COMPATIBLE WHOLE-STAND AND DIAMETER DISTRIBUTION MODELS
FOR LOBLOLLY PINE PLANTATIONS

by

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INTRODUCTION

Loblolly pine (*Pinus taeda* L.) is the most intensively managed tree species in the South. Because harvesting methods, utilization standards and management practices are changing, foresters need detailed stand information. Information by tree diameter classes is needed since merchantability limits and guidelines for management practices such as thinning are frequently defined in terms of tree diameter at breast height (dbh). The product mix (pulpwood, sawlogs, veneer logs), defined by dbh limits, which is obtained at harvest is largely a result of the management practices implemented. Thus models to predict detailed information which can also be combined for more "strategic" planning purposes are needed.

Two major approaches that have been applied in the past for estimating the yield of loblolly pine are stand average (whole stand) models and diameter distribution models. Stand average models are equations which predict the yield per unit area of the whole stand or some specified portion as a function of age, stand density and site index. Diameter distribution models provide estimates of the number
of trees by dbh classes. The stand yield per unit area is then obtained by multiplying the number of trees in each dbh class by the volume of a tree of class midpoint dbh, then summing the volume over all diameter classes. The diameter distribution has been modelled in recent years by probability density functions (pdf's) such as the Weibull and beta. The parameters of the pdf are estimated for each sample plot in the data set by procedures such as maximum likelihood or the method of moments. Parameter prediction equations are then constructed, usually by linear regression procedures, and these parameter values related to stand characteristics such as age, mean stand height and density. These parameter prediction equations are then used to predict the diameter distribution for a given set of stand conditions.

Forest managers need information of different levels of resolution or stand detail for various types of management decisions. The stand average models and diameter distribution models used to provide these different levels of resolution have been constructed independently, and even when constructed from the same data set do not necessarily produce the same estimate of stand yield for a given set of stand conditions (Daniels, et al. 1979). One logical approach to insuring consistency between stand average and
diameter distribution based yield estimates would be to predict the stand average attributes of interest for a specified set of stand conditions and use these estimates as the basis to "recover" the parameters of the underlying diameter distribution. This approach was termed a "parameter recovery" model by Hyink (1980a). The primary advantages of such a system would be mathematical compatibility of the stand average model and the diameter distribution-based yield model, the ability to partition total yield by diameter classes, and consistency among the various stand yield estimates.
OBJECTIVES

The primary objective of this study was:

1. To construct and evaluate parameter recovery models for predicting the diameter distribution of loblolly pine plantations using the beta and Weibull probability density functions.

Secondary objectives were:

2. To determine the most appropriate set of stand average prediction equations to use.

3. To determine how to estimate the end point(s) of the pdf and the sensitivity of the estimate.
PREVIOUS WORK

Stand Average Yield Models

In the early 1900's, yield prediction methods were limited primarily to normal yield tables such as USDA Miscellaneous Publication No. 50 (1929). These tables provide, for a given species, the per acre yield of wood in some specified volume unit as a function of age and site index. Stand density was assumed to be full or normal stocking. However, the definition of a normal stand was somewhat nebulous (Nelson and Bennett 1965), and in practice few stands were normally stocked. The subjectivity involved in this approach led to interest in variable density yield tables.

MacKinney, et al. (1937) were the first to use a regression approach to construction of variable density yield tables from observed data. The equation they used was the Pearl-Reed growth function. MacKinney and Chaiken (1939) used a logarithmic equation with independent variables age, site index, stand density index and stand composition. Several recent studies have used a multiple
regression approach similar to that of MacKinney and Chaiken to predict yield. Schumacher and Coile (1960) presented a yield model for natural stands of the four major southern pine species and Coile and Schumacher (1964) provide yield models for thinned and unthinned plantations of slash and loblolly pine. Burkhart, et al. (1972a, 1972b) used this method to predict yield for natural stands and plantations of loblolly pine.

Buckman (1962) and Clutter (1963) proposed that stand growth models and stand yield models be mathematically compatible. That is, the yield model should be the integral of the growth model so that yield at any time is simply the summation of growth up to that time. Sullivan and Clutter (1972) generalized this concept and constructed a simultaneous growth and yield model for loblolly pine which provided numerically consistent growth and yield predictions. Given the age, site index and basal area of a stand at a given time, the model predicts the yield increment to some future time; and if the projection interval is zero it becomes simply a yield model. This growth and yield model has been applied to loblolly pine (Brender and Clutter 1970, Sullivan and Williston 1977, and Murphy and Sternitzke 1979), shortleaf pine (Murphy and Beltz 1981), slash pine (Bennett 1970), and yellow-poplar (Beck and Della-Bianca 1972 and 1975).
All of the above yield models have been constructed by regression using observed stand characteristics such as age, site index and basal area or number of trees per unit area. The empirical models predict well within the range of the data and, in general, are consistent with observed biological behavior and trends.

**Diameter Distribution Models**

A diameter distribution model provides estimates of the number of trees per unit area by dbh classes. By estimating the volume of a tree with dbh equal to the dbh class midpoint, multiplying by the number of trees per unit area in that class, then summing these volumes over all dbh classes, an estimate of total stand volume per unit area may be obtained. Many different mathematical functions have been used to describe diameter distributions and Lenhart (1968) provides an excellent review of attempts to mathematically model diameter distributions prior to 1965.

Clutter and Bennett (1965) used the beta probability density function (pdf) to model diameter distributions in old-field slash pine. The model was
\begin{align*}
f_X(x; \alpha, \beta) &= \begin{cases} 
\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)(D_{max}-D_{min})} \cdot \left( \frac{x-D_{min}}{(D_{max}-D_{min})} \right)^{\alpha-1} \cdot \left( 1 - \frac{x-D_{min}}{D_{max}-D_{min}} \right)^{\beta-1} 
\quad \text{if } \alpha, \beta > 0, 0 < D_{min} < x < D_{max}; \\
0, & \text{otherwise;} 
\end{cases}
\end{align*}

where

\begin{itemize}
\item $x = \text{tree dbh},$
\item $D_{min} = \text{minimum dbh in the stand},$
\item $D_{max} = \text{maximum dbh in the stand},$
\item $\alpha, \beta = \text{parameters of the pdf},$
\item $\Gamma(y) = \int_0^{\infty} e^{-t} t^{y-1} \, dt,$ the gamma function.
\end{itemize}

Integrating $f_X$ between two diameters, say $D_l$ and $D_u$, where $D_{min} \leq D_l < D_u \leq D_{max}$ gives the proportion of trees in that diameter class. Multiplying this by the total number of trees per acre gives the number of trees per acre in that class. In this model $D_{min}$ and $D_{max}$ were predicted as functions of stand age, site index and number of trees per acre. However, $\alpha$ and $\beta$ were constants estimated from the data, which means the shape of the distribution was assumed to be constant over all ages and stand conditions. This inability to change shape for varying stand conditions nullifies one of the advantages of using the beta pdf. Subsequent work by Lenhart (1968), Lenhart and Clutter
(1971), Burkhart (1971), Burkhart and Strub (1974) and Bennett and Clutter (1968) continued the study of the beta pdf as a diameter distribution model in loblolly and slash pine. McGee and Della-Bianca (1967) also used this approach for yellow-poplar. In each instance, some function of stand characteristics was used to predict Dmin, Dmax and the other pdf parameters.

Strub and Burkhart (1975) demonstrated a method for determining the volume for any desired portion of the diameter distribution using a local volume equation and a specified pdf. As noted above, integrating \( f_x \) between two diameters gives the proportion of trees in that class. By applying a local volume equation \( g(x) \), which predicts the volume of a tree as a function of dbh and stand characteristics, the total volume per unit area for that dbh class is given by

\[
\text{VOL} = N \int_{D_l}^{D_u} g(x) f(x; \theta) \, dx
\]

where

\[
f(x; \theta) = \text{the pdf, with parameter vector } \theta,
\]

\[
N = \text{number of trees per acre.}
\]
This reduces the imprecision and bias inherent in using class midpoint diameter for volume estimates.

The Weibull pdf was first used as a diameter distribution model by Bailey (1972). Some of the desirable properties possessed by the Weibull are flexibility of shape, parameters that are more easily related to stand characteristics, and existence of a closed form integral of the pdf (Bailey and Dell 1973). The pdf and cumulative distribution function (cdf) for the three-parameter Weibull distribution are

\[
f_X(x; a, b, c) = \begin{cases} 
\left( \frac{c}{b} \right) \left( \frac{x-a}{b} \right)^{c-1} \exp \left[ - \left( \frac{x-a}{b} \right)^c \right], \\
0, & \text{otherwise;}
\end{cases} \quad (3)
\]

\[
F_X(x) = 1 - \exp(-((x - a)/b)^c), \quad (4)
\]

where

\( x = \) tree dbh,

\( a = \) location parameter,

\( b = \) scale parameter,

\( c = \) shape parameter.
The two-parameter Weibull results from the transformation $y = (x - a)$ which gives

$$f_Y(y; b, c) = \begin{cases} 
(c/b) \frac{(y/b)^c}{y} \exp \left\{-\frac{(y/b)^c}{y}\right\}, & y, b, c \geq 0; \\
0, & \text{otherwise}
\end{cases}$$

(5)

$$F_Y(y) = 1 - \exp \left\{-\frac{(y/b)^c}{y}\right\}. \quad (6)$$

The Weibull distribution has been used to construct yield tables for loblolly pine (Smalley and Bailey 1974a and Feduccia, et al. 1979), shortleaf pine (Smalley and Bailey 1974b), slash pine (Dell, et al. 1979) and longleaf pine (Lohrey and Bailey 1977).

Another distribution, the Johnson's $S_B$, was proposed by Hafley and Schreuder (1977) as a model for diameter distribution. They compared six distributions in terms of flexibility of skewness and kurtosis and for fitting the diameter distribution. The six distributions were beta, $S_B$, Weibull, normal, lognormal and gamma. In terms of flexibility of skewness and kurtosis, the $S_B$ was the most flexible followed by the beta, Weibull, gamma and lognormal. The normal has only one shape, thus, no flexibility. In a comparison of the six distributions for modelling diameter distribution, they found the $S_B$ to be consistently better.
than the others, followed by the beta and Weibull functions. In another article Schreuder and Hafley (1977) demonstrated the bivariate Johnson's $S_B$ distribution for modelling diameter-height relationships.

The essential feature in using pdf's to model diameter distribution is the ability to predict the pdf parameter values for a given set of stand conditions. Those pdf's discussed above utilized parameter prediction equations developed from sample plot data. For each sample plot, values for the parameters were estimated usually using maximum likelihood procedures for the Weibull or the method of moments for the beta. Once the parameter estimates for each plot had been computed, regression equations were constructed relating the parameter estimates to stand characteristics such as age, site index and number of trees per acre. These regression equations are referred to as parameter prediction equations. Parameter prediction equations typically have $R^2$ values in the range of 0.1 to 0.3, indicating that perhaps we do not have an adequate understanding of the relationships of the parameters to changing stand conditions.

In order to avoid the problems caused by the poor parameter prediction equations, Bailey, et al. (1981) predicted the 24th, 63rd, and 93rd percentiles of the
Weibull pdf from stand characteristics then used these statistics to solve for the three parameters of the pdf. The $R^2$ values for the percentile regression equations were much better than for the parameter prediction equations and the resulting pdf modelled the diameter distribution exceedingly well.

Thus far, stand average models and diameter distribution models have been constructed independently with no guarantee that, for a given set of stand conditions, the predicted total yield per unit area would be the same for both methods. Hyink (1980a, 1980b) suggested a method to "recover" the parameters of the diameter distribution pdf from estimates of stand attributes given by stand average models. In general, at any time $t$, the yield table constructed by the diameter distribution model may be represented as

$$ y_i = N_t \int_{D_l}^{D_u} g_i(x) f(x; \theta) \, dx, $$

(7)

where

- $x =$ tree dbh,
- $N_t =$ number of trees per acre surviving at time $t$,
- $D_l, D_u =$ lower and upper limits of integration, respectively, for that particular $g_i(x)$,
- $f(x; \theta) =$ the pdf,
- $\theta =$ the parameter vector $\theta_1, \theta_2, \ldots, \theta_k$. 

\[ g_i(x) = \text{the } i^{th} \text{ function of tree dbh,} \]
\[ y_i = \text{the per unit area value of the } i^{th} \text{ stand attribute defined by } g_i(x). \]

Now assume there exists a set of functions \( g_1(x), g_2(x), \ldots, g_k(x) \) and the values of the corresponding stand attributes \( y_1, y_2, \ldots, y_k \). The stand attributes may be basal area per unit area, total cubic volume per unit area, volume to a 4-inch top diameter outside bark per unit area, or a statistic relevant to the distribution being considered. The function \( g_i(x) \) gives the value of the \( i^{th} \) stand attribute for a tree of diameter \( x \). Then the set of \( k \) yield equations formed is

\[
\begin{align*}
Y_1 &= N_t \int_{D_1}^{D_u} g_1(x) f(x; \theta) \, dx, \\
Y_2 &= N_t \int_{D_1}^{D_u} g_2(x) f(x; \theta) \, dx, \\
&\quad \vdots \\
Y_k &= N_t \int_{D_1}^{D_u} g_k(x) f(x; \theta) \, dx. 
\end{align*}
\]

Thus there are \( k \) equations in \( k \) unknowns for which a solution exists for each of the \( k \) parameters, provided this is a consistent set of equations.
Many different sets of equations can be constructed for any number of pdf's. The primary objective of this study was to define appropriate sets of equations for the Weibull and beta pdf's and to construct the parameter recovery models for unthinned and thinned loblolly pine plantations.
METHODS

Parameter Recovery Model Structure

The general diameter distribution yield function was shown to be

\[ y_i = N_t \int_{D_l}^{D_u} g_i(x) f(x; \theta) \, dx. \]

For any \( g_i(x) \), integration of this function over the range of diameters, \( x \), gives the total per unit area value of the stand attribute defined by \( g_i(x) \). Some of the many possible stand attributes that could be used are basal area per acre, total cubic volume per acre, board-foot volume per acre or average diameter of trees in the stand. In order to solve for the pdf parameters there must be as many stand attribute equations as there are parameters. Two basic sets of equations were defined for this research. One, consisting of the non-central moments of the random variable \( x \), \( E(x^i) \), is termed the moment-based parameter recovery system. The other, consisting of one or more volume equations in combination with non-central moment equations, is termed the volume-based parameter recovery system.
To define the moment-based parameter recovery system, consider the case where $g_i(x) = x^i$. The equation for the $i^{th}$ non-central moment of $x$ is

$$E(x^i) = \int x^i f(x; \theta) \, dx = y_i/N.$$  

Basically, the moment-based parameter recovery system is the method of moments technique for pdf parameter estimation (Mendenhall and Scheaffer 1973). The first and second non-central moments of $x$ (or dbh) are estimated by the average diameter of the stand and the basal area per acre. Higher moments, though they have no equivalent applied forestry usage, may be estimated in a similar fashion. Specifically

$$E(x) \text{ is estimated by } \frac{\sum_{j=1}^{n} x_j}{N} = \bar{x},$$

$$E(x^2) \text{ is estimated by } \frac{BA \text{ per acre}}{0.005454 N} = \frac{\sum_{j=1}^{n} x_j^2}{N} = x^2,$$

$$E(x^k) \text{ is estimated by } \frac{\sum_{j=1}^{n} x_j^k}{N} = x^k.$$

(9)
Thus stand average estimates of the first \( k \) moments results in \( k \) equations with \( k \) unknown parameters from which solutions for the parameters may be obtained.

The volume-based parameter recovery system may use equations for the non-central moments of \( x \), but it also contains at least one stand volume equation. Only one volume equation was used in this study; the combined variable equation for total cubic-foot volume per acre,

\[
g(x) = a_0 + a_1 x^2 h(x) 
\]

where

\( a_0, a_1 \) = regression coefficients of the volume equation estimated from sample data,

\( x \) = tree dbh,

\( h(x) \) = tree height as a function of stand age, site index, number of trees per acre and dbh (discussed in more detail later),

\( g(x) \) = tree volume equation.

Again there are \( k \) equations defining stand attributes, one of which is total cubic foot volume per acre, which are used to solve for the \( k \) parameters.

Once the pdf parameters have been estimated then other stand average attributes, such as total dry weight per acre
or merchantable volume per acre, can be computed for the
diameter classes of interest.

As stated earlier, the pdf's used in this study were
the Weibull and the beta. Below are the parameter recovery
models for these distributions.

Moment-Based Parameter Recovery Models

Two-Parameter Weibull. The first two non-central
moments of the two-parameter Weibull pdf (equation 5) are
estimated as in equation 9 to give the system

\[ \bar{x} = \int_{0}^{\infty} x f(x; b, c) \, dx = b \Gamma(1 + 1/c), \quad (11) \]
\[ \bar{x}^2 = \int_{0}^{\infty} x^2 f(x; b, c) \, dx = b^2 \Gamma(1 + 2/c). \quad (12) \]

Then the estimated variance of the distribution is given by

\[ s^2 = \bar{x}^2 - \bar{x}^2 = b^2 [\Gamma(1 + 2/c) - \Gamma^2(1 + 1/c)], \quad (13) \]

and the coefficient of variation is estimated by

\[ c.v. = s/\bar{x} = \frac{\Gamma(1+2/c) - \Gamma^2(1+1/c)}{\Gamma(1+1/c)}^{1/2}, \quad (14) \]

which is a function of \( c \) only. Given the estimates \( \bar{x} \) and
\( \bar{x}^2 \), it is then possible to solve for \( c \). Given this solution
for \( c \), then \( b \) is solved from

\[ \bar{x} = b \Gamma(1 + 1/c). \quad (15) \]
Three-Parameter Weibull. The system of equations for the three-parameter Weibull (equation 3) is

\[ \overline{x} = \int_a^\infty x f(x; a, b, c) \, dx = b \Gamma(1 + 1/c) + a, \quad (16) \]

\[ \overline{x^2} = \int_a^\infty x^2 f(x; a, b, c) \, dx = b^2 \Gamma(1 + 2/c) + 2ab \Gamma(1 + 1/c) + a^2, \quad (17) \]

\[ \overline{x^3} = \int_a^\infty x^3 f(x; a, b, c) \, dx = b^3 \Gamma(1 + 3/c) + 3ab^2 \Gamma(1 + 2/c) + 3a^2b \Gamma(1 + 1/c) + a^3. \quad (18) \]

Solution of the three non-linear equations in three unknowns then gives estimates of the parameters \( a, b, \) and \( c, \) given values for the stand average attributes \( \overline{x}, \overline{x^2}, \) and \( \overline{x^3}. \) There is an alternative to this which reduces the problem to the two-parameter Weibull, which was ultimately used due to convergence problems. Since the parameter \( a \) corresponds to the smallest possible dbh in the stand (Bailey and Dell 1973), minimum dbh is predicted and \( a \) is set equal to some function of this value. Then the transformation \( y = (x - a) \) reverts the problem to the two-parameter Weibull system described above.
Two-Parameter Beta. When the first two non-central moments of the beta pdf (1) are estimated by \( \bar{x} \) and \( \bar{x}^2 \), the resulting system is

\[
\bar{x} = \frac{D_{\text{max}}}{\int_{x_{\text{min}}}^{x_{\text{max}}} x f(x; \alpha, \beta) \, dx}, \quad (19)
\]

\[
\bar{x}^2 = \frac{D_{\text{max}}}{\int_{x_{\text{min}}}^{x_{\text{max}}} x^2 f(x; \alpha, \beta) \, dx}, \quad (20)
\]

where \( D_{\text{min}} \) and \( D_{\text{max}} \) are constants observed or predicted for a given set of stand characteristics. The analytic solution to this set of equations is (Johnson and Kotz 1970)

\[
\alpha = \left[ \frac{(\bar{x} - D_{\text{min}})^2}{(D_{\text{max}} - D_{\text{min}})^2} \right] \left( 1 - \frac{\bar{x} - D_{\text{min}}}{D_{\text{max}} - D_{\text{min}}} \right) - \frac{\bar{x} - D_{\text{min}}}{D_{\text{max}} - D_{\text{min}}}, \quad (21)
\]

\[
\alpha + \beta = \left[ \frac{(\bar{x} - D_{\text{min}})^2}{(D_{\text{max}} - D_{\text{min}})^2} \right] \left( 1 - \frac{\bar{x} - D_{\text{min}}}{D_{\text{max}} - D_{\text{min}}} \right) - 1.0 \quad (22)
\]

This is simply the method of moments technique for parameter estimation as used by Burkhart and Strub (1974) when using observed plot data.

Four-Parameter Beta. What is called the four-parameter beta here is the result of including \( D_{\text{max}} \) and \( D_{\text{min}} \) as
parameters to be estimated rather than observed or predicted values for a given set of stand characteristics. The system of equations to be solved then becomes

$$
\bar{x} = \int_{D_{\text{min}}}^{D_{\text{max}}} x f(x; D_{\text{min}}, D_{\text{max}}, \alpha, \beta) \, dx,
$$

$$
\bar{x^2} = \int_{D_{\text{min}}}^{D_{\text{max}}} x^2 f(x; D_{\text{min}}, D_{\text{max}}, \alpha, \beta) \, dx,
$$

$$
\bar{x^3} = \int_{D_{\text{min}}}^{D_{\text{max}}} x^3 f(x; D_{\text{min}}, D_{\text{max}}, \alpha, \beta) \, dx,
$$

$$
\bar{x^4} = \int_{D_{\text{min}}}^{D_{\text{max}}} x^4 f(x; D_{\text{min}}, D_{\text{max}}, \alpha, \beta) \, dx. \quad (23)
$$

The analytic solution to these equations is from Johnson and Kotz (1970). First, estimates of the central moments ($\hat{\mu}_k$) are computed from the estimated non-central moments,

$$
\hat{\mu}_2 = \bar{x^2} - \bar{x}^2
$$

$$
\hat{\mu}_3 = \bar{x^3} - 3\bar{x^2} \bar{x} + 2\bar{x}^3
$$

$$
\hat{\mu}_4 = \bar{x^4} - 4\bar{x^3} \bar{x} + 6\bar{x^2} \bar{x}^2 - 3\bar{x}^4.
$$

Then compute the coefficients of skewness and kurtosis respectively as

$$
\sqrt{\beta_1} = \hat{\mu}_3/(\hat{\mu}_2)^{3/2},
$$

$$
\beta_2 = \hat{\mu}_4/(\hat{\mu}_2)^2.
$$
And let \( r = \frac{6(\beta_2 - \beta_1 - 1)}{(6 + 3\beta_1 - 2\beta_2)} \).

Then solve for the parameters \( \alpha \) and \( \beta \) from

\[
\alpha, \beta = \frac{1}{2} r \left( 1 \pm (r+2) \sqrt{\frac{\beta_1}{(r+2)^2 \beta_2 + 16(r+1)}} \right)^{-1}
\]  

(24)

where

\[ \alpha < \beta \text{ for } \sqrt{\beta_1} > 0 \text{ or } \alpha > \beta \text{ for } \sqrt{\beta_1} < 0. \]

Then compute

\[
(D_{\text{max}} - D_{\text{min}}) = \frac{1}{2} \sqrt{\hat{\mu}_2} \sqrt{(r+2)^2 \beta_2 + 16(r+1)}
\]

(25)

and

\[
D_{\text{min}} = \frac{x - \frac{\alpha(D_{\text{max}} - D_{\text{min}})}{\alpha + \beta}}{\alpha + \beta}
\]

(26)

**Volume-Based Parameter Recovery Models**

**Two-Parameter Weibull.** Three sets of two equations each were investigated for recovering parameters for the two-parameter Weibull. In each set, one equation is the diameter distribution yield equation for total cubic-foot volume outside bark (TVOB),

\[
\text{TVOB} = N \int_0^\infty \left( a_0 + a_1 x^2 h(x) \right) f(x; b, c) \, dx.
\]

(27)

As the second equation in the system average diameter (equation 11) is used in one instance and the estimated second non-central moment in the second case (equation 12).
In both cases the resulting set of two equations in two unknowns was solved by numerical solution techniques.

An alternative is to use the estimated coefficient of variation (equation 14) along with the volume equation. Using this technique, the parameter $c$ is solved for in equation 14 and this value for $c$ is used in equation 27 to solve for the parameter $b$.

**Three-Parameter Weibull.** The set of equations to solve for the three parameters in this model are equations 16, 17 and 27. This resulted in similar convergence problems, therefore, the alternative described for the moment-based model was investigated. The parameter $a$ was estimated as a function of the minimum dbh of the stand then the problem transformed to the two-parameter Weibull by $y = (x - a)$ for the c.v. equation, and the set of equations solved as described above

\[
c.v. = \sqrt{\frac{\Gamma(1+2/c) - \Gamma^2(1+1/c)}{\Gamma(1+1/c)}}
\]  

(28)

\[
TVOB = N \int_{a}^{\infty} (a_0 + a_1 x^2 h(x)) f(x; a, b, c) \, dx.
\]  

(29)

**Two-Parameter Beta.** Two sets of two equations were defined for this pdf. The volume equation in each case was of the form
\[ \text{Dmax} \]

\[ \text{TVOB} = N \int_{\text{Dmin}}^{\text{Dmax}} (a_0 + a_1 x^2 h(x)) f(x; \alpha, \beta) \, dx. \quad (30) \]

The second equation was for $\bar{x}$ (equation 19) in one case and $\bar{x^2}$ (equation 20) in the other. The problem of non-convergence to a solution for $\alpha$ and $\beta$ occurred for both sets of equations.

**Data Sets**

**Virginia Unthinned Plantations**

Stand average models and parameter recovery models were constructed and tested using data from 189 temporary 0.1 acre circular sample plots from plantations in the Piedmont and Coastal Plain of Virginia and the Coastal Plain of Delaware, Maryland and North Carolina (Burkhart, et al. 1972b). These plantations were unthinned, free of disease and insect damage, and relatively free of wildlings. On each plot, dbh was measured for all trees in the 1-inch class and above, and total height was measured for at least one, but usually two trees per one-inch dbh class. These sample height trees were used to develop a regression equation for each plot of the form

\[ \log_{10} H = b_0 + b_1 (1/D) \quad (31) \]

where
\[ H = \text{total tree height}, \]
\[ D = \text{diameter at breast height}. \]

This provided the estimate of height for each tree on the plot for estimation of the volume per tree.

Two trees on each plot were felled and cut into 4-foot sections and measured for dbh, total height, diameter outside bark (ob) at stump and at each 4-foot section up to an approximate 2-inch top diameter (ob). Volume was estimated, by use of Smalian's formula on each section for each tree. These sample tree volumes were used to develop the equation for total cubic-foot volume (ob) described in equation 10.

\[ V = 0.34864 + 0.00232 D^2 H. \]  \hspace{1cm} (32)

where

\[ V = \text{tree volume outside bark (cubic feet)}, \]
\[ D = \text{diameter at breast height (inches)}, \]
\[ H = \text{total tree height (feet)}. \]

The sum of these estimated tree volumes for a given plot, multiplied by ten, is the estimate of total cubic foot (ob) volume per acre.

Six to eight dominant and codominant trees were selected for determining the age (A) of the stand and
estimating the average height of dominant and codominant trees ($H_d$).

Summary statistics for this data set are as follows:

<table>
<thead>
<tr>
<th></th>
<th>minimum</th>
<th>maximum</th>
<th>mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>9</td>
<td>35</td>
<td>16.6</td>
</tr>
<tr>
<td>$H_d$</td>
<td>24.3</td>
<td>87.5</td>
<td>46.3</td>
</tr>
<tr>
<td>$N$</td>
<td>300</td>
<td>2900</td>
<td>751.9</td>
</tr>
<tr>
<td>$\overline{D}$</td>
<td>2.8</td>
<td>11.1</td>
<td>6.1</td>
</tr>
<tr>
<td>$BA$</td>
<td>72.0</td>
<td>277.3</td>
<td>151.8</td>
</tr>
<tr>
<td>TVOB</td>
<td>941.2</td>
<td>9965.2</td>
<td>3183.8</td>
</tr>
</tbody>
</table>

The plot summary statistics used to develop the stand average models described later are age ($A$), average height of dominant and codominant trees ($H_d$), number of trees per acre ($N$), total cubic-foot volume (ob) per acre (TVOB), basal area per acre ($BA$), average dbh of the stand ($\overline{D}$ or $\overline{x}$), average squared dbh ($D^2$ or $x^2$), average cubed dbh ($D^3$ or $x^3$), average fourth power of dbh ($D^4$ or $x^4$), minimum dbh ($D_{\text{min}}$) and maximum dbh ($D_{\text{max}}$).

**Georgia Unthinned Plantations**

This data set was used to test the models constructed using the previously-mentioned data set. The data are from 191 sample plots of varying size, depending on the planting spacing, in old-field loblolly pine plantations in the
Piedmont region of Georgia (Lenhart and Clutter 1971). These plantations were unthinned, relatively free of disease and insect damage, and relatively free of wildlings. The following information was recorded: age, plot dimensions, original spacing of trees, number of trees by 1-inch dbh classes, and average height of dominant and codominant trees. Computation of number of trees per acre then completed the information necessary for use with the parameter recovery models.

**Virginia Thinned Plantation Data**

Data for thinned loblolly pine plantations was obtained from the Virginia Division of Forestry (VDF). The measurements are from 82 0.2-acre permanent plots from old-field plantations in the Virginia Piedmont and Coastal Plain. These 82 plots were measured immediately after thinning and on at least one occasion subsequent to that. The total number of plot measurements was 199.

On each plot, the dbh of each tree was recorded by 1-inch classes with trees in the 1- and 2-inch classes grouped into a single class. Average height of the dominant and codominant trees \( H_d \) was estimated on each plot. This was used to compute site index from Devan's (1979) site index equations. Average dbh and basal area was computed using class midpoint values as follows:
\[
\bar{D} = \frac{\sum_{j=1}^{k} n_j x_j}{N} \quad \text{and} \quad \sum_{j=1}^{k} n_j x_j^2
\]

where

- \( n_j = \text{number of trees in the } j^{th} \text{ dbh class,} \)
- \( x_j = \text{class midpoint of the } j^{th} \text{ dbh class; 1.5, 3.0, 4.0, etc.} \)
- \( D = \text{average dbh (inches),} \)
- \( BA = \text{basal area per acre (ft}^2). \)

Dmax for this data set was established as the upper class limit for the largest dbh class containing at least one observation. Dmin presented somewhat of a problem since on many plots there were observations in one or two classes below and apart from the main body of observations (Figure 1). Since the thinning conducted on most plots was thinning from below, these would seem to be wildlings (rather than planted trees) or small trees of no economic value to those doing the thinning. Dmin was established as the lower class limit of the smallest dbh class contiguous with the main body of observations (e.g., 5.5 and 6.5 inches, respectively, in Figure 1). Then \( \bar{D} \) and BA were recomputed using the new number of trees, for use when observed values of Dmin were used.
Figure 1. Observed diameter distribution of two plots from the Virginia thinned plantation data set.
Stand Average Models

The equations defining these stand average models (Table 1) were developed from the Virginia unthinned plantation data set using linear regression techniques in the Statistical Analysis System (SAS).

Early attempts to use separate and independent equations for D and BA led to severe computational problems in the parameter recovery models. Given BA, then

$$D^2 = \frac{BA \text{ per acre}}{0.005454 N}$$  \hspace{1cm} (33)

Then to compute the coefficient of variation for the Weibull requires

$$c.v. = \frac{(D^2 - \bar{D}^2)^{1/2}}{\bar{D}}$$  \hspace{1cm} (34)

and the parameter solution equations for the beta, equations 21 and 22, also involve a $(D^2 - \bar{D}^2)$ term. The estimates of $D$ and $D^2$ using independent regression equations often led to cross-overs which gave negative variance (i.e., $(D^2 - \bar{D}^2) < 0$). In order to condition this term to be greater than zero the form $\ln(D^2 - \bar{D}^2)$ was used. Basal area computed from the transformation of this model gave reasonably good predictions. ($R^2$ (BA) when predicting $\ln(BA)$ was 0.79).
Table 1. Stand-level prediction equations for average dbh, variance of dbh, basal area, minimum dbh, maximum dbh, total tree height, and total cubic-foot volume (o.b.) developed from the Virginia unthinned plantation data set.

(a) \( \bar{D} = 2.966205 + 0.0545204 \, (H_d) - 0.5241566 \, (A) \, (N/10,000) + 18.10461 \, (H_d/N) \)
\[ R^2 = 0.92 \quad s_y = 0.35 \]

(b) \( \ln (\bar{D}^2 - \bar{D}^2) = 0.3465732 + 0.7167411 \ln (H_d) + 1.401712 \ln (1/A) - 0.4067656 \ln (N) \)
\[ R^2 = 0.26 \quad s_y = 0.39 \]

Use predicted \( \bar{D} \) and \( \ln (\bar{D}^2 - \bar{D}^2) \) to get BA as

(c) \( BA = (\exp (\ln (\bar{D}^2 - \bar{D}^2)) + \bar{D}^2) \, (0.005454154) \, (N) \)
\[ R^2 (BA) = 0.75 \quad s_y \, (BA) = 17.22 \]

(d) \( D_{\min} = 0.02752554 + 0.04735365 \, (H_d) - 0.18440156 \, (A) \, (N/10,000) + 16.75045 \, (H_d/N) \)
\[ R^2 = 0.77 \quad s_y \, = 0.57 \]

(e) \( D_{\max} = 3.6095355 + 0.1016991 \, (H_d) - 0.40091529 \, (A) \, (N/10,000) + 18.28637 \, (H_d/N) \)
\[ R^2 = 0.87 \quad s_y = 0.64 \]

(f) \( \log_{10} (H) = 0.3834532 + 0.8152608 \, \log_{10} (H_d) - 0.2398250 \, (1/A) + 0.0354982 \, (N/10,000) - 0.7598067 \, (1/D) \)
\[ R^2 (H) = 0.92 \quad s_y \, (H) = 3.54 \]

(g) \( \log_{10} (TVOB) = 2.484578 - 5.451979 \, (1/A) + 0.2995715 \, (H_d/A) + 0.00945814 \, (N/100) + 0.00885679 \, (A) \, (\log_{10} (N)) \)
\[ R^2 (TVOB) = 0.92 \quad s_y \, (TVOB) = 360.39 \]
Table 1. Stand-level prediction equations for average dbh, variance of dbh, basal area, minimum dbh, maximum dbh, total tree height, and total cubic-foot volume (o.b.) developed from the Virginia unthinned plantation data set (continued).

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{D}$ = average tree dbh (inches),</td>
<td></td>
</tr>
<tr>
<td>$\bar{D}^2$ = average squared tree dbh (inches²),</td>
<td></td>
</tr>
<tr>
<td>BA = basal area per acre (square feet),</td>
<td></td>
</tr>
<tr>
<td>$D_{\text{min}}$ = minimum dbh (inches),</td>
<td></td>
</tr>
<tr>
<td>$D_{\text{max}}$ = maximum dbh (inches),</td>
<td></td>
</tr>
<tr>
<td>$H$ = total tree height (feet),</td>
<td></td>
</tr>
<tr>
<td>$\text{TVOB}$ = total cubic-foot volume (outside bark) per acre,</td>
<td></td>
</tr>
<tr>
<td>$A$ = stand age (years),</td>
<td></td>
</tr>
<tr>
<td>$H_d$ = average height of dominant and codominant trees in the stand (feet),</td>
<td></td>
</tr>
<tr>
<td>$N$ = number of trees per acre,</td>
<td></td>
</tr>
<tr>
<td>$R^2(Z)$ = pseudo - $R^2$ value for the transformed variable.</td>
<td></td>
</tr>
</tbody>
</table>
Many models were examined for $D_{\text{min}}$ and $D_{\text{max}}$ and all of the better ones gave similar results. So the equational form used for $\bar{D}$ was also fitted for $D_{\text{min}}$ and for $D_{\text{max}}$. This insures that, in the case of the beta, the proper relationships among $D_{\text{min}}$, $\bar{D}$, and $D_{\text{max}}$ hold; specifically, $D_{\text{min}} < \bar{D} < D_{\text{max}}$.

The equation for total tree height is a slight modification of one presented by Bennett and Clutter (1968) in which $\log_{10}(H_d)$ is used instead of their site index (SI) term. The total cubic-foot volume equation is from Burkhart, et al. (1972b). When new coefficients were fitted they were slightly different from the published values probably due to changes in the computer routines used. The new coefficients were used solely for reasons of consistency in fitting all models with the same computer algorithm.

Stand average equations for the Virginia thinned plantation data are presented in Table 2. These equations predict stand attributes given the initial age, value of the attribute at that initial age and the age at which the prediction is to be made. Equation (a) was developed by Clutter and Jones (1980). Lemin (1981) used the Virginia thinned plantation data to solve for the coefficients shown in Table 2. Lemin also constructed and fit the equations to predict average dbh and basal area per acre.
Table 2. Stand-level prediction equations for number of trees per acre, average dbh and basal area per acre developed for the Virginia thinned plantation data set (Lemin 1981).

(a) \[ N_2 = \left[ N_1^{a_1} + 0.75795 \times 10^{-5} (A_2^{a_2} - A_1^{a_2}) \right]^{1/a_1} \]

(b) \[ \overline{D}_2 = \left[ -b_1 + b_2 (SI) + b_1 \left( A_2/A_1 \right) - b_2 (SI) \left( A_2/A_1 \right) \right. \]
\[ + \left. \left( A_2/A_1 \right) (\overline{D}_1)^{b_3} \right]^{1/b_3} \]

(c) \[ BA_2 = \left[ c_1 + c_2 (SI) - (A_2/A_1) (c_1 + c_2 (SI) - BA_1^{c_3}) \right]^{1/c_3} \]

where

- \( A_1, A_2 = \) initial age and age at which prediction is made, respectively,
- \( N_1, N_2 = \) initial number of trees per acre and predicted number of trees at \( A_2 \), respectively,
- \( \overline{D}_1, \overline{D}_2 = \) initial average dbh and predicted average dbh at \( A_2 \), respectively,
- \( BA_1, BA_2 = \) initial basal area and predicted basal area at \( A_2 \), respectively,
- \( SI = \) site index (base age 25),
- \( a_1 = -0.658083 \)
- \( a_2 = 1.7801871 \)
- \( b_1 = 18.504338 \)
- \( b_2 = 0.235287 \)
- \( b_3 = 1.847780 \)
- \( c_1 = -3025.240542 \)
- \( c_2 = -14.148832 \)
- \( c_3 = 1.872085 \)
Procedures

Computer Solution Routines

Computer programs to solve for the pdf parameters as described above were written in FORTRAN Level-G for use on Virginia Tech's IBM 3032 computer (e.g., Appendices I and II). Subroutines from the International Mathematical and Statistical Library (IMSL) were used for evaluating the gamma function (GAMMA), integration (DCADRE), iterative solution of one equation in one unknown (ZBRENT), K-S test statistic (NKSL), chi-square statistics (MDCH) and solution of n equations in n unknowns (ZSYSTEM). Another subroutine, HYBRID, was also tried in the attempts to solve n equations in n unknowns.

Weibull

It was mentioned previously that if the parameter $a$ in the three-parameter Weibull pdf is set equal to some constant value then it is possible to transform the problem to solution of parameters $b$ and $c$ of the two-parameter Weibull. Attempts to solve for all three-parameters of the Weibull simultaneously were unsuccessful, so this solution

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2 I thank Dr. Layne Watson, Dept. of Computer Science, VPI & SU, for making HYBRID available for this research. HYBRID was developed for the Minpack Project at Argonne National Laboratory by B.S. Garbow, K.E. Hillstrom and J.J. More.
technique was used. (This eliminates the need for distinguishing between the two-parameter and three-parameter Weibull since $a$ can be any value greater than or equal to zero. Thus, no further distinction is made in this paper.)

It was then necessary to determine a way to estimate the value of $a$. Bailey and Dell (1973) state that $a$ can be considered the smallest possible diameter in the stand. An approximation to this smallest possible diameter is given by $D_{min}$, the minimum observed diameter on the sample plots. This value is positively biased since $D_{min}$ is always greater than or equal to the true smallest diameter in the stand. Thus the value of $a$ should most likely be $0 < a < D_{min}$.

Five values for $D_{min}$ were selected and a sensitivity analysis conducted. Values for $a$ examined were $0$, $(1/3) D_{min}$, $(1/2) D_{min}$, $(2/3) D_{min}$, $D_{min}$. Using the Virginia unthinned plantation data, observed stand attributes were used to solve for the parameters $b$ and $a$, given $a$, in both the moment-based parameter recovery model and the volume-based model. Solution techniques are described below.

**Moment-based Weibull.** Observed values for $D_{min}$, $\bar{D}$ and $\bar{D}^2$ were used to compute the coefficient of variation

$$c.v. = \frac{(\bar{Y}^2 - \bar{Y}^2)^{1/2}}{\bar{Y}}$$

where $\bar{Y} = \bar{D} - a$,

$$\bar{Y}^2 = \bar{D}^2 - 2\bar{D}a + a^2.$$

Then $a$ was solved, using ZBRENT and GAMMA, in
\[ c.v. = \frac{[\Gamma(1+2/c) - \Gamma^2(1+1/c)]^{1/2}}{\Gamma(1+1/c)} \]

Given the estimated value of \( c \), \( b \) was then solved from

\[ \overline{D} = b \Gamma(1 + 1/c). \]

**Volume-based Weibull.** Observed values of \( D_{\min}, \overline{D}, D^2 \) and \( TVOB \) were used to solve for \( b \) and \( c \), given \( a \), in this model. The coefficient of variation is estimated in the same manner as previously described. Given \( c \), \( b \) was estimated using ZBRENT and DCADRE from

\[
TVOB = N \int_{a}^{\infty} (0.34864 + 0.0232x^2 h(x)) f(x; a, b, c) \, dx
\]

where

- \( x = \) tree dbh,
- \( h(x) = \) the tree height equation \((f)\) in Table 1 with \( h(x) = H \) at some specified value of \( x \). (Note that for a given stand, \( A, H_d, \) and \( N \) are all constants).

Once parameter estimates were made on each of the 189 plots, the recovered diameter distribution given by these parameter values was compared to the observed diameter distribution on each plot by means of the Kolmogorov-Smirnov
goodness-of-fit test statistic (discussed in detail later). These statistics were used to compare the different model formulations. In addition, maximum likelihood estimates of $b$ and $c$ were computed for the 189 plots at each value of $a$. These were then compared with the estimates given by the parameter recovery model using the nonparametric sign test (discussed later) to determine how close the parameter recovery estimates were to maximum likelihood estimates. The K-S test was also performed for the maximum likelihood estimates.

When the feasibility of the model was established and the best value for the parameter $a$ selected, the parameter recovery models were then completed by incorporating the appropriate stand average models to predict values for $\bar{D}$, $\ln(D^2 - \bar{D}^2)$ and TVOB for a given set of stand conditions. Again, using the stand conditions, $A$, $H_d$ and $N$, for each of the 189 plots, parameter values were estimated in the manner described above. The K-S test statistic was used to compare recovered and observed diameter distributions, differences between observed and recovered basal area per acre, and total cubic-foot volume per acre were computed.

The Georgia unthinned plantation data was used as an independent test on the best model from the previous tests and comparisons. The stand characteristics $A$, $H_d$, and $N$
were used to predict stand average attributes using equations from Table 1 and the parameters estimated. In this case however, the Chi-square ($\chi^2$) goodness-of-fit test statistic was used to compare the recovered diameter distribution with the observed since the observations were grouped by 1-inch dbh classes.

To determine whether the Weibull parameter recovery model was better than a previously used parameter prediction model, a comparison was made with the Smalley and Bailey (1974a) model. Maximum likelihood estimates of $a$, $b$ and $c$ were made on each of the 189 plots of the Virginia data set using a computer program developed by Cao. Using these parameter values, the parameter prediction equations were fitted with linear regression procedures in SAS, using the Virginia unthinned data set (Table 3). These equations were then used to predict the parameters on the 191 plots of the Georgia unthinned data set. The $\chi^2$ statistic was computed as before. The parameter recovery model and parameter prediction model were compared using the $\chi^2$ probabilities or computed p-values.

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2 Quang Van Cao. Personal communication.
Table 3. Smalley and Bailey's (1974a) parameter prediction models for estimating $a$, $b$, and $c$ for the three-parameter Weibull pdf with regression coefficients fitted using the Virginia unthinned data set.

| Equation | Model Description | $R^2$ | $s_{y|x}$ |
|----------|-------------------|-------|-----------|
| (a) $a = -3.280636 + 0.1121528 (H_d)$ | $R^2 = 0.59$ | $s_{y|x} = 1.03$ |
| (b) $a + b = -6.510177 - 0.00049015 (N)$ | $+ 3.042222 (\ln H_d) + 1274.7279 (1/N)$ | $R^2 = 0.91$ | $s_{y|x} = 0.37$ |
| (c) $c = 15.00924 - 0.1720857 (A) - 1.281707(\ln N)$ | $R^2 = 0.33$ | $s_{y|x} = 1.05$ |

where $a, b, c = \text{estimated Weibull parameters},$

$A = \text{stand age}$

$H_d = \text{average height of dominant and codominant trees (feet)},$

$N = \text{number of trees per acre surviving at age A}.$
Two-Parameter Beta

The initial step with the beta pdf was to determine what the endpoints of the pdf should be. The observed value of $D_{\min}$ is a positively biased estimate of the smallest diameter in the stand; and $D_{\max}$ is a negatively biased estimate of the largest diameter. A sensitivity analysis was performed with three values for the endpoints of the pdf; $(D_{\min}, D_{\max})$, $(D_{\min}-1.0, D_{\max}+1.0)$ and $(D_{\min}-0.6, D_{\max}+0.6)$. Extending the range 1.0 inches past $D_{\min}$ and $D_{\max}$ was a somewhat arbitrary choice; however, but there was a reason for the choice of 0.6 inches. When fitting a linear regression model the parameters are estimated as

$$\hat{\beta} = (X'X)^{-1} X'y$$

where

- $\hat{\beta}$ = vector of parameters (coefficients),
- $X$ = matrix of observations of the independent variables, say $(A, H_d, N)$,
- $y$ = vector of observations of the dependent variable, $D_{\min}$ or $D_{\max}$.

The prediction interval for some vector of the independent variables, $x_0$, is found by

$$y + t_\alpha s_{y,x} \sqrt{x_0' \left(X'X\right)^{-1} x_0' + 1}.$$  \hspace{1cm} (36)
This provides a means of adjusting $D_{\text{min}}$ (or $D_{\text{max}}$) based on the prediction interval for the particular regression equation for $D_{\text{min}}$ (or $D_{\text{max}}$) being used. For this study endpoints of the pdf were set one standard deviation below $D_{\text{min}}$ and one above $D_{\text{max}}$ (i.e., $t_\alpha = 1.0$). For equations (d) and (e) in Table 1, this value was always in the range (0.57 to 0.65), so for the initial study using observed values of $D_{\text{min}}$ and $D_{\text{max}}$, for which $s_{y,x}$ does not exist, the endpoints were moved 0.6 inches below $D_{\text{min}}$ and 0.6 inches above $D_{\text{max}}$.

**Moment-based Beta.** Observed values for $D_{\text{min}}$, $D_{\text{max}}$, $\bar{D}$ and $\bar{D}^2$ were used to solve for $\alpha$ and $\beta$ as in equations 21 and 22. (Note $\bar{D} = \bar{x}$ and $\bar{D}^2 = \bar{x}^2$). The variable $\bar{D}^2$ was computed from $\ln(\bar{D}^2 - D^2)$ predicted by equation (b) in Table 1.

**Volume-based Beta.** Observed values for $D_{\text{min}}$, $D_{\text{max}}$, $TV_{\text{OB}}$ and $\bar{D}$ or $\bar{D}^2$ were used in two versions of this model. One version was

\[
\bar{D}^2 = \bar{x}^2 = \frac{D_{\text{max}}}{\int_{D_{\text{min}}}^{D_{\text{max}}} x^2 f(\alpha, \beta) \, dx},
\]

\[
TV_{\text{OB}} = N \int_{D_{\text{min}}}^{D_{\text{max}}} (a_0 + a_1 x^2 h(x)) f(\alpha, \beta) \, dx.
\]

The solution of two equations in two unknowns was then tried using ZSYSTEM and HYBRID. Convergence to a solution was achieved in fewer than 50 percent of the cases. The equation

\[
\frac{D_{\text{max}}}{D_{\text{min}}} = \frac{1}{\int_{D_{\text{min}}}^{D_{\text{max}}} x f(\alpha, \beta) \, dx}
\]
was substituted for $D^2$. Convergence to a solution was not found in this case either. As a result only the moment-based model was investigated further.

Once parameter estimates for the moment-based parameter recovery model were made on the 189 plots, the recovered diameter distribution was compared to the observed by means of the K-S statistic. These statistics were then used to compare the different formulations of the model. The model was then completed by incorporating the appropriate stand average prediction equations from Table 1 and used to recover the parameters $\alpha$ and $\beta$ for the 189 plots. The K-S test statistic was again computed for these distributions, along with differences between observed and recovered basal area per acre and total cubic-foot volume per acre.

As with the Weibull pdf, the Georgia unthinned data set was used as an independent test and the $\chi^2$ statistic computed.

To ascertain whether the beta parameter recovery model was better than a previously-used parameter prediction model, a comparison was made using the model proposed by Burkhart and Strub (1974). Moment estimates of the parameters had been computed previously using the moment-based model with observed $D_{\text{min}}$, $D_{\text{max}}$, $\bar{D}$ and $D^2$. Using these
parameter values, regression equations were fitted using the Virginia unthinned data set (Table 4). These equations were then used to predict the parameters on the 191 plots of the Georgia unthinned plantation data set. The $\chi^2$ probabilities used to compare the parameter recovery model with the parameter prediction model.

Four-parameter Beta

The analytic solution of the moment-based four-parameter beta was made using observed stand attributes from the Virginia unthinned data set. The estimates of the non-central moments were computed from the measured diameters using the equations

$$\overline{D} = \frac{1}{N} \sum_{i=1}^{N} D_i$$

$$\overline{D^2} = \frac{1}{N} \sum_{i=1}^{N} D_i^2$$

$$\overline{D^3} = \frac{1}{N} \sum_{i=1}^{N} D_i^3$$

$$\overline{D^4} = \frac{1}{N} \sum_{i=1}^{N} D_i^4$$
Table 4. Burkhart and Strub's (1974) parameter prediction equations for estimating $\alpha$ and $\beta$ for the beta pdf with regression coefficients fitted using the Virginia unthinned plantation data set.

(a) \[ \alpha = 2.596167 - 11.18173 \frac{A}{N} - 1.242511 \frac{A}{(H_d/10000)} \]  
\[ R^2 = 0.04 \quad s_{y \cdot x} = 1.00 \]

(b) \[ \beta = 1.233228 + 0.7523354 \frac{A}{N} + 0.2769411 \frac{N}{(H_d/10000)} \]  
\[ R^2 = 0.11 \quad s_{y \cdot x} = 0.88 \]

where

$\alpha$, $\beta$ = beta parameters,

$A$ = stand age,

$H_d$ = average height of dominant and codominant trees (feet),

$N$ = number of trees per acre surviving at age $A$. 
The parameters $D_{\text{min}}$, $D_{\text{max}}$, $\alpha$ and $\beta$ were then estimated as in equations 24, 25 and 26.

**Kolmogorov-Smirnov Goodness-of-Fit Statistic**

Evaluation of the recovered diameter distribution model was based on its ability to estimate the actual diameter distribution on sample plot data. For the Virginia unthinned plantation data, used to develop the parameter recovery models, the Kolmogorov-Smirnov goodness-of-fit statistic (Massey 1951) was used to compare actual and recovered distributions. The test statistic is

$$k = \text{maximum } | F_0(x) - S_n(x)|$$

(37)

where

- $F_0(x) =$ hypothesized cumulative distribution function defined by the recovered parameters,
- $S_n(x) =$ the actual distribution of the dbh of the $n$ trees on the plot.

The statistic $k$ is used to test the hypothesis

- $H_0: F_0(x) = H(x)$
- $H_1: F_0(x) \neq H(x)$

at some specified significance level, $\alpha$, where $H(x)$ is the unknown population distribution function. The critical value $k_\alpha(n)$ is the value for which $\text{Pr} \{ \text{max } | F_0(x) - S_n(x)| > k_\alpha(n) \} = \alpha$. 


\( k_a(n) = \alpha \). Massey (1951) states that when the parameters have been estimated from the sample, as they have in this case, the distribution of \( K \) is not known, but the effect would be to reduce the critical value \( k_a(N) \). Thus when the computed value \( k \) exceeds the critical value \( k_a(n) \), the difference between the hypothesized distribution and the actual distribution is significant, but an exact probability level cannot be computed. Consequently, no valid statistical tests are possible with these K-S statistics, however, they should prove useful for comparison of the various model formulations on a plot by plot basis. If on a plot, one set of parameter estimates gives a lower value of \( k \) than another then that model can be said to be a better estimate of the distribution for that plot.

Summaries of the K-S statistics and the associated p-values (\( p = \Pr[K > k] \)) were also computed.

**Sign Test**

The sign test is a nonparameteric test for differences between paired observations of a sample (Hollander and Wolfe 1973). The paired observations were the maximum likelihood parameter estimates on a plot and the parameter recovery estimates, say \( X_i \) and \( Y_i \), respectively. Then the difference is
\[ Z_i = Y_i - X_i = M + e_i \]

where

\[ M = \text{population median of } Z, \]
\[ e_i = \text{random variability.} \]

The test is \( H_0: M = 0 \)
\( H_1: M \neq 0 \)

at some level of significance, \( \alpha \).

To compute the test statistic \( B^* \), first compute

\[ S_i = \begin{cases} 1 & \text{if } Z > 0 \\ 0 & \text{if } Z < 0 \end{cases} \]

and

\[ B = \sum_{i=1}^{n} S_i. \]

Then

\[ B^* = \frac{B-(n/2)}{(n/4)^{1/2}}. \]

\( B^* \) is distributed asymptotically \( N(0, 1) \) when \( H_0 \) is true.

The p-value is determined from normal distribution tables as

\[ \alpha(B^*) = 2 \Pr(Z > b^*) \]

where \( b^* \) is the observed value of the random variable \( B^* \). The hypothesis \( H_0 \) is rejected if \( \alpha(B^*) < \alpha \).
Chi-square Goodness-of-Fit Statistic

A different evaluation criterion was needed for the Georgia unthinned plantation data and the Virginia thinned plantation data because the diameter measurements were grouped by 1-inch classes. Plots for which the expected frequency was greater than 1.0 in 6 or more classes were used to compute the $\chi^2$ statistic as:

$$\chi^2 = \sum_{i=1}^{k} \left[ \frac{(E_i - O_i)^2}{E_i} \right],$$

where

$$E_i = N \int_{D_{i-1}}^{D_i} f(x; \theta) \, dx,$$

the expected frequency of trees in the $i^{th}$ dbh class,

$$O_i = \text{the observed frequency in the } i^{th} \text{ dbh class},$$

$$k = \text{number of dbh classes}.$$  

The hypothesis to be tested is

$$H_0: F_0(x) = H(x)$$

$$H_1: F_0(x) \neq H(x)$$

at some significance level $\alpha$. Again no valid test is possible in this instance since often fewer than five observations were made in one or more classes. The IMSL subroutine MDCH was used to compute the $\chi^2$ values and
$\Pr[ x^2 \geq x^2 ]$ for each plot, where $x^2$ is the computed $x^2$ value.
RESULTS AND DISCUSSION

Weibull Parameter Recovery Model

When observed values $D_{\text{min}}$, $\bar{D}$, and $\bar{D}^2$ (and $\text{TVOB}$ for the volume-based model) were used to solve for the parameters $b$ and $c$ for each of the five values for $a$, the algorithm converged to a solution on all 189 plots of the Virginia unthinned data set for both the moment-based and volume-based models. For all five moment-based models there were six plots for which the computed p-value for the K-S statistic was less than 0.05. For the volume-based models, the number of p-values less than 0.05 was as follows:

- 14 for $a = 0$,
- 13 for $a = 1/3 \, D_{\text{min}}$,
- 11 for $a = 1/2 \, D_{\text{min}}$,
- 10 for $a = 2/3 \, D_{\text{min}}$,
- 12 for $a = D_{\text{min}}$.

(Remember however that the true p-value is less than 0.05). Plots where the computed p-value were low had observed diameter distributions similar to those in Figures 2 and 3 which are difficult, if not impossible, to model with a continuous, unimodal pdf such as the Weibull or beta. The
Figure 2. Observed diameter distribution and the distribution produced by the moment-based Weibull parameter recovery model.
Figure 3. Observed diameter distribution and the distribution produced by the moment-based beta parameter recovery model.
K-S goodness-of-fit statistics indicated that when given observed values for Dmin, and $\overline{D^2}$ (and TVOB for volume-based models), both the moment-based and volume-based models give good results for any of the five values for the parameter $\alpha$.

Summaries of the K-S statistic, k, and the associated p-values are provided for the moment-based models in Table 5 and for the volume-based model in Table 6. In Table 5 the model with $\alpha = D_{\text{min}}$ was the poorest with the highest mean value for k and the lowest mean p-value. The other models were better and had very similar values with $\alpha = (1/2) D_{\text{min}}$ being slightly better than the others. In Table 6, the model using $\alpha = (2/3) D_{\text{min}}$ was the best followed closely by $\alpha = (1/2) D_{\text{min}}$. Again $\alpha = D_{\text{min}}$ gave the poorest results and $\alpha = 0$ was only slightly better.

A final plot-by-plot comparison of $\alpha = (1/2) D_{\text{min}}$ with the alternative values is summarized in Table 7. It shows that it had the lower valued K-S statistic on a larger number of plots than each of the others for both the moment-based and volume-based models.

The moment-based and volume-based Weibull parameter recovery models with the parameter $\alpha$ set equal to $(1/2) D_{\text{min}}$ was used for further investigation and refinement of the models.
Table 5. Summary statistics for the 189 values of the K-S statistic, \( k \), and the computed p-value\(^1\) using the moment-based Weibull parameter recovery model\(^2\) at each of five values for the parameter \( a \).

<table>
<thead>
<tr>
<th>Value of ( a )</th>
<th>( k ) minimum</th>
<th>( k ) maximum</th>
<th>( k ) mean</th>
<th>( k ) minimum</th>
<th>( k ) maximum</th>
<th>( k ) mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0476</td>
<td>0.1857</td>
<td>0.0973</td>
<td>0.0044</td>
<td>0.9961</td>
<td>0.5531</td>
</tr>
<tr>
<td>(1/3)Dmin</td>
<td>0.0479</td>
<td>0.1847</td>
<td>0.0963</td>
<td>0.0084</td>
<td>0.9989</td>
<td>0.5591</td>
</tr>
<tr>
<td>(1/2)Dmin</td>
<td>0.0435</td>
<td>0.1860</td>
<td>0.0959</td>
<td>0.0134</td>
<td>0.9998</td>
<td>0.5591</td>
</tr>
<tr>
<td>(2/3)Dmin</td>
<td>0.0443</td>
<td>0.1889</td>
<td>0.0961</td>
<td>0.0185</td>
<td>1.0000</td>
<td>0.5532</td>
</tr>
<tr>
<td>Dmin</td>
<td>0.0535</td>
<td>0.1956</td>
<td>0.1059</td>
<td>0.0053</td>
<td>0.9975</td>
<td>0.4573</td>
</tr>
</tbody>
</table>

\(^1\)p-value = Pr[\( K > k \)] where \( k = \max | F_0(x) - S_n(x) | \).

\(^2\)Stand average attributes were the observed values on each plot.
Table 6. Summary statistics for the 189 values of the K-S statistic, $k$, and the computed p-value\(^1\) using the volume-based Weibull parameter recovery model\(^2\) at each of five values for the parameter $\alpha$.

<table>
<thead>
<tr>
<th>Value of $\alpha$</th>
<th>minimum $k$</th>
<th>maximum $k$</th>
<th>mean $k$</th>
<th>minimum p-value</th>
<th>maximum p-value</th>
<th>mean p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0543</td>
<td>0.2141</td>
<td>0.1088</td>
<td>0.0022</td>
<td>0.9881</td>
<td>0.4448</td>
</tr>
<tr>
<td>$(1/3)D_{\text{min}}$</td>
<td>0.0520</td>
<td>0.2072</td>
<td>0.1069</td>
<td>0.0043</td>
<td>0.9762</td>
<td>0.4563</td>
</tr>
<tr>
<td>$(1/2)D_{\text{min}}$</td>
<td>0.0475</td>
<td>0.2056</td>
<td>0.1063</td>
<td>0.0048</td>
<td>0.9828</td>
<td>0.4591</td>
</tr>
<tr>
<td>$(2/3)D_{\text{min}}$</td>
<td>0.0452</td>
<td>0.2035</td>
<td>0.1061</td>
<td>0.0029</td>
<td>0.9902</td>
<td>0.4582</td>
</tr>
<tr>
<td>$D_{\text{min}}$</td>
<td>0.0544</td>
<td>0.1999</td>
<td>0.1091</td>
<td>0.0007</td>
<td>0.9948</td>
<td>0.4330</td>
</tr>
</tbody>
</table>

\(^1\)p-value = $\Pr[K > k]$ where $k = \max | F_o(x) - S_n(x) |$.

\(^2\)Stand average attributes were the observed values on each plot.
Table 7. Summary of a plot-by-plot comparison of the K-S statistics for moment-based and volume-based Weibull parameter recovery models with $\alpha = (1/2)D_{\text{min}}$ vs. the other four values of $\alpha$. Summary statistic is the number of plots on which that model had the smaller valued K-S statistic.

<table>
<thead>
<tr>
<th>Moment-based Weibull</th>
<th>$\alpha = (1/2)D_{\text{min}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>91</td>
</tr>
<tr>
<td>$(1/3)D_{\text{min}}$</td>
<td>93</td>
</tr>
<tr>
<td>$(2/3)D_{\text{min}}$</td>
<td>88</td>
</tr>
<tr>
<td>$D_{\text{min}}$</td>
<td>51</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Volume-based Weibull</th>
<th>$\alpha = (1/2)D_{\text{min}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>83</td>
</tr>
<tr>
<td>$(1/3)D_{\text{min}}$</td>
<td>93</td>
</tr>
<tr>
<td>$(2/3)D_{\text{min}}$</td>
<td>87</td>
</tr>
<tr>
<td>$D_{\text{min}}$</td>
<td>78</td>
</tr>
</tbody>
</table>

1Stand average attributes were the observed values on each plot.
As a standard against which to compare the parameter recovery models, the maximum likelihood estimates of \( b \) and \( c \) were computed for each of the five values of \( a \) on each of the 189 plots. The number of plots which had computed p-values less than 0.05 were as follows:

- 10 for \( a = 0 \),
- 5 for \( a = (1/3) \) Dmin,
- 3 for \( a = (1/2) \) Dmin,
- 3 for \( a = (2/3) \) Dmin,
- 9 for \( a = D\text{min} \).

The same trend shows here that did for the volume-based model. Summary values for the K-S statistic and p-value are shown in Table 8. The moment-based Weibull has lower mean K-S values and higher mean p-values than maximum likelihood for all values of \( a \). The mean values for the volume-based model are more nearly equal those of maximum likelihood. It appears that the moment-based and volume-based model each do equally well across the range of values for \( a \) while the maximum likelihood does well for \( a = (2/3) \) Dmin and \( (1/2) \) Dmin, but poorer for the extreme values of \( a \). The results for maximum likelihood also support the conclusion that \( a = (1/2) \) Dmin is a reasonable choice for the value of the parameter \( a \).

Table 9 summarizes the differences between parameter recovery estimates and maximum likelihood estimates of \( b \) and
Table 8. Summary statistics for the 189 values of the K-S statistic, \( k \), and the computed p-value using the maximum likelihood estimates of \( b \) and \( c \) at each of the five values for the parameter \( a \).

<table>
<thead>
<tr>
<th>Value of ( a )</th>
<th>minimum</th>
<th>( k ) maximum</th>
<th>mean</th>
<th>minimum</th>
<th>( k ) maximum</th>
<th>mean</th>
<th>p-value maximum</th>
<th>mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0408</td>
<td>0.2592</td>
<td>0.1215</td>
<td>0.0115</td>
<td>0.8375</td>
<td>0.3162</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1/3)Dmin</td>
<td>0.0445</td>
<td>0.1940</td>
<td>0.1100</td>
<td>0.0200</td>
<td>0.9259</td>
<td>0.4131</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1/2)Dmin</td>
<td>0.0466</td>
<td>0.1836</td>
<td>0.1031</td>
<td>0.0267</td>
<td>0.9864</td>
<td>0.4827</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2/3)Dmin</td>
<td>0.0452</td>
<td>0.1908</td>
<td>0.1016</td>
<td>0.0175</td>
<td>0.9894</td>
<td>0.4995</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dmin</td>
<td>0.0519</td>
<td>0.2302</td>
<td>0.1098</td>
<td>0.0031</td>
<td>0.9947</td>
<td>0.4246</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 9. Mean values of the maximum likelihood estimates and summary statistics of the differences between maximum likelihood estimates and parameter recovery estimates of the Weibull parameters $b$ and $c$. Difference is PRE minus MLE.

<table>
<thead>
<tr>
<th>Value of $a$</th>
<th>$b$</th>
<th></th>
<th></th>
<th></th>
<th>$c$</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mean</td>
<td>mean difference</td>
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<td>mean</td>
<td>mean difference</td>
<td>standard deviation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Moment-based</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>6.542</td>
<td>0.094</td>
<td>0.057</td>
<td>4.329</td>
<td>1.241</td>
<td>1.068</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1/3)Dmin</td>
<td>5.482</td>
<td>0.072</td>
<td>0.054</td>
<td>3.769</td>
<td>0.757</td>
<td>0.792</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1/2)Dmin</td>
<td>4.951</td>
<td>0.059</td>
<td>0.041</td>
<td>3.486</td>
<td>0.526</td>
<td>0.667</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2/3)Dmin</td>
<td>4.408</td>
<td>0.052</td>
<td>0.031</td>
<td>3.124</td>
<td>0.375</td>
<td>0.564</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dmin</td>
<td>3.282</td>
<td>0.036</td>
<td>0.038</td>
<td>2.309</td>
<td>0.186</td>
<td>0.392</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Volume-based</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>0</td>
<td>6.542</td>
<td>0.122</td>
<td>0.128</td>
<td>4.329</td>
<td>1.241</td>
<td>1.068</td>
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<tr>
<td>(1/3)Dmin</td>
<td>5.482</td>
<td>0.100</td>
<td>0.126</td>
<td>3.769</td>
<td>0.757</td>
<td>0.792</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1/2)Dmin</td>
<td>4.951</td>
<td>0.087</td>
<td>0.118</td>
<td>3.486</td>
<td>0.526</td>
<td>0.667</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2/3)Dmin</td>
<td>4.408</td>
<td>0.081</td>
<td>0.112</td>
<td>3.124</td>
<td>0.375</td>
<td>0.564</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dmin</td>
<td>3.282</td>
<td>0.063</td>
<td>0.118</td>
<td>2.309</td>
<td>0.186</td>
<td>0.392</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
e. In all instances, except one, the mean parameter recovery estimates exceeded the mean maximum likelihood estimates. The sign test conducted to determine if significant differences exist in the median differences resulted in significant differences in all cases, with p-values < 0.001. The parameter recovery estimates appear to be positively biased, but the amount of this bias is small for the middle values of \( a \). Notice that the same differences (Table 9) for \( a \) result because it is the same value for both the moment-based and volume-based models, being recovered from the same equation in each instance.

The parameter recovery models compare very well with maximum likelihood under the constraint of a constant valued \( a \), and should produce good approximations of the diameter distribution in cases where an underlying unimodal, continuous distribution can be assumed.

To complete the moment-based parameter recovery model stand average equations were incorporated to predict \( \text{Dmin} \), \( \bar{D} \) and \( \ln(D^2-\bar{D}^2) \). The computer program in Appendix I also includes construction of a table of number of trees, basal area per acre and total cubic-foot volume (ob) per acre using the recovered parameters. The volume-based program was similar but contained an additional stand average equation to predict total cubic-foot volume per acre. In
both cases, the parameter $a$ was set equal to one-half the predicted value for $D_{\text{min}}$.

Using the stand conditions $A, H_d, N$ on each sample plot in the Virginia unthinned plantation data set, stand average attributes were predicted and the parameters recovered. The differences between observed values for total basal area per acre and total cubic volume per acre and those values obtained by solving the equations

$$BA = 0.005454 N \int_a^\infty x^2 f(x; a, b, c) \, dx,$$

$$TVOB = N \int_a^\infty (0.34864 + 0.0232 x^2 h(x)) f(x; a, b, c) \, dx,$$

were computed using the recovered parameters (Table 10). Both models predict very well for total basal area per acre. As expected the volume-based model predicted total cubic volume per acre better than the moment-based, since volume was a stand attribute used to solve for the parameters. When K-S statistics were computed, 73 out of 189 had a computed p-value, $Pr[K > k]$, less than 0.05 for the moment-based model and 79 for the volume-based model. The summary statistics for the K-S statistic and computed p-values are in Table 11.

Again noting that this cannot be considered a valid statistical test, it appears that the models can
Table 10. Mean observed values and the mean differences and mean absolute differences between observed and recovered basal area per acre and total cubic-foot volume per acre for the moment-based and volume-based Weibull parameter recovery models.\(^1\)

<table>
<thead>
<tr>
<th>Model</th>
<th>Basal Area (ft(^2)/acre)</th>
<th>Total Volume (ft(^3)/acre)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Moment-based Weibull</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean difference</td>
<td>+0.6</td>
<td>+46.8</td>
</tr>
<tr>
<td>Mean absolute difference</td>
<td>12.8</td>
<td>273.7</td>
</tr>
<tr>
<td><strong>Volume-based Weibull</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean difference</td>
<td>-0.6</td>
<td>+18.2</td>
</tr>
<tr>
<td>Mean absolute difference</td>
<td>12.5</td>
<td>263.5</td>
</tr>
<tr>
<td>Mean observed value</td>
<td>151.8</td>
<td>3183.8</td>
</tr>
</tbody>
</table>

\(^1\)Both models used predicted values of stand attributes and \(a = (1/2)d_{\text{min}}\).
Table 11. Summary statistics for the 189 values of the K-S statistic, k, and the computed p-value using the completed moment-based and volume-based Weibull parameter recovery models.¹

<table>
<thead>
<tr>
<th>Model</th>
<th>k minimum</th>
<th>k maximum</th>
<th>k mean</th>
<th>p-value minimum</th>
<th>p-value maximum</th>
<th>p-value mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moment-based</td>
<td>0.0510</td>
<td>0.4010</td>
<td>0.1576</td>
<td>0.0000</td>
<td>0.9697</td>
<td>0.2459</td>
</tr>
<tr>
<td>Volume-based</td>
<td>0.0555</td>
<td>0.4029</td>
<td>0.1578</td>
<td>0.0000</td>
<td>0.9908</td>
<td>0.2405</td>
</tr>
</tbody>
</table>

¹Dmin was predicted by a stand average prediction equation as were the other stand attributes and α = (1/2)Dmin.
successfully describe the diameter distribution in 60 percent or more of the cases. Successfully predicting the diameter distribution depends first on the assumption of an underlying continuous, unimodal pdf, and also on the appropriateness of the stand average equations. Violation of both these requirements are evident in the plots with low K-S values. For plots with higher K-S values, the parameters recovered for both the moment-based model and the volume-based were not very much different and the graph of the recovered diameter distributions were quite similar (e.g., Figure 4). Using the same stand average prediction equations in both models, the same values for \( a \) and \( c \) will result, with only parameter \( b \) being different. (The \( b \) parameter is defined by \( \bar{D} \) in the moment-based model and TVOB in the volume-based model.) It seems reasonable that in case of discrepancies between the two models, the moment-based model should give a better description of the diameter distribution since the estimated moments of the pdf are used to estimate the parameters and also because the variable (dbh) used to develop the model is a measured variable. In contrast, the volume-based model parameter \( b \) is recovered from an equation in which the tree height (\( h(x) \)) must be predicted; and the stand attribute TVOB is an estimated variable from tree measurements and regressions of other variables.
Figure 4. Observed diameter distribution and the distribution produced by the moment-based (a) and volume-based (b) Weibull parameter recovery models using predicted stand average attributes for two plots from the Virginia unthinned data set.
Figure 4. Observed diameter distribution and the distribution produced by the moment-based (a) and volume-based (b) Weibull parameter recovery models using predicted stand average attributes for two plots from the Virginia unthinned data set (continued).
Since these two models produce similar results over the range of stand conditions though, the decision on which model to use depends upon the objective: diameter distribution or distribution of volume by dbh.

Georgia Unthinned Plantation Data

As an additional test the Weibull parameter recovery models were used to estimate the parameters for the 191 plots on the Georgia unthinned plantation data set and the results compared to those of the Smalley and Bailey (1974a) parameter prediction model. The stand average models fitted by regression using the Virginia unthinned plantation data (Table 1) were used to predict stand average attributes, given A, H_d, N, in the parameter recovery models. The parameter prediction equations fitted using the Virginia data set (Table 3) were used in Smalley and Bailey's model. The \( \chi^2 \) statistic was computed, rather than the K-S, since tree dbh's were grouped by 1-inch classes.

For the moment-based model 30 plots, out of the 132 for which a \( \chi^2 \) statistic was computed, had p-values less than 0.05 (27.7%), and 29 out of 134 (21.6%) less than 0.05 for the volume-based model, and 49 out of 129 (38.0%) for Smalley and Bailey's model. The volume-based model had the highest average p-value, 0.3842, followed by the moment-
based model with 0.3498 and Smalley and Bailey's model with 0.3351.

Differences between observed and predicted total basal area per acre and total cubic volume per acre are shown in Table 12. The volume-based model and Smalley and Bailey's model have smaller differences than the moment-based model. Mean absolute differences are not greatly different. Note that the mean absolute differences compare favorably with $s_{y|x}$ for the stand average regression models (Table 1).

The results of this comparison indicate that for this data set the volume-based model is better overall than the others. It had the smallest percentage of $\chi^2$ values less than 0.05 as well as more accurate predictions for total basal area and total volume, with the average p-value only slightly lower than the others. The moment-based model predicts the diameter distribution nearly as well as the volume-based model but has a positive bias for stand total basal area and volume. Overall the parameter recovery models predict the diameter distribution as well as the parameter prediction model.
Table 12. Mean observed values and the mean differences and mean absolute differences between observed and predicted\(^1\) basal area per acre and total cubic-foot volume per acre for the moment-based and volume-based Weibull parameter recovery model and Smalley and Bailey's parameter prediction model using the Georgia unthinned plantation data set.

<table>
<thead>
<tr>
<th>Model</th>
<th>Basal Area (ft(^2)/acre)</th>
<th>Total Volume (ft(^3)/acre)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moment-based Weibull</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean difference</td>
<td>+5.3</td>
<td>+120.2</td>
</tr>
<tr>
<td>Mean absolute difference</td>
<td>14.9</td>
<td>368.9</td>
</tr>
<tr>
<td>Volume-based Weibull</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean difference</td>
<td>-2.2</td>
<td>-55.3</td>
</tr>
<tr>
<td>Mean absolute difference</td>
<td>16.1</td>
<td>360.3</td>
</tr>
<tr>
<td>Smalley and Bailey</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean difference</td>
<td>-1.5</td>
<td>-56.2</td>
</tr>
<tr>
<td>Mean absolute difference</td>
<td>16.8</td>
<td>395.3</td>
</tr>
<tr>
<td>Mean observed values</td>
<td>151.3</td>
<td>3002.5</td>
</tr>
</tbody>
</table>

\(^1\)Predicted values were computed by integration of the pdf with the estimated parameters of the specified model.
Beta Parameter Recovery Model

Two-parameter Beta

Observed values of Dmin, Dmax, D and D^2 were used to solve for parameters α and β of the moment-based and volume-based beta distribution. Three values for the endpoints of the distribution examined were (Dmin, Dmax), (Dmin-0.6, Dmax+0.6), and (Dmin-1.0, Dmax+1.0). The two volume-based models (p. 43) required numerical solution of two equations in two unknowns. For these models convergence to a solution for α and β occurred for fewer than 35 percent of the plots for both models, and often the solutions achieved were illogical for these data. Perturbation of the stand attribute values gave no improvement in convergence. It appears that these functions are much too complex to be solved by the two algorithms tried here. More sophisticated (and expensive) search techniques would be required to consistently achieve a solution for the parameters.

The remainder of this section deals with the moment-based beta parameter recovery model. This model achieved a solution for all 189 plots for each set of endpoint values. Once parameter estimates were obtained the K-S goodness-of-fit statistic was computed for each plot. There were two plots where the computed p-value was less than 0.05 for endpoints (Dmin, Dmax) and 3 plots for the others. Again
the plots with low p-values were those with distributions difficult to model with a continuous, unimodal pdf (Figures 2 and 3). Therefore it was concluded that the three models can recover the diameter distribution, given actual values of $D_{\min}$, $D_{\max}$, $\bar{D}$ and $\bar{D}^2$.

In this case no comparison was made with maximum likelihood estimates of $\alpha$ and $\beta$. The parameter recovery model is basically the method of moments technique, and Strub (1972) found little difference between the two.

Summaries of the 189 K-S statistics and p-values are shown in Table 13. The middle value has the smallest average K-S value and the highest mean p-value. The range $(D_{\min}, D_{\max})$, however, is almost as good.

A plot-by-plot comparison of the K-S statistics for $(D_{\min}-0.6, D_{\max}+0.6)$ with the others is shown in Table 14. This confirms the conclusion that $(D_{\min}-0.6, D_{\max}+0.6)$ is the best range of the three examined and was used in the further investigation and testing of this model.

To complete the moment-based beta parameter recovery model, stand average equations were incorporated to predict $D_{\min}$, $D_{\max}$, $\bar{D}$, and $\ln(D^2 - \bar{D}^2)$ (Table 1). The computer program in Appendix II also includes construction of a table of number of trees per acre, basal area per acre and total cubic-foot volume (ob) per acre by 1-inch dbh classes using
Table 13. Summary statistics for the 189 values of the K-S statistic, \( k \), and the computed p-value\(^1\) using the moment-based beta parameter recovery model at each of three values for the limits of integration.\(^2\)

<table>
<thead>
<tr>
<th>Endpoints of pdf</th>
<th>( k ) minimum</th>
<th>( k ) maximum</th>
<th>( k ) mean</th>
<th>p-value minimum</th>
<th>p-value maximum</th>
<th>p-value mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>(( D_{\text{min}}-1.0, D_{\text{max}}+1.0 ))</td>
<td>0.0492</td>
<td>0.2173</td>
<td>0.0953</td>
<td>0.0071</td>
<td>0.9988</td>
<td>0.5617</td>
</tr>
<tr>
<td>(( D_{\text{min}}-0.6, D_{\text{max}}+0.6 ))</td>
<td>0.0420</td>
<td>0.1812</td>
<td>0.0917</td>
<td>0.0161</td>
<td>0.9999</td>
<td>0.5965</td>
</tr>
<tr>
<td>(( D_{\text{min}}, D_{\text{max}} ))</td>
<td>0.0438</td>
<td>0.1771</td>
<td>0.0918</td>
<td>0.0197</td>
<td>0.9997</td>
<td>0.5946</td>
</tr>
</tbody>
</table>

\(^1\)p-value = \( \Pr[K > k] \) where \( k = \max | F_0(x) - S_n(x) | \).

\(^2\)Stand average attributes were the observed values on each plot.
Table 14. Summary of a plot-by-plot comparison of the K-S statistics for the moment-based beta with pdf endpoints \((D_{\text{min}}-0.6, D_{\text{max}}+0.6)\) vs. the two other sets of endpoints. Summary statistic is the number of plots on which that model had the smaller valued K-S statistic.

<table>
<thead>
<tr>
<th>Endpoints</th>
<th>((D_{\text{min}}-0.6, D_{\text{max}}+0.6))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((D_{\text{min}}-1.0, D_{\text{max}}+1.0))</td>
<td>88</td>
</tr>
<tr>
<td>((D_{\text{min}}, D_{\text{max}}))</td>
<td>74</td>
</tr>
</tbody>
</table>
the recovered parameters. It should be noted that in the completed model the quantity subtracted from Dmin and added to Dmax was not a constant 0.6, but rather the value computed by equation 36 (p. 42).

Using the stand conditions A, H_d, N on each sample plot in the Virginia unthinned plantation data set, stand average attributes Dmin, Dmax, \( \overline{D} \) and \( \overline{D}^2 \) were predicted by the regression equations and the parameters recovered. Table 15 presents a summary of the results. There were 69 plots which had a computed p-value less than 0.05. These results are comparable to those of the Weibull (Tables 10 and 11). In the case of the beta, not only must \( \overline{D} \) and \( \overline{D}^2 \) be accurately predicted, but also Dmin and Dmax. In cases where stand average attributes were accurately predicted, the recovered and observed diameter distributions were very similar (Figure 5).

**Georgia Unthinned Plantation Data.** As an additional test the moment-based beta parameter recovery model was used to estimate the parameters for the 191 plots in the Georgia unthinned plantation data set. The results were compared to the Burkhart and Strub (1973) parameter prediction model. The parameter prediction equations for Dmin, Dmax, \( a \) and \( \beta \) (Tables 1 and 3) were fitted by regression using the Virginia unthinned plantation data set. Stand average
Table 15. Summary statistics for the 189 values of the K-S statistic, k, and the computed p-values; mean observed values, and the mean difference and mean absolute difference for observed and recovered total basal area per acre and total cubic-foot volume per acre using the moment-based beta parameter recovery model.¹

<table>
<thead>
<tr>
<th></th>
<th>Basal Area (ft²/acre)</th>
<th>Total Volume (ft³/acre)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mean difference</strong></td>
<td>+1.3</td>
<td>+62.0</td>
</tr>
<tr>
<td><strong>Mean absolute difference</strong></td>
<td>12.9</td>
<td>276.4</td>
</tr>
<tr>
<td><strong>Mean observed value</strong></td>
<td>151.8</td>
<td>3183.8</td>
</tr>
</tbody>
</table>

¹Used predicted values of stand average attributes.
Figure 5. Observed diameter distribution and the distribution produced by the moment-based beta using predicted stand average attributes.
models fitted with the Virginia unthinned data set (Table 1) were used for the parameter recovery model. The parameters were estimated with each model and the recovered distribution compared to the observed with the $\chi^2$ statistic.

The parameter recovery model resulted in 26 out of 127 (20.5%) computed p-values less than 0.05 and the parameter prediction model had 22 out of 109 (20.2%). The mean p-values were 0.3591 and 0.3649 respectively. The differences between observed and predicted stand basal area and total cubic volume are shown in Table 16. Overall both models appear to give very similar results and it is difficult from this study to say one is superior to the other.

Four-Parameter Beta

A computer program was constructed to solve the four-parameter beta according to the solution algorithm given previously. Then for observed values of $\bar{D}$ and $D^2$, and the values of $D^3$ and $D^6$ computed from the dbh measurements, in the Virginia unthinned plantation data set, the four parameters were estimated.

The algorithm produced a solution for less than 25 percent of the plots. In an instance where a solution was achieved the range (Dmax-Dmin) was often much greater than the observed range.
Table 16. Mean observed values and the mean differences and mean absolute differences between observed and predicted\(^1\) basal area per acre and total cubic-foot volume per acre for the moment-based beta parameter recovery model and Burkhart and Strub's (1974) parameter prediction model using the Georgia unthinned plantation data set.

<table>
<thead>
<tr>
<th>Model</th>
<th>Basal Area (ft(^2)/acre)</th>
<th>Total Volume (ft(^3)/acre)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moment-based beta</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean difference</td>
<td>+6.0</td>
<td>+134.3</td>
</tr>
<tr>
<td>Mean absolute difference</td>
<td>15.0</td>
<td>372.2</td>
</tr>
<tr>
<td>Burkhart and Strub</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean difference</td>
<td>+5.0</td>
<td>+104.3</td>
</tr>
<tr>
<td>Mean absolute difference</td>
<td>14.3</td>
<td>339.9</td>
</tr>
<tr>
<td>Mean observed values</td>
<td>151.3</td>
<td>3002.5</td>
</tr>
</tbody>
</table>

\(^1\)Predicted values were computed by integration of the pdf with the estimated parameters of the specified model.
The failure to achieve solutions for the parameters was at least partially due to rounding off the dbh measurements to the nearest 0.1 inch. This caused much imprecision in the higher powers of D which resulted in such things as \( \sqrt{-x} \) terms in the computational procedure. For example, if the measured dbh was recorded as 6.6 inches, the actual "true" dbh is said to be in the range 6.55 \( \leq \) 6.6 \( < \) 6.65. If the "true" dbh were 6.550 inches then \( D^b \) is 1840.6245 and if 6.649 then \( D^b \) is 1954.4535.

Since tree stems are irregular, such precise measurements are not feasible. A four-parameter recovery model appears to be impractical for predicting diameter distributions.

Comparison of Weibull and Beta P.R.M.

The previous analyses indicate that the moment-based Weibull, volume-based Weibull and moment-based beta are very similar in their ability to predict diameter distributions in loblolly pine plantations. The K-S and \( \chi^2 \) statistic summaries are nearly equal. The beta generally results in fewer values less than 0.05, but the volume-based model generally has better predictions for stand total basal area and total cubic volume.
Another comparison to be made is the computer time necessary to estimate parameters. The times below are those required to recover the parameter and to compute a table of number of trees per acre, basal area per acre, total cubic-foot volume per acre by 1-inch dbh classes (e.g., Appendices I and II) for the 189 plots of the Virginia unthinned plantation data set. The moment-based Weibull required 0.15 minutes of CPU time, the volume-based Weibull 0.55 minutes, and the beta 0.24 minutes on the IBM 3032 at Virginia Tech. The numerical integration needed to solve for the parameter $b$ in the TVOB equation probably accounts for the increased time required for that model.

**Virginia Thinned Plantation Study**

The moment-based Weibull and beta parameter recovery models have been shown to be good predictors of the diameter distribution in unthinned plantations. Thinned stands, however, may present special problems when trying to predict the diameter distribution since the very act of thinning alters the distribution; often in ways that make it difficult to model with a continuous unimodal pdf. Bimodal, truncated or discontinuous distributions may result (Figure 1). The parameter recovery models would seem to be a more efficacious means of predicting the distribution than
parameter prediction models, since they are estimated using predicted stand attributes $\bar{D}$ and BA which recent studies have shown can be predicted very well in thinned stands (Clutter and Jones 1980, Lemin 1981). For each of the 199 plot measurements in the Virginia thinned plantation data set observed values of stand attributes were used to recover the parameters for the moment-based Weibull and beta models.

**Weibull**

Estimates of the parameters $b$ and $a$ were made for each of three values of $a$: $(1/3)D_{\text{min}}$, $(1/2)D_{\text{min}}$, and $(2/3)D_{\text{min}}$. (Note the description of $D_{\text{min}}$ on p. 29). Solutions were achieved for all 199 plots for each value of $a$. Because data were grouped by 1-inch classes, the $x^2$ statistic was used to compare recovered and observed diameter distributions. For the 140 plots for which $x^2$ could be computed, there were 12 plots (8.6%) with p-values less than 0.05 for $a = (1/3)D_{\text{min}}$ and 11 out of 143 (7.7%) for $(1/2)D_{\text{min}}$ and 9 out of 144 (6.3%) for $(1/3)D_{\text{min}}$. As in the unthinned plantations the plots where the p-values were low were most often those exhibiting the problems mentioned above (Figure 6, (a) and (b)). Problems on other plots may be due to improper or poorly estimated relationship between $\bar{D}$ and $D^2$. It appears that for those plots for which the underlying
assumptions of unimodality and continuity can be safely made, the Weibull parameter recovery model approximates the diameter distribution reasonably well (Figure 6, (c) and (d)).

To complete the moment-based Weibull parameter recovery model for thinned stands equations to predict \( N_2, \) \( BA_2 \) and \( \overline{D}_2 \) at \( A_2 \) given \( N_1, \) \( BA_1, \) and \( \overline{D}_1, \) at \( A_1 \) (Table 2) were incorporated into the model. The value of \( a \) was \((1/2)D_{\text{min}}\) where \( D_{\text{min}} \) was predicted from equation (d) in Table 1. If a plot had three measurements then the observed values at the first measurement were used to predict stand average attributes at the second measurement and the parameters recovered. Then the observed values at the second measurement were used to predict stand attributes at the third and the parameters recovered at that point. There were 116 predictions possible. Of those, it was impossible to recover parameters on 16 plots because of the problem that \((\overline{D}^2 < \overline{D}^2)\) due to the use of the two independent regression equations (Table 2). For the remaining 100 predictions where the parameters were estimated the mean difference in basal area between observed and predicted was 1.49 ft/acre per plot and mean absolute difference was 7.07 ft/acre per plot. The \( \chi^2 \) statistic was then computed and 30 out of 81 computed (37.0%) had p-values less than 0.05.
Figure 6. Observed diameter distribution and that produced by the moment-based Weibull parameter recovery model for the Virginia thinned plantation data.
Figure 6. Observed diameter distribution and that produced by the moment-based Weibull parameter recovery model for the Virginia thinned plantation data (continued).
The moment-based Weibull appears to have potential for use in thinned plantations of loblolly pine. The major problem to be overcome is the problem of the relationship between predicted $\overline{D}$ and $\overline{D^2}$. Equations to condition the two so that ($\overline{D^2} > \overline{D^2}$) must be developed.

**Beta**

When observed values of $D_{min}$, $D_{max}$ $D$ and $BA$ were used to recover parameters $\alpha$ and $\beta$, for the three sets of endpoints used previously solutions were achieved on all 199 plots. The $\chi^2$ statistic had a p-value less than 0.05 on 12 out of 144 plots (8.3%) for $(D_{min}, D_{max})$ and 9 out of 150 (6.0%) for the others. All estimated values of $\alpha$ and $\beta$ were within the range expected based on the previous work. These results are comparable to those of the Weibull parameter recovery model.

However when stand average prediction equations for $D_{min}$, $D_{max}$, $\overline{D}$ and $BA$ were incorporated into the model severe problems occurred in addition to the problems of ($\overline{D^2} < \overline{D^2}$), estimates of $\alpha$ and $\beta$ less than 1.0 and greater than 12.0 resulted. Values less than 1.0 give J-shaped distributions and those greater than 12.0 give extremely peaked distributions. Problems also occurred with $\overline{D}$ being extremely close to $D_{min}$ or $D_{max}$.
In an attempt to alleviate this problem, new coefficients were computed for the Dmin and Dmax equations using the thinned data set values. This helped some (Figure 7, (a) and (b)). There were plots where $\overline{D^2} < \overline{D^2}$, and plots with extremely large values of $\alpha$ and $\beta$ (Figure 7, (c) and (d)). When $\chi^2$ values were computed 34 out of 77 (44.2%) plots had p-values less than 0.05. Use of the moment-based parameter recovery model to predict diameter distributions of thinned stands will be dependent on constructing good prediction equations for Dmin and Dmax, as well as for predicting $(\overline{D^2} - \overline{D^2})$. Predicting Dmin and Dmax will be especially difficult since thinning alters these values abruptly.
Figure 7. Observed diameter distribution and that produced by the moment-based beta parameter recovery model for the Virginia thinned plantation data.
Figure 7. Observed diameter distribution and that produced by the moment-based beta parameter recovery model for the Virginia thinned plantation data (continued).
CONCLUSIONS

Three diameter distribution parameter recovery models were constructed to predict the diameter distribution of unthinned stands of loblolly pine, given predicted values of stand average attributes. These models compare well with previously-published parameter prediction models and have the advantage of compatibility with the stand average models. For the moment-based model, the stand total basal area per acre resulting from integration of the pdf equals that from the stand average model as does volume in the volume-based model. Thus the numerical consistency between the stand average and diameter distribution models was achieved. With these recovered parameters, stand attributes can be partitioned by dbh classes and other stand attributes such as dry weight or merchantable volume may be computed.

For thinned stands, the moment-based Weibull shows promise of being a good model for predicting the diameter distribution when certain problems are resolved. Stand average equations to predict compatible estimates of average dbh and basal area are needed to insure a solution for the parameter recovery model. The moment-based beta, however,
will present much more difficulty in thinned stands than unthinned. Most critical will be specification of Dmin and Dmax immediately after thinning and prediction during the intervals between thinnings (and harvest). The compatible estimates of average dbh and basal area are also necessary for this model. The major obstacles to use of parameter recovery models in thinned plantations are inadequate stand average models. Further research will be needed to solve these problems.

The diameter distribution parameter recovery technique described in this paper has been shown to provide an accurate description of the diameter distribution when accurate predictions of stand average attributes are made and the assumption of an underlying unimodal, continuous pdf is met. It provides the forest manager with an alternative means of predicting the diameter distribution, with the additional advantage of numerical consistency between the stand average values and the diameter distribution values.
LITERATURE CITED


APPENDICES
APPENDIX I

Program listing and sample output for the moment-based Weibull parameter recovery model using predicted stand average attributes.
APENDIX I (continued)
**WELDUL FORTAN AL 11/21/81 16:59 WAREAL**

```fortran
* 0.00334*V2 = *M11/100*0
C COMPUTE PRECISED UAVE, USUM, AND IVU01.
CALL UMIN(AC,MC,U1M1,U1M1)
CALL AVGUL(AC,MC,U1AVG,U1AVG)
CALL UMAX(AC,MC,U2AVG,U2AVG,HA)
C
A = UMIN * 0.5
IF(ALEL = 0.5) GO TO 31
UAVG = 0.0
GO TO 32
31 XAVG = UAVE - A
32 XAVG = XAVG - 1.0*UAVE + A*TA
XAVG = XAVG*XAVE
P(A) = 1.0/(XAVG**A)**XAVE
C PIC(n) IS USED TO SEARCH FOR THE VALUES OF THE FUNCTION OF INTEREST.
C SUCH THAT ONE IS POSITIVE AND ONE IS NEGATIVE.
C THEN ZERO INTERSECTS TO THE SOLUTION.
CALL PIC(N1,IC2)
IF(C1.LE.0.0,GO TO 22,IC1,IC0,IC.0) GO TO 1000
M11 = 30
CALL ZERENF(U0,0.001,U1,L1,L1,N1,ER1)
C L = L2
D = XAVE / GAMMA(1.0+L1/0.6)
C
WHILE(L1,14) PIC10,AL,AL,11
19 FORMATT(15)' /I/ STAND CHARACTERISTICS: *X, *PLUT. NUM. *=S", 15/1/ WS: 009710
$X2, *AUL = S1,5.1, J2, X, NUMBER OF TRIES PER ACRE = *F, J2, 5.2,2.2, WS: 009220
$XX = X**1.5, IC2 + 1.0, X, USE FOR 2.5, JS1: 009710
$X = X**1.5, IC2 + 1.0, X, USE FOR 2.5, JS1: 009710
$X = X**1.5, IC2 + 1.0, X, USE FOR 2.5, JS1: 009710
WRITE(6,16) XAVE
GO TO 14
16 FORMATT(15)'/I/ PRECISED STAND AVERAGE: XYZ: 5A, AVERAGE XYZ: S1,5.1, IC2, B,
$X = S1,5.1, IC2, B, B, B, B, USE FOR 2.5, JS1: 009710
$X = S1,5.1, IC2, B, B, B, B, USE FOR 2.5, JS1: 009710
$X = S1,5.1, IC2, B, B, B, B, USE FOR 2.5, JS1: 009710
39 FORMATT(15)'/I/ WELDUL PARAMETER ESTIMATE: *S, A1, A = *F, 1,0, 1A, $X = *F, 1,0, 1A,
WRITE(6,19) XAVE
49 FORMATT(15)'/I/ XYZ PRECISED UAVE, USUM, B, AND VOLUME *,
$UL, KIDSUM, *=S1, 1, IC2, B, USE FOR 2.5, JS1: 009710
$UL, KIDSUM, *=S1, 1, IC2, B, USE FOR 2.5, JS1: 009710
$UL, KIDSUM, *=S1, 1, IC2, B, USE FOR 2.5, JS1: 009710
C COMPUTE THE PRECISED DISTRIBUTIONS.
USUM = 0.0
USUM = 0.0
USUM = 0.0
1 = A + 0.5
UL = A + 0.01
U0 = 1 + 0.2
10 UCL = 1.1*(UL**L1)**X2 = L1**XU0**L1**X2
*IF ( (UL, XAVE,XAVE,2) ) GO TO 53
```

APPENDIX I (continued)
WEIBULL FORTRAN 11/21/81 16:19 WAREW 272 REC'S 01:50 PRINTED 11/21/81 16:19

WALL = N + U * (WUL / UUL) + UU * 0.01

WALL = N + U + UU * 0.01

USUM = USUM + WALL

USUM = USUM + U + UU

VSUM = VSUM + WALL

09 FORMAT (10E16.6,2F15.6,6F16.6)

I = 1 + 1

UU = I + 0.5

UL = I - 0.5

UU = 10

08 FORMAT (10E16.6,2F15.6,6F16.6)

1000 CONTINUE

STOP

END

THIS SUBROUTINE COMPUTES PU GIVEN SI OR SI GIVEN PU.

EQUATIONS ARE FROM DEAN; JAMES S. SEYFV. BASE-AGE INVARIENT

PULPWOOD SIE INDEX CURVES FOR CEDAR IN WASHINGTON STATE.

UNPUBLISHED M.S.

1: SI = EXPI/0.0018 + 1.1 / AGE - 0.0411

2: SI = 0.0018 + (11.0 / AGE) - 0.0411

$ = 0.0018 + (11.0 / AGE) + (1.0 / AGE)

IF (M <= 0.0) GO TO 2

IF (M <= 0.0) GO TO 2

IF (M <= 0.0) GO TO 2

SI = EXPI

RETURN

2: Y = (ALU SI) * CL + C2

HU = EXPI

RETURN

END

PRESENT THE MINIMUM DUN OF THE STAND.

SUBROUTINE UMINU(NU,ND,NUMIN)

REAL NU

UMIN = 0.215253 + 0.0453098444 - 0.19480164 - 4.0 + 4.0

RETURN

END

PRESENT THE AVERAGE DUN OF THE STAND.

SUBROUTINE AVGDUN(NU,ND,AVG)

REAL NU

AVG = 2.96049 + 0.02520 + 0.0642136 - 0.19480164 - 4.0 + 4.0

RETURN

END

APPENDIX I (continued)
END

C
C PREDICT THE BASAL AREA PER ACRE.
C SUBROUTINE HAPRE(AUL,NU,NI,UAVG,DAVG,BA)
C REAL NI
C AULF = 0.340532 + 0.716711*ALUG(NU) + 1.401712/ACE - 0.404762
C $ = AULG(NIT)
C DIF = EXPL(AULF)$
C UAVG = DIF + UAVG
C DA = UAVG $ NI $ 0.945415E-2
C RETURN
C END

C
C PREDICT TREE HEIGHT FOR STAND CONSTANTS A, Nc, AND Nt, AND THE
C VARIABLE DNH, USED IN THE VOLUME COMPUTATION.
C REAL FUNCTION IRT(HYDN)
C REAL4 NI
C COMMON/AREA2/HIUN
C RETURN = 9.0
C TEST1 = HIUN - 0.750007 $ DBH
C IF (TEST1LE0.05) RETURN
C RETURN = 10.0 + TEST
C RETURN
C END

C
C THIS SUBROUTINE SELECTS TWO STARTING VALUES FOR THE PARAMETERS
C SUCH THAT ONE FOR EACH OF THE VALUES IS UP OPPOSITE
C ALGORITHM SIUN.
C SUBROUTINE PEST(A,W)
C REAL A
C C1 = 3.0
C C2 = 3.0
C TEST1 = FUNCTION
C TEST2 = TEST1
C IF (TEST1LE0.0 AND TEST2LE1.0) GO TO 100
C TEST1 = 0.0
C TEST2 = TEST1
C RETURN
C 100 C1 = C1 - 0.5
C TEST1 = FUNCTION
C TEST2 = TEST1
C GO TO 10
C C2 = C2 + 1.0
C TEST1 = FUNCTION
C TEST2 = TEST1
C GO TO 10
C 100 A = C1
C B = C2
C RETURN
C END

C
C SOLVE THE COEFFICIENT OF VARIATION EQUATION FOR THE PARAMETER C.
C REAL FUNCTION FCHITA(
C APPENDIX 1 (continued)
REAL A
CUMUL=1/1+UNI-UAVG,A,B,C,PAR(2)
GAM = GAMMA(1.0+L/A)
GAM2 = GAM*GAM
GAMX = GAM*(1.0+2.0/A)
FLP1 = (SUM([GAM2-GAM])/GAM)-PAR(1)
RETURN
END

REAL FUNCTION CUMUL(UNI)
REAL UIN,UNI
CUMUL=1/1+UNI-UAVG,A,B,C,PAR(2)
CUMUL = 0.0
XX = 1.0
XY = L * ALINT(UWH-A1/B)
IF(XX.LT.4.0) RETURN
IF(XX.LT.10.0) XX = 30
XX = CAPT([UWH-A1/B])
30 CUMUL = 0.00000000000000001*XY+CUMUL+CUMUL+BU1+BU1+1.0+1.0+XX
RETURN
END

REAL FUNCTION VOL1(UNI)
REAL UNI
VOL1 = 1.0
XX = 1.0
XY = L * ALINT(UWH-A1/B)
IF(XX.LT.4.0) RETURN
IF(XX.LT.10.0) XX = 30
XX = CAPT([UWH-A1/B])
30 VOL1 = 0.3400+1.0232+UWH+UWH+UWH+UWH+UWH+UWH+1.0+1.0+XX
RETURN
END

//G0.53YIN UD *
1 15.0 42.3 600.0
2 15.0 41.8 550.0
3 12.0 35.2 530.0
4 12.0 35.0 600.0
5 19.0 26.1 620.0
6 17.0 40.8 750.0
7 25.0 61.0 450.0

//
STAND CHARACTERISTICS:

| PLOT NO. = | 1 |
| NUM. T. PER ACRE = | 800.0 |
| AVG. D. (FT) = | 42.3 |
| SITE INDEX = | 67.2 |

PREDICTED STAND AVERAGE ATTRIBUTES:

<table>
<thead>
<tr>
<th>ATTRIBUTE</th>
<th>VALUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>AVERAGE DBH (INCHES)</td>
<td>9.6</td>
</tr>
<tr>
<td>BASAL AREA PER ACRE (SQFT)</td>
<td>143.4</td>
</tr>
<tr>
<td>MINIMUM DBH (INCHES)</td>
<td>2.9</td>
</tr>
</tbody>
</table>

WEIBULL PARAMETER ESTIMATES:

<table>
<thead>
<tr>
<th>PARAMETER</th>
<th>VALUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.471325</td>
</tr>
<tr>
<td>B</td>
<td>4.571327</td>
</tr>
<tr>
<td>C</td>
<td>3.761973</td>
</tr>
</tbody>
</table>

PREDICTED DIAMETER, BASAL AREA AND VOLUME DISTRIBUTIONS:

<table>
<thead>
<tr>
<th>DBH</th>
<th>NUM. TREES</th>
<th>BASAL AREA</th>
<th>TOT. CUFT. VOLUME (UB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>2.90</td>
<td>0.08</td>
<td>0.08</td>
<td>1.89</td>
</tr>
<tr>
<td>3.05</td>
<td>1.03</td>
<td>1.03</td>
<td>1.03</td>
</tr>
<tr>
<td>4.38</td>
<td>10.59</td>
<td>10.59</td>
<td>195.52</td>
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<tr>
<td>5.17</td>
<td>30.08</td>
<td>30.08</td>
<td>352.49</td>
</tr>
<tr>
<td>6.78</td>
<td>46.08</td>
<td>46.08</td>
<td>564.85</td>
</tr>
<tr>
<td>14.06</td>
<td>37.77</td>
<td>37.77</td>
<td>708.25</td>
</tr>
<tr>
<td>4.76</td>
<td>14.16</td>
<td>14.16</td>
<td>268.94</td>
</tr>
<tr>
<td>4.76</td>
<td>2.09</td>
<td>2.09</td>
<td>40.18</td>
</tr>
<tr>
<td>0.19</td>
<td>0.10</td>
<td>0.10</td>
<td>1.93</td>
</tr>
</tbody>
</table>

SUM | 600.0 | 143.40 | 2669.38 |
STAND CHARACTERISTICS

PLOT NO. = 2

AGE = 12.0

NUMBER OF TREES PER ACRE = 730.0

AVG. HI. OF UDBM. AND JUDBM. (FEET) = 47.8

SITE INDEX (BASE AGE 25) = 74.0

PREDICTED STAND AVERAGE ATTRIBUTES:

AVERAGE DBH (INCHES) = 6.2

BASAL AREA PER ACRE (SUFT.) = 159.0

MINIMUM DBH (INCHES) = 3.4

WIEBULL PARAMETER ESTIMATES:

A = 1.716824  B = 4.940282  C = 3.828483

PREDICTED DIAMETER, BASAL AREA AND VOLUME DISTRIBUTIONS

<table>
<thead>
<tr>
<th>DBH</th>
<th>NUM. TREES</th>
<th>BASAL AREA</th>
<th>TDF. CUFT.</th>
<th>VOLUME (UB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.63</td>
<td>0.02</td>
<td>0.44</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>13.99</td>
<td>0.77</td>
<td>15.81</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>62.18</td>
<td>5.71</td>
<td>113.53</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>143.90</td>
<td>20.08</td>
<td>400.31</td>
<td></td>
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<td>6</td>
<td>207.61</td>
<td>41.01</td>
<td>827.77</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>184.33</td>
<td>48.09</td>
<td>1000.48</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>92.16</td>
<td>31.52</td>
<td>624.36</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>23.76</td>
<td>9.74</td>
<td>204.68</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>2.35</td>
<td>1.23</td>
<td>26.24</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>0.08</td>
<td>0.05</td>
<td>1.14</td>
<td></td>
</tr>
</tbody>
</table>

SUM  | 730.0      | 159.01     | 3245.25    |
STAND CHARACTERISTICS:

PLT NO. = 3
AGE = 12.0
NUMBER OF TREES PER ACRE = 630.0
AVG. H. OF DOM. AND CODOM. (FLET) = 53.2
SITE INDEX (BASE AGE 25) = 68.0

PREDICTED STAND AVERAGE ATTRIBUTES:
AVERAGE DBH (INCHES) = 53.3
BASE AREA PER ACRE (SFT) = 102.6
MINIMUM DBH (INCHES) = 2.6

WEIBULL PARAMETER ESTIMATES:
A = 1.255354
B = 4.469822
C = 3.780189

PREDICTED диАМЕTER, BASE AREA AND VOLUME DISTRIBUTIONS

<table>
<thead>
<tr>
<th>DBH</th>
<th>NO. TREES</th>
<th>BASE AREA</th>
<th>TOT. CUFT. VOLUME (UB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>4.41</td>
<td>0.12</td>
<td>2.56</td>
</tr>
<tr>
<td>3</td>
<td>37.05</td>
<td>2.02</td>
<td>33.97</td>
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<tr>
<td>4</td>
<td>113.80</td>
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</tr>
<tr>
<td>5</td>
<td>189.22</td>
<td>26.09</td>
<td>397.45</td>
</tr>
<tr>
<td>6</td>
<td>176.42</td>
<td>34.74</td>
<td>520.45</td>
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<tr>
<td>7</td>
<td>160.27</td>
<td>22.96</td>
<td>345.08</td>
</tr>
<tr>
<td>8</td>
<td>134.06</td>
<td>6.19</td>
<td>95.45</td>
</tr>
<tr>
<td>9</td>
<td>1.41</td>
<td>0.59</td>
<td>9.23</td>
</tr>
<tr>
<td>10</td>
<td>0.03</td>
<td>0.02</td>
<td>0.25</td>
</tr>
<tr>
<td>SUM</td>
<td>630.0</td>
<td>102.65</td>
<td>1572.99</td>
</tr>
</tbody>
</table>
STAND CHARACTERISTICS:

PLOT NO. = 4
AUE = 26.0
NUMBER OF TREES PER ACRE = 600.0
AVG. HI. OF UOM. AND CAUMUM. (FEET) = 58.8
SITE INDEX (BASE AGE 25) = 62.0

PREDICTED STAND AVERAGE ATTRIBUTES:

AVERAGE DBH (INCHES) = 6.9
BASAL AREA PER ACRE (SF/FT) = 176.4
MINIMUM DBH (INCHES) = 4.1

WEIBULL PARAMETER ESTIMATES:  A = 2.069371  B = 5.297381  C = 3.839191

PREDICTED DIAMETER, BASE AREA AND VOLUME DISTRIBUTIONS

<table>
<thead>
<tr>
<th>DBH</th>
<th>NO. TREES</th>
<th>BASAL AREA</th>
<th>TOT. CUFT. VOLUME (DB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.04</td>
<td>0.00</td>
<td>0.03</td>
</tr>
<tr>
<td>4</td>
<td>4.29</td>
<td>0.24</td>
<td>5.50</td>
</tr>
<tr>
<td>5</td>
<td>61.13</td>
<td>11.40</td>
<td>256.60</td>
</tr>
<tr>
<td>6</td>
<td>147.68</td>
<td>29.39</td>
<td>676.92</td>
</tr>
<tr>
<td>7</td>
<td>174.19</td>
<td>47.95</td>
<td>1127.62</td>
</tr>
<tr>
<td>8</td>
<td>129.29</td>
<td>48.15</td>
<td>1150.77</td>
</tr>
<tr>
<td>9</td>
<td>63.55</td>
<td>27.23</td>
<td>667.26</td>
</tr>
<tr>
<td>10</td>
<td>15.16</td>
<td>4.03</td>
<td>197.08</td>
</tr>
<tr>
<td>11</td>
<td>1.04</td>
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<td>29.92</td>
</tr>
<tr>
<td>12</td>
<td>0.07</td>
<td>0.05</td>
<td>1.30</td>
</tr>
</tbody>
</table>

SUM  600.0  176.39  4171.36
STAND CHARACTERISTICS:

PLOT NO. = 5  
AGE = 19.0  
NUMBER OF TREES PER ACRE = 620.0  
AVG. H. OF DDM. AND GDLOM. (FEET) = 56.7  
SITE INDEX (BASE AGE 25) = 10.7

PREDICTED STAND AVERAGE ATTRIBUTES:

AVERAGE DBH (INCHES) = 7.1  
BASE AREA PER ACRE (SQUAFT) = 177.1  
MINIMUM DBH (INCHES) = 4.3

WEIBULL PARAMETER ESTIMATES:  
A = 2.137418  
B = 5.478882  
C = 3.914329

PREDICTED DIAMETER, BASAL AREA AND VOLUME DISTRIBUTIONS

<table>
<thead>
<tr>
<th>DBH</th>
<th>NO. TREES</th>
<th>BASAL AREA</th>
<th>DIT. CUFT. VOLUME (MB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.02</td>
<td>0.00</td>
<td>0.01</td>
</tr>
<tr>
<td>3</td>
<td>19.95</td>
<td>2.58</td>
<td>41.40</td>
</tr>
<tr>
<td>6</td>
<td>123.36</td>
<td>24.61</td>
<td>563.98</td>
</tr>
<tr>
<td>7</td>
<td>164.37</td>
<td>44.10</td>
<td>1028.72</td>
</tr>
<tr>
<td>8</td>
<td>144.22</td>
<td>50.02</td>
<td>1165.67</td>
</tr>
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<td>9</td>
<td>77.00</td>
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<td>22.40</td>
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<td>3.10</td>
<td>1.98</td>
<td>45.65</td>
</tr>
<tr>
<td>12</td>
<td>0.17</td>
<td>0.13</td>
<td>3.26</td>
</tr>
</tbody>
</table>

SUM  | 620.0     | 177.05     | 4168.36                |
**STAND CHARACTERISTICS:**

- PLOT NO. = 0
- AGE = 17.0
- NUMBER OF TREES PER ACRE = 750.0
- AVG. HT. OF DOM. AND CODOM. (FEET) = 48.8
- SITE INDEX (BASE AGE 25) = 67.9

**PREDICTED STAND AVERAGE ATTRIBUTES:**

- AVERAGE DBH (INCHES) = 6.1
- BASAL AREA PER ACRE (SFT.) = 160.9
- MINIMUM DBH (INCHES) = 3.4

**WEIBULL PARAMETER ESTIMATES:**

- \( A = 1.7120448 \)
- \( B = 4.886240 \)
- \( C = 3.795600 \)

<table>
<thead>
<tr>
<th>DBH</th>
<th>NO. TREES</th>
<th>BASAL AREA</th>
<th>TOT. CUFT. VOLUME (08)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
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<td>0.02</td>
<td>0.20</td>
</tr>
<tr>
<td>3</td>
<td>15.32</td>
<td>0.04</td>
<td>17.60</td>
</tr>
<tr>
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<td>0.07</td>
<td>0.04</td>
<td>0.48</td>
</tr>
</tbody>
</table>

**SUM**

- 750.0
- 160.95
- 3352.18
STAND CHARACTERISTICS:

- PLOT NO. = 1
- AGE = 60
- NUMBER OF TREES PER ACRE = 950.0
- AVERAGE HEIGHT OF GROWN TREES = 61.6
- SITE INDEX (BASE AGE = 50) = 91.6

PREDICTED STAND AVERAGE ATTRIBUTES:
- AVERAGE DIAMETER = 6.2
- BASAL AREA PER ACRE (SUM) = 111.4
- MINIMUM DBH THRESHOLD = 7.3

WETBULL PARAMETER ESTIMATES:
- $a = 2.038040$
- $b = 0.145344$
- $c = 9.322282$

<table>
<thead>
<tr>
<th>DBH</th>
<th>NO. TREES</th>
<th>BASAL AREA</th>
<th>FULL LUI</th>
<th>VOLUME (LB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
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<td>0.01</td>
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<tr>
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<td>0.32</td>
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<td>9.98</td>
<td>93.25</td>
<td></td>
</tr>
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<td>10.30</td>
<td>119.13</td>
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<td>4.97</td>
<td>74.02</td>
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<td>0.48</td>
<td>0.48</td>
<td>11.48</td>
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</tr>
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</table>

SUM: 950.0  111.40  4350.19
APPENDIX II

Program listing and sample output for the moment based beta parameter recovery model using predicted stand average attributes.
1000 CONTINUE
STOP
END

THIS SUBROUTINE COMPUTES HD GIVEN SI OR SI GIVEN MU.

EQUATION IS FROM DEVAN, JAMES B. 1979. BASE-AGE INVARIANT

POLYMORPHIC SITE INDEX CURVES FOR Loblolly Pine. M.S. THESIS.

VPI AND SU.

SUBROUTINE SI(HD, HAGE, SM)

IMPLICIT REAL*4 (A-H, I-Z)

INTEGER*2 SM

INFEGK#2 MU*H/5/S/31*/

CL = UNXPl9501700 + [1.1000/AGE - 0.0490])

C2 = -.2379*U00*(L1-1.000)*19.900*U00*(6.04000 - 1.0000/AGE)

D = 56.761200 + (0.00000600 - (1.0000/AGE4(1.0000/AGE))

IF(SM.EQ.5) GO TO 2

YU = (ULXPMU) - C2/CL

SI = ULXPMU
RETURN

2Y = (ULXG(SI)*C1) + CL

MU = DEEPY
RETURN
END

COMPUTE THE ONE STANDARD DEVIATION PREDICTION INTERVAL FOR DMIN

AND UMAX.

SUBROUTINE RNGADJ(AUJL, ADJU)

IMPLICIT REAL*4 (A-H, I-Z)

REAL*4 X1, X2

DIMENSION XL(4,4), ADJ(4,4), XG(4)

COMMON/AREA/AAGE, HD, SIG, NI

COMMON/AREA/AAGE, HD, SIG, NI

BET01110
BET01120
BET01130
BET01140
BET01150
BET01160
BET01170
BET01180
BET01190
BET01200
BET01210
BET01220
BET01230
BET01240
BET01250
BET01260
BET01270
BET01280
BET01290
BET01300
BET01310
BET01320
BET01330
BET01340
BET01350
BET01360
BET01370
BET01380
BET01390
BET01400
BET01410
BET01420
BET01430
BET01440
BET01450
BET01460
BET01470
BET01480
BET01490
BET01500
BET01510
BET01520
BET01530
BET01540
BET01550
BET01560
BET01570
BET01580
BET01590
BET01600
BET01610
BET01620
BET01630
BET01640
BET01650

APPENDIX II (continued)
C XU(J) = 1.000
C XU(J) = 0
C XU(J) = AGE*NT/1000.000
C ADJ(J) = HU/NT
C SYX = 0.5726494*U0
C SYX = 0.0662595*U0
C
C DU 100 J = 1
C ADJ(J) = XPRN(J,J) + ADJ(J)
C DU 200 K = Z
C ADJ(J) = ADJ(J) + XPRN(K,J) * XU(J)
C
C 200 CONTINUE
C 100 CONTINUE
C
C ADJ = ADJ(J) * XU(J)
C ADJ = ADJ(J) * XU(J)
C ADJ = ADJ(J) * XU(J)
C RETURN
C
C C PREDICT THE AVERAGE UHU.
C SUBROUTINE AVGUA(DAVG)
C IMPLICIT REAL*8 (A-H,O-Z)
C REAL*4 NT
C COMMON/AREA/AGE*HU,S1N)
C UAVE = 0.04540*HU + 0.034204*HU*HU = 0.52415*HU
C $18.1046044*HU/NT
C RETURN
C
C C C PREDICT THE BASAL AREA AND U2AVG.
C SUBROUTINE BAPKE(DAVG,U2AVG,BA)
C IMPLICIT REAL*8 (A-H,O-Z)
C REAL*4 NT
C COMMON/AREA/AGE*HU,S1N)
C UAVE = 0.04540*HU + 0.034204*HU*HU = 0.52415*HU
C $18.1046044*HU/NT
C RETURN
C
C C C PREDICT THE MINIMUM UHU OF THE STAND.
C SUBROUTINE UMIN(U2AVG,ULIM)
C IMPLICIT REAL*8 (A-H,O-Z)
C REAL*4 NT
C
APPENDIX II (continued)
C C PREDICT THE MAXIMUM DBH OF THE STAND.
SUBROUTINE UMMAX(DAVG,UMAX)
[IMPLICIT REAL*8 (A-H,O-Z)]
REAL*8 NT
COMMON/ARACL/AGE,NU,S1,NT
UMAX = 5.6095359500*10016691590*10011045*AGE*NT +
13.62969786690*AGE*NT
RETURN
END

C FUNCTION FOR PREDICTING NUMBER OF TACSE OF DBH CLASS.
DOUBLE PRECISION FUNCTION UUGIST(DBH)
[IMPLICIT REAL*8 (A-H,O-Z)]
REAL*8 NT
COMMON/AKAC3/A,B,UL,DU
TERM1 = (DBH-UL)/(UU-UL)
UUGIST = (UGAMMA(A+B)/UU-UL)*UGAMMA(A)*UGAMMA(B))**TERM1**
(A-1.000)**(1.000-TERM1)**(B-1.000)
RETURN
END

C FUNCTION FOR PREDICTING BASEL AREA PER ACRE BY DBH CLASS.
DOUBLE PRECISION FUNCTION BUSTI(DBH)
[IMPLICIT REAL*8 (A-H,O-Z)]
REAL*8 NT
COMMON/AKAC3/A,B,UL,DU
TERM1 = (DBH-UL)/(UU-UL)
BUSTI = (DBH*DBH*UGAMMA(A+B)/UU-UL)*UGAMMA(A)*UGAMMA(B))**TERM1**
(A-1.000)**(1.000-TERM1)**(B-1.000)
RETURN
END

C FUNCTION FOR PREDICTING VOLUME BY DBH CLASS.
DOUBLE PRECISION FUNCTION VUIST(DBH)
[IMPLICIT REAL*8 (A-H,O-Z)]
REAL*8 NT
COMMON/AKAC3/A,B,UL,DU
EXTERNAL TERM1
TERM1 = (DBH-UL)/(UU-UL)
VUIST = (DBH*DBH*DBH*DBH+DBH**REENT(DBH))*UGAMMA(A+B)/
((UU-UL)*UGAMMA(A)*UGAMMA(B))**TERM1**(A-1.000)**(1.000-TERM1)**
(B-1.000)
RETURN
END

C
C
C PALEIIT TREE HEIGHT GIVEN STAND CONSTANTS A, H0, AND N1, AND THE
C VARIABLE DBH, USED IN THE VOLUME FUNCTION.
C
C DOUBLE PRECISION FUNCTION TREEH(TUBM)
C IMPLICIT REAL*8 (A-H,O-Z)
C REAL*3 N1
C COMMON/AREA2/HILN
C TST = 3.000
C TEST = HILN-0.759697190/DBH
C IF(TEST.LE.0.99900) RETURN
C TST = 10.000 ** TEST
C RETURN
C
END

/*U.D.SYSIN DD *
0.130572232 -0.003337133 -0.00621120 0.519993930
-0.06322713 -0.00020930 -0.00192883 -0.0572069
-0.06322713 -0.00020930 -0.00192883 -0.0572069
0.019993930 -0.0572069 -0.00192883 -0.0572069

0 15.0 42.3 600.0
1 15.0 47.8 750.0
2 12.0 35.2 630.0
3 22.0 36.8 660.0
4 15.0 56.7 750.0
5 17.0 48.6 750.0
*/
**STAND CHARACTERISTICS:**

- **PLT NR.** = 1
- **AGE** = 15.0
- **AVERAGE DIA. OF DBM. AND CDBM. (FEET)** = 42.3
- **SITE INDEX (BASE AGE 25)** = 67.2
- **NUMBER OF TREES PER ACRE SURVIVING** = 800.0

**PREDICTED STAND AVERAGE ATTRIBUTES:**

- **AVERAGE DBH (INCHES)** = 5.0
- **BASAL AREA PER ACRE (SF)** = 142.7
- **MINIMUM DBH (INCHES)** = 2.4
- **MAXIMUM DBH (INCHES)** = 8.4

**BETA PARAMETER ESTIMATES:**

- **A** = 3.49034
- **B** = 3.46474
- **U = 2.12041**
- **DU = 6.108641**
- **UMIN ADJUSTMENT** = 0.57477
- **UMAX ADJUSTMENT** = 0.64241

**PREDICTED DIAMETER, BASAL AREA, AND VOLUME DISTRIBUTIONS.**

<table>
<thead>
<tr>
<th>DBH</th>
<th>NO. TREES</th>
<th>BASAL AREA</th>
<th>TOTAL 100</th>
<th>VOLUME (LUFT)</th>
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</thead>
<tbody>
<tr>
<td>2</td>
<td>0.5</td>
<td>0.02</td>
<td>0.36</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>33.9</td>
<td>1.98</td>
<td>36.24</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>129.0</td>
<td>11.78</td>
<td>217.41</td>
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</tr>
<tr>
<td>5</td>
<td>215.0</td>
<td>29.67</td>
<td>544.89</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>225.3</td>
<td>43.73</td>
<td>810.20</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>148.2</td>
<td>39.03</td>
<td>731.86</td>
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</tr>
<tr>
<td>8</td>
<td>41.9</td>
<td>12.94</td>
<td>302.32</td>
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<tr>
<td>9</td>
<td>41.7</td>
<td>9.71</td>
<td>13.55</td>
<td></td>
</tr>
</tbody>
</table>

| SUM | 800.0     | 142.75     | 2657.03   |
STAND CHARACTERISTICS:

PLOT NO. = 2
AGE = 15.0
AVERAGE H.D. OF TALLER AND COLUMN (FEET) = 47.8
SITE INDEX (BASE AGE 25) = 74.0
NUMBER OF TREES PER ACRE SURVIVING = 730.0

PREDICTED STAND AVERAGE ATTRIBUTES:
AVERAGE DBH (INCHES) = 9.2
BASE AREA PER ACRE (SQRFT) = 158.3
MINIMUM DBH (INCHES) = 3.2
MAXIMUM DBH (INCHES) = 9.2

UPLA PARAMETER ESTIMATES:

A = 3.310851
B = 3.446428
UL = 2.61061
UO = 4.87214
UMIN Adjustment = 0.57331
UMAX Adjustment = 0.64301

PREDICTED DIAMETER, BASE AREA, AND VOLUME DISTRIBUTIONS:

<table>
<thead>
<tr>
<th>DBH</th>
<th>NO. TREES</th>
<th>BASE AREA</th>
<th>TOTAL (DB) VOLUME (CU.FT)</th>
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<tbody>
<tr>
<td>3</td>
<td>8.9</td>
<td>0.53</td>
<td>10.76</td>
</tr>
<tr>
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<td>71.2</td>
<td>6.59</td>
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</tr>
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<td>134.8</td>
<td>21.50</td>
<td>423.33</td>
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<td>147.3</td>
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<td>784.95</td>
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<td>173.1</td>
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<td>941.41</td>
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<td>8</td>
<td>98.4</td>
<td>33.93</td>
<td>704.68</td>
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<td>9</td>
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<td>10.67</td>
<td>224.14</td>
</tr>
<tr>
<td>10</td>
<td>0.4</td>
<td>0.19</td>
<td>3.95</td>
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<tr>
<td>SUM</td>
<td>730.0</td>
<td>158.26</td>
<td>3299.44</td>
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STAND CHARACTERISTICS:

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<th>Plot No. = 3</th>
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<tbody>
<tr>
<td>Age = 12.0</td>
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<tr>
<td>Avg. ht. of dom. and coupm. (feet) = 33.2</td>
</tr>
<tr>
<td>Site Index (base age 25) = 68.0</td>
</tr>
<tr>
<td>Number of trees per acre surviving = 630.0</td>
</tr>
</tbody>
</table>

PREDICTED STAND AVERAGE ATTRIBUTES:

<table>
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<tr>
<th>Attribute</th>
<th>Value</th>
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</thead>
<tbody>
<tr>
<td>Average dbh (inches)</td>
<td>5.3</td>
</tr>
<tr>
<td>Basal area per acre (S.F.)</td>
<td>102.3</td>
</tr>
<tr>
<td>Minimum dbh (inches)</td>
<td>2.3</td>
</tr>
<tr>
<td>Maximum dbh (inches)</td>
<td>7.6</td>
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BETA PARAMETER ESTIMATES:

<table>
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<th>Parameter</th>
<th>Value</th>
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<tr>
<td>B = 2.924352</td>
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</tr>
<tr>
<td>DL = 1.70535</td>
<td></td>
</tr>
<tr>
<td>DU = 6.29212</td>
<td></td>
</tr>
<tr>
<td>UM = 0.57763</td>
<td></td>
</tr>
<tr>
<td>OM = 0.64560</td>
<td></td>
</tr>
</tbody>
</table>

PREDICTED DIAMETER, BASAL AREA, AND VOLUME DISTRIBUTIONS.

<table>
<thead>
<tr>
<th>DBH</th>
<th>NO. TREES</th>
<th>BASAL AREA</th>
<th>TOTAL (sq ft)</th>
<th>VOLUME (cu ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
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<td>0.09</td>
<td>1.75</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4.2</td>
<td>2.27</td>
<td>38.21</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>118.8</td>
<td>10.71</td>
<td>166.75</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>174.0</td>
<td>24.33</td>
<td>374.00</td>
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<td>9.4</td>
<td>5.01</td>
<td>517.35</td>
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<td>49.2</td>
<td>23.92</td>
<td>390.63</td>
<td></td>
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<tr>
<td>8</td>
<td>14.5</td>
<td>4.71</td>
<td>72.51</td>
<td></td>
</tr>
<tr>
<td>SUM</td>
<td>630.0</td>
<td>102.26</td>
<td>1567.20</td>
<td></td>
</tr>
</tbody>
</table>
**STAND CHARACTERISTICS**

PLOT NO. = 4  
AGE = 22.0  
AVG. HT. OF DMH. AND GUDM. (FEET) = 56.8  
SITE INDEX (BASE AGE 25) = 63.0  
NUMBER OF TREES PER ACRE SURVIVING = 860.0

**PREDICTED STAND AVERAGE ATTRIBUTES:**  
AVERAGE DMH (INCHES) = 6.8  
Basal area per acre (SQF) = 175.5  
MINIMUM DMH (INCHES) = 3.4  
MAXIMUM DMH (INCHES) = 10.4

**BETA PARAMETER ESTIMATES:**  
A = 3.012813  
B = 3.570408  
UL = 3.31485  
DL = 1.02162  
DMIN ADJUSTMENT = 0.57517  
DMAX ADJUSTMENT = 0.64397

**PREDICTED DIAMETER, BASAL AREA, AND VOLUME DISTRIBUTIONS.**

<table>
<thead>
<tr>
<th>DMH</th>
<th>NO. TREES</th>
<th>BASAL AREA</th>
<th>TOTAL (Db) VOLUME (CUFT)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.1</td>
<td>0.01</td>
<td>0.19</td>
</tr>
<tr>
<td>4</td>
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</tr>
<tr>
<td>5</td>
<td>97.1</td>
<td>13.62</td>
<td>305.14</td>
</tr>
<tr>
<td>6</td>
<td>134.7</td>
<td>30.64</td>
<td>707.62</td>
</tr>
<tr>
<td>7</td>
<td>165.2</td>
<td>44.20</td>
<td>1039.27</td>
</tr>
<tr>
<td>8</td>
<td>129.2</td>
<td>49.77</td>
<td>1069.97</td>
</tr>
<tr>
<td>9</td>
<td>00.9</td>
<td>29.75</td>
<td>725.98</td>
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<tr>
<td>10</td>
<td>18.0</td>
<td>9.51</td>
<td>233.33</td>
</tr>
<tr>
<td>11</td>
<td>0.5</td>
<td>0.31</td>
<td>7.75</td>
</tr>
<tr>
<td>SUM</td>
<td>860.0</td>
<td>175.51</td>
<td>4144.25</td>
</tr>
</tbody>
</table>
STAND CHARACTERISTICS:

- PLOT NO. = 5
- AGE = 19.0
- AVG. HT. OF DUM. AND COLUM. (FEET) = 56.7
- SITE INDEX (BASE AGE 25) = 70.7
- NUMBER OF TREES PER ACRE SURVIVING = 750.0

PREDICTED STAND AVERAGE ATTRIBUTES:
- AVERAGE DBH (INCHES) = 6.7
- BASAL AREA PER ACRE (SOFT.) = 189.1
- MINIMUM DBH (INCHES) = 3.7
- MAXIMUM DBH (INCHES) = 10.2

DELTA PARAMETER ESTIMATES:
- A = 3.15709
- B = 3.736754
- UL = 3.13047
- UU = 10.83355
- UMIN ADJUSTMENT = 0.57757
- UMAX ADJUSTMENT = 0.64553

PREDICTED DIAMETER, BASAL AREA, AND VOLUME DISTRIBUTIONS:

<table>
<thead>
<tr>
<th>DBH</th>
<th>NO. TREES</th>
<th>BASAL AREA</th>
<th>TOTAL (OB) VOLUME (CUFT)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.4</td>
<td>0.05</td>
<td>1.16</td>
</tr>
<tr>
<td>4</td>
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<td>5.68</td>
<td>82.70</td>
</tr>
<tr>
<td>5</td>
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<td>17.33</td>
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<td>0.34</td>
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<tr>
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<td>0.09</td>
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<tr>
<td>SUM</td>
<td>750.0</td>
<td>189.12</td>
<td>4465.68</td>
</tr>
</tbody>
</table>
**STAND CHARACTERISTICS:**

- **PLANT NO.:** 6
- **AGE:** 17.0
- **AVG. HI. OF DUM. ANU CUDUM. (FEET):** 48.8
- **SITE INDEX (BASE AGE 25):** 67.9
- **NUMBER OF TREES PER ACRE SURVIVING:** 750.0

**PREDICTED STAND AVERAGE ATTRIBUTES:**

- **AVERAGE DUM (INCHES):** 8.1
- **BASAL AREA PER ACRE (SF/AF):** 160.2
- **MINIMUM DUM (INCHES):** 3.2
- **MAXIMUM DUM (INCHES):** 9.3

**BETA PARAMETER ESTIMATES:**

- **A:** 3.290937
- **B:** 3.544330
- **UL:** 2.61015
- **ULU:** 9.89680
- **UMIN ADJUSTMENT:** 0.57502
- **UMAX ADJUSTMENT:** 0.64258

**PREDICTED DIAMETER, BASAL AREA, AND VOLUME DISTRIBUTIONS.**

<table>
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<tr>
<th>DBH</th>
<th>NO. TREES</th>
<th>BASAL AREA</th>
<th>TOTAL DBH VOLUME (CUFT)</th>
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<td>7.12</td>
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**SUM**: 750.0 160.16 3335.48
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COMPATIBLE WHOLE-STAND AND DIAMETER DISTRIBUTION MODELS FOR LOBLOLLY PINE PLANTATIONS

by

JAMES R. FRAZIER

(ABSTRACT)

A method was developed to approximate the diameter distribution of unthinned plantations of loblolly pine from whole stand predictions of stand attributes. The beta probability density function (pdf) and the Weibull pdf were used as models for the diameter distribution function for estimating a stand attribute (such as average diameter at breast height (dbh) or total volume per acre) and the whole stand estimation of the attribute was defined. The given estimates of k stand attributes from whole stand models, the k parameter of the pdf were estimated (recovered).

Two types of parameter recovery models were constructed. The first used equations for the non-central moments of dbh. For the beta pdf, equations for the predicted first moment (average dbh) and second moment (basal area per acre/(0.005454 N)) were used to compute the analytic solution for parameter $a$ and $\beta$. The endpoints of the pdf were defined to be the predicted minimum dbh minus 0.6 inches and maximum dbh plus 0.6. In the case of the Weibull pdf, the parameter $\alpha$ was solved from the coefficient
of variation which itself was computed from the predicted first and second non-central moments. Then given this value of \( a, b \) was solved for using the predicted average dbh. The parameter \( a \) was set equal to one-half predicted minimum dbh.

The second type of parameter recovery model used volume as one of the stand attributes used to solve for the parameters. Due to failure of the numerical solution algorithms to consistently converge to a solution for the parameters, the beta pdf could not be used. For the Weibull though solutions were always possible. The parameters \( a \) and \( c \) were estimated as before and \( b \) was then solved using the diameter distribution yield equation for total volume per acre.

The three models described above achieved a solution for the parameters 100% of the time. Comparison of the three using Kolmogorov-Smirnov statistics showed all three to be nearly equal in their ability to approximate the diameter distribution of stands. When compared to other conventional diameter distribution prediction methods (Burkhart and Strub 1974 and Smalley and Bailey 1974a), the parameter recovery models proved as good as or better than these other methods.

In unthinned loblolly pine plantations the parameter recovery models proved to be a feasible alternative for
predicting diameter distributions. The major advantage of the models is the numerical compatibility of the whole stand estimates of stand attributes and the diameter distribution estimates. Thus given whole stand estimates, such as basal area per acre or total cubic-foot volume per acre, the distribution of these attributes by diameter classes can be obtained.