohomology ring of  $R$ .*
- (c) *All the simple  $R$ -modules have complexity one, and there exists a non-nilpotent element of positive degree in the Hochschild cohomology ring of  $R$ .*  $\square$

**Remark 4.3.** It would be interesting to know whether a selfinjective algebra  $R$  with the property that it has complexity one over its enveloping algebra, is necessarily itself a periodic algebra. Specifically, is the existence of a non-nilpotent element of positive degree in the Hochschild cohomology simply a consequence? Let us assume for a moment that the ground field is algebraically closed. Using geometric methods, Dugas has proved in [3] that if a simple module over a selfinjective algebra has complexity one, then that simple module must be periodic. Hence all the simple modules having complexity one over  $R$ , are periodic. In the finite representation case, he also proved in [4] that this is equivalent to the algebra being periodic. Not much is known in the case of infinite representation type. Note that we also have the following result: “If all the simple modules are periodic, then the algebra is selfinjective” by [7].

We recall from [7] that if every simple  $R$ -module is periodic, and if the base field is algebraically closed, then there exists an automorphism  $\varphi$  of  $R$  such that the  $R^e$ -modules  $\Omega_{R^e}^n R$  and  ${}_1 R_\varphi$  are isomorphic for some integer  $n \geq 1$ . We have the following:

**Proposition 4.4.** *Let  $R$  be a selfinjective algebra over an algebraically closed field and assume that all the simple modules are periodic. Let  $\varphi$  be an automorphism of  $R$  such that the  $R^e$ -modules  $\Omega_{R^e}^n R$  and  ${}_1R_\varphi$  are isomorphic for some integer  $n \geq 1$ . The following statements are equivalent.*

- (a) *The Hochschild cohomology ring of  $R$  has a non-nilpotent element of positive degree.*
- (b)  *$R$  is a periodic algebra.*
- (c) *The  $R$ -bimodules  $R$  and  ${}_1R_{\varphi^t}$  are isomorphic for some integer  $t \geq 1$ .*
- (d) *The automorphism  $\varphi^t$  is inner for some integer  $t \geq 1$ .*

**Proof.** (a) implies (b): This follows from [Corollary 4.2](#) since all the simple modules have complexity one.  
 (b) implies (c): This follows immediately from the fact that  $\Omega_{R^e}^n R \cong {}_1R_\varphi$ .  
 (c) implies (a): Straightforward.  
 (c) equivalent with (d): Well-known, see [\[2, Theorem 55.11\]](#).  $\square$

As a corollary of [\[7, Theorem 1.4\]](#) and the remark following that result, we also have the following:

**Proposition 4.5.** *Let  $R$  be a finite dimensional algebra over a finite field. Then all simple  $R$ -modules are periodic if and only if  $R$  is a periodic algebra.  $\square$*

Now we show there are situations where periodic modules give rise to non-nilpotent elements that are actually in the *graded center* of their Ext-algebra, and consequently, by [Theorem 2.5](#), give rise to non-nilpotent elements in the Hochschild cohomology ring of the algebra itself. Recall that an algebra  $R$  satisfies the **Fg** condition (see [\[5\]](#) for instance), if its Ext-algebra  $\mathcal{E}_R$  is a finitely generated  $H$ -module for some commutative Noetherian graded subalgebra  $H \subseteq \text{HH}^*(R)$  with  $H^0 = \text{HH}^0(R)$ . There is a homomorphism of graded rings  $\gamma_M: \text{HH}^*(R) \rightarrow \text{Ext}_R^*(M, M)$  so  $\text{Ext}_R^*(M, M)$  becomes an  $H$ -module via the action of  $\gamma_M$ . If  $M$  is a periodic  $R$ -module, then its Ext-algebra is nonzero, so, by [\[9, Proposition 2.1\]](#), its annihilator in  $H$  is a proper ideal. By [\[5, Proposition 2.1\]](#), the variety of  $M$ ,  $V_H(M)$  has Krull dimension one, since  $M$  being periodic, has complexity one. So there are nonzero homogeneous elements in  $H$  of arbitrary large degrees.

**Proposition 4.6.** *Let  $R$  be a selfinjective algebra over an algebraically closed field satisfying **Fg**. If  $M$  is a periodic module, then there exists a non-nilpotent element of positive degree in the graded center of the Ext-algebra of  $M$  and also, there exists a non-nilpotent element of positive degree in the Hochschild cohomology ring of  $R$ .*

**Proof.** Assume that  $M$  is a periodic module. By the above remarks the dimension of the variety of  $M$  is one, and there are nonzero homogeneous elements in  $H$  of arbitrary large degrees. Let  $\eta$  be a homogeneous element of degree  $n \geq 1$  in the Hochschild cohomology ring  $\text{HH}^*(R)$  represented as a homomorphism  $\eta: \Omega_{R^e}^n R \rightarrow R$ . We can form the following pushout,

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Omega_{R^e}^n R & \longrightarrow & P^{n-1} & \longrightarrow & \Omega_{R^e}^{n-1} R \longrightarrow 0 \\
 & & \downarrow \eta & & \downarrow & & \parallel \\
 0 & \longrightarrow & R & \longrightarrow & M_\eta & \longrightarrow & \Omega_{R^e}^{n-1} R \longrightarrow 0
 \end{array}$$

where

$$0 \rightarrow \Omega_{R^e}^n R \rightarrow P^{n-1} \rightarrow \dots \rightarrow P^1 \rightarrow P^0 \rightarrow R \rightarrow 0$$

is the start of a minimal projective resolution of  $R$  over  $R^e$ . By [5, Proposition 4.3], the variety of the module  $M \otimes_R M_\eta$  is given by the ideal defining the variety of  $M$  and the element  $\eta$ . As in the proof of [5, Theorem 5.3], we may choose  $\eta$  such that the variety of  $M \otimes_R M_\eta$  is trivial. Hence  $M \otimes_R M_\eta$  is a projective  $R$ -module, again by [5, Theorem 2.5].

The image of  $\eta$  under the homomorphism  $\gamma_M: \text{HH}^*(R) \rightarrow \text{Ext}_R^*(M, M)$  is given by

$$\gamma_M(\eta): 0 \rightarrow M \rightarrow M \otimes_R M_\eta \rightarrow M \otimes_R P^{n-2} \rightarrow \dots \rightarrow M \otimes_R P^0 \rightarrow M \rightarrow 0$$

and  $\gamma_M(\eta)$  is contained in the graded center of  $\text{Ext}_R^*(M, M)$  by [9, Corollary 1.3]. Since  $M \otimes_R M_\eta$  is projective, the extension  $\gamma_M(\eta)$  is given by the first  $n$  terms in a minimal projective resolution of  $M$  (and the period of  $M$  is a divisor of  $n$ ). Therefore  $\gamma_M(\eta)$  is a non-nilpotent element in the graded center of  $\text{Ext}_R^*(M, M)$ . Consequently  $\eta \in \text{HH}^*(R)$  is also a non-nilpotent element. This completes the proof.  $\square$

We have the following immediate consequence of our discussion so far:

**Proposition 4.7.** *Let  $R$  be a selfinjective algebra over an algebraically closed field. Assume that  $R$  satisfies the **Fg** condition. Then  $R$  is periodic if and only if all the simple  $R$ -modules are periodic.  $\square$*

We give now an example showing that the existence of a periodic module  $M$  over a selfinjective (even Koszul) algebra  $R$  does not necessarily imply the existence of a non-nilpotent element of positive degree in the graded center of its Ext-algebra. Obviously the **Fg** condition is not satisfied in this case, and the graded center of the Ext-algebra of  $M$  is spanned as a vector space by the identity map  $1_M$ .

**Example 4.8.** Let

$$R = \mathbb{k} \langle x \begin{array}{c} \curvearrowright \\ 1 \\ \curvearrowleft \end{array} y \rangle / \langle x^2, xy + qyx, y^2 \rangle$$

be the Koszul dual of the quantum plane, where  $\mathbb{k}$  is a field and  $q$  is nonzero and not a root of unity in  $\mathbb{k}$ . Let  $M$  be the  $R$ -module given by the representation

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{array}{c} \curvearrowright \\ \mathbb{k}^2 \\ \curvearrowleft \end{array} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Then  $R$  is a selfinjective algebra of dimension four, and  $M$  is an indecomposable periodic  $R$ -module of period one. As a  $\mathbb{k}$ -vector space, the endomorphism ring of  $M$ , is spanned by the set  $\{1_M, f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\}$ . The homomorphism

$$\mu = \begin{pmatrix} 1 & 0 \\ 0 & -q \end{pmatrix}$$

is an isomorphism  $\mu: \Omega_R^1 M \rightarrow M$ . Consider the pushout

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_R^1 M & \xrightarrow{\iota} & P^0 & \xrightarrow{\pi} & M \longrightarrow 0 \\ & & \downarrow \mu & & \parallel & & \parallel \\ 0 & \longrightarrow & M & \xrightarrow{\iota\mu^{-1}} & P^0 & \xrightarrow{\pi} & M \longrightarrow 0 \end{array}$$

and denote the lower exact sequence again by  $\mu$ . Then, this sequence, together with the set  $\{1_M, f\}$ , generate  $\text{Ext}_R^*(M, M)$  as an algebra over  $\mathbb{k}$ . The relations for these generators are  $f^2 = 0$  and  $\mu f + qf\mu = 0$ . One can then show that the graded center of  $\text{Ext}_R^*(M, M)$  is just the one dimensional  $\mathbb{k}$ -space spanned by  $1_M$ .

Our next aim is to describe another setting where having non-nilpotent elements in the Ext-algebra of a simple module implies the existence of non-nilpotent elements of positive degree in the Hochschild cohomology ring of the algebra. We start with the following preparatory result.

**Proposition 4.9.** *Let  $S$  be a graded algebra such that  $S$  is a finitely generated module over  $Z_{\text{gr}}(S)$ . Assume that  $S_{\geq 1}$  has a non-nilpotent element. Then there exists a non-nilpotent element in  $Z_{\text{gr}}(S)_{\geq 1}$ .*

**Proof.** Let  $\{f_1, \dots, f_n\}$  be a finite set of homogeneous generators for  $S$  as a module over  $Z_{\text{gr}}(S)$ . Let  $m = \max\{\deg(f_i)\}_{i=1}^n$ . Let  $z$  be a non-nilpotent element in  $S_{\geq 1}$ . Choose  $t$  such that  $\deg(z^t) > m$ . Then

$$z^t = f_1x_1 + \dots + f_nx_n$$

for some  $x_i$  in  $Z_{\text{gr}}(S)$  with all the nonzero  $x_i$  in  $Z_{\text{gr}}(S)_{\geq 1}$ . Then, for each  $j \geq 1$ , we have  $z^{tj} \neq 0$  and it can be written as

$$z^{tj} = \sum_{i_1, i_2, \dots, i_n} w_{i_1, i_2, \dots, i_n}(f_1, \dots, f_n)x_1^{i_1}x_2^{i_2} \dots x_n^{i_n}$$

where  $w_{i_1, i_2, \dots, i_n}(f_1, \dots, f_n)$  are linear combinations of words over  $\mathbb{Z}$  in the set  $\{f_1, \dots, f_n\}$ . Since  $j$  can be arbitrary large, it follows that there exists some  $k$  such that  $x_k$  is non-nilpotent in  $Z_{\text{gr}}(S)_{\geq 1}$ . This completes the proof.  $\square$

Letting  $S = \mathcal{E}_R$  be the Ext-algebra of  $R$ , the next result follows immediately, using the fact that there is a surjective homomorphism from the Hochschild cohomology ring of a Koszul algebra  $R$  onto the graded center of the Koszul dual  $\mathcal{E}_R$ .

**Corollary 4.10.** *Let  $R$  be a Koszul algebra such that  $\mathcal{E}_R$  is a finitely generated  $Z_{\text{gr}}(\mathcal{E}_R)$ -module. If there exists a non-nilpotent element in  $(\mathcal{E}_R)_{\geq 1}$ , then exists a non-nilpotent element of positive degree in the Hochschild cohomology ring of  $R$ .  $\square$*

Let  $M$  be an indecomposable linear module over a Koszul algebra  $R$ . We have seen in [Proposition 3.7](#) that if  $n \geq 1$ , and if  $\eta: P_M^n \rightarrow M$  represents an element of  $\Delta_M^n$  with the property that one of its liftings is a monomorphism, then all subsequent liftings are also monomorphisms. We have the following:

**Theorem 4.11.** *Let  $R$  be a Koszul algebra and let  $M$  be an indecomposable linear  $R$ -module. Let  $n \geq 1$  and let  $\eta: P_M^n \rightarrow M$  represent an element of  $\Delta_M^n$ . Assume that for some  $j$ , the lifting  $l^j$  of  $\eta$  having entries in  $R_0$  is a monomorphism. Then there exists a positive integer  $n_0$  such that there is an isomorphism  $P_M^{n_0n+j} \rightarrow P_M^{kn+j}$  of degree  $-kn + n_0n$  for each  $k \geq n_0$ . Consequently,  $M$  is eventually periodic. In particular, if  $R$  is also selfinjective, then  $M$  is periodic of period dividing  $n$ .*

**Proof.** Using [Proposition 3.7](#) and the notation therein, each lifting  $l^m: P_M^{n+m} \rightarrow P_M^m$  for  $m \geq j$  is a monomorphism of degree  $-n$ , and therefore each induced map  $f^{m+1}: \Omega^{n+m+1}M \rightarrow \Omega^{m+1}M$  is also a monomorphism of degree  $-n$  for  $m \geq j$ . We have for each  $k \geq 1$ , chains of monomorphisms of degree  $-n$  given by our liftings:

$$P_M^{kn+j} \hookrightarrow P_M^{(k-1)n+j} \hookrightarrow \dots \hookrightarrow P_M^{n+j} \hookrightarrow P_M^j,$$

and

$$\Omega^{kn+j}M \hookrightarrow \Omega^{(k-1)n+j}M \hookrightarrow \dots \hookrightarrow \Omega^{n+j}M \hookrightarrow \Omega^jM.$$

In the finite dimensional case, the result is now obvious, so we prove the general case. By Lemma 3.4 we have that

$$P_M^{(k-1)n+j} \simeq P_M^{kn+j} \oplus P'(k)$$

for each  $k \geq 1$ . Since the top degree of all the modules  $P_M^{kn+j}$  is finite dimensional, it follows that for some positive integer  $n_0$  the module  $P'(k)$  is zero for all  $k \geq n_0$ , and that we have an isomorphism  $P_M^{n_0n+j}((k - n_0)n) \cong P_M^{kn+j}$  for each  $k \geq n_0$ . But  $l^{n_0n+j}$  being an isomorphism of degree  $-n$  implies that  $f^{s-n} : \Omega^s M \rightarrow \Omega^{s-n} M$  is a monomorphism for  $s \geq n_0n + j$ . By applying the Snake Lemma to

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^{(n_0+1)n+j+1} M & \longrightarrow & P_M^{(n_0+1)n+j} & \longrightarrow & \Omega^{(n_0+1)n+j} M \longrightarrow 0 \\ & & \downarrow & & \downarrow l^{n_0n+j} & & \downarrow \\ 0 & \longrightarrow & \Omega^{n_0n+j+1} M & \longrightarrow & P_M^{n_0n+j} & \longrightarrow & \Omega^{n_0n+j} M \longrightarrow 0 \end{array}$$

we see that  $\Omega^{(n_0+1)n+j} M \rightarrow \Omega^{n_0n+j} M$  is both a monomorphism and an epimorphism. It follows that  $M$  is eventually periodic. If  $R$  is selfinjective,  $M$  being eventually periodic implies that there is a degree  $-n$  isomorphism  $\Omega^n M \rightarrow M$  so  $M$  is in fact periodic.  $\square$

Let us also recall the following well-known result.

**Lemma 4.12.** *Let  $R = R_0 \oplus R_1 \oplus R_2 \oplus \dots$  be a positively  $\mathbb{Z}$ -graded algebra with  $R_0 = \mathbb{k} \times \dots \times \mathbb{k}$  and let  $M$  and  $N$  be finitely generated graded modules generated in degree 0. Suppose  $f : M \rightarrow N$  is an  $R$ -isomorphism and write  $f = \sum_i f_i$ , where  $f_i : M \rightarrow N$  is a degree  $i$  homomorphism from  $M$  to  $N$ . Then  $(f_0)|_{M_0} : M_0 \rightarrow N_0$  is an isomorphism.  $\square$*

Let  $R$  be a Koszul algebra and let  $M$  be a linear module. Let  $(P_M^\bullet, d_M^\bullet)$  be a linear resolution of  $M$ , and let  $n \geq 1$ . Denote by  $\pi_n$  the projection of  $P_M^n$  onto  $\Omega^n M$  induced by the differential  $d_M^n$ . It is clear that having a map  $\eta : P_M^n \rightarrow M$  such that  $\eta d_M^{n+1} = 0$  is equivalent to having a map  $\bar{\eta} : \Omega^n M \rightarrow M$  such that  $\bar{\eta} \pi_n = \eta$ . So we may think of  $\eta$  and  $\bar{\eta}$  as representing the same element in  $\text{Ext}_R^n(M, M)$ . We can summarize our discussion:

**Theorem 4.13.** *Let  $R$  be a Koszul algebra and let  $M$  be an indecomposable linear  $R$ -module of infinite projective dimension. Consider the following statements:*

- (1) *The module  $M$  is periodic.*
- (2) *There exist  $n \geq 1$  and  $\eta \in \Delta_M^n$  such that some lifting of  $\eta$  is an isomorphism.*
- (3) *There exist  $n \geq 1$  and  $\eta \in \Delta_M^n$  such that some lifting of  $\eta$  is a monomorphism.*
- (4) *The module  $M$  is eventually periodic.*

Then

$$(1) \implies (2) \iff (3) \implies (4)$$

Moreover, if  $R$  is selfinjective, then (1)–(4) are all equivalent.

**Proof.** To see that (1) implies (2), assume that  $f : \Omega^n M \rightarrow M$  is an isomorphism of  $R$ -modules for some  $n \geq 1$ . Shifting and applying Lemma 4.12, we see that  $(f_0)|_{(\Omega^n M)_n} : (\Omega^n M)_n \rightarrow M_n$  is an isomorphism of

degree  $-n$ . Now  $f_0$  represents an element  $\eta \in \Delta_M^n$ . Since the restriction  $(f_0)|_{(\Omega^n M)_n} : (\Omega^n M)_n \rightarrow M_n$  is an isomorphism, the matrix of the first lifting,  $l^0 : P_M^n \rightarrow P_M^0$  is an isomorphism.

It is clear that (2) implies (3). On the other hand, the proof of [Theorem 4.11](#) shows that (3) implies (2). [Theorem 4.11](#) also shows that (3) implies (4).

Finally, if  $R$  is selfinjective, (4) implies (1) and the proof is complete.  $\square$

We have the following consequence.

**Corollary 4.14.** *Let  $R$  be a Koszul algebra and let  $M$  be an indecomposable linear  $R$ -module. Consider the following two properties for the module  $M$ :*

- (i)  $M$  is eventually periodic.
- (ii)  $\Delta_M^n \neq 0$  for some  $n > 0$ .

Then we have the following:

- (1) (i) implies (ii).
- (2) If all Betti numbers of  $M$  are equal to 1, then (i) and (ii) are equivalent.

**Proof.** (1) Assume that  $M$  is eventually periodic of period  $n > 0$ , so there is an integer  $n_0$  such that there is a degree  $-n$  isomorphism  $\Omega^{n_0+n} M \rightarrow \Omega^{n_0} M$ . By looking at  $\Omega^{n_0} M$  instead of  $M$  we may assume, without loss of generality, that  $M$  is itself periodic of period  $n$ . So we have a commutative diagram

$$\begin{array}{ccc}
 P_M^n & \longrightarrow & \Omega^n M \\
 \downarrow l & & \downarrow f \\
 P_M^0 & \xrightarrow{d_M^0} & M
 \end{array}$$

where  $f$  and  $l$  are degree  $-n$  isomorphisms. In fact,  $l$  can be viewed as a matrix with entries in  $R_0$ . Letting  $\eta = d_M^0 l$  and  $l^0 = l$  we see that  $\eta \in \Delta_M^n$  and is nonzero.

(2) By (1) we have that (i) implies (ii). So assume now that there exists a nonzero element  $\eta$  of degree  $n > 0$  in  $\Delta_M$ . From the preceding results, each lifting  $l^i$  of  $\eta$  is a matrix with entries in  $R_0$ , and the Betti numbers of  $M$  being equal to 1 imply that  $l^0$  is an isomorphism. Therefore  $M$  is eventually periodic by [Theorem 4.13](#).  $\square$

The following result requires an element of  $\Delta_M^n$  all of whose liftings are nonzero. Recall that if  $R$  is a selfinjective Koszul algebra, then, by [Corollary 3.3](#), if for some  $n \geq 1$  there exists a nonzero  $\eta \in \Delta_M^n$ , then all the liftings  $l^i$  of  $\eta$  are nonzero.

**Proposition 4.15.** *Let  $R$  be a Koszul algebra and let  $M$  be an indecomposable linear  $R$ -module such that the  $i$ -th Betti number of  $M$  is 1, for some  $i \geq 1$ . If, for some  $n < i$ ,  $\Delta_M^n$  contains an element all of whose liftings are nonzero, then  $M$  is eventually periodic. If, in addition,  $R$  is selfinjective and  $\Delta_M^n \neq 0$  for some  $n < i$ , then  $M$  is periodic of period dividing  $n$ .*

**Proof.** Suppose that  $\eta$  is a nonzero element of  $\Delta_M^n$  and  $n < i$ . Let  $(P_M^\bullet, d_M^\bullet)$  be a linear resolution of  $M$  and let  $l^j$  be liftings of  $\eta$  as above. Let  $s = i - n$ . Then  $l^s : P_M^{s+n} \rightarrow P_M^s$  and  $P_M^{s+n} = P_M^i$  has rank 1. Since the liftings of  $\eta$  are assumed to be nonzero,  $l^s : P_M^i \rightarrow P_M^s$  is a monomorphism. The result now follows from [Theorem 4.11](#).  $\square$



We have the following example:

**Example 4.16.** First assume that  $M$  is a linear module over a connected Koszul algebra such that the Betti numbers of  $M$  are eventually 1. Let  $n_0$  be such that the rank of each  $P_M^k$  is 1 for each  $k \geq n_0$ . Let  $n \geq n_0$  and let  $0 \neq \eta \in \Delta_M^n$ . We may think of  $\eta$  as being a degree  $-n$  map from  $\Omega^n M$  to  $M$  that does not factor through a projective module and which is given by a matrix whose entries are all in  $\mathbb{k}$ . But each syzygy  $\Omega^n M$  of  $M$  is cyclic so  $\eta$  has to be onto as it maps  $(\Omega^n M)_n$  onto  $M_n$ . In other words,  $\Delta_M^n \neq 0$  if and only if there is an onto map  $\Omega^n M \rightarrow M$ .

Let us apply this to the following situation. Let  $R = \mathbb{k}\langle x, y \rangle / \langle x^2, xy - qyx, y^2 \rangle$  where  $0 \neq q \in \mathbb{k}$ . Let  $\bar{x}$  and  $\bar{y}$  denote the residue classes in  $R$  of  $x$  and  $y$  respectively, and let  $M$  be the cokernel of the multiplication by  $\bar{x} + \bar{y}$  viewed as a map  $R \rightarrow R$ . A linear resolution of  $M$  is given by the following:

$$\dots \rightarrow R(-n) \xrightarrow{\bar{x} + (-q)^n \bar{y}} R(-n + 1) \xrightarrow{\bar{x} + (-q)^{n-1} \bar{y}} \dots \rightarrow R(-1) \xrightarrow{\bar{x} + \bar{y}} R \rightarrow M \rightarrow 0.$$

This means that for each  $n \geq 1$ ,  $\Omega^n M \cong \text{coker}(\bar{x} + (-q)^n \bar{y})$ . If  $q$  is not a root of unity, then  $\Omega^n M \not\cong \Omega^m M$  for all  $m \neq n$ , and they are all two-dimensional so  $\Delta_M^n = 0$  for all  $n > 0$ . Hence  $\Delta_M = \mathbb{k}$  in this case. If  $(-q)^n = 1$ , then  $\Omega^n M \cong M$  and it is easy to see that  $\Delta_M$  is infinite dimensional, but finitely generated as a  $\mathbb{k}$ -algebra by its degree  $0 \leq k \leq n$  parts.

Let  $R = R_0 \oplus R_1 \oplus \dots$  be a Koszul  $\mathbb{k}$ -algebra where  $R_0 = \mathbb{k} \times \dots \times \mathbb{k}$ . Let  $M$  and  $N$  be linear  $R$ -modules. In an analogous way to the diagonal subalgebra which was introduced in Section 2, we define the  $n$ -th graded part  $\Delta^n(M, N)$  of the diagonal module  $\Delta(M, N)$  of  $\text{Ext}_R^*(M, N)$  as

$$\Delta^n(M, N) = \text{Ext}_R^n(M, N)_{-n}.$$

Then the whole diagonal module  $\Delta(M, N)$  of  $\text{Ext}_R^*(M, N)$  is defined as

$$\Delta(M, N) = \bigoplus_{n \geq 0} \Delta^n(M, N).$$

Using this definition and the results on liftings from Section 3, we show that having some Betti numbers equal to 1 forces further occurrences of 1 as a Betti number, in some cases.

**Proposition 4.17.** *Let  $R$  be a Koszul algebra. Let  $M$  and  $N$  be two linear  $R$ -modules with the following properties:*

- (i) *no syzygy of  $N$  has a nonzero projective direct summand,*
- (ii) *there exists an element  $\eta$  in  $\Delta^i(M, N)$  for some  $i \geq 0$  and there exists an element  $\theta$  in  $\text{Ext}_R^n(N, L)$  for some  $n \geq 1$  and some linear  $R$ -module  $L$ , such that the Yoneda product  $\theta\eta \neq 0$ ,*
- (iii)  *$\beta_m(M) = \beta_n(N) = 1$  for some  $m$  with  $i \leq m < n + i$ .*

Then  $\beta_{m-i}(N) = 1$ .

**Proof.** Let

$$P_M^\bullet: \dots \rightarrow P_M^1 \rightarrow P_M^0 \rightarrow M \rightarrow 0$$

and

$$P_N^\bullet: \dots \rightarrow P_N^1 \rightarrow P_N^0 \rightarrow N \rightarrow 0$$

be linear projective resolutions of  $M$  and  $N$ , respectively. Suppose  $\eta$  in  $\Delta^i(M, N)$  is represented by  $\eta: P_M^i \rightarrow N$ , and  $\theta$  in  $\text{Ext}_R^n(N, L)$  is represented by  $\theta: P_N^n \rightarrow L$ , where we assume that the Yoneda product  $\theta\eta$  is nonzero. The element  $\theta\eta$  can be represented by the composition  $P_M^{n+i} \xrightarrow{\ell^n(\eta)} P_N^n \xrightarrow{\theta} L$ . Since the product  $\theta\eta$  is nonzero, the above composition of maps is nonzero, and in particular,  $\ell^n(\eta)$  is nonzero. Since  $\eta$  is in  $\Delta^i(M, N)$ , all the liftings  $\ell^j(\eta)$  are given by matrices in  $R_0$ . By assumption  $\beta_n(N) = 1$ , so we infer that  $\ell^n(\eta)$  is surjective. By Proposition 3.8 all the previous liftings  $\ell^j(\eta)$  for  $j = n - 1, n - 2, \dots, 1, 0$  are also surjective. In particular  $\ell^{m-i}(\eta): P_M^m \rightarrow P_N^{m-i}$  is surjective where  $m - i$  is in the set  $\{0, 1, \dots, n - 1\}$ . Since  $\beta_m(M) = 1$ , it follows that  $\beta_{m-i}(N) = 1$ .  $\square$

We have the following result:

**Theorem 4.18.** *Let  $R = R_0 \oplus R_1 \oplus \dots$  be an indecomposable Koszul algebra. Assume that all the simple modules have infinite projective dimension and that no syzygy of a simple module has a nonzero projective direct summand. Assume also, that for each simple module, 1 occurs at least twice as a Betti number. Then every simple module has a syzygy which is simple and periodic.*

**Proof.** Fix a simple module  $S$ . By assumption there exists some  $n \geq 1$  such that  $\beta_n(S) = 1$ . Let  $T$  be a simple  $R$ -module such that  $\text{Ext}_R^n(S, T) \neq (0)$ . Let

$$P_S^\bullet: \dots \rightarrow P_S^1 \rightarrow P_S^0 \rightarrow S \rightarrow 0$$

and

$$P_T^\bullet: \dots \rightarrow P_T^1 \rightarrow P_T^0 \rightarrow T \rightarrow 0$$

be linear projective resolutions of  $S$  and  $T$ , respectively. Let  $\eta$  be a nonzero element in  $\text{Ext}_R^n(S, T)$ . Note that  $\eta \in \Delta^n(S, T)$  since  $T$  is a simple module. We can represent  $\eta$  as a homomorphism denoted again by  $\eta: P_S^n \rightarrow T$ , and let  $l^i: P_S^{n+i} \rightarrow P_T^i$  be the liftings of  $\eta$  as shown in the following diagram:

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & P_S^{n+i} & \longrightarrow & \dots & \longrightarrow & P_S^{n+1} & \longrightarrow & P_S^n & & \\
 & & \downarrow l^i & & & & \downarrow l^1 & & \downarrow l^0 & \searrow \eta & \\
 \dots & \longrightarrow & P_T^i & \longrightarrow & \dots & \longrightarrow & P_T^1 & \longrightarrow & P_T^0 & \longrightarrow & T
 \end{array}$$

Since  $\eta \in \Delta^n(S, T)$ , we may assume that each lifting is given by a matrix with entries in  $R_0$ . The lifting  $l^0$  is surjective, since  $\eta$  is nonzero. Since  $\beta_n(S) = 1$ ,  $l^0$  is also a monomorphism and therefore it is an isomorphism. Then it follows from Proposition 3.7 that all the liftings  $l^j$  for  $j \geq 1$  are monomorphisms (and thus nonzero). By assumption there exists some  $m \geq 1$  such that the Betti number  $\beta_m(T) = 1$ . Hence the lifting  $l^m$  is surjective and therefore all previous liftings  $l^0, l^1, \dots, l^{m-1}$  are surjective from Proposition 3.8. Hence all the liftings  $l^0, l^1, \dots, l^{m-1}, l^m$  are isomorphisms. Since  $m \geq 1$ , it follows that  $\Omega^n S \cong T$ . The number of simple modules is finite (up to shift) and the above is true for any simple module, so we can conclude that each simple module have a simple periodic syzygy.  $\square$

It is possible that all simple modules of an indecomposable (even local) selfinjective algebra are periodic but not all their Betti numbers are equal to 1. In fact, if the algebra is additionally Koszul, then that is the case except in very few examples, as the following consequence of the previous theorem shows:

**Proposition 4.19.** *Let  $R$  be a selfinjective Koszul algebra where all the simple modules are periodic. Then:*

- (1) *all the Betti numbers for the simple modules are 1.*
- (2)  *$R \cong \mathbb{k}\tilde{\mathbb{A}}_m/J^2$  for some  $m \geq 1$ , where  $\tilde{\mathbb{A}}$  has circular orientation and  $J$  is the ideal generated by the arrows. In particular, all simple modules have the same period.*

**Proof.** (1) Let  $n \geq 1$  be minimal such that  $\Omega^n T \cong T$  for all simple modules  $T$ . Let  $S$  be a simple  $R$ -module, and let

$$\eta: 0 \rightarrow S \rightarrow P_S^{n-1} \rightarrow \dots \rightarrow P_S^1 \rightarrow P_S^0 \rightarrow S \rightarrow 0$$

be exact with  $P_S^i$  projective. First we claim that proving the following is enough for the first part:

Assume that for a given  $i$  with  $1 \leq i \leq n - 1$ , there exists  $t \geq 1$  and a simple module  $T$  such that  $\eta^t \cdot \text{Ext}_R^{n-i}(T, S) \neq (0)$ .

Suppose this is proven. Then we can recycle the arguments from Proposition 4.17, and we obtain that  $\beta_i(S) = 1$ . Since we can let  $i$  vary freely in the interval  $[1, \dots, n - 1]$ , we infer that all the Betti numbers for  $S$  are 1 and the claim in (1) follows.

Now we prove the above claim. The element  $\eta$  is non-nilpotent in  $\text{Ext}_R^*(S, S)$ , and we have

$$0 \neq \eta^{t+1} = \eta^t \cdot \underbrace{(0 \rightarrow S \rightarrow P_S^{n-1} \rightarrow \dots \rightarrow P_S^{n-i} \rightarrow \Omega^{n-i} S \rightarrow 0)}_{\theta_{n-i}} \cdot \underbrace{(0 \rightarrow \Omega^{n-i} S \rightarrow P_S^{n-i-1} \rightarrow \dots \rightarrow P_S^0 \rightarrow S \rightarrow 0)}_{\theta'_i}$$

for all  $t \geq 0$ . In particular,  $\eta^t \cdot \theta_{n-i} \neq 0$  for all  $t \geq 1$  and for all  $1 \leq i \leq n - 1$ .

We want to show  $\eta^t \cdot \text{Ext}_R^{n-i}(R_0, S) \neq (0)$  for some  $t \geq 1$ . So suppose for contradiction that  $\eta^t \cdot \text{Ext}_R^{n-i}(R_0, S) = (0)$  for all  $t \geq 1$ . Then we claim the following:

$$\eta^u \cdot \text{Ext}_R^{n-i}(B, S) = (0) \text{ for all modules } B \text{ of Loewy length at most } u.$$

For  $u = 1$  it is true by assumption.

Assume the claim is true for  $u$ , and let  $B$  be a module of Loewy length  $u + 1$ . Consider the exact sequence

$$0 \rightarrow B\mathfrak{r} \rightarrow B \rightarrow B/B\mathfrak{r} \rightarrow 0.$$

This gives rise to the exact sequences

$$\text{Ext}_R^{jn-i}(B/B\mathfrak{r}, S) \xrightarrow{\delta_j} \text{Ext}_R^{jn-i}(B, S) \xrightarrow{\epsilon_j} \text{Ext}_R^{jn-i}(B\mathfrak{r}, S)$$

for  $j \geq 1$ . Let  $\nu$  be an element in  $\text{Ext}_R^{n-i}(B, S)$ . Then by assumption

$$\epsilon_{u+1}(\eta^u \cdot \nu) = \eta^u \cdot \epsilon_1(\nu) = 0,$$

so that  $\eta^u \cdot \nu = \delta_{u+1}(\mu)$  for some  $\mu$  in  $\text{Ext}_R^{(u+1)n-i}(B/B\mathfrak{r}, S) \cong \text{Ext}_R^{n-i}(B/B\mathfrak{r}, S)$ , since  $\Omega^n(B/B\mathfrak{r}) \cong B/B\mathfrak{r}$ . Hence  $\eta \cdot \mu = 0$  and therefore  $\eta^{u+1} \cdot \nu = 0$ . This completes the proof of the above claim.

Hence, if  $\eta^t \cdot \text{Ext}_R^{n-i}(R_0, S) = (0)$  for all  $t \geq 1$ , then  $\eta^N \cdot \text{Ext}_R^{n-i}(X, S) = (0)$  for all finitely generated  $R$ -modules  $X$  where  $N$  is the Loewy length of  $R$ . However, this is not true for  $X = \Omega_R^{n-i}S$ , so this gives a contradiction. This completes the proof of (1).

(2) Since  $R_0 = \mathbb{k} \times \cdots \times \mathbb{k}$  and  $R$  is a Koszul algebra, we have that  $R \cong \mathbb{k}Q/I$  where  $Q$  is a finite quiver and  $I$  is a quadratic admissible ideal in the path algebra  $\mathbb{k}Q$ . The fact that the two first Betti numbers are 1 for all simple modules, implies that there is exactly one arrow starting at each vertex in  $Q$ . Since  $R$  is indecomposable and all simple modules have infinite projective dimension, the quiver  $Q$  must be  $\tilde{A}_m$  for some  $m \geq 1$  and there is a relation starting at each vertex. Since all relations are quadratic, they are generated by all paths of length 2. Consequently,  $R \cong \mathbb{k}\tilde{A}_m/J^2$  for some integer  $m \geq 1$ , where  $J$  is the ideal generated by the arrows. It is easy to see that all the simple modules have period  $m$  in this case.  $\square$

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