STABILITY ANALYSIS OF SPATIALLY DEPENDENT NONLINEAR REACTOR SYSTEMS

by

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TABLE OF CONTENTS

I. INTRODUCTION: SPACE TIME REACTOR KINETICS

The purpose of this work is to simplify the numerical computations required in the stability analysis of a nonlinear space-time nuclear reactor study. The generalized mean value theorem is used to eliminate the necessity of solving the nonlinear, time independent equilibrium state problem.

Before embarking on the question of stability, it is felt that some review of the space-time reactor kinetics together with the solution approaches is necessary.

Reactor kinetics is concerned with the time behavior of a nuclear reactor whose state is described by several dynamical variables. The essential property which determines the state of the reactor is the neutron distribution in the reactor core whose nuclear and geometric properties vary in time. The knowledge of the neutron distribution usually requires the solution to coupled partial differential equations, which describe other conservation laws, such as energy conservation and conservation of various nuclear species. The neutron distribution formulated in the most precise manner is impractical, if not impossible, to solve; therefore various approximations have to be made. Formal time dependent solutions of the linear transport equation are discussed in Bell and Glasstone,[(1)] where the notion of criticality is introduced in this manner. A detailed description of various physical phenomena and their interrelationships influencing the temporal behavior of neutrons is given by Z. Akcasu, et al.,[(2)] where a formulation of reactor kinetics with space-time variations in temperature and the rates of gain and loss of neutron species is developed. We shall assume here that the multigroup diffusion approximation model of the reactor is capable of describing the kinetic behavior of the system; there is no rigorous justification for the above statement, and we shall rely on the fact that there is no experimental evidence which conclusively invalidates this assumption.

In the matrix form, the multigroup diffusion equations may be written:

$$V^{-1}\frac{\partial\phi}{\partial t} = (\nabla\cdot D\nabla - A + (1-\beta)XF^T)\phi + \sum_i \lambda_i \chi_i C_i \qquad (1.1)$$

and

$$\frac{\partial C_i}{\partial t} = \beta_i F^T \phi - \lambda_i C_i \qquad (1.2)$$

where, for G groups

$$V = \text{diag}\,(v_1, v_2, \ldots, v_g, \ldots, v_G)$$

$$\phi = \text{col}(\phi_1, \phi_2, \ldots, \phi_g, \ldots, \phi_G)$$

$$D = \text{diag}\,(D_1, D_2, \ldots, D_G)$$

$$F = \text{col}\,(\nu\Sigma_{f1}, \nu\Sigma_{f2}, \ldots, \nu\Sigma_{fG})$$

$$X = \text{col}\,(X_1, X_2, \ldots, X_G)$$

$$\chi_i = \text{col}\,(\chi_{i1}, \chi_{i2}, \ldots, \chi_{iG})$$

$$A = \begin{bmatrix} \Sigma_1, & \Sigma_{12}, & \ldots, & \Sigma_{1G} \\ \Sigma_{21}, & \Sigma_2, & \ldots, & \Sigma_{2G} \\ \Sigma_{G1}, & \Sigma_{G2}, & \ldots, & \Sigma_G \end{bmatrix}$$

v_g = speed of neutrons in group g

ϕ_g = flux of group g neutrons

D_g = diffusion coefficient for group g neutrons

Σ_g = total macroscopic interaction cross-section for group g

$\Sigma_{gg'}$ = cross-section for removal of neutrons from g' to group g

X_g = fraction of prompt neutrons that appear in energy group g

χ_{ig} = fraction of delayed neutrons emitted by precursor i to group g

C_i = concentration of the ith precursor group

β_i = fraction of prompt neutrons emitted by fission which appear in energy group g

$\nu\Sigma_{fg}$ = cross-section for production of fission neutrons by group g neutrons.

The parameters D_g, Σ_g, $\Sigma_{gg'}$ depend on the temperature distribution in the reactor and the distribution of nuclear species.

The atomic concentrations change because of the continual occurrence of nuclear reactions. For example, the production of fission fragments with high absorption cross-section such as Xe^{135} leads to an important temporal variation in thermal reactor. Another reason for the change in atomic concentrations is their dependence on temperature, that is, they decrease with increase in temperature, due to thermal expansion and can have significant effect upon reactor operation. The microscopic cross-sections are sensitive to temperature variation, as they depend on the relative speed of the neutron and nucleus. This phenomenon is called Doppler effect and has an important influence on reactor operation and safety.

It follows from the above remarks that a more complete formulation of reactor kinetics with feedback requires a description of the temporal and spatial variations of the temperature in the reactor, and the knowledge of the rates of gain and loss of nuclear species.

In a power reactor, large amounts of thermal energy generated within the fuel must be efficiently removed from the reactor core to prevent dangerously high temperatures.

The equation which describes the energy balance in the reactor core is given by:[2]

$$\rho(\bar{r},T)\; C_p(\bar{r},T)\; \frac{\partial T(\bar{r},t)}{\partial t} = \nabla \cdot k(\bar{r},T)\; \nabla T(\bar{r},T) - \rho(\bar{r},T)\; C_p(\bar{r},T)\; \bar{V}(\bar{r}) \cdot \nabla T(\bar{r},t) + \varepsilon \sum_{g'=1}^{G} \Sigma_{fg'}(\bar{r},T)\; \phi_{g'}(\bar{r},t) \tag{1.3}$$

from which we obtain the temperature distribution in the core. $\rho(\bar{r},T)$ is the mass density of the core evaluated at position $\bar{r}$ and temperature T $C_p(\bar{r},T)$ is the specific heat capacity of the core. They are to be evaluated in the fuel and coolant respectively. $k(\bar{r},T)$ is the thermal conductivity, $\bar{V}$ is the velocity of the coolant, and ε is heat energy in appropriate units.

Because large amounts of energy have to be removed, the reactor core is composed of small fuel pins surrounded by the constantly moving coolant. This heterogeneous property may result in sharp temperature differences between fuel and coolant. The study of this property in detail is desirable, but analyzing an extremely large number of simultaneous equations is impractical. To simplify the analysis without completely ignoring the heterogeneous property of the core, a procedure given by C. Hsu[3] is used to formulate the equations for temperature distributions in the fuel and coolant. In this procedure the reactor core is assumed to be composed of many identical unit cells, each consisting of a fuel pin and its surrounding equivalent coolant channel.

Balance equations for the production, decay, and burnup of nuclear species may be written in the following form. Let $N_i(\bar{r},t)$ be the number density of some nuclide indicated by i. Then the rate at which N_i changes with time may be written as:

$$\frac{\partial N_i}{\partial t} = \text{production rate} - \text{destruction rate} - \text{decay rate} \tag{1.4}$$

The nuclides may be considered to be formed and lost only as a result of fission, neutron capture and radioactive decay. Equation (1.4) may be expressed as:

$$\frac{\partial N_i}{\partial t} = \sum_j \gamma_{ji}\, \sigma_{fj}\, N_j\, \phi + \sigma_{\gamma=i-1}\, N_{i-1} + \lambda_{i'}\, N_{i'}$$

$$- \sigma_{f,i}\, N_i\, \phi - \sigma_{\gamma,i}\, N_i\, \phi - \lambda_i\, N_i \tag{1.5}$$

where the subscript i-1 denotes the concentration of nuclides which can be converted into type i by neutron capture; that is to say, if i denotes a nuclide with mass and atomic numbers (A,Z), then (i-1) denotes nuclide (A-1,Z). N_i' denotes the concentration of nuclei which yield those of type i by positron decay, i.e., with composition (A, Z-1), and N_j is the concentration of fissionable isotopes and γ_{ji} is the probability that a type i nuclide will be formed as a fission product by absorption of neutrons by nuclide of type j.

It is clear from the above comments that a general problem of finding the temperature distribution and the distribution of various nuclear species without reference to a specific system may be too difficult for analytical study, and the procedure that is usually followed is to break up the problem and study only certain phenomena that seem to have an important bearing on safe design. Magnitudes of the time scales involved describing certain phenomenon are usually taken into account, thus eliminating other phenomena which do not contribute significantly to a given problem. The general problem of dynamic behavior of the reactor core is much more complicated as mentioned above, and a complete description would be outside the scope of this thesis. Engineering disciplines of heat transfer, fluid mechanics, and stress analysis have to be taken into account. Much of the calculational details of the first two disciplines are semiempirical and much dependent on the reactor type.

Because of the complexity of the problem, the earliest works in literature employed various simplifying models and assumptions. Equations of motion were averaged in some prescribed manner[2,4] over the spatial domain of the reactor core, resulting in a set of ordinary differential equations called the "point kinetics" model. Only the dominant features of the time behavior of neutron population, such as the variation of total number of neutrons or the total power generation in the reactor, may be obtained from the point kinetics model. Because of the trend to increasing core size of the reactors that are being designed now, point kinetics may no longer be a good approximation. In a large thermal or fast reactor, local changes such as control rod motion perturbs the flux from the fundamental mode distribution so that deviations from the point kinetics model occur.

This situation has led to the development of a number of methods which account for the spatial effects with more or less accuracy and for the past several years, a number of different approaches for predicting the space time behavior of large power reactors have been under development. These include direct solutions of the space and time differenced multigroup diffusion equations as well as modal[5] and nodal approximations to such solutions.

Numerical comparisons between point kinetics and space dependent kinetic analyses of reactor transients[6,7] suggest that, even in reactors which would generally be considered tightly coupled, failure to account for changes in flux shape during a transient can lead to significant errors in predicted behavior. For many cases finding an equivalent point model is difficult, and a full space-time treatment will have to be applied to the actual three dimensional case being analyzed.

There is thus a growing demand for accurate, cheap methods for predicting spatial changes in flux shape during transients. Three methods are:

(a) Finite Difference Approach. Replace the continuous space and time variables by finite difference meshes, i.e., approximate the spatial derivatives by finite difference expressions involving state variables at a given time at neighboring spatial mesh points, and to replace the time derivative by a finite difference expression involving state variables at some position at neighboring time mesh points. Computer programs have been developed for the study of the space time behavior, however, a practical program for a large three dimensional problem (involving few energy groups and several hundred thousand spatial mesh points) still seems to be beyond the capacity of the present generation of computers.

(b) Modal Expansion Approximation.[(5)] The essential idea which characterizes modal approximations is to express all, or a portion, of the spatial part of the state variables in known functions, which are found by solving static equations and belong to a number of operators representing various static conditions. Variations of this approximation were developed depending on the type of problem considered.

(c) Nodal Approximation. As the name suggests, a nodal method looks at the state variable behavior within a number of large subregions, or nodes, into which the reactor is partitioned. The nodal equations then usually take the form of coupled ordinary differential equations,[(8)] the average power level in a given node being determined both by feedback effects within that node and the interaction with neighboring nodes through "coupling coefficients." The most difficult problem in this approach is the derivation of coupling constants which properly account for leakage of neutrons and flow of energy from one node to another. This can be done in a formal manner if one knows a detailed (many spatial mesh points) solution for the problem considered. But having to obtain such a solution somewhat defeats the original motivation (reduction of computation costs) for turning to the nodal approximation in the first place. There are systematic procedures[9] for obtaining the coupling coefficients from the properties of the node itself by "stiching together" the detailed node solutions into an overall power shape.

The following conclusions may be drawn from the various numerical approaches:

a) There is convincing numerical evidence that the point kinetic equations, which ignore changes in space, fail to predict correctly the behavior for certain transients. It may be generally assumed that the time dependent group diffusion theory can adequately predict space dependent transient phenomena.

b) Finite difference schemes to multigroup diffusion equations in a full three dimensional program is very expensive. Running time considerations will limit the use of such a code to a problem involving ten or twenty thousand spatial mesh points.

c) Nodal methods hold promise of providing another practical tool. However, the essential problem in determining the coupling coefficients in an economic manner has yet to be solved.

When the complete solution is not known, stability analysis is achieved mainly by determination of criteria for bounded solutions and asymptotic stability.

Safety and Stability

Together with the development of solution methods, various approaches to the stability problem of nuclear reactors were under investigation for the past several years. Some conceive stability analysis in the context of explicit space-time solutions,[10] as was discussed above; others find the upper and lower bound of solutions and make conjectures about the exact solutions,[11] or by using purely analytical procedures combined with numerical methods[12] in specific cases.

The earliest work to appear in the stability literature was the analysis of the point model with linear temperature feedback and no delayed neutrons by J. Chernick.[13] The same model was examined by Ergen and Weinberg[14] by constructing the Hamiltonian of an analogous mechanical system. Welton[15] treated the point reactor with delayed neutrons by constructing the Hamiltonian for the dissipative system. Akcasu and Dalfes[16] used electrical analogues and extended the model to include heterogenities and arbitrary temperature feedback. Levin and Nohel[17] investigated the stability of point model with space dependent temperature distribution in one dimension by intro-differential equations. Their procedure was generalized to three dimensional temperature distribution by Helliwell.[18] Recently, Hsu[3] and Kastenberg[19] studied stability of space-time dependent systems by Liapunov and comparison methods.

The subject of reactor safety is very broad and it involves many remote engineering disciplines. Safety requires that a reactor be designed, tested, and operated to the highest possible standards. Most safety analyses start with some failure or malfunction of equipment, which gives rise to an accident sequence which can follow many paths. The likelihood of following any one path depends on the performance of many items of a power plant, and from the knowledge of this performance a probability can be assigned to each path. Stability analysis is a particular aspect of overall safety assessment.

For example, a typical loss of coolant accident may be initiated by a loss of pumping power. The reactor power will be determined by the effects of fuel and coolant temperatures, and we may view such an accident as a perturbation on some of the state variables in a given model. We shall also assume that the system is free of control, i.e., its time evolution proceeds according to its own intrinsic constraints and feedbacks. In that sense, stability analysis is a deterministic approach as opposed to probabilistic method of safety assessment. The results of stability analyses are conditions on system parameters and on the allowable sizes of perturbations which will result in stable response.

II. STABILITY ANALYSIS OF REACTOR DYNAMICS BY THE METHOD OF LIAPUNOV

The concept of stability may be intuitively familiar, and any introduction may seem superfluous. Yet on closer analysis, some discussion of the basic ideas seems necessary. The purpose of this chapter is to define precisely what is meant by dynamical system and stability in a sense of Liapunov. Subsequent chapters will rely on the definitions and concepts introduced here. The definitions follow closely the ones given by Zubov.[24]

The foundations of axiomatic theory of stability was triggered by the work of A. M. Liapunov[20] where it was first applied to ordinary differential equations. Liapunov reduced the problem from one of stability of the undisturbed system to the one of stability of an equilibrium state. He was able to relate the question of stability to the question of existence of a function that possesses certain definite properties. From this function one may obtain estimates of a domain of allowable initial disturbances which will disapper with passage of time.

Stated in very simple terms, a system is said to be stable if in the motion following the application of one arbitrary small perturbation the resulting disturbances remain arbitrarily small throughout the period of investigation. This intuitive idea will be defined more precisely later. Much of Liapunov's theory was applied to mechanical and electrical systems,[29] such as particle motion or rigid body motion. The term "point reactor" model actually came from the analogy of point particle motion where by the motion of a point particle, we mean the change in its position in the course of time. The motion of such point particle is known if its generalized coordinates are given as functions of time. Liapunov's method may be viewed as a generalization of the concept of the Hamiltonian (or total energy) in classical mechanics.

The harmonic oscillator serves as an example to bring out these ideas. Without damping the total energy remains constant, but with a frictional force the system dissipates energy and motion will eventually cease. In the language of stability theory, one would say that the trivial equilibrium state of the harmonic oscillator with damping is asymptotically stable. Since the total energy in the above example is proportional to the sum of squares of generalized coordinates and momenta, the Hamiltonian is positive definite, and this is one of the features of the generalized Liapunov's method. Unfortunately, for a general dynamical system, such analogies between the classical concept of energy and Liapunov's function are difficult to formulate.

An example[(4)] which may be "Hamiltonionized" is a point reactor model with temperature feedback.[(14)] Neglecting the delayed neutrons and other processes, equations for the power and temperature are

$$\frac{dP(t)}{dt} = -\frac{\alpha T(t)}{\ell} P(t) \tag{2.1}$$

$$\frac{dT}{dt} = k(P - P_o) - f(T) \; . \tag{2.2}$$

where P is the reactor power, ℓ is the average neutron lifetime, T is the temperature, k is the reciprocal of reactor's heat capacity, and f(T) is a specified function of temperature. The equilibrium temperature is taken as zero and the equilibrium power P_o. Introducing a new variable defined by

$$x = -\ln \frac{P}{P_o} \tag{2.3}$$

we have

$$\dot{x} \equiv \frac{dx}{dt} = \frac{\alpha}{\ell} T, \text{ and letting } m = \frac{\ell}{\alpha k} , \tag{2.4}$$

we are able to put the equations of motion in the form

$$m\ddot{x} = P_o(e^{-x} - 1) - \frac{1}{k} f(\frac{\ell}{\alpha} x) . \tag{2.5}$$

Equation (2.5) appears to describe a motion or a particle of mass m whose position is x(t), and the solution can be determined once the initial position and initial speed are specified. The last term represents a nonconservative force that depends upon the velocity of the particle.

Consider for a moment the conservative system (f=0) for which Hamilton's principle of least action is

$$\delta\left(\int_{t_1}^{t_2} L(x, \dot{x}, t)\, dt\right) = 0 \tag{2.6}$$

where L is the Lagrangian

$$L = \frac{1}{2} m\dot{x}^2 - P_o(x + e^{-x} + 1) \quad , \tag{2.7}$$

and the Hamiltonian is given by

$$H = p\dot{x} - L = \frac{p^2}{2m} + P_o(x + e^{-x} - 1) \quad , \tag{2.8}$$

where $p = \frac{\partial L}{\partial \dot{x}}$ is the generalized momentum. The time derivative of the Hamiltonian is

$$\frac{dH}{dt} = \dot{x}\left(m\dot{x} + P_o(1 - e^{-x})\right) \quad . \tag{2.9}$$

For a conservative system $\frac{dH}{dt} = 0$, and the motion is represented by closed trajectories in p-x plane (or P-T plane of the original equation). For the nonconservative system

$$\frac{dH}{dt} = -\frac{\alpha}{\ell k} T\, f(T) \quad . \tag{2.10}$$

In order for the motion to be asymptotically stable, $\frac{dH}{dt}$ must be negative implying

$$\frac{\alpha}{\ell k} T\, f(T) > 0 \quad . \tag{2.11}$$

This inequality, when obeyed, will guarantee asymptotic stability of the equilibrium state ($P = P_o$, $T = 0$).

Such an approach seems very attractive from a physical point of view because it not only helps us to have broader insight into reactor dynamics but also dictates the construction of the Liapunov function.

The above ideas were extended[26] to a reactor with arbitrary finite number of temperature feedback coefficients. This may not be done in general even for a model which is described by a set of ordinary differential equations, and for models that are described by partial differential equations, mechanical analogies such as the one above are more difficult to formulate. The reason for employing such ideas is due to the fact that Liapunov's theory of stability provides us with an approach rather than a systematic method, and gives no unique way to construct Liapunov functions. As a result the conditions for asymptotic stability using Liapunov's approach should not be viewed as unique conditions, but rather as conditions belonging to a certain class of Liapunov functions.

Hahn[21] considers a problem dealing with a set of ordinary differential equations which are formulated in the form of Hamilton's equations of motion and derives the condition for asymptotic stability for that system. Although the methods of constructing Liapunov functions permit one to establish that these functions exist, the methods may not be powerful enough to be of use in studying actual reactor systems.

A considerably more difficult problem is the construction of a Liapunov functional for systems of nonlinear partial differential equations, and recent attempts to solve this problem were made.[22,23,27] As a result, a wide variety of Liapunov functions (or functionals) for specific cases have been accumulated but for the present, there is no reliable, simple, well-detailed algorithm developed. Since the author did some investigation along those lines, it should be reported here that such approaches proved to be unfruitful when attempted to be applied to reactor systems. The difficulty of this approach is believed to lie in the fact that the equations contain first order time derivative and a transformation as given by Eq. (2.3) will not result in equation of motion such as Eq. (2.5).

All mathematical models with which we shall be concerned may usually be written in the form

$$\frac{\partial \Phi(\bar{r},t)}{\partial t} = F(\bar{r},\Phi) \; ; \; \bar{r} \varepsilon R \tag{2.12}$$

where $\Phi(\bar{r},t)$ describes physical quantities, $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_n)$ and $F(\bar{r},\Phi)$ is a specified operation on Φ. F may contain differential operators and nonlinear terms. We assume that the spatial domain R is fixed and Γ will denote its boundary. We make up a space X,of functions, elements of which are $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ and $\phi_i(\bar{r})$ are functions with continuous spatial derivatives specified by F in R. We assume that for any $\phi \varepsilon X$ we can construct a solution to Eq. (2.12) $\Phi(\phi,t)$ such that $\Phi(\phi,t)$ is defined for all $0 \leq t < \infty$.

<u>Definition 1</u> (Dynamical system[20])

The solutions $\Phi(\phi,t)$ may be viewed as a single parameter transformation such that for every vector $\phi \varepsilon X$, $\Phi(\phi,t) \varepsilon X$ with the following properties:

(a) $\Phi(\phi,t)$ is continuous at any point (t,ϕ) in both of its arguments t and ϕ, i.e., for any $\varepsilon>0$, it is possible to increase the quantities δ_1 and $\delta_2 > 0$ such that when $|t_1 - t_2| < \delta_1$ and $||\phi_1 - \phi_2|| < \delta_2$, we have $||\Phi(\phi_1,t_1) - \Phi(\phi_2,t_2)|| < \varepsilon$,
where the norm $||\cdot||$ is the L_2 norm.

(b) $\Phi(\Phi(\phi,t_1),t_2) = \Phi(\phi,t_1 + t_2)$

(c) For any function $\phi(\bar{r})$ there exists a unique solution $\Phi(\phi,t)$ such that $\Phi(\phi,0) = \phi$.

We shall denote this initial condition by ϕ_o. The equation of motion with its solutions obeying properties (a), (b), and (c) (and boundary conditions) is called a <u>dynamical system</u>.

<u>Definition 2</u>. (Equilibrium state)

The functions for which

$$\Phi(\phi,t) = \phi \qquad \text{for all } 0 \le t < \infty$$

will be called an <u>equilibrium state</u> of the dynamical system and will be denoted by Φ_{eq}. An equivalent definition of the equilibrium state frequently met is the solution to Eq. (2.12) with the left hand side equal to zero. We shall denote the equilibrium state by Φ_{eq}.

<u>Definition 3</u>. (Stability in the sense of Liapunov)

The equilibrium state $\Phi_{eq}(\bar{r})$ is called stable in the sense of Liapunov if for any $\varepsilon>0$ it is possible to indicate $\delta>0$ such that when

$$||\Phi(\phi,0) - \Phi_{eq}|| < \delta, \text{ we have}$$

$$||\Phi(\phi,t) - \Phi_{eq}|| < \varepsilon \quad \text{for } t \geq 0 .$$

Furthermore, if

$$||\Phi(\phi,t) - \Phi_{eq}(\bar{r})|| \to 0 \text{ as } t\to\infty, \tag{2.13}$$

then the equilibrium state is said to be asymptotically stable.

We are now equipped to state the main stability theorems. It is always possible and convenient to transform our variables in terms of deviations from the equilibrium solution and rephrase the theorems in terms of stability of the trivial equilibrium solution.

$$\psi = \Phi - \Phi_{eq}(\bar{r}) \tag{2.14}$$

<u>Theorem 1</u> (Zubov,[24] Hsu[3])

For a trivial equilibrium state to be stable in the sense of Liapunov, it is necessary and sufficient that in a sufficiently small neighborhood of zero there exists a specified functional $V(\psi)$ with the following properties:

(a) $V(\psi)$ is positive definite with respect to $||\psi||$; i.e., for any sufficiently small $c_1>0$ it is possible to indicate $c_2>0$ such that for $||\psi|| > c_1$, $V(\psi) > c_2$ for all $t > 0$, and $V(\psi)\to 0$ as $||\psi||\to 0$.

(b) $V(\psi)$ is continuous with respect to $||\psi||$; i.e., for any $\varepsilon>0$ there exists a $\delta>0$ such that $V(\psi)<\varepsilon$ whenever $||\psi||<\delta$.

(c) $V(\psi)$ is non-increasing when evaluated along systems trajectory for all $t>0$ provided that $||\psi||<\delta$, $\delta>0$. Furthermore, if $V(\psi)\to 0$ as $t\to\infty$, the system will be asymptotically stable.

Theorem 1 has been proven by C. Hsu.(3) A theorem which will be applied here is due to Massera(25) and is usually referred to as Liapunov's second method. Since it is so frequently used, a simple proof is given here.

Theorem 2. (Massera(25))

If there exists a functional $V(\psi)$ satisfying the properties:

(a) $V(\psi)$ is positive definite, that is $V(\psi) > a(||\psi||)$ when $a(||\psi||)$ is continuous and increasing function of $||\psi||$ when $||\psi||>0$, and $a(0) = 0$.

(b) $\frac{dV}{dt} \leq -c(V)$ where $c(V)$ has the same properties as $a(V)$. Then the trivial equilibrium solution $\psi=0$ is asymptotically stable.

Proof:

Along the trajectory of motion through some initial value $\psi_o(\bar{r})$, we have

$$\int_{V(\psi_o)}^{V(\psi)} \frac{dV}{c(V)} \leq -t \ .$$

As $t\to\infty$, the right hand side of the inequality goes to $-\infty$ which is only possible if $V(\psi)\to 0$ and from (a) it follows that $||\psi||\to 0$.

Since Liapunov's method may be applied to linear as well as non-linear equations, an application to the linear nonseparable diffusion equation will now be considered. The coefficients in the diffusion equation are allowed to depend on position and time in some prescribed fashion. It will be shown how it is possible to obtain conditions which when obeyed, the solution will tend asymptotically to the equilibrium value.

The one speed diffusion equation with boundary and initial conditions is:

$$\frac{1}{v}\frac{\partial\phi(\bar{r},t)}{\partial t} = \nabla\cdot D(\bar{r},t)\ \nabla\phi(\bar{r},t) + \nu\Sigma_f(\bar{r},t)\ \phi(\bar{r},t) - \Sigma_a(\bar{r},t)\ \phi(\bar{r},t)$$

$$\phi(\bar{r},t) = 0 \quad \bar{r}\varepsilon\Gamma$$

$$\phi(\bar{r},0) = \phi_o(\bar{r}) \ . \tag{2.15}$$

Since we are allowing D, $\nu\Sigma_f$ and Σ_a to change in time, the only physically possible equilibrium solution will be $\phi_{eq} = 0$, the trivial solution. Analysis of such a system will therefore correspond to the stability of the shutdown state of the reactor. Together with Eq. (2.15) we are assuming that D is differentiable and $\nabla^2\phi$, $\partial\phi/\partial t$ exist. Consider a Liapunov functional:

$$V(\phi) = \frac{1}{2}\int_R \phi^2(\bar{r},t)\ d^3\bar{r} = \frac{1}{2}||\phi||^2 \tag{2.16}$$

which is positive definite, then

$$\frac{dV}{dt} = \int_R \phi(\bar{r},t)\,\frac{\partial\phi(\bar{r},t)}{\partial t}\,d^3\bar{r}\ . \tag{2.17}$$

Assuming that

$$D(\bar{r},t) \geq f(t) \tag{2.18}$$

for all $\bar{r}\varepsilon R$ where f(t) is a continuous function specified for $t\geq 0$, we obtain

$$\frac{dV}{dt} \leq f(t)\ v \int_R d^3\bar{r}\ \phi(\bar{r},t)\ \nabla^2\phi(\bar{r},t)\ + \\ v\int_R (\nu\Sigma_f(\bar{r},t) - \Sigma_a(\bar{r},t))\ \phi^2(\bar{r},t)\ d^3\bar{r}\ . \tag{2.19}$$

Integrating the first term by parts and using the boundary condition and the fact that

$$\int_R \nabla\phi\cdot\nabla\phi\ d^3\bar{r} \geq \frac{2}{d^2}\int_R \phi^2(\bar{r},t)\ d^3\bar{r}\ , \tag{2.20}$$

where $d = \max_{\bar{r}_1,\bar{r}_2\varepsilon R}|\bar{r}_1 - \bar{r}_2|$ is the size of the core, we finally attain

$$\frac{dV}{dt} < -\int_R v\left(\frac{2}{d^2}\ f(t) + \Sigma_a(\bar{r},t) - \nu\Sigma_f(\bar{r},t)\right)\ \phi^2\ d^3\bar{r}\ . \tag{2.21}$$

When the inequality

$$v\left(\frac{2}{d^2}\ f(t) + \Sigma_a(\bar{r},t) - \nu\Sigma_f(\bar{r},t)\right) > \gamma(t)\ \text{for all}\ \bar{r}\varepsilon R \tag{2.22}$$

and all $0 \leq t < \infty$ is satisfied where $\gamma(t)$ has the property

$$\lim_{t\to\infty} \int_0^t \gamma(t')\, dt' = \infty \,, \tag{2.23}$$

then the trivial equilibrium is stable under all initial perturbations. An estimate of the solution will be

$$||\phi(\bar{r},t)|| \leq ||\phi_o(\bar{r})|| \exp\left[-\int_0^t \gamma(t')\, dt'\right]. \tag{2.24}$$

In a realistic situation, we generally do not know the cross-sections and diffusion coefficient as functions of time explicitly. They do depend on time through various feedback mechanisms. The cross-sections are sensitive to reactor temperature which in turn depends on the neutron flux. This results in a set of coupled nonlinear partial differential equations. A more general formulation which corresponds more closely to the physical situation would involve the multigroup diffusion approximation, and the temperature distribution in the reactor core such as described in the introductory chapter.

To be more specific, a thermal reactor described by two group equations[28] with temperature feedback will be taken as our next example. The equations are:

$$\frac{\partial\phi_1(\bar{r},t)}{\partial t} = v_1\, D_1\nabla^2\phi_1 - v_1\Sigma_1\phi_1 + v_1\nu\Sigma_f(T)\, \phi_2(\bar{r},t)\,, \tag{2.25}$$

$$\frac{\partial\phi_2(\bar{r},t)}{\partial t} = v_2\, D_2\nabla^2\phi_2(\bar{r},t) - v_2\Sigma_2(T)\, \phi_2(\bar{r},t) + v_2\, p(T)\, \Sigma_1\, \phi_1(\bar{r},t), \tag{2.26}$$

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p} \nabla^2 T(\bar{r},t) - \frac{h}{\rho c_p} (T(\bar{r},t) - T_c) + \frac{\varepsilon}{\rho c_p} \Sigma_f(T)\ \phi_2(\bar{r},t)\ , \quad (2.27)$$

where ϕ_1 and ϕ_2 are the fast and thermal fluxes, T is the core temperature, and T_c is an average temperature of the coolant; p is the resonance escape probability. It is assumed that the diffusion coefficients and the fast neutron absorption cross-section are insensitive to temperature variations. The reactor core is homogenized in a certain sense because we are assuming the temperature of the coolant as constant in space. Liapunov's method was shown[3] to encompass a more detailed model with coolant temperature changing in space and time. This simple model is used only for the reason of comparing it with other methods. Equation (2.27) then, is an approximation to the energy balance equation (1.3). The boundary conditions are given by:

$$\phi_1(\bar{r},t) = \phi_2(\bar{r},t) = 0; \quad \bar{r}\varepsilon\Gamma \quad (2.28)$$

$$T(\bar{r},t) = 0; \quad \bar{r}\varepsilon\Gamma\ . \quad (2.29)$$

Assuming that Σ_f, Σ_2, and p vary with temperature in a linear fashion given by

$$\begin{aligned} \Sigma_f &= \Sigma_f^\circ + \alpha_f\ T\ , \\ \Sigma_2 &= \Sigma_2^\circ + \alpha_2\ T\ , \\ p &= p^\circ + \alpha_p\ T\ , \end{aligned} \quad (2.30)$$

the equations of motion become

$$\frac{\partial\phi_1}{\partial t} = v_1 D_1 \nabla^2 \phi_1(\bar{r},t) - v_1\Sigma_1\phi_1 + v_1\nu\Sigma_f^\circ\ \phi_2 + v_1\nu\alpha_f\ T(\bar{r},t)\ \phi_2(\bar{r},t), \quad (2.31)$$

$$\frac{\partial\phi_2}{\partial t} = v_2 D_2 \nabla^2 \phi_2(\bar{r},t) - v_2\Sigma_2^\circ\phi_2 + v_2 p^\circ\Sigma_1\phi_1 - v_2\alpha_2\phi_2\ T + v_2\alpha_p\Sigma_1\phi_1 T, \quad (2.32)$$

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\ \nabla^2 T(\bar{r},t) - \frac{h}{\rho c_p}\ (T - T_c) + \frac{\varepsilon}{\rho c_p}\ \Sigma_f^\circ\phi_2 + \frac{\varepsilon\alpha_f}{\rho c_p}\ \phi_2 T. \quad (2.33)$$

The equilibrium state is obtained from the time independent solutions

$$D_1\nabla^2\phi_{1_{eq}} - \Sigma_1\phi_{1_{eq}} - \nu\Sigma_f\phi_{2_{eq}} + \nu\alpha_f\ T_{eq}\ \phi_{2_{eq}} = 0, \quad (2.34)$$

$$D_2\nabla^2\phi_{2_{eq}} - \Sigma_2^\circ\phi_{2_{eq}} + p^\circ\Sigma_1\phi_{1_{eq}} + \alpha_2\phi_{2_{eq}}\ T_{eq} + \alpha_p\Sigma_1\phi_{1_{eq}}\ T_{1_{eq}} = 0, \quad (2.35)$$

$$k\nabla^2 T_{eq} - h(T_{eq} - t_c) + \varepsilon\Sigma_f^\circ\phi_{2_{eq}} + \varepsilon\ \alpha_f\ \phi_{2_{eq}}\ T_{eq} = 0. \quad (2.36)$$

Formulating the equations about the equilibrium state in terms of

$$\psi_1(\bar{r},t) = \phi_1(\bar{r},t) - \phi_{1_{eq}}(\bar{r}),$$

$$\psi_2(\bar{r},t) = \phi_2(\bar{r},t) - \phi_{2_{eq}}(\bar{r}), \quad (2.37)$$

$$\psi_3(\bar{r},t) = T(\bar{r},t) - T_{eq}(\bar{r}),$$

we have

$$\frac{\partial \psi_1}{\partial t} = v_1 D_1 \nabla^2 \psi_1 - v_1 \Sigma_1 \psi_1 + v_1 (\nu \Sigma_f^\circ - \nu \alpha_f \; T_{eq}(\bar{r}))\psi_2$$

$$+ v_1 \; \nu\alpha_f \; \phi_{2_{eq}}(\bar{r})\psi_3 + v_1 \; \nu\alpha_f \; \psi_2 \; \psi_3 \; , \tag{2.38}$$

$$\frac{\partial \psi_2}{\partial t} = v_2 D_2 \nabla^2 \psi_2 - v_2 (\Sigma_2^\circ - \alpha_2 \; T_{eq}(\bar{r}))\psi_2 + v_2 (p^\circ + \alpha_p \; T_{eq})\Sigma_1 \psi_1$$

$$+ v_2 (\alpha_p \Sigma_1 \phi_{1_{eq}} - \alpha_2 \phi_{2_{eq}})\psi_3 - v_2 \alpha_2 \psi_3 \psi_2 + v_2 \alpha_p \Sigma_1 \psi_1 \psi_3 \; , \tag{2.39}$$

$$\frac{\partial \psi_3}{\partial t} = \frac{k}{\rho c_p} \nabla^2 \psi_3 - \frac{1}{\rho c_p} (h - \varepsilon \alpha_f \phi_{2_{eq}})\psi_3 + \frac{\varepsilon}{\rho c_p} (\Sigma_f^\circ \; \varepsilon \alpha_f \; T_{eq})\psi_2$$

$$+ \frac{\varepsilon \alpha_f}{\rho c_p} \psi_2 \; \psi_3 \; . \tag{2.40}$$

In matrix notation, the equations are of the form of Eq. (2.12)

$$\frac{\partial \psi}{\partial t} = \mathscr{L}(\bar{r})\psi + g(\psi); \qquad \bar{r} \varepsilon R \tag{2.41}$$

where

$$\psi = \begin{bmatrix} \psi_1(\bar{r},t) \\ \psi_2(\bar{r},t) \\ \psi_3(\bar{r},t) \end{bmatrix} \tag{2.42}$$

and $\mathscr{L}(\bar{r})$

$$= \begin{bmatrix} v_1(D_1 \nabla^2 - \Sigma_1) & ; \; v_1(\nu\Sigma_f^\circ + \nu\alpha_f T_{eq}(\bar{r})) & ; \; v_1 \nu\alpha_f \phi_{2_{eq}}(\bar{r}) \\ v_2(p^\circ + \alpha_p T_{eq}(\bar{r}))\Sigma_1 ; & v_2(D_2 \nabla^2 - \Sigma_2^\circ - \alpha_2 T_{eq}(\bar{r})); & v_2(\alpha_p \Sigma_1 \phi_{1_{eq}}(\bar{r}) - \alpha_2 \phi_{2_{eq}}(\bar{r})) \\ 0 & ; \; \frac{\varepsilon}{\rho c_p}(\Sigma_f^\circ + \alpha_f T_{eq}(\bar{r})) & ; \; \frac{k}{\rho c_p}\nabla^2 - (h - \varepsilon\alpha_f)/\rho c_p \end{bmatrix} \tag{2.43}$$

is a linear operator, and

$$g(\psi) = \begin{bmatrix} v_1 \nu \alpha_f \psi_2 \psi_3 \\ v_1 \alpha_p \Sigma_1 \psi_1 \psi_3 - v_2 \alpha_2 \psi_2 \psi_3 \\ \frac{\varepsilon \alpha_f}{\rho c_p} \psi_2 \psi_3 \end{bmatrix} \tag{2.44}$$

is the nonlinearity due to temperature feedback. Although the operator $\mathscr{L}(\overline{r})$ is linear, its entries depend on solutions to equations (2.34 - 2.36), and we have to assume that these solutions exist for further stability analysis.

Consider Liapunov functional of the form

$$V(\psi) = \frac{1}{2} \langle \psi, \psi \rangle \equiv \frac{1}{2} ||\psi||^2 , \tag{2.45}$$

where

$$||\psi||^2 = \int_R d^3\overline{r} (\psi_1^2 + \psi_2^2 + \psi_3^2) .$$

Then

$$\frac{dV}{dt} = \frac{1}{2} \left\langle \psi, \frac{\partial \psi}{\partial t} \right\rangle + \frac{1}{2} \left\langle \frac{\partial \psi}{\partial t}, \psi \right\rangle$$

$$= \frac{1}{2} \left\langle \psi, (\mathscr{L}\psi + g) \right\rangle + \frac{1}{2} \left\langle (\mathscr{L}\psi + g), \psi \right\rangle$$

$$= \langle g, \psi \rangle + \frac{1}{2} \langle \psi, \mathscr{L}\psi \rangle + \frac{1}{2} \mathscr{L}\psi, \psi \rangle$$

$$= \langle g, \psi \rangle + \left\langle \psi, \frac{1}{2}(\mathscr{L} + \mathscr{L}^+)\psi \right\rangle , \tag{2.46}$$

where $\mathscr{L}^+$ is the adjoint of $\mathscr{L}$. The operator $\frac{1}{2}(\mathscr{L}+\mathscr{L}^+)$ is self adjoint, and consequently the eigenvalues γ_n of

$$\frac{1}{2}(\mathscr{L}+\mathscr{L}^+)\xi_n = \gamma_n\xi_n \tag{2.47}$$

are real. Denoting the largest eigenvalue by γ_o we have

$$\frac{dV}{dt} \leq \gamma_o \left\langle \psi,\psi \right\rangle + \left\langle g,\psi \right\rangle \tag{2.48}$$

and using the result of theorem 2, the condition for asymptotic stability is

$$\gamma_o \left\langle \psi,\psi \right\rangle + \left\langle g,\psi \right\rangle < 0 \ , \tag{2.49}$$

which defines the domain of allowable perturbations which will eventually disappear in time. We are free to choose the initial conditions to be perturbations of a given amplitude, that is, we may choose

$$\psi(\bar{r},o) = \psi_o(\bar{r}) = \begin{bmatrix} A_1 \ \psi_{10}(\bar{r}) \\ A_2 \ \psi_{20}(\bar{r}) \\ A_3 \ \psi_{30}(\bar{r}) \end{bmatrix} \tag{2.50}$$

with $\left\langle \psi_{io},\psi_{io} \right\rangle = 1$, $i = 1, 2, 3$ for convenience, A_i, $(i = 1, 2, 3)$ is the amplitude of the deviation from equilibrium state. The inequality (2.49) will give us the maximum allowable amplitudes, i.e.,

$$\gamma_o(A_1^2 + A_2^2 + A_3^2) + v_1A_1A_2A_3 \int_R \nu\alpha_f\psi_{10}\psi_{20}\psi_{30}\ d^3\bar{r}$$

$$- v_2A_2^2A_3 \int_R \alpha_2\psi_{20}^2\psi_{30}\ d^3\bar{r} + v_2A_1A_2A_3 \int_R \alpha_p\Sigma_1\psi_{10}\psi_{20}\psi_{30}\ d^3\bar{r}$$

$$+ A_2A_3^2\varepsilon \int_R \frac{\alpha_f}{\rho c_p}\,\psi_{20}\psi_{30}^2\ d^3\bar{r} > 0\ . \tag{2.51}$$

One may look at this result as a problem of design for the temperature coefficients α_2, α_f, and α_p if the expected perturbations are given. For example, manufacturers of reactor equipment may specify sizes of perturbations that one may expect while under normal operation; from these data it may be possible to evaluate perturbations on the group fluxes and temperature.

Other examples with numerical results have been studied by C. Hsu(3) where the model includes delayed neutrons and the reactor core is a homogeneous slab. As a final example in this chapter, the phenomenon of spatial xenon oscillations will be examined. The reason for including the Liapunov's formulation of this problem is for future comparison with the semigroup method.

In a thermal reactor, the local production of xenon and its consumption may lead to an instability problem. When a reactor is operating at constant power and the flux is increased in one region of the core and simultaneously decreased in another region, the xenon concentration will initially decrease in the region of decreased flux and vice versa. The reason for that is due to the fact that xenon destruction rate changes instantaneously with changes in neutron flux, while xenon production depends upon iodine concentration and hence upon the local flux history over the past few hours. This shift in xenon concentration is such as to increase the multiplication properties in the region of increased flux and decrease the multiplication properties in the region of decreased flux; this may reverse the flux tilt. Because the xenon oscillations occur at constant power, they may go unnoticed and represent something of a hazard to safe operation of a reactor. Conceivably, they may lead to dangerously high local power and local temperature peaking, which in turn may result in premature materials failure. In the model, delayed neutron effects may be neglected due to time constants involving xenon and iodine production.

The dynamic equations describing the neutron, iodine, and xenon concentration, respectively, are of the form

$$\frac{\partial N(\bar{r},t)}{\partial t} = vD\nabla^2 N + v(\nu\Sigma_f - \Sigma_a)N - v\sigma_x XN \tag{2.52}$$

$$\frac{\partial I}{\partial t} = v\gamma_I \Sigma_f N - \lambda_I I \tag{2.53}$$

$$\frac{\partial X}{\partial t} = \lambda_I I - \lambda_x X - v\sigma_x XN \tag{2.54}$$

Formulating the equations in terms of deviations from equilibrium state $N_{eq}(\bar{r})$, $I_{eq}(\bar{r})$, $X_{eq}(\bar{r})$, we obtain equations identical in form to Eq. (2.41) with

$$\mathscr{L}(\bar{r}) = \begin{bmatrix} vD\nabla^2 + v(\nu\Sigma_f - \Sigma_a) - v\sigma_x X_{eq}(\bar{r}); & 0 & ; & -v\sigma_x N_{eq}(\bar{r}) \\ v\gamma_I \Sigma_f & ; & -\lambda_I & ; & 0 \\ v\sigma_x X_{eq}(\bar{r}) & ; & \lambda_I & ; & -(\lambda_x + v\sigma_x N_{eq}) \end{bmatrix} \tag{2.55}$$

and

$$g(\psi) = - v\sigma_x \begin{bmatrix} \psi_1\psi_3 \\ 0 \\ \psi_1\psi_3 \end{bmatrix} \qquad (2.56)$$

where

$$\psi_1 = N(\bar{r},t) - N_{eq}(\bar{r}) ,$$

$$\psi_2 = I(\bar{r},t) - I_{eq}(\bar{r}) , \text{ and}$$

$$\psi_3 = X(\bar{r},t) - X_{eq}(\bar{r}) .$$

The stability result follows exactly the procedure outlined by Eqs. (2.45-2.51), and is given by

$$0 < \gamma_o(A_1^2 + A_2^2 + A_3^2) - v\sigma_x A_1A_3 \int (A_1\Psi_{10} + A_3\psi_{30})\psi_{10}\psi_{30}\, d^3\bar{r} . \qquad (2.57)$$

A_i and ψ_{io} have the same meaning as in Eq. (2.51) and γ_o corresponds to the operator $\mathscr{L}$ given by Eq. (2.55).

III. STABILITY ANALYSIS BY SEMIGROUP METHOD

As a motivation for the use of semigroup approach to stability analysis, consider the one speed diffusion equation with constant coefficients.

$$\frac{\partial\phi(\bar{r},t)}{\partial t} = vD\nabla^2\phi(\bar{r},t) + v(\nu\Sigma_f - \Sigma_a)\,\phi(\bar{r},t) + q(\bar{r},t);\ \bar{r}\varepsilon R \tag{3.1}$$

R is the interior of the reactor core. With the initial and boundary conditions

$$\phi(\bar{r},o) = \phi_o(\bar{r}) \tag{3.2}$$

$$\phi(\bar{r},t) = 0, \quad \bar{r}\varepsilon\Gamma \tag{3.3}$$

where Γ is the boundary of the reactor core. It is well known that the solution to this problem may be obtained in series expansion of the form

$$\phi(\bar{r},t) = \sum_n a_n(t)\psi_n(\bar{r}) \tag{3.4}$$

where $\psi_n(\bar{r})$ are eigenfunctions of the Helmholtz equation

$$\nabla^2\psi_n(\bar{r}) + \lambda_n\,\psi_n(\bar{r}) = 0\ ; \quad \psi_n(\bar{r}')\big|_{\bar{r}'\varepsilon\Gamma} = 0 \tag{3.5}$$

with convenient normalization $\langle \psi_i(\bar{r}), \psi_j(\bar{r}) \rangle = \delta_{ij}$. The eigenvalue λ_n will always be positive for solutions that vanish on the boundary. Expanding the inhomogenous source term in a similar fashion, and using orthogonality, the solution to Eq. (3.1) is

$$\phi(\bar{r},t) = \sum_n e^{k_n t} \psi_n(\bar{r}) \langle \psi_n(\bar{r}'), \phi_o(\bar{r}') \rangle + \\ + \int_o^t dt' \sum_n e^{k_n(t-t')} \psi_n(\bar{r}) \langle \psi_n(\bar{r}'), q(\bar{r}'t') \rangle \tag{3.6}$$

where

$$k_n = v\left[(\nu\Sigma_f - \Sigma_a) - D\lambda_n\right] \cdot \tag{3.7}$$

Equation (3.6) may be written in the form

$$\phi(\bar{r},t) = \int_R G(\bar{r},\bar{r}',t)\ \phi_o(\bar{r}')\ d^3\bar{r}' + \int_o^t dt' \int_R G(\bar{r},\bar{r}';\ t-t') \\ q(\bar{r}',t')\ d^3\bar{r}'. \tag{3.8}$$

In the event of a nonlinear equation

$$\frac{\partial\phi}{\partial t} = vD\nabla^2\phi(\bar{r},t) + v(\nu\Sigma_f - \Sigma_a)\ \phi(\bar{r},t) + g(\phi(\bar{r},t)) \tag{3.9}$$

with same boundary and initial conditions (3.2) and (3.3), the problem may be rewritten as an integral equation

$$\phi(\bar{r},t) = \int_R G(\bar{r},\bar{r}'; t)\, \phi_o(\bar{r}')\, d^3\bar{r}' + \int_o^t dt' \int_R G(\bar{r},\bar{r}'; t-t')$$

$$g(\phi(\bar{r}',t'))\, d^3\bar{r}' \quad . \tag{3.10}$$

The solution to Eqs. (3.1) or (3.9) possesses the following property which leads to group theoretical considerations. If the neutron distribution $\phi(\bar{r},t)$ in the core at time $t>0$ is uniquely determined by the initial distribution $\phi_o(\bar{r})$, then $\phi(\bar{r},t)$ can also be obtained by first computing the neutron distribution $\phi(\bar{r},t_o)$ at some intermediate time $t_o>t$, and then computing it at time t regarding the state at time t_o as a new initial state. If we denote the solution $\phi(\bar{r},t)$ associated with given initial value $\phi_o(\bar{r})$ by $\phi(\bar{r},t;\ \phi_o)$, the above statement may be expressed as

$$\phi(\bar{r},t;\ \phi_o) = \phi(\bar{r},t-t_o;\ \phi(\bar{r},t_o;\ \phi_o)). \tag{3.11}$$

It is essential for our study that there exists unique solution and that

$$\lim_{t\to 0+} \phi(\bar{r},t;\ \phi_o) = \phi_o(\bar{r}) \tag{3.12}$$

For each $t>0$, $\phi(\bar{r},t;\ \phi_o)$ may be considered as a transformation on ϕ_o defined by

$$\phi(\bar{r},t,\ \phi_o) \equiv \Big[S(t)\phi_o\Big](\bar{r}) + \int_o^t dt'\ S(t-t')\ g(\phi)\ . \tag{3.13}$$

Property (11) can now be restated in the form

$$S(t_o+t)\phi_o = S(t_o)\ S(t)\phi_o \tag{3.14}$$

which is called semigroup property, and sometimes the operator S(t) is called the transition operator and the diffusion operator

$$\mathscr{L} = vD\bar{\nabla}^2 + v(\nu\Sigma_f - \Sigma_a) \tag{3.15}$$

is called generator of the semigroup. The semigroup approach to stability analysis uses an important property of the transition operator.

The diffusion operator in the above example is obviously unbounded but S(t) is bounded. To demonstrate its boundedness, consider

$$\begin{aligned}
||S(t)\phi_o|| &= \left[\int_R d^3\bar{r}\left(\int_R d^3\bar{r}'\ G(\bar{r},\bar{r}';\ t)\ \phi_o\right)^2\right]^{1/2} \\
&= \left[\int_R d^3\bar{r}\left(\sum_n e^{(v(\nu\Sigma_f-\Sigma_a) - D\lambda_n)t}\int_R d^3\bar{r}'\ \psi_n(r)\ \psi_n(\bar{r}')\phi_o(\bar{r}')\right)^2\right]^{1/2} \\
&< e^{(v(\nu\Sigma_f-\Sigma_a) - D\lambda_1)t}\left[\int_R d^3\bar{r}\left(\int d^3\bar{r}'\sum_n\psi_n(\bar{r})\ \psi_n(\bar{r}')\ \phi_o(\bar{r}')\right)^2\right]^{1/2} \\
&= e^{\gamma t}\left[\int_R d^3\bar{r}\left(\int_R d^3\bar{r}'\ \delta(\bar{r}-\bar{r}')\ \phi_o(\bar{r}')\right)^2\right]^{1/2} \\
&= e^{\gamma t}\ ||\phi_o||,
\end{aligned}$$

or

$$||S(t)\phi_o|| \leq ||S(t)|| \, ||\phi_o|| \; ; \quad ||S(t)|| \leq e^{\gamma t}$$

where

$$\gamma = v(\nu\Sigma_f - \Sigma_a) - D\lambda_1 \text{ and } \lambda_1 < \lambda_2 < \ldots < \lambda_n \ldots . \tag{3.16}$$

Now consider the connection of S(t) to the diffusion operator $\mathscr{L}$ given by eq. (3.15)

$$\frac{\phi(\bar{r},t+t;\ \phi_o) - \phi(\bar{r},t;\ \phi_o)}{t'} = \frac{S(t+t')\phi_o - S(t)\phi_o}{t'} . \tag{3.17}$$

Using semigroup property eq. (14)

$$\frac{\phi(\bar{r},t+t';\ \phi_o) - \phi(\bar{r},t;\ \phi_o)}{t'} = S(t)\left(\frac{S(t') - I}{t'}\right)\phi_o$$

$$= \left(\frac{S(t') - I}{t'}\right)S(t)\phi_o = \left(\frac{S(t') - I}{t'}\right)\phi(\bar{r},t;\ \phi_o) . \tag{3.18}$$

We may define an operator U called the generator of the semigroup $\{S(t);\ 0 \lesssim t < \infty\}$ by

$$U\ \phi_o = \lim_{t'\to 0^+}\left(\frac{S(t') - I}{t'}\right)\phi_o \tag{3.19}$$

and if we take the limit as $t\to 0$ of the homogenous diffusion equation (1) we see that the operator U is equal to $\mathscr{L}$.

The problem considered above is a special case of a quite more general situation (e.g., multigroup diffusion approximation), and it served only to illustrate the concept of a semigroup of an operator. In practice, it is usually difficult to invert a general partial differential equation into an integral equation, that is, the mathematical problem of determining actual expression for Green's function $G(\bar{r},\bar{r}',t)$ may be difficult. The fact that stability analysis does not require us to look for exact solutions, but rather their behavior in time, the abstract approach in the language of semigroup serves as a convenient tool. To be more precise, one only has to show whether or not a general linear operator $\mathscr{L}$ is an infinitesimal generator of semigroup.

A necessary and sufficient condition for an operator $\mathscr{L}$ to have the above property is given by the theorem due to Hille and Yosida.[36] In view of the context of the theorem, a more precise definition of a semigroup or operator is given below.

Definition: If $S(t)$ is an operator function on $0 \leq t < \infty$ satisfying the following conditions:

$$S(t_1 + t_2) = S(t_1)\, S(t_2)\ , \tag{3.20}$$

and

$$S(0) = I, \tag{3.21}$$

where I is the identity operator,

then $\{S(t);\ 0 \le t < \infty\}$ is called semigroup of operator. The semigroup is said to be of class C_o if it satisfies one further condition

$$\lim_{t \to 0^+} S(t)f = f \tag{3.22}$$

Property (3.22) is sometimes referred to as strong continuity.

Theorem 1. (Hille-Yosida)

A necessary and sufficient condition for a closed operator U with dense domain D(U) and range in Banach space X to generate semigroup of class C_o $\{S(t)\ 0 \le t < \infty\}$ is that there exist real numbers M and γ such that for every real $\lambda > \gamma$, λ belonging to the resolvent set of U (the resolvent being set of all s for which $R(s;\ U) = (sI - U)^{-1}$ exists with domain of R dense in X, and R bounded) and

$$||R(\lambda;\ U)||^n \le M(\lambda - \gamma)^{-n}. \quad n = 1, 2, \ldots \tag{3.23}$$

In this case

$$||S(t)|| \le Me^{\gamma t}; \quad \text{for all } t \ge 0 . \tag{3.24}$$

The theorem holds in real or complex Banach space and it gives a much more general situation than is actually needed for problems that will be attempted here; that is, the operator $\mathscr{L}$ being the linear portion of multigroup approximation generates solutions that belong to the solutions of system of parabolic differential equations. The latter was proven to generate a semigroup.[31]

Results of Hille and Yosida are in a sense abstract generalization of Laplace transform approach to initial value problems for linear systems.[32] To illustrate that, consider a linear operator equation of the form

$$\frac{\partial\phi(\bar{r},t)}{\partial t} = \mathscr{L}(\bar{r})\ \phi(\bar{r},t) \tag{3.25}$$

with

$$\phi(\bar{r},0) = \phi_o(\bar{r}) \ . \tag{3.26}$$

Taking the Laplace transform of (3.25), we have

$$(sI - \mathscr{L}(\bar{r}))\ \tilde{\phi}(\bar{r},s) = \phi_o(\bar{r}) \tag{3.27}$$

where s is the transform variable.

In order for the inverse Laplace transform of $R(s,\mathscr{L})$ to be of the form $\exp(t\mathscr{L})$, we have

$$R(s,\mathscr{L})\ \phi_o(\bar{r}) = \int_0^\infty e^{-st}\ e^{t\mathscr{L}}\phi_o(\bar{r})\ dt, \text{ and} \tag{3.28}$$

$$||R(s,\mathscr{L})\phi_o|| \le \int_0^\infty e^{-st}\ ||e^{t\mathscr{L}}\phi_o||\ dt \tag{3.29}$$

and if $||e^{t\mathscr{L}}\phi_o|| \le e^{\omega t}\ ||\phi_o||$, then the above integral converges for $Re(s) > \omega$ and

$$||R(s,\mathscr{L})|| \le (Re(s) - \omega)^{-1} \ . \tag{3.30}$$

This last inequality is exactly condition (3.20) of Hille-Yosida theorem for $M = n = 1$. In fact the following corrollary to theorem 1 was proven by P. Butzer.[33]

If $\mathscr{L}$ is a closed linear operator with domain $D(\mathscr{L})$ dense in X and if $R(s,\mathscr{L})$ exists for all s larger than some real ω and satisfies the inequality

$$||R(s,\mathscr{L})|| \leq (s - \omega)^{-1} \tag{3.31}$$

then $\mathscr{L}$ is an infinitesimal generator of a semigroup $\{S(t);\ 0 \leq t < \infty\}$ of class C_o such that

$$||S(t)|| \leq e^{\omega t} \text{ for all } t \geq 0 \ . \tag{3.32}$$

According to the above arguments, there exists a unique operator valued function symbolized by $e^{t\mathscr{L}}$. The interpretation of such exponential function is straightforward for a bounded operator $\mathscr{L}$, and we may write

$$S(t) = e^{t\mathscr{L}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} (\mathscr{L})^n \tag{3.33}$$

For $\mathscr{L}$ that is bounded this series converges. In reactor kinetics the operator $\mathscr{L}$ will not be bounded in most cases and the interpretation of $e^{t\mathscr{L}}$ is less direct. Perhaps, one way to think of it then is in terms of the resolvent operator $(sI - \mathscr{L})^{-1}$:

$$e^{t\mathscr{L}}\phi_o = S(t)\ \phi_o = \frac{1}{2\pi i}\ \lim_{\alpha\to\infty}\int_{\beta-i\alpha}^{\beta+i\alpha} e^{st}(sI - \mathscr{L})^{-1}\ \phi_o\ ds\ . \qquad (3.34)$$

It is important to realize that in stability analysis we need not have an explicit expression for S(t) but only know of its existence, and the norm of S(t) may be obtained from the spectrum of the operator $\mathscr{L}$. The constant ω has to be in the spectrum of $\mathscr{L}$ since s = ω.

Stability Study

In general, the system of equations occurring in nuclear dynamics may be written in the form

$$\frac{\partial\phi(\bar{r},t)}{\partial t} = \mathscr{L}(\bar{r})\ \phi(\bar{r},t) + g(\phi(\bar{r},t)) \qquad (3.35)$$

where ϕ is a vector whose components are variables that describe the reactor system. They may be the multigroup neutron distributions, delayed neutron precursors distributions, temperature distributions, etc. The dimension of this vector depends usually on how detailed the model is. $\mathscr{L}$ is a matrix whose elements depend on position and may contain differential operators, and g is the nonlinearity due to feedback mechanisms. In order for the problem to be well defined, we must specify initial and boundary conditions:

$$\begin{aligned} &\phi(\bar{r},0) = \phi_o(\bar{r}), \\ &B(\phi(\bar{r},t)) = 0; \quad \bar{r}\varepsilon\Gamma\ . \end{aligned} \qquad (3.36)$$

An equilibrium state ϕ_{eq} of Eq. (3.35) is defined to be the solution to the following problem.

$$\mathscr{L}(\bar{r})\ \phi_{eq}(\bar{r}) + g(\phi_{eq}) = 0 \tag{3.37}$$

$$B(\phi_{eq}(\bar{r})) = 0;\ \bar{r}\varepsilon\Gamma \tag{3.38}$$

Since the partial differential equation (3.35) is to represent a physical system, we assume that solution exists and is uniquely determined and, furthermore, it depends continuously on the initial data $\phi_o(\bar{r})$ as described by Eq. (3.12). In essence, we are excluding any ambiguous and contradictory properties in the physical situation. As shown in the introductory example a compact way of expressing the solution to Eq. (3.35), in which the initial and boundary conditions are incorporated, is by an integral equation

$$\phi(\bar{r},t) = S(t)\ \phi_o(\bar{r}) + \int_0^t S(t - t')\ g(\phi(\bar{r},t'))\ dt'. \tag{3.39}$$

One may then obtain asymptotic behavior of solution by finding bounds on terms of Eq. (3.39).

For certain reactor safety problems, it is necessary to study the stability of the shutdown state which corresponds to the trivial equilibrium solution $\phi_{eq}(\bar{r}) = 0$ if analysis of nontrivial equilibrium state introduces additional computation as will be shown later.

Stability Theorems

Theorem 2: (Stability of linear system due to C. Hsu.[3]) If $\mathscr{L}(\bar{r})$ is an infinitesimal generator of a semigroup and its spectrum $\sigma(\mathscr{L})$ satisfies the condition

$$\mathrm{Re}(\sigma(\mathscr{L})) \le \gamma; \quad \gamma < 0 , \tag{3.40}$$

then the trivial solution of Eq. (3.35) with g = 0 is asymptotically stable; moreover, the solution will satisfy an estimate of the form

$$||\phi(\bar{r},t)|| \le e^{\gamma t} \, ||\phi_o(\bar{r})|| \, . \tag{3.41}$$

The proof is given by C. Hsu.[3] Note that γ in theorem 2 corresponds to ω in Hille-Yosida theorem. The spectrum of $\mathscr{L}$ is the set of all complex numbers λ for which $(\lambda I-\mathscr{L})^{-1}$ does not exist so that γ will be on the boundary of the spectrum.

In general, we shall always require asymptotic stability of a linearized problem, which corresponds to small perturbations for which nonlinearities may be neglected. Physically, one would not expect the system to be stable under large perturbations if it is not stable even for the smallest ones. The use of this theorem will come into analysis of nonlinear systems.

Theorem 3: (C. Hsu[3])

In addition to the conditions of theorem 2, if

(a) $g(0) = 0$, and

(b) $||g(\phi)|| \le \rho(t) \; ||\phi||$

then the trivial equilibrium state of Eq. (3.35) with initial and boundary conditions (3.36) will be asymptotically stable if

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \rho(t')\, dt' + \gamma < 0 \ . \tag{3.42}$$

Condition (b) of the above theorem dictates the restriction on the nature of nonlinearities, and it assumes an a priori bound on the nonlinear term. This restricts the applicability to systems which can be bounded in this way. C. Hsu[34] has obtained asymptotic stability condition for a restriction on the nonlinear term of the form

$$||g(\phi)|| < \rho(t)\ ||\phi||^{1+\alpha} \qquad \alpha > 0 \ . \tag{3.43}$$

Since C. Hsu states the theorem without proof, the theorem will be proven here. First, we need the following lemma.

Lemma 1. (Due to Bihari[35])

Let $Y(t)$, $F(t)$ be positive continuous functions in $a \le t \le b$ and $k \neq 0$, $M \ge 0$, $W(u)$ a non-negative, non-decreasing-continuous function for $u \ge 0$. Then the inequality

$$Y(t) \le k + M \int_a^t F(t')\, W(Y(t'))\, dt' \qquad a \le t \le b \tag{3.44}$$

implies the inequality

$$Y(t) < \Omega^{-1}\left(\Omega(k) + M \int_a^t F(t')\, dt' \right) \tag{3.45}$$

where

$$\Omega(u) = \int_{u_o}^{u} \frac{du'}{W(u')}\ u_o > 0 \ ; \qquad u \ge 0 \ . \tag{3.46}$$

Proof of this lemma will be found in the appendix.

Theorem 4: In addition to the conditions of theorem 2 and if inequality (3.43) holds, then the trivial equilibrium solution to (3.35) is asymptotically stable for all initial conditions that satisfy

$$||\phi_o(\bar{r})|| < \left(\alpha \int_0^\infty \rho(t')\ e^{\alpha\gamma t'}\ dt'\right)^{-1/\alpha} \tag{3.47}$$

and furthermore the solution is bounded from above by

$$||\phi|| < e^{\gamma t}\left(||\phi_o||^{-\alpha} - \alpha \int_0^t \rho(t')\ e^{\alpha\gamma t'}\ dt'\right)^{-1/\alpha} . \tag{3.48}$$

Proof:

Since $\mathscr{L}(\bar{r})$ is a generator of a semigroup S(t) and $||S(t)|| \le e^{\gamma t}$, by the conditions of theorem 2, using triangle inequality on Eq. (3.39)

$$\begin{aligned} ||\phi(\bar{r},t)|| &\le ||S(t)\phi_o|| + \int_0^t dt'\ ||S(t-t')\ g(\phi)|| \\ &\le ||S(t)||\ ||\phi_o|| + \int_0^t dt'\ ||S(t-t')||\ ||g(\phi)|| \\ &\le e^{\gamma t}\ ||\phi_o|| + \int_0^t dt'\ e^{\gamma(t-t')}\ \rho(t')\ ||\phi(\bar{r},t')||^{1+\alpha} . \end{aligned} \tag{3.49}$$

Applying the result of lemma 1, we have

$$||\phi|| \le e^{\gamma t}\ \Omega^{-1}\left(\Omega(||\phi_o||) + \int_0^t e^{\alpha\gamma t'}\ \rho(t')\ dt'\right) . \tag{3.50}$$

In our case $\Omega(u) = \int_{u_o}^{u} \frac{du'}{u'^{(1+\alpha)}} = \frac{1}{\alpha}\left(\frac{1}{u_o^{\alpha}} - \frac{1}{u^{\alpha}}\right)$ and $\Omega^{-1}(\Omega(u)) = u$ defines the inverse function to be:

$$\Omega^{-1}(z) = \left(\frac{1}{u_o^{\alpha}} - \alpha z\right)^{-1/\alpha}, \text{ and}$$

$$||\phi|| < e^{\gamma t}\left(||\phi_o||^{-\alpha} - \alpha \int_0^t e^{\alpha\gamma t'} \rho(t')\, dt'\right)^{-1/\alpha}. \tag{3.51}$$

In order to have non-negative and bounded $||\phi||$ for all $t > 0$, we must have

$$||\phi_o|| < \left(\alpha \int_0^{\infty} \rho(t')\, e^{\alpha\gamma t'}\, dt'\right)^{-1/\alpha} \tag{3.52}$$

which proves the theorem.

In the event of more complicated bounds on the nonlinearity such as the one studied by C. Hsu[34] has its computational limitations because it is not always possible to find the inverse function $\Omega^{-1}(z)$ as it was above in Eq. (3.51).

In connection with the semigroup approach, a method for obtaining stability conditions is proposed in this work through the use of generalized mean value theorem for a vector function. It will be shown that conditions for asymptotic stability may be derived in terms of derivatives of the system nonlinearities. The conditions may be more conservative when compared with other methods; however, this approach leads to computational simplifications especially for problems that require analysis of nontrivial equilibrium state. We need the following lemma to prove stability theorem.

Lemma 2: (Generalized mean value theorem[36])

Let $g(u_1, u_2, \ldots, u_n)$ (or in short $g(u)$) be a vector valued function, that is $g(u) = \text{col}\,(g_1(u), g_2(u), \ldots, g_n(u))$. Assume that g possesses a Jacobian at every point of u. Then

$$||g(u) - g(v)|| \leq ||J(u^*)||\ ||u - v|| \tag{3.53}$$

where

$$J_{ij}(u^*) = \left(\frac{\partial g_i}{\partial u_j}\right)_{u^*} \qquad i, j=1, 2, \ldots, n \tag{3.54}$$

is the Jacobian matrix evaluated at some point u* which lies between vectors u and v, and

$$||J(u^*)|| = \left(\max_{r\varepsilon R} \sum_{ij} |\ J_{ij}(u^*)|^2\right)^{1/2} . \tag{3.55}$$

Proof of this is found in the appendix.

Theorem 5:

Let $\phi(\vec{r},t)$ be the solution to Eq. (3.35) with $\phi_{eq}(\vec{r})$ the solution to Eq. (3.37) and $\mathscr{L}(\vec{r})$ the generator of a semigroup S(t) with

$$||S(t)|| \leq e^{\gamma t}, \qquad \gamma < 0 .$$

Then a necessary condition for asymptotic stability is

$$\gamma + ||J(\phi_{eq} + \phi_o(\vec{r}))|| < 0 . \tag{3.56}$$

Proof: From Eq. (3.55) it follows that $||J(u)||$ is non-decreasing with respect to

$$\max_{\vec{r}\varepsilon R} ||u|| = ||u||_\infty .$$

Since if $||u_1||_\infty \leq ||u_2||_\infty$, then $||J(u_1)|| \leq ||J(u_2)||$.

By physical requirement, $\phi_{eq}(\vec{r})$ and $\phi(\vec{r},t)$ must be positive or zero for all $\vec{r}\varepsilon R$ and all $t \geq 0$, so

$$||\phi^*||_\infty = ||\phi_{eq}(\vec{r})(1 - \xi) + \xi\phi(\vec{r},t)||_\infty,\ 0 \leq \xi \leq 1$$

$$< ||\phi_{eq}(\vec{r}) + \phi(\bar{r},t)||_\infty$$

or

$$||\phi^*||_\infty < ||\phi_{eq}(\vec{r})||_\infty + ||\phi(\vec{r},t)||_\infty .$$

From the condition of the theorem

$$||\phi(\vec{r},t)|| \leq ||\phi_o(\vec{r})|| ,$$

and

$$||J(\phi^*)|| < ||J(\phi_{eq}(\vec{r}) + \phi_o(\vec{r}))|| . \tag{3.57}$$

Now set

$$\psi(\vec{r},t) = \phi(\vec{r},t) - \phi_{eq}(\vec{r})$$

so the equation of motion becomes

$$\frac{\partial\psi}{\partial t}(\bar{r},t) = L(\bar{r})\psi(\bar{r},t) + g(\phi(\bar{r},t)) - g(\phi_{eq}(\bar{r})) \tag{3.58}$$

and this may formally be written as an integral equation for $\psi(\bar{r},t)$, i.e.,

$$\psi(\bar{r},t) = S(t)\psi_o(\bar{r}) + \int_0^t S(t-\tau)[g(\phi) - g(\phi_{eq})]\, d\tau.$$

The proof now proceeds in the same fashion as was used in theorem 4, so that one obtains

$$||\psi(\bar{r},t)|| < e^{\gamma t}\, ||\phi_o|| + \int_0^t e^{\gamma(t-\tau)}\, ||g(\phi(\bar{r},\tau)) - g(\phi_{eq}(\bar{r})||\, d\tau. \tag{3.59}$$

Using the result of lemma 2 and Eq. (3.57), we have

$$||\psi|| < e^{\gamma t}||\psi_o|| + ||J(\phi_{eq}(\bar{r}) + \phi_o(\bar{r})||\int_0^t d\tau\, e^{\gamma(t-\tau)}||\psi(\bar{r},\tau)|| \;. \tag{3.60}$$

Employing the Bellman-Gronwall inequality, we obtain (see appendix)

$$||\psi|| < ||\psi_o||\, e^{\left\{\gamma + ||J(\phi_{eq}(\bar{r}) + \phi_o(\bar{r}))||\right\}t} \;. \tag{3.61}$$

From this inequality, it is seen that Eq. (3.56) must hold for asymptotic stability.

There are several remarks that can be made concerning theorem 5. They are:

i) $||J(\phi_{eq}(\bar{r}) + \phi_o(\bar{r}))||$ will depend on the maximum values of $\phi_{eq}(\bar{r})$ and $\phi_o(\bar{r})$ for $\bar{r} \varepsilon R$. The exact solution to Eq. (3.37) need not be known and only an estimate of $||\phi_{eq}||_\infty$ is needed.

ii) The operator $\mathscr{L}(\bar{r})$ does not contain the equilibrium solution $\phi_{eq}(\bar{r})$ and consequently the eigenvalue problem

$$\mathscr{L}\psi_n(\bar{r}) = \gamma_n\psi_n \tag{3.62}$$

will be easier to solve than that which contains the equilibrium solution.

iii) Since the Jacobian matrix contains the derivatives of the nonlinearities, condition (3.56) gives us the domain of allowable perturbations in terms of the derivatives of the nonlinearities. Just as before, γ is on the boundary of the spectrum of $\mathscr{L}$, i.e.,

$$\mathrm{Re}\ \sigma(\mathscr{L}) < \gamma \ ; \qquad \gamma < 0 \ .$$

Condition (3.56) may be overly restrictive, but it does simplify the practical calculation.

Examples and Comparisons

An illustration of the semigroup method employing the generalized mean value theorem is that of xenon oscillations. Stacey[37] has obtained the condition for asymptotic stability using Liapunov's method to be:

$$\mu_o + 2C_1 \frac{\int_R \psi_1^2 \psi_3^2 d^3\overline{r}}{\int_R (\psi_1^2+\psi_2^2+\psi_3^2) d^3 r} < 0 \ ,$$

where ψ_1, ψ_2, and ψ_3 are the deviations of the neutron, iodine, and xenon distribution from their equilibrium values $N_{eq}(\overline{r})$, $I_{eq}(\overline{r})$, $X_{eq}(\overline{r})$, and $C_1 = v\sigma_x$. The μ_o is the largest eigenvalue (negative) of the self adjoint operator

$$\frac{1}{2}(L_1 + L_1^+)\ u_n = \mu_n u_n(\overline{r}) \tag{3.64}$$

where

$$L_1(\overline{r}) = \begin{bmatrix} a_1\nabla^2+b_1 - c_1 X_{eq}(r) & 0 & -C_1 N_{eq}(\overline{r}) \\ a_2 & -\lambda_I & 0 \\ -C_1 X_{eq}(\overline{r}) & \lambda_I & -(\lambda_x + C_1 N_{eq}(\overline{r})) \end{bmatrix} \tag{3.65}$$

Note that $\frac{1}{2}(L_1 + L_1^+)$ contains the equilibrium distributions of the neutrons and xenon atoms. Inequality (3.63) defines a domain of perturbation for which a stable response will be obtained.

The above problem can also be handled using the semigroup methods as developed above. Using the results of theorem 5, the operator $\mathscr{L}(\overline{r})$

as it appears in theorem 5 is

$$\mathscr{L} = \begin{bmatrix} a_1\nabla^2+b_1 & 0 & 0 \\ a_2 & -\lambda_I & 0 \\ 0 & \lambda_I & -\lambda_x \end{bmatrix} \tag{3.66}$$

and

$$J(\phi^*) = -C_1 \begin{bmatrix} X^* & 0 & N^* \\ 0 & 0 & 0 \\ X^* & 0 & N^* \end{bmatrix} \tag{3.67}$$

with $C_1 = v\sigma_x$. The N^* and X^* are the neutron and xenon concentration evaluated at some intermediate state between N_{eq}, X_{eq}, and $N(\bar{r},t)$, $X(\bar{r},t)$. The constant γ which appears in the stability condition (3.56) corresponds to the largest (negative) real part of the eigenvalue of Eq. (3.62), i.e.,

$$\gamma = \max_n \mathrm{Re}\{\gamma_n\} < 0 \ . \tag{3.68}$$

The condition for stability now is

$$\gamma + 2C_1 \max_{\bar{r}\varepsilon R} \left[(X_{eq}(\bar{r}) + X_o(\bar{r}))^2 + (N_{eq}(\bar{r}) + N_o(\bar{r}))^2\right]^{1/2} < 0 \tag{3.69}$$

which again defines a domain of allowable perturbations of the neutron and xenon distributions. Note that the result (3.69) is dependent on the equilibrium distribution and in fact guarantees global asymptotic stability. Therefore, we may expect this result to be more conservative than the one obtained by Liapunov's method. Numerical examples will be furnished in a later chapter.

IV. STABILITY ANALYSIS USING THE METHOD OF COMPARISON FUNCTIONS

In this chapter, stability criteria for the equilibrium states of multigroup diffusion approximation with temperature feedback are determined by the method of comparison functions. The essential tool in this analysis is the theorem of Westphal-Prodi,[38] and its extensions developed by W. Kastenberg.[39]

Comparison functions are used to approximate the solutions to the equations which will be considered, furthermore, the solutions will be bounded by these comparison functions. A bound on the maximum value of a transient may be obtained or one can specify the maximum value of a transient and obtain a bound on the initial conditions necessary not to exceed the prescribed bound.

Before proceeding further, it is convenient to introduce notation in the four dimensional space $(\bar{r},t)$ or (x_1, x_2, x_3, t)[38]. The spatial domain R is a bounded open region with Γ being its boundary.

$$\bar{R} = R \cup \Gamma , \tag{4.1}$$

$$\bar{r} \varepsilon \bar{R} . \tag{4.2}$$

Let $\bar{B}$ be the four dimensional domain with $\bar{r}\varepsilon\bar{R}$ and $0 \le t \le T$. E will denote the interior points of this hypercylinder, and B_T will denote the set of points $(\bar{r},T)$ where $\bar{r}\varepsilon R$ and B the remainder of the boundary of $\bar{B}$. Thus

$$\overline{B} = E\ U\ B_T\ U\ B \tag{4.3}$$

As in the previous chapters we are concerned with a dynamic system of the form

$$\frac{\partial\phi}{\partial t} = F(\overline{r},\phi); \quad \overline{r}\varepsilon R; \quad 0 \leq t \leq \infty \tag{4.4}$$

with boundary and initial conditions

$$\phi(\overline{r},t) = 0; \quad \overline{r}\varepsilon\Gamma \tag{4.5}$$

$$\phi(\overline{r},o) = \phi_o(\overline{r}). \tag{4.6}$$

The operator F in Eq. (4.4) is not quite as general as it was considered in Liapunov's and semigroup approach and furthermore stability analysis takes on a different form for a scalar equation as opposed to systems of equations. We shall first take Eq. (4.4) to be scalar equation with $F(\overline{r},\phi)$ restricted to the following operator

$$F(\overline{r},\phi) = a(\overline{r})\nabla^2\phi(\overline{r},t) + \sum_{i=1}^{3} b_i(\overline{r}) \frac{\partial}{\partial x_i} \phi(\overline{r},t)$$

$$+ C(\overline{r})\phi(\overline{r},t) + g(\overline{r},\phi) \tag{4.7}$$

where $g(\overline{r},\phi)$ represents the nonlinear term. There are limitations on $g(\overline{r},\phi)$ which will be brought out when needed. Two further restrictions on F are that

$$a(\bar{r})\xi^2(\bar{r}) \tag{4.8}$$

is positive definite for ξ $(\bar{r})$ real, and F is assumed to be continuous when viewed as a function of

$$\nabla^2\phi, \frac{\partial}{\partial x_i}\phi, \phi \ . \tag{4.9}$$

Under the above assumptions, we have the theorem 1.

Theorem 1: (Westphal-Prodi[38])

Let $u(\bar{r},t)$, $v(\bar{r},t)$ be functions continuous in $\bar{B}$ with partial derivatives $\frac{\partial u}{\partial t}$, $\frac{\partial}{\partial x_i}u$, $\nabla^2 u$, $\frac{\partial v}{\partial t}$, $\frac{\partial}{\partial x_i}v$, and $\nabla^2 v$ continuous in $E \cup B_T$. Moreover in $E \cup B_T$ let the following inequalities hold

$$\frac{\partial u}{\partial t} > F(\bar{r},u) \quad (\bar{r},t) \in E \cup B_T \ , \tag{4.10}$$

and

$$\frac{\partial v}{\partial t} \leq F(\bar{r},v) \quad (\bar{r},t) \in E \cup B_T \tag{4.11}$$

where F has the properties (4.7-4.9). Then, if

$$v(\bar{r},t) < u(\bar{r},t) \quad \text{on B, we have} \tag{4.12}$$

$$v(\bar{r},t) < u(\bar{r},t) \quad \text{in } \bar{B}. \tag{4.13}$$

The translation and a sketch of the proof of the theorem is given by R. Narasimhan.[38] Kastenberg[39] extended the theorem to include equality in relations (4.10), (4.12), and (4.13) with an additional restriction on F given by

$$F(\overline{r},u) - F(\overline{r},v) \leq M(u - v) \tag{4.14}$$

for $u \geq v \geq 0$, and M is a positive constant.

The objective of this extension was to ensure that if $u = v$ on B, v could not be greater than u in E. Relation (4.14) however limits us to the analysis of systems with negative definite nonlinearities. That is, $g(\overline{r},\phi) < 0$ for $\phi \neq 0$, and $g(\overline{r},\phi) = 0$ for $\phi = 0$.

For a bare homogenous reactor model described by the nonlinear diffusion equation, the only interesting stability assertions could be made about the trivial equilibrium solution to the nonlinear diffusion equation when the reactor is linearly stable (subcritical) or about the nontrivial equilibrium solution when the linearized model is supercritical. Since the use of theorem 1 and its extension is a constructive method for asserting stability conditions, i.e., it depends on the constructions of comparison functions, an example will be furnished.

The adiabatic model of a reactor core will be examined where the neutron distribution obeys the diffusion equation given by

$$\frac{1}{v}\frac{\partial\phi}{\partial t} = D\nabla^2\phi - \Sigma_a(T)\ \phi(\overline{r},t) + \nu\Sigma_f\ \phi(\overline{r},t) \tag{4.15}$$

with D, $\nu\Sigma_f$ constant, and absorption cross-section varying linearly with the temperature of the core, i.e.,

$$\Sigma_a(T) = \Sigma_a^\circ + \alpha_a(T - T_c) \tag{4.16}$$

where T_c is an average core temperature. Adiabatic means that all heat loss during an excursion is negligible so that the temperature distribution is given by

$$\rho c_p \frac{\partial T(\bar{r},t)}{\partial t} = \varepsilon \Sigma_f \; \phi(\bar{r},t) \tag{4.17}$$

where ρ, c_p, ε were defined in the paragraph below Eq. (1.3). The boundary and initial conditions are

$$\begin{aligned} \phi(r,t) &= T(\bar{r},t) = 0 \text{ for } \bar{r}\varepsilon\Gamma \;, \\ \phi(\bar{r},o) &= \phi_o(\bar{r}) \,, \\ T(\bar{r},o) &= T_o(\bar{r}) \,. \end{aligned} \tag{4.18}$$

Letting

$$\psi(\bar{r},t) = T(\bar{r},t) - T_c(\bar{r}) \tag{4.19}$$

we have

$$\frac{\partial \psi}{\partial t} = \frac{\varepsilon \Sigma_f}{\rho c_p} \phi \tag{4.20}$$

and integrating (4.20)

$$\psi(\bar{r},t) = \psi_o(\bar{r}) + \frac{\varepsilon \Sigma_f}{\rho c_p} \int_0^t \phi(\bar{r},t')\, dt' \tag{4.21}$$

where

$$\psi_o(\bar{r}) = T(\bar{r},o) - T_c(\bar{r}) \;.$$

Substitution of (4.16) and (4.21) into (4.15) yields

$$\frac{\partial \phi}{\partial t} = vD\nabla^2\phi + k_o^2\,\phi - v\alpha_a\psi_o(\bar{r})\phi - c_1\phi \int_0^t \phi(\bar{r},t')\,dt' , \tag{4.22}$$

where

$$k_o^2 = v(\nu\Sigma_f - \Sigma_a^\circ) > 0 \; ; \qquad k_o^2 < +\infty$$

$$c_1 = \frac{v\alpha_a\varepsilon\Sigma_f}{\rho c_p} > 0 \; ; \qquad \alpha_a > 0 \, . \tag{4.23}$$

Equation (4.22) may be written in the form of Eq. (4.7), with

$$a(\bar{r}) = vD$$

$$C(r) = k_o^2 - v\alpha_a\psi_o(\bar{r}), \text{ and}$$

$$g(\bar{r},\phi) = -c_1\phi \int_0^t \phi(\bar{r},t')\,dt' \, .$$

Proposition 1

If for the system described by (4.22) and (4.23) there exists a number $K > 0$ such that

$$\max_{t>0} ||\phi(\bar{r},t)||_\infty \leq K \, . \tag{4.24}$$

Then there exists a function $f(\bar{r}) \geq 0$, $\bar{r}\varepsilon\bar{R}$ such that for

$$0 \leq \phi(\bar{r},o) \leq f(\bar{r}); \quad \bar{r}\varepsilon\bar{R}$$

$$\lim_{t\to\infty} |\phi(\bar{r},t)| = 0 \tag{4.25}$$

i.e., $\phi = 0$ is the asymptotically stable equilibrium.

Proof:

Consider the comparison function

$$u(\bar{r},t) = A\, e^{-\gamma t}\, \xi_o(\bar{r}) \tag{4.26}$$

where A and γ are positive constants to be determined, and $\xi_o(\bar{r})$ is the fundamental eigenfunction to

$$vD\nabla^2\xi_n(\bar{r}) = \lambda_n\xi_n(\bar{r}) \;; \qquad \xi_n(\bar{r}) = 0 \qquad \bar{r}\varepsilon\Gamma \tag{4.27}$$

with the corresponding largest eigenvalue λ_o, and

$$||\xi_o(\bar{r})||_\infty = 1 .$$

The comparison function has the following properties:

(a) $u(\bar{r},o) = A\, \xi_o(\bar{r}) \qquad \bar{r}\varepsilon\bar{R}$

(b) $u(\bar{r},t) = 0 \qquad \bar{r}\varepsilon\Gamma \qquad t \geq 0$

(c) $\lim\limits_{t\to\infty} u(\bar{r},t) = 0 \qquad \bar{r}\varepsilon\bar{R}$

(d) $\max\limits_{t\geq 0} ||u(\bar{r},t)||_\infty = A$.

Now

$$\frac{\partial u}{\partial t} - F(\bar{r},u) = -\gamma\, Ae^{-\gamma t}\xi_o(\bar{r}) - vDe^{-\gamma t}\nabla^2\xi_o - k_o^2 u$$

$$+ v\alpha_a\psi_o u + c_1A^2\xi_o e^{-\gamma t}\int_0^t e^{-\gamma t'}\, dt' . \tag{4.28}$$

Using (4.27) and integrating

$$\frac{\partial u}{\partial t} - F(\bar{r},u) = u\left(-\gamma - \lambda_o - k_o^2 + v\alpha_a\psi_o + c_1 A\xi_o(\bar{r})/\gamma \right.$$

$$\left. * (1 - e^{-\gamma t})\right), \tag{4.29}$$

since $v\alpha_a\psi_o(\bar{r}) > 0$, $\bar{r}\varepsilon R$, choose $\gamma > 0$ such that for an $\bar{r}\varepsilon R$

$$\gamma \leq v\alpha_a\psi_o(\bar{r}) , \tag{4.30}$$

Furthermore, we have

$$c_1 A\xi_o(\bar{r}) \; (1 - e^{-\gamma t})/\gamma > 0 \text{ in } E \cup B_T \tag{4.31}$$

and we may always choose A large enough so that

$$c_1 A\xi_o(\bar{r}) \; (1 - e^{-\gamma t})/\gamma - \lambda_o \geq k_o^2 \; . \tag{4.32}$$

A will always be finite since $k_o^2 < +\infty$. Finally, we may conclude that

$$\frac{\partial u}{\partial t} - F(\bar{r},u) \geq 0 \text{ in } E \cup B_T. \tag{4.33}$$

Similarly consider the comparison function

$$v(\bar{r},t) = 0 \text{ in } \bar{B} \; . \tag{4.34}$$

$$\frac{\partial v}{\partial t} - F(\bar{r},v) = 0 \text{ in } E \cup B_T \tag{4.35}$$

Employing property (b) of u, i.e., the fact that $\phi(\bar{r},t) = 0$, $\bar{r}\varepsilon\Gamma$, $t > 0$ yields

$$v(\bar{r},t) = \phi(\bar{r},t) = u(\bar{r},t) = 0; \ \bar{r}\varepsilon\Gamma, \ t > 0 \tag{4.36}$$

and from property (a) of $u(\bar{r},t)$ it follows that for

$$0 \leq \phi(\bar{r},o) \leqslant A\xi_o(\bar{r}) \tag{4.37}$$

we have

$$v(\bar{r},t) \leq \phi(\bar{r},t) \leq u(\bar{r},t) \text{ on } B . \tag{4.38}$$

Since F satisfies relation (4.33) and since

$$\begin{aligned} F(\bar{r},u) - F(\bar{r},v) &= vD\nabla^2 u + k_o^2 u - v\alpha_a\psi_o(\bar{r})\, u - c_1 u \int_0^t u \, dt' \\ &= k_o^2 u + vD\lambda_o u - v\alpha_a\psi_o(\bar{r})u - c_1 u \int_0^t u \, dt' \\ &\leq k_o^2 u , \end{aligned} \tag{4.39}$$

i.e., F satisfies the condition (4.14), we may use the extension to Westphal-Prodi theorem and conclude that $\phi(\bar{r},t)$ is bounded from above by u and from below by v for all time in the reactor and its surface, i.e.,

$$v(\overline{r},t) \le \phi(\overline{r},t) \le u(\overline{r},t) \text{ in } \overline{B} \ . \tag{4.40}$$

Since $u(\overline{r},t)$ is a monotomically decreasing function of time, choose A such that $A \geq K$, and so that it satisfies relation (4.32). Then by property (d) of $u(\overline{r},t)$, the maximum value of u will be greater than or equal to K. Then set

$$f(\overline{r}) = A\xi_o(\overline{r}); \quad \overline{r}\varepsilon R \tag{4.41}$$

to be the bound on the initial condition necessary to provide that $\phi(\overline{r},t)$ does not exceed K. Then by property (c) of $u(\overline{r},t)$, it follows that

$$\lim_{t\to\infty} |\phi(\overline{r},t)| = 0; \quad \overline{r}\varepsilon\overline{R} \tag{4.42}$$

and from (4.37)

$$||\phi(\overline{r},o)||_\infty \le A \ . \tag{4.43}$$

From condition (4.24) of proposition 1

$$\max_{t>0} ||\phi(\overline{r},t)|| \le K \tag{4.44}$$

and proposition 1 is proven.

Proposition 2:

If for the system given by (4.22) and (4.23) there exists a function $f(\overline{r}) \geq 0$ $\overline{r\varepsilon R}$ such that

$$0 \leq \phi(\bar{r},o) \leq f(\bar{r}); \quad \bar{r}\varepsilon\bar{R}$$

then there exists a number K > 0 such that for

$$\max_{t>0} ||\phi(\bar{r},t)||_{\infty} \leq K ,$$

$$\lim_{t\to\infty} |\phi(\bar{r},t)| = 0 .$$

Proof:

A portion of the proof will be identical up to and including relation (4.35) of proposition 1. Choose A such that for a given $f(\bar{r})$

$$A\xi_o(\bar{r}) \geq f(\bar{r}) \text{ for } \bar{r}\varepsilon\bar{R}.$$

Employing property (b) of u

$$v(\bar{r},t) = \phi(\bar{r},t) = u(\bar{r},t) = 0 \text{ for } \bar{r}\varepsilon\Gamma, \ t > 0,$$

and from property (a) of u, it follows that for

$$0 \leq \phi(\bar{r},o) \leq A\xi_o(\bar{r}),$$

and since F satisfies relations (4.33) and (4.14), we have

$$v \leq \phi \leq u \text{ in } \bar{B}.$$

Since u is a monotomically decreasing function of time, its maximum value is A. Choose $K \leq A$, then

$$\max_{t>0} ||\phi(\bar{r},t)||_{\infty} \leq K$$

and from (c) of $u(\bar{r},t)$ we have

$$\lim_{t\to\infty} |\phi(\bar{r},t)| = 0 .$$

The adiabatic model assumes that no energy is extracted from the core by either conduction or coolant, and it is applicable to certain kinds of experimental reactors. For other reactors one may assume failure of energy extraction mechanism, and the subsequent time behavior is given by Eqs. (4.15) and (4.17). Also a situation where we have a reactor at zero flux and zero temperature at at time $t = 0$, a pulse of neutrons is introduced. In all instances, because of negative temperature feedback (i.e., as temperature increases, absorption cross-section increases), the reactor will shut itself down, and the only equilibrium solution is the zero solution.

The above example is a peculiar one in a sense that we were able to put a system of two equations into one equation, which generally cannot be done. Another restriction is that the nonlinearity must be negative definite.

Contrary to the semigroup and Liapunov's approaches, we shall not be looking at a system of equations in their aggregate form, that is, matrix form, but rather in the form given by

$$\frac{\partial \phi_n(\bar{r},t)}{\partial t} = F_n(\bar{r}, \phi_1, \ldots, \phi_n, \ldots, \phi_N); \quad n = 1, 2, \ldots, N \; . \tag{4.45}$$

To abbreviate the notation, we will write

$$F_n(\bar{r}, \phi_1, \ldots, \phi_n, \ldots, \phi_N) = F_n(\bar{r},\phi) \tag{4.46}$$

The operators $F_n(\bar{r},\phi)$ are given by

$$F_n(\bar{r},\phi) = \mathscr{L}_n(\bar{r})\phi_n(\bar{r},t) + \sum_{m\neq n}^{N} a_{nm}(\bar{r}) \; \phi_m(\bar{r},t) + g_n(\bar{r},\phi) \tag{4.47}$$

with

$$\mathscr{L}_n(\bar{r})\phi_n(\bar{r},t) = a_n(\bar{r})\nabla^2\phi_n(\bar{r},t) + \sum_{i=1}^{3} b_{ni}(\bar{r}) \frac{\partial}{\partial x_i} \phi_n(\bar{r},t)$$

$$+ C_n(\bar{r}) \; \phi_n(\bar{r},t) \; . \tag{4.48}$$

The coefficients $a_n(\bar{r})$, $b_{ni}(\bar{r})$, $C_n(\bar{r})$, and $a_{nm}(\bar{r})$ are assumed to be continuous functions for $\bar{r}\varepsilon R$. In the multigroup formulation of reactor kinetics, the entities $a_{nm}(\bar{r})$ will represent the linear coupling coefficients between equations. For example, they may be the group transfer cross-sections and coefficients of neutron fluxes appearing in the energy balance equations in the core. The nonlinearities are expressed in the last term $g_n(\bar{r},\phi)$.

We will say that $g_n(\bar{r},\phi)$ is bounded if there exists some positive finite constant K_n such that

$$|g_n(\bar{r},\phi)| \leq K_n; \quad (\bar{r},t) \in E \cup B_T \tag{4.49}$$

for $\phi_n(\bar{r},t) < \varepsilon_n$, n = 1, 2, . . ., N, i.e., when ϕ_n are bounded, and $g_n(\bar{r},\phi)$ is positive definite if

$$g_n(\bar{r},\phi) > 0 \quad (\bar{r},t) \in E \cup B_T \tag{4.50}$$

and a similar definition for negative definite $g_n(\bar{r},\phi)$. The extension of Westphal-Prodi theorem, the system given by Eqs. (4.45-4.48) is given by Mlak[40] with the restriction

$$F_n(\bar{r},u) \leq F_n(\bar{r},v) \ ; \ n = 1, 2, \ldots, N \tag{4.51}$$

for $u_n = v_n$, $u_i \leq v_i$, $i \neq n$. Physically this restricts us to the analysis of positive definite nonlinearities and the only interesting application would be a reactor that is linearly subcritical. This limitation is circumvented by applying a theorem due to Szarski,[41] where it is suggested that a comparison system $G_n(\bar{r},u)$ is to be employed.

Theorem 3:[28,41]

Let $u_n(\bar{r},t)$ and $v_n(\bar{r},t)$; n = 1, 2, . . ., N, be continuous functions in $\bar{B}$ with partial derivatives continuous. Let

$$\frac{\partial v_n}{\partial t} - G_n(\bar{r},v) > \frac{\partial u_n}{\partial t} - F_n(\bar{r},u); (\bar{r},t) \in E \cup B_T \tag{4.52}$$

with

$$G_n(\bar{r},x) \geq F_n(\bar{r},y) \tag{4.53}$$

for $x_n = y_n$ and $x_i \geq y_i$; $i \neq n$. Then if $v_n > u_n$ on B, $v_n > u_n$ in $\bar{B}$. The translation of that proof is given by Kastenberg.[28] While formulating Eq. (4.48) about, the equilibrium state $\phi_{eq} = (\phi_{1_{eq}}, \ldots, \phi_{n_{eq}})$, the coefficients of the operator $\mathscr{L}_n(\bar{r})$ as well as the coupling coefficients $a_{nm}(\bar{r})$ will become functions of the equilibrium state, and we may always phrase stability theorems concerning the stability of trivial equilibrium by letting,

$$\psi_n = \phi_n(\bar{r},t) - \phi_{n_{eq}}(\bar{r}) \ . \tag{4.54}$$

<u>Theorem 4:</u>[28]

If λ_n, the largest eigenvalues of

$$\mathscr{L}_n \, \xi_n^{(i)}(\bar{r}) = \lambda_n^{(i)} \, \xi_n^{(i)}(\bar{r}) \tag{4.55}$$

$n = 1, 2, \ldots, N$ are negative and if there exist positive numbers η_n such that the initial conditions $\psi_n(\bar{r},o)$ satisfy

$$||\psi_n(\bar{r},o)||_\infty \leq \eta_n; \qquad \bar{r}\varepsilon\bar{R} \tag{4.56}$$

and $g_n(r,\psi)$ are bounded. Then there exist constants $\bar{K}_n$ and A_n dependent on η_n such that a sufficient condition for the trivial equilibrium state to be asymptotically stable is for

$$\lambda_n - \overline{K}_n - \sum_{m\neq n}^{N} ||a_{nm}||_\infty \; ||\xi_n^*/\xi_n^*||_\infty \; A_m/A_n > 0 \tag{4.57}$$

where the functions $\xi_m^*(\overline{r})$, are the fundamental eigenfunctions of Eq. (4.55) for a slightly enlarged region R_ε such that $\overline{R}$ is contained in R_ε. (We drop the superscript (i) on ξ_n and subscript ∞ on norm because we will always refer to the fundamental eigenfunctions and L_∞ norm throughout this chapter.) Since the proof of the theorem is constructive, that is, the constants $\overline{K}_n$ and A_n are obtained in the course of the proof, the detailed proof following that of Kastenberg[28] will be given.

Proof:

Consider the comparison functions

$$v_n(\overline{r},t) = A_n \; \xi_n^*(\overline{r}) \; e^{-\varepsilon_n t} \tag{4.58}$$

where $\xi_n^*(\overline{r}) > 0 \quad \overline{r}\varepsilon\overline{R} \subset R_\varepsilon$; $\xi_n^*(\overline{r}) = 0$; $\overline{r}\varepsilon\Gamma_\varepsilon$ is an eigenfunction corresponding to the largest (negative) eigenvalue of Eq. (4.55) for a slightly enlarged region R_ε. The corresponding eigenvalue is $(\lambda_n + \varepsilon_n)$, $\varepsilon_n > 0$. A_n and ε_n are constants to be determined. In the region R under consideration, $v_n(\overline{r},t)$ has the following properties

(a) $v_n(\overline{r},t) > 0 \quad \overline{r}\varepsilon\overline{R} \quad t \geq 0$

(b) $\sup_{t\geq 0} ||v_n(\overline{r},t)|| = A_n; \quad (\overline{r},t) \;\varepsilon\; \overline{B}$

(c) $v_n(\overline{r},o) = A_n \; \xi_n^*(\overline{r}) \quad \overline{r}\varepsilon\overline{R}$

(d) $\lim_{t\to\infty} v_n(\bar{r},t) = 0; \quad \bar{r}\varepsilon\bar{R}.$

Since $\xi_n^*(\bar{r})$ is continuous and positive in $\bar{R}$ and since the initial conditions satisfy (4.56), we may choose the constants A_n such that

$$|\psi_n(\bar{r},o)| < A_n\,\xi_n^*(\bar{r}) \qquad n = 1, 2, \ldots, N \tag{4.59}$$

and A_n depend on η_n, and since $g_n(\bar{r},\psi)$ is bounded, there exists some positive definite functional $g_{n+}(\bar{r},\psi)$ such that

$$|g_n(\bar{r},\psi)| \le g_{n+}(\bar{r},\psi) < K_n . \tag{4.60}$$

In order to use theorem 3, consider the comparison system given by

$$\frac{\partial v_n}{\partial t} - F_{n+}(\bar{r},v) = \frac{\partial v_n}{\partial t} - \mathscr{L}_n v_n - \sum_{m\neq n}^{N} ||a_{nm}(\bar{r})||\, v_m$$

$$- g_{n+}(\bar{r},v) \quad \text{in } E \cup B_T , \tag{4.61}$$

since $-|\lambda_n| = \lambda_n$. Also

$$\frac{1}{v_n}\left(\frac{\partial v_n}{\partial t} - F_{n+}(\bar{r},v)\right) = |\lambda_n| - 2\varepsilon_n - \sum_{m\neq n} ||a_{nm}||\,\frac{A_m}{A_n}\,\frac{\xi_m^*}{\xi_n^*}\,e^{-(\varepsilon_m-\varepsilon_n)t}$$

$$-\frac{g_{n+}(\bar{r},v)}{v_n(r,t)} .$$

Choosing all ε_n to be equal to the largest one which is equal to ε,

$$\frac{1}{v_n}\left(\frac{\partial v_n}{\partial t} - F_{n+}(\bar{r},v)\right) \geq |\lambda_n| - 2\varepsilon - \sum_{m \neq n} ||a_{nm}(\bar{r})|| \frac{A_m}{A_n} \frac{\xi^*_m(\bar{r})}{\xi^*_n(r)}$$

$$- \frac{g_{n+}(\bar{r},v)}{v_n(r,t)}$$

and since $g_{n+}(\bar{r},v) \leq K_n$ from (4.49)

$$\frac{g_{n+}(\bar{r},v)}{v_n} \leq \frac{K_n}{v_n} \equiv \bar{K}_n \tag{4.62}$$

and since

$\frac{\xi^*_m(\bar{r})}{\xi^*_n(r)} \leq ||\frac{\xi^*_m(\bar{r})}{\xi^*_n(r)}||$, we have

$$\frac{1}{v_n}\left(\frac{\partial v_n}{\partial t} - F_{n+}(\bar{r})\right) < |\lambda_n| - 2\varepsilon - \sum_{m \neq n} ||a_{mn}|| \, ||\frac{A_m}{A_n}|| \, ||\frac{\xi^*_m}{\xi^*_n}|| - \bar{K}_n . \tag{4.63}$$

If we choose

$$\theta_n > \varepsilon > 0 \tag{4.64}$$

where

$$\theta_n = |\lambda_n| - \bar{K}_n - \sum_{m \neq n} ||a_{nm}|| \, ||\frac{A_m}{A_n}|| \, ||\frac{\xi^*_m}{\xi^*_n}|| > 0 \tag{4.65}$$

we will have

$$\frac{1}{v_n}\left(\frac{\partial v_n}{\partial t} - F_{n+}(\bar{r},\bar{v})\right) > 0 . \tag{4.66}$$

Using property (a) of $v_n(\overline{r},t)$, we obtain

$$\frac{\partial v_n}{\partial t} - F_{n+}(\overline{r},\overline{v}) > 0 \quad \text{in } E \cup B_T \tag{4.67}$$

and since $\psi_n(\overline{r},t)$ is a solution to

$$\frac{\partial \psi_n}{\partial t} - F_n(\overline{r},\psi) = 0 ,$$

and

$$\begin{aligned} F_{n+}(\overline{r},z) - F_n(\overline{r},y) &= \mathscr{L}_n(\overline{r})(z_n - y_n) \\ &+ \sum_{m \neq n} ||a_{nm}||\ z_m - \sum_{m \neq n} a_{nm}\ y_m \\ &+ g_{n+}(\overline{r},z) - g_n(\overline{r},y) \\ &\geq \mathscr{L}_n(\overline{r})(z_n - y_n) + \sum_{m \neq n} ||a_{nm}||(z_m - y_m) \\ &+ g_{n+}(\overline{r},z) - g_n(\overline{r},y) \\ &\geq 0 , \end{aligned}$$

we have

$$F_{n+}(\overline{r},z) \geq F_n(\overline{r},y) \text{ for } z_n = y_n,\ z_m \geq y_m \ . \tag{4.68}$$

Also because of property (c) of $v(\overline{r},t)$,

$$\psi_n(\bar{r},o) < v_n(\bar{r},o) \quad \bar{r}\varepsilon\bar{R} \tag{4.69}$$

and since $\psi_n(\bar{r},t) = 0 \quad \bar{r}\varepsilon\Gamma \quad t > 0$, and property (a) of $v_n(\bar{r},t)$, we have

$$\psi_n(\bar{r},t) < v_n(\bar{r},t) \quad \bar{r}\varepsilon\Gamma \quad t > 0, \text{ or } (\bar{r},t)\ \varepsilon\ B\ . \tag{4.70}$$

Using theorem 3, we conclude that

$$\psi_n(\bar{r},t) < v_n(\bar{r},t); \quad (\bar{r},t)\ \varepsilon\ \bar{B}\ . \qquad n = 1, 2, \ldots, N \tag{4.71}$$

Similarly, we take the comparison functions

$$u_n(\bar{r},t) = - v_n(\bar{r},t) \qquad n = 1, 2, \ldots, N \tag{4.72}$$

which obey the following properties:

(a) $u_n(\bar{r},t) < 0 \ ;\ \bar{r}\varepsilon\bar{R},\ t \geq 0$

(b) $\min\limits_{(\bar{r},t)\varepsilon\bar{B}} u_n(\bar{r},t) = - A_n \ ;\ \bar{r}\varepsilon\bar{R},\ t \geq 0$

(c) $u_n(\bar{r},o) = - A_n\ \xi_n^*(\bar{r}) \ ;\ \bar{r}\varepsilon\bar{R}$

(d) $\lim\limits_{t\to\infty} u_n(\bar{r},o) = 0 \ ;\ \bar{r}\varepsilon\bar{R}$.

Employing the comparison system with a negative nonlinearity $-g_{n+}(\bar{r},u)$

$$\frac{\partial u_n}{\partial t} - F_{n-}(\bar{r},u) = \frac{\partial u_n}{\partial t} - \mathscr{L}_n u_n + \sum_{m \neq n} ||a_{nm}|| \; u_m + g_{n+}(\bar{r},u) \qquad (4.73)$$

we obtain

$$\frac{\partial u_n}{\partial t} - F_{n-}(\bar{r},u) < 0 \quad \text{in } E \cup B_T .$$

By similar arguments leading to (4.71), we obtain

$$\psi_n(\bar{r},t) > u_n(\bar{r},t); \; (\bar{r},t) \in \bar{B} . \qquad (4.74)$$

Thence the solution is bounded from below and above

$$u_n(\bar{r},t) < \psi_n(\bar{r},t) < v_n(\bar{r},t) \; ; \quad n = 1, 2, \ldots, N \qquad (4.75)$$

and from properties (d) of v_n and u_n, it follows that

$$\lim_{t\to\infty} |\psi_n(\bar{r},t)| = \lim_{t\to\infty} |\phi_n(\bar{r},t) - \phi_{n_{eq}}(\bar{r})| = 0 \; ; n = 1, 2, \ldots, N$$

and the equilibrium state $\phi_{eq} = (\phi_{1_{eq}} \; . \; . \; . \; \phi_{N_{eq}})$ is asymptotically stable.

The stability criterion (4.65) was applied to[28] a two group diffusion model with temperature feedback as it was developed in the chapter on Liapunov's method (Eqs. 2.27-2.29). The contributions to the stability criterion are the following: (a) determination of the eigenvalues λ_n of the linearized decoupled problem, (b) choice of the bounds on the initial conditions using relation (4.59), and (c) choosing bounds on the nonlinear term using relation (4.49) and making sure that these choices satisfy condition (4.65).

For the multigroup dynamic models it is possible to obtain conditions for asymptotic stability in terms of derivatives of the system nonlinearities. As it will become apparent, this approach will lead to slightly different interpretation of results and will result in computational simplification. Proceeding in a similar manner as in the semigroup approach, use is made of the mean value theorem for a function of several variables. We shall start by considering Eqs. (4.45) through (4.48) with bounded nonlinearity and nontrivial equilibrium state $\phi_{eq}(\bar{r}) = (\phi_{1_{eq}}(\bar{r}) \ . \ . \ . \ \phi_{N_{eq}}(\bar{r}))$ being the solution to

$$\mathscr{L}_n \ \phi_{n_{eq}}(\bar{r}) + \sum_{m \neq n}^{N} a_{nm}(\bar{r}) \ \phi_{m_{eq}}(\bar{r}) + g_n(\bar{r}, \ \phi_{eq}) = 0 \tag{4.76}$$

$$n = 1, 2, \ . \ . \ ., N.$$

The equations of motion about the equilibrium state are

$$\frac{\partial \psi_n}{\partial t} = \mathscr{L}_n \ \psi_n(\bar{r},t) + \sum_{m \neq n}^{N} a_{nm} \ \psi_m(\bar{r},t) + g_n(\bar{r},\phi) - g_n(\bar{r},\phi_{eq}) \ . \tag{4.77}$$

Using the mean value theorem for a function of several variables

$$\begin{aligned} g_n(\bar{r},\phi) - g_n(\bar{r},\phi_{eq}) &= \sum_{k=1}^{N} \left(\frac{\partial g_n}{\partial \phi_k}\right)_{\phi^*} (\phi_k - \phi_{k_{eq}}) \\ &= \sum_{k=1}^{N} J_{nk}(\phi^*) \ \psi_K(\bar{r},t) \end{aligned} \tag{4.78}$$

where $\phi^*(\bar{r},t)$ is on a line between $\phi_{eq}(\bar{r})$ and $\phi(\bar{r},t)$. We are assuming in (4.78) that $g_n(\bar{r},\phi)$ are continuous and that $\left(\frac{\partial g_n}{\partial \phi_k}\right)$ are continuous.

The equations of motion become

$$\frac{\partial \psi_n}{\partial t} = \mathscr{L}_n \psi_n(\bar{r},t) + \sum_{m\neq n}^{N} a_{nm}(\bar{r})\ \psi_m(\bar{r}) + \sum_{m=1}^{N} J_{nm}\ \psi_m(\bar{r}) \tag{4.79}$$

They appear to be linear, however, they are not because the derivatives J_{nm} are functions of ϕ_n^* which in turn depend on ψ_n. In order to obtain stability results, employ comparison functions

$$v_n = A_n\ \xi_n^*(\bar{r})\ e^{-\varepsilon_n t} \tag{4.80}$$

where $\xi_n^*(\bar{r})$ are the fundamental eigenfunctions of $\mathscr{L}_n(\bar{r})$ in a slightly enlarged region, i.e.,

$$\mathscr{L}_n\ \xi_n^*(\bar{r}) = (\lambda_n + \varepsilon_n)\ \xi_n^*(\bar{r})\ . \tag{4.81}$$

Consider comparison system

$$\begin{aligned}
\frac{\partial v_n(\bar{r},t)}{\partial t} - F_{n+}(\bar{r},v) &= \frac{\partial v_n}{\partial t} - \mathscr{L}_n v_n - \sum_{m\neq n} ||a_{nm}(\bar{r})||_\infty\ v_m \\
&\quad - J_{nn}\ v_n - \sum_{m\neq n} ||J_{nm}||\ v_m \\
&= v_n(\bar{r},t)\Bigg\{ - (\lambda_n + 2\varepsilon_n) - \sum_{m\neq n} ||a_{nm}(\bar{r})||\ \frac{v_m}{v_n} \\
&\quad - J_{nn} - \sum_{m\neq n} ||J_{nm}||\ \frac{v_m}{v_n} \Bigg\}.
\end{aligned} \tag{4.82}$$

$$\frac{1}{v_n}\left(\frac{\partial v_n}{\partial t} - F_{n+}(\bar{r},v)\right) > -(\lambda_n + 2\varepsilon) - \sum ||a_{nm}(\bar{r})|| \frac{A_m}{A_n} ||\frac{\xi^*_m}{\xi^*_n}||$$

$$- J_{nn} - \sum ||J_{nm}|| \frac{A_m}{A_n}|| \frac{\xi^*_m}{\xi^*_n}|| \tag{4.83}$$

where $\varepsilon > \varepsilon_n$; $n = 1, 2, \ldots, N$) and all ε_n's are chosen to be equal. Since $\lambda_n = -|\lambda_n|$, and choosing

$$0 < \varepsilon < \theta_n \text{ with} \tag{4.84}$$

$$\theta_n = |\lambda_n| - \sum_{n\neq m} ||a_{nm}|| \frac{A_m}{A_n}|| \frac{\xi^*_m}{\xi^*_n} || - J_{nn} - \sum_{n\neq m} ||J_{nm}|| \frac{A_m}{A_n}|| \frac{\xi^*_m}{\xi^*_n}|| > 0$$

we have (4.85)

$$\frac{\partial v_n}{\partial t} - F_{n+} \geq 0 . \tag{4.86}$$

Since $v_n(\bar{r},t)$ is the same function as in theorem 4, using property (c) it follows that

$$\psi_n(\bar{r},o) < v_n(\bar{r},o); \quad \bar{r}\varepsilon\bar{R} \tag{4.87}$$

and since $\psi_n(\bar{r},t) = 0 \quad \bar{r}\varepsilon\Gamma$; $t > 0$, and $v_n(\bar{r},t) > 0$ for $\bar{r}\varepsilon R$, $t > 0$, then by property (a)

$$\psi_n(\bar{r},t) < v_n(\bar{r},t) \text{ on } B . \tag{4.88}$$

$$F_{n+}(\overline{r},x) - F_n(\overline{r},y) = \mathscr{L}_n(x_n - y_n) + \sum_{m\neq n}^{N} ||a_{nm}||\ x_m$$

$$- \sum a_{nm}\ y_m + J_{nn}(x_n - y_n)$$

$$+ \sum_{m\neq n} ||J_{nm}||\ x_m - \sum_{m\neq n} J_{nm}\ y_m$$

$$\geq \mathscr{L}_n(x_n - y_n) + J_{nn}(x_n - y_n) + \sum_{m\neq n} (||a_{nm}|| + ||J_{nm}||)\ (x_m - y_m)$$

$$\geq 0 \qquad \text{for } x_n = y_n; \quad x_m \geq y_m; \quad m\neq n\ . \tag{4.89}$$

Using theorem 3, we conclude that

$$\psi_n(\overline{r},t) < v_n(\overline{r},t) \text{ in } \overline{B} \tag{4.90}$$

and similarly considering comparison function

$$u_n(\overline{r},t) = -\ v_n(\overline{r},t) \tag{4.91}$$

and the comparison system

$$\frac{\partial u_m}{\partial t} - F_{n-}(\overline{r},y) = \frac{\partial u_n}{\partial t} - \mathscr{L}_n\ u_n + \sum_{m\neq n} ||a_{nm}||\ u_m$$

$$- J_{nn}\ u_n + \sum_{m\neq n} ||J_{nm}||\ u_m \tag{4.92}$$

we obtain

$$\frac{\partial u_n}{\partial t} - F_{n-}(\bar{r},u) < 0 \tag{4.93}$$

and finally

$$u_n(\bar{r},t) < \psi_n(\bar{r},t) < v_n(\bar{r},t) \ . \tag{4.94}$$

Using property (d), we conclude that

$$\lim_{t\to\infty} |\psi_n(\bar{r},t)| = \lim_{t\to\infty} |\phi_n(\bar{r},t) - \phi_{n_{eq}}(\bar{r})| = 0$$

i.e., $\phi_{eq}(\bar{r})$ is asymptotically stable equilibrium state if (4.84) holds. The stability criterion (4.84) may be used in different ways. One can choose the initial conditions $A_1, \ldots, A_N$, and test the bounds on derivatives or one may choose the domain of perturbations and solve for the domain of allowable initial conditions.

As an illustration of this method, we shall reconsider the two group equations with temperature feedback. In the nomenclature of this approach, the equations are given by

$$\frac{\partial\psi_1}{\partial t} = v_1 D_1 \nabla^2\psi_1 - v_1\Sigma_1\psi_1 + v_1\nu\Sigma_f^\circ\psi_2 + \sum_{i=1}^{3} J_{1i}\psi_i \tag{4.95}$$

$$\frac{\partial\psi_2}{\partial t} = v_2 D_2 \nabla^2\psi_2 - v_2\Sigma_2^\circ\psi_2 + v_2 p^\circ\Sigma_1\psi_1 + \sum_{i=1}^{3} J_{2i}\psi_i \tag{4.96}$$

$$\frac{\partial\psi_3}{\partial t} = \frac{k}{\rho c_p}\nabla^2\psi_3 - \frac{h}{\rho c_p}\psi_3 + \frac{\varepsilon}{\rho c_p}\Sigma_f^\circ\psi_2 + \sum_{i=1}^{3} J_{3i}\psi_i \ . \tag{4.97}$$

The linear decoupled operators $\mathscr{L}_n$ are

$$\mathscr{L}_1 = v_1 D_1 \nabla^2 - v_1 \Sigma_1$$

$$\mathscr{L}_2 = v_2 D_2 \nabla^2 - v_2 \Sigma_2^\circ \qquad (4.98)$$

$$\mathscr{L}_3 = \frac{k}{\rho c_p} \nabla^2 - \frac{h}{\rho c_p}$$

Nonvanishing coupling coefficients and the derivatives of nonlinearities are listed below:

$$a_{12} = v_1 \ \nu\Sigma_f^\circ$$

$$a_{21} = v_1 p^\circ \Sigma_1$$

$$a_{32} = \frac{\varepsilon}{\rho c_p} \Sigma_f^\circ$$

$$(4.99)$$

$$J_{12} = v_1 \ \nu\alpha_f T \qquad J_{22} = -v_2 \alpha_2 T$$

$$J_{13} = v_1 \ \nu\alpha_f \phi_2 \qquad J_{23} = -v_3 \alpha_2 \phi_2 + v_2 \alpha_p \phi_1$$

$$J_{21} = v_2 \alpha_p \Sigma_1 T \qquad J_{32} = \frac{\varepsilon\alpha_f}{\rho c_p} T$$

$$J_{33} = \frac{\varepsilon\alpha_f}{\rho c_p} \phi_2$$

Applying stability condition (4.84), we obtain

$$|\lambda_1| > v_1 \left\{ \left(||\nu\Sigma_f^\circ|| + ||\nu\alpha_f T|| \right) \frac{A_2}{A_1} || \frac{\xi_2}{\xi_1} || + ||\nu\alpha_f \phi_2|| \ \frac{A_3}{A_1} || \frac{\xi_3}{\xi_2} || \right\} \qquad (4.100)$$

$$|\lambda_2| + v_2\alpha_2 T > v_2 \left\{\left(||p^\circ\Sigma_1|| + ||\Sigma_1\alpha_p T||\right)\frac{A_1}{A_2}|| \frac{\xi_1}{\xi_2}|| + ||\alpha_p\phi_1 - \alpha_2\phi_2|| \frac{A_3}{A_2}|| \frac{\xi_3}{\xi_2}||\right\} \tag{4.101}$$

$$|\lambda_3| - \frac{\varepsilon\alpha_f}{\rho c_p}\phi_2 > \frac{\varepsilon}{\rho c_p}\left(||\Sigma_f^\circ|| + ||\alpha_f T||\right)\frac{A_2}{A_3}|| \frac{\xi_2}{\xi_2}|| \tag{4.102}$$

The eigenvalues λ_1, λ_2, λ_2 correspond to the operators $\mathscr{L}_i$ i=1, 2, 3 given by Eq. (4.98), which are Helmholtz operators if we take D_1, D_2, Σ_1, Σ_2° to be constant. This restriction, however, does not remove most of the space dependence which is buried in the various temperature coefficients of cross-sections, and the space dependent feature of temperature feedback mechanism is preserved. In this case, the conditions (4.100-4.102) reduce further to the following stability conditions.

$$B^2 D_1 + \Sigma_1 > (||\nu\Sigma_f^\circ|| + ||\nu\alpha_f T||)\frac{A_2}{A_1} + ||\nu\alpha_f\phi_2||\frac{A_3}{A_1} \tag{4.103}$$

$$B^2 D_2 + \Sigma_2^\circ > (||p^\circ||\Sigma_1 + \Sigma_1||\alpha_p T||)\frac{A_1}{A_2} + ||\alpha_p\phi_1 - \alpha_1\phi_2||\frac{A_3}{A_1} \tag{4.104}$$

$$(B^2 + \frac{h}{k})k - \varepsilon\alpha_f\phi_2 > \varepsilon(||\Sigma_f^\circ|| + ||\alpha_f T||)\frac{A_2}{A_3}\ , \tag{4.105}$$

where B^2 is the geometrical buckling of the core.

One may apply the above relations in two ways. First, choose the initial conditions A_1, A_2, A_3 and test for the bounds on the derivatives, i.e., possible perturbations, or one can fix the values of the derivatives and solve for allowable initial conditions.

V. NUMERICAL EXAMPLE

In an attempt to compare all of the above discussed stability approaches, one has to choose a particular model for which all methods will be applicable. In general, we may not study stability domains of a given reactor model via all existing methods. The method by comparison functions is applicable only to parabolic type of equations.

Consider a homogeneous sphere with a positive power feedback coefficient for which the nonlinear diffusion equation reads

$$\frac{\partial\phi(x,t)}{\partial t} = \nabla_x^2\ \phi(x,t) + k\ \phi(x,t) + \alpha_p\ \phi^2(x,t); \qquad 0 < x < \pi \tag{5.1}$$

where

$$\nabla_x^2 = \frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial}{\partial x}\right)$$

$$x = r\sqrt{\frac{\Sigma_a}{D}}\ ; \qquad t = t'\ \sigma\ \Sigma_a, \qquad k = \frac{\nu\Sigma_f}{\Sigma_a} - 1$$

with the boundary condition

$$\phi(\pi,t) = 0; \qquad \phi'(0,t) = 0. \tag{5.2}$$

α_p is called the power feedback reactivity coefficient. Physically this type of feedback can represent mechanism acting instantaneously as the flux changes. The distance x is measured in units of diffusion length, and time t is measured in units of neutron lifetimes. Initial conditions are not specified directly, since we will be interested in the behavior of solutions for various initial disturbances. The linearized system must be asymptotically stable as is required by all stability theorems, because if the system is not stable for very small perturbations, one would not expect it to be stable for larger perturbations. The above statement implies that the largest eigenvalue of

$$\nabla_x^2 \psi_n(x) + k \psi_n(x) = \lambda_n \psi_n(x); \quad 0 < x < \pi \tag{5.3}$$

$$\psi_n(\pi) = 0, \ \psi_n^1(0) = 0 \tag{5.4}$$

must be negative. The solutions to Eq. (5.3) with boundary conditions (5.4) are of the form

$$\frac{\sin (k - \lambda_n)x}{(k - \lambda_n)x} \quad \text{with}$$

$$\lambda_n = k - n^2; \quad n = 1, 2, \ldots \tag{5.5}$$

n^2 corresponds to B_n^2, the buckling, and it takes on integer values because of the choice of the size of the sphere.

Since λ_1 is the largest eigenvalue, setting $\lambda_1 < 0$ implies that $k - 1 < 0$, or in other words, the linear reactor is subcritical. This means that we will be concerned with the trivial equilibrium solution of Eq. (5.1) and we will get conditions under which the shutdown state of the reactor will be stable if we introduce perturbations.

Provided that we interpret the results of each approach properly, we should be able to arrive at equivalent conclusions concerning the allowable values of perturbations.

(a) Liapunov's Method

Defining a Liapunov functional to be

$$V = \frac{1}{2} \int_0^{\pi} \phi^2(x,t)\ 4\pi\ x^2\ dx \ ,$$

according to theorem 2 of Chapter 2, we arrive at a sufficient condition for asymptotic stability to be

$$\int_0^{\pi} 4\pi\ x^2\ dx\ \phi^3(x,t) < \frac{|\lambda_1|}{\alpha_p} \int_0^{\pi} 4\pi\ x^2\ dx\ \phi^2(x,t) \tag{5.6}$$

where λ_1 is given by Eq. (5.5). Relation (5.6) may be looked upon as a domain allowable initial conditions since it is valid for all $t \geq 0$.

(b) Semigroup Method

(i) Using the results of theorem 4 of the chapter on semigroup approach, the solution is asymptotically bounded for all initial conditions that satisfy

$$||\phi(x,o)||_{\infty} < \left(\int_0^{\infty} \alpha_p\ e^{-|\lambda_1| t}\ dt \right)^{-1}$$

or rewriting it in another form

$$\max_{0 \leq x \leq \pi} |\phi(x,o)| < \frac{|\lambda_1|}{\alpha_p} \ . \tag{5.7}$$

(ii) Using the results of theorem 5 of the chapter on semi-group method, we have

$$\max_{0 \le x \le \pi} |\phi(x,o)| < \frac{1}{2} \frac{|\lambda_1|}{\alpha_p} . \tag{5.8}$$

Note that condition (5.8) is more conservative than condition (5.7).

(c) Comparison Functions Method

(i) Using the results of theorem 3 of Chapter 4

$$\max_x |w^2(x,t)| < \max_x |w(x,t)| \frac{|\lambda_1|}{\alpha_p} \tag{5.9}$$

where w(x,t) is the comparison function that bounds the solution. Again, relation (5.9) is to be viewed as the condition on initial values.

(ii) Using the approach involving derivatives of non-linearities

$$\max_x |\phi(x,o)| < \frac{1}{2} \frac{|\lambda_1|}{\alpha_p} . \tag{5.10}$$

The initial perturbations considered were of the following form

$$\phi = A \quad , \quad A \cos \frac{x}{2}, \quad A(1 - \frac{x}{\pi}); \quad 0 \le x \le \pi$$

where A is the amplitude of perturbation. The upper bounds on $A \frac{\alpha_p}{|\lambda_1|}$ are tabulated below. For example, Liapunov's method yields that $A \frac{\alpha_p}{|\lambda_1|}$ be less than 1.549 for a perturbation given by $A \cos \frac{x}{2}$.

Method	Relation Used	$\phi = A$	$\phi = A \cos \frac{x}{2}$	$\phi = A(1 - \frac{x}{\pi})$
Liapunov	5.6	1.000	1.549	2.000
Semigroup	5.7	1.000	1.000	1.000
Semigroup	5.8	0.500	0.500	0.500
Comparison	5.9	1.000	1.000	1.000
Comparison	5.10	0.500	0.500	0.500

TABLE 1: Comparison of Results for Homogeneous Sphere with Positive Power Feedback.

A similar type of calculation was made for a homogeneous slab reactor of width 2π, for initial perturbations of the form

Method	Relation Used	$\phi = A$	$\phi = A \cos \frac{x}{2}$	$\phi = A(1 - \frac{\|x\|}{\pi})$
Liapunov	5.6	1.000	1.549	1.333
Semigroup	5.7	1.000	1.000	1.000
Semigroup	5.8	0.500	0.500	0.500
Comparison	5.9	1.000	1.000	1.000
Comparison	5.10	0.500	0.500	0.500

TABLE 2: Homogeneous Slab with Positive Power Feedback.

Although the example is a very simple one, it illustrates the ideas presented in this thesis and at the same time tedious numerical calculations are avoided.

For a more complicated problem with nontrivial equilibrium calculations, using relations (5.6), (5.7), (5.9) is much more involved. In fact, one has to solve the nonlinear boundary value problem

$$\nabla_x^2 \phi_{eq}(x) + k \phi_{eq}(x) + \alpha_p \phi_{eq}^2(x) = 0 ,$$

$$\phi_{eq}(\pi) = 0 \qquad \phi'_{eq}(0) = 0 ,$$

and then, solve for largest eigenvalue of the following equation

$$\nabla_x^2 \psi_n(x) + (k + 2 \phi_{eq}(x)) \psi_n(x) = \gamma_n \psi_n(x) .$$

Sometimes the feedback coefficients of the state variables depend on the values of neutron flux or temperature.

In our example, we could have had

$$\alpha_p = \alpha_1 + \alpha_2 \phi(x,t) , \tag{5.11}$$

and in that case, the nonlinearity in the diffusion equation is of the form

$$g(\phi) = \alpha_1 \phi^2 + \alpha_2 \phi^3 ; \qquad \alpha_1, \alpha_2 > 0 . \tag{5.12}$$

The conventional approach by semigroup analysis is difficult to apply in this case, because it is difficult to find a $\rho(t)$ such that

$$||g(\phi)|| \le \rho(t) \; ||\phi||^{1+\alpha} . \tag{5.13}$$

In that case, theorems 3 and 4 of semigroup approach are of no use, however, theorem 5 may be used and the condition for asymptotic stability is given by

$$2\ \alpha_1 \max_x |\phi(x,o)| + 3\ \alpha_2 \max_x |\phi^2(x,o)| < |\lambda_1| \tag{5.14}$$

where λ_1 is the same eigenvalue as explained above.

Among other approaches as candidates for analyzing this problem is Liapunov's method, which gives us

$$\frac{||\phi g(\phi)||^2_{L_2}}{||\phi||^2_{L_2}} < |\lambda_1| \tag{5.15}$$

Again, consider the slab reactor of width 2π and consider a perturbation of the form

$$\phi(x,o) = A \qquad \pi \leq x \leq \pi . \tag{5.16}$$

Liapunov's method yields

$$A_L < \frac{\alpha_1}{2\alpha_2} \ (1 + \frac{4\alpha_2}{\alpha_1^2} |\lambda_1|)^{1/2} - 1 \tag{5.17}$$

where the subscript L denotes the result was obtained by Liapunov's method from relation (5.6), and semigroup method yields

$$A_s < \frac{\alpha_1}{3\alpha_2} \ (1 + \frac{3\alpha_2}{\alpha_1^2} |\lambda_1|)^{1/2} - 1 . \tag{5.18}$$

If we fix λ_1 which corresponds to keeping the "criticality constant" and look at the ratio of A_L/A_s as a function of $\frac{\alpha_2}{\alpha_1^2}$, we obtain

$$\sqrt{3} < \frac{A_L}{A_s} < 2 \tag{5.19}$$

which again shows that Liapunov's method gives a better result; but for nontrivial equilibrium, Liapunov's method presents greater computational difficulties than the semigroup.

CONCLUDING REMARKS

The object of this thesis is to extend the study of nonlinear reactor kinetics. The ideas developed are applied to reactor core models described by space-time dependent equations. Stability criteria obtained depend on the methods used whenever more than one method is applicable to a given situation. Several remarks can be made with regard to applicability of each approach.

1. Liapunov's method seems to encompass a wider class of problems within the definition of dynamical system. The system of equations may include linear and nonlinear equations. There are no restrictions on the nonlinearities. The difficulty with this approach is in the construction of Liapunov functional. Different suitable functionals will yield different results.
2. Semigroup approach requires that the linear operator be an infinitesinal generator of a semigroup. It may be generally assumed that this holds true. A more serious difficulty lies in the a priori bounding of the nonlinear terms in a certain way. As illustrated in theorem 4, one can specify a bound on the solution and obtain a criterion on initial condition necessary not to exceed the specification. The advantage of semigroup approach over Liapunov's method is that it provides us with an estimate of the solution.
3. The comparison function method has its limitations primarily due to the fact that each nonlinear equation in the system has to have the time independent solutions belonging to an

elliptic equation. This makes the space-time equation a parabolic type. This is because the theorems require that the eigenvalues of the time independent operator decrease as the size of the region increases. The use of different comparison functions may result in a different stability criteria. The comparison functions bound the solution from below and above, hence an estimate is obtained.

Numerical examples reveal that the stability criteria obtained depend on the method used whenever more than one method is applicable to a given situation. Although Liapunov's method tends to smooth out some of the spatial effects because of the use of L_2 norm, it does give the least conservative result. When applicable, the semigroup and comparison function methods tend to preserve more of the spatial effect by the use of L_∞ norm. That is, disturbances and initial conditions have to satisfy the stability criteria pointwise throughout the region of interest. In general it is difficult to predict which approach will yield the best results, and a natural procedure to follow would amount to choosing the most easily applicable approach.

The existing methods prior to this work involved a two stage calculation in order to establish stability or instability of nontrivial equilibrium. The two stages are the following:

(a) Solution to nonlinear boundary value problem, i.e., solution for nontrivial equilibrium state.

(b) Solution for an eigenvalue of linear operator whose parameters depend on solutions to (a).

Use of the mean value theorems makes the calculation much easier, that is, it eliminates step (a) partially or completely. Consequently, step (b) is greatly simplified.

For the comparison function method we have a complete elimination of step (a), and at the same time stability criteria are much more flexible with regard to their interpretation. The criteria carry information about solutions, initial conditions, and parameters that enter the governing equations of motion. Furthermore, one may suppress little of the space dependence of parameters, retaining much of the feedback spatial effects thus making step (b) even simpler. In this trend of simplification, the essential nonlinear space time effects are preserved.

The use of generalized mean value theorem in the semigroup method gives a necessary condition for asymptotic stability. Here we have the partial elimination of step (a) where an estimate of the upper bound of the static solution is required. In both extensions presented here, the price paid for this simplification results in smaller sizes of allowable perturbations when compared with previous results.

One possible extension lies in the area of comparison function approach. The method is limited to strictly parabolic types of equations, and this fact makes it less flexible when compared with other methods. The trend of this extension is partially exhibited here in the adiabatic reactor model, where a nonparabolic equation was in the system, however it was linear, and the nonlinearity was negative. The roots of such investigation would probably be in the Westphal-Prodi theorem.

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APPENDIX

Bellman-Gronwall Inequality

If the functions u(t) and v(t) are non-negative for $t \geq 0$ and if $a \geq 0$, then the inequality

$$(1) \quad u(t) \leq a + \int_0^t v(t')\, u(t')\, dt' \qquad t \geq 0$$

implies

$$(2) \quad u(t) \leq a \exp \int_0^t v(t')\, dt' \ .$$

Proof: From (1) we have

$$\frac{u(t)\ v(t)}{a + \int_0^t v(t')\, u(t')\, dt'} \leq v(t)$$

integrating this from 0 to t, we have

$$\ln\left(\frac{a + \int_0^t v(t')\, u(t')\, dt'}{a}\right) \leq \int_0^t v(t')\, dt'$$

which gives us

$$a + \int_0^t dt'\, u(t')\, v(t') \leq a \exp \int_0^t v(t')\, dt' \ .$$

Using (1), the result follows.

Bihari Inequality (Lemma 1 of chapter on semigroups)

Let $u(t)$ and $v(t)$ be positive continuous functions for $0 \leq t < \infty$, $a \geq 0$, and let $w(u)$ be a non-negative non-decreasing function for $u \geq 0$. Then the inequality

$$(3) \quad u(t) \leq a + \int_0^t v(t')\, w(u(t'))\, dt'$$

implies the inequality

$$(4) \quad u(t) \leq \Omega^{-1}\left(\Omega(a) + \int_0^t v(t')\, dt'\right)$$

where

$$(5) \quad \Omega(u) = \int_{u_o}^{u} \frac{du'}{w(u')}\ ; \qquad u_o > 0, \qquad u \geq 0$$

and $\Omega^{-1}(u)$ means the inverse function of $\Omega(u)$.

Proof:

$\Omega^{-1}(u)$ exists because of the monotonocity of $\Omega(u)$. Differentiating (3) yields:

$$(6) \quad w(u) \geq \frac{(du/dt)}{v(t)}\ ,$$ and differentiating (3), we have

$$(7) \quad \frac{d\Omega(u)}{du} = \frac{1}{w(u)} \quad .$$

Combining (6) with (7), we have

$$d\Omega(u(t')) \le v(t')\,dt',$$

and integrating this last inequality from 0 to t yields

$$\Omega(u(t)) - \Omega(u(o)) \le \int_0^t v(t')\,dt' \quad .$$

But u(o) = a from (3); therefore

$$u(t) \le \Omega^{-1}\left[\Omega(a) + \int_0^t v(t')\,dt'\right],$$

which is the desired result.

<u>Generalized Mean Value Theorem</u> (Lemma 2 of semigroup chapters)

Consider the real valued function

$$\sum_{i=1}^{n} e_i\,(g_i(u) - g_i(v)), \text{ where}$$

$$\begin{aligned} u &= (u_1(x,t), u_2(x,t), \ldots, u_n(x,t)) \ ; \qquad x \in R \\ v &= (v_1(x,t), v_2(x,t), \ldots, v_n(x,t)) \ ; \qquad 0 \le t < \infty \end{aligned}$$

Let us now apply the mean value theorem to the above function of several real variables.

$$\sum_{i=1}^{n} e_i\,(g_i(u) - g_i(v)) = \sum_{i=1}^{n} \sum_{j} J_{ij}(u^*)\,(u_j - v_j)$$

where u* lies between u and v and

$$J_{ij}(u^*) = \frac{\partial g_i(u^*)}{\partial u_j}$$

$$\int_{x\varepsilon R} dx \sum_{i=1}^{n} |e_i \; g_i(u) - g_i(v)| = \int dx \sum_{i=1}^{n} |e_i \sum_j J_{ij}(u^*) \; (u_j - v_j)|$$

$$\leq \int dx \sum_{k=1}^{n} |e_k| \sum_{ij} |J_{ij}(u^*) \; (u_j - v_j)|$$

$$\leq \int dx \left(\sum_{k=1}^{n} |e_k|^2\right)^{1/2} \left(\sum_{ij} |J_{ij}(u^*) \; (u_j - v_j)|^2\right)^{1/2}$$

and using Schwartz's inequality

$$\leq \left(\int dx \sum_k |e_k|^2\right)^{1/2} \left(\int dx \sum_{ij} |J_{ij}(u^*) \; (u_j - v_j)|^2\right)^{1/2} .$$

Since

$$||e|| = 1 = \left(\int dx \sum_k |e_k|^2\right)^{1/2} ,$$

we have

$$\int dx \sum_i |e_i \; g_i(u) - g_i(v)| \leq \left(\int dx \sum_{ij} |J_{ij}(u^*) \; (u_j - v_j)|^2\right)^{1/2}$$

$$\leq \max_{x\varepsilon R} \left(\sum_{ij} |J_{ij}(u^*)|^2\right)^{1/2} \left(\int dx \sum_k |u_j - v_j|^2\right)^{1/2}$$

$$= ||J(u^*)|| \; ||u - v|| .$$

Choosing the unit vector

$$e = \frac{g(u) - g(v)}{||g(u) - g(v)||} ,$$

we have

$$||g(u) - g(v)|| \le ||J(u^*)||\ ||u - v|| \quad ,$$

where

$$||J(u^*)|| \quad \max_{x \varepsilon R}\left(\sum_{ij} |J_{ij}(u^*)|^2\right)^{1/2} \quad .$$

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STABILITY ANALYSIS OF SPATIALLY DEPENDENT NONLINEAR REACTOR SYSTEMS

by

Lech Mync

(ABSTRACT)

The space-time behavior of a neutron distribution governed by nonlinear multigroup diffusion approximation is considered in this thesis. Stability criteria for equilibrium states of various reactor feedback models are determined by the methods of Liapunov, semigroup and comparison functions. Comparison of the three approaches are made with respect to applicability to various models as well as computational difficulties associated with the three methods. The models chosen serve as illustrative examples of stability analysis; they also complement the existing examples in literature.

The primary objective of this work is to simplify computational difficulties by the use of generalized mean value theorem for functionals, and functions of several variables. The results are expressed in the form of a theorem for the semigroup method, where a necessary condition for asymptotic stability is proven. It is applied to the problem of xenon oscillations. The use of the generalized mean value theorem in connection with the method by comparison function is also shown to lead to computational simplification. The result is applied to two energy group reactor models with temperature feedback.

A simple numerical example and a comparison of the three methods, together with their variations, is given. The results show that the proposed method of calculating stability conditions leads to more conservative conditions, that is, smaller domains of allowable perturbations. The calculational procedure is, however, simplified in that the equilibrium nonlinear problem does not have to be solved.