THE BEST TRUNCATION POINT FOR THE ESTIMATED SPECTRAL DENSITY FUNCTION OF A STATIONARY TIME SERIES

by

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CHAPTER I

INTRODUCTION AND LITERATURE REVIEW

1.1 INTRODUCTION

A set of random observations arranged chronologically is called a time series. Time series are observed in connection with quite diverse phenomena and by a wide variety of researchers; such as

(i) the economist trying to predict yearly corn prices;

(ii) the physicist studying ambient noise level at a given point in the river;

(iii) the engineer studying the internal noise of a certain machine;

Techniques of time series analysis have long been used in economics, physics and engineering sciences. Time series has also become an essential tool in space technology, since a wide variety of problems involving communication and control systems generate data which may be analyzed as time series.

We denote a time series by a set of observed values $\{y(t),\ t \in T\}$, where T is the index set. In terms of the index set T, there are two important cases:

(i) The discrete case, in which T is the set of positive integers,

$$T = \{1,2,\ldots\}$$

(ii) The continuous case, in which T is the infinite interval,

$$T = \{t: \; 0 \leq t \leq \infty\}.$$

In the present study we shall consider only the discrete case.

We shall treat a time series as an observation made on a family of random variables $\{Y(t),\ t \in T\}$, that is, for each $t \in T$, $Y(t)$ is a random variable taking the value $y(t)$. The family of random variables $\{Y(t),\ t \in T\}$ is called a stochastic process. An observed time series $\{y(t),\ t \in T\}$ is thus regarded as a realization or observation of a stochastic process $\{Y(t),\ t \in T\}$.

Two of the aims in analyzing a time series are:

(i) to understand the mechanism generating the time series;

(ii) to predict the behavior of the time series in the future.

There are essentially two main approaches to time series analysis:

(i) periodogram analysis;

(ii) autoregressive analysis.

The periodogram analysis was first introduced by Schuster [20] in 1898. The problem of tests of significance in periodogram analysis [7], [10], played a central role in the early stage of the development of time series analysis. The autoregressive approach was pioneered by Yule [24],[25],[26] in the early 1920's and has been surveyed and used extensively by Kendall [13].

In the 1930's and 1940's, the probabilistic theory of stationary time series was developed as a result of prediction theory [5],[11],[22]; stationarity implying that the distribution of the series is invariant under the group of translations on the time index T. Wold [23] interpreted the autoregressive approach in terms of the theory of stationary processes.

Much of the recent statistical research on time series analysis has been concerned with questions of statistical inference on stationary time series [1],[3],[12],[15],[18],[21]. One of the areas that has been studied by several authors is the problem of estimating the spectral density function of a stationary time series [1],[15],[18],[21], where the spectral density function is the Fourier transform of the covariance function in a stationary time series. Once the spectral density has been estimated the scientist is able to investigate the physical mechanism generating the time series, to determine the behavior of the dynamic linear system and possibly to simulate the time series for the purpose of prediction.

The periodogram method is often used to estimate the spectral density function of a stationary time series. However, this method does not yield a consistent estimator [16]. (An estimator is called a consistent estimator if the mean square error of the estimator tends to zero as the sample size tends to infinity.) Several researchers [1], [3], [15], [16], [18], [21], have proposed alternatives to the periodogram method which do yield consistent estimators of the spectral density function.

One problem which must be faced in estimating the spectral density is the choice of a sample size. A researcher may want to analyze only a part of the sample data [15], [18], [19], and may want to determine the truncation point which yields an estimator of the spectral density function which has minimum mean square error.

In this study we shall classify the weight functions (to be defined) which yield a consistent estimator of the spectral density function and we shall also formulate a new expression for the best truncation function.

In the remainder of this chapter, we shall survey and review the problem of obtaining a consistent estimator of the spectral density function and determine the best truncation function. The classification of the weight functions which give a consistent estimator of the spectral density function will be presented in Chapter II. The best truncation function for an algebraic weight function will be obtained in Chapter

III. The asymptotic form of the mean square error of the spectral density estimator obtained under best truncation will be examined. The relative asymptotic efficiency of this estimator will also be examined. In Chapter IV extensions will be presented which illustrate the best truncation function for other types of weight functions.

In the first part of the dissertation the assumption is made that the covariance function of a stationary time series is known. In the final chapter, the case of an unknown covariance function will be considered.

1.2 PERIODOGRAM ANALYSIS

The estimation of the spectral density function of a stationary time series has been studied extensively by several authors [1],[3],[15],[16],[18],[21]. We shall let $\{y(t)\}$ or $\{y_t\}$ denote the stationary time series under consideration, where the domain of the index t is understood to be the set of all positive integers $(1,2,\ldots,)$. With respect to the aims of our study, it is sufficient to assume stationary to the second order, that is

$$Ey(t) = 0, \tag{1.2.1}$$

and

$$Ey(t)y(t+\nu) = R(\nu). \tag{1.2.2}$$

The function $R(\nu)$, which does not depend on t and is a function only of the time interval is called the covariance function of the process $y(t)$. It can be written, Cramér [4], by

$$R(\nu) = \int_{-\pi}^{\pi} e^{i\omega\nu} dF(\omega), \tag{1.2.3}$$

where $F(\omega)$ is a bounded, non-decreasing function and is called the spectral distribution of the time series $y(t)$. If the sequence $\{R(\nu)\}$ is an absolutely summable function over its domain, then $F(\omega)$ possesses everywhere a derivative $f(\omega)$, called the spectral density function of the time series $y(t)$ given by

$$f(\omega) = \frac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} e^{-i\omega\nu} R(\nu). \tag{1.2.4}$$

Equation (1.2.4) was introduced by Schuster [20] in 1898, and is called a periodogram.

Let y_t, $t = 1,2,\ldots,T$ be a sample of size T (or realization of length T). An estimate of the spectral density function is given by

$$\hat{f}_T(\omega) = \frac{1}{2\pi} \sum_{\nu=1-T}^{T-1} e^{-i\omega\nu} R_T(\nu), \tag{1.2.5}$$

where $R_T(\nu)$ is the sample covariance function defined by

$$R_T(\nu) = \frac{1}{T-|\nu|} \sum_{t=1}^{T-|\nu|} y_t y_{t+|\nu|}, \quad |\nu| \leq T-1 \tag{1.2.6}$$

However, the estimator in (1.2.5) is not consistent.

An estimator is called a consistent estimator if

$$\lim_{T\to\infty} E \int_{-\pi}^{\pi} [\hat{f}_T(\omega) - f(\omega)]^2 d\omega = 0. \tag{1.2.7}$$

Another estimator which has been considered is given by

$$f_T(\omega) = \frac{1}{2\pi} \sum_{\nu=1-T}^{T-1} e^{-i\omega\nu} K_T(\nu) R_T(\nu), \tag{1.2.8}$$

where $K_T(\nu)$ assumes non-negative constant values, and satisfies certain criterion of consistency. $K_T(\nu)$ is called a <u>weight function</u> or <u>kernel function</u>. One basic and important problem is to choose the appropriate weight function, $K_T(\nu)$.

The first (non-constant) weight function ever introduced is the simple truncated function

$$K_T(\nu) = \begin{cases} 1 - \frac{|\nu|}{T}, & |\nu| \leq m(T) \\ 0, & |\nu| > m(T) \end{cases} \tag{1.2.9}$$

where m(T) is a positive function satisfying the following conditions:

$$\lim_{T\to\infty} m(T) = \infty, \lim_{T\to\infty} \frac{m(T)}{T} = 0, \text{ and } m(T) \leq T \tag{1.2.10}$$

Bartlett [1],[3], considered a <u>rectangular weight function</u> of the form

$$K_T(\nu) = \begin{cases} 1, & |\nu| \leq m(T) \\ 0, & |\nu| > m(T) \end{cases} \tag{1.2.11}$$

where m(T) satisfies the conditions in (1.2.10).

Bartlett also introduced the famous smoothed periodogram which has a weight function of the form

$$K_T(\nu) = \begin{cases} 1 - \frac{|\nu|}{m(T)}, & |\nu| \leq m(T) - 1 \\ 0, & |\nu| \geq m(T) \end{cases} \tag{1.2.12}$$

where m(T) satisfies condition (1.2.10). He compared the truncated and smoothed periodograms, and concluded that as an estimator, the smoothed periodogram was superior to the truncated periodogram.

Tukey [21], investigated the following weight function

$$K_T(\nu) = \begin{cases} 1-2a+2a \cos \frac{|\nu|}{m(T)} \pi, & |\nu| \leq m(T) \\ 0, & |\nu| > m(T) \end{cases} \tag{1.2.13}$$

where m(T) satisfies (1.2.10). In his study two different values of a, $a = \frac{1}{2}$ and $a = 0.23$ were considered; unfortunately no justification was given for choosing these numerical values.

Lomnicki and Zaremba [15], introduced the following weight function

$$K_T(\nu) = \frac{R^2(\nu)}{ER_T^2(\nu)}, \quad |\nu| = 0,1,2,\ldots,(T-1), \qquad (1.2.14)$$

where $R_T(\nu)$ is the sample covariance function defined by equation (1.2.6). They labeled (1.2.14) the <u>ideal weight function</u>. The mean square error of the spectral density function estimator defined by equation (1.2.8) using the ideal weight function is a minimum and is equal to

$$I = \sum_{\nu=1-T}^{T-1} \frac{R^2(\nu)}{ER_T^2(\nu)} + 2 \sum_{\nu=T}^{\infty} R^2(\nu). \qquad (1.2.15)$$

Although the ideal weight function leads to a minimum mean square error density estimator it requires knowledge of the unknown covariance function and thus can not be used in practice. They continued their study by constructing the following weight function

$$K_T(\nu) = \frac{p^{|\nu|}}{a(T-|\nu|)^{-1}+p^{|\nu|}}, \quad |\nu| = 0,1,2,\ldots,(T-1) \quad (1.2.16)$$

where p is any positive number smaller than 1 and a any positive number. For a given p, a can be chosen in the following manner:

$$a = \sum_{\nu=-\infty}^{\infty} p^{|\nu|} = \frac{1+p}{1-p} . \qquad (1.2.17)$$

Lomnicki and Zaremba also show that the weight function in (1.2.16), with a chosen as in (1.2.17) and for a certain value of p, is very similar to the ideal weight function. However, from a practical point of view it is very difficult to choose the value p. Ideally the choice of the parameter p should reflect the average rate of decrease of the covariance function $\{R(\nu)\}$.

Recently, Parzen [16], proposed the following weight function which yield a consistent spectral density estimator.

$$K_T(\nu) = \begin{cases} 1 - \left(\frac{|\nu|}{m(T)}\right)^q, & |\nu| \leq m(T) - 1 \\ 0 & |\nu| \geq m(T) \end{cases} \qquad (1.2.18)$$

where m(T) satisfies equation (1.2.10) and q is called a characteristic exponent. When q = 1, (1.2.18) yields a smoothed periodogram.

In Chapter II, we shall classify the weight functions which give consistent estimators of the spectral density function into two different types.

1.3 TRUNCATION FUNCTION

Recall that in this study the spectral density function of a time series is being estimated by

$$f_T(\omega) = \frac{1}{2\pi} \sum_{\nu=1-T}^{T-1} e^{-i\nu\omega} K_T(\nu) R_T(\nu), \qquad (1.3.1)$$

where $R_T(\nu)$ is the sample covariance function defined by equation (1.2.6), and $K_T(\nu)$ is the weight function to be chosen in such a way so as to give a consistent estimate of the spectral density function. The weight functions we mentioned in section 1.2 are related to (1.2.10) by the class of functions m(T) which satisfy the following conditions:

$$\lim_{T\to\infty} m(T) = \infty, \ \lim_{T\to\infty} \frac{m(T)}{T} = 0, \text{ and } m(T) \leq T \qquad (1.3.2)$$

Functions which satisfy these conditions will be called truncation functions. In this section we shall review the literature of choosing the truncation function which gives the minimum mean square error of the estimated spectral density function for a given weight function. Lomnicki and Zaremba [15] have found the best truncation function for the truncated periodogram and the smoothed periodogram. That is, the best truncation point m'(T) for both types is the asymptotic solution of the following equation in ν.

$$(T-\nu) \text{ Var } R_T(\nu) = (T+\nu)\ R^2(\nu),$$

where Var $R_T(\nu)$ is the variance of the sample covariance function $R_T(\nu)$.

The usefulness of their proposed truncation point was illustrated by considering a first-order Markov process as an example.

In the remainder of this section we shall review some of the work of Parzen [18] and Pickands [19] concerning the best truncation function problem.

1.3.1 PARZEN'S APPROACH

In this subsection we shall review the work done by Parzen in 1958. In his study [18], Parzen proposed two different types of weight functions which are of the following general forms:

(1) Exponential weight functions

$$K_T(\nu) = h(A_T e^{\alpha|\nu|}), \ \alpha > 0$$
$$\text{where } A_T = AT^{-b}, \ A > 0, \ b > 0. \tag{1.3.3}$$

(2) Algebraic weight functions

$$K_T(\nu) = h(B_T|\nu|),$$
$$\text{where } B_T = BT^{-b}, \ B > 0, \ b > 0. \tag{1.3.4}$$

Parsen considered two different types of covariance functions of the time series. These two types were the exponential type of covariance function of the form $e^{-\rho|\nu|}$, $\rho > 0$ and the algebraic type of covariance function of the form $C_\beta|\nu|^{-\beta}$, $C_\beta > 0$, $\beta \geq 1$.

Parzen investigated the integrated mean square error of the spectral density function estimator obtained by using

$$I[f_T(\omega)] = E\int[f_T(\omega) - f(\omega)]^2 d\omega, \qquad (1.3.5)$$

by considering the above two types of weight functions and assuming the corresponding covariance function. Although he investigated truncation functions he only gave the best truncation function for the specific algebraic weight function discussed below.

Consider the algebraic type weight function where the function h(u) satisfies the following condition

$$h^{(p)} = \lim_{u\to 0} \frac{1-h(u)}{|u|^p} < \infty, \qquad (1.3.6)$$

with $b = \frac{1}{2p+1}$ in (1.3.4) and $p > 0$. The integrated mean square error of the estimator obtained using this weight function is

$$T^{2p/(2p+1)} I[f_T(\omega)] = \frac{1}{B} S_0 S(h) + B^{2p}|h^{(p)}|^2 S_{2p} \qquad (1.3.7)$$

where

$$S_{2p} = \frac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} \nu^{2p} R^2(\nu), \qquad (1.3.8)$$

$$S(h) = \int_{-\infty}^{\infty} h^2(u)\,du. \qquad (1.3.9)$$

For fixed value of T, expression (1.3.7) is a function of B. Let I_T be the minimum integrated mean square error over all possible choices of B. The following expression for the relative minimum integrated mean square error can be obtained

$$\left(\frac{I_T}{S_0}\right)^{1+\frac{1}{2p}} = \frac{1}{T} D(p) T(h) \quad \frac{S_{2p}}{S_0}^{\frac{1}{2p}} \qquad (1.3.10)$$

where

$$D(p) = (1 + 2p)^{\frac{1}{2p}}(1 + \frac{1}{2p}), \qquad (1.3.11)$$

$$T(h) = h^{(p)\frac{1}{p}} S(h). \qquad (1.3.12)$$

The minimum value of the integrated mean square error is obtained at a value B which satisfies the following equation.

$$\left(\frac{1}{B}\right)^{2p+1} = 2p \frac{S_{2p}}{S_o} \frac{|h^{(p)}|^2}{S(h)} \qquad (1.3.13)$$

Equivalently, the constant B_T in (1.3.4) satisfies

$$\left(\frac{1}{B_T}\right)^{2p+1} = T\left(\frac{1}{B}\right)^{2p+1} = 2pT \frac{S_{2p}}{S_o} \frac{|h(p)|^{(2p+1)/p}}{T(h)} . \qquad (1.3.14)$$

The function h(u) which vanishes for $|u| > 1$ is a truncated algebraic type weight function. The best truncation function m'(T) which achieves the minimum relative integrated mean square error can be derived from (1.3.14) as follows:

$$m'(T) = \frac{1}{B_T} = \left(2pT \frac{S_{2p}}{S_o}\right)^{\frac{1}{2p+1}} \frac{|h^{(p)}|^{\frac{1}{p}}}{T(h)^{1/(2p+1)}} . \qquad (1.3.15)$$

1.3.2 PICKANDS' APPROACH

Pickands, in a recent paper [19], considered the following form of the estimated spectral density function

$$\hat{f}(\omega) = \frac{1}{2\pi} \sum_{\nu=1-T}^{T-1} a(\nu,T)\hat{R}(\nu)e^{i\omega\nu} \qquad (1.3.16)$$

where $a(\nu,T) = a(-\nu,T)$ does not depend on the observations y_t, $t = 1,2,\ldots,T$, and $\hat{R}(\nu)$ is an estimate of the covariance function defined by

$$\hat{R}(\nu) = \frac{1}{T-|\nu|} \sum_{t=1}^{T-|\nu|} (y_t-\bar{y})(y_{t+|\nu|} - \bar{y}) \;, \qquad (1.3.17)$$

where

$$\bar{y} = \frac{1}{T} \sum_{t=1}^{T} y_t \;. \qquad (1.3.18)$$

He defined the integrated mean square error of the estimator

$$I = 2\pi \; E\int_{-\pi}^{\pi} [\hat{f}(\omega) - f(\omega)]^2 d\omega; \qquad (1.3.19)$$

and obtained an ideal weight function, $b(\nu,T)$, which possesses minimum integrated mean square error, I_{min}. These functions are given by

$$b(\nu,T) = R(\nu) \; E\,\hat{R}(\nu)/ \; E\,\hat{R}_2(\nu) \;, \qquad (1.3.20)$$

and

$$I_{min} = \sum_{\nu=1-T}^{T-1} R^2(\nu) \; \mathrm{Var}\,\hat{R}(\nu)/ \; E\,\hat{R}^2(\nu). \qquad (1.3.21)$$

He considered the truncated periodogram estimator of the spectral density function of the form

$$\hat{f}^*(\omega) = \frac{1}{2\pi} \sum_{\nu=-m(T)}^{m(T)} \left(1 - \frac{|\nu|}{T}\right) \hat{R}(\nu) e^{i\omega\nu}, \qquad (1.3.22)$$

where $m(T)$ satisfies the condition (1.2.10). He obtained the best truncation function $m'(T)$ which yields an estimator with asymptotic relative efficiency [as defined by equation (1.3.22)] of unity, that is, the ratio I_{min} to the integrated mean square error of the estimator in equation (1.3.22) is one in the limit as $T \to \infty$. Pickands obtained the above results under the condition that

$$-\ln R^2(\nu) = -C\nu^{\gamma} \tag{1.3.23}$$

for some $C > 0$ and $\gamma < \infty$, and the best truncation function he obtained is of the form given by the following equation

$$m'(T) = C^{-\frac{1}{\gamma}}(\ln T)^{\frac{1}{\gamma}} . \tag{1.3.24}$$

In the present study we shall investigate the best truncation function of a consistent spectral density function estimator. The method of finding the best truncation function of a given algebraic type of weight function, which has different conditions in comparison with that of Parzen's algebraic type of weight function is presented in Chapter III. Furthermore, various applications of the theory developed in Chapter III to certain other weight functions are presented in Chapter IV.

CHAPTER II

CONSISTENT ESTIMATES OF THE SPECTRAL DENSITY FUNCTION OF A LINEAR PROCESS

2.1 INTRODUCTION

The problem of estimating the spectral density function of a stationary time series is of considerable importance both from the theoretical and application point of views. A consistent estimator of the spectral density is desired in most applications. The basic problem of obtaining a consistent estimator of the spectral density function is to choose a weight function as defined in Chapter I. In this chapter we shall examine a type of weight function which gives a consistent estimator.

In section 2.2 we state the conditions, obtained by Lomnicki and Zaremba [15], under which the weight function yields a consistent estimator of a spectral density function of a linear process. However, Lomnicki and Zaremba did not mention what kind of weight function satisfies their conditions. In section 2.3 we shall examine the type of weight functions which satisfy such conditions and, furthermore, show that the weight function in this category can be split into two classes. Specific weight functions are displayed in section 2.4 which illustrate the usefulness of our results.

2.2 CONDITIONS FOR THE ESTIMATE TO BE CONSISTENT

The work we review in this section is mainly the product of Lomnicki and Zaremba [15]. They investigated a weight function which gives a consistent estimator of the spectral density function of a linear process.

Definition 2.2.1 A stochastic process X_t will be called a linear stochastic process if the process is defined by

$$X_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j} , \qquad (2.2.1)$$

where

$$\sum_{j=0}^{\infty} |a_j| < \infty , \qquad (2.2.2)$$

and $\{\varepsilon_j\}$ is a sequence of uncorrelated random variables with zero means and having the same variance σ^2. It is assumed that the moments of the variable $\{\varepsilon_j\}$ satisfy the following conditions.

$$E(\varepsilon_i^2) = \sigma^2, \ E(\varepsilon_i^3) = \mu_3, \ E(\varepsilon_i^4) = \mu_4 = 3\sigma^4 + \kappa_4, \text{ for all } i$$

$$E(\varepsilon_i^2 \varepsilon_j^2) = \sigma^4, \text{ where } i \neq j; \qquad (2.2.3)$$

$$E(\varepsilon_i \varepsilon_j \varepsilon_k) = 0, \text{ unless } i = j = k;$$

$$E(\varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l) = 0, \text{ unless two pairs of indices coincide}$$

where κ_4 is the 4^{th} cumulant of $\{\varepsilon_i\}$.

From the properties of $\{\varepsilon_i\}$, we have that

$$E(X_t) = 0 , \tag{2.2.4}$$

and

$$R(\nu) = E(X_t X_{t+|\nu|}) = \sigma^2 \sum_{j=0}^{\infty} a_j a_{j+|\nu|} . \tag{2.2.5}$$

From equation (2.2.2), it has been shown that (Leppink [8])

$$\sum_{\nu=-\infty}^{\infty} |R(\nu)| < \infty . \tag{2.2.6}$$

For convenience we write the spectral density function as

$$f(\omega) = \sum_{\nu=-\infty}^{\infty} R(\nu) \cos 2\pi\nu\omega, \ (-\tfrac{1}{2} \leq \omega \leq \tfrac{1}{2}). \tag{2.2.7}$$

Let x_t, $t = 1,2,\ldots,T$, be a sample of size T from the linear process X_t. An estimate of the covariance function $R(\nu)$ of the linear process is given by

$$R_T(\nu) = \frac{1}{T-|\nu|} \sum_{t=1}^{T-|\nu|} x_t x_{t+|\nu|}, \ |\nu| \leq T-1. \tag{2.2.8}$$

It is clear that

$$R_T(-\nu) = R_T(\nu), \ \nu = 0,1,2,\ldots,(T-1). \tag{2.2.9}$$

It can be seen that $R_T(\nu)$ is an unbiased estimate of $R(\nu)$ since

$$E\, R_T(\nu) = R(\nu), \ |\nu| = 0,1,2,\ldots,(T-1). \tag{2.2.10}$$

It can be shown that (Hanan [6])

$$E\, R_T^2(\nu) = R^2(\nu) + \frac{R^2(\nu)}{T-|\nu|}\frac{\kappa_4}{\sigma^4} + \frac{1}{T-|\nu|}\sum_{p=|\nu|+1-T}^{T-|\nu|-1} \left(1-\frac{|p|}{T-|\nu|}\right)$$

$$\cdot [R^2(p) + R(p+|\nu|)\ R(p-|\nu|)],\quad |\nu| \le (T-1). \qquad (2.2.11)$$

If we assume that

$$R_2 = \sum_{\nu=-\infty}^{\infty} R^2(\nu) < \infty , \qquad (2.2.12)$$

then

$$\lim_{T\to\infty} E\ R_T^2(\nu) = R^2(\nu) \text{ for fixed } \nu. \qquad (2.2.13)$$

This indicates that $R_T^2(\nu)$ is an asymptotically unbiased estimate of $R^2(\nu)$ for fixed ν. The variance of $R_T(\nu)$ is given by

$$\operatorname{Var}\ R_T(\nu) = \frac{R^2(\nu)}{T-|\nu|}\frac{\kappa_4}{\sigma^4} + \frac{1}{T-|\nu|}\sum_{p=|\nu|+1-T}^{T-|\nu|-1}\left(1 - \frac{|p|}{T-|\nu|}\right)$$

$$\cdot [R^2(p) + R(p+|\nu|)\ R(p-|\nu|)],\quad |\nu| \le T-1. \qquad (2.2.14)$$

Furthermore, Lomnicki and Zaremba [15] gave the following result,

$$(T-|\nu|)\ \operatorname{Var}\ R_T(\nu) \le \frac{\kappa_4}{\sigma^4} R^2(0) + 2\sum_{\nu=-\infty}^{\infty} R^2(\nu),$$

$$|\nu| \le T-1. \qquad (2.2.15)$$

An estimate $f_T(\omega)$ of $f(\omega)$ is

$$f_T(\omega) = \sum_{\nu=1-T}^{T-1} k_T(\nu) R_T(\nu) \cos 2\pi\nu\omega,\quad (-\tfrac{1}{2} \le \omega \le \tfrac{1}{2}), \qquad (2.2.16)$$

where $k_T(\nu)$, the weight function, is a non-negative constant not exceeding 1 and satisfying the following condition

$$k_T(\nu) = k_T(-\nu),\quad \nu = 1,2,\dots,(T-1). \qquad (2.2.17)$$

The mean square error of the estimate, [15], is

$$I_T = E\int_{-\frac{1}{2}}^{\frac{1}{2}} [f_T(\omega) - f(\omega)]^2 d\omega, \tag{2.2.18}$$

$$= \sum_{\nu=1-T}^{T-1} k_T^2(\nu) \operatorname{Var} R_T(\nu) + \sum_{\nu=1-T}^{T-1} [1-k_T(\nu)]^2 R^2(\nu) +$$

$$2 \sum_{\nu=T}^{\infty} R^2(\nu). \tag{2.2.19}$$

Definition 2.2.2 The estimator $f_T(\omega)$ given by equation (2.2.16) of $f(\omega)$ will be called strongly consistent in the mean over the interval $[a,b]$, if

$$\lim_{T\to\infty} E\{\int_a^b [f_T(\omega) - f(\omega)]^2 d\omega\} = 0. \tag{2.2.20}$$

Lomnicki and Zaremba [15] proved the following important theorem which will be used frequently in this study.

Theorem 2.2.3 Let X_t be the linear process defined by equation (2.2.1). If equation (2.2.12) holds, then the estimator $f_T(\omega)$ defined by equation (2.2.16) is strongly consistent in the mean if and only if

(i) $0 \leq k_T(\nu) \leq 1, \quad |\nu| \leq T-1;$ (2.2.21)

(ii) $\lim_{T\to\infty} k_T(\nu) = 1$, for fixed ν; (2.2.22)

(iii) $\lim_{T\to\infty} \sum_{\nu=1-T}^{T-1} \frac{k_T^2(\nu)}{T-|\nu|} = 0.$ (2.2.23)

This theorem gives the conditions under which the weight function $k_T(\omega)$ yields a consistent estimator $f_T(\omega)$ of $f(\omega)$.

We can rewrite equation (2.2.18) as follows

$$I_T = E\{\int_{-\frac{1}{2}}^{\frac{1}{2}} [f_T(\omega)-f(\omega)]^2 d\omega\} = E\{\int_{-\frac{1}{2}}^{\frac{1}{2}} [(f_T(\omega)-E\ f_T(\omega))+(Ef_T(\omega)-$$

$$f(\omega))]^2 d\omega\}$$

$$= E\{\int_{-\frac{1}{2}}^{\frac{1}{2}} [f_T(\omega) - E\ f_T(\omega)]^2 d\omega + E\{\int_{-\frac{1}{2}}^{\frac{1}{2}} [Ef_T(\omega)-f(\omega)]^2 d\omega\}$$

$$= \sigma^2\ f_T(\omega) + b^2\ f_T(\omega)\ , \tag{2.2.24}$$

where

$$\sigma^2 [f_T(\omega)] = E\{\int_{-\frac{1}{2}}^{\frac{1}{2}} [f_T(\omega)-Ef_T(\omega)]^2 d\omega\}, \tag{2.2.25}$$

is called the <u>variance of the estimate</u> $f_T(\omega)$, and

$$b^2 [f_T(\omega)] = E\{\int_{-\frac{1}{2}}^{\frac{1}{2}} [Ef_T(\omega)-f(\omega)]^2 d\omega\}, \tag{2.2.26}$$

is called the bias of the estimate $f_T(\omega)$. However,

$$\sigma^2 [f_T(\omega)] = E\{\int_{-\frac{1}{2}}^{\frac{1}{2}} [\sum_{\nu=1-T}^{T-1} k_T(\nu)(R_T(\nu)-R(\nu))\cos 2\pi\nu\omega]^2 d\omega\}$$

$$= \sum_{\nu=1-T}^{T-1} \sum_{\nu=1-T}^{T-1} k_T(\nu_1)k_T(\nu_2)E(R_T(\nu_1)-R(\nu_1))$$

$$\cdot (R_T(\nu_2)-R(\nu_2))$$

$$\cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos 2\pi\nu_1\omega \cos 2\pi\nu_2\omega\ d\omega$$

$$= \sum_{\nu=1-T}^{T-1} k_T^2(\nu)\ E(R_T(\nu)-R(\nu))^2$$

$$= \sum_{\nu=1-T}^{T-1} k_T^2(\nu)\ \mathrm{Var}\ R_T(\nu)\ , \tag{2.2.27}$$

because

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos 2\pi\nu_1\omega \cos 2\pi\nu_2\omega d\omega = \begin{cases} 1, & \text{if } \nu_1 = \nu_2 \\ 0, & \nu_1 \neq \nu_2 \end{cases} .$$

From equation (2.2.19) we have

$$b^2[f_T(\omega)] = \sum_{\nu=1-T}^{T-1} [1-k_T(\nu)]^2 R^2(\nu) + 2\sum_{\nu=1-T}^{T-1} R^2(\nu). \qquad (2.2.28)$$

2.3 CLASSIFICATION OF THE WEIGHT FUNCTIONS

The condition given by equation (2.2.23) is a critical one in Lomnicki and Zaremba's results. We now give two general classes of weight functions $k_T(\nu)$ which satisfy the condition given by equation (2.2.23) under the assumption that the conditions expressed by equations (2.2.21) and (2.2.22) are satisfied.

Theorem 2.3.1 Suppose that the conditions given by equation (2.2.21) and (2.2.22) are satisfied. If $k_T(\nu)$ is defined to be nonnegative in the interval $[-m(T), m(T)]$ and 0 outside the interval, where $m(T)$ is a truncation function which assumes values smaller than T and satisfies equation (1.3.2). Then, the condition given by equation (2.2.23) is satisfied.

Proof: Since $0 \leq k_T(\nu) \leq 1$, we have

$$\lim_{T\to\infty} \sum_{\nu=1-T}^{T-1} \frac{k_T^2(\nu)}{T-|\nu|} = \lim_{T\to\infty} \sum_{\nu=-m(T)}^{m(T)} \frac{k_T^2(\nu)}{T-|\nu|} \leq \lim_{T\to\infty} \sum_{\nu=-m(T)}^{m(T)} \frac{1}{T-|\nu|}$$

$$\leq \lim_{T\to\infty} \frac{1}{T-m(T)} \sum_{\nu=-m(T)}^{m(T)} 1 = \lim_{T\to\infty} \frac{2m(T)+1}{T-m(T)} = 0. \qquad (2.3.1)$$

The second class of weight functions is given in the following theorem.

Theorem 2.3.2 Suppose that the conditions given by equations (2.2.21) and (2.2.22) are satisfied, and $k_T(\nu)$ is a function of the form $h(x)$ for some x

$$k_T(\nu) = \begin{cases} h\left(\frac{|\nu|}{m(T)}\right), & |\nu| \leq T-1 \\ 0, & |\nu| \geq T, \end{cases} \tag{2.3.2}$$

where $m(T)$ satisfies equation (1.3.2). If

(i) $\lim_{T\to\infty} \frac{m(T)\ln T}{T} = 0$ for any $m(T)$ satisfying equation (1.3.2), (2.3.3)

and

(ii) $h(x) \leq \frac{1}{x}$ for $x \geq 1$, (2.3.4)

then the condition given by equation (2.2.23) will be satisfied.

Proof: We can write

$$\lim_{T\to\infty} \sum_{\nu=1-T}^{T-1} \frac{k_T^2(\nu)}{T-|\nu|} = \lim_{T\to\infty} \sum_{\nu=-m(T)}^{m(T)} \frac{k_T^2(\nu)}{T-|\nu|} + 2 \lim_{T\to\infty} \sum_{\nu=m(T)+1}^{T-1} \frac{h^2\left(\frac{\nu}{m(T)}\right)}{T-\nu} . \tag{2.3.5}$$

The first term of the right-hand side of equation (2.3.5) is zero by the result of Theorem 2.3.1 The second term of the right-hand side of equation (2.3.5) can be written as follows

$$2 \lim_{T\to\infty} \sum_{\nu=m(T)+1}^{T-1} \frac{h^2(\frac{\nu}{m(T)})}{T-\nu} \leq 2 \lim_{T\to\infty} \int_{m(T)}^{T-1} \frac{h(\frac{\nu}{m(T)})}{T-\nu} d\nu$$

$$= 2 \lim_{T\to\infty} \frac{m(T)}{T} \int_1^{\frac{T-1}{m(T)}} \frac{h^2(x)}{1 - \frac{xm(T)}{T}} dx, \quad x = \frac{\nu}{m(T)}$$

$$\leq 2 \lim_{T\to\infty} \frac{m(T)}{T} \int_1^{\frac{T-1}{m(T)}} \frac{dx}{x^2(1 - \frac{m(T)x}{T})} \leq 2 \lim_{T\to\infty} \frac{m(T)}{T}$$

$$\cdot \int_1^{\frac{T-1}{m(T)}} \frac{dx}{x(1 - \frac{m(T)}{T}x)}$$

$$= 2 \lim_{T\to\infty} \frac{m(T)}{T} \int_1^{\frac{T-1}{m(T)}} [\frac{1}{x} + \frac{m(T)}{T} (1 - \frac{m(T)}{T} x)^{-1}] dx$$

$$= 2 \lim_{T\to\infty} \frac{m(T)}{T} [\ell n\ x - \ell n\ (1 - \frac{m(T)}{T} x)]_1^{\frac{T-1}{m(T)}}$$

$$= 2 \lim_{T\to\infty} \frac{m(T)}{T} [\ell n \frac{T-1}{m(T)} + \ell n\ T + \ell n\ (1 - \frac{m(T)}{T})] = 0,$$

which proves the theorem.

2.4 EXAMPLES

We now discuss some specific weight functions which satisfy the conditions of Theorem 2.3.1 or Theorem 2.3.2. Two well-known examples are

(1) Truncated Periodogram

$$k_T(\nu) = \begin{cases} 1 - \frac{|\nu|}{T}, & |\nu| \leq m(T) \\ 0, & |\nu| > m(T) \end{cases} \qquad (2.4.1)$$

where m(T) satisfies equation (1.3.2).

(2) Smoothed Periodogram

$$k_T(\nu) = \begin{cases} 1 - \frac{|\nu|}{m(T)}, & |\nu| \leq m(T) - 1 \\ 0, & |\nu| \geq m(T) \end{cases} \qquad (2.4.2)$$

where m(T) satisfies equation (1.3.2).

The above two functions which are known to yield consistent estimators of $f(\omega)$ satisfy Theorem 2.3.1. Another $k_T(\nu)$ which satisfies also Theorem 2.3.1 is given in the following example.

(3) $$k_T(\nu) = \begin{cases} e^{-|\nu|/T}, & |\nu| \leq m(T) \\ 0, & |\nu| > m(T) \end{cases} \qquad (2.4.3)$$

where m(T) satisfies equation (1.3.2).

The following two examples give $k_T(\nu)$ satisfying Theorem 1.3.2.

(4) $$k_T(\nu) = \begin{cases} e^{-|\nu|/m(T)}, & |\nu| \leq T-1 \\ 0, & |\nu| > T-1 \end{cases} \qquad (2.4.4)$$

where m(T) satisfies equation (2.3.1) and (2.3.3). This function has the value between 0 and 1 included in the interval $[1 - T, T - 1]$ and is 0 as $T \to \infty$. For the condition given by equation (2.3.4), we consider for

$x = \frac{|\nu|}{m(T)} \geq 1$

$$\frac{1}{x} - e^{-x} = \frac{e^x - 1}{x\, e^x} > 0.$$

which implies that equation (2.3.4) is satisfied. Hence the estimator $f_T(\omega)$ is consistent for any $m(T)$ satisfying the stated conditions.

The last example is

$$K_T(\nu) = \begin{cases} \dfrac{1}{\left(1 + \frac{|\nu|}{m(T)}\right)^k}, & |\nu| \leq T-1 \\ 0, & |\nu| \geq T \end{cases} \qquad (2.4.5)$$

where $m(T)$ satisfies the conditions given by equations (1.3.2) and (2.3.3), and $k \geq 1$. Obviously, this function satisfies the conditions given by equations (2.2.21) and (2.2.22), furthermore, with $x = \frac{|\nu|}{m(T)}$ it is easy to see that $\frac{1}{x} \geq \frac{1}{(1+x)^k}$ for $k \geq 1$ and $x \geq 1$.

2.5 SUMMARY

The consistency of an estimator $f_T(\omega)$ of $f(\omega)$ depends on the weight function $k_T(\nu)$ it contains. If we consider those weight functions $k_T(\nu)$ satisfying the conditions of Theorems 2.3.1 and 2.3.2, then we will achieve the goal of having a consistent estimator $f_T(\omega)$. Note that the function given by equation (2.3.2) of Theorem 2.3.2 requires stronger restrictions on m(T) than the functions which satisfy Theorem 2.3.1, this is due to the fact that the function given by equation (2.3.2) has a large number of terms after $|\nu| > m(T)$.

CHAPTER III

THE BEST TRUNCATION FUNCTION

3.1 INTRODUCTION

It was shown in Chapter II that the form of the consistent estimator of the spectral density function depends on the weight function which in turn, is related to the truncation function. The mean square error defined by equation (2.2.18) of the estimator obtained by using a given weight function will be different for different truncation functions. A truncation function which yields a minimum mean square error estimator will be called a best truncation function.

In this chapter we shall study the formulation of the best truncation function and the relative efficiency of the spectral density function estimator obtained by using a given weight function and the best truncation function.

In section 3.2 we shall review some results of Parzen [18], which relate to aims of this study. Specifically, Parzen developed a formula which yields the best truncation function for an algebraic type of weight function under certain conditions. In section 3.3 we shall derive the formulae which yield the best truncation function for an algebraic type of weight function under conditions different than Parzen's. In order to study this algebraic type of weight function, we shall begin by examining the conditions under which the related estimator of the spectral density function is consistent. We shall also obtain asymptotic expressions for the variance and the bias of the estimator. We will show that the best truncation function for this algebraic weight function depends on one term of the bias of the estimator which we shall denote by $C(m(T))$. $C(m(T))$ is an important part of determining the best truncation function. In section 3.5, we shall study three different types of $C(m(T))$. In section 3.4, we shall investigate the asymptotic relative efficiency of the spectral density function estimator formed by using the best truncation point.

3.2 PRELIMINARIES

Parzen [18], studied time series with two different types of covariance functions. The first one is the exponential type, and the second one is the algebraic type.

Definition 3.2.1 The covariance function $R(\nu)$ will be said to be of the exponential type or to decrease exponentially with coefficient $\rho > 0$, if the following three conditions hold:

(i) for some constant R_0 and for all ν

$$|R(\nu)| \leq R_0 e^{-\rho|\nu|}, \tag{3.2.1}$$

(ii) for almost all u in $0 < |u| < 1$

$$\lim_{\nu\to\infty} e^{\rho\nu}|R(u\nu)| = \infty; \tag{3.2.2}$$

(iii) for any constant $C > 0$

$$\lim_{\nu\to\infty} \frac{1}{\nu} \sum_{|u|<\nu} \frac{1}{1+Ce^{2\rho\nu}R^2(u)} = 0. \tag{3.2.3}$$

Definition 3.2.2 The covariance function $R(\nu)$ will be said to be of the algebraic type or to decrease algebraically of degree $\beta > 1$, if for large ν it is of the form $C_\beta|\nu|^{-\beta}$, that is, for some finite positive constant C_β,

$$\lim_{\nu\to\infty} \nu^\beta|R(\nu)| = C_\beta. \tag{3.2.4}$$

Parzen defined the sample covariance function as follows:

$$R_T'(\nu) = \frac{1}{T}\sum_{t=1}^{T-|\nu|} x_t x_{t+|\nu|}, \tag{3.2.5}$$

and for the algebraic type of covariance function he considered estimators of the spectral density function of the form

$$f_T'(\omega) = \sum_{\nu=1-T}^{T-1} h\left(\frac{|\nu|}{m(T)}\right)R_T'(\nu)\cos 2\pi\nu\omega, \quad \left(-\frac{1}{2}\leq\omega\leq\frac{1}{2}\right), \tag{3.2.6}$$

where $m(T)$, the truncation function, satisfies

$$\lim_{T\to\infty} m(T) = \infty, \ \lim_{T\to\infty}\frac{m(T)}{T} = 0, \text{ and } m(T) \leq T \tag{3.2.7}$$

The function $h(x)$, defined for all x, is even and is assumed to satisfy the following conditions for some positive constants p, ε, H_0, H_p and H:

$$|h(x)| \leq H_0 \quad , \quad \text{for all } x, \tag{3.2.8}$$

$$|1-h(x)| \leq H_p|x|^p, \quad |x| \leq 1, \tag{3.2.9}$$

$$|x^{\frac{1}{2}+\varepsilon}h(x)| \leq H \quad , \quad |x| \geq 1. \tag{3.2.10}$$

A function $h(x)$ satisfying conditions (3.2.8), (3.2.9), and (3.2.10) will be said to be of type p, and the corresponding estimator $f_T'(\omega)$ will be said to be of algebraic type p. Parsen showed that for the algebraic type covariance function

$$\lim_{T\to\infty} T^{1-\frac{1}{2p}} \sigma^2[f_T'(\omega)] = R_2 \int_{-\infty}^{\infty} h^2(x)dx, \tag{3.2.11}$$

where $\sigma^2[f_T'(\omega)]$ is the variance of the estimate $f_T'(\omega)$, and

$$R_2 = \sum_{\nu=-\infty}^{\infty} R^2(\nu) < \infty. \tag{3.2.12}$$

Let $I[m(T)]$ denote the mean square error of the estimator of the spectral density function obtained by using $m(T)$ as the truncation function. If for a given weight function there is a function $m'(T)$ satisfying condition (3.2.7) such that if $m(T)$ satisfies condition (3.2.7), then $I[m'(T)] \leq I[m(T)]$, will be called a <u>best truncation function</u>. Parzen [18], showed that the best truncation function for the type p algebraic estimator is of the form

$$m'(T) = (2pT\, R_2^{-1}R_{2p}^{*})^{\frac{1}{1+2p}}|h^{(p)}|^{\frac{1}{p}}[T(h)]^{-\frac{1}{2p+1}}, \tag{3.2.13}$$

where

$$R^*_{2p} = \sum_{\nu=-\infty}^{\infty} \nu^{2p} R^2(\nu), \quad p \geq 1, \tag{3.2.14}$$

$$|h^{(p)}| = \lim_{x \to 0} \frac{1-h(x)}{|x|^p}, \tag{3.2.15}$$

and

$$T(h) = |h^{(p)}|^{\frac{1}{p}} \int_{-\infty}^{\infty} h^2(x)dx. \tag{3.2.16}$$

For a given weight function denote the minimum mean square error of the estimators of the spectral density function by

$$I_{0pt} = \min E\int_{-\infty}^{\infty} [f_T(\omega) - f(\omega)]^2 d\omega. \tag{3.2.17}$$

Parzen [18], obtained the following two theorems concerning the minimum mean square errors.

Theorem 3.2.3 If $R(\nu)$ decreases exponentially with coefficient $\rho > 0$, then

$$\lim_{T\to\infty} \frac{T}{\ln T} I_{0pt} = \frac{1}{\rho} R_2, \tag{3.2.18}$$

where R_2 is defined in equation (3.2.12).

Theorem 3.2.4 If $R(\nu)$ decreases algebraically of degree $\beta > 1$, then

$$\lim_{T\to\infty} T^{(1-\frac{1}{2\beta})} I_{0pt} = C_\beta^{\frac{1}{\beta}} R_2^{1-\frac{1}{2\beta}} \int_{-\infty}^{\infty} (1+u^{2\beta})^{-1} du, \tag{3.2.19}$$

where $C_\beta > 0$.

3.3 FORMULATION OF THE BEST TRUNCATION FUNCTION

Recall that

$$R'_T(\nu) = \frac{T-|\nu|}{T} R_T(\nu) = (1 - \frac{|\nu|}{T}) R_T(\nu). \qquad (3.3.1)$$

and consider the algebraic weight function

$$K_T(\nu) = \begin{cases} (1 - \frac{|\nu|}{T}) h(\frac{|\nu|}{m(T)}), & |\nu| \leq T-1 \\ 0, & |\nu| \geq T \end{cases} \qquad (3.3.2)$$

for some function h(x), where m(T) satisfies equation (3.2.7).

In this section we shall prove conditions on h(x) which guarantee a strongly consistent spectral density estimator. We will then proceed to display the best truncation functions.

The following are conditions on h(x) under which the estimator $f_T(\omega)$, given by

$$f_T(\omega) = \sum_{\nu=1-T}^{T-1} K_T(\nu) R_T(\nu) \cos 2\pi\nu\omega, \quad (-\tfrac{1}{2} \leq \omega \leq \tfrac{1}{2}) \qquad (3.3.3)$$

is a strongly consistent estimator of the spectral density function $f(\omega)$.

<u>Theorem 3.3.1</u> Let X_t be a linear process defined by equation (2.2.1). If equation (3.2.12) holds, then the estimator $f_T(\omega)$ given in (3.3.3) is a strongly consistent estimator in the mean if:

(i) $0 \leq h(x) \leq 1$, for all x; (3.3.4)

(ii) $\lim_{x \to 0} h(x) = 1$; (3.3.5)

(iii) $\int_0^\infty h^2(x)dx = F < \infty$, (3.3.6)

for any $m(T)$ which satisfies equation (3.2.7).

<u>Proof</u>: It is sufficient to prove that equations (2.2.21), (2.2.22), and (2.2.23) of Theorem 2.2.3 are satisfied if equations (3.3.4), (3.3.5), and (3.3.6) hold.

(i) Equation (3.3.4) implies $0 \leq h(\frac{|\nu|}{m(T)}) \leq 1$ for all ν, hence $0 \leq K_T(\nu) \leq 1$ for all ν, that is, equation (2.2.21) holds.

(ii) Expression (3.3.5) implies $\lim_{T \to \infty} h(\frac{|\nu|}{m(T)}) = 1$ for fixed ν, hence, $\lim_{T \to \infty} K_T(\nu) = 1$ for fixed ν, that is, equation (2.2.22) holds.

(iii) Consider the expression

$$\lim_{T \to \infty} \sum_{\nu=1-T}^{T-1} \frac{K_T^2(\nu)}{T-|\nu|} = \lim_{T \to \infty} \sum_{\nu=1-T}^{T-1} \frac{T-|\nu|}{T^2} h^2 \left(\frac{|\nu|}{m(T)}\right)$$

$$= \lim_{T \to \infty} \frac{1}{T} \sum_{\nu=1-T}^{T-1} h^2\left(\frac{|\nu|}{m(T)}\right) - \lim_{T \to \infty} \frac{1}{T^2} \sum_{\nu=1-T}^{T-1} |\nu| h^2\left(\frac{|\nu|}{m(T)}\right).$$

(3.3.7)

The first term of the right-hand side of equation (3.3.7) can be written

$$\lim_{T\to\infty} \frac{1}{T} \sum_{\nu=1-T}^{T-1} h^2\left(\frac{|\nu|}{m(T)}\right) = \lim_{T\to\infty} \frac{1}{T} h^2(0) + 2 \lim_{T\to\infty} \frac{1}{T} \sum_{\nu=1}^{T-1} h^2\left(\frac{\nu}{m(T)}\right)$$

$$\leq \lim_{T\to\infty} \frac{1}{T} h^2(0) + 2 \lim_{T\to\infty} \frac{1}{T} \int_0^{T-1} h^2\left(\frac{\nu}{m(T)}\right) d\nu$$

$$= \lim_{T\to\infty} \frac{1}{T} h^2(0) + 2 \lim_{T\to\infty} \frac{m(T)}{T} \int_0^{\frac{T-1}{m(T)}} h^2(x)\,dx = 0, \qquad (3.3.8)$$

by $x = \frac{\nu}{m(T)}$ and the condition (3.3.6).

The second term of the right-hand side of equation (3.3.7) can be written

$$\lim_{T\to\infty} \frac{1}{T^2} \sum_{\nu=1-T}^{T-1} |\nu| h^2\left(\frac{|\nu|}{m(T)}\right) \leq \lim_{T\to\infty} \frac{1}{T} \sum_{\nu=1-T}^{T-1} h^2\left(\frac{|\nu|}{m(T)}\right) = 0,$$

$$(3.3.9)$$

by the preceeding result. Combining the results of equations (3.3.8) and (3.3.9) gives (2.2.23), which completes the proof.

The following theorems give the asymptotic forms of the variance and the bias of the estimator $f_T(\omega)$. Since these forms are dependent on $m(T)$, we shall use the notation $\sigma_m^2[f_T(\omega)]$ for the variance of the estimator $f_T(\omega)$, and $b_m^2[f_T(\omega)]$ for the bias of the estimator $f_T(\omega)$.

<u>Theorem 3.3.2</u> Let X_t be the linear process defined by equation (2.2.1). If

(i) $h(x)$ satisfies the conditions of Theorem 3.3.1, and $m(T)$ satisfies equation (3.2.7), and

(ii) $$\sum_{\nu=-\infty}^{\infty} |R(\nu)| < \infty \text{ and } \sum_{\nu=-\infty}^{\infty} |\nu| R^2(\nu) < \infty, \tag{3.3.10}$$

then

$$\sigma_m^2[f_T(\omega)] = \sum_{\nu=1-T}^{T-1} \frac{T-|\nu|}{T^2} h^2\left(\frac{|\nu|}{m(T)}\right) \sum_{p=|\nu|+1-T}^{T-|\nu|-1} R^2(p) + O(T^{-1}), \tag{3.3.11}$$

where $O(T^{-1})$ does not depend on $m(T)$.

Proof: From equations (2.2.14) and (2.2.27), we have

$$\sigma_m^2[f_T(\omega)] = \sum_{\nu=1-T}^{T-1} \left(1-\frac{|\nu|}{T}\right) h^2\left(\frac{|\nu|}{m(T)}\right) \operatorname{Var} R_T(\nu)$$

$$= \frac{\kappa_4}{\sigma^4} \sum_{\nu=1-T}^{T-1} \frac{T-|\nu|}{T^2} h^2\left(\frac{|\nu|}{m(T)}\right) R^2(\nu) + \sum_{\nu=1-T}^{T-1} \frac{T-|\nu|}{T} h^2\left(\frac{|\nu|}{m(T)}\right)$$

$$\cdot \sum_{p=|\nu|+1-T}^{T-|\nu|-1} \left(1-\frac{|p|}{T-|\nu|}\right) [R^2(p) + R(p+|\nu|) R(p-|\nu|)]$$

$$= \frac{1}{T^2} \frac{\kappa_4}{\sigma^4} \sum_{\nu=1-T}^{T-1} (T-|\nu|) h^2\left(\frac{|\nu|}{m(T)}\right) R^2(\nu)$$

$$+ \sum_{\nu=1-T}^{T-1} \frac{T-|\nu|}{T^2} h^2\left(\frac{|\nu|}{m(T)}\right) \sum_{p=|\nu|+1-T}^{T-|\nu|-1} \left(1-\frac{|p|}{T-|\nu|}\right) R^2(p)$$

$$+ \sum_{\nu=1-T}^{T-1} \frac{T-|\nu|}{T^2} h^2\left(\frac{|\nu|}{m(T)}\right) \sum_{p=|\nu|+1-T}^{T-|\nu|-1} \left(1-\frac{|p|}{T-|\nu|}\right) [R(p+|\nu|) R(p-|\nu|)]. \tag{3.3.12}$$

(a) The first term of the right-hand side of equation (3.3.12) can be written as

$$\frac{1}{T^2}\frac{\kappa_4}{\sigma^4}\sum_{\nu=1-T}^{T-1}(T-|\nu|)h^2\left(\frac{|\nu|}{m(T)}\right)R^2(\nu) \le \frac{1}{T}\frac{\kappa_4}{\sigma^4}\sum_{\nu=1-T}^{T-1}h^2\left(\frac{|\nu|}{m(T)}\right)R^2(\nu)$$

$$-\frac{1}{T}\frac{k_4}{\sigma^4}\sum_{\nu=1-T}^{T-1}R^2(\nu) = O(T^{-1}). \qquad (3.3.13)$$

(b) The middle term of the right-hand side of equation (3.3.12) can be written as

$$\sum_{\nu=1-T}^{T-1}\frac{T-|\nu|}{T^2}h^2\left(\frac{|\nu|}{m(T)}\right)\sum_{p=|\nu|+1-T}^{T-|\nu|-1}\left(1-\frac{|p|}{T-|\nu|}\right)R^2(p)$$

$$= \sum_{\nu=1-T}^{T-1}\frac{T-|\nu|}{T^2}h^2\left(\frac{|\nu|}{m(T)}\right)\sum_{p=|\nu|+1-T}^{T-|\nu|-1}R^2(p)$$

$$- \sum_{\nu=1-T}^{T-1}\frac{1}{T^2}h^2\left(\frac{|\nu|}{m(T)}\right)\sum_{p=|\nu|+1-T}^{T-|\nu|-1}|p|R^2(p)$$

$$= \sum_{\nu=1-T}^{T-1}\frac{T-|\nu|}{T^2}h^2\left(\frac{|\nu|}{m(T)}\right)\sum_{p=|\nu|+1-T}^{T-|\nu|-1}R^2(p) + O(T^{-1}). \qquad (3.3.14)$$

because

$$\sum_{\nu=1-T}^{T-1}\frac{1}{T^2}h^2\left(\frac{|\nu|}{m(T)}\right)\sum_{p=|\nu|+1-T}^{T-|\nu|-1}|p|R^2(p) \le \sum_{\nu=1-T}^{T-1}\frac{1}{T^2}h^2\left(\frac{|\nu|}{m(T)}\right)\sum_{p=-\infty}^{\infty}|p|R^2(p)$$

$$\leq \sum_{p=-\infty}^{\infty} |p| R^2(p) \sum_{\nu=1-T}^{T-1} \frac{1}{T^2} = \frac{2T-1}{T^2} \sum_{p=-\infty}^{\infty} |p| R^2(p) = O(T^{-1}).$$

(c) The last term of the right-hand side of equation (3.3.12) can be written as

$$\sum_{\nu=1-T}^{T-1} \frac{T-|\nu|}{T^2} h^2(\frac{|\nu|}{m(T)}) \sum_{p=|\nu|+1-T}^{T-|\nu|-1} (1-\frac{|p|}{T-|\nu|}) [R(p+|\nu|)R(p-|\nu|)]$$

$$\leq \sum_{\nu=1-T}^{T-1} \frac{T-|\nu|}{T^2} h^2(\frac{|\nu|}{m(T)}) \sum_{p=|\nu|+1-T}^{T-|\nu|-1} |R(p+|\nu|)R(p-|\nu|)|$$

$$\leq \sum_{\nu=1-T}^{T-1} \frac{T-|\nu|}{T^2} h^2(\frac{|\nu|}{m(T)}) \sum_{p=-\infty}^{\infty} |R(p+|\nu|)||R(p-|\nu|)|$$

$$\leq \sum_{\nu=1-T}^{T-1} \frac{1}{T} \sum_{p=-\infty}^{\infty} |R(p+|\nu|)||R(p-|\nu|)|$$

$$= \frac{1}{T} \sum_{\nu=1-T}^{T-1} \sum_{p=-\infty}^{\infty} |R(p)||R(p-2|\nu|)| \leq \frac{1}{T} \sum_{\nu=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} |R(\nu)||R(p-2|\nu|)|$$

$$= \frac{1}{T} \sum_{p=-\infty}^{\infty} |R(p)| \sum_{\nu=-\infty}^{\infty} |R(p-2|\nu|)| \leq \frac{1}{T} \{ \sum_{p=-\infty}^{\infty} |R(p)| \}^2 = O(T^{-1}).$$

(3.3.15)

Substituting (a), (b), and (c) into equation (3.3.12) gives the desired result.

Theorem 3.3.3 Let X_t be the linear process defined by equation (2.2.1). If conditions (i) and (ii) of Theorem 3.3.2 hold, then

$$b_m^2[f_T(\omega)] = \sum_{\nu=1-T}^{T-1} [1-h(\frac{|\nu|}{m(T)})]^2 R^2(\nu) + 2\sum_{\nu=T}^{\infty} R^2(\nu) + O(T^{-1}), \tag{3.3.16}$$

where $O(T^{-1})$ does not depend on $m(T)$.

Proof: The proof follows directly from equation (2.2.28), since

$$b_m^2[f_T(\omega)] = \sum_{\nu=1-T}^{T-1} [1-h(\frac{|\nu|}{m(T)})(1-\frac{|\nu|}{T})]^2 R^2(\nu) + 2\sum_{\nu=T}^{\infty} R^2(\nu)$$

$$= \sum_{\nu=1-T}^{T-1} [1-h(\frac{|\nu|}{m(T)}) + \frac{|\nu|}{T} h(\frac{|\nu|}{m(T)})]^2 R^2(\nu) + 2\sum_{\nu=T}^{\infty} R^2(\nu)$$

$$= \sum_{\nu=1-T}^{T-1} [1-h(\frac{|\nu|}{m(T)})]^2 R^2(\nu) + \frac{2}{T}\sum_{\nu=1-T}^{T-1} h(\frac{|\nu|}{m(T)})(1-h(\frac{|\nu|}{m(T)}))|\nu| R^2(\nu)$$

$$+ \frac{1}{T^2}\sum_{\nu=1-T}^{T-1} h^2(\frac{|\nu|}{m(T)})|\nu|^2 R^2(\nu) + 2\sum_{\nu=T}^{\infty} R^2(\nu)$$

$$= \sum_{\nu=1-T}^{T-1} [1-h(\frac{|\nu|}{m(T)})]^2 R^2(\nu) + 2\sum_{\nu=T}^{\infty} R^2(\nu) + O(T^{-1}), \tag{3.3.17}$$

(3.3.17) holds since

$$\frac{2}{T}\sum_{\nu=1-T}^{T-1} h^2\left(\frac{|\nu|}{m(T)}\right)\left[1-h\left(\frac{|\nu|}{m(T)}\right)\right]|\nu|R^2(\nu) \leq \frac{2}{T}\sum_{\nu=1-T}^{T-1}|\nu|R^2(\nu)=O(T^{-1});$$

(3.3.18)

and

$$\frac{1}{T^2}\sum_{\nu=1-T}^{T-1} h^2\left(\frac{|\nu|}{m(T)}\right)\nu^2R^2(\nu) \leq \frac{1}{T}\sum_{\nu=1-T}^{T-1}|\nu|R^2(\nu) = O(T^{-1}). \quad (3.3.19)$$

The following theorems show that the best truncation point depends on the bias of the estimator $f_T(\omega)$. More precisely, it depends on the term $C(m(T))$ which will be defined in Theorem 3.3.6. First we shall examine the term $\frac{T}{m(T)}\ \sigma_m^2[f_T(\omega)]$ which approaches a constant in the limit as T tends to infinity for any $m(T)$ satisfying condition (3.2.7).

<u>Theorem 3.3.4</u> If $h(x)$, $m(T)$ and $R(\nu)$ satisfy the conditions of Theorem 3.3.2, then

$$\lim_{T\to\infty}\frac{T}{m(T)}\ \sigma_m^2[f_T(\omega)] = 2FR_2 = M, \qquad (3.3.20)$$

where R_2 and F are defined by equations (3.2.12) and (3.3.6), respectively.

<u>Proof</u>: Using equation (3.3.10), we have

$$\lim_{T\to\infty}\frac{T}{m(T)}\sigma_m^2[f_T(\omega)] = \lim_{T\to\infty}\frac{1}{m(T)}\sum_{\nu=1-T}^{T-1}\left(1-\frac{|\nu|}{T}\right)h^2\left(\frac{|\nu|}{m(T)}\right)\sum_{p=|\nu|+1-T}^{T-|\nu|-1}R^2(p)$$

$$+\lim_{T\to\infty}O(m^{-1}(T)) = 2\lim_{T\to\infty}\frac{1}{m(T)}\sum_{\nu=1}^{T-1}\left(1-\frac{\nu}{T}\right)h^2\left(\frac{\nu}{m(T)}\right)\sum_{p=\nu+1-T}^{T-\nu-1}R^2(p)$$

$$+ \lim_{T\to\infty} \frac{1}{m(T)} h^2(0) \sum_{p=1-T}^{T-1} R^2(p) + \lim_{T\to\infty} o(m^{-1}(T))$$

$$= 2 \lim_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=1}^{T-1} (1-\frac{\nu}{T}) h^2(\frac{\nu}{m(T)}) \sum_{p=\nu+1-T}^{T-\nu-1} R^2(p). \tag{3.3.21}$$

We now show that

$$\text{(i)} \quad \lim_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=1}^{T-1} (1-\frac{\nu}{T}) h^2(\frac{\nu}{m(T)}) \sum_{p=\nu+1-T}^{T-\nu-1} R^2(p)$$

$$\leq \lim_{T\to\infty} \frac{1}{m(T)} \sum_{p=-\infty}^{\infty} R^2(p) \sum_{\nu=1}^{T-1} (1 - \frac{\nu}{T}) h^2(\frac{\nu}{m(T)})$$

$$\leq \lim_{T\to\infty} \frac{1}{m(T)} \sum_{p=-\infty}^{\infty} R^2(p) \sum_{\nu=1}^{T-1} h^2(\frac{\nu}{m(T)})$$

$$\leq R_2 \lim_{T\to\infty} \frac{1}{m(T)} \int_0^{T-1} h^2(\frac{\nu}{m(T)}) d\nu = R_2 F.$$

Thus,

$$\limsup_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=1}^{T-1} (1 - \frac{\nu}{T}) h^2(\frac{\nu}{m(T)}) \sum_{p=\nu+1-T}^{T-\nu-1} R^2(p) \leq R_2 F. \tag{3.3.22a}$$

However, on the other hand we have

$$\text{(ii)} \quad \lim_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=1}^{T-1} (1 - \frac{\nu}{T}) h^2(\frac{\nu}{m(T)}) \sum_{p=\nu+1-T}^{T-\nu-1} R^2(p)$$

$$\geq \lim_{T\to\infty} \frac{1}{m(T)} \int_1^T (1 - \frac{\nu}{T}) h^2(\frac{\nu}{m(T)}) \sum_{p=\nu+1-T}^{T-\nu-1} R^2(p) d\nu$$

$$= \lim_{T\to\infty} \int_{1/m(T)}^{T/m(T)} (1 - \frac{m(T)x}{T}) h^2(x) \sum_{p=m(T)x+1-T}^{T-m(T)x-1} R^2(p) dx$$

$$= \lim_{T\to\infty} \int_{1/m(T)}^{T/m(T)} (1 - \frac{m(T)x}{T}) h^2(x)\overline{R}^2(T-m(T)x-1)dx,$$

where $\overline{R}_2(\omega) = \sum_{\nu=-\omega}^{\omega} R^2(\omega)$, and $\lim_{\omega\to\infty} \overline{R}_2(\omega) = R_2$.

Let $T_1^* > 0$ be an arbitrarily chosen number, then

$$\frac{m(T)T_1^*}{T} \leq \frac{1}{T_1^*}$$

is true for sufficiently large values of T, and furthermore,

$$\int_{1/m(T)}^{T/m(T)} (1 - \frac{m(T)x}{T}) h^2(x)\overline{R}_2(T-m(T)x-1)dx$$

$$\geq \int_{1/m(T)}^{T_1^*} (1 - \frac{m(T)x}{T}) h^2(x)\overline{R}_2(T-m(T)x-1)dx$$

$$\geq (1 - \frac{1}{T_1^*}) \int_{1/m(T)}^{T_1^*} h^2(x)\overline{R}_2(T-T_1^*m(T)-1)dx.$$

Hence,

$$\liminf_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=1}^{T-1} (1 - \frac{\nu}{T}) h^2(\frac{\nu}{m(T)}) \sum_{p=\nu+1-T}^{T-\nu-1} R^2(p)$$

$$\geq (1 - \frac{1}{T_1^*}) R_2 \int_0^{T_1^*} h^2(x)dx.$$

Since $T_1^* > 0$ is arbitrarily chosen, we have

$$\liminf_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=1}^{T-1} (1 - \frac{\nu}{T}) h^2(\frac{\nu}{m(T)}) \sum_{p=\nu+1-T}^{T-\nu-1} R^2(p) \geq R_2 \int_0^{T_1^*} h^2(x)dx$$

$$= R_2 \int_0^{\infty} h^2(x)dx - R_2 \int_{T_1^*}^{\infty} h^2(x)dx.$$

Also, since $\int_0^\infty h^2(x)dx = F < \infty$, for any $\varepsilon > 0$, we can choose T_2^* such that

$$\int_{T_2^*}^\infty h^2(x)dx \le \frac{\varepsilon}{R_2} .$$

If we choose $T^* = \max(T_1^*, T_2^*)$, then

$$\liminf_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=1}^{T-1} (1 - \frac{\nu}{T}) h^2(\frac{\nu}{m(T)}) \sum_{p=\nu+1-T}^{T-\nu-1} R^2(p) \ge FR_2. \tag{3.3.22b}$$

From equations (3.3.22a) and (3.3.22b), we can write

$$\lim_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=1}^{T-1} (1 - \frac{\nu}{T}) h^2(\frac{\nu}{m(T)}) \sum_{p=\nu+1-T}^{T-\nu-1} R^2(p) = FR_2,$$

and the proof of the theorem is complete.

The following lemma will be needed in a later proof.

<u>Lemma 3.3.5</u> If

$$\sum_{\nu=-\infty}^{\infty} |\nu| R^2(\nu) < \infty,$$

then

$$\lim_{T\to\infty} \frac{T}{m(T)} \sum_{\nu=T}^{\infty} R^2(\nu) = 0, \tag{3.3.23}$$

where $m(T)$ satisfies condition (3.2.7).

<u>Proof</u>: It is clear that

$$\lim_{T\to\infty} \frac{T}{m(T)} \sum_{\nu=T}^{\infty} R^2(\nu) = \lim_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=T}^{\infty} T\, R^2(\nu)$$

$$\leq \lim_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=T}^{\infty} \nu R^2(\nu) = 0.$$

The following theorems examine the term $\frac{T}{m(T)} b^2[f_T(\omega)]$ as T approaches infinity.

<u>Theorem 3.3.6</u> If the conditions of Theorem 3.3.2 hold, then

$$\lim_{T\to\infty} \frac{T}{m(T)} b_m^2[f_T(\omega)] = \lim_{T\to\infty} \frac{T}{m(T)} C(m(T)), \tag{3.3.24}$$

where

$$C(m(T)) = \sum_{\nu=1-T}^{T-1} [1-h(\frac{|\nu|}{m(T)})]^2 R^2(\nu), \tag{3.3.25}$$

and m(T) satisfies condition (3.2.7).

<u>Proof:</u> From equation (3.3.15), we can write

$$\lim_{T\to\infty} \frac{T}{m(T)} b_m^2[f_T(\omega)] = \lim_{T\to\infty} \frac{T}{m(T)} \sum_{\nu=1-T}^{T-1} [1-h(\frac{|\nu|}{m(T)})]^2 R^2(\nu)$$

$$+ 2 \lim_{T\to\infty} \frac{T}{m(T)} \sum_{\nu=T}^{\infty} R^2(\nu) + \lim_{T\to\infty} O(m^{-1}(T))$$

$$= \lim_{T\to\infty} \frac{T}{m(T)} C(m(T)),$$

by equation (3.3.23). This completes the proof.

For the purpose of finding $\lim_{T\to\infty} \frac{T}{m(T)} C(m(T))$, we shall use the following formula, for $0 \leq h(\frac{|\nu|}{m(T)}) \leq 1$, $|\nu| \geq T$

$$\lim_{T\to\infty} \frac{T}{m(T)} C(m(T)) = \lim_{T\to\infty} \frac{T}{m(T)} \sum_{\nu=-\infty}^{\infty} [1-h(\frac{|\nu|}{m(T)})]^2 R^2(\nu). \tag{3.3.26}$$

The right-hand side of (3.3.26) is equal to

$$\lim_{T\to\infty} \frac{T}{m(T)} \sum_{\nu=1-T}^{T-1} [1-h(\frac{|\nu|}{m(T)})]^2 R^2(\nu) - 2 \lim_{T\to\infty} \frac{T}{m(T)} \sum_{\nu=T}^{\infty} [1-h(\frac{\nu}{m(T)})] R^2(\nu)$$

$$= \lim_{T\to\infty} \frac{T}{m(T)} \sum_{\nu=1-T}^{T-1} [1-h(\frac{|\nu|}{m(T)})]^2 R^2(\nu),$$

because

$$\lim_{T\to\infty} \frac{T}{m(T)} \sum_{\nu=T}^{\infty} [1-h(\frac{\nu}{m(T)})]^2 R^2(\nu) \le \lim_{T\to\infty} \frac{T}{m(T)} \sum_{\nu=T}^{\infty} R^2(\nu) = 0.$$

Furthermore, we can write

$$I[m(T)] = \sigma_m^2[f_T(\omega)] + b_m^2[f_T(\omega)]$$

$$= \sum_{\nu=1-T}^{T-1} \frac{T-|\nu|}{T^2} h^2(\frac{|\nu|}{m(T)}) \sum_{p=|\nu|+1-T}^{T-|\nu|-1} R^2(p)$$

$$+ \sum_{\nu=1-T}^{T-1} [1-h(\frac{|\nu|}{m(T)})]^2 R^2(\nu)$$

$$+ 2\sum_{\nu=T}^{\infty} R^2(\nu) + O(T^{-1}) = M(m(T)) + C(m(T)), \tag{3.3.27}$$

where

$$M(m(T)) = \sum_{\nu=1-T}^{T-1} \frac{T-|\nu|}{T^2} h^2(\frac{|\nu|}{m(T)}) \sum_{p=|\nu|+1-T}^{T-|\nu|-1} R^2(p)$$

$$+ 2\sum_{\nu=T}^{\infty} R^2(\nu) + O(T^{-1}). \tag{3.3.28}$$

This form of $I[m(T)]$ will be used in later developments.

The following theorem follows immediately from Theorem 3.3.4 and Lemma 3.3.5.

<u>Theorem 3.3.7</u> If the conditions of Theorem 3.2.2 hold, then

$$\lim_{T\to\infty} \frac{T}{m(T)} M(m(T)) = M, \tag{3.3.29}$$

for any m(T) satisfying equation (3.2.7) and M defined by equation (3.3.19).

From equations (3.3.27) and (3.3.29), it is seen that the best truncation function minimizes the term C(m(T)). There are three different forms of C(m(T)) we shall be concerned with. Theorems 3.3.8 through 3.3.12 will give the best truncation function m'(T) for the first two forms, and Theorems 3.3.13 through 3.3.20 will give the best truncation function for the third form.

<u>Theorem 3.3.8</u> If the conditions of Theorem 3.2.2 hold, and

(i) if $C(m(T)) = Nm^{-(\alpha-1)}(T)(1+o(1)),$ (3.3.30a)

where $N > 0$ and $\alpha > 1$, and we let

$$m'(T) = [(\alpha-1)NM^{-1}T]^{\frac{1}{\alpha}}; \tag{3.3.31a}$$

(ii) if $C(m(T)) = N\ell n(Bm(T))m^{-(\alpha-1)}(T)(1+o(1)),$ (3.3.30b)

where $N > 0$, $B > 0$ and $\alpha > 1$, and we let

$$m'(T) = [\alpha^{-1}(\alpha-1)NM^{-1}T\ell nT]^{\frac{1}{\alpha}}. \tag{3.3.31b}$$

Then

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m'(T)] = \frac{\alpha M}{\alpha-1}, \qquad (3.3.32)$$

where M is defined by equation (3.3.19).

<u>Proof:</u> From equation (3.3.27), we have

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m'(T)] = \lim_{T\to\infty} \frac{T}{m'(T)} M(m'(T)) + \lim_{T\to\infty} \frac{T}{m'(T)} C(m'(T))$$

$$= M + \frac{M}{\alpha-1} = \frac{\alpha M}{\alpha-1} ;$$

since

(i) $$\lim_{T\to\infty} \frac{T}{m'(T)} C(m'(T)) = N \lim_{T\to\infty} \frac{T}{m'^{\alpha}(T)}(1+o(1)) = N\frac{1}{(\alpha-1)NM^{-1}} = \frac{M}{\alpha-1};$$

and

(ii) $$\lim_{T\to\infty} \frac{T}{m'(T)} C(m'(T)) = N \lim_{T\to\infty} \frac{T\ell nBm'(T)}{m'^{\alpha}(T)}(1+o(1))$$

$$= N \frac{\alpha M}{(\alpha-1)N} \lim_{T\to\infty} [\frac{\ell n\{B[\alpha^{-1}(\alpha-1)NM^{-1}T\ell nT]^{\frac{1}{\alpha}}\}}{\ell nT}(1+o(1))]$$

$$= \frac{\alpha M}{\alpha-1} \lim_{T\to\infty} [\frac{\ell nB}{\ell nT} + \frac{1}{\alpha} \frac{\ell n\alpha^{-1}(\alpha-1)NM^{-1}}{\ell nT} + \frac{1}{\alpha} \frac{\ell nT}{\ell nT}$$

$$+ \frac{1}{\alpha} \frac{\ell n\ell nT}{\ell nT}](1+o(1))$$

$$= \frac{M}{\alpha-1} .$$

In order to prove that m'(t) in equation (3.3.31) is the best truncation function, we shall consider other types of truncation functions, and find their asymptotic relative efficiency with respect to the best truncation function, $m'(T)$. First we shall investigate the behavior of an arbitrary truncation function which approach infinity more slowly than does the best truncation function $m'(T)$.

<u>Theorem 3.3.9</u> If $m'(T)$ is chosen by Theorem 3.3.8 and $m_1(T)$ satisfies equation (3.2.7) and $\lim_{T\to\infty} \frac{m_1(T)}{m'(T)} = 0$, then $E[m_1(T)] = \lim_{T\to\infty} \frac{I[m'(T)]}{I[m_1(T)]} = 0$; that is, the asymptotic relative efficiency of $m_1(T)$ with respect to $m'(T)$ is zero.

<u>Proof</u>:

(i) If $C(m(T)) = Nm^{-(\alpha-1)}(T)(1+o(1))$

and

(ii) $m'(T) = [(\alpha-1)NM^{-1}T]^{\frac{1}{\alpha}}$,

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m_1(T)] = \lim_{T\to\infty} \frac{m_1(T)}{m'(T)} \frac{T}{m_1(T)} M(m_1(T))$$

$$+ N \lim_{T\to\infty} \frac{T}{m_1^{\alpha-1}(T)m'(T)}(1+o(1))$$

$$= M \lim_{T\to\infty} \frac{m_1(T)}{m'(T)} + N \lim_{T\to\infty} \left[\frac{m'(T)}{m_1(T)}\right]^{\alpha-1} \frac{T}{m'^{\alpha}(T)}(1+o(1))$$

$$= 0 + \infty = \infty,$$

by equation (3.3.29).

(ii) If $C(m(T)) = N\ln(Bm(T))m^{-(\alpha-1)}(T)(1+o(1))$

and

$$m'(T) = [\alpha^{-1}(\alpha-1)NM^{-1}T\ln T]^{\frac{1}{\alpha}},$$

we have

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m_1(T)] = \lim_{T\to\infty} \frac{m_1(T)}{m'(T)} \frac{T}{m_1(T)} M(m_1(T))$$

$$+ N \lim_{T\to\infty} \frac{T\ln(Bm'(T))}{m_1^{\alpha-1}(T)m'(T)} \left(\frac{m'(T)}{m_1(T)}\right)^{\alpha-1}$$

$$\cdot \frac{\ln Bm_1(T)}{\ln Bm'(T)} (1+o(1))$$

$$= \frac{M}{\alpha-1} \lim_{T\to\infty} \left[\frac{m'(T)}{m_1(T)}\right]^{\alpha-1} \frac{\ln Bm_1(T)}{\ln Bm'(T)} (1+o(1)).$$

There are two possible cases:

(a) If $\lim_{T\to\infty} \frac{m_1(T)}{m'(T)} = 0$, and $0 < \lim_{T\to\infty} \frac{\ln m_1(T)}{\ln m'(T)} \leq 1$, then

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m_1(T)] = \infty.$$

(b) If $\lim_{T\to\infty} \frac{m_1(T)}{m'(T)} = 0$, but $\lim_{T\to\infty} \frac{\ln B\, m_1(T)}{\ln B\, m'(T)} = 0$, then

there exists a constant k such that $0 < k < \alpha-1$ and

$$\lim_{T\to\infty} \frac{m'^{\alpha-1-k}(T)}{m_1^{\alpha-1}(T)} > 0, \text{ then}$$

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m_1(T)] = \frac{M}{\alpha-1} \lim_{T\to\infty} \frac{m'^{\alpha-1-k}(T)}{m_1^{\alpha-1}(T)} \frac{m'^{k}(T)}{\ell nBm'(T)} \ell n(Bm_1(T))$$

$$\cdot(1 + o(1))$$

$$= \infty.$$

For both cases (i) and (ii)

$$E[m_1(T)] = \lim_{T\to\infty} \frac{\frac{T}{m'(T)} I[m'(T)]}{\frac{T}{m'(T)} I[m_1(T)]} = \frac{\frac{\alpha M}{\alpha-1}}{\infty} = 0.$$

We now discuss truncation functions which tend to infinity faster than $m'(T)$.

<u>Theorem 3.3.10</u> If $m'(T)$ is chosen by Theorem 3.3.8, $m_1(T)$ satisfies equation (3.2.7) and $\lim_{T\to\infty} \frac{m_1(T)}{m'(T)} = \infty$, then $E[m_1(T)] = 0$.

<u>Proof</u>:

(i) If $C(m(T))$ and $m'(T)$ are given by (3.3.20a) and (3.3.31a), respectively, then

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m_1(T)] = \lim_{T\to\infty} \frac{m_1(T)}{m'(T)} \frac{T}{m_1(T)} M(m_1(T))$$

$$+ N \lim_{T\to\infty} \frac{T}{m'(T)m_1^{\alpha-1}(T)} (1 + o(1))$$

$$= \infty + 0 = \infty$$

(ii) If $C(m(T))$ and $m'(T)$ are given by equations (3.3.30b) and (3.3.31b), respectively, then

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m_1(T)] = M \lim_{T\to\infty} \frac{m_1(T)}{m'(T)} + N \lim_{T\to\infty} \frac{T \ell n B m_1(T)}{m'(T) m_1^{\alpha-1}(T)} (1 + o(1))$$

$$= M \lim_{T\to\infty} \frac{m_1(T)}{m'(T)} + N \lim_{T\to\infty} \left[\frac{m'(T)}{m_1(T)}\right]^{\alpha-1} \frac{T \ell n T}{m'^{\alpha}(T)} \cdot \frac{\ell n B m_1(T)}{\ell n\ T} (1 + o(1))$$

$$= \infty + 0 = \infty,$$

since $\lim_{T\to\infty} \frac{\ell n B m_1(T)}{\ell n\ T} = \theta$, $0 \leq \theta \leq 1$ by equation (3.2.7). Hence, for both cases $E[m_1(T)] = 0$.

Finally, we consider a truncation function which is a linear function of $m'(T)$.

Theorem 3.3.11 If $m'(T)$ is chosen as in Theorem 3.3.8, and

$$m_1(T) = \theta\, m'(T) + \ell(T),\ \theta > 0 \tag{3.3.33}$$

where

$$\lim_{T\to\infty} \frac{\ell(T)}{m'(T)} = 0, \tag{3.3.34}$$

then

$$E[m_1(T)] = \left[(1 - \frac{1}{\alpha})\theta + \frac{1}{\alpha\theta^{\alpha-1}}\right]^{-1} = \begin{cases} 1, & \theta = 1 \\ <1, & \theta \neq 1 \end{cases} \tag{3.3.35}$$

Proof:

(a) For the result of the first part of equation (3.3.35) we proceed as follows:

(i) If $C(m'(T))$ and $m'(T)$ are given by equations (3.3.30a) and (3.3.31a), respectively, than

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m_1(T)] = \lim_{T\to\infty} \frac{m_1(T)}{m'(T)} \frac{T}{m_1(T)} M(m_1(T))$$

$$+ N \lim_{T\to\infty} \frac{T}{m'(T)m_1^{\alpha-1}(T)} (1 + o(1))$$

$$= \theta M + N \lim_{T\to\infty} [\frac{m'(T)}{m_1(T)}]^{\alpha-1} \frac{T}{m'^{\alpha}(T)} (1 + o(1))$$

$$= \theta M + \frac{M}{(\alpha-1)\theta^{\alpha-1}} = \frac{(\alpha-1)\theta^{\alpha}+1}{(\alpha-1)\theta^{\alpha-1}} M,$$

because from equation (3.3.33)

$$\lim_{T\to\infty} \frac{m'(T)}{m_1(T)} = \frac{1}{\theta} .$$

(ii) If $C(m(T))$ and $m'(T)$ are given by equations (3.3.30b) and (3.3.31b), respectively, then

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m_1(T)] = \theta M + N \lim_{T\to\infty} \frac{T \ell n B m_1(T)}{m'(T)m_1^{\alpha-1}(T)} (1 + o(1))$$

$$= \theta M + N \lim_{T\to\infty} [\frac{m'(T)}{m_1(T)}]^{\alpha-1} \frac{T \ell n B m_1(T)}{m'^{\alpha}(T)} (1 + o(1))$$

$$= \theta M + N \lim_{T\to\infty} [\frac{m'(T)}{m_1(T)}]^{\alpha-1} \frac{T \ell n \frac{m_1(T)}{m'(T)} + T \ell n B m'(T)}{m'^{\alpha}(T)} (1 + o(1))$$

$$= \theta M + \frac{1}{\theta^{\alpha-1}} \frac{M}{\alpha-1} = \frac{(\alpha-1)\theta^{\alpha}+1}{(\alpha-1)\theta^{\alpha-1}} M.$$

Thus, for the both cases

$$E[m_1(T)] = [\lim_{T\to\infty} \frac{I[m_1(T)]}{I[m'(T)]}]^{-1} = \left[\frac{\frac{(\alpha-1)\theta^{\alpha}+1}{(\alpha-1)\theta^{\alpha-1}} M}{\frac{\alpha M}{\alpha-1}} \right]^{-1}$$

$$= [\frac{(\alpha-1)\theta^{\alpha}+1}{\alpha\theta^{\alpha-1}}]^{-1} = [(1 - \frac{1}{\alpha})\theta + \frac{1}{\alpha\theta^{\alpha-1}}]^{-1} .$$

(b) For the proof of the second part of equation (3.3.35), we write

$$g(\theta) = (1 - \frac{1}{\alpha})\theta + \frac{1}{\alpha\theta^{\alpha-1}} , \quad \alpha > 1, \quad \theta > 0$$

then

$$g'(\theta) = (1 - \frac{1}{\alpha}) - (1 - \frac{1}{\alpha}) \frac{1}{\theta^{\alpha}} = 0, \text{ at } \theta = 1;$$

and

$$g''(\theta) = \alpha(1 - \frac{1}{\alpha}) \frac{1}{\theta^{\alpha+1}} > 0, \qquad \text{at } \theta = 1.$$

This indicates that $g(\theta)$ has a minimum at $\theta = 1$, that is, $g(1) = (1 - \frac{1}{\alpha}) + \frac{1}{\alpha} = 1$, and $g(\theta) > 1$ for $\theta \neq 1$. Therefore,

$$E[m_1(T)] = [g(\theta)]^{-1} = \begin{cases} 1, & \theta = 1 \\ <1, & \theta \neq 1 \end{cases}$$

In view of Theorems 3.3.8 t0 3.3.11, we can conclude the following results:

Theorem 3.3.12 If $m'(T)$ is chosen as in Theorem 3.3.8, then $m'(T)$ is the best truncation function.

We now display the best truncation function for the third form of $C(m(T))$.

Theorem 3.3.13 Suppose the conditions of Theorem 3.3.2 hold. If

$$C(m(T)) = N\, e^{-2\rho s m(T)} m^{-(\alpha-1)}(T)(1+o(1)), \qquad (3.3.36)$$

for any $N > 0$, $\rho > 0$, $\alpha \geq 1$ and $s > 0$, and if

$$m'(T) = \frac{1}{2\rho s} \ln(M^{-1}T), \qquad (3.3.37)$$

where M is defined in (3.3.19), then

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m'(T)] = M. \qquad (3.3.38)$$

Proof: From equations (3.3.27) and (3.3.29), we have

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m'(T)] = \lim_{T\to\infty} \frac{T}{m'(T)} M(m'(T)) + N \lim_{T\to\infty} \frac{T}{m'^{\alpha}(T)} e^{-2\rho s m'(T)}$$

$$\cdot\, (1+o(1)) = M + N \lim_{T\to\infty} \frac{(2\rho s)^{\alpha} T}{(\ln M^{-1}T)^{\alpha}} \frac{1}{M^{-1}T}(1+o(1)) = M.$$

We now investigate different truncation functions in order to show that $m'(T)$ as given in equation (3.3.37) is a best truncation function for $C(m(T))$ as defined in (3.3.36). First we examine an arbitrary truncation function $m_1(T)$ which has the property $\lim_{T\to\infty} \frac{m_1(T)}{m'(T)} = \infty$.

<u>Theorem 3.3.14</u> Suppose $m'(T)$ is chosen as in Theorem 3.3.13. $m_1(T)$ satisfies condition (3.2.7) and $\lim_{T\to\infty} \frac{m_1(T)}{m'(T)} = \infty$, then $E[m_1(T)] = 0$.

<u>Proof</u>:

$$\lim_{T\to\infty} \frac{T}{m_1(T)} I[m'(T)] = \lim_{T\to\infty} \frac{m'(T)}{m_1(T)} \frac{T}{m'(T)} M(m'(T))$$

$$+ N \lim_{T\to\infty} \frac{T}{m_1(T)} \frac{1}{M^{-1}T} \frac{1}{m'^{\alpha}(T)} (1 + o(1))$$

$$= 0,$$

$$\lim_{T\to\infty} \frac{T}{m_1(T)} I[m_1(T)] = M + N \lim_{T\to\infty} \frac{T}{m_1(T)} \frac{1}{e^{2\rho s m_1(T)}} (1+o(1))$$

$$= M + N \lim_{T\to\infty} \frac{T}{e^{2\rho s m'(T)}} \frac{1}{m_1^{\alpha}(T)} \frac{e^{2\rho s m'(T)}}{e^{2\rho s m_1(T)}} (1+o(1))$$

$$= M + 0 = M,$$

then

$$E[m_1(T)] = \frac{0}{M} = 0.$$

Next we examine a truncation function $m_1(T)$ which has the property $\lim_{T\to\infty} \frac{m_1(T)}{m'(T)} = 0$.

<u>Theorem 3.3.15</u> If $m'(T)$ is chosen as in Theorem 3.3.13, and $m_1(T)$ satisfies equation (3.2.7) and $\lim_{T\to\infty} \frac{m_1(T)}{m'(T)} = 0$, then $E[m_1(T)] = 0$.

<u>Proof</u>:

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m_1(T)] = M \lim_{T\to\infty} \frac{m_1(T)}{m'(T)} + N \lim_{T\to\infty} \frac{Te^{-2\rho s m_1(T)}}{m_1^{\alpha-1}(T)m'(T)} (1+o(1)).$$

Consider the term

$$\lim_{T\to\infty} \ell n \frac{Te^{-2\rho s m_1(T)}}{m_1^{\alpha-1}(T)m'(T)} = \lim_{T\to\infty} [\ell n\, T - \ell n\, m'(T) - 2\rho s\, m_1(T) - (\alpha-1)\ell n m_1(T)]$$

$$= \lim_{T\to\infty} m'(T)\left[\frac{\ell n\, T}{m'(T)} - \frac{\ell n m'(T)}{m'(T)} - 2\rho s \frac{m_1(T)}{m'(T)} - (\alpha-1)\frac{\ell n\, m_1(T)}{m'(T)}\right]$$

$$= \infty \cdot 2\rho s = \infty,\ \rho > 0,\ s > 0;$$

then $\lim_{T\to\infty} \frac{T}{m'(T)} I[m_1(T)] = 0 + \infty = \infty$; therefore, $E[m_1(T)] = 0$.

There are four different linear functions of $m'(T)$, which we shall consider one by one in Theorems 3.3.16 through 3.3.19.

<u>Theorem 3.3.16</u> If $m'(T)$ is chosen as in Theorem 3.3.13, and $m_1(T) = \theta m'(T) + \ell(T)$, $\theta > 1$, where $\ell(T)$ satisfies condition (3.3.34), then $E[m_1(T)] = \frac{1}{\theta}$.

Proof:

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m_1(T)] = M + N \lim_{T\to\infty} \frac{T}{m'(T)} \frac{e^{-2\rho s m_1(T)}}{m_1^{\alpha-1}(T)} (1+o(1))$$

$$= M + N \lim_{T\to\infty} \frac{1}{m'(T)} \frac{e^{2\rho s m'(T)}}{e^{2\rho s m_1(T)}} \frac{T}{e^{2\rho s m'(T)}} \frac{1}{m_1^{\alpha-1}(T)} (1+o(1))$$

$$= M + 0 = \theta M,$$

therefore, $E[m_1(T)] = \frac{M}{\theta M} = \frac{1}{\theta}$.

Theorem 3.3.17 If $m'(T)$ is chosen as in Theorem 3.3.13, and $m_1(T) = \theta m'(T) + \ell(T)$, $0 < \theta < 1$, where $\ell(T)$ satisfies condition (3.3.34), then $E[m_1(T)] = 0$.

Proof:

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m_1(T)] = M + N \lim_{T\to\infty} \frac{T}{m'(T)} \frac{e^{-2\rho s m_1(T)}}{m_1^{\alpha-1}(T)} (1+o(1)).$$

Consider the term

$$\lim_{T\to\infty} \ell n \frac{T}{m'(T)} \frac{e^{-2\rho s m_1(T)}}{m_1^{\alpha-1}(T)} = \lim_{T\to\infty} m'(T) [\frac{2\rho s \ell nT}{\ell nM^{-1}T} - 2\rho s \frac{m_1(T)}{m'(T)} - \frac{\ell nm'(T)}{m'(T)} - (\alpha-1) \frac{\ell nm_1(T)}{m'(T)}]$$

$$= \infty \cdot 2\rho s(1-\theta) = \infty, \quad \rho > 0,\ s > 0,\ 0 < \theta < 1,$$

Thus, we have

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m_1(T)] = \theta M + \infty = \infty,$$

and therefore,

$$E[m_1(T)] = \frac{M}{\infty} = 0.$$

Theorem 3.3.18 If $m'(T)$ is chosen as in Theorem 3.3.13, and $m_1(T) = m'(T) + \ell(T)$, where $\ell(T) > 0$ and $\ell(T)$ satisfies condition (3.3.34), then $E[m_1(T)] = 1$.

Proof:

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m_1(T)] = M + N \lim_{T\to\infty} \frac{T}{m'(T)} \frac{e^{-2\rho s m_1(T)}}{m_1^{\alpha-1}(T)} (1+o(1))$$

$$= M + N \lim_{T\to\infty} [\frac{T}{e^{2\rho s m'(T)}} \frac{1}{m'(T)} \frac{1}{m^{\alpha-1}(T)}$$

$$\cdot \frac{1}{e^{2\rho s \ell(T)}} (1+o(1))]$$

$$= M + 0 = M,$$

hence, $E[m_1(T)] = \frac{M}{M} = 1.$

Theorem 3.3.19 If $m'(T)$ satisfies the conditions of Theorem 3.3.13, and $m_1(T) = m'(T) - \ell(T)$, where $\ell(T) > 0$ and $\ell(T)$ satisfies condition (3.3.34), then

$$[m_1(T)] = \begin{cases} 1 \\ \frac{1}{1+GN} \\ 0 \end{cases} \quad \text{if } \lim_{T\to\infty} \frac{e^{2\rho s \ell(T)}}{m'(T) m_1^{\alpha-1}(T)} = \begin{cases} 0 \\ G \\ \infty \end{cases}, \qquad (3.3.34)$$

where $G > 0$.

Proof:

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m_1(T)] = M + N \lim_{T\to\infty} \frac{T}{m'(T)} \frac{e^{-2\rho s m_1(T)}}{m_1^{\alpha-1}(T)} (1+o(1))$$

$$= M + N \lim_{T\to\infty} \frac{T}{e^{2\rho s m'(T)}} \frac{e^{2\rho s \ell(T)}}{m'(T) m_1^{\alpha-1}(T)} (1+o(1))$$

$$= \begin{cases} M \\ M(1+GN) \\ \infty \end{cases} \text{if } \lim_{T\to\infty} \frac{e^{2\rho s \ell(T)}}{m'(T) m_1^{\alpha-1}(T)} = \begin{cases} 1 \\ G \\ \infty \end{cases},$$

and equation (3.3.39) follows.

Using Theorems 3.3.14 through 3.3.19, we find $E[m'(T)] \leq 1$ for any truncation function $m_1(T)$; we summarize these results in the following theorem.

Theorem 3.3.20 If $m'(T)$ is chosen as in Theorem 3.3.13, then $m'(T)$ is the best truncation function for the function $C(m'(T))$ given in equation (3.3.36).

In this section we have displayed two different types of best truncation functions; however, we have not examined the efficiency of the spectral density function estimator obtained by using the best truncation function. We shall examine this problem in the next section.

3.4 ASYMPTOTIC EFFICIENCY OF THE ESTIMATORS OBTAINED USING TRUNCATION FUNCTIONS

In this section we study the asymptotic relative efficiency of the estimator obtained using the optimal truncation functions. Parzen gave two different asymptotic forms of I_{opt}, the minimal mean square error, with a given covariance function. We shall study the estimators using the algebraic and exponential covariance functions. Behavior under the exponential type covariance function is considered in the following theorem.

Theorem 3.4.1 Suppose the covariance function $R(\nu)$ decreases exponentially with $\rho > 0$,

(i) If $m'(T)$ is chosen by Theorem 3.3.8, then

$$E[m'(T)] = \lim_{T\to\infty} \frac{I_{opt}}{I[m'(T)]} = 0 \tag{3.4.1}$$

and

(ii) If $m'(T)$ is chosen by Theorem 3.3.13, then

$$E[m'(T)] = \frac{S}{F} , \tag{3.4.2}$$

where $S > 0$ and F is defined in equation (3.3.6).

Proof:

(i) From equation (3.2.18) we have

$$\lim_{T\to\infty} \frac{T}{m'(T)} I_{0pt} = \lim_{T\to\infty} \frac{\ell nT}{m'(T)} \frac{T}{\ell nT} I_{0pt} = 0 \cdot \frac{1}{\rho} R^2 = 0,$$

where m'(T) is given in equation (3.3.31). From Theorem 3.3.8

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m'(T)] = \frac{\alpha M}{\alpha-1} \quad ,$$

hence,

$$E[m'(T)] = \lim_{T\to\infty} \frac{\frac{T}{m'(T)} I_{0pt}}{\frac{T}{m'(T)} I[m'(T)]} = \frac{0}{\frac{\alpha M}{\alpha-1}} = 0.$$

(ii) Similarly we have

$$\lim_{T\to\infty} \frac{T}{m'(T)} I_{0pt} = 2\rho s \frac{1}{\rho} R_2 = 2sR_2 \ ,$$

where m'(T) is given in equation (3.3.37), and R_2 is defined in equation (3.2.12). From Theorem 3.3.13, we have

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m'(T)] = M,$$

thus,

$$E[m'(T)] = \frac{2sR_2}{2R_2F} = \frac{s}{F} \ .$$

From the second part of Theorem 3.4.1, we can see that $F = \int_0^\infty h^2(x)dx$ plays an important role in determining E[m'(T)]. If F can be chosen equal to S, then the value E[m'(T)] will be equal to 1.

In the following theorem, results are presented for the algebraic type covariance function.

Theorem 3.4.2 Suppose the covariance function $R(\nu)$ decreases algebraically of degree $\beta > 1$, if $m'(T)$ is chosen by Theorem 3.3.8, then

$$E[m'(T)] = \begin{cases} H[(\alpha-1)NM^{-1}]^{-\frac{1}{\alpha}} \lim\limits_{T\to\infty} T^{\frac{1}{2\beta}-\frac{1}{\alpha}}, & \text{if } m'(T) \text{ is (3.3.31a)} \quad (3.4.3a) \\ H[\alpha^{-1}(\alpha-1)NM^{-1}]^{-\frac{1}{\alpha}} \lim\limits_{T\to\infty} T^{\frac{1}{2\beta}}(T\ln T)^{-\frac{1}{\alpha}}, & \text{if } m'(T) \text{ is (3.3.31b)} \quad (3.4.3b) \end{cases}$$

where

$$H = \frac{\alpha-1}{2F\alpha} C_\beta^{\frac{1}{\beta}} R_2^{-\frac{1}{2\beta}} \int_{-\infty}^{\infty}(1+u^{2\beta})^{-1}du. \qquad (3.4.4)$$

F is defined in equation (3.3.6) and R_2 is defined in equation (3.2.12).

Proof: From (3.2.19) we have

$$\lim_{T\to\infty} \frac{T}{m'(T)} I_{0pt} = C_\beta^{\frac{1}{\beta}} R_2^{1-\frac{1}{2\beta}} \int_{-\infty}^{\infty}(1+u^{2\beta})^{-1}du \lim_{T\to\infty} \frac{T^{\frac{1}{2\beta}}}{m'(T)}$$

which is the same as

$$\frac{\alpha M}{\alpha-1} H[(\alpha-1)NM^{-1}]^{-\frac{1}{\alpha}} \lim_{T\to\infty} T^{\frac{1}{2\beta}-\frac{1}{\alpha}}$$

if $m'(T)$ satisfies equation (3.3.31a); and is the same as

$$\frac{\alpha M}{\alpha-1} H[\alpha^{-1}(\alpha-1)NM^{-1}]^{-\frac{1}{\alpha}} \lim_{T\to\infty} T^{\frac{1}{2\beta}}(T\ell nT)^{-\frac{1}{\alpha}}$$

if m'(T) satisfies equation (3.3.31b), which completes the proof.

It can be seen that E[m'(T)] is zero if $\alpha > 2\beta$. Unfortunately, it will be shown that this is usually the case for algebraic type covariance functions.

3.5 SEVERAL TYPES OF C(m(T)) FUNCTIONS

In section 3.3 it was shown that the best truncation function is dependent on the term C(m(T)). In this section we shall investigate the asymptotic forms of C(m(T)), where m(T) satisfies equation (3.2.7). The asymptotic forms of C(m(T)) obtained in Theorems 3.5.1 and 3.5.3 are precisely those forms studied in Theorems 3.3.8 and 3.3.13. Thus, we have obtained the optimal truncation functions for each of these forms.

Theorem 3.5.1 Suppose h(x) satisfies the conditions of Theorem 3.3.1, then for some $A > 0$, and $p \geq \frac{1}{2}$, h(x) can be written as

$$h(x) = 1 - Ax^p + o(x^p). \qquad (3.5.1)$$

and the following hold.

(i) If $R^*_{2p} = \lim_{T\to\infty} \sum_{\nu=-m(T)}^{m(T)} |\nu|^{2p} R^2(\nu)$

$$= \sum_{\nu=-\infty}^{\infty} |\nu|^{2p} R^2(\nu) < \infty, \qquad (3.5.2a)$$

then

$$C(m(T)) = A^2 m^{-2p}(T) R^*_{2p}(1+o(1)). \qquad (3.5.3a)$$

(ii) If

$$R^{**}_{2p} = \lim_{T\to\infty} \frac{1}{\ln Bm(T)} \sum_{\nu=-m(T)}^{m(T)} |\nu|^{2p} R^2(\nu) < \infty, \qquad (3.5.2b)$$

where $B > 0$, then

$$C(m(T)) = A^2 \ln(Bm(T)) m^{-2p}(T) R^{**}_{2p}(1+o(1)). \qquad (3.5.3b)$$

<u>Proof</u>: (i) From equation (3.5.1), we have

$$\lim_{x\to 0} [1 - h(x)]\, x^{-p} = A. \qquad (3.5.4)$$

From condition (3.5.2a), for any $\varepsilon_1 > 0$, there exists a $\ell_1(T) > 0$ such that $\ell_1(T) \to \infty$ as $m(T) \to \infty$ and $\lim_{T\to\infty} \frac{\ell_1(T)}{m(T)} = 0$, and

$$\ell_1^{2p}(T) \sum_{\nu=\ell_1(T)+1}^{m(T)} R^2(\nu) \le \sum_{\nu=\ell_1(T)+1}^{m(T)} |\nu|^{2p} R^2(\nu) \le \varepsilon_1.$$

This indicates

$$\sum_{\nu=\ell_1(T)+1}^{m(T)} R^2(\nu) = o(\ell_1^{-2p}(T)).$$

Also, for any $\varepsilon_2 > 0$, there exists a $\ell_2(T) > 0$, such that $\ell_2(T) \to \infty$ as $m(T) \to \infty$ and $\lim_{T\to\infty} \frac{\ell_2(T)}{m(T)} = 0$ and

$$m^{2p}(T)\cdot o(\ell^{-2p}(T)) = \frac{m^{2p}(T)}{\ell^{2p}(T)}[\ell_2^{2p}(T)\cdot o(\ell^{-2p}(T))] \leq \frac{\varepsilon_2}{2}.$$

Let $\ell(T) = \max(\ell_1(T), \ell_2(T))$, that is $\ell(T) \to \infty$ as $m(T) \to \infty$ and $\lim_{T\to\infty} \frac{\ell(T)}{m(T)} = 0$, then

$$m^{2p}(T) \sum_{\nu=1-T}^{T-1} [1-h(\tfrac{|\nu|}{m(T)})]^2 R^2(\nu) = m^{2p}(T) \sum_{\nu=-\ell(T)}^{\ell(T)} [1-h(\tfrac{|\nu|}{m(T)})]^2 R^2(\nu)$$

$$+ 2m^{2p}(T) \sum_{\nu=\ell(T)+1}^{m(T)} [1-h(\tfrac{\nu}{m(T)})]^2 R^2(\nu)$$

$$+ 2m^{2p}(T) \sum_{\nu=m(T)}^{T-1} [1-h(\tfrac{\nu}{m(T)})]^2 R^2(\nu). \qquad (3.5.5)$$

(a) The first term of the right-hand side of equation (3.5.5) can be written

$$m^{2p}(T) \sum_{\nu=-\ell(T)}^{\ell(T)} [1-h(\tfrac{|\nu|}{m(T)})]^2 R^2(\nu)$$

$$= \sum_{\nu=-\ell(T)}^{\ell(T)} |\nu|^{2p}[(\tfrac{|\nu|}{m(T)})^{-p}(1-h(\tfrac{|\nu|}{m(T)}))]^2 R^2(\nu)$$

$$= A^2 R^*_{2p}(1 + o(1)), \quad \text{as } m(T) \to \infty.$$

(b) The middle term of the right-hand side of equation (3.5.5) can be written

$$2m^{2p}(T) \sum_{\nu=\ell(T)+1}^{m(T)} \left[1-h\left(\frac{\nu}{m(T)}\right)\right]^2 R^2(\nu) \leq 2m^{2p}(T) \sum_{\nu=\ell(T)+1}^{m(T)} R^2(\nu)$$

$$= 2m^{2p}(T) \cdot o(\ell^{-2p}(T)) \leq \varepsilon_2,$$

for all $\varepsilon_2 > 0$, hence

$$2m^{2p}(T) \sum_{\nu=\ell(T)+1}^{m(T)} \left[1-h\left(\frac{\nu}{m(T)}\right)\right]^2 R^2(\nu) = o(1), \text{ as } m(T) \to \infty.$$

(c) The last term of the right-hand side of equation (3.5.5) can be written

$$2m^{2p}(T) \sum_{\nu=m(T)+1}^{T-1} \left[1-h\left(\frac{\nu}{m(T)}\right)\right]^2 R^2(\nu) \leq 2m^{2p}(T) \sum_{\nu=m(T)+1}^{T-1} R^2(\nu)$$

$$\leq 2 \sum_{\nu=m(T)+1}^{T-1} |\nu|^{2p} R^2(\nu) = o(1), \text{ as } m(T) \to \infty.$$

Substituting (a), (b), and (c) into equation (3.5.5) yields

$$m^{2p}(T)C(m(T)) = A^2 R^*_{2p}(1 + o(1)),$$

or

$$C(m(T)) = A^2 R^*_{2p} m^{-2p}(T)(1 + o(1)).$$

(ii) Using equation (3.5.2b), for any $\varepsilon_1 > 0$, there exists a $\ell_1(T) > 0$, such that $\ell_1(T) \to \infty$ as $m(T) \to \infty$ and $\lim_{T\to\infty} \frac{\ell_1(T)}{m(T)} = 0$, and

$$2\ell_1^{2p}(T)(\ell nBm(T))^{-1} \sum_{\nu=\ell_1(T)+1}^{m(T)} R^2(\nu)$$

$$\leq 2(\ell nBm(T))^{-1} \sum_{\nu=\ell_1(T)+1}^{m(T)} |\nu|^{2p} R^2(\nu) \leq \varepsilon_1 .$$

This indicates that

$$2(\ell nBm(T))^{-1} \sum_{\nu=\ell_1(T)+1}^{m(T)} R^2(\nu) = o(\ell_1^{-2p}(T)).$$

For any $\varepsilon_2 > 0$, there exists a $\ell_2(T) > 0$ such that $\ell_2(T) \to \infty$ as $m(T) \to \infty$, and $\lim_{T\to\infty} \frac{\ell_2(T)}{m(T)} = 0$, and

$$m^{2p}(T)\cdot o(\ell_2^{-2p}(T)) = \frac{m^{2p}(T)}{\ell_2^{2p}(T)}[\ell_2^{2p}(T)\cdot o(\ell^{-2p}(T))] \leq \frac{\varepsilon_2}{2} .$$

Let $\ell(T) = \max[\ell_1(T), \ell_2(T)]$, and consider

$$m^{2p}(T)(\ell nBm(T))^{-1}C(m(T)) = m^{2p}(T)(\ell nBm(T))^{-1} \sum_{\nu=1-T}^{T-1} [1-h(\frac{|\nu|}{m(T)})]^2 R^2(\nu)$$

$$= m^{2p}(T)(\ell nBm(T))^{-1} \sum_{\nu=-\ell(T)}^{\ell(T)} [1-h(\frac{|\nu|}{m(T)})]^2 R^2(\nu)$$

$$+ 2m^{2p}(T)(\ell nBm(T))^{-1} \sum_{\nu=\ell_1(T)+1}^{m(T)} [1-h(\frac{\nu}{m(T)})]^2 R^2(\nu)$$

$$+ \; 2m^{2p}(T)(\ell nBm(T))^{-1} \sum_{\nu=m(T)+1}^{T-1} [1-h(\frac{\nu}{m(T)})]^2 R^2(\nu). \qquad (3.3.6)$$

(a) The first term of the right-hand side of equation (3.3.6) can be written

$$m^{2p}(T)(\ell nBm(T))^{-1} \sum_{\nu=-\ell(T)}^{\ell(T)} [1-h(\frac{|\nu|}{m(T)})]^2 R^2(\nu)$$

$$= (\ell nBm(T))^{-1} \sum_{\nu=-\ell(T)}^{\ell(T)} |\nu|^{2p} [(\frac{|\nu|}{m(T)})^{-p}(1-h(\frac{|\nu|}{m(T)})]^2 R^2(\nu)$$

$$= A^2 R^*_{2p}(1 + o(1)), \text{ as } m(T) \to \infty.$$

(b) The middle term of the right-hand side of equation (3.3.6) can be written

$$2m^{2p}(T)(\ell nBm(T))^{-1} \sum_{\nu=\ell(T)+1}^{m(T)} [1-h(\frac{\nu}{m(T)})]^2 R^2(\nu)$$

$$\leq 2m^{2p}(T)(\ell nBm(T))^{-1} \sum_{\nu=\ell(T)+1}^{m(T)} R^2(\nu)$$

$$= m^{2p}(T) \cdot o(\ell^{-2p}(T)) \leq \varepsilon_2,$$

hence,

$$2m^{2p}(T)(\ell nBm(T))^{-1} \sum_{\nu=\ell(T)+1}^{m(T)} [1-h(\frac{\nu}{m(T)})]^2 R^2(\nu) = o(1),$$

$$\text{as } m(T) \to \infty.$$

(c) The last term of the right-hand side of equation (3.5.6) can be written

$$2m^{2p}(T)(\ln Bm(T))^{-1}\sum_{\nu=m(T)+1}^{T-1}[1-h(\frac{\nu}{m(T)})]^2R^2(\nu)$$

$$\le 2m^{2p}(T)(\ln Bm(T))^{-1}\sum_{\nu=m(T)+1}^{T-1}R^2(\nu)$$

$$\le 2(\ln Bm(T))^{-1}\sum_{\nu=m(T)+1}^{T-1}|\nu|^{2p}R^2(\nu) = o(1), \text{ as } m(T) \to \infty.$$

Substituting (a), (b) and (c) into equation (3.5.6) gives

$$C(m(T)) = A^2R^{**}_{2p}\ln(Bm(T))m^{-2p}(T)(1+o(1)).$$

which completes the proof.

By the above theorem, equation (3.3.31) can be written as

$$\text{(i)}\quad m'(T) = [2pA^2R^*_{2p}M^{-1}T]^{\frac{1}{2p+1}}, \tag{3.5.7a}$$

which is the same as Parzen's result displayed in equation (3.2.13).

$$\text{(ii)}\quad m'(T) = [2p(2p+1)^{-1}A^2R^{**}_{2p}M^{-1}T\ln T]^{\frac{1}{2p+1}}, \tag{3.5.7b}$$

where M is defined in (3.3.19).

We prove that E[m'(T)] is zero for algebraic type covariance functions in the following corollary.

Corollary 3.5.2 If $C(m(T))$ is of the form in (3.5.3) and if $R(\nu)$ decreases algebraically of degree $\beta > 1$, then $E[m'(T)] = 0$.

Proof: (i) From (3.4.3a) of Theorem 3.4.2, we know that $E[m'(T)]$ is dependent on the term

$$\lim_{T\to\infty} T^{\frac{1}{2\beta} - \frac{1}{\alpha}},$$

where $\alpha = 2p + 1$. From the condition (i) of Theorem 3.5.1 we have

$$\sum_{\nu=-\infty}^{\infty} |\nu|^{2p} \frac{C_\beta^2}{(|\nu|+1)^{2\beta}} (1+o(1)) < \infty.$$

This implies $2\beta - 2p > 1$ or $2\beta > 2p + 1$. Consequently, $2\beta - \alpha > 0$, hence

$$\lim_{T\to\infty} T^{\frac{1}{2\beta}-\frac{1}{\alpha}} = \lim_{T\to\infty} T^{-(2\beta-\alpha)/2\alpha\beta} = 0,$$

which implies $E[m'(T)] = 0$.

(ii) From condition (ii) of Theorem 3.5.1, we have $2\beta = 2p + 1$. From equation (3.4.3b) of Theorem 3.4.2, $E[m'(T)]$ is dependent on the term

$$\lim_{T\to\infty} T^{\frac{1}{2\beta}} (T\ell nT)^{-\frac{1}{2p+1}},$$

which is 0, therefore $E[m'(T)] = 0$.

Let us consider a special choice for $h(x)$. Define

$$h(x) = \begin{cases} 1, & |x| \leq s \\ 0, & |x| > s \end{cases} \tag{3.5.8}$$

where $s > 0$. Then it follows that

$$C(m(T)) = 2 \sum_{\nu=sm(T)+1}^{T-1} R^2(\nu), \tag{3.5.9}$$

and for exponential type covariance function we have the following theorem concerning the term $C(m(T))$.

<u>Theorem 3.5.3</u> If $h(x)$ is defined in equation (3.5.8), and $R(\nu)$ decreases exponentially with coefficient $\rho > 0$, then

(i) $C(m(T)) = Ne^{-2\rho sm(T)}(1+o(1))$, (3.5.10)

where $0 < N < \dfrac{2R_0^2}{e^{2\rho}(1-e^{-2\rho})}$, and R_0 is defined in equation (3.2.1).

(ii) $E[m'(T)] = 1.$ (3.5.11)

<u>Proof</u>: (i) From equation (3.2.1) we have

$$C(m(T)) = 2 \sum_{\nu=sm(T)+1}^{T-1} R^2(\nu) \leq 2\, R_0 \sum_{\nu=sm(T)+1}^{T-1} e^{-2\rho\nu}$$

$$= \frac{2R_0^2}{1-e^{-2\rho}} \left(e^{-2\rho(sm(T)+1)} - e^{-2\rho T}\right)$$

$$= \frac{2R_0^2}{e^{2\rho}(1-e^{-2\rho})}\, e^{-2\rho s m(T)}(1+o(1)).$$

Hence, we can write

$$C(m(T)) = Ne^{-2\rho s m(T)}(1+o(1)),$$

where N satisfies the inequalities given in the theorem.

(ii) From equation (3.3.6) we have

$$F = \int_0^s dx = s. \tag{3.5.12}$$

From the second part of Theorem 3.4.1, we have

$$E[m'(T)] = \frac{s}{F} = \frac{s}{s} = 1.$$

which completes the proof.

If $R(\nu)$ decreases algebraically with $\beta > 1$, then

$$C(m(T)) = \frac{2C_\beta^2}{2\beta-1}(sm(T))^{-(2\beta-1)}(1+o(1)), \qquad \beta > 1$$

since for large values of ν, $R(\nu)$ is of the form $C_\beta \nu^{-\beta}$. Thus,

$$m'(T) = [2C_\beta^2 s^{-(2\beta-1)} M^{-1} T)^{\frac{1}{2\beta}} \tag{3.5.13}$$

by Theorem 3.3.8 with $\alpha = 2\beta$. From equation (3.4.3a) of Theorem 3.4.2, we have

$$E[m'(T)] = \frac{2\beta-1}{2\beta}\, C_\beta^{1/\beta} R_2^{-\frac{1}{2}\beta}\, \frac{\int_{-\infty}^{\infty}(1+u^{2\beta})^{-1}du}{2F}\left(\frac{2FR_2 s^{2\beta-1}}{2C_\beta^2}\right)^{\frac{1}{2\beta}}$$

$$= \frac{2\beta-1}{4\beta} \int_{-\infty}^{\infty} (1+u^{2\beta})^{-1}\, du, \tag{3.5.14}$$

since F = s in (3.5.12).

3.6 SUMMARY

In this chapter algebraic type weight functions h(x) are studied. These functions satisfy conditions different than those given by Parzen. Conditions are given under which these weight functions yield consistent estimators of the spectral density function. These conditions are assumed throughout the remainder of the chapter.

For a given weight function the asymptotic form of the variance of the spectral density function estimator is shown to be a fixed quantity regardless of the truncation function, while the asymptotic form of the bias of the spectral density function estimator varies with different truncation functions. Moreover, the determination of the best truncation function m'(T) relies on a single term of the asymptotic bias, which we call C(m(T)).

There are three different types of C(m(T)) which arise in considering algebraic and exponential covariance functions and thus there are three different formulae for the best truncation functions. It is interesting to point out that whenever the form of the weight function is known, the best truncation function can be determined once the form of the covariance function is assumed. The asymptotic form of the mean square error $I[m'(T)]$ of the estimated spectral density function obtained by using the best truncation function is also found, which yields the asymptotic relative efficiency $E[m'(T)]$ of the best truncation function. The results of this chapter are summarized in the following two tables.

TABLE 3.6.1

$m'(T)$ and $E[m'(T)]$ assuming $R(\nu)$ decreases exponentially with coefficient $\rho > 0$

$h(x)$	$C(m(T))$	$m'(T)$	$I[m'(T)]$	I_{opt}	$E[m'(T)]$
$h(x)=1, \lvert x\rvert \le s$ $=0, \lvert x\rvert > s$ $s > 0$	$Ne^{-2\rho s m(T)}$ $\cdot(1+o(1))$	$\frac{1}{2\rho s}\ln M^{-1}T$	$\frac{m'(T)}{T}M(1+o(1))$	$\frac{\ln T}{T}\frac{R_2}{\rho}$ $\cdot(1+o(1))$	1
$h(x)$ $=1=Ax^p+0(x^p)$ $A > 0$ $\rho \ge \frac{1}{2}$	(1) A^2m^{-2p} $\cdot(T)R^*_{2p}$ $\cdot(1+o(1))$ (2) $Ne^{-2\rho s m(T)}$ $\cdot m^{-(\alpha-1)}(T)$ $\cdot(1+o(1))$ $\alpha \ge 1$ $s > 0$	(1) $(2pA^2R^*_{2p}m^{-1}T)$ $\cdot 1/(2pA^2R^*_{2p}$ $\cdot m^{-1}T)^{\frac{1}{2p+1}}$ (2) $\frac{1}{2\rho s}\ \ln M^{-1}T$	(1) $\frac{m'(T)}{T}(\frac{2p+1}{2p})$ $M(1+o(1))$ (2) $\frac{m'(T)}{T}M(1+o(1))$	(1) $\frac{\ln T}{T}\ \frac{R_2}{\rho}(1+o(1))$ (2) $\frac{\ln T}{T}\ \frac{R_2}{\rho}(1+o(1))$	(1) 0 (2) $\frac{s}{F}$

TABLE 3.6.2

$m'(T)$ and $E[m'(T)]$ Assuming $R(\nu)$ decreases Algebraically of Degree $\beta > 1$

$h(x)$	$C(m(T))$	$m'(T)$
$h(x) = 1,\ \lvert x\rvert \le s$ $= 0,\ \lvert x\rvert > s$ $s > 0$	$\frac{2C_\beta^2}{(2\beta-1)s^{2\beta-1}}\, m^{-(2\beta-1)}(T)$ $\cdot(1+o(1))$ $\beta > 1$	$\left(\frac{2C_\beta^2 T}{s^{2\beta-1}M}\right)^{\frac{1}{2}\beta}$
$h(x)=1-Ax^p+o(x^p)$ $A > 0$ $p \ge \frac{1}{2}$	(1) $A^2R^{**}_{2p}m^{-2p}(T)\,\ell n Bm(T)$ $\cdot(1+o(1))$ if $\beta=p+\frac{1}{2},\ p > \frac{1}{2}$ (2) $A^2R^{*}_{2p}m^{-2p}(T)(1+o(1))$ if $\beta>p+\frac{1}{2},\ p \ge \frac{1}{2}$	(1) $\left(\frac{2pA^2}{2p+1}\, M^{-1}R^{**}_{2p}T\ \ell n\ T\right)^{1/(2p+1)}$ (2) $(2pA^2R^{*}_{2p}M^{-1}T)^{1/(2p+1)}$

TABLE 3.6.2 - Continued

m'(T) and E[m'(T)] Assuming R(ν) Decreases Algebraically of Degree β > 1

$I[m'(T)]$	I_{opt}	$E[m'(T)]$
$\frac{m'(T)}{T}\ \frac{2\beta}{2\beta-1}\ M(1+o(1))$	$T^{-(1-\frac{1}{2\beta})}\ C_\beta^{\frac{1}{\beta}}\ R_2^{1-\frac{1}{2\beta}} \int_{-\infty}^{\infty} (1+u^{2\beta})^{-1} \cdot du(1+o(1))$	$\frac{2\beta-1}{4\beta} \int_{-\infty}^{\infty} (1+u^{2\beta})^{-1} du$
(1) $\frac{m'(T)}{T}\ \frac{2p+1}{2p}\ M(1+o(1))$	(1) $T^{-(1-\frac{1}{2\beta})}\ C_\beta^{\frac{1}{\beta}}\ R_2^{1-\frac{1}{2\beta}} \int_{-\infty}^{\infty} (1+u^{2\beta})^{-1} \cdot du(1+o(1))$	(1) 0
(2) $\frac{m'(T)}{T}\ \frac{2p+1}{2p}\ M(1+o(1))$	(2) $T^{-(1-\frac{1}{2\beta})}\ C_\beta^{\frac{1}{\beta}}\ R_2^{1-\frac{1}{2\beta}} \int_{-\infty}^{\infty} (1+u^{2\beta})^{-1} \cdot du(1+o(1))$	(2) 0

CHAPTER IV

EXTENSIONS

4.1 INTRODUCTION

In this chapter we shall discuss how the results of Section 3.3 can be used to find the best truncation function for various types of weight functions. Two special weight functions and one general weight function are considered. Of course, all the truncation functions considered will satisfy the condition (3.2.7).

The first weight function we shall discuss is labeled $Q_T(\nu)$ and is the product of a weight function which gives a consistent estimator of the spectral density function and the term $(1 - \frac{|\nu|}{T})$. The second weight function we shall discuss is labeled $S_T(\nu)$ and is the product of $Q_T(\nu)$ and a function of $(\frac{|\nu|}{T})$.

The results of Section 3.3 can also be used to find the best truncation function for other estimators of the spectral density function. For example, if the product of $(1 - \frac{|\nu|}{T})$ and any function of the form $\frac{\psi(|\nu|)}{m(T)}$ is used as the weight function, the best truncation function can be found by the methods developed in this dissertation.

We shall display one example using each weight function and shall compare the best truncation functions and their asymptotic mean square errors for a common covariance function.

4.2 CASE I: $Q_T(\nu) = (1 - \frac{|\nu|}{T}) K_T(\nu)$

In this section we shall use the theorems developed in section 3.3 to find the best truncation function for any function $K_T(\nu)$, which yields a consistent estimator of the spectral density function. The estimator $f_T(\omega)$ of the spectral density $f(\omega)$ is

$$f_T(\omega) = \sum_{\nu=1-T}^{T-1} K_T(\nu) R_T(\nu) \cos 2\pi\nu\omega. \quad (-\tfrac{1}{2} \leq \omega \leq \tfrac{1}{2}) \tag{4.2.1}$$

The mean square error of the estimator $f_T(\omega)$ corresponding to $K_T(\nu)$ will be

$$I[K_T(\nu)] = \sum_{\nu=1-T}^{T-1} K_T^2(\nu) \operatorname{Var} R_T(\nu) + \sum_{\nu=1-T}^{T-1} [1-K_T(\nu)]^2 R^2(\nu)$$

$$+ 2 \sum_{\nu=T}^{\infty} R^2(\nu). \tag{4.2.2}$$

Defining the weight function $Q_T(\nu)$ as

$$Q_T(\nu) = \begin{cases} (1 - \frac{|\nu|}{T}) K_T(\nu), & |\nu| \leq T-1 \\ 0, & |\nu| \geq T \end{cases} \tag{4.2.3}$$

the mean square error of the estimate $f_T(\omega)$ corresponding to $Q_T(\nu)$ is

$$I[Q_T(\nu)] = \sum_{\nu=1-T}^{T=1} Q_T^2(\nu) \operatorname{Var} R_T(\nu) + \sum_{\nu=1-T}^{T-1} [1-Q_T(\nu)]^2 R^2(\nu)$$

$$+ 2 \sum_{\nu=T}^{\infty} R^2(\nu). \tag{4.2.4}$$

It is interesting to note that $Q_T(\nu)$ will give a consistent estimator $f_T(\omega)$ if $K_T(\nu)$ does.

Theorem 4.2.1 If $K_T(\nu)$ satisfies the conditions of Theorem 2.2.3, then the spectral density estimator associated with $Q_T(\nu)$ is strongly consistent.

Proof: It is sufficient to prove that $Q_T(\nu)$ satisfies conditions (2.2.21) to (2.2.23) of Theorem 2.2.3.

(i) Since $0 \leq K_T(\nu) \leq 1$, $0 \leq (1 - \frac{|\nu|}{T})K_T(\nu) \leq 1$, $|\nu| \leq T-1$, which satisfies (2.2.21).

(ii) Since $\lim_{T\to\infty} K_T(\nu) = 1$ for fixed ν,

$$\lim_{T\to\infty} Q_T(\nu) = \lim_{T\to\infty} (1 - \frac{|\nu|}{T})K_T(\nu) = 1$$

for fixed ν, condition (2.2.22) follows.

(iii) $$\lim_{T\to\infty} \sum_{\nu=1-T}^{T-1} \frac{Q_T^2(\nu)}{T-|\nu|} = \lim_{T\to\infty} \sum_{\nu=1-T}^{T-1} (1-\frac{|\nu|}{T})^2 \frac{K_T^2(\nu)}{T-|\nu|} \leq$$

$$\lim_{T\to\infty} \sum_{\nu=1-T}^{T-1} \frac{K_T^2(\nu)}{T-|\nu|} = 0,$$ since $K_T(\nu)$ satisfies (2.2.23).

Hence $Q_T(\nu)$ satisfies (2.2.23).

The proof is complete.

The following theorem states that the mean square error $I[Q_T(\nu)]$ is asymptotically equivalent to the mean square error $I[K_T(\nu)]$ provided certain conditions are satisfied.

Theorem 4.2.2 Let X_t be a linear process defined by (2.2.1), and let $K_T(\nu)$ satisfy the conditions of Theorem

2.2.3. If

(i) $\sum_{\nu=-\infty}^{\infty} |\nu| R^2(\nu) < \infty,$ (4.2.5)

(ii) $\lim_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=1-T}^{T-1} \frac{|\nu| K_T^2(\nu)}{T-|\nu|} = 0$ for any $m(T)$ satisfying condition (3.2.7), then (4.2.6)

$$\lim_{T\to\infty} \frac{T}{m(T)} I[Q_T(\nu)] = \lim_{T\to\infty} \frac{T}{m(T)} I[K_T(\nu)]. \quad (4.2.7)$$

Proof: From equations (4.2.3) and (4.2.4) we have

$$I[Q_T(\nu)]$$

$$= \sum_{\nu=1-T}^{T-1} (1-\frac{|\nu|}{T})^2 K_T^2(\nu) \mathrm{Var} R_T(\nu) + \sum_{\nu=1-T}^{T-1} [1-(1-\frac{|\nu|}{T}) K_T(\nu)]^2 R^2(\nu)$$

$$+ 2 \sum_{\nu=T}^{\infty} R^2(\nu)$$

$$= \sum_{\nu=1-T}^{T-1} K_T^2(\nu) \mathrm{Var} R_T(\nu) + \sum_{\nu=1-T}^{T-1} [1-K_T(\nu)]^2 R^2(\nu) + 2 \sum_{\nu=T}^{\infty} R^2(\nu)$$

$$- 2 \sum_{\nu=1-T}^{T-1} \frac{|\nu|}{T} K_T^2(\nu) \mathrm{Var}\, R_T(\nu) + \sum_{\nu=1-T}^{T-1} \frac{|\nu|^2}{T^2} K_T^2(\nu) \mathrm{Var}\, R_T(\nu)$$

$$+ 2 \sum_{\nu=1-T}^{T-1} \frac{|\nu|}{T} [1-K_T(\nu)] K_T(\nu) R^2(\nu) + \sum_{\nu=1-T}^{T-1} \frac{|\nu|^2}{T^2} K_T^2(\nu)\ R^2(\nu)$$

$$= I[K_T(\nu)] - 2 \sum_{\nu=1-T}^{T-1} \frac{|\nu|}{T} K_T^2(\nu) \mathrm{Var} R_T(\nu)$$

$$+ \sum_{\nu=1-T}^{T-1} \frac{|\nu|^2}{T^2} K_T^2(\nu) \mathrm{Var} R_T(\nu) + 2 \sum_{\nu=1-T}^{T=1} \frac{|\nu|}{T} [1-K_T(\nu)] K_T(\nu) R^2(\nu)$$

$$+ \sum_{\nu=1-T}^{T-1} \frac{|\nu|^2}{T^2} K_T^2(\nu) R^2(\nu),$$

multiplying both sides by $\frac{T}{m(T)}$ and taking the limit yields

$$\lim_{T\to\infty} \frac{T}{m(T)} I[Q_T(\nu)]$$

$$= \lim_{T\to\infty} \frac{T}{m(T)} I[K_T(\nu)] - \lim_{T\to\infty} \frac{2}{m(T)} \sum_{\nu=1-T}^{T-1} |\nu| K_T^2(\nu) \mathrm{Var} R_T(\nu)$$

$$+ \lim_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=1-T}^{T-1} \frac{|\nu|^2}{T} K_T^2(\nu) \mathrm{Var} R_T(\nu)$$

$$+ \lim_{T\to\infty} \frac{2}{m(T)} \sum_{\nu=1-T}^{T-1} [1-K_T(\nu)] |\nu| K_T(\nu) R^2(\nu)$$

$$+ \lim_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=1-T}^{T-1} \frac{|\nu|^2}{T} K_T^2(\nu) R^2(\nu) \tag{4.2.8}$$

(a) The second and the third terms of the right-hand side of equation (4.2.8) satisfy

$$\lim_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=1-T}^{T-1} \frac{|\nu|^2}{T} K_T^2(\nu) \mathrm{Var} R_T(\nu)$$

$$\leq \lim_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=1-T}^{T-1} |\nu| K_T^2(\nu) \mathrm{Var} R_T(\nu)$$

$$\leq [\frac{\kappa_4}{\sigma^4} R^2(0) + 2R_2] \lim_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=1-T}^{T-1} \frac{|\nu| K_T^2(\nu)}{T-|\nu|} = 0,$$

by equations (2.2.15) and (4.2.6).

(b) The last two terms of equation (4.2.8) are less than

$\lim_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=1-T}^{T-1} |\nu| R^2(\nu)$, which approaches zero as $T\to\infty$ by (4.2.5).

Equation (4.2.7) follows.

The relevance of Theorem 4.2.2 is that if $K_T(\nu)$ is a function of $\frac{|\nu|}{m(T)}$, then we can use the formulae in section 3.3 to find the best truncation point $m(T)$ for $K_T(\nu)$ which is also the best truncation point of $Q_T(\nu)$. In the next two theorems we shall consider weight functions for which condition (4.2.6) is satisfied.

Theorem 4.2.3 If $K_T(\nu)$ has finite support $[-m(T), m(T)]$; for any $m(T)$ which satisfies (3.2.7), and if $K_T(\nu)$ satisfies the conditions of Theorem 2.2.3, then condition (4.2.6) holds, that is

$$\lim_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=1-T}^{T-1} \frac{|\nu| K_T^2(\nu)}{T-|\nu|} = 0.$$

Proof: $$\lim_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=1-T}^{T-1} \frac{|\nu| K_T^2(\nu)}{T-|\nu|}$$

$$= \lim_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=-m(T)}^{m(T)} \frac{|\nu| K_T^2(\nu)}{T-|\nu|}$$

$$\leq \lim_{T\to\infty} \sum_{\nu=-m(T)}^{m(T)} \frac{K_T^2(\nu)}{T-|\nu|} = 0,$$

since $K_T(\nu)$ gives a consistent estimator of $f(\omega)$.

Theorem 4.2.4 Suppose $K_T(\nu)$ is a function of the form

$$K_T(\nu) = \begin{cases} h(\frac{|\nu|}{m(T)}), & |\nu| \leq T-1 \\ \\ 0, & |\nu| \geq T \end{cases} \qquad (4.2.9)$$

note $K_T(\nu)$ is not identical to zero outside the interval $[-m(T), m(T)]$. If $K_T(\nu)$ satisfies the conditions of Theorem 2.2.3 and

(i) $\lim\limits_{T\to\infty} \frac{m(T)\ell nT}{T} = 0$ for any $m(T)$ satisfying condition (3.2.7);

(ii) $h(x)$ is non-increasing function and $h(x) \leq \frac{1}{x}$ for $x \geq 1$,

then condition (4.2.6) holds.

<u>Proof</u>: $\lim\limits_{T\to\infty} \frac{1}{m(T)} \sum\limits_{\nu=1-T}^{T-1} \frac{|\nu| K_T^2(\nu)}{T-|\nu|}$

$$= \lim_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=-m(T)}^{m(T)} \frac{|\nu| K_T^2(\nu)}{T-|\nu|}$$

$$+ 2 \lim_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=m(T)+1}^{T-1} \frac{\nu K_T^2(\nu)}{T-\nu} . \qquad (4.2.10)$$

The first term of the right-hand side of equation (4.2.10) is zero by the result of Theorem 4.2.3, the second term of the right-hand side of equation (4.2.10) is equal to

$$2\lim_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=m(T)+1}^{T-1} \frac{\nu K_T^2(\nu)}{T-\nu} = 2\lim_{T\to\infty} \frac{1}{m(T)} \sum_{\nu=m(T)+1}^{T-1} \frac{\nu h^2(\frac{\nu}{m(T)})}{T-\nu}$$

$$\leq 2\lim_{T\to\infty} \frac{1}{m(T)} \int_{m(T)}^{T-1} \frac{\nu h^2(\frac{\nu}{m(T)})}{T-\nu} d\nu = 2\lim_{T\to\infty} \int_{1}^{\frac{T-1}{m(T)}} \frac{xh^2(x)}{T-m(T)x} m(T)\,dx$$

$$= 2\lim_{T\to\infty} \frac{m(T)}{T} \int_{1}^{\frac{T-1}{m(T)}} \frac{xh^2(x)}{1-\frac{m(T)}{T}x} dx \leq 2\lim_{T\to\infty} \frac{m(T)}{T} \int_{1}^{\frac{T-1}{m(T)}} \frac{1}{x(1-\frac{m(T)}{T})x)} dx$$

$$= 2\lim_{T\to\infty} \frac{m(T)}{T} \int_{1}^{\frac{T-1}{m(T)}} \left(\frac{1}{x} + \frac{m(T)}{T} \frac{1}{1-\frac{m(T)}{T}x}\right) dx$$

$$= 2\lim_{T\to\infty} \frac{m(T)}{T} \left[\ell n x - \ell n(1-\frac{m(T)}{T} x)\right]_{1}^{\frac{T-1}{m(T)}}$$

$$= 2\lim_{T\to\infty} \frac{m(T)}{T} \left[\ell n \frac{T-1}{m(T)} - \ell n \frac{1}{T} + \ell n(1-\frac{m(T)}{T})\right]$$

$$\leq 2\lim_{T\to\infty} \frac{m(T)}{T} \left[\ell n \frac{T}{m(T)} + \ell n T\right] = 0.$$

We now present two examples to illustrate how Theorem 4.2.2 can be applied. The first example is the smoothed periodogram.

Example 1: Let

$$K_T(\nu) = \begin{cases} 1 - \frac{|\nu|}{m(T)}, & |\nu| \leq m(T) - 1 \\ 0, & |\nu| \geq m(T) \end{cases}$$

where m(T) satisfies equation (3.2.7). This function satisfies Theorems 2.2.3 and 4.2.2. Let $h(x) = 1-x$, $x\geq 1$; $h(x) = 0$, $x\geq 1$. It follows that $F = \int_0^1 (1-x)^2 dx = \frac{1}{3}$ and $M = \frac{2}{3} R_2$. From Theorem 3.3.8

(i) if $C(m(T)) = m^{-2}(T)R_2^*(1+o(1))$, $R_2^* < \infty$, then $m'(T) = [3R_2^{-1}R_2^*T]^{1/3}$;

(ii) if $C(m(T)) = m^{-2}(T)\ell n(Bm(T))R_2^{**}(1+o(1))$, $R_2^{**} < \infty$, and $B > 0$, then $m'(T) = [R_2^{-1}R_2^{**}T\ell nT]^{1/3}$.

For both cases $\lim\limits_{T\to\infty} \dfrac{T}{m'(T)} I[m'(T)] = R_2$.

The second example follows:

<u>Example 2</u>: Let

$$K_T(\nu) = \begin{cases} h_1\left(\dfrac{|\nu|}{m(T)}\right)\left(1 - \dfrac{|\nu|}{m(T)}\right), & |\nu| \le m(T)-1 \\ \\ 0, & |\nu| \ge m(T) \end{cases}$$

for some function $h_1(x)$, where $h_1(x)$ can be written as

$$h_1(x) = 1 - Ax^p + o(x^p), \quad A > 0,\ p \ge 1.$$

The density estimator obtained by using this $K_T(\nu)$ is a strongly consistent estimator provided $0 \le h_1(x) \le 1$ for all x. Let

$$\begin{aligned} h(x) = (1-x)h_1(x) &= 1 - (1+A)x + o(x), & p = 1 \\ &= 1 - x + o(x), & p \ge 1 \end{aligned}$$

then $M = 2FR_2$ with $F = \int_0^1 (1-x)^2 h_1(x)dx$.

(i) For $p = 1$, if $C(m(T)) = (1+A)^2R_2^*m^{-2}(T)(1 + o(1))$, $R_2^* < \infty$, then $m'(T) = [2M^{-1}(1+A)^2R_2^*T]^{1/3}$.

(ii) For $p > 1$, if $C(m(T)) = R_2^*m^{-2}(T)(1 + o(1))$,

$R_2^* < \infty$, then $m'(T) = [2M^{-1}R_2^*T]^{1/3}$

For both cases $\lim_{T\to\infty} \frac{T}{m(T)} I[m(T)] = \frac{3M}{2}$.

4.3 CASE II: $S_T(\nu) = g(\frac{|\nu|}{T})K_T(\nu)$

In this section we shall generalize the function $Q_T(\nu)$ to the following function

$$S_T(\nu) = \begin{cases} g(\frac{|\nu|}{T})K_T(\nu), & |\nu| \le T-1 \\ 0, & |\nu| \ge T \end{cases} \quad (4.3.1)$$

for any function $g(y)$ satisfying

(i) $0 \le g(Y) \le 1$ for all y; (4.3.2)

(ii) $g(y) = 1 - Ay^p + o(y^p)$, $A > 0$, $p \ge 1$, (4.3.3)

The mean square error of the estimator $f_T(\omega)$ corresponding to $S_T(\nu)$ is

$$I[S_T(\nu)] = \sum_{\nu=1-T}^{T-1} S_T^2(\nu)\mathrm{Var}R_T(\nu) + \sum_{\nu=1-T}^{T-1} [1-S_T(\nu)]^2\bar{R}^2(\nu)$$

$$+ 2\sum_{\nu=T}^{\infty} R^2(\nu) \quad (4.3.4)$$

If $K_T(\nu)$ satisfies the conditions of Theorem 2.2.3, then $S_T(\nu)$ yields a strongly consistent spectral density estimator. The mean square error $I[S_T(\nu)]$ is asymptotically equivalent to the mean square error $I[K_T(\nu)]$ under the conditions of Theorem 4.2.2.

Theorem 4.3.1 If the conditions of Theorem 4.2.2 hold, then

$$\lim_{T\to\infty} \frac{T}{m(T)} I[S_T(\nu)] = \lim_{T\to\infty} \frac{T}{m(T)} I[K_T(\nu)] \qquad (4.3.5)$$

for any $m(T)$ satisfying condition (3.2.7).

Proof: Substituting equation (4.3.3) into equation (4.3.4) yields

$$I[S_T(\nu)] = \sum_{\nu=1-T}^{T-1} [1-A(\frac{|\nu|}{T})^p + o((\frac{|\nu|}{T})^p)]^2 K_T^2(\nu)\mathrm{Var}R_T(\nu)$$

$$+ \sum_{\nu=1-T}^{T-1} [1-(1-A(\frac{|\nu|}{T})^p + o((\frac{|\nu|}{T})^p))K_T(\nu)]^2 R^2(\nu)$$

$$+ 2\sum_{\nu=T}^{\infty} R^2(\nu)$$

$$= I[K_T(\nu)] - 2\sum_{\nu=1-T}^{T-1} [A(\frac{|\nu|}{T})^p - o((\frac{|\nu|}{T})^p)] K_T^2(\nu)\mathrm{Var}R_T(\nu)$$

$$+ \sum_{\nu=1-T}^{T-1} [A(\frac{|\nu|}{T})^p - o((\frac{|\nu|}{T})^p)]^2 K_T^2(\nu)\mathrm{Var}R_T(\nu)$$

$$+ 2\sum_{\nu=1-T}^{T-1} K_T(\nu)[1-K_T(\nu)][A(\frac{|\nu|}{T})^p - o((\frac{|\nu|}{T})^p)]R^2(\nu)$$

$$+ \sum_{\nu=1-T}^{T-1} [A(\frac{|\nu|}{T})^p - o((\frac{|\nu|}{T})^p)]^2 K_T^2(\nu) R^2(\nu)$$

$$= I[K_T(\nu)] - 2\sum_{\nu=1-T}^{T-1} A(\frac{|\nu|}{T})^p[1-A^{-1}(\frac{|\nu|}{T})^{-p}$$

$$\cdot o((\frac{|\nu|}{T})^p)]^2 K_T^2(\nu)\mathrm{Var}R_T(\nu)$$

$$+ \sum_{\nu=1-T}^{T-1} A^2 (\tfrac{|\nu|}{T})^{2p} [1-A^{-1} (\tfrac{|\nu|}{T})^{-p}$$

$$\cdot o((\tfrac{|\nu|}{T})^p)]^2 K_T^2(\nu) \mathrm{Var} R_T(\nu)$$

$$+ 2 \sum_{\nu=1-T}^{T-1} A (\tfrac{|\nu|}{T})^p K_T(\nu) [1-K_T(\nu)] [1-(\tfrac{|\nu|}{T})^{-p}$$

$$\cdot o((\tfrac{|\nu|}{T})^p)] R^2(\nu)$$

$$+ \sum_{\nu=1-T}^{T-1} A^2 (\tfrac{|\nu|}{T})^{2p} [1-A^{-1} (\tfrac{|\nu|}{T})^{-p}$$

$$\cdot o((\tfrac{|\nu|}{T})^p)]^2 K_T^2(\nu) R^2(\nu).$$

Since $p \geq 1$, the conclusion follows from Theorem 4.2.2.

The consequence of Theorem 4.5.1 is that the best truncation point $m'(T)$ for $Q_T(\nu)$ is also the best truncation point for $S_T(\nu)$. Therefore the formulae of section 3.3 can be used to find the best truncation function for $S_T(\nu)$. The following example is used to illustrate this point.

Example 1: Let

$$S_T(\nu) = \begin{cases} g(\frac{|\nu|}{T})(1-\frac{|\nu|}{m(T)}), & |\nu| \leq m(T)-1 \\ 0, & |\nu| \geq m(T) \end{cases}$$

where $m(T)$ satisfies (3.2.7).

Let $h(x) = 1-x$, $x < 1$; 0, $x \geq 1$, then the answers of Example 1 in section 4.2 also apply to this example.

4.4 GENERAL CASE: $K_T(\nu) = (1-\frac{|\nu|}{T})g(\frac{\psi(|\nu|)}{m(T)})$

There are many different types of the weight function each of which has a related truncation function. It may be necessary to find the best truncation function for any one of them. Conditions under which the best truncation function can be found are presented without proof, since the proofs are similar to those in section 3.3.

Consider the general weight function expressable as

$$K_T(\nu) = \begin{cases} (1-\frac{|\nu|}{T})g(\frac{\psi(|\nu|)}{m(T)}), & |\nu| \leq T-1 \\ 0, & |\nu| \geq T \end{cases} \tag{4.4.1}$$

where $\psi(|\nu|)$ is any function of $|\nu|$ independent of $m(T)$ which satisfies (3.2.7). If the function $g(x)$ satisfies

(i) $0 \leq g(\frac{\psi(|\nu|)}{m(T)}) \leq 1$ for all ν; (4.4.2)

(ii) $\lim\limits_{T\to\infty} g(\frac{\psi(|\nu|)}{m(T)}) = 1$ for fixed ν; (4.4.3)

(iii) $\lim\limits_{T\to\infty} \frac{1}{m(T)} \int_0^{T-1} g^2(\frac{\psi\ |\nu|}{m(T)})d\nu = F_1 < \infty$, (4.4.4)

then Theorem 3.3.1, guarantees that the estimator $f_T(\omega)$ obtained by using the weight function displayed in (4.4.1) is strongly consistent where $f_T(\omega)$ is defined in (3.3.3). It should be noted that if $\psi(|\nu|) = e^{\alpha|\nu|}$, $\alpha > 0$, then $K_T(\nu)$ is the exponential type weight function defined by Parzen, and if $\psi(|\nu|) = |\nu|$, then $K_T(\nu)$ is the algebraic type weight function discussed in chapter 3, and conditions

(4.4.2) to (4.4.4) become conditions (3.3.4) to (3.3.6) respectively.

Furthermore if condition (3.3.10) holds, then from results of section 3.3, for any m(T) satisfying (3.2.7) we have

$$\lim_{T\to\infty} \frac{T}{m(T)}\sigma_m^2[f_T(\omega)] = 2F_1R_2 = M_1, \text{ say} \tag{4.4.5}$$

and

$$\lim_{T\to\infty} \frac{T}{m(T)}b_m^2[f_T(\omega)] = \lim_{T\to\infty} \frac{T}{m(T)}D(m(T)), \tag{4.4.6}$$

where

$$D(m(T)) = \sum_{\nu=1-T}^{T-1} [1-g(\frac{\psi(|\nu|)}{m(T)}]^2R^2(\nu), \tag{4.4.7}$$

and R_2 is defined in (3.2.12). Then the best truncation function is determined by the term D(m(T)), and the formulae in section 3.3 can be applied. The following two examples illustrate this idea.

Example 1: Let

$$g(\frac{\psi(|\nu|)}{m(T)}) = \begin{cases} 1 - \frac{\psi(|\nu|)}{m(T)}, & |\nu| \leq m(T)-1 \\ \\ 0, & |\nu| \geq m(T) \end{cases} \tag{4.4.8}$$

then $D(m(T)) = \sum_{\nu=-m(T)}^{m(T)} \frac{\psi^2(|\nu|)}{m^2(T)} R^2(\nu)$, and if $R_2^S =$

$\lim_{T\to\infty} \frac{1}{m^S(T)} \sum_{\nu=-m(T)}^{m(T)} \psi^2(|\nu|)R^2(\nu) < \infty$, $0 \leq S < 2$, then by

equation (3.3.31a) $m'(T) = [(2-S)R_2^S M_1^{-1}T]^{\frac{1}{3-S}}$, and

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m'(T)] = \frac{(3-S)M_1}{2-S} .$$

Note that if $\psi(|\nu|) = |\nu|$, $g(x)$ is the smoothed periogram weight function and then $S = 0$, $M_1 = M$ and $R_2^S = R_2^*$. The best truncation function and asymptotic form of the mean square error are the same as in Example 1 of section 4.2.

Example 2: Let

$$g\left(\frac{\psi(|\nu|)}{m(T)}\right) = \begin{cases} 1 - \dfrac{\psi(|\nu|)}{e^{m(T)}}, & |\nu| \le T-1 \\ 0, & |\nu| \ge T \end{cases} \tag{4.4.9}$$

then $D(m(T)) = \sum_{\nu=1-T}^{T-1} e^{-2m(T)}\psi^2(|\nu|)R^2(\nu)$. If we have

$D(m(T)) = Nm^{-\beta}(T)e^{-2pm(T)}(1 + o(1))$, $\beta \ge 0$, $p > 0$, $N > 0$; then by equation (3.3.37)

$$m'(T) = \frac{1}{2p}\ell n M_1^{-1}T, \text{ and } \lim_{T\to\infty} \frac{T}{m'(T)} I[m'(T)] = M_1 .$$

4.5 COMPARISON OF THE THREE CASES

In this section we shall consider one example from each case and compare the best truncation points and the asymptotic mean square errors. Through this section we shall make the following assumption that the covariance function is same for three cases and that equation (3.3.31a) has been used to find the best truncation functions in each case.

(1) Consider the smoothed periodogram example from Case I, that is, let

$$K_T(\nu) = \begin{cases} 1 - \frac{|\nu|}{m(T)}, & |\nu| \leq m(T)-1 \\ 0, & |\nu| \geq m(T) \end{cases}$$

then $F = \int_0^1 (1-x)^2 dx = \frac{1}{3}$, $M = \frac{2}{3}R_2$. If $R_2^* < \infty$,

$m'(T) = [3R_2^{-1}R_2^* T]^{1/3}$ and $\lim\limits_{T\to\infty} \frac{T}{m'(T)} I[m'(T)] = R_2$.

(2) Consider the following example from Case II

$$S_T(\nu) = \begin{cases} 1 - (\frac{|\nu|}{T})^2 (1 - \frac{|\nu|}{m(T)}), & |\nu| \leq m(T)-1 \\ 0, & |\nu| \geq m(T) \end{cases}$$

from the result of section 4.3, we have $h(x) = 1-x$ for $x < 1$ and 0 for $x \geq 1$, then $F = \frac{1}{3}$ and $M = \frac{2}{3}R_2$. If $R_2^* < \infty$, $m'(T) = [3R_2^{-1}R_2^* T]^{1/3}$ and $\lim\limits_{T\to\infty} \frac{T}{m(T)} I[m'(T)] = R_2$.

These two examples yield the same results, since in both cases $h(x) = 1-x$.

(3) Consider the following example from Case III

$$g\left(\frac{\psi(|\nu|)}{m(T)}\right) = \begin{cases} 1 - \frac{e^{-|\nu|}}{m(T)}, & |\nu| \leq m(T) \\ 0, & |\nu| > m(T) \end{cases}$$

then from (4.4.4)

$$F_1 = \lim_{T\to\infty} \frac{1}{m(T)} \int_0^{m(T)} (1 - \frac{1}{m(T)} e^{-\nu})^2 d\nu$$

$$= \lim_{T\to\infty} \frac{1}{m(T)} [\nu + \frac{2}{m(T)} e^{-\nu} - \frac{1}{2} \frac{1}{m^2(T)} e^{-2\nu}]_0^{m(T)} = 1.$$

and $M_1 = 2R_2$. The term $D(m(T))$ is obtained as follows:

$$D(m(T)) = \sum_{\nu=1-T}^{T-1} \frac{e^{-2|\nu|}}{m^2(T)} R^2(\nu) = \frac{1}{m^2(T)} \sum_{\nu=1-T}^{T-1} (e^{-|\nu|}R(\nu))^2$$

$$= \frac{1}{m^2(T)} R_2'(1 + o(1)),$$

where $R_2' = \sum_{\nu=-\infty}^{\infty} (e^{-|\nu|}R(\nu))^2 < \infty$ since $R_2^* < \infty$. Thus

$$m'(T) = [2R_2'R_2^{-1}T]^{1/3}, \text{ and}$$

$$\lim_{T\to\infty} \frac{T}{m'(T)} I[m'(T)] = \frac{3}{2}R_2.$$

$m'(T)$ in Case III is smaller than $m'(T)$ for Cases I and II, conversely the asymptotic mean square error of the estimator $f_T(\omega)$ is greater in Case III than Cases I and II. Among three weight functions we prefer the smoothed periodogram, because it leads to an estimator $f_T(\omega)$ with small asymptotic mean square error.

4.6 CONCLUSION

There are two important types of weight functions which are related to the truncation function. One of these is of the form $(\frac{|\nu|}{T})$ defined in the interval described by the truncation function, (2) the other is of a form $(\frac{|\nu|}{m(T)})$.

The formulae derived in section 3.3 can be used to find the best truncation point and the asymptotic mean square error of the estimator of the spectral density function for each of these functions separately, or for the product of them.

Furthermore, we can use the ideas of section 3.3 to find the best truncation function and the asymptotic mean square error of the estimator for other types of weight functions.

CHAPTER V

ESTIMATION WITH AN UNKNOWN COVARIANCE FUNCTION

5.1 INTRODUCTION

In the first four chapters of this dissertation knowledge of the covariance function was assumed to be known. If the covariance function is unknown, the sample covariance function can be used in its place. Some properties of the sample covariance function have been given in Section 2.2.

Since the sum of squares of the covariance function plays an important role in the determination of the best truncation function, we shall first investigate the properties of its estimator. Then, the estimate of the best truncation point is given and illustrated by one example which consists of analysis of a computer simulated time series.

5.2 THE ESTIMATE OF R_2

$R_2 = \sum_{\nu=-\infty}^{\infty} R^2(\nu)$ plays an important role in the determination of the best truncation function. In this section we shall consider an estimator of R_2. From a theorem for Fourier Series, we can write

$$R_2 = \sum_{\nu=-\infty}^{\infty} R^2(\nu) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f^2(\omega)d\omega, \qquad (5.2.1)$$

where $f(\omega)$ is the spectral density function defined in (2.2.7).

If the sequence $\{R(\nu)\}$ is unknown, one estimator, $R_{2,T}$ of $R_{2,T}$ based on a sample of size T is

$$R_{2,T} = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_T^2(\omega)\,d\omega, \qquad (5.2.2)$$

where $f_T(\omega)$ is the estimate of $f(\omega)$ defined in (2.2.16). Lemma 5.2.1 gives conditions under which $R_{2,T}$ will converge to R_2 in probability.

<u>Lemma 5.2.1</u> If

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} [f_T(\omega) - f(\omega)]^2 d\omega \to 0 \quad \text{in probability}, \qquad (5.2.3)$$

for a spectral density function estimator $f_T(\omega)$, then

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f_T^2(\omega)\,d\omega \to \int_{-\frac{1}{2}}^{\frac{1}{2}} f^2(\omega)\,d\omega \quad \text{in probability}. \qquad (5.2.4)$$

In other works, $R_{2,T} \to R_2$ in probability.

<u>Proof:</u> We can write

$$f_T(\omega) = f(\omega) + f_T(\omega) - f(\omega),$$

then

$$f_T^2(\omega) = f^2(\omega) + 2f(\omega)[f_T(\omega) - f(\omega)] + [f_T(\omega) - f(\omega)]^2.$$

Applying the Schwarz inequality

$$-\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} f^2(\omega)\,d\omega\right]^{\frac{1}{2}} \cdot \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} [f_T(\omega) - f(\omega)]^2 d\omega\right]^{\frac{1}{2}}$$

$$\leq \int_{-\frac{1}{2}}^{\frac{1}{2}} f(\omega)[f_T(\omega) - f(\omega)]\,d\omega$$

$$\leq [\int_{-\frac{1}{2}}^{\frac{1}{2}} f^2(\omega)d\omega]^{\frac{1}{2}} \cdot [\int_{-\frac{1}{2}}^{\frac{1}{2}} [f_T(\omega) - f(\omega)]^2 d\omega]^{\frac{1}{2}},$$

so

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f^2(\omega)d\omega + \int_{-\frac{1}{2}}^{\frac{1}{2}} [f_T(\omega)-f(\omega)]^2 d\omega - 2[\int_{-\frac{1}{2}}^{\frac{1}{2}} f^2(\omega)d\omega]^{\frac{1}{2}}$$

$$\cdot [\int_{-\frac{1}{2}}^{\frac{1}{2}} [f_T(\omega)-f(\omega)]^2 d\omega]^{\frac{1}{2}} \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} f_T^2(\omega)d\omega$$

$$\leq \int_{-\frac{1}{2}}^{\frac{1}{2}} f^2(\omega)d\omega + \int_{-\frac{1}{2}}^{\frac{1}{2}} [f_T(\omega)-f(\omega)]^2 d\omega + 2[\int_{-\frac{1}{2}}^{\frac{1}{2}} f^2(\omega)d\omega]^{\frac{1}{2}}$$

$$\cdot [\int_{-\frac{1}{2}}^{\frac{1}{2}} [f_T(\omega)-f(\omega)]^2 d\omega]^{\frac{1}{2}} \qquad (5.2.5)$$

But by assumption, all of the terms, except the first term in each of the three parts converges to zero in probability, thus

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f_T^2(\omega)d\omega \to \int_{-\frac{1}{2}}^{\frac{1}{2}} f^2(\omega)d\omega \quad \text{in probability.}$$

The following lemma yields almost sure convergence and is thus stronger than Lemma 5.2.1.

Lemma 5.2.2 If $\int_{-\frac{1}{2}}^{\frac{1}{2}}[f_T(\omega) - f(\omega)]^2 d\omega \to 0$ a.s. for some spectral density function estimator $f_T(\omega)$, then $\int_{-\frac{1}{2}}^{\frac{1}{2}} f_T^2(\omega)d\omega \to \int_{-\frac{1}{2}}^{\frac{1}{2}} f^2(\omega)d\omega$ a.s. In other words, $R_{2,T} \to R_2$ a.s.

Proof: The proof is precisely the same as that for Lemma 5.2.1 except that almost sure convergence is considered instead of convergence in probability.

If $f_T(\omega)$ is the consistent estimator of $f(\omega)$, then the following lemma shows that $R_{2,T}$ is an asymptotic unbiased estimator of R_2.

<u>Lemma 5.2.3</u> If $E\int_{-\frac{1}{2}}^{\frac{1}{2}}[f_T(\omega) - f(\omega)]^2 d\omega \to 0$ as $T \to \infty$ for some spectral density function estimator $f_T(\omega)$, then

$$E\int_{-\frac{1}{2}}^{\frac{1}{2}} f_T^2(\omega)d\omega \to \int_{-\frac{1}{2}}^{\frac{1}{2}} f^2(\omega)d\omega \quad \text{as } T \to \infty. \tag{5.2.6}$$

In other words, $R_{2,T}$ is asymptotic unbiased estimate of R_2.

<u>Proof</u>: Define a new random variable

$$X = [\int_{-\frac{1}{2}}^{\frac{1}{2}}[f_T(\omega) - f(\omega)]^2 d\omega]^{\frac{1}{2}}, \tag{5.2.7}$$

then

$$\mathrm{Var}(X) = E\int_{-\frac{1}{2}}^{\frac{1}{2}}[f_T(\omega)-f(\omega)]^2 d\omega - [E(\int_{-\frac{1}{2}}^{\frac{1}{2}}[f_T(\omega)-f(\omega)]^2 d\omega)^{\frac{1}{2}}]^2 \geq 0.$$

This indicates that

$$E\int_{-\frac{1}{2}}^{\frac{1}{2}}[f_T(\omega)-f(\omega)]^2 d\omega \geq [E(\int_{-\frac{1}{2}}^{\frac{1}{2}}[f_T(\omega)-f(\omega)]^2 d\omega)^{\frac{1}{2}}]^2$$

The right-hand side of the inequality tends to zero as the left-hand side of the inequality does. Combining this result with taking expectation of the inequality (5.2.5), we get

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f^2(\omega)d\omega + E\int_{-\frac{1}{2}}^{\frac{1}{2}}[f_T(\omega)-f(\omega)]^2 d\omega - 2[\int_{-\frac{1}{2}}^{\frac{1}{2}} f^2(\omega)d\omega]^{\frac{1}{2}}$$

$$\cdot E[\int_{-\frac{1}{2}}^{\frac{1}{2}}[f_T(\omega)-f(\omega)]^2 d\omega]^{\frac{1}{2}} \leq E\int_{-\frac{1}{2}}^{\frac{1}{2}} f_T^2(\omega)d\omega$$

$$\leq \int_{-\frac{1}{2}}^{\frac{1}{2}} f^2(\omega)d\omega + E\int_{-\frac{1}{2}}^{\frac{1}{2}} [f_T(\omega)-f(\omega)]^2 d\omega + 2[\int_{-\frac{1}{2}}^{\frac{1}{2}} f^2(\omega)d\omega]^{\frac{1}{2}}$$

$$\cdot E[\int_{-\frac{1}{2}}^{\frac{1}{2}} [f_T(\omega)-f(\omega)]^2 d\omega]^{\frac{1}{2}}.$$

As $T \to \infty$, we have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f^2(\omega)d\omega \leq E\int_{-\frac{1}{2}}^{\frac{1}{2}} f_T^2(\omega)d\omega \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} f^2(\omega)d\omega.$$

Applying the Tchebycheff inequality, we get the following lemma and corollary.

<u>Lemma 5.2.4</u> If $E\int_{-\frac{1}{2}}^{\frac{1}{2}} [f_T(\omega) - f(\omega)]^2 \to 0$ as $T \to \infty$, then

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} [f_T(\omega) - f(\omega)]^2 d\omega \to 0 \quad \text{in probability.}$$

<u>Corollary 5.2.5</u> If $E\int_{-\frac{1}{2}}^{\frac{1}{2}} [f_T(\omega) - f(\omega)]^2 \to 0$ as $T \to \infty$, then

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f_T^2(\omega)d\omega \to \int_{-\frac{1}{2}}^{\frac{1}{2}} f^2(\omega)d\omega \quad \text{in probability.}$$

In other words, $R_{2,T} \to R_2$ in probability.

This corollary implies that if the estimator $f_T(\omega)$ is strongly consistent, then $R_{2,T}$ converges to R_2 in probability.

5.3 <u>THE ESTIMATE OF THE BEST TRUNCATION POINT</u>

In this section the estimate of the best truncation point is considered when the covariance function is unknown. Suppose we have the following $h(x)$,

$$h(x) = 1 - A\,x^p + o(x^p),\ A > 0,\ p \geq \frac{1}{2} \qquad (5.3.1)$$

the truncation function in Theorem 3.3.8 can be rewritten as

$$m'(T) = (R_2^{-1}R_{2p}^*)^{\frac{1}{2p+1}}(pA^2F^{-1}T)^{\frac{1}{2p+1}} = \theta^{\frac{1}{2p+1}}(pA^2F^{-1}T)^{\frac{1}{2p+1}}, \qquad (5.3.2)$$

where

$$F = \int_0^\infty h^2(x)dx,$$

$$R_{2p}^* = \sum_{\nu=-\infty}^{\infty} |\nu|^{2p}R^2(\nu),$$

and

$$\theta = (R_{2p}^* R_2^{-1}). \qquad (5.3.3)$$

If the sequence $\{R(\nu)\}$ is unknown, one estimator $\widehat{m'(T)}$ of $m'(T)$ is

$$\widehat{m'(T)} = \hat{\theta}^{\frac{1}{2p+1}}\ (pA^2F^{-1}T)^{\frac{1}{2p+1}}, \qquad (5.3.4)$$

where $\hat{\theta}$ is an estimate of θ. There are two different methods of defining $\hat{\theta}$.

i) Assign any positive value to $\hat{\theta}$. The rationale for this method is that it does not require determining the sample covariance sequence, and thus is quite simple to use in practice.

ii) From the equations displayed in Section 3.3 and the results of Sections 4.2 and 4.3, the sample covariance can be approximated by the following formula

$$R_T(\nu) = \frac{1}{T} \sum_{t=1}^{T-|\nu|} x_t x_{t+|\nu|}, \quad |\nu| \leq T-1 \tag{5.3.5}$$

since $R^2(\nu)$ is a neglible quantity when ν is large because R_2 and R^*_{2p} are convergent. Thus, we let

$$\hat{\theta} = R^{-1}_{2,T} \cdot R^*_{2p,T} , \tag{5.3.6}$$

where $R_{2,T}$ and $R^*_{2p,T}$ are estimates for R_2 and R^*_{2p}, respectively,

$$R_{2,T} = \sum_{\nu=1-T}^{T-1} R^2_T(\nu), \tag{5.3.7}$$

$$R^*_{2p,T} = \sum_{\nu=1-T}^{T-1} |\nu|^{2p} R^2_T(\nu). \tag{5.3.8}$$

For these two cases, the asymptotic relative efficiency of the estimator $f_T(\omega)$ of $f(\omega)$ based on $\widehat{m'(T)}$ will be less than the relative efficiency given by Theorem 3.3.11. Of course, if $\hat{\theta}$ is closed to θ, then $\widehat{m'(T)}$ gives a good estimate.

We turn to a numerical example. Consider

$$h(x) = \begin{cases} 1 - x, & |x| < 1 \\ 0 \quad , & |x| \geq 1 \end{cases}$$

with $F = \int_0^1 (1-x)^2 dx = \frac{1}{3}$, $A = 1$, $p = 1$, and consider a linear process of the form

$$x_t = \frac{1}{2} x_{t-1} + \varepsilon_t,$$

with $a_o = 1$, $a_1 = \frac{1}{2}, \ldots, a_k = (\frac{1}{2})^k \ldots$, (see (2.2.17)), then

$$R(\nu) = \frac{4}{3}\left(\frac{1}{2}\right)^{\nu}\sigma^2,$$

where σ^2 is the variance of ε_t. We have

$$R_2 = \sigma^4 \sum_{\nu=-\infty}^{\infty} \left(\frac{4}{3}\right)^2 \left(\frac{1}{2}\right)^{2\nu} = \frac{80}{27}\sigma^4,$$

$$R_2^* = \sigma^4 \sum_{\nu=-\infty}^{\infty} \left(\frac{4}{3}\right)^2 \nu^2 \left(\frac{1}{2}\right)^{2\nu} = 2\left(\frac{4}{3}\right)^2 \sigma^4 \sum_{\nu=1}^{\infty} \nu^2 \left(\frac{1}{4}\right)^{\nu}$$

$$= \frac{640}{243}\sigma^4,$$

then $\theta = \frac{8}{9}$. If a random sample of size $T = 480$ is drawn from the above linear process with ε_t taking the value -49, -48,...,48,49, with equal probabilities, then $E(\varepsilon_t) = 0$, $\sigma^2 = \frac{2450}{3}$. The observed are x_t displayed in Table 5.3.1 and are rounded off to integer values. The sample covariance sequence $\{R_T(\nu)\}$ is tabled in Table 5.3.2.

Then $m'(T)$ can be found

$$m'(T) = \left(\frac{8}{9}\right)^{\frac{1}{3}}(3\cdot 480)^{\frac{1}{3}} = 11.$$

The two different methods of estimating give the following answers

1) Let $\hat{\theta} = 1$, then $\widehat{m'(T)} = (3\cdot 480)^{\frac{1}{3}} = 12.$

2) $R_{2,T} = 4{,}370{,}590$, $R_{2,T}^* = 72{,}199{,}700{,}480$, $\hat{\theta} = 16{,}519.4375$, and $\widehat{m'(T)} = 280.$

5.4 Summary

The properties of the estimate $R_{2,T}$ are discussed in this chapter. It is shown that if the estimator $f_T(\omega)$ of $f(\omega)$ is consistent, then $R_{2,T}$ will be an asymptotic unbiased estimator and will converge to R_2 in probability.

There are two different methods for obtaining the estimate of the best truncation point. The first method is to choose a positive value for $\hat{\theta}$. The value assigned should depend on the nature of the covariance function. If the covariance function decreases slowly, the $\hat{\theta}$ value chosen should be large, conversely the $\hat{\theta}$ value should be approximately 1 if the covariance function converges quickly to zero.

Table 5.3.1 Experimental Series

t	x_t	t	x_t	t	x_t	t	x_t	t	x_t
1	-44	25	56	49	-20	73	62	97	-52
2	-52	26	55	50	25	74	1	98	-74
3	21	27	35	51	52	75	-25	99	-17
4	16	28	43	52	7	76	-27	100	-21
5	53	29	71	53	-24	77	- 8	101	-30
6	37	30	-12	54	-21	78	15	102	-16
7	30	31	14	55	-23	79	-40	103	10
8	- 7	32	41	56	-10	80	-46	104	0
9	13	33	18	57	9	81	-20	105	-44
10	0	34	48	58	38	82	-26	106	- 8
11	25	35	4	59	11	83	-61	107	-15
12	29	36	26	60	55	84	16	108	-45
13	4	37	35	61	57	85	-21	109	-34
14	-26	38	46	62	35	86	-35	110	-13
15	-33	39	55	63	36	87	-43	111	-34
16	30	40	-16	64	59	88	6	112	-22
17	47	41	-47	65	0	89	34	113	33
18	46	42	20	66	2	90	-28	114	53
19	7	43	34	67	3	91	- 3	115	51
20	-43	44	17	68	-41	92	33	116	40
21	-34	45	51	69	-46	93	11	117	2
22	-30	46	28	70	-24	94	32	118	-27
23	-51	47	58	71	-37	95	15	119	-62
24	23	48	47	72	28	96	-23	120	9

Table 5.3.1 Experimental Series - Continued

t	x_t	t	x_t	t	x_t	t	x_t	t	x_t
121	35	145	28	169	61	193	-47	217	- 5
122	-30	146	14	170	-16	194	0	218	21
123	7	147	-24	171	- 5	195	-48	219	13
124	-29	148	25	172	7	196	-48	220	- 4
125	-16	149	- 4	173	-20	197	2	221	-24
126	-19	150	29	174	-22	198	16	222	-12
127	26	151	33	175	-50	199	- 3	223	-29
128	31	152	51	176	17	200	-29	224	-44
129	42	153	36	177	-25	201	-58	225	-34
130	35	154	11	178	28	202	-26	226	29
131	- 1	155	50	179	- 8	203	- 7	227	46
132	18	156	64	180	35	204	5	228	51
133	-15	157	70	181	23	205	- 7	229	61
134	-21	158	30	182	49	206	-15	230	- 5
135	-50	159	26	183	52	207	20	231	4
136	-31	160	15	184	30	208	14	232	31
137	-55	161	33	185	50	209	6	233	18
138	1	162	-11	186	17	210	42	234	21
139	-11	163	13	187	-23	211	48	235	-17
140	34	164	- 1	188	-32	212	59	236	28
141	-17	165	31	189	24	213	- 3	237	3
142	32	166	36	190	27	214	-18	238	1
143	31	167	-13	191	-31	215	-30	239	42
144	2	168	38	192	-33	216	- 9	240	52

Table 5.3.1 Experimental Series - Continued

t	x_t	t	x_t	t	x_t	t	x_t	t	x_t
241	50	265	65	289	7	313	-28	337	-42
242	50	266	75	290	11	314	-14	338	- 2
243	17	267	73	291	17	315	-48	339	-50
244	-25	268	32	292	58	316	-62	340	9
245	-57	269	4	293	55	317	-51	341	8
246	-51	270	-12	294	25	318	- 8	342	-11
247	7	271	-35	295	62	319	22	343	8
248	48	272	-42	296	-11	320	-17	344	30
249	17	273	-20	297	27	321	6	345	45
250	52	274	13	298	-30	322	7	346	- 9
251	60	275	28	299	30	323	-37	347	-13
252	2	276	23	300	57	324	-10	348	15
253	-16	277	55	301	25	325	-29	349	57
254	-55	278	-13	302	60	326	-47	350	5
255	-43	279	23	303	44	327	-30	351	50
256	25	280	-36	304	-17	328	-12	352	19
257	27	281	- 2	305	38	329	-43	353	9
258	21	282	29	306	-20	330	3	354	- 6
259	-19	283	3	307	21	331	-23	355	-32
260	-14	284	48	308	19	332	14	356	-59
261	- 6	285	53	309	6	333	- 1	357	- 9
262	- 5	286	39	310	27	334	-46	358	-37
263	33	287	28	311	-17	335	-42	359	-12
264	52	288	-17	312	-31	336	-14	360	30

Table 5.3.1 Experimental Series - Continued

t	x_t	t	x_t	t	x_t	t	x_t	t	x_t
361	52	385	0	409	-43	433	21	457	-27
362	59	386	-39	410	-67	434	28	458	-18
363	40	387	-33	411	11	435	30	459	-27
364	- 6	388	17	412	-30	436	15	460	1
365	-11	389	5	413	20	437	-25	461	-16
366	-38	390	8	414	-36	438	22	462	-21
367	3	391	-19	415	0	439	3	463	-35
368	42	392	4	416	-31	440	50	464	-59
369	35	393	-34	417	- 6	441	56	465	-34
370	-18	394	-32	418	11	442	-10	466	-46
371	-46	395	-29	419	40	443	-13	467	23
372	- 8	396	-31	420	31	444	-34	468	- 9
373	38	397	-30	421	26	445	-35	469	29
374	49	398	-25	422	46	446	0	470	-15
375	8	399	-42	423	16	447	3	471	13
376	-43	400	1	424	- 4	448	4	472	17
377	- 7	401	-13	425	29	449	-38	473	12
378	- 8	402	-16	426	7	450	-48	474	- 9
379	16	403	13	427	-12	451	-46	475	42
380	29	404	-19	428	33	452	-53	476	40
381	43	405	-55	429	37	453	-38	477	-25
382	59	406	-38	430	28	454	-37	478	-62
383	-16	407	-27	431	49	455	- 7	479	-10
384	28	408	1	432	8	456	-20	480	43

Table 5.3.2 Sample Covariance of Experimental Series

ν	R(ν)	ν	R(ν)	ν	R(ν)	ν	R(ν)	ν	R(ν)
0	1067.254	24	-97.663	48	-93.987	72	-74.935	96	73.346
1	550.327	25	-25.552	49	-54.212	73	-69.762	97	43.829
2	300.166	26	11.822	50	-40.168	74	-73.541	98	33.346
3	126.433	27	34.981	51	16.789	75	-55.883	99	-43.220
4	32.558	28	64.956	52	40.528	76	-38.867	100	-71.017
5	30.661	29	75.885	53	34.278	77	10.034	101	-62.709
6	35.129	30	-12.614	54	82.067	78	34.689	102	-102.000
7	105.046	31	-83.764	55	59.069	79	52.862	103	-55.284
8	181.727	32	-131.323	56	96.243	80	30.005	104	-70.815
9	110.911	33	-83.347	57	59.615	81	51.466	105	-39.349
10	133.402	34	-82.360	58	41.057	82	81.665	106	-33.549
11	115.468	35	-70.919	59	17.131	83	63.630	107	-60.743
12	91.493	36	-62.882	60	18.447	84	52.080	108	3.411
13	111.293	37	-68.491	61	-54.812	85	41.603	109	0.440
14	68.191	38	-20.359	62	-64.552	86	-32.615	110	42.029
15	40.275	39	12.089	63	-72.512	87	23.402	111	92.035
16	27.080	40	-12.611	64	-77.941	88	3.281	112	64.642
17	19.409	41	-34.019	65	-35.484	89	-26.619	113	38.711
18	29.886	42	-53.021	66	59.915	90	-35.491	114	-4.489
19	36.494	43	-48.223	67	112.231	91	-38.103	115	-22.125
20	5.742	44	-53.279	68	102.971	92	-32.051	116	57.377
21	36.272	45	-95.179	69	12.382	93	-32.402	117	51.500
22	-4.227	46	-89.808	70	-14.086	94	-6.892	118	109.794
23	-81.324	47	-121.258	71	-58.000	95	71.225	119	80.311

Table 5.3.2 Sample Covariance of Experimental Series - Continued

ν	R(ν)	ν	R(ν)	ν	R(ν)	ν	R(ν)	ν	R(ν)
120	90.415	144	- 2.169	168	-127.178	192	91.230	216	- 8.549
121	128.015	145	- 3.994	169	- 73.630	193	78.119	217	10.968
122	129.508	146	17.343	170	- 62.280	194	22.595	218	29.110
123	117.540	147	49.092	171	- 26.760	195	- 30.297	219	57.205
124	123.219	148	5.116	172	4.433	196	- 52.842	220	4.666
125	100.450	149	- 22.385	173	- 25.386	197	- 58.555	221	22.130
126	113.369	150	7.597	174	- 43.649	198	- 29.559	222	44.832
127	59.625	151	18.617	175	- 29.437	199	- 0.616	223	43.319
128	- 30.965	152	42.838	176	- 50.018	200	49.703	224	20.182
129	- 34.067	153	5.721	177	- 46.078	201	51.903	225	- 32.231
130	- 58.863	154	- 0.458	178	- 6.459	202	36.607	226	- 64.099
131	3.260	155	- 7.991	179	21.834	203	72.661	227	- 52.983
132	24.465	156	- 55.670	180	21.941	204	92.017	228	- 4.250
133	104.007	157	-129.336	181	- 25.668	205	80.053	229	61.495
134	170.263	158	-180.938	182	- 31.384	206	51.922	230	54.809
135	130.200	159	-163.644	183	- 31.911	207	4.948	231	73.859
136	135.667	160	-128.142	184	- 47.622	208	- 1.433	232	63.440
137	118.650	161	- 68.992	185	- 49.534	209	- 0.520	233	48.903
138	104.473	162	- 23.663	186	- 96.118	210	- 33.716	234	29.801
139	111.611	163	- 32.480	187	-115.544	211	4.614	235	36.161
140	69.007	164	- 28.599	188	- 80.473	212	2.860	236	25.524
141	46.663	165	- 53.960	189	- 28.822	213	9.703	237	35.407
142	0.917	166	-100.865	190	68.226	214	- 5.655	238	11.309
143	- 11.630	167	-125.359	191	69.192	215	3.553	239	56.028

Table 5.3.2 Sample Covariance of Experimental Series - Continued

ν	R(ν)	ν	R(ν)	ν	R(ν)	ν	R(ν)	ν	R(ν)
240	24.745	264	- 18.608	288	1.081	312	9.041	336	- 4.821
241	9.981	265	- 12.889	289	7.206	313	- 11.221	337	- 4.242
242	- 13.084	266	- 15.331	290	38.002	314	33.968	338	- 32.858
243	- 39.685	267	6.643	291	- 14.212	315	69.149	339	- 29.229
244	- 24.233	268	54.928	292	- 71.539	316	61.278	340	- 5.085
245	- 10.970	269	23.248	293	- 47.322	317	46.151	341	- 4.256
246	16.057	270	11.923	294	- 77.937	318	36.135	342	- 2.340
247	42.030	271	27.901	295	- 54.752	319	37.360	343	- 13.229
248	57.199	272	3.317	296	- 30.898	320	- 3.894	344	18.033
249	39.495	273	20.998	297	- 39.533	321	- 41.691	345	19.816
250	36.653	274	13.962	298	- 50.112	322	- 30.427	346	18.908
251	6.922	275	24.070	299	- 60.341	323	0.785	347	- 15.199
252	27.622	276	12.835	300	- 35.095	324	21.377	348	- 42.743
253	- 1.613	277	6.538	301	- 10.808	325	31.702	349	- 54.712
254	- 14.523	278	- 9.749	302	- 7.523	326	23.131	350	- 53.447
255	- 3.225	279	- 8.537	303	- 12.208	327	35.200	351	- 37.200
256	32.357	280	- 43.537	304	- 2.156	328	- 6.466	352	- 7.941
257	95.043	281	- 46.033	305	- 35.650	329	- 8.447	353	0.056
258	119.107	282	- 46.043	306	- 15.673	330	- 5.562	354	- 6.846
259	81.513	283	- 48.251	307	- 12.502	331	1.192	355	36.070
260	97.195	284	- 52.110	308	- 20.856	332	- 31.125	356	60.793
261	44.545	285	- 2.369	309	- 37.185	333	- 5.138	357	27.543
262	- 3.456	286	2.569	310	- 37.158	334	6.252	358	- 11.002
263	17.248	287	- 8.981	311	- 8.150	335	13.503	359	- 35.585

Table 5.3.2 Sample Covariance of Experimental Series - Continued

ν	R(ν)	ν	R(ν)	ν	R(ν)	ν	R(ν)	ν	R(ν)
360	-46.906	384	26.154	408	3.731	432	-25.662	456	5.631
361	-30.477	385	13.329	409	13.815	433	-17.617	457	1.081
362	-18.385	386	6.956	410	-0.060	434	-18.758	458	1.283
363	4.150	387	6.913	411	-8.987	435	-7.119	459	-11.283
364	-0.102	388	-12.008	412	3.571	436	-2.538	460	-15.185
365	-5.773	389	-16.727	413	-3.867	437	-0.771	461	-14.150
366	-0.947	390	-12.463	414	-9.746	438	-14.252	462	10.058
367	-9.587	391	6.631	415	-10.056	439	-40.250	463	20.025
368	18.342	392	-9.777	416	-16.485	440	-26.858	464	22.871
369	16.592	393	-3.748	417	-32.998	441	-0.002	465	-0.071
370	-0.935	394	17.262	418	-49.458	442	14.181	466	-3.060
371	-10.481	395	39.766	419	-42.279	443	21.900	467	-3.304
372	10.587	396	37.829	420	-50.306	444	3.031	468	4.175
373	6.283	397	18.575	421	-31.210	445	-19.773	469	6.779
374	37.064	398	-13.356	422	5.573	446	-21.823	470	3.983
375	30.623	399	-28.948	423	1.233	447	-12.623	471	-1.927
376	25.348	400	-39.808	424	3.662	448	-4.675	472	-5.746
377	20.671	401	-23.733	425	-16.327	449	-6.671	473	-7.737
378	-1.800	402	2.904	426	-39.037	450	1.875	474	-9.133
379	3.146	403	5.671	427	-44.504	451	9.146	475	0.744
380	-0.581	404	-8.894	428	-11.510	452	-14.971	476	10.004
381	6.234	405	-38.502	429	13.221	453	-20.021	477	8.648
382	13.123	406	-30.206	430	0.879	454	-8.467	478	-3.742
383	36.069	407	-9.994	431	-12.756	455	7.515	479	-3.942

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THE BEST TRUNCATION POINT FOR THE ESTIMATED SPECTRAL DENSITY FUNCTION OF A STATIONARY TIME SERIES

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Abstract

In many applications of time series analysis, the scientist estimates the spectral density function of the process. One type of spectral density estimator is obtained by using the periodogram representation of the spectral density function. However, if the estimator is to be consistent, a weight function which satisfies certain conditions must be added to the periodogram estimator. If the weight function is a truncated function, the estimator of the spectral density function is labeled a truncated estimator. For this type of estimator, a truncation point can be chosen which yields a minimum mean square estimator among all truncated estimators.

In this study we are concerned with two problems; i) finding consistent estimators of the spectral density function, and ii) determining the best truncation point. Two different types of weight functions, both of which give consistent estimators of the spectral density function, are presented in this dissertation. Both of the weight functions have related truncation functions.

For the problem of determining the best truncation function one particular weight function is considered. Using this weight function, the asymptotic forms of the variance and the bias of the estimator are found. It is discovered that the best truncation function is dependent on one term of the bias of the estimator; this term is denoted by $C(m(T))$. There are three different forms of $C(m(T))$; each of these three forms can be used to obtain the best truncation function.

There are several different type of covariance functions for a time series. The asymptotic relative efficiency of the spectral density estimator is dependent on the covariance function of the process. Using the best truncation point for a given weight function the asymptotic relative efficiency is derived for two covariance functions.

The proposed method of finding the best truncation point is applied to other types of weight functions. Finally, the problem of estimating the best truncation function when the covariance function is unknown is discussed.