ORDER STATISTICS FOR A DISCRETE PARENT DISTRIBUTION

by

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TABLE OF CONTENTS

I. INTRODUCTION

1.1 Background

Considerable amount of research has been done in order statistics where the underlying population is continuous. The book by Sarhan and Greenberg (1962) covers most major areas of work in the continuous case. Although often the distribution is appropriately regarded as continuous, there are cases where the real distribution is discrete or when due to rounding of the data we are dealing in fact with a discrete case. For a continuous parent distribution the probability of equality of two or more variates is zero, and the ordering is unique with probability one. In the discrete case we have possible lack of a unique ordering leading to ties. The literature touches very little on the problem of order statistics in the discrete case, although this case is applicable to many pratical problems.

It is the aim of this thesis to unify the distribution theory for continuous and discrete populations and to point out the analogy between the results in the two cases. Also, we derive many recurrence relations between moments of order statistics in the discrete case and establish the same results as given before in the continuous case. In such situations there is no need for the assumption that the underlying distribution is continuous, and these relations are in fact valid for both the continuous and discrete cases.

We now review briefly previous work dealing with order statistics for a discrete parent.

Abdel-Aty (1954) gives results for the distribution of the rth order statistic, the extreme ordered variable, the distribution of the range, and the joint distribution of two order statistics $X_{(i)}$, $X_{(j)}$ for $j > i$ (where $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ are the variates arranged in ascending order of magnitude). He also obtained as a corollary the distribution of range for samples from a discrete rectangular population, previously given by Rider (1951).

Burr (1955) deals with the distribution of range from a discrete population.

Siotani (1957) gives results for the distribution, mean and variance of the extremes and of the range in the discrete case.

Khatri (1962) derives a simple formula for the joint distribution of the $X_{(i)}$ and $X_{(j)}$ $(j > i)$ for discrete populations.

Melnick (1964) derives a few recurrence relations between moments of order statistics in the discrete case and considers moments of ranked poisson variates.

1.2 Summary of Contents

In Chapter II of this thesis we find the distribution of the rth order statistic $X_{(r)}$ by first deriving the cumulative probability distribution from first principles, from which the probability distribution comes directly. We note that the c.d.f. of $X_{(r)}$ is the same for continuous and discrete parents. This approach is easier than the usual method, of first deriving the p.d.f., and then the c.d.f.

Also, to find the joint distribution of the $X_{(r)}$, $X_{(s)}$ $(s > r)$, we first derive the c.d.f. The advantage of our procedure is that this formula for the c.d.f. can be easily computed.

To get the moments of $X_{(r)}$, we use convenient formulae involving the tails of the c.d.f. of $X_{(r)}$ rather than the p.d.f. of $X_{(r)}$, the former being readily derived from general results for discrete distributions. This procedure gives us directly the factorial moments from which the raw moments are deduced. Applications to three discrete distributions are given.

We consider some results on certain uncorrelated statistics and prove the validity of these results in the discrete case as well as in the continuous case.

In Chapter III we derive many recurrence relations between moments of order statistics in the discrete case and get the same results as previously given by Govindarajulu (1963) in the continuous case.

Theorems 5 and 7 enable us to express the moments of order statistics in a sample of N in terms of the moments of order statistics in a sample of M where $N \geq M$ and $N \leq M$ respectively.

Theorem 6 is useful in giving the moments of $X_{(r)}$ in terms of the simpler moments of the extremes.

Several relations between the moments of the range are derived for a discrete parent leading to the same results as established by Sillitto (1951, 1964) in the continuous case.

Simple recurrence formulae among the expected values of sample quasi-ranges from an arbitrary population are listed which were derived by Govindarajulu (1963), and are again valid for both continuous and discrete populations.

II. DISTRIBUTION THEORY AND MOMENTS OF ORDER STATISTICS

2.1 Definitions

Let X be a discrete random variable which may take the values $x_0, x_1, x_2, \ldots$, with probabilities $p(x_0), p(x_1), p(x_2), \ldots$, respectively.

In the usual case where the x's are the integers 0, 1, 2, ..., we require

$$p(x) \geq 0 \quad \text{and} \quad \sum_{x=0}^{\infty} p(x) = 1 .$$

When x takes only the finite number of values 0, 1, 2, ..., a we interpret

$$p(a+b) = 0 \quad \text{for} \quad b = 1, 2, \ldots$$

In a sample of size N we order the X's in ascending order of magnitude so that

$$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)} \leq \cdots \leq X_{(N)} .$$

Let

$P(X=x) = p(x)$ the discrete probability mass function.

$P(X \leq x) = P(x)$ the corresponding c.d.f.

$P(X > x) = 1-P(x)$.

$$\Delta_{y,x} = P(y) - P(x) \qquad y \geq x . \tag{2.1.1}$$

$P(x-1) = 0$ when $x = 0$.

$$P(x) = \sum_{i=0}^{x} p(i) ,$$

$p_r(x)$ is the probability mass function of the rth order statistic.

$$P_r(x) = \sum_{i=0}^{x} p_r(i) \qquad , \; p_r(x) = P_r(x) - P_r(x-1).$$

$\mu'_r = E(X^r)$ is the rth raw moment of x.

$\mu'_1 = \mu = E(X)$.

$\mu_k = E(X-E(X))^k$ is the kth moment about the mean.

$\mu_{[k]} = E[X(X-1) \ldots (X-k+1)]$ is the kth factorial moment.

$p_{r,s}(x,y) = P\{X_{(r)} = x, \; X_{(s)} = y\}$ the joint distribution of the rth and sth ordered variables $(r < s)$.

$P_{r,s}(x,y) = P\{X_{(r)} \leq x, X_{(s)} \leq y\}$ the corresponding c.d.f.

2.2 Distribution of the rth order statistic

We shall start first by deriving the c.d.f. of the rth order statistic:

$$P_r(x) = \Pr \{\text{at least } r \text{ of the } X_i \text{ are less than or equal to } x\}$$

$$= \sum_{i=r}^{N} \binom{N}{i} P^i(x) \, [1 - P(x)]^{N-i}$$

$$= I_{P(x)} (r, N-r+1) . \qquad 2.2.1$$

This is the incomplete B-function and can easily be evaluated from tables (K. Pearson, 1934).

Note that result (2.2.1) is the same as the corresponding result in the continuous case.

Hence,

$$p_r(x) = I_{P(x)}(r, N-r+1) - I_{P(x-1)}(r, N-r+1) \tag{2.2.2}$$

$$r = 1, 2, \ldots, N \text{ and } x = 0, 1, 2, \ldots, a .$$

Equation (2.2.2) is directly used when $r \geq \frac{1}{2}(N+1)$, while if $r < \frac{1}{2}(N+1)$, then

$$p_r(x) = I_{1-P(x-1)}(N-r+1, r) - I_{1-P(x)}(N-r+1, r) \tag{2.2.3}$$

will be the appropriate formula for computational work. On the other hand, $p_r(x)$ can be written as follows:

Suppose in a given sample of N we have

ℓ observations each less than a given value x

m " " equal to the given value x

$N-\ell-m$ " " greater than the given value x

Then the probability of obtaining a sample so classified is

$$\frac{N!}{\ell!\, m!\, (N-\ell-m)!} \quad [P(x-1)]^{\ell}\, p^{m}(x)\, [1 - P(x)]^{N-\ell-m} . \tag{2.2.4}$$

Therefore

$$p_r(x) = \sum_{\ell=0}^{r-1} \sum_{m=r-\ell}^{N-\ell} \frac{N!}{\ell!\, m!\, (N-\ell-m)!} \quad [P(x-1)]^{\ell}\, p^{m}(x)$$

$$\cdot\ [1-P(x)]^{N-\ell-m} \quad \text{for} \quad x = 0, 1, 2, \ldots, a\ . \tag{2.2.5}$$

Using (2.2.1) and tables of the Incomplete Beta-Function, percentage points for $x_{(r)}$ can be obtained.

As a special case of (2.2.1); for the distributions of the extremes, we have

$$P_N(x) = P^N(x)\ . \tag{2.2.6}$$

$$P_1(x) = 1 - [1 - P(x)]^N\ . \tag{2.2.7}$$

So that,

$$p_N(x) = P^N(x) - [P(x-1)]^N\ . \tag{2.2.8}$$

$$p_1(x) = [1 - P(x-1)]^N - [1 - P(x)]^N\ . \tag{2.2.9}$$

2.3 Joint distribution of two or more order statistics

The joint distribution of the rth and sth ordered variables ($r < s$) can be obtained as follows (Abdel-Aty, 1954)

Suppose a sample of N is classified so that there are

ℓ	observations	each	less than a given value x
m	"	"	equal to the given value x
k	"	"	greater than x and less than y
t	"	"	equal to y
$N-\ell-m-k-t$	"	"	greater than y

The probability of obtaining such a sample is

$$\frac{N!}{\ell!\, m!\, k!\, t!\, (N-\ell-m-k-t)!}\; P^{\ell}(x-1)\; p^{m}(x)\; \Delta^{k}_{y-1,x}\; p^{t}(y)$$

$$\cdot\,[1-P(y)]^{N-\ell-m-k-t} \,. \qquad (2.3.1)$$

If the ordered variables $X_{(r)}$ and $X_{(s)}$ are equal to x and y respectively, then

$$p_{r,s}(x,y) = P\{X_{(r)} = x,\ X_{(s)} = y\}$$

$$= \sum_{\ell=0}^{r-1} \binom{N}{\ell}\; P^{\ell}(x-1) \sum_{k=0}^{s-r-1} \binom{N-\ell}{k}\; \Delta^{k}_{y-1,x} \sum_{m=r-\ell}^{s-\ell-k-1} \binom{N-\ell-k}{m}\; p^{m}(x)$$

$$\cdot \sum_{t=s-\ell-k-m}^{N-\ell-k-m} \binom{N-\ell-k-m}{t}\; p^{t}(y)\; [1-P(y)]^{N-t-\ell-k-m} \quad (r < s),\ (x < y) \qquad (2.3.2)$$

where $P(x-1) = 0$ when $x = 0$.

$\Delta_{y,x}$ is defined in (2.1.1) and $\Delta^{0}_{u,v} = 1$ all u, v.

When $X_{(r)} = X_{(s)}$, we have

$$P\{X_{(r)} = X_{(s)} = x\} = \sum_{\ell=0}^{r-1} \binom{N}{\ell}\; P^{\ell}(x-1) \sum_{m=r-\ell}^{N-\ell} \binom{N-\ell}{m}$$

$$\cdot\; p^{m}(x)\; [1-P(x)]^{N-m-\ell} \qquad (2.3.3)$$

which is the same result previously derived in (2.2.5) .

Formula (2.3.2) can be written as : (Khatri, 1962)

$$p_{r,s}(x,y) = c_{rs} \sum_{\ell=0}^{r-1} \sum_{k=0}^{s-r-1} \sum_{m=r-\ell}^{s-\ell-k-1} \sum_{t=s-\ell-k-m}^{N-\ell-k-m} \binom{r-1}{\ell} \binom{N-s}{N-t-\ell-k-m}$$

$$\cdot \frac{(s-r-1)!}{k!\,(s-1-m-\ell-k)!\,(m+\ell-r)!} \quad P^{\ell}(x-1)\,\Delta^{k}_{y-1,x}\,p^{m}(x)\,p^{t}(y)$$

$$\cdot \, [1-P(y)]^{N-t-\ell-k-m} \int_0^1 \int_0^1 z^{r-1-\ell}(1-z)^{m+\ell-r}\, u^{t+m+\ell+k-s}$$

$$\cdot \, (1-u)^{s-1-m-\ell-k} \quad d\,z\,d\,u \qquad (2.3.4)$$

where $c_{rs} = \frac{N!}{(r-1)!\,(s-r-1)!\,(N-s)!}$.

Interchanging the summation and the integral signs and putting

$$w = z\,p(x) + P(x-1) \quad , \quad v = P(y) - u\,p(y)$$

we find that

$$p_{r,s}(x,y) = c_{rs} \int_{P(x-1)}^{P(x)} \int_{P(y-1)}^{P(y)} w^{r-1}\,(v-w)^{s-r-1}(1-v)^{N-s}\,d\,v\,d\,w \qquad (2.3.5)$$

$$r < s \, , \quad x < y \, .$$

When $x = y$

we obtain similarly,

$$p_{r,s}(x,y) = c_{rs} \int \int w^{r-1} (v - w)^{s-r-1} (1 - v)^{N-s} \, d\,v\,d\,w$$

where the integration is now over

$$P(x-1) \leq w \leq v \leq P(x).$$

Since in (2.3.5) $x \leq v$ is automatically satisfied, we have the general result:

$$p_{r,s}(x,y) = c_{rs} \int \int w^{r-1} (v - w)^{s-r-1} (1 - v)^{N-s} \cdot d\,v\,d\,w \qquad (2.3.6)$$

and the integration being over $w \leq v$,

$$P(x-1) \leq w \leq P(x) \; , \; P(y-1) \leq v \leq P(y).$$

Note that region of integration does not involve r, s or N.

Also we have

$$P_{r,s}(x,y) = \sum_{r=0}^{x} \sum_{s=r}^{y} p_{r,s}(x,y)$$

$$= c_{rs} \int_{0}^{P(x)} \int_{w}^{P(y)} w^{r-1} (v - w)^{s-r-1} (1 - v)^{N-s} \cdot d\,v\,d\,w$$

$$= c_{rs} \int_{P(x)}^{P(y)} \int_{0}^{P(x)} w^{r-1} (v - w)^{s-r-1} (1 - v)^{N-s} \cdot d\,w\,d\,v$$

$$+ c_{rs} \int_0^{P(x)} \int_0^{v} w^{r-1} (v - w)^{s-r-1} (1 - v)^{N-s} \, d\,w\,d\,v$$

Now in the second term put $w = v\,x$ and simplify, we get

$$P_{r,s}(x,y) = c_{rs} \int_{P(x)}^{P(y)} \int_0^{P(x)} w^{r-1} (v - w)^{s-r-1} (1 - v)^{N-s} \, d\,w\,d\,v$$

$$+ s\binom{N}{s} \int_0^{P(x)} v^{s-1} (1 - v)^{N-s} \, d\,v \,. \tag{2.3.7}$$

Formula (2.3.7) can be compared to the corresponding result for the continuous case.

Generalization is clear, for example

$$P_{r,s,k}(x,y,z) = \frac{N!}{(r-1)!\,(s-r-1)!\,(k-s-1)!\,(N-k)!}$$

$$\int_0^{P(x)} dw \int_w^{P(y)} dv \int_v^{P(z)} w^{r-1} (v - w)^{s-r-1} (u-v)^{k-s-1} (1-u)^{N-k} \, du$$

$$k > s > r. \tag{2.3.8}$$

Formulas (2.3.6) and (2.3.7) are used for theoretical reasons but do not help us for computations. However we can deduce directly the joint c.d.f. from first principles as follows:

We have for $x < y$

$$P_{r,s}(x,y) = \sum_{i=r}^{N} \sum_{j=s-i}^{N-i} \Pr \left\{ \text{exactly } i \ X_i \leq x, \text{ exactly } j \ X_i \text{ with } x < X_i \leq y \right\} \text{ unless } i \geq s, \text{ when } j \text{ starts at zero.}$$

Thus

$$P_{r,s}(x,y) = \sum_{i=r}^{N} \sum_{s=\max(0,s-i)}^{N-i} \frac{N!}{i!\ j!\ (N-i-j)!} \quad P^i(x) \cdot \left[P(y)-P(x)\right]^j \quad [1-P(y)]^{N-i-j} \quad . \tag{2.3.9}$$

Note that formula (2.3.9) can easily be computed. This argument is valid also in the continous case.

The bivariate probability function $p_{r,s}(x,y)$ follows since

$$p_{r,s}(x,y) = P_{r,s}(x,y) - P_{r,s}(x-1,y) - P_{r,s}(x,y-1) + P_{r,s}(x-1,y-1) \ . \tag{2.3.10}$$

In particular we shall derive the joint distribution of $x_{(1)}$ and $x_{(N)}$.

From (2.3.2) we get

$$p_{1,N}(x,y) = \sum_{k=0}^{N-2} \binom{N}{k} \Delta^{k}_{y-1,x} \sum_{m=1}^{N-k-1} \binom{N-k}{m} p^{m}(x) \; \bar{p}^{N-k-m}(y) \quad (2.3.11)$$

So that for $y > x$,

$$p_{1,N}(x,y) = \Delta^{N}_{y,x-1} + \Delta^{N}_{y-1,x} - \Delta^{N}_{y,x} - \Delta^{N}_{y-1,x-1} . \quad (2.3.12)$$

For $x = y$, we have

$$P\left\{X_{(1)} = X_{(N)} = x\right\} = p^{N}(x) . \quad (2.3.13)$$

For samples of $N = 2$ and 3, we have for example

$$\begin{aligned} P\left\{X_{(1)} = x,\ X_{(2)} = y \mid N = 2\right\} &= 2\ p(x)\ p(y),\ y > x \\ P\left\{X_{(1)} = X_{(2)} = x \mid N = 2\right\} &= p^{2}(x) \end{aligned} \quad (2.3.14)$$

and

$$\begin{aligned} P\left\{X_{(1)} = x,\ X_{(3)} = y \mid N = 3\right\} &= 3\ p(x)\ p(y)\left(\Delta_{y,x} + \Delta_{y-1,x-1}\right) \\ P\left\{X_{(1)} = X_{(3)} = x \mid N = 3\right\} &= p^{3}(x). \end{aligned} \quad (2.3.15)$$

2.4 Distribution of the range

Using equations (2.3.12) and (2.3.13) we get for the distribution of range $R = x_{(N)} - x_{(1)}$ (Abdel-Aty, 1954)

$$F(R) = \sum_{x=0}^{a-R} \left\{ \Delta^N_{x+R,\,x-1} + \Delta^N_{x+R-1,\,x} - \Delta^N_{x+R,\,x} - \Delta^N_{x+R-1,\,x-1} \right\} \quad (2.4.1)$$

where $R \geq 1$.

For $R = 0$

$$F(0) = \sum_{x=0}^{a} p^N(x) \; . \quad (2.4.2)$$

The probability that the sample range will be equal to the population range is

$$F(a) = 1 + (1 - p(0) - p(a))^N - (1 - P(0))^N - (1 - p(a))^N \; . \quad (2.4.3)$$

In a symmetrical population, (2.4.1) and (2.4.3) can be reduced to the forms

$$F(R) = \sum_{x=0}^{a-R} \left\{ \Delta^N_{x+R,\,x-1} + \Delta^N_{x+R-1,\,x} - 2\,\Delta^N_{x+R,\,x} \right\} \quad (2.4.4)$$

for $R \geq 1$.

$$F(a) = 1 + (1 - 2p(0))^N - 2\,(1 - p(0))^N. \tag{2.4.5}$$

Example:

Consider the distribution of range in samples from a discrete rectangular population. The set of possible values of the discrete random variable are the consecutive numbers $x = 0, 1, 2, \ldots, a - 1$.

Hence

$$F(0) = \sum_{x=0}^{a-1} \left(\frac{1}{a}\right)^N = \frac{1}{a^{N-1}} \tag{2.4.6}$$

and

$$F(R) = \sum_{x=0}^{a-1-R} \left(\frac{1}{a}\right)^N \left[(R + 1)^N + (R - 1)^N - 2\,R^N \right]$$

$$= \frac{a-R}{a^N} \left[(R + 1)^N + (R - 1)^N - 2\,R^N \right] \tag{2.4.7}$$

and the probability of the sample range being equal to the population range is

$$F(\,a - 1\,) = 1 + \left(\frac{a-2}{a}\right)^N - 2\left(\frac{a-1}{a}\right)^N . \tag{2.4.8}$$

2.5 Moments and product moments of order statistics

For a discrete parent $p(x)$ $(x = 0, 1, 2, \ldots)$ the kth raw moment of $X_{(r)}$ is

$$E(X_{(r)}^k) = \sum_{x=0}^{\infty} x^k \, p_r(x)$$

where $p_r(x) = I_{P(x)}(r, N-r+1) - I_{P(x-1)}(r, N-r+1)$.

Somewhat more convenient formulae involving the tails $(1 - P_r(x))$ rather than $p_r(x)$ are readily derived from general results for discrete distributions.

Let

$$P(X = j) = p(j) \quad , \quad P(X > j) = q(j)$$

Following Feller (1957, p. 249) we have

$$q(x) = p(x+1) + p(x+2) + \ldots, x \geq 0$$

$$= 1 - P(x). \tag{2.5.1}$$

Define the generating functions of the sequences

$$\{p(j)\} \text{ and } \{q(x)\}$$

$$P(s) = \sum_{x=0}^{\infty} p(x)\, s^x,$$

$$Q(s) = \sum_{x=0}^{\infty} q(x)\ s^x \quad .$$

Clearly, for $|s| < 1$, k differentiations of P(s) give

$$P^{(k)}(s) = \sum_{x=k}^{\infty} x \cdots (x - k + 1)\ p(x)\ s^{x-k} \ .$$

If the kth factorial moment $\mu_{[k]}$ of X exists we may set $s = 1$, and have

$$\mu_{[k]} = P^{(k)}(1) \tag{2.5.2}$$

Feller proves that for $|s| < 1$

$Q(s) \ . \ (1-s) = 1-P(s).$

From which on differentiation k times and using Leibnitz's theorem we obtain

$$Q^{(k)}(s) \ .(1 - s) - k \ \ Q^{(k-1)}(s) = -P^{(k)}(s) \quad .$$

When $\mu_{[k]}$ exists we deduce from (2.5.2)

$$\mu_{[k]} = k\ Q^{(k-1)}(1) \quad .$$

In particular

$$\mu_{[1]} = \mu = \sum_{x=0}^{\infty} \left[1 - P(x) \right] .$$

$$\mu_{[2]} = E\left[X(X-1)\right] = 2 \sum_{x=0}^{\infty} x\, q(x) \qquad (2.5.3)$$

$$= 2 \sum_{x=0}^{\infty} x \left[1 - P(x) \right] .$$

and so on.

To apply these results to the moments of $X_{(r)}$ we need only replace $P(x)$ by $P_r(x)$.

For the raw moments we have

$$\mu'_2 = \mu_{[2]} + \mu = 2\,Q'(1) + Q(1) .$$

$$\mu'_3 = \mu_{[3]} + 3\mu'_2 - 2\mu = 3Q''(1) + 6Q'(1) + Q(1).$$

Similarly we can get the higher raw moments.

From the relation $\mu_r = \mu'_r - r\,\mu'_{r-1}\,\mu + \ldots (-)^i \binom{r}{i} \mu'_{r-i}\,\mu^i + \ldots$

$(-)^{r-1}(r-1)\,\mu^r$ we get the moments about the mean.

For example

$$\mu_2 = \mu_2' - \mu^2 \quad = \quad 2\,Q'(1) + Q(1) - Q^2(1)$$

which is the variance .

$$\mu_3 = \mu_3' - 3\mu_2'\,\mu + 2\mu^3$$

$$= 3\,Q''(1) + 6\,Q'(1) - 6\,Q'(1)\,Q(1) + Q(1) - 3\,Q^2(1) + 2\,Q^3(1). \tag{2.5.4}$$

$$\mu_4 = 4\,Q'''(1) + 18\,Q''(1) - 12\,Q''(1)\,Q(1) + 14\,Q'(1) - 24\,Q'(1)\,Q(1) + 12\,Q'(1)\,Q^2(1) + Q(1) - 4\,Q^2(1) + 6\,Q^3(1) - 3\,Q^4(1).$$

So that we can get

$$B_1 = \frac{\mu_3^2}{\mu^3} \quad , \quad \gamma_1 = \frac{\mu_3}{\mu_2^{3/2}} \quad \text{coefficients of skewness .}$$

$$B_2 = \frac{\mu_4}{\mu_2^2} \quad , \quad \gamma_2 = \frac{\mu_4}{\mu_2^2} - 3 \quad \text{coefficients of kurtosis .}$$

First we consider the case when x has a finite range:

$x = 0, 1, \dots , a$.

Since $P(a) = 1$

we have

$$E\left(X_{(r)}\right) = \sum_{x=0}^{a} \left[1 - P_r(x)\right]$$

$$= \sum_{x=0}^{a} \left[1 - I_{P(x)}(r, N - r + 1) \right] \qquad (2.5.5)$$

which is the corresponding form to the formula in the continuous case viz,

$$E\left(X_{(r)}\right) = \int_0^a \left[1 - F_r(x)\right] dx - \int_{-a}^{0} F_r(x)\, dx \;,$$

$$\mathrm{var}\left(X_{(r)}\right) = 2 \sum_{x=0}^{a} x(1 - P_r(x)) + E\left(X_{(r)}\right) \left\{1 - E\left(X_{(r)}\right)\right\}$$

$$= 2 \sum_{x=0}^{a} \left[1 - I_{P(x)} (r, N - r + 1)\right]$$

$$+ E\left(X_{(r)}\right) \left\{1 - E\left(X_{(r)}\right)\right\} \;. \qquad (2.5.6)$$

In general for the t^{th} moment of $X_{(r)}$

$$E\left(X^t_{(r)}\right) = \sum_{x=0}^{a} x^t \left[I_{P(x)} (r, N - r + 1) - I_{P(x-1)}(r, N-r+1) \right].$$

$$= a^t - \Big\{ I_{P(0)} + (2^t - 1)\, I_{P(1)} + (3^t - 2^t)\, I_{P(2)} + \cdots$$

$$+ \left[a^t - (a-1)^t \right] I_{P(a-1)} \Big\} . \qquad (2.5.7)$$

For $t = 3$, for example

$$E\left(X^3_{(r)}\right) = a^3 - \sum_{x=1}^{a} \left[1 + 3x\,(x-1) \right] I_{P(x-1)}(r, N - r + 1) . \qquad (2.5.8)$$

In particular for the moments of the extremes

$$E\left(X_{(1)}\right) = \sum_{x=0}^{a} \left[1 - P_1(x) \right]$$

$$= \sum_{x=0}^{a} \left[1 - P(x) \right]^N . \qquad (2.5.9)$$

$$\mathrm{var}\left(X_{(1)}\right) = 2 \sum_{x=0}^{a} x \left[1 - P(x) \right]^N + E\left(X_{(1)}\right) \left\{ 1 - E\left(X_{(1)}\right) \right\} . \qquad (2.5.10)$$

In general

$$E\left(X^t_{(1)}\right) = \sum_{x=0}^{a} x^t \left\{ \left[1 - P(x-1) \right]^N - \left[1 - P(x) \right]^N \right\} . \qquad (2.5.11)$$

Similarly

$$E\left(X_{(N)}\right) = \sum_{x=0}^{a} \left[1 - P^N(x)\right] . \tag{2.5.12}$$

$$\mathrm{var}\left(X_{(N)}\right) = 2\sum_{x=0}^{a} x\left[1 - P^N(x)\right] + E\left(X_{(N)}\right)\left\{1 - E\left(X_{(N)}\right)\right\} . \tag{2.5.13}$$

In general

$$E\left(X_{(N)}^t\right) = \sum_{x=0}^{a} x^t \left[P^N(x) - P^N(x-1)\right] . \tag{2.5.14}$$

The mean of the range

$$R = x_{(N)} - x_{(1)}$$

can be obtained from (2.5.9) and (2.5.12)

$$\begin{aligned} E(R) &= E\left(X_{(N)}\right) - E\left(X_{(1)}\right) \\ &= \sum_{x=0}^{a}\left[1 - P^N(x) - \left\{1 - P(x)\right\}^N\right] \end{aligned} \tag{2.5.15}$$

which is the corresponding form to Tippett's formula for the continuous case (Tippett, 1925), which states that

$$E(R) = \int_{-a}^{a} \left\{ 1 - P^N(x) - \left[1 - P(x) \right]^N \right\} dx \; .$$

To get the variance of R :

Set r = 1, s = N in (2.3.5), we have for $x < y$

$$p_{1,N}(x,y) = N(N-1) \int_{P(x-1)}^{P(x)} \int_{P(y-1)}^{P(y)} (v - w)^{N-2} \, d\,v\,d\,w$$

$$= \left[P(y) - P(x-1) \right]^N - \left[P(y) - P(x) \right]^N - \left[P(y-1) - P(x-1) \right]^N$$

$$+ \left[P(y-1) - P(x) \right]^N \; . \qquad (2.5.16)$$

Hence

$$E(R^2) = \sum_{y=1}^{\infty} \sum_{x=0}^{y-1} (y-x)^2 \left\{ \left[P(y) - P(x-1) \right]^N - \left[P(y-1) - P(x-1) \right]^N \right\}$$

$$- \sum_{y=1}^{\infty} \sum_{x=0}^{y-1} (y-x)^2 \left\{ \left[P(y) - P(x) \right]^N - \left[P(y-1) - P(x) \right]^N \right\} \; . \qquad (2.5.17)$$

Following Siotani (1957) , we first take the upper limit finite a.

Then the first sum in (2.5.17) , on separating out x = 0, equals

$$\sum_{y=1}^{a} y^2 \left\{ \left[P(y) \right]^N - \left[P(y-1) \right]^N \right\} + \sum_{y=2}^{a} \sum_{x=1}^{y-1} (y-x)^2$$

$$\cdot \left\{ \left[P(y) - P(x-1) \right]^N - \left[P(y-1) - P(x-1) \right]^N \right\}$$

with $x' = x - 1$ the double sum becomes

$$\sum_{y=2}^{a} \sum_{x'=0}^{y-2} (y-x'-1)^2 \left\{ \left[P(y) - P(x') \right]^N - \left[P(y-1) - P(x') \right]^N \right\}$$

$$= \sum_{y=1}^{a} \sum_{x'=0}^{y-1} (y-x'-1)^2 \left\{ \left[P(y) - P(x') \right]^N - \left[P(y-1) - P(x') \right]^N \right\}$$

since $x' = y - 1$ adds nothing to the sum.

Combining this with the second sum of (2.5.17) (on replacing x' by x) we have

$$E(R^2) = \sum_{y=1}^{a} y^2 \left\{ \left[P(y) \right]^N - \left[P(y-1) \right]^N \right\}$$

$$- \sum_{y=1}^{a} \sum_{x=2}^{y-1} \left\{ 2\,(y-x) - 1 \right\} \left\{ \left[P(y) - P(x) \right]^N - \left[P(y-1) - P(x) \right]^N \right\}. \tag{2.5.18}$$

Now simplifying the first sum and separating out $y = a$ in the second sum of (2.5.18) we find by similar argument as above that,

$$E(R^2) = \sum_{y=1}^{a} (2y-1) \left\{ 1 - \left[P(y-1) \right]^N \right\}$$

$$- \sum_{x=0}^{a-1} \left[2(a-x) -1 \right] \left[1 - P(x) \right]^N + 2 \sum_{y=1}^{a} \sum_{x=0}^{y-1} \left[P(y-1) - P(x) \right]^N$$

$$= 2 \sum_{y=1}^{a} \sum_{x=0}^{y-1} \left\{ 1 - \left[P(y-1) \right]^N - \left[1 - P(x) \right]^N \right.$$

$$\left. + \left[P(y-1) - P(x) \right]^N \right\}$$

$$- \sum_{y=1}^{a} \left\{ 1 - \left[P(y-1) \right]^N - \left[1 - P(y-1) \right]^N \right\}$$

$$= 2 \sum_{y=0}^{a} \sum_{x=0}^{y} \left\{ 1 - P^N(y) - \left[1 - P(x) \right]^N + \left[P(y) - P(x) \right]^N \right\}$$

$$- \sum_{y=0}^{a} \left\{ 1 - P^N(y) - \left[1 - P(y) \right]^N \right\}. \qquad (2.5.19)$$

Therefore,

$$\operatorname{var}(R) = 2 \sum_{x=0}^{a-1} \sum_{y=0}^{x} \left[1 - P^N(x) - \left\{1 - P(y)\right\}^N + \left\{P(x) - P(y)\right\}^N \right]$$

$$- E(R) \left\{1 + E(R)\right\} \tag{2.5.20}$$

which is the corresponding form to the continuous case (Tippett, 1925), which states that

$$\operatorname{var}(R) = 2 \int_{x=-a}^{a} \int_{y=-a}^{x} \left[1 - P^N(x) - \left\{1 - P(y)\right\}^N + \left\{P(x) - P(y)\right\}^N \right]$$

$$\cdot\ d\,x\,d\,y - \left[E(R)\right]^2. \tag{2.5.21}$$

Now let

$R' = X_{(s)} - X_{(r)}$ (Khatri, 1962) .

$$E(R') = \sum_{x=0}^{a-1} \left\{ P_r(x) - P_s(x) \right\} \tag{2.5.22}$$

where $$P_m(x) = m\binom{N}{m} \int_0^{P(x)} v^{m-1}(1-v)^{N-m}\,dv$$

for $m = r, s$.

Equation (2.5.22) can be written as

$$E(R') = \sum_{x=0}^{a-1} \sum_{m=1}^{s-r} \binom{N}{s-m} \left\{P(x)\right\}^{s-m} \left\{1 - P(x)\right\}^{N-s+m} . \tag{2.5.23}$$

When $s = N$ and $r = 1$, we get relation (2.5.15).

Now if $a = +\infty$, to get the moments of the order statistics we have to prove first the following Lemma.

Lemma

If the original distribution has finite moments up to order t, then the moments of order statistics exist up to order t.

Proof

$$E\left(X_{(r)}^{t}\right) = \sum_{x=0}^{a} \sum_{\ell=0}^{r-1} \sum_{m=r-\ell}^{N-\ell} x^t \frac{N!}{\ell!\ m!\ (N-\ell-m)!}$$

$$\cdot P^{\ell}(x-1)\ p^m(x)\ (1 - P(x))^{N-\ell-m}$$

$$< \sum_{x=0}^{a} \sum_{\ell=0}^{r-1} \sum_{m=r-\ell}^{N-\ell} x^t \frac{N!}{\ell!\ m!\ (N-\ell-m)!} p(x)$$

$$< \sum_{x=0}^{\infty} \sum_{\ell=0}^{r-1} \sum_{m=r-\ell}^{N-\ell} x^t \frac{N!}{\ell!\ m!\ (N-\ell-m)!} p(x)$$

$$= c\ E(X^t)$$

where C is a finite constant

$$C = \sum_{\ell=0}^{r-1} \sum_{m=r-\ell}^{N-\ell} \frac{N!}{\ell!\ m!\ (N-\ell-m)!}$$

for all $r = 1, 2, \dots, N$.

From (2.5.5) and (2.5.6) , we can write directly the mean and variance of $X_{(r)}$ when $a = +\infty$ as follows:

$$E\left(X_{(r)}\right) = \sum_{x=0}^{\infty} \left[1 - I_{P(x)}\ (r, N - r + 1)\right] . \tag{2.5.24}$$

$$\operatorname{var}\left(X_{(r)}\right) = 2 \sum_{x=0}^{\infty} \left[1 - I_{P(x)}\ (r, N - r + 1)\right] .$$

$$+ E\left(X_{(r)}\right) \left[1 - E\left(X_{(r)}\right)\right] . \tag{2.5.25}$$

<u>The product moments of the order statistics</u>

$$E\left(X_{(r)}^{i}\ X_{(s)}^{j}\right) = \sum_{y=0}^{a} \sum_{x=0}^{y} x^{i}\ y^{j}\ P\left\{X_{(r)} = x,\ X_{(s)} = y\right\} \tag{2.5.26}$$

where $P\left\{X_{(r)} = x,\ X_{(s)} = y\right\}$ as defined in (2.3.2).

(2.5.26) can be calculated by specifying the population.

In particular for $r = 1$ and $s = N$

$$E\left(X^i_{(1)}\, X^j_{(N)}\right) = \sum_{y=0}^{a}\ \sum_{x=0}^{y} x^i y^j\ P\left\{X_{(1)} = x,\ X_{(N)} = y\right\}$$

$$= \sum_{y=0}^{a}\ \sum_{x=0}^{y} x^i y^j \left(\Delta^N_{y,x-1} + \Delta^N_{y-1,x} - \Delta^N_{y,x} - \Delta^N_{y-1,x-1}\right),\quad x < y.$$

Also we have

$$\mathrm{Cov}\left(X_{(r)},\, X_{(s)}\right) = E\left(X_{(r)}X_{(s)}\right) - E\left(X_{(r)}\right) \cdot E\left(X_{(s)}\right)$$

Therefore,

$$\mathrm{Cov}\left(X_{(1)},\, X_{(N)}\right) = \sum_{y=0}^{a}\ \sum_{x=0}^{y} x\, y \left(\Delta^N_{y,x-1} + \Delta^N_{y-1,x} - \Delta^N_{y,x} - \Delta^N_{y-1,x-1}\right)$$

$$\cdot \left\{\sum_{x=0}^{a-1}\left[1 - P(x)\right]^N \cdot \sum_{x=0}^{a-1}\left[1 - P^N(x)\right]\right\},\quad x < y. \tag{2.5.27}$$

2.6 Examples

i The rectangular distribution R(1, a)

$$p(x) = \frac{1}{a} \qquad x = 1, 2, \ldots, a$$

$$= 0 \qquad \text{elsewhere} .$$

$$P_r(x) = I_{\frac{x}{a}} \; (r, N - r + 1) . \tag{2.6.1}$$

$$p_N(x) = P^N(x) - P^N(x-1) = \left(\frac{x}{a}\right)^N - \left(\frac{x-1}{a}\right)^N \tag{2.6.2}$$

which is the probability function of the largest number in N drawings from a bowl containing balls numbered 1 to a when random sampling with replacement is used [Feller (1957) p. 211].

$$p_1(x) = \left(1 - \frac{x-1}{a}\right)^N - \left(1 - \frac{x}{a}\right)^N . \tag{2.6.3}$$

For the joint c.d.f

$$P\left\{X_{(1)} = x ,\; X_{(N)} = y\right\} = \left(\frac{y-x+1}{a}\right)^N + \left(\frac{y-1-x}{a}\right)^N - 2\left(\frac{y-x}{a}\right)^N \quad \text{when } y > x$$

(2.6.4)

$$= a^{-N} \qquad \text{if } x = y .$$

$$E\left(X_{(r)}\right) = \sum_{x=0}^{a-1} \left[1 - I_{\frac{x}{a}}(r, N - r + 1)\right]. \tag{2.6.5}$$

$$\operatorname{var}\left(X_{(r)}\right) = 2 \sum_{x=0}^{a-1} x\left[1 - I_{\frac{x}{a}}(r, N - r + 1)\right] + E\left(X_{(r)}\right)$$

$$\cdot \left\{1 - E\left(X_{(r)}\right)\right\} . \tag{2.6.6}$$

In particular, for the extremes we have

$$E\left(X_{(N)}\right) = \sum_{x=0}^{a-1} \left\{1 - \left(\frac{x}{a}\right)^N\right\}. \tag{2.6.7}$$

It follows that for large a

$$E\left(X_{(N)}\right) \simeq \frac{N}{N+1}\, a \, .$$

The observed maximum in a sample is used to estimate the unknown true number a . This method was used during the last war to estimate enemy production [Feller (1957) p. 212].

$$E\left(X_{(1)}\right) = \sum_{x=0}^{a-1} \left(1 - \frac{x}{a}\right)^N. \tag{2.6.8}$$

ii <u>The Poisson distribution with parameter λ</u>

$$p(x) = \frac{e^{-\lambda}\lambda^k}{k!} \qquad x = 0, 1, 2, \ldots$$

$$P(x) = \sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= 1 - G_\lambda (x + 1)$$

where $G_\lambda(x+1) = 1 - \sum_{k=0}^{x-1} \frac{e^{-\lambda}\lambda^k}{k!}$ is the incomplete gamma function with parameter x (Siotani, 1956).

$$P_r(x) = I_{1-G_\lambda(x+1)} (r, N - r + 1) . \qquad (2.6.9)$$

$$p_N(x) = P^N(x) - P^N(x-1) \qquad (2.6.10)$$

$$= \left[1 - G_\lambda(x+1)\right]^N - \left[1 - G_\lambda(x)\right]^N \quad x = 0, 1, 2, \ldots \qquad (2.6.11)$$

$$p_1(x) = \left[1 - P(x-1)\right]^N - \left[1 - P(x)\right]^N \tag{2.6.12}$$

$$= \left[G_\lambda(x) \right]^N - \left[G_\lambda(x+1) \right]^N \quad x = 0,1,2, \ldots \tag{2.6.13}$$

Formulas (2.6.10), (2.6.12) are computed by the "Tables of the Poisson Distribution" (Kitagawa, 1951) and (2.6.11), (2.6.13) are for using the "Tables of the Incomplete Gamma-Function" (Pearson, 1951)

$$E\left(X_{(r)}\right) = \sum_{x=0}^{\infty} \left[1 - I_{1-G_\lambda(x+1)}(r, N-r+1) \right]. \tag{2.6.14}$$

$$E\left(X_{(N)}\right) = \sum_{x=0}^{\infty} \left[1 - \left(1 - G_\lambda(x+1)\right)^N \right]. \tag{2.6.15}$$

$$E\left(X_{(1)}\right) = \sum_{x=0}^{\infty} \left[G_\lambda(x+1) \right]^N. \tag{2.6.16}$$

iii <u>The binomial distribution with parameters ρ and n</u>

$$p(x) \equiv b(x; n, \rho) = \binom{n}{x} \rho^x (1-\rho)^{n-x} \qquad x = 0,1,2, \ldots n .$$

$$P(x) \equiv B(x; n, \rho) = \sum_{k=0}^{x} b(k; n, \rho)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \rho^{k} \quad (1-\rho)^{n-k} - \sum_{k=x+1}^{n} \binom{n}{k} \quad \rho^{k} \quad (1-\rho)^{n-k}$$

$$= 1 - I_{\rho} \; (x+1, \; n-x). \tag{2.6.17}$$

$$p_r(x; \, n, \rho \,) \quad = \quad I_{P(x)} \; (r, \; N - r + 1) - I_{P(x-1)} \; (r, \; N - r + 1) \tag{2.6.18}$$

where P(x) is defined in (2.6.17).

$$p_N(x) \; = \; \left\{ B(x; \, n, \rho \,) \right\}^N - \; \left\{ B(x-1; \, n, \rho \,) \right\}^N \tag{2.6.19}$$

$$= \left[\, 1 - I_{\rho}(x+1, \; n - x \right]^N - \left[1 - I_{\rho}(x, \; n-x + 1) \right]^N \; . \tag{2.6.20}$$

$$p_1(x) = \left\{ 1 - B(x-1 \, ; \, n, _{\rho} \,) \right\}^N - \left\{ \, 1 - B(x; \, n, _{\rho} \,) \right\}^N \tag{2.6.21}$$

$$= \; I_{\rho}^{N} \; (x; \, n - x + 1) - I_{\rho}^{N} \, (x + 1 \; ; \, n - x) \; . \tag{2.6.22}$$

The numerical values of $p_N(x)$, $p_1(x)$ can be easilty obtained from (2.6.19), (2.6.21) by using the "Tables of the Binomial Probability Distribution" or from (2.6.20), (2.6.22) by using the "Tables of the Incomplete Beta-Function" (Pearson, 1948) for fixed values of N, ρ, and n.

From (2.6.18) the following result can be easily deduced (Abdel-Aty, 1954).

In two binomial populations, bin (n, ρ) and bin (n, ρ'), where $\rho + \rho' = 1$

$p_r(x)$ in the bin (n, ρ) = p_{N-r+1} $(n - x)$ in the bin (n, ρ').

Also

$$E\left(X_{(r)}\right) = \sum_{x=0}^{n-1} \left[1 - I_{P(x)}(r, N - r + 1) \right] \qquad (2.6.23)$$

where $P(x)$ is defined in (2.6.17).

$$E\left(X_{(N)}\right) = \sum_{x=0}^{n-1} \left[1 - \left\{ 1 - I_{\rho}(x + 1, n - x) \right\}^{N} \right]. \qquad (2.6.24)$$

$$E\left(X_{(1)}\right) = \sum_{x=0}^{n-1} I_{\rho}^{N}(x + 1, n - x). \qquad (2.6.25)$$

2.7 Certain uncorrelated order statistics

Let X be a discrete random variable having the probability mass function $p(x)$. In a sample of size N, we order the X's in ascending order of magnitude and if the distribution is symmetric about the origin, we have (Lloyd, 1952)

$$X_{(1)} \leq X_{(2)} \leq X_{(r)} \leq \cdots \leq X_{(N)}.$$

Also

$$-X_{(N)} \leq -X_{(N-1)} \leq \cdots \leq -X_{(1)} .$$

Since the latter set may be regarded as increasingly ordered observations on the variate $-X$, which by symmetry has the same distribution as X. It follows that the joint distribution of $\left(X_{(1)}, X_{(2)}, \ldots, X_{(N)}\right)$ and $\left(-X_{(N)}, -X_{(N-1)}, \ldots, -X_{(1)}\right)$ coincide.

Therefore

$$p_r(x) = p_{N-r+1}(-x)$$

and (2.7.1)

$$p_{r,s}(x,y) = p_{N-r+1,\, N-s+1}(-x,-y)$$

Now if the distribution is symmetric about the point C, we get

$$p_r(c-x) = p_{N-r+1}(c+x) \tag{2.7.2}$$

for every real x.

Definitions (Hogg, 1960)

1. The statistic $T\left(X_{(1)}, X_{(2)}, \ldots, X_{(N)}\right)$ is an odd location statistic if, for all real $x_{(1)}, x_{(2)}, \ldots, x_{(N)}$ we have

i $$T\left(x_{(1)}+h, x_{(2)}+h, \ldots, x_{(N)} + h\right) = T\left(x_{(1)}, x_{(2)}, \ldots, x_{(N)}\right) + h \text{ for every } h. \tag{2.7.3}$$

ii $$T\left(-x_{(1)}, -x_{(2)}, \ldots, -x_{(N)}\right) = -T\left(x_{(1)}, x_{(2)}, \ldots, x_{(N)}\right). \tag{2.7.4}$$

2. The statistic $S\left(X_{(1)}, X_{(2)}, \ldots, X_{(N)}\right)$ is an even location-free statistic if, for all real $x_{(1)}, x_{(2)}, \ldots x_{(N)}$ we have

i $S\left(x_{(1)} + h, x_{(2)} + h, \ldots , x_{(N)} + h\right) = S\left(x_{(1)} ,\right.$

$\left. x_{(2)}, \ldots , x_{(N)}\right)$ for every h. (2.7.5)

ii $S\left(-x_{(1)}, -x_{(2)}, \ldots , -x_{(N)}\right) = S\left(x_{(1)}, x_{(2)}, \ldots , x_{(N)}\right)$. (2.7.6)

Some examples of odd location statistics are the sample mean, the sample median, and the sample mid-range. The sample variance, the sample range, the sample quasi ranges, and the sample mean deviation from the sample median are examples of even location-free statistics.

Theorem

In a distribution that is symmetric about the point c, if the correlation coefficient of an odd location statistic $T\left(X_{(1)}, X_{(2)}, \ldots , X_{(N)} \right)$ and an even location-free statistic $S\left(X_{(1)}, X_{(2)}, X_{(N)} \right)$ exists, it is equal to zero.

Proof

It is observed that the existence of the correlation coefficient of T and S implies the existence of the means, the variances, and the covariance of these two statistics.

We shall prove first that $E(T) = c$.

Let $E(T) - c = D$.

$$D = \sum_0^\infty \sum_0^\infty \cdots \sum_0^\infty \left[T\left(x_{(1)}, x_{(2)}, \ldots, x_{(N)}\right) - c \right]$$

$$\cdot\, p_1(x)\; p_2(x) \ldots p_N(x) \quad . \qquad (2.7.7)$$

Make the transformation $x = y + c$

then, by (2.7.3) we have

$$D = \sum_0^\infty \cdots \sum_0^\infty T\left(y_{(1)}, y_{(2)}, \ldots, y_{(N)}\right) \quad p_1(y+c)$$

$$\cdot\, p_2(y+c) \ldots p_N(y+c) \quad . \qquad (2.7.8)$$

Change the variable $y = -z$

so that

$$D = \sum_0^\infty \sum_0^\infty T\left(-z_{(1)}, -z_{(2)}, \ldots , -z_{(N)}\right) p_1(-z+c) \ldots$$

$$\cdot \; p_N(-z+c)$$

making use of (2.7.2) and (2.7.4), we get

$$D = -\sum_{0}^{\infty} \cdots \sum_{0}^{\infty} T(z_{(1)}, z_{(2)}, \ldots, z_{(N)}) \; p_N(z+c)$$

$$\cdot \; p_{N-1}(z+c) \ldots p_1(z+c) \; . \qquad (2.7.9)$$

From (2.7.8) and (2.7.9), we deduce that

$$D = -D \qquad \text{or} \qquad D = 0 \; .$$

Therefore

$$E(T) = c \; .$$

Now we shall show that the covariance, and hence the correlation coefficient, is zero.

Let V denote the covariance of T and S.

Then

$$V = \sum_{0}^{\infty} \cdots \sum_{0}^{\infty} \left[T\left(x_{(1)}, \ldots, x_{(N)}\right) - c \right] S\left(x_{(1)}, \ldots\right.$$

$$\left., x_{(N)}\right) \; p_1(x) \;\ldots\; p_N(x) \;.$$

Using exactly the same transformations and argument given above, making use of (2.7.2) , (2.7.3), (2.7.4), (2.7.5), and (2.7.6) we can prove that

$$V = 0 .$$

Aiyar (1963) gives this interesting result which holds for both the continuous and discrete distributions. " For an unsymmetrical population the mean and the range of a random sample is uncorrelated if the sum of the elements of the first column (or row) is equal to the sum of the elements of the last column (or row) of the variance-covariance matrix of the ordered observations."

Special cases

Ostle and Steck (1959) give these two results:

i- The mean and sample range are uncorrelated .

ii- The sample range and midrange are uncorrelated .

Moreover, they show by an example that symmetry in the distribution is not necessary for zero correlation between the mean and the range of the sample. Also they point out that the sign of $E(\bar{X} R)$ does not depend on the sign of the skewness of the underlying distribution.

III. RECURRENCE FORMULAE, CERTAIN RELATIONSHIPS BETWEEN MOMENTS OF ORDER STATISTICS

3.1 Relationships between the moments and product moments of order statistics

Suppose we have N observations independently and identically distributed from a discrete population. In this chapter it is convenient to use a notation emphasizing dependence on the sample size N .

Define

$$\mu^{(i)}_{r:N} = E\left(X^{i}_{(r)} \middle| N\right) = E(X_{r:N})$$

$$\mu^{(i,j)}_{r,s:N} = E\left(X^{i}_{r:N} \quad X^{j}_{s:N}\right)$$

$$\mu^{(1)}_{r:N} = \mu_{r:N} \quad , \quad \mu^{(1,1)}_{r,s:N} = \mu_{r,s:N}$$

Then

$$\mu_{r,s:N} = \mu_{s,r:N} \quad , \quad \mu^{(i+j)}_{r:N} = \mu^{(i,j)}_{r,r:N} \quad .$$

Govindarajulu (1963) gives many recurrence relations between moments of order statistics in the continuous case. Dealing with the discrete case we have:

<u>Theorem 1</u>

a) $$\sum_{r=1}^{N} \mu^{(i)}_{r:N} = N \, E \, (X^{i})$$

b) $$\sum_{r=1}^{N-1} \sum_{s=r+1}^{N} E\left(X^{i}_{r:N} \; X^{i}_{s:N} \right) = \frac{1}{2} \, N(N-1) \left[E(X^{i}) \right]^{2}$$

c) $$\sum_{r=1}^{N} \sum_{s=1}^{N} \sigma_{r,s:N} = N \, \sigma^{2}$$

where $$\sigma_{r,s:N} = E \left[\left(X_{r:N} - \mu_{r:N} \right) \left(X_{s:N} - \mu_{s:N} \right) \right] .$$

<u>Proof</u>

$$\left(\sum_{r=1}^{N} X^{k}_{r:N} \right)^{m} = \left(\sum_{r=1}^{N} X^{k}_{r} \right)^{m}$$

(where X_r is the rth unordered variate) .

Since the L. H. S. is only a rearrangement of the R. H. S. Taking expectations we obtain with (k, m) in turn equal to $(i,1)$, $(i,2)$ and $(2i, 1)$

$$\sum_{r=1}^{N} \mu_{r:N}^{(i)} = N\, E(X^{i}) \tag{3.1.1}$$

$$\sum_{r=1}^{N} \sum_{s=1}^{N} E\left(X^{i}_{r:N} \; X^{i}_{s:N}\right) = N\, E(X^{2i}) + N(N-1) \cdot \left[E(X^{i})\right]^{2} \tag{3.1.2}$$

$$\sum_{r=1}^{N} \mu_{r:N}^{(2i)} = N\, E(X^{2i}) . \tag{3.1.3}$$

Whence, subtracting (3.1.3) from (3.1.2) and dividing by 2 ,

$$\sum_{r=1}^{N-1} \sum_{s=r+1}^{N} E\left(X^{i}_{r:N} \; X^{i}_{s:N}\right) = \frac{1}{2}\, N(N-1)\left[E(X^{i})\right]^{2} . \tag{3.1.4}$$

Now by squaring the relation

$$\sum_{r=1}^{N}\left(X_{r:N} - \mu_{r:N}\right) = \sum_{r=1}^{N}\left(X_r - \mu\right)$$

we get

$$\sum_{r=1}^{N}\sum_{s=1}^{N} \sigma_{r,s:N} = N\,\sigma^2 .$$

Note that these proofs and results apply to both continuous and discrete parents.

Corollary 1

a) $$\sum_{r=1}^{N} \mu_{r:N} = N\,\mu .$$

and $$\sum_{r=1}^{N} \mu_{r:N} = 0 \quad \text{if} \quad \mu = 0 .$$

b) $$\sum_{r=1}^{N-1}\sum_{s=r+1}^{N} \mu_{r,s:N} = \frac{1}{2} N(N-1)\ \mu^2 .$$

These results are special cases of Theorem 1 when $i = 1$.

Lemma 1

$$E\left(X_{(1)}^{i}\ \ X_{(2)}^{i}\ \middle|\ N=2\right) = \left[E(\ X^{i})\right]^{2} \qquad (3.1.5)$$

Proof.

By definition and using (2.3.14) , we have

$$E\left(X_{(1)}^{i}\ \ X_{(2)}^{i}\ \middle|\ N=2\right) = \sum_{y=1}^{a}\ \sum_{x=0}^{y-1}\ 2\ x^{i}\ y^{i}\ p(x)\ p(y)$$

$$+ \sum_{x=0}^{a}\ x^{2i}\ p^{2}(x)$$

$$= \sum_{y=0}^{a}\ \sum_{x=0}^{a}\ x^{i}\ y^{i}\ p(x)\ p(y)$$

$$= \left[E(\ X^{i})\right]^{2}.$$

Theorem 2

$$\sum_{\substack{r=1 \\ r\neq s}}^{N}\ \sum_{s=1}^{N}\ E(\ X_{r:N}^{i}\ \ X_{s:N}^{j}\ ) = N(N-1)\ \ E(\ X^{i})\ \ E(X^{j}) \qquad (3.1.6)$$

Proof.

$$\sum_{r=1}^{N} \sum_{s=1}^{N} \left(X^i_{r:N} X^j_{s:N} \right) = \sum_{r=1}^{N} \sum_{s=1}^{N} \left(X^i_r X^j_s \right) \tag{3.1.7}$$

and

$$\sum_{r=1}^{N} \left(X^{i+j}_{r:N} \right) = \sum_{r=1}^{N} X^{i+j}_r \tag{3.1.8}$$

By subtracting (3.1.8) from (3.1.7), we get

$$\sum_{r=1}^{N} \underset{r \neq s}{\sum_{s=1}^{N}} \left(X^i_{r:N} X^j_{s:N} \right) = \sum_{r=1}^{N} \underset{r \neq s}{\sum_{s=1}^{N}} \left(X^i_r X^j_s \right) .$$

Taking expectations and simplifying R. H. S., we have

$$\sum_{r=1}^{N} \underset{r \neq s}{\sum_{s=1}^{N}} E\left(X^i_{r:N} X^j_{s:N} \right) = \underset{r \neq s}{N(N-1)} \; E\left(X^i_r X^j_s \right)$$

$$= N(N-1) \; E(X^i) \; E(X^j) .$$

The proof is valid for both continuous and discrete parents.

If $i = j$ we get Theorem 1(b) .

Theorem 3

$$(N-r)\ \mu^{(i)}_{r:N} + r\ \mu^{(i)}_{r+1:N} = N\ \mu^{(i)}_{r:N-1} \tag{3.1.9}$$

where

$r = 1, 2, \ldots, N-1$ and $i = 1, 2, \ldots$

Proof.

$$\mu^{(i)}_{r:N} = \sum_{x=0}^{a} x^i \left[I_{P(x)}(r, N-r+1) - I_{P(x-1)}(r, N-r+1) \right]. \tag{3.1.10}$$

Also it is known that

$$(a+b)\, I_y(a+b) \doteq a\, I_y(a+1, b) + b\, I_y(a, b+1). \tag{3.1.11}$$

Putting $a = r$, $b = N - r$, we rewrite (3.1.10) in the form

$$(N-r)\, I_y(r, N-r+1) = N\, I_y(r, N-r) - r\, I_y(r+1, N-r). \tag{3.1.12}$$

Multiply both sides of (3.1.10) by $(N - r)$ and substitute for $I_{P(x)}$, $I_{P(x-1)}$ by using (3.1.12), we get

$$(N - r)\, \mu^{(i)}_{r:N} = N \sum_{x=0}^{a} x^i \left[I_{P(x)}\,(r, N - r) - I_{P(x-1)}\,(r, N - r) \right]$$

$$- r \sum_{x=0}^{a} x^i \left[I_{P(x)}\,(r + 1, N - r) - I_{P(x-1)}\,(r + 1, N - r) \right]$$

$$= N\, \mu^{(i)}_{r:N-1} - r\, \mu^{(i)}_{r+1:N} .$$

Hence,

$$(N - r)\, \mu^{(i)}_{r:N} + r\, \mu^{(i)}_{r+1:N} = N\, \mu^{(i)}_{r:N-1}$$

which is the same result as given by Cole (1951) in the continuous case.

Using this result recurrently, we can generate the $\mu^{(i)}_{r:N}$ $(r = 1, 2, \ldots, N)$ if the $\mu^{(i)}_{r:N-1}$ $(r = 1, 2, \ldots, N-1)$ and any one of the $\mu^{(i)}_{r:N}$ are available.

Comment

Since the proof of Theorem 3 depends only on a property of th incomplete B-function, it is clear that the same recurrence relation links also the p.d.f.'s , c.d.f.'s and in fact the expected values (if these exist) of any function $g\left(X_{r:N}\right)$.

Thus we have [Srikantan (1962)]

$$(N - r)\ Eg\left(X_{r:N}\right) + r\ Eg\left(X_{r+1:N}\right) = N\ Eg\left(X_{r:N-1}\right) . \tag{3.1.13}$$

Corollary 3.1

For N even

(a) $$\frac{1}{2}\left(\mu^{(i)}_{\frac{N}{2}+1:N} + \mu^{(i)}_{\frac{N}{2}:N} \right) = \mu^{(i)}_{\frac{N}{2}:N-1} \tag{3.1.14}$$

(b) $$E(M) = \mu_{\frac{N}{2}:N-1}$$

where M is the sample median . (3.1.15)

Proof.

(a) Follows from Theorem 3 .

(b) In (a) put $i = 1$.

Corollary 3.2

If the distribution is symmetric about the origin, and N is even, then

$$\mu^{(i)}_{\frac{N}{2}:\,N-1} = \mu^{(i)}_{\frac{N}{2}:\,N} \qquad i \text{ even}$$

$$= 0 \qquad i \text{ odd} . \qquad (3.1.16)$$

Proof.

$$\mu^{(i)}_{\frac{N}{2}+1:N} = (-1)^{i}\ \mu_{\frac{N}{2}:\,N} \qquad \text{by symmetry} .$$

Hence, from (3.1.14)

$$\mu^{(i)}_{\frac{N}{2}:\,N-1} = \mu^{(i)}_{\frac{N}{2}:\,N} \qquad i \text{ even}$$

$$= 0 \qquad i \text{ odd} .$$

Theorem 4

For an arbitrary distribution and for $1 \leq r \leq s \leq N$

$$(r-1)\mu_{r,s:N} + (s-r)\mu_{r-1,s:N} + (N-s+1)\mu_{r-1,s-1:N}$$

$$= N\mu_{r-1,s-1:N-1} \qquad (3.1.17)$$

Proof.

Using equation (2.3.5), we have

$$N\mu_{r-1,s-1:N-1} = \frac{N!}{(r-2)!\,(s-r-1)!\,(N-s)!} \sum_{x=0}^{a} \sum_{y=x}^{a} x\,y$$

$$\cdot \int\int w^{r-2}\,(v-w)^{s-r-1}\,(1-v)^{N-s}\,d\,v\,d\,w \qquad (3.1.18)$$

where the integration is to be carried out over the region

$v \geq w$, $P(x) \geq w \geq P(x-1)$ and $P(y) \geq v \geq P(y-1)$.

Now multiply the integrand $\left[w^{r-2} (v-w)^{s-r-1} (1-v)^{N-s} \right]$ by unity,

write $1 = w + (v-w) + (1-v)$

and split up the integral as the sum of three integrals. The result follows directly, which is the same result as in the continuous case. Using this result recurrently, we can generate all the $\mu_{r,s:N}$ $(r < s \,;\, r, s = 1, 2, \ldots , N)$ if the $\mu_{r,s:N-1}$ $(r < s \,;\, r, s = 1, 2, \ldots , N-1)$ and any $N - 1$ of the $\mu_{r,s:N}$ are available.

Note too that if $r = s$, Theorem 4 gives Theorem 3 with $i = 2$.

Theorem 5

$$(N - r)_m \; \mu^{(t)}_{r:N} = \sum_{i=0}^{m} (-r)_i \, (N)_{m-i} \, \binom{m}{i} \, \mu^{(t)}_{r+i:N-m+i} \quad . \qquad (3.1.19)$$

Proof.

Theorem 3 gives

$$(N - r - 1)\ \mu^{(t)}_{r:N-1} = (N - 1)\ \mu^{(t)}_{r:N-2} - r\ \mu^{(t)}_{r+1:N-1} \qquad (3.1.20)$$

and

$$(N - r - 1)\ \mu^{(t)}_{r+1:N} = N\ \mu^{(t)}_{r+1:N-1} - (r + 1)\ \mu^{(t)}_{r+2:N} . \qquad (3.1.21)$$

Substituting for $\mu^{(t)}_{r:N-1}$ and $\mu^{(t)}_{r+1:N}$ in (3.1.9) , we have

$$(N - r)\ (N - r - 1)\ \mu^{(t)}_{r:N} = N(N - 1)\ \mu^{(t)}_{r:N-2} - 2\ r\ N\ \mu^{(t)}_{r+1:N-1}$$

$$+\ r(r + 1)\ \mu^{(t)}_{r+2:N} . \qquad (3.1.22)$$

Write the corresponding expressions for $\mu^{(t)}_{r:N-2}$, $\mu^{(t)}_{r+1:N-1}$,

and $\mu^{(t)}_{r+2:N}$ as we did in (3.1.20) , (3.1.21) .

Substituting in (3.1.22) , we get

$$(N - r)_3 \ \mu^{(t)}_{r:N} = (N)_3 \ \mu^{(t)}_{r:N-3} - 3\, r\, (N)_2$$

$$\mu^{(t)}_{r+1:N-2} + 3\,(-r)_2 \ N \ \mu^{(t)}_{r+2:N-1} + (-r)_3 \ \mu^{(t)}_{r+3:N}$$

where

$$(N)_c = N(N - 1) \ldots (N - c + 1) \quad (\ c \ \text{ any positive integer}) .$$

Repeating the same process $(m-1)$ times, we find that

$$(N - r)_m \ \mu^{(t)}_{r:N} = \sum_{i=0}^{m} (- r)_i \ (N)_{m-i} \binom{m}{i} \ \mu^{(t)}_{r+i:N-m+i}$$

This enables us to express the moments of order statistics in a sample of N in terms of the moments of order statistics in a sample of M where $N \geq M$.

Theorem 6

$$\text{(a)} \quad \mu^{(t)}_{r:N} = \sum_{j=r}^{N} \binom{j-1}{r-1} \binom{N}{j} (-1)^{j-r} \ \mu^{(t)}_{j:j} .$$

(b) $$\mu^{(t)}_{r:N} = \sum_{i=N-r+1}^{N} \binom{N}{i} \binom{i-1}{N-r} (-1)^{i-N+r-1} \mu^{(t)}_{1:i}$$

Proof.

(a) Putting $m = N-r$ in (3.1.19), we have

$$(N - r)!\ \mu^{(t)}_{r:N} = \sum_{i=0}^{N-r} (-1)_i\ (N)_{N-r-i} \binom{N-r}{i} \mu^{(t)}_{r+i:r+i} \tag{3.1.23}$$

or

$$\mu^{(t)}_{r:N} = \sum_{j=r}^{N} \binom{j-1}{r-1} \binom{N}{j} (-1)^{j-r} \mu^{(t)}_{j:j} . \tag{3.1.24}$$

(b) $$\mu^{(t)}_{r:N} = \sum_{x=0}^{a} x^t \int_{P(x-1)}^{P(x)} \frac{y^{r-1} (1-y)^{N-r}}{(r-1)!\ (N-r)!} N!\ d\,y . \tag{3.1.25}$$

Expanding y^{r-1} as a binomial series in $(1-y)$ and simplify.

Thus,

$$\mu^{(t)}_{r:N} = \sum_{j=0}^{r-1} \sum_{x=0}^{a} x^t \int_{P(x-1)}^{P(x)} \frac{(-1)^j \binom{r-1}{j} (1-y)^{N-r+j}}{(r-1)! \ (N-r)!} N! \ d\,y.$$

Putting $i = j + N - r + 1$,

Making use of the fact that

$$\mu^{(t)}_{1:i} = \sum_{x=0}^{a} x^t \int_{P(x-1)}^{P(x)} (1-y)^{i-1} \ i \ d\,y$$

and simplify we get,

$$\mu^{(t)}_{r:N} = \sum_{i=N-r+1}^{N} \binom{N}{i} \binom{i-1}{N-r} (-1)^{i-N+r-1} \ \mu^{(t)}_{1:i}. \qquad (3.1.26)$$

We can deduce also (3.1.24) directly from (3.1.25) by expanding $(1-y)^{N-r}$ as a binomial series in y.

Theorem 6 is useful because it enables us to express the moments of $X_{r:N}$ in terms of the simpler moments of the extremes.

Alternatively for later use we can write $\mu^{(t)}_{r:N}$ in the form (Section 2.5)

$$\mu^{(t)}_{r:N} = \sum_{x=0}^{a-1} \Phi(x) \left[1 - I_{P(x)}(r, N - r + 1)\right] \tag{3.1.27}$$

where $\Phi(x)$ is a polynomial in x .

For example $\Phi(x) = 1$ if $t = 1$

$= (2x+1)$ if $t = 2$.

Hence,

$$\mu^{(t)}_{r:N} = \sum_{x=0}^{a-1} \Phi(x) \int_{P(x)}^{1} \frac{v^{r-1} (1-v)^{N-r} \; N!}{(r-1)! \; (N-r)!} \, d\,v \; .$$

Expanding $(1-v)^{N-r}$ as a binomial series in v, we get

$$\mu^{(t)}_{r:N} = \sum_{x=0}^{a-1} \sum_{k=0}^{N-r} \Phi(x) \; (-1)^k \frac{N!}{(r-1)! \; (N-r)!} \binom{N-r}{k}$$

$$\cdot \int_{P(x)}^{1} v^{r-1+k} \, d\,v$$

$$= \sum_{k=0}^{N-r} (-1)^k \binom{k+r-1}{k} \binom{N}{k+r} \sum_{x=0}^{a-1} \Phi(x) \left(1 - P^{r+k}(x)\right)$$

$$= \sum_{k=0}^{N-r} (-1)^k \binom{k+r-1}{k} \binom{N}{k+r} \mu^{(t)}_{r+k:r+k} \tag{3.1.28}$$

putting $r + k = j$ we get (a) .

Similarly if we expand v^{r-1} as a a binomial series in $(1-v)$ we get (b).

<u>Theorem 7</u>

$$\binom{N}{m} \mu_{q:m} = \sum_{r=0}^{N-m} \binom{N-q-r}{m-q} \binom{q+r-1}{r} \mu_{q+r:N}, N \geq m . \tag{3.1.29}$$

Proof .

$$\mu_{q+r:N} \doteq \sum_{x=0}^{a-1} \left[1 - I_{P(x)} (q + r, N - q - r + 1) \right].$$

So that R. H. S. of (3.1.29) $= \sum_{x=0}^{a-1} \sum_{r=0}^{N-m} \binom{N-q-r}{m-q} \binom{q+r-1}{r}$

$$. \left[1 - \int_0^{P(x)} \frac{v^{q+r-1} (1-v)^{N-q-r} N!}{(q+r-1)! (N-q-r)!} d v \right]$$

$$= \sum_{x=0}^{a-1} \int_{P(x)}^{1} \sum_{r=0}^{N-m} \binom{N-q-r}{m-q} \binom{q+r-1}{r} \frac{v^{q+r-1} (1-v)^{N-q-r} N!}{(q+r-1)! \ (N-q-r)!} \ d\,v \ .$$

But

$$\int_{P(x)}^{1} \sum_{r=0}^{N-m} \binom{N-q-r}{m-q} \binom{q+r-1}{r} \frac{v^{q+r-1} (1-v)^{N-q-r} N!}{(q+r-1)! \ (N-q-r)!} \ d\,v$$

$$= \int_{P(x)}^{1} \sum_{r=0}^{N-m} \frac{(N-m)! \ v^{r} \ (1-v)^{N-m-r}}{(N-r-m)! \ \ r!} \cdot \frac{v^{q-1} \ (1-v)^{m-q} \ N!}{(m-q)! \ (q-1)! \ (N-m)!} \ d\,v$$

$$= \int_{P(x)}^{1} \frac{v^{q-1} \ (1-v)^{m-q} \ N!}{(m-q)! \ (q-1)! \ (N-m)!} \ d\,v$$

$$= \binom{N}{m} \left[1 - I_{P(x)} \ (q, \ m - q + 1) \right].$$

Therefore

$$\text{R. H. S. of } (3.1.29) = \binom{N}{m} \sum_{x=0}^{a-1} \left\{ 1 - I_{P(x)} \ (q, \ m - q + 1) \right\}$$

$$= \binom{N}{m} \ \mu_{q:m}$$

which is the same result as given by Sillitto (1964) in the continuous case.

This result enables us to express the moments of order statistics in a sample of m in terms of the moments of order statistics in a sample of N where $N \geq m$.

We notice that Theorem 7 is the reverse of Theorem 5 .

Theorem 8

If the arbitrary distribution is symmetric about the origin, then

$$\mu^{(i,\ j,\ k,\ldots)}_{r,\ s,\ t,\ \ldots:\ N} = (-1)^{i+j+k+\ \ldots}\ \mu^{(i,\ j,\ k,\ldots)}_{N-r+1,\ N-s+1,\ N-t+1,\ldots:N} \tag{3.1.30}$$

and in particular

$$\mu^{(i)}_{r:N} = (-1)^{i}\ \mu^{(i)}_{N-r+1:N},\ r = 1,\ 2,\ \ldots,\ [\tfrac{N}{2}]\ .$$

(3.1.31)

$$\mu^{(i,j)}_{r,s:N} = (-1)^{i+j}\ \mu^{(i,j)}_{N-s+1,\ N-r+1:N} \qquad 1 \leq r \leq s \leq N\ .$$

Proof.

These results follow directly from (2.7.1).

<u>Theorem 9</u>

For any distribution symmetric about zero the matrix $((\mu_{r,s:N}))$ is doubly symmetric (i.e. symmetric with respect to the two major diagonals) and the distinct elements in $((\mu_{r,s:N}))$ are those lying in any wedge-shaped region bounded by the two major diagonals.

Hence the number of distinct elements in $((\mu_{r,s:N}))$ is $N(N+2)/4$ if N is even and is $(N+1)^2/4$ if N is odd.

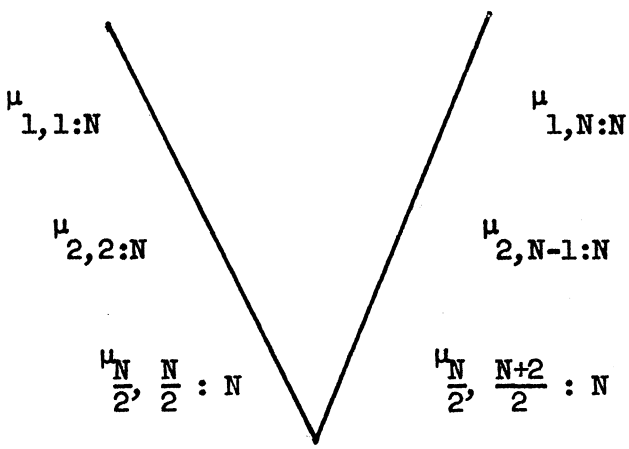

Figure (1)

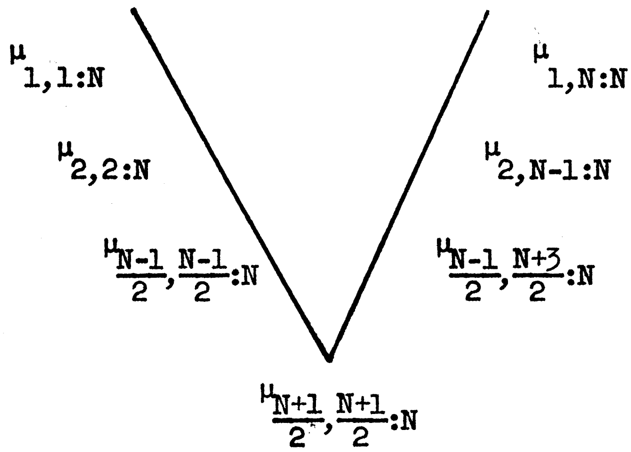

Figure (2)

Corollary 9.1

From Theorem 9, and Figures (1), (2) we obtain the following table

	Even N	Odd N
$\mu^{(2)}_{r:N}$	$N/2$	$(N+1)/2$
$\mu_{r,s:N}$ $(r\neq s)$	$N^2/4$	$(N^2-1)/4$
Total	$N(N+2)/4$	$\left[\frac{(N+1)}{2}\right]^2$

Theorem 10

For an arbitrary distribution symmetric about zero, the number of distinct and independent constraints among the distinct $\mu_{r,s:N}$ $(r \neq s)$ imposed by the recurrence formula in Theorem 4 is $\left[(N-1)^2 - 1\right] / 4$ if N is even and $(N-1)^2/4$ if N is odd.

The proofs of Theorems 9, 10 are the same as given by Govindarajulu (1963) in the continuous case.

Theorem 11

In order to find the first, second and mixed (linear) moments of order statistics in a sample of size N drawn from an arbitrary population symmetric about zero, given these moments for all sample sizes less than N, we have to evaluate *at most one single sum and* $\frac{(N-2)}{2}$ *double sums if* N *is even,* and one single sum and $(N-1)/2$ double sums if N is odd.

Proof.

Assume first, second and mixed (linear) moments are known for all sample sizes less than N.

To compute $\mu_{r:N}$ consider formula (3.1.28) with $t = 1$, such that

$$\mu_{r:N} = \sum_{k=0}^{N-r} (-1)^k \binom{k+r-1}{k} \binom{N}{k+r} \sum_{x=0}^{a-1} \left(1 - P(x)^{r+k}\right)$$

and all the inner sums $(k = 0, 1, 111 , N-r-1)$ would have been computed previously except for $\sum_{x=0}^{a-1} \left(1 - P^{N}(x)\right)$.

Hence we have to evaluate at most one sum when N is even and none when N is odd since

$$\mu_{\frac{(N+1)}{2}:N} = 0 \ .$$

Again, put $t = 2$ in (3.1.28) , we get

$$\mu^{(2)}_{r:N} = \sum_{k=0}^{N-r} (-1)^k \binom{K+r-1}{k} \binom{N}{K+r} \sum_{x=0}^{a-1}$$

$$\cdot \ (2x + 1) \left(1 - P^{r+k}(x)\right).$$

All of the inner sums would be available except

$$\sum_{x=0}^{a-1} (2x + 1) \left(1 - P^N(x)\right) .$$

Hence there would be at most one sum to be evaluated when N is odd and none when N is even, since

$$\mu^{(2)}_{\frac{N}{2}:N} = \mu^{(2)}_{\frac{N}{2}:N-1} \quad , \quad \text{when } N \text{ is even.}$$

To obtain

$$\mu_{r,s:N} = \frac{N!}{(r-1)!\,(s-r-1)!\,(N-s)!} \sum_{x} \sum_{\substack{y \\ x<y}} \int \int w^{r-1}$$

$$\cdot\ (v-w)^{s-r-1} (1-v)^{N-s} \quad d\,v\,d\,w$$

we follow a similar argument as given by Govindarajulu (1963) .

3.2 Relationships between the moments of the range

Notation [Sillitto (1951)].

Let $E(R|N) = w_N$

and

$$E\left(X_{(r+1)} - X_{(r)} \mid N \right) = \chi_{N,r} \ .$$

Theorem 12

$$w_N = \frac{1}{N} \left[\chi_{N,1} + \chi_{N,N-1} \right] + w_{N-1} \ .$$

Proof.

$$w_N = \sum_{x=0}^{a-1} \left[1 - P^N(x) - \left\{1 - P(x)\right\}^N \right]$$

so that

$$w_{N-1} = \sum_{x=0}^{a-1} \left[1 - P^{N-1}(x) - \left\{1 - P(x)\right\}^{N-1} \right].$$

Therefore

$$w_N = \sum_{x=0}^{a-1} \left\{ \left(1 - P(x)\right)^{N-1} P(x) + P^{N-1}(x) \left(1 - P(x)\right)\right\} + w_{N-1} .$$

Now, putting $s = r+1$ in equation (2.5.23), we get

$$\chi_{N,r} = \sum_{x=0}^{a-1} \binom{N}{r} \left\{P(x)\right\}^r \left\{1 - P(x)\right\}^{N-r} \tag{3.2.3}$$

$$\chi_{N,1} = \sum_{x=0}^{a-1} N \; P(x) \left\{1 - P(x)\right\}^{N-1}$$

$$\chi_{N,N-1} = \sum_{x=0}^{a-1} N \left\{ P(x) \right\}^{N-1} \left\{1 - P(x) \right\} .$$

Therefore

$$w_N = \frac{1}{N} \left[\chi_{N,1} + \chi_{N,N-1} \right] + w_{N-1} . \tag{3.2.2}$$

Similarly w_{N-1} can be expressed in terms of

$\chi_{N-1,1}$, $\chi_{N-1,N-2}$ and w_{N-2} .

Theorem 13

$$\frac{N}{r}\,\chi_{N-1,\,r-1} - \frac{N-r+1}{r}\,\chi_{N,r-1} = \chi_{N,r}\,. \tag{3.2.3}$$

Proof.

From (3.2.1) , we have

$$\frac{N}{r}\,\chi_{N-1,r-1} - \frac{N-r+1}{r}\,\chi_{N,r-1} = \sum_{x=0}^{a-1} \binom{N}{r}$$

$$\cdot \left\{P(x)\right\}^{r-1} \left\{1 - P(x)\right\}^{N-r}$$

$$- \sum_{x=0}^{a-1} \binom{N}{r} \left\{P(x)\right\}^{r-1} \left\{1 - P(x)\right\}^{N-r+1}$$

$$= \sum_{x=0}^{a-1} \binom{N}{r} \left\{P(x)\right\}^{r} \left\{1 - P(x)\right\}^{N-r}$$

$$= \chi_{N,r}\,.$$

Theorem 14

$$\chi_{N,r} = \frac{N!\,(r-v)!}{(N-v)!\,r!}\Bigg[\chi_{N-v,r-v} - v\,\frac{N-r+1}{N-v+1}\,\chi_{N-v+1,r-v}$$

$$+ \binom{v}{2}\frac{(N-r+1)(N-r+2)}{(N-v+1)(N-v+2)}\,\chi_{N-v+2,r-v}\cdots$$

$$+ (-1)^{v}\,\frac{(N-r+1)(N-r+2)\,\ldots\,(N-r+v)}{(N-v+1)(N-v+2)\,\ldots\,(N-1)\,N}\,\chi_{N,r-v}\Bigg] \qquad (3.2.4)$$

Proof.

From (3.2.3), we have

$$\chi_{N-1,r-1} = \frac{N-1}{r-1}\,\chi_{N-2,r-2} - \frac{N-r+1}{r-1}\,\chi_{N-1,r-2}$$

and

$$\chi_{N,r-1} = \frac{N}{r-1}\,\chi_{N-1,r-2} - \frac{N-r+2}{r-1}\,\chi_{N,r-2}\,.$$

Therefore

$$\chi_{N,r} = \frac{N(N-1)}{r(r-1)}\,\chi_{N-2,r-2} - 2\,\frac{N(N-r+1)}{r(r-1)}\,\chi_{N-1,r-2}$$

$$+ \frac{(N-r+1)(N-r+2)}{r(r-1)} \quad \chi_{N,r-2} \; .$$

Repeating the same process $(v-1)$ times, we get the result.

In the special case when $v=r-1$, then (3.2.4) reduces to

$$\chi_{N,r} = \binom{N}{r} \left\{ \frac{\chi_{N-r+1,1}}{N-r+1} - (r-1) \frac{\chi_{N-r+2,1}}{N-r+2} + \binom{r-1}{2} \right.$$

$$\left. \cdot \frac{\chi_{N-r+3,1}}{N-r+3} \; \dots + (-1)^{r-1} \; \frac{\chi_{N,1}}{N} \right\} . \qquad (3.2.5)$$

<u>Theorem 15</u>

$$- \sum_{i=0}^{r} (-1)^i \binom{r}{i} \; w_{N-r+i} = \frac{r!\,(N-r)!}{N!} \left(\chi_{N,N-r} + \chi_{N,r} \right) . \qquad (3.2.6)$$

Proof.

$$\text{L. H. S.} = -\sum_{x=0}^{a-1} \left\{ \sum_{i=0}^{r} (-1)^i \binom{r}{i} \sum_{i=0}^{r} (-1)^i \binom{r}{i} \; P(x)^{N-r+i} \right.$$

$$\left. - \sum_{i=0}^{r} (-1)^i \binom{r}{i} \left(1 - P(x)^{N-r+i} \right) \right\}$$

$$= \sum_{x=0}^{a-1} - \left\{ (1-1)^r - \left(1 - P(x)\right)^r \overset{N-r}{P(x)} - \left[1 - \left(1 - P(x)\right)\right]^r \right.$$

$$\left. \cdot \left(1 - \overset{N-r}{P(x)}\right) \right\}$$

$$= \sum_{x=0}^{a-1} \left\{ \left(1 - P(x)\right)^r \overset{N-r}{P(x)} + \overset{r}{P}(x) \left(1 - P(x)\right)^{N-r} \right\}$$

$$= \frac{r!\,(N-r)!}{N!} \left(\chi_{N,N-r} + \chi_{N,r} \right).$$

In a symmetrical population we have

$$\chi_{N,N-r} = \chi_{N,r} \quad .$$

Therefore in a symmetrical population (3.2.6) reduces to

$$- \sum_{i=0}^{r} (-1)^i \binom{r}{i} w_{N-r+i} = 2 \frac{r!\,(N-r)!}{N!} \chi_{N,r} \quad .$$

<u>Corollary 15.1</u>

$$w_3 = \frac{3}{2} w_2 \tag{3.2.7}$$

Proof.

Putting $r = N - 1$ in (3.2.6), we get :

For N odd

$$\frac{(N-1)!}{N!} \left(\chi_{N,1} + \chi_{N,N-1} \right) = - w_N + \binom{N-1}{1} w_{N-1}$$

$$- \binom{N-1}{2} w_{N-2} + \ldots + \binom{N-1}{1} w_2 \; .$$

Hence using relation (3.2.2), we have

$$w_N = \frac{1}{2} \left\{ N \, w_{N-1} - \binom{N-1}{2} w_{N-2} + \ldots + \binom{N-1}{1} w_2 \right\} . \qquad (3.2.8)$$

An equivalent formula is

$$(N-2) \, w_{N-1} = \binom{N-1}{2} w_{N-2} - \binom{N-1}{3} w_{N-3} + \ldots + \binom{N-1}{1} w_2 \; . \qquad (3.2.9)$$

As a special case of (3.2.8), (3.2.9)

$$w_3 = \frac{3}{2} \, w_2$$

which is the same result as given by Sillitto (1951) in the continuous case.

K. R. Nair (1950) used the symbol $j_{(r)}$ for the difference between the sum of the largest r and the smallest r members of a sample of N

$$\left[\, 1 \leq r \leq \tfrac{1}{2}\, N \quad \text{or} \quad \tfrac{1}{2}\,(N-1) \quad \text{according as } N \text{ is even or odd} \right]$$

which was suggested by Jones (1946) as a measure of dispersion in large samples. I shall use ${}_N D_{(r)}$ to denote this statistic in a sample of N.

Theorem 16

$$E\left({}_N D_{(r)} \right) = \sum_{s=0}^{r-1} (-1)^{r-s-1} \binom{N-s-2}{r-s-1} \binom{N}{s} w_{N-s}$$

$$0 \leq s \leq r-1 \qquad (3.2.10)$$

Proof.

$${}_N D_{(r)} = (x_N + x_{N-1} + \dots + x_{N-r+1}) - (x_r + x_{r-1} + \dots x_1)$$

$$= (x_N - x_1) + (x_{N-1} - x_2) + \dots + (x_{N-r+1}) - x_r) .$$

Therefore

$$E\left({}_N D_{(r)}\right) = w_N + \left[w_N - \chi_{N,N-1} - \chi_{N,1} \right] + \ldots + \left\{ w_N \right.$$

$$- \left(\chi_{N,N-1} + \chi_{N,1} \right) - \left(\chi_{N,N-2} + \chi_{N,2} \right) \cdots$$

$$\left. - \left(\chi_{N,N-r+1} + \chi_{N,r-1} \right) \right\} .$$

By means of (3.2.6) , we get the result, which is the same as that given by Sillitto (1951) in the continuous case.

Define $Z_{N,r} = x_{r+1} - x_r$ in a sample of N and $m'_N =$ the mean deviation from the median in the sample.

<u>Theorem 17</u>

$$E\left(m'_{2k+1}\right) = E\left(m'_{2k}\right) = \frac{1}{2} \sum_{j=1}^{k} \frac{1}{2j-1} \chi_{2j,j} . \qquad (3.2.11)$$

Proof.

If N is odd

$$m'_N = \frac{1}{N} \quad \left(x_{\frac{N+1}{2}} - x_1\right) + \left(x_{\frac{N+1}{2}} - x_2\right) + \cdots$$

$$+ \left(x_{\frac{N+1}{2}} - x_{\frac{N-1}{2}} \right) + \left(x_{\frac{N+3}{2}} - x_{\frac{N+1}{2}} \right) + \dots + \left(x_{N-1} \right.$$

$$\left. - x_{\frac{N+1}{2}} \right) + \left(x_N - x_{\frac{N+1}{2}} \right) = \frac{1}{N} \left\{ Z_{N,1} + 2Z_{N,2} \right.$$

$$+ \dots + \frac{1}{2}(N-1)\ Z_{N,\frac{N-1}{2}} + \frac{1}{2}\ (N-1)\ Z_{N,\frac{N+1}{2}} + \dots$$

$$\left. + 2\,Z_{N,N-2} + Z_{N,N-1} \right\} .$$

If N is even

$$m'_N = \frac{1}{N} \left\{ Z_{N,1} + 2\,Z_{N,2} + \dots + \left(\frac{N}{2} - 1 \right) Z_{N,\frac{N}{2}-1} \right.$$

$$\left. + \frac{N}{2}\ Z_{N,\frac{N}{2}} + \left(\frac{N}{2} - 1 \right) Z_{N,\frac{N}{2}+1} + \dots + 2\,Z_{N,N-2} + Z_{N,N-1} \right\} .$$

By taking expectations and using (3.2.3), we have

$$E\left(m'_{2k-1} \right) = E\left(m'_{2k} \right) - \frac{1}{2(2k-1)}\ \chi_{2k,k} \qquad (3.2.12)$$

and

$$E\left(m'_{2k+1}\right) = E\left(m'_{2k}\right) \tag{3.2.13}$$

By applying (3.2.12) repeatedly on the R. H. S. of itself, making use of (3.2.13) , we get the result.

<u>Theorem 18</u>

$$w_m = \sum_{r=1}^{N-1} \chi_{N,r} - \frac{(N-m)!}{N!} \sum_{r=1}^{N-m} \frac{(N-r)!}{(N-m-r)!} \left(\chi_{N,r} + \chi_{N,N-r} \right)$$

$$2 \leq m \leq N \tag{3.2.14}$$

Proof.

Putting $r = 1, 2, \ldots, N-1$ in (3.2.6) , we get $N-1$ independent equations relating $w_2, w_3, \ldots, w_N$ and $\chi_{N,1}, \chi_{N,2}, \ldots, \chi_{N,N-1}$.

By solving these equations, Sillitto (1951) deduced the result.

If $m = 2$

$$w_2 = \frac{2}{N(N-1)} \Big\{ (N-1)\ \chi_{N,1} + 2(N-2)\ \chi_{N,2} + \cdots$$

$$+ r(N-r)\ \chi_{N,r} + \cdots + (N-1)\ \chi_{N,N-1} \Big\} \qquad (3.2.15)$$

If we put $Z_{N,r}$ instead of its expectation $(\chi_{N,r})$ for $r = 1, 2, \ldots , N-1$ in (3.2.15) we get a statistics g_m which may be regarded as statistics that are generalizations of Gini's coefficient of mean difference.

That is $E(g_m) = w_m$.

We notice that the expectations of the g_m statistics are independent of the number of observations unlike Jones's $j_{(r)}$ statistics, or the mean deviation from the median.

<u>Theorem 19</u>

$$\chi_{m,q} = \binom{m}{q} \sum_{r=0}^{N-m} \binom{N-m}{r} \chi_{N,q+r} \Big/ \binom{N}{q+r} . \qquad (3.2.16)$$

Proof.

$$\chi_{N,r} = \sum_{x=0}^{a-1} \binom{N}{r} \left\{ P(x) \right\}^{r} \left\{ 1 - P(x) \right\}^{N-r}. \tag{3.2.17}$$

Hence,

$$\binom{m}{q} \sum_{r=0}^{N-m} \binom{N-m}{r} \quad \chi_{N,q+r} \Big/ \binom{N}{q+r} = \sum_{x=0}^{a-1} \sum_{r=0}^{N-m}$$

$$\cdot \frac{m!}{q!\ (m-q)!} \quad \frac{(n-m)!}{r!\ (N-m-r)!} \quad \frac{(q+r)!\ (N-q-r)!}{N!}$$

$$\cdot \frac{N!}{(q+r)!\ (N-q-r)!} \quad P(x)^{q+r} \quad \left(1 - P(x)\right)^{N-q-r}$$

$$= \sum_{x=0}^{a-1} \frac{m!}{q!\ (m-q)!} \quad P^{q}(x) \quad \left(1 - P(x)\right)^{m-q} \sum_{r=0}^{N-m}$$

$$\cdot \frac{(N-m)!}{r!\ (N-m-r)!} \quad P^{r}(x) \quad \left(1 - P(x)\right)^{N-m-r}$$

$$= \chi_{m,q} \; .$$

Lemma 2

$$P_{r,N}(x) = \sum_{i=r}^{N} \binom{N}{i} P(x)^{i} \left[1 - P(x)\right]^{N-i} \tag{3.2.18}$$

$$= \sum_{j=0}^{N-r} \binom{N}{j-r} P(x)^{j+r} \left[1 - P(x)\right]^{N-j-r} \tag{3.2.19}$$

From (3.2.18), we have

$$P_{r,N}(x) = \sum_{i=0}^{N} \binom{N}{i} P(x)^{i} \left[1 - P(x)\right]^{N-i} - \sum_{i=0}^{r-1}$$

$$\cdot \binom{N}{i} P(x)^{i} \left[1 - P(x)\right]^{N-i} .$$

First sum = 1

In the second sum put $i = r-j$, we get

$$P_{r,N}(x) = 1 - \sum_{j=1}^{r} \binom{N}{r-j} P(x)^{r-j} \left[1 - P(x)\right]^{N-r+j} . \tag{3.2.20}$$

<u>Theorem 20</u>

Let $w_2\,(x_{m,m})$ denote the expectation of the difference between pairs of largest members in samples of m, then

$$w_2\,(x_{m,m}) = 2 \sum_{r=1}^{m} \binom{m}{r} \chi_{2m,2m-r} \Big/ \binom{2m}{r}. \tag{3.2.21}$$

$$w_2\,(x_{m,1}) = 2 \sum_{r=1}^{m} \binom{m}{r} \chi_{2m,r} \Big/ \binom{2m}{r}. \tag{3.2.22}$$

Proof.

$$w_2\,(x_{m,m}) = \sum_{x=0}^{a-1} \left\{ 1 - \left[1 - P_{m,m}(x) \right]^2 - P^2_{m,m}\,(x) \right\}$$

$$= 2 \sum_{x=0}^{a-1} P_{m,m}\,(x) \left(1 - P_{m,m}(x) \right) .$$

Now by (3.2.19) and (3.2.20) , we have

$$P_{m,m}(x) = P^m(x)$$

and

$$1 - P_{m,m}(x) = \sum_{r=1}^{m} \binom{m}{m-r} P^{2m-r}(x)\left(1 - P(x)\right)^{r}.$$

Hence,

$$w_2\,(x_{m,m}) = 2 \sum_{x=0}^{a-1} \sum_{r=1}^{m} \binom{m}{r} P^{2m-r}(x)\left(1 - P(x)\right)^{r}.$$

So that, by (3.2.17), we have

$$w_2\,(x_{m,m}) = 2 \sum_{r=1}^{m} \binom{m}{r} \quad \chi_{2m,2m-r} \Big/ \binom{2m}{r} .$$

Similarly

$$w_2\,(x_{m,1}) = 2 \sum_{r=1}^{m} \binom{m}{r} \quad \chi_{2m,r} \Big/ \binom{2m}{r} .$$

Relations (3.2.21) and (3.2.22) enable us to express the expectation of the difference between pairs of extreme members in samples of m in terms of the expectations of differences between successive members in samples of $2m$, and by using

(3.2.2) , in terms of expectations of differences between successive members of samples of size greater than 2m .

Similarly Sillitto (1964) deduced the expectation of the difference between pairs of medians of samples of (2t-1) in the continuous case, and it holds also in the discrete case .

$$w_2\left(x_{2t-1,t}\right) = \sum_{r=0}^{t-2} c_r \left(\chi_{4t-2,t+r} + \chi_{4t-2,3t-2-r}\right)$$

$$+ \chi_{4t-2,2t-1} \tag{3.2.23}$$

where $c_r = 2 \sum_{u=0}^{r} \binom{t+r}{u}\binom{3t-2-r}{2t-1-u} \Big/ \binom{4t-2}{2t-1}$.

Theorem 21

The expectation of the range in a sample of two is a measure of the dispersion of a variate.

By means of equations (3.2.15) and (3.2.23) , it is possible to determine whether a population is such that the distribution of the medians of samples of $2t-1$ has a smaller dispersion than the population distribution.

Proof.

From equation (3.2.15), we have

$$w_2(x) = \frac{2}{(4t-2)(4t-3)} \sum_{r=1}^{4t-3} r(4t-2-r) \quad \chi_{4t-2,r} \tag{3.2.24}$$

which can be compared with equation (3.2.23).

For example, from equation (3.2.23), we get

$$w_2(x_{3,2}) = \frac{2}{5}\chi_{6,2} + \chi_{6,3} + \frac{2}{5}\chi_{6,4}$$

and from equation (3.2.24)

$$w_2(x) = \frac{1}{3}\chi_{6,1} + \frac{8}{15}\chi_{6,2} + \frac{3}{5}\chi_{6,3} + \frac{8}{15}\chi_{6,4}$$

$$+ \frac{1}{3}\chi_{6,5} .$$

Therefore the distribution of medians of samples of three has a smaller dispersion than the distribution of individual observations if

$$w_2(x) - w_2\left(x_{3,2}\right) = \frac{1}{2}\ \chi_{6,1} + \frac{2}{15}\ \chi_{6,2}$$

$$-\frac{2}{5}\ \chi_{6,3} + \frac{2}{15}\ \chi_{6,4} + \frac{1}{3}\ \chi_{6,5}$$

is positive.

3.3 Quasi - Ranges

Govindarajulu (1963) gives the following results in the continuous case, which hold also in the discrete case.

Notation

$$W_{r:N} = X_{N-r:N} - X_{r+1:N} \qquad r = 0, 1 \ldots ,$$

$$\left[\frac{(N-2)}{2}\right].$$

$$w_{r:N} = E\left(W_{r:N}\right) = \mu_{N-r:N} - \mu_{r+1:N} \qquad r = 0, 1, \ldots ,$$

$$\left[\frac{(N-2)}{2} \right] .$$

$W_{0:N}$ is the sample range.

$$w_{0:N} = E\left(W_{0:N} \right) .$$

$$a_{r,s:N} = E(W_{r:N} \; W_{s:N}) \qquad 0 \leq r \leq s \leq \left[\frac{(N-2)}{2} \right] .$$

$$a^{(2)}_{r:N} = a_{r,r:N} \qquad r = 0, 1, \ldots , \left[\frac{(N-2)}{2} \right] .$$

$\rho_{r,s:N}$ = the correlation between $X_{r:N}$ and $X_{s:N}$

$$(1 \leq r \leq s \leq N) .$$

For an arbitrary distribution

1.- $$(N- r) w_{r-1:N} + r w_{r:N} = N w_{r-1:N-1} \quad (r = 0, 1, \ldots, \left[(\frac{N-2}{2}) \right] . \tag{3.3.1}$$

Dividing both sides by N, equation (3.3.1) can be used for working downwards in numerical evaluation of the expected values of the sample quasi-ranges, without serious accumulation of rounding errors.

Now, for an arbitrary distribution symmetric about zero, we have the following results:

2. $w_{r:N} = 2\mu_{N-r:N} \qquad r = 0, 1, \ldots, \left[\frac{(N-2)}{2}\right].$ (3.3.2)

3.i $a_{r,s:N} = 2\left[\mu_{r+1,s+1:N} - \mu_{r+1,N-s:N}\right].$ (3.3.3)

3.ii $\mathrm{var}\left(W_{r:N}\right) = 2\,\mathrm{var}\left(X_{N-r:N}\right)\left[1 - \rho_{r+1,N-r:N}\right].$ (3.3.4)

4. $\mathrm{Cov}\left(W_{r:N}, W_{s:N}\right) = 2\left[\mathrm{Cov}\left(X_{r+1:N}, X_{s+1:N}\right)\right.$

$\left. - \mathrm{Cov}\left(X_{r+1:N}, X_{N-s:N}\right)\right].$ (3.3.5)

5. $\rho\left(W_{r:N}, W_{s:N}\right) = \frac{\rho_{r+1,\, s+1:N} - \rho_{r+1,\, N-s:N}}{\sqrt{(1-\rho_{r+1,N-r:N})(1-\rho_{s+1,N-s:N})}}.$ (3.3.6)

The results of 2, 3 and 4 can be used to prepare tables of the moments and product moments of quasi-ranges in samples drawn from populations symmetric about zero, provided tables of these for the corresponding order statistics are available.

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VII. CANDIDACY

To my wife and my daughter.

ORDER STATISTICS FOR A DISCRETE PARENT DISTRIBUTION

Raafat S. Mishriky

ABSTRACT

This paper provides a systematic study of order statistics drawn from discrete parent distributions. New procedures are followed for the derivation of the distribution of the r^{th} order statistic $X_{(r)}$ and of the joint distribution of $X_{(r)}$, $X_{(s)}$ $(s > r)$, that is, we first derive the cumulative probability distribution, from which the probability distribution comes directly. This approach is easier than the usual method, moreover the formulae for the c.d.f. derived in this way can be easily computed.

To get the moments of $X_{(r)}$, we use convenient formulae involving the tails of the c.d.f. of $X_{(r)}$ rather than the p.d.f. of $X_{(r)}$. The moments are then readily derived from general results for discrete distributions. We show the analogy between the results in the continuous and discrete cases. Applications to three discrete distributions are given.

We consider some results on uncorrelated statistics which were established in the continuous case and show that the same results hold also for the discrete case. Many recurrence relations between moments of order statistics are derived in the discrete case yielding the same results as previously given by Govindarajulu (1963) and Sillitto (1951, 1964) in the continuous case.